Forecasting

in Economics, Business, Finance and Beyond

Time Series: A Components Perspective
CH 5: Trend and Seasonality

Time-Series Data

- Numerical data obtained at regular time intervals
- The time intervals can be annually, quarterly, monthly, weekly, daily, hourly, etc.
- Example:

Year:	2005	2006	2007	2008	2009
Sales:	75.3	74.2	78.5	79.7	80.2

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Time Plot

A time-series plot (time plot) is a two-dimensional plot of time series data

the vertical axis measures the variable of interest

the horizontal axis corresponds to the time periods

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Time-Series Components

Time Series

Trend Component	Seasonal Component	Cyclical Component	Irregular /Random Component
Overall, persistent, long-term movement	Regular periodic fluctuations, usually within a 12-month period	Repeating swings or movements over more than one year	Erratic or residual fluctuations

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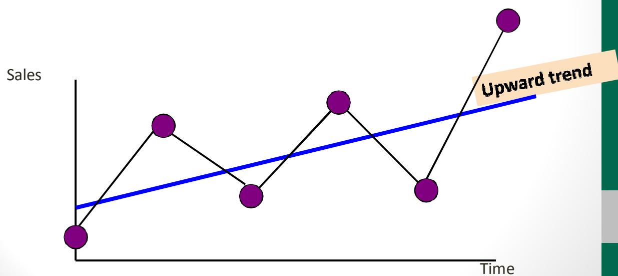
Trend Component

- Long-run increase or decrease over time
  (overall upward or downward movement)
- Data taken over a long period of time



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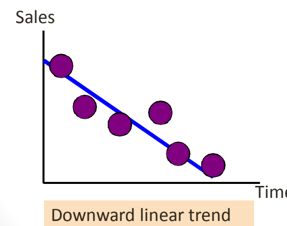
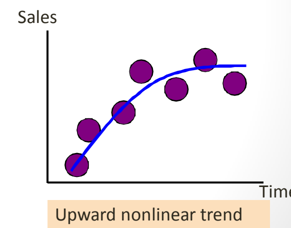
Trend Component

(continued)

- Trend can be upward or downward
- Trend can be linear or non-linear





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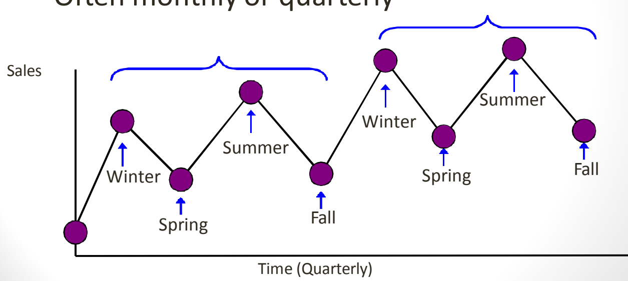
Seasonal Component

- Short-term regular wave-like patterns
- Observed within 1 year
- Often monthly or quarterly



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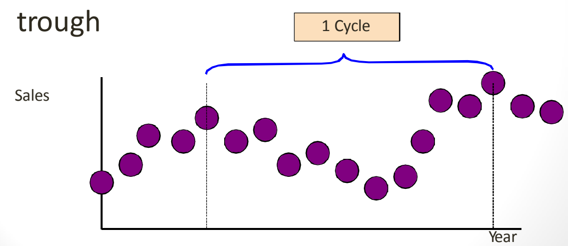
Cyclical Component

- Long-term wave-like patterns
- Regularly occur but may vary in length
- Often measured peak to peak or trough to trough



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Irregular/Random Component

- Unpredictable, random, “residual” fluctuations
- “Noise” in the time series
- The truly irregular component may not be estimated – however, the more predictable random component can be estimated – and is usually the emphasis of time series analysis via the usual stationary time series models such as AR, MA, ARMA etc after we filter out the trend, seasonal and other cyclical components

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Two simplified time series models

- In the following, we present two classes of simplified time series models
    1. <u>Non-seasonal Model</u> with Trend
    2. <u>Classical Decomposition Model</u> with Trend and Seasonal Components
- The usual procedure is to first filter out the trend and seasonal component – then fit the random component with a stationary time series model to capture the correlation structure in the time series
- If necessary, the entire time series (with seasonal, trend, and random components) can be re-analyzed for better estimation, modeling and prediction.

