Review

AR(1)

AR(1) Models: Review

Realizations are wandering or oscillating depending on whether the root of the characteristic equation is positive or negative, respectively.

Autocorrelations are damped exponentials or damped oscillating exponentials depending on whether the root of the characteristic equation is positive or negative, respectively.

Spectral densities have a peak at $f = 0$ or $f = .5$ depending on whether the root of the characteristic equation is positive or negative, respectively.

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Autoregressive Model of Order (Lag) 2

Autoregressive Model of Order 2: AR(2)

$$X_t = \beta + \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + a_t$$

where $\beta = (1 - \varphi_1 - \varphi_2)\mu$

Notes:

- This model "looks like" a multiple regression model with two independent variables.
    - But in this case, the "independent variables" are values of the dependent variable at the two previous time periods.
- It specifies that the value at time t is a linear combination of values at the two previous time periods plus a random noise component that enters the model at time *t*.

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AR(2) | Stationarity

Facts about Stationary AR(2) Processes

Expected value

$E[X_t] = \mu$ (as in the AR(1) case)

Variance

$$\sigma_X^2 = \frac{\sigma_a^2}{1 - \varphi_1\rho_1 - \varphi_2\rho_2}$$

Autocorrelations and spectral density

ρ_k

$S_X(f)$

The fundamental behavior of the autocorrelations and spectral density depends on whether the roots of the characteristic equation are real or complex.

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AR(2) | Zero Mean Form

AR(2) Processes: Zero Mean Form

Zero mean form

Note that if $\mu = 0,$ the AR(2) model takes the form

$$X_t = \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + a_t$$

- as in the AR(1) case, this is called the “zero mean” form of the model and we sometimes use it for convenience
- alternative form of zero mean AR(2) model

  $$X_t - \varphi_1 X_{t-1} - \varphi_2 X_{t-2} = a_t$$

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AR(2) | Backshift Operator

AR(2): Backshift Operator Notation

Backshift operator notation for AR(2)

Earlier we defined $BX_t = X_{t-1}$.

More generally we can define $\boxed{B^k X_t = X_{t-k}}$

Using this more general definition of the backshift operator, we can rewrite:

$$X_t - \varphi_1 X_{t-1} - \varphi_2 X_{t-2} = a_t$$

as $$X_t - \varphi_1 B X_t - \varphi_2 B^2 X_t = a_t$$

or $$(1 - \varphi_1 B - \varphi_2 B^2) X_t = a_t$$

or $$\varphi(B) X_t = a_t$$

where $\varphi(B)$ is the operator

$$\varphi(B) = 1 - \varphi_1 B - \varphi_2 B^2$$

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AR(2) | Characteristic Equation and Stationarity

AR(2) Characteristic Equation and Stationarity

Light board

Define key result

Example:

$X_t + .2X_{t-1} + .7X_{t-2} = a_t$

$X_t + .2BX_t + .7B^2X_t = a_t$

$1 + .2z + .7z^2 = 0$ be sure and define characteristic equation for AR(2) and so on

AR(2) Key Result about Stationarity

Key result

An AR(2) model is stationary if and only if the roots of the characteristic equation are greater than 1 in absolute value (lie outside the unit circle).

AR(2) Characteristic Polynomial and Equation

Light board?

Consider re-naming:

Characteristic polynomial and characteristic equation

4. Corresponding to the polynomial operator $1-\varphi_1 B-\varphi_2 B^2$ is the **characteristic polynomial** $1-\varphi_1 z-\varphi_2 z^2$ obtained by replacing the operator B with the numeric value z (which may be real or complex)

5. $1-\varphi_1 z-\varphi_2 z^2=0$ is called the **characteristic equation**
   - this equation will have two roots, denoted r_1 and r_2
   - either both r_1 and r_2 are real or they appear as complex conjugate pairs, $r_1=a+bi$ and $r_2=a-bi$

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AR(2) | Real Roots

AR(2) Key Result about Stationarity

Key result

An AR(2) model is stationary if and only if the roots of the characteristic equation are greater than 1 in absolute value.