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Non-seasonal Models with Trend

$$\boxed{\underset{\text{Stochastic process}}{X_t} = \underset{\text{trend}}{m_t} + \underset{\text{random noise}}{Y_t}}$$

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Non-seasonal Models with Trend

There are two basic methods for estimating/eliminating trend:

Method 1: Trend estimation
(first we estimate the trend either by moving average smoothing or regression analysis – then we remove it)

Method 2: Trend elimination by differencing

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SOME COMMON TREND FUNCTIONS

m_t	restrictions	trend name
$m_t = a_0 + a_1 t$	$a_1 \neq 0$	linear trend
$m_t = a_0 + a_1 t + a_2 t^2$	$a_2 \neq 0$	quadratic trend
$m_t = a_0 + a_1 t + a_2 t^2 + \ldots + a_p t^p$	$a_p \neq 0$	polynomial trend of degree p
$m_t = a_0 e^{a_1 t}$	$a_0, a_1 \neq 0$	exponential trend
$m_t = e^{a_0 + a_1 b^t}$	$0 < b < 1$	Gompertz trend
$m_t = \dfrac{a_0}{1 - a_1 e^{-a_2 t}}$	$a_0, a_1 > 0$	logistic trend

Sometimes m_t can be determined by theory, or at least the standard theory may suggest the form that m_t should take. Otherwise plot the data and use one of the methods given on the following slides.

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Estimating Trend

Use Least Squares.

Pick the form of the trend and let t be the explanatory variable.

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Method 1: Trend Estimation by Regression Analysis

- Estimate a trend line using regression analysis

Year	Time Period (t)	Sales (X)
2004	0	20
2005	1	40
2006	2	30
2007	3	50
2008	4	70
2009	5	65

- Use time (t) as the independent variable:

$$\hat{X}_t = \hat{\beta}_0 + \hat{\beta}_1 t$$

In least squares linear, non-linear, and exponential modeling, time periods are numbered starting with 0 and increasing by 1 for each time period.

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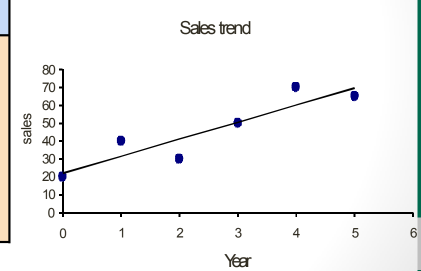
Least Squares Regression

Without knowing the exact time series random error correlation structure, one often resorts to the ordinary least squares regression method, not optimal but practical.

Year	Time Period (t)	Sales (X)
2004	0	20
2005	1	40
2006	2	30
2007	3	50
2008	4	70
2009	5	65

The estimated linear trend equation is:

$$\hat{X}_t = 21.905 + 9.571t$$



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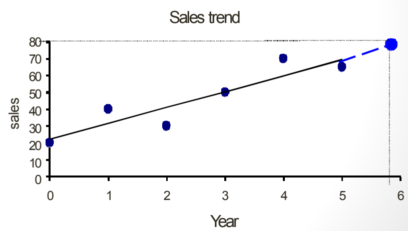
Linear Trend Forecasting

One can even perform trend forecasting at this point – but bear in mind that the forecasting may not be optimal.

- Forecast for time period 6 (2010):

Year	Time Period (t)	Sales (X)
2004	0	20
2005	1	40
2006	2	30
2007	3	50
2008	4	70
2009	5	65
2010	6	??

$$\hat{X}_t = 21.905 + 9.571 * \boxed{6} = 79.33$$



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Smoothing Using a Linear Filter

Consider the filter given by

$$\theta_j = \begin{cases} \dfrac{1}{2q+1} & -q \le j \le q;\ j \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

We apply this filter to the time series x_t as follows:

$$W_t = \sum_{j=-q}^{q} \theta_j X_{t-j} = \frac{1}{2q+1} \sum_{j=-q}^{q} X_{t-j}$$

The effect is to smooth the extreme values in the time series, thus lessening their effect.

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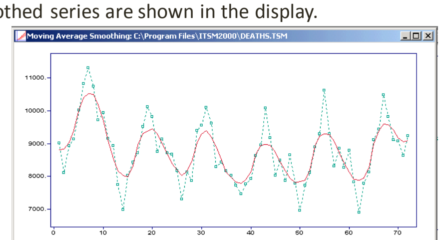
Smoothing Using a Linear Filter

Example: Open the DEATHS data in ITSM by clicking **File → Project → Open** and selecting the file **DEATHS.TSM**. We will smooth this with a filter using q = 2. Click **Smooth → Moving Ave.** and enter the order q = 2. Enter 1 for each of the theta values. Note that ITSM rescales the theta values correctly; i.e., θ_j = 0.2 in this example. Click OK. The original time series and the smoothed series are shown in the display.



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Smoothing Using a Linear Filter

We can use the original time series X_t or the filtered time series W_t to estimate trend, and it can be shown that

$$\text{Var}[W_t] \to 0 \quad \text{as} \quad q \to \infty$$

In practice, a larger window (q) on the filter decreases the number of values of *t* for which m_t may be estimated, so you trade decreased range for increased accuracy.

I tend to use a linear filter ONLY to better see the form of the trend in a time series.

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Smoothing Using a Linear Filter

Linear filters can be designed to pass higher degree polynomial trend without distortion. In addition, we can convolute several filters to obtain another filter.

Convoluting a filter means to pass the filter through a filter.

e.g. $\left\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\right\}\otimes\left\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\right\}=\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\{1,1,1\}\otimes\{1,1,1\}$

$$=\left(\frac{1}{9}\right)\{1,2,3,2,1\}=\left\{\frac{1}{9},\frac{2}{9},\frac{3}{9},\frac{2}{9},\frac{1}{9}\right\}$$

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Smoothing Using a Linear Filter

Tetley (1946) used a 15-point symmetric filter (called Spenser's 15-point filter) to smooth mortality statistics and obtain life tables. This filter is as follows:

$$\{\theta_{-7},\theta_{-6},\ldots,\theta_{-1},\theta_0,\theta_1,\ldots,\theta_6,\theta_7\}=\{\ldots,\theta_0,\theta_1,\ldots,\theta_6,\theta_7\}$$
$$=\tfrac{1}{320}\{\ldots,74,67,46,21,3,-5,-6,-3\}$$

and can be constructed using the convolutions

$$\left\{\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right\}\otimes\left\{\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right\}\otimes\left\{\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\right\}\otimes\left\{\frac{-3}{4},\frac{3}{4},1,\frac{3}{4},\frac{-3}{4}\right\}$$

This filter allows cubic trends to pass unaffected.

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Method 2: Trend Elimination by Differencing

Define the lag-1 **difference operator**,

$$\nabla X_t = X_t - X_{t-1} = (1-B)X_t$$

where B is the **backshift** operator, $BX_t = X_{t-1}$.

$$\nabla^k X_t = \nabla(\nabla^{k-1}X_t) \text{ and } \nabla^1 X_t = \nabla X_t$$

Trend Elimination by Differencing

If the operator ∇ is applied to a linear trend function:

$$\boldsymbol{m_t = \beta_0 + \beta_1 t}$$

Then we obtain the constant function:

$$\boldsymbol{\nabla m_t = m_t - m_{t-1} = \beta_0 + \beta_1 t - \beta_0 - \beta_1(t-1) = \beta_1.}$$

In the same way any polynomial trend of degree k can be removed by the operator: ∇^k

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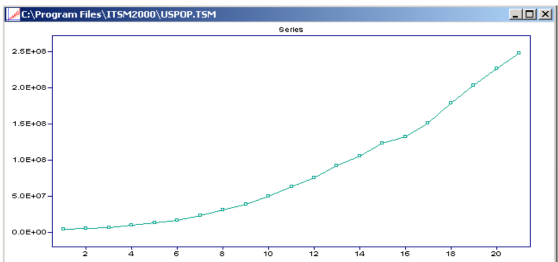
Trend Elimination by Differencing

Example: Open the USPOP data in ITSM by clicking **File** $\rightarrow$ **Project** $\rightarrow$ **Open** and selecting the file **USPOP.TSM**. You should see the following graph:



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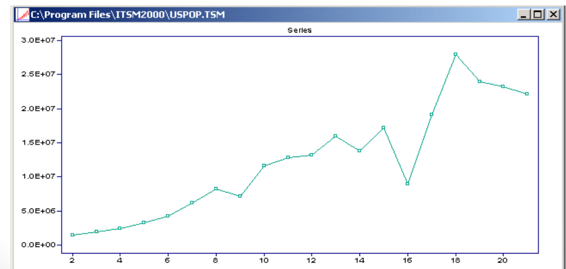
Trend Elimination by Differencing

Example: We can see the series ∇x_t by clicking **Transform** $\rightarrow$ **Difference**, entering a lag of 1, and clicking the OK button. You should see the following graph:



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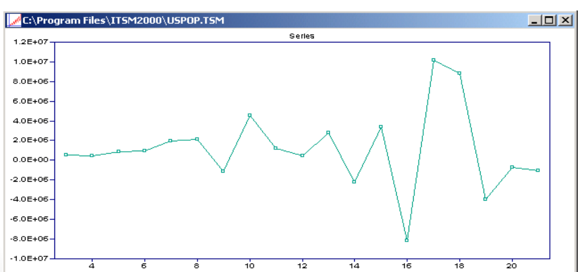
Trend Elimination by Differencing

Example: We can see the series $\nabla^2 x_t$ by repeating the above steps; i.e., by clicking **Transform** $\rightarrow$ **Difference**, entering a lag of 1, and clicking the OK button. You should see the following graph:



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Trend Elimination by Differencing

Example: Note that the resulting series does not have a constant variance. This can be remedied by taking the natural log of each x_t prior to differencing. This transformation is a special case of a **Box – Cox** transformation (named after two famous statisticians). Box – Cox transformations are used to stabilize variance when the variance changes over time.

The Box – Cox transformation is given by

$$f_\lambda(y) = \begin{cases} \dfrac{y^\lambda - 1}{\lambda} & \lambda \neq 0 \\ \ln(y) & \lambda = 0 \end{cases}$$

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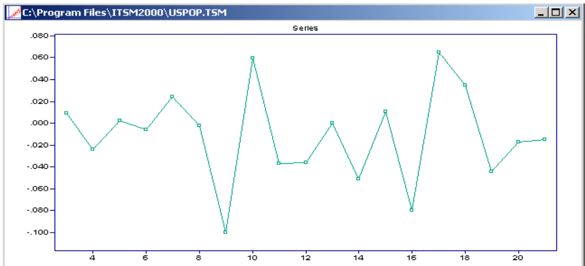
Trend Elimination by Differencing

- **Example:** Click **Transform** → **Undo Differencing** twice to return to the original data, then click **Transform** → **Box – Cox** and enter the parameter $\lambda = 0$ (or use the slider). Now redo the two lag 1 differences. You should see the following graph:



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Classical Decomposition Model with Trend and Seasonality

$$\boxed{X_t = m_t + s_t + Y_t}$$

Stochastic process — trend — seasonal component — Cycles + random noise

The "cycles + random noise" includes any patterns patterns that we can model that are not captured by trend and seasonality.

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Classical Decomposition Model (Seasonal Model) with trend and seasonality

$$X_t = m_t + s_t + Y_t, \qquad t = 1, \cdots, n,$$

where

$$EY_t = 0, \; s_{t+d} = s_t, \; and \; \textstyle\sum_{j=1}^{d} s_j = 0.$$

Classical Decomposition Model

Method 1: Filtering: First we estimate and remove the trend component by using a moving average filter; then we estimate and remove the seasonal component by using suitable periodic averages.

Method 2: Differencing: First we remove the seasonal component by differencing. We then remove the trend by differencing as well.

Method 3: Joint-fit method: Alternatively, we can fit a combined polynomial linear regression and harmonic functions to estimate and then remove the trend and seasonal component simultaneously.