Are the following AR(2) models stationary?

a. $X_t - .2X_{t-1} - .48X_{t-2} = a_t$
b. $X_t - 1.6X_{t-1} + .15X_{t-2} = a_t$
c. $X_t - 1.6X_{t-1} + .8X_{t-2} = a_t$

Note: It is essentially impossible to tell by simple examination of the coefficients. We will use the key result above to make the determination.

Are These AR(2) Models Stationary?

a. $X_t - .2X_{t-1} - .48X_{t-2} = a_t$

Operator notation: $(1 - .2B - .48B^2)X_t = a_t$

Characteristic equation: $1 - .2z - .48z^2 = 0$

What are the roots?

Factoring we get $(1 - .8z)(1 + .6z) = 0$

the roots are $r_1 = 1/.8 = 1.25$ and $r_2 = -1/.6 = -1.33$

Both roots are >1 ***in absolute* value,**

so the model **is stationary**

Are These AR(2) Models Stationary?

b. $X_t - 1.6X_{t-1} + .15X_{t-2} = a_t$

Operator notation: $(1 - 1.6B + .15B^2)X_t = a_t$

Characteristic equation: $1 - 1.6z + .15z^2 = 0$

What are the roots?

Factoring we get $(1 - 1.5z)(1 - .1z) = 0$

the roots are $r_1 = 1/1.5 = .67$ and $r_2 = 1/.1 = 10$

Since $\boldsymbol{r_1 = .67 < 1}$ **in *absolute* value,**

the model **is not stationary**

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AR(2) | Imaginary Roots

Light board on imaginary roots and quadratic formula

With full example

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AR(2) | Example with Imaginary Roots

Are These AR(2) Models Stationary?

ICC

c. $X_t - 1.6X_{t-1} + .8X_{t-2} = a_t$

Operator notation: $(1 - 1.6B + .8B^2)X_t = a_t$

Characteristic equation: $1 - 1.6z + .8z^2 = 0$

What are the roots?

This quadratic can't be factored as in the other two cases, so we use the quadratic formula:

$$az^2 + bz + c = 0 \text{ has roots } \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Modulus

The roots are $r, r^* = \dfrac{1.6 \pm \sqrt{1.6^2 - 4(.8)(1)}}{2(.8)} = 1 \pm .5i$

In both cases, $|r| = \sqrt{1 + .5^2} = 1.12$

Since $|r| > 1$ the model **is stationary**

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AR(2) | Two Real Roots

Case 1: Characteristic Equation for AR(2) Model Has Two Real Roots

a. $X_t - .2X_{t-1} - .48X_{t-2} = a_t$ stationary with two real roots

Operator notation: $(1 - .2B - .48B^2)X_t = a_t$

Characteristic equation:

Factored form

$$1 - .2z - .48z^2 = (1 - .8z)(1 + .6z) = 0$$

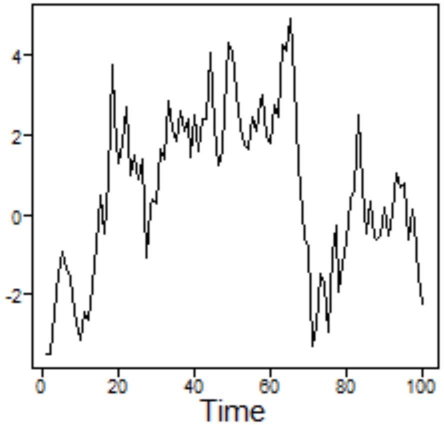
Note: The operator form $(X_t - .2B - .48B^2)X_t = a_t$ can also be factored as $(1 - .8B)(1 + .6B)X_t = a_t$

To understand the behavior of the AR(2) model above, we first consider the two AR(1) models associated with the two first order factors