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Method 1: Filtering

We can eliminate seasonality of lag *d* by using a symmetric filter such as

$$\theta = \left\{ \theta_{\frac{-(d-1)}{2}}, \ldots, \theta_{-1}, \theta_0, \theta_1, \ldots, \theta_{\frac{-(d-1)}{2}} \right\}$$

where

$$\theta_j = \begin{cases} 1/d \quad \forall\ j = \frac{-(d-1)}{2}, \ldots, -1, 0, 1, \ldots, \frac{d-1}{2} & \text{if } d \text{ is odd} \\ 1/d \quad \forall\ j = \frac{-d}{2} + 1, \ldots, -1, 0, 1, \ldots, \frac{d}{2} - 1 & \\ 1/2d \quad \text{for } j = \frac{-d}{2} \text{ or } j = \frac{d}{2} & \text{if } d \text{ is even} \end{cases}$$

For example, if d = 3 use $\theta = \left\{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\}$

and if d = 4 use $\theta = \left\{ \frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8} \right\}$

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Method 2: Differencing

Define the lag-d differencing operator ∇_d *as:*

$$\nabla_d X_t = X_t - X_{t-d} = (1 - B^d) X_t$$

We can transform a seasonal model to a non-seasonal model:

$$\begin{aligned} \nabla_d X_t &= m_t + s_t + Y_t - m_{t-d} - s_{t-d} - Y_{t-d} \\ &= m_t - m_{t-d} + Y_t - Y_{t-d} \end{aligned}$$

Differencing can then be further applied to eliminate the trend component.

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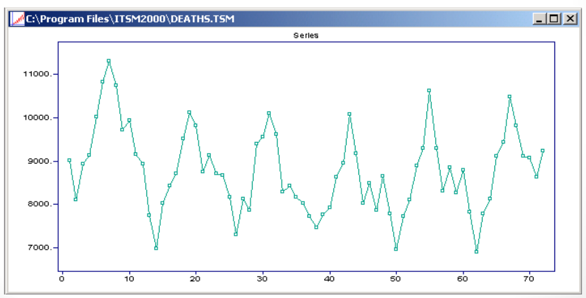
Method 2: Differencing

Example: Once again we will use the DEATHS data as an example. Open the DEATHS data in ITSM by clicking **File** → **Project** → **Open** and selecting the file **DEATHS.TSM**. You should see the following graph:



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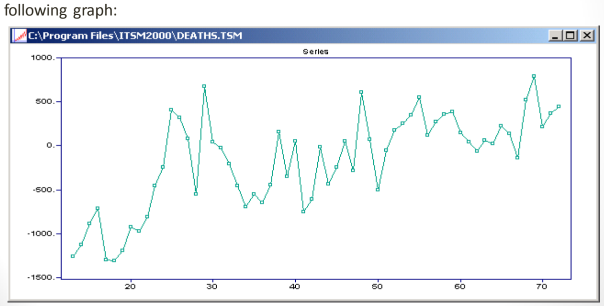
Method 2: Differencing

Example: This data exhibits quadratic trend and seasonality at lag 12. This suggests that applying ∇_{12} to the data should eliminate the seasonality and reduce the trend from quadratic to linear. We can check this by clicking **Transform** → **Difference**, entering a lag of 12, and clicking the OK button. You should see the following graph:



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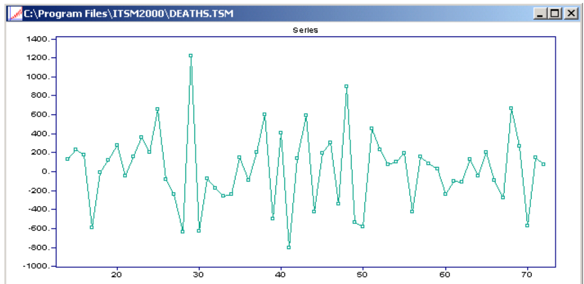
Method 2: Differencing

Example: Note the expected linear trend. We can eliminate this by applying ∇ to the series $\nabla_{12}X_t$. Click **Transform** → **Difference**, entering a lag of 1, and clicking the OK button. You should see the following graph of $\nabla\nabla_{12}X_t$:



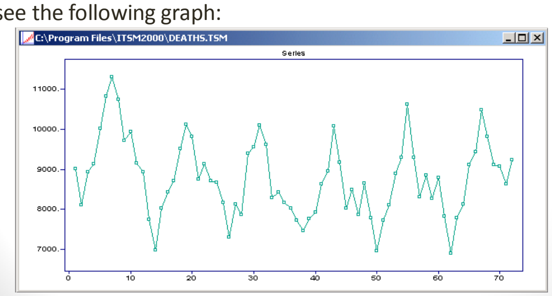
Method 2: Differencing

Example: Note that the above differencing operations have cost us some data points. In general, applying ∇_d costs us the first d data points of the original time series, since the first difference for which we have data is $X_{d+1} - X_1$. For the DEATHS data, applying $\nabla\nabla_{12}X_t$ cost us the first 13 data points.