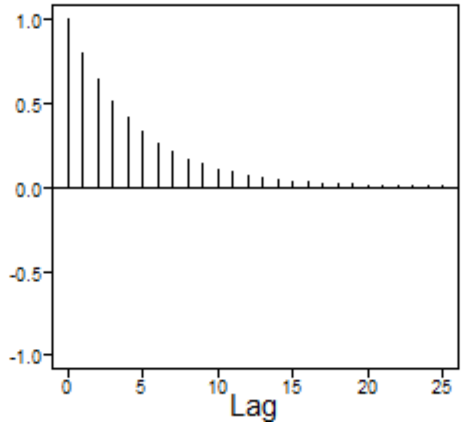
$(1 - .8B)X_t = a_t$ and $(1 + .6B)X_t = a_t$

$$\boxed{X_t - .8X_{t-1} = a_t}$$

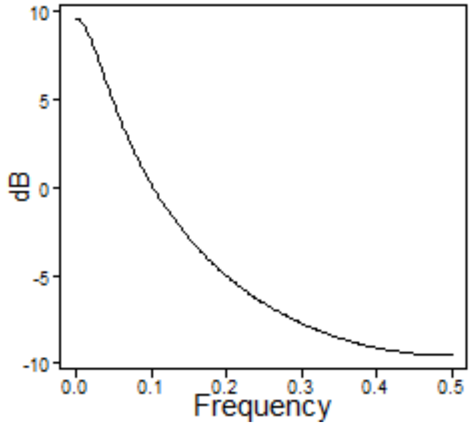
Realization from model above



Autocorrelations



Spectral density



In operator form: $(X_t - .8B)X_t = a_t$

Characteristic equation: $1 - .8z = 0$ and $r = 1.25$

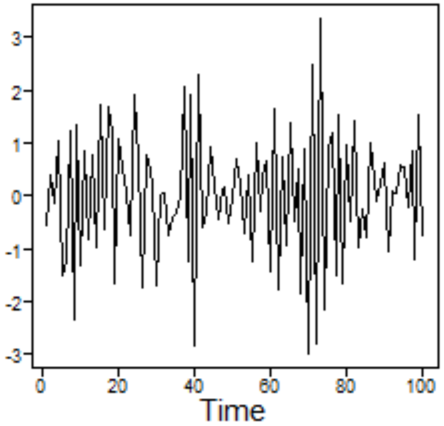
ρ_k is damped exponential

$S_X(f)$ has peak at zero

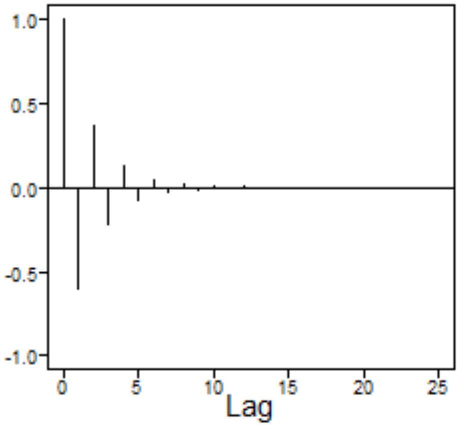
Realization has wandering, aperiodic behavior

$$X_t + .6X_{t-1} = a_t$$

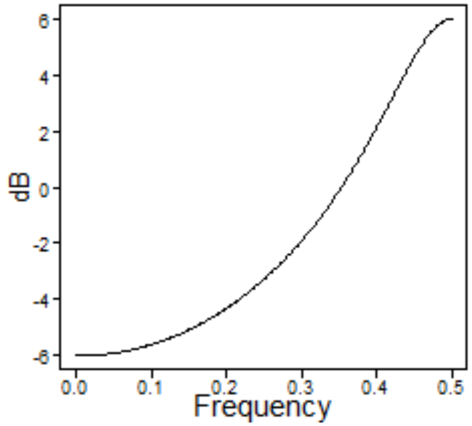
Realization from model above



Autocorrelations



Spectral density



In operator form: $(X_t + .6B)X_t = a_t$

Characteristic equation: $1 + .6z = 0$ and $r = -1.33$

ρ_k is a quickly damping oscillating exponential

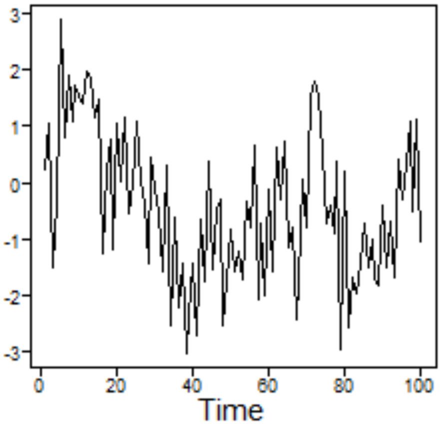
$S_X(f)$ has peak at .5 indicative of the up-down behavior