Method 3: Joint Modeling

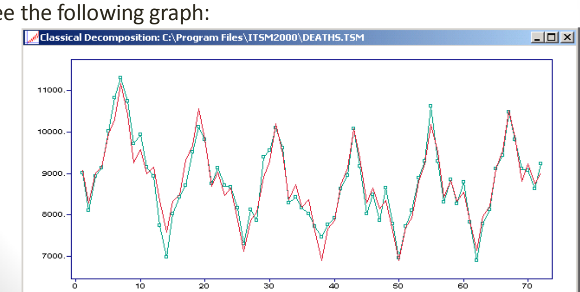
One can also fit a joint model to analyze both trend and seasonal components simultaneously.

Example: Open the DEATHS data in ITSM by clicking **File** → **Project** → **Open** and selecting the file **DEATHS.TSM**. You should see the following graph:



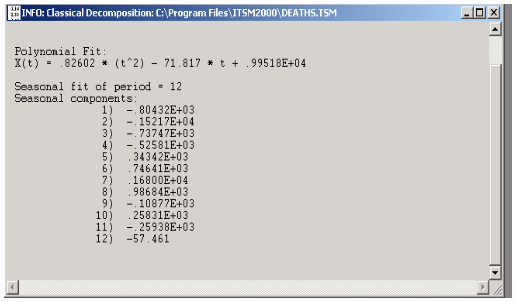
Method 3: Joint Modeling

Example: Click **Transform** → **Classical** to open a dialog box. Click the **Seasonal Fit** checkbox and enter 12 as the period *d*. Make sure the **Polynomial Fit** checkbox is checked, and also that the **Quadratic Trend** checkbox is checked. Click the **OK** button. To see the results, click **Transform** → **Show Classical Fit**. You should see the following graph:



Method 3: Joint Modeling

Example: You can view the parametric forms of $\hat{m}_t$ and $\hat{s}_k$ by clicking the  icon in the upper left corner and selecting **Info**. You should see the following window:



```
INFO: Classical Decomposition: C:\Program Files\ITSM2000\DEATHS.TSM

Polynomial Fit:
X(t) = .82602 * (t^2) - 71.817 * t + .99518E+04

Seasonal fit of period = 12
Seasonal components:
               1)  -.80432E+03
               2)  -.15217E+04
               3)  -.73747E+03
               4)  -.52581E+03
               5)  .34342E+03
               6)  .74641E+03
               7)  .16800E+04
               8)  .98684E+03
               9)  -.10877E+03
              10)  .25831E+03
              11)  -.25938E+03
              12)  -57.461
```

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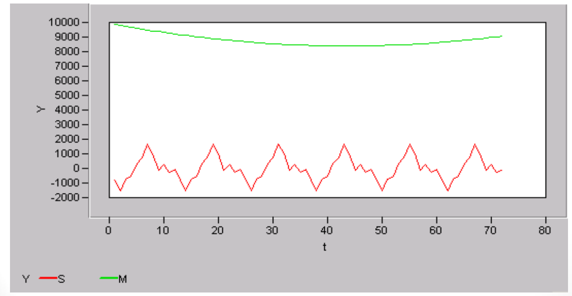
Method 3: Joint Modeling

Example: This graph shows the estimated components

$$\hat{m}_t = M;\ \hat{s}_t = S$$



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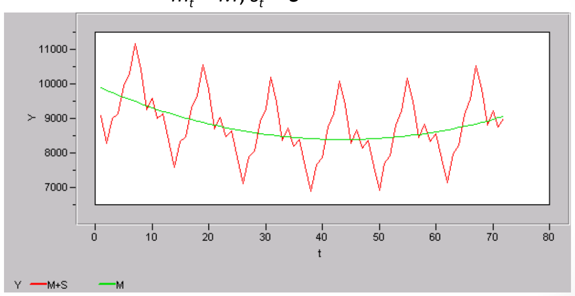
Method 3: Joint Modeling

Example: This graph shows the combined components

$$\hat{m}_t = M;\ \hat{s}_t = S$$



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