Realization has up-downup (i.e. cycle length 2 behavior)

Returning to Stationary AR(2) Model with Two Real Roots (One Positive and One Negative)

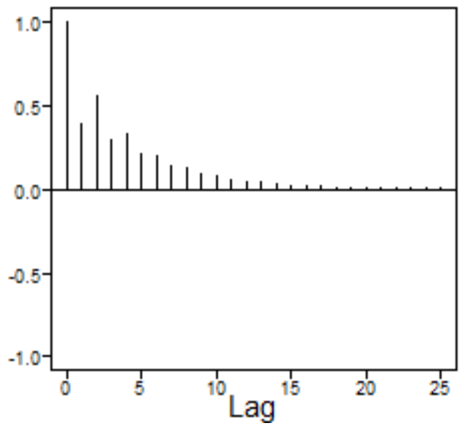
a. $$X_t - .2X_{t-1} - .48X_{t-2} = (1 - .2B - .48B^2)X_t$$
$$= (1 - .8B)(1 + .6B)X_t$$
$$= a_t$$

Realization from model above



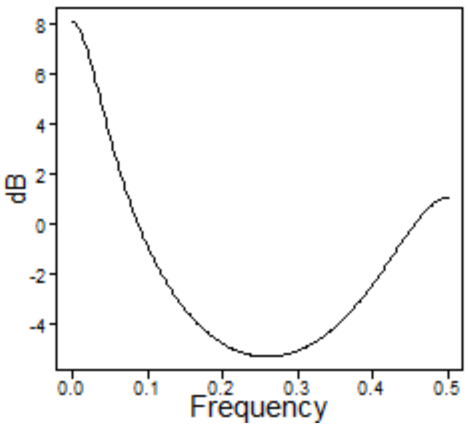
Wandering (due to $1 - .8B$) and high-frequency behavior (due to $1 + .6B$)

Autocorrelations



Damped exponential (due to $1 - .8B$) with a hint of oscillatory behavior for small lags (due to $1 + .6B$)

Spectral density

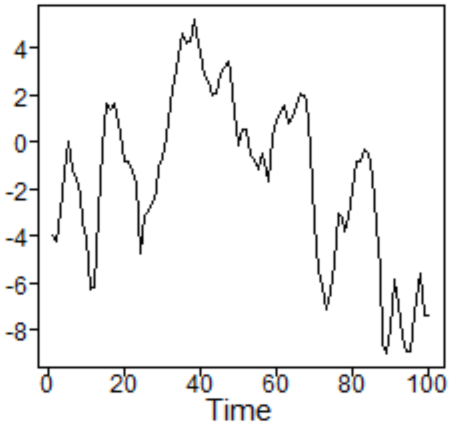


Very easy to see peak at zero (due to $1 - .8B$) and peak at $.5$ (due to $1 + .6B$)

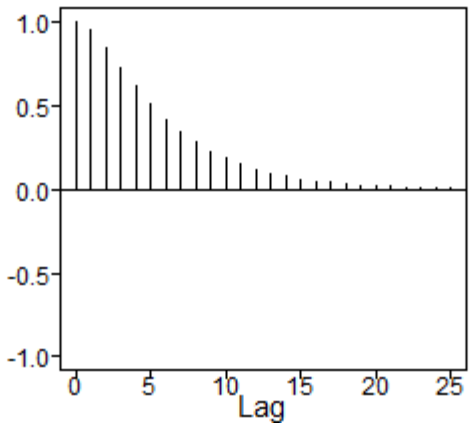
Stationary AR(2) Model with Two Positive Real Roots

$$\begin{aligned} X_t - 1.4X_{t-1} + .48X_{t-2} &= (1 - 1.4B + .48B^2)X_t \\ &= (1 - .8B)(1 - .6B)X_t \\ &= a_t \end{aligned}$$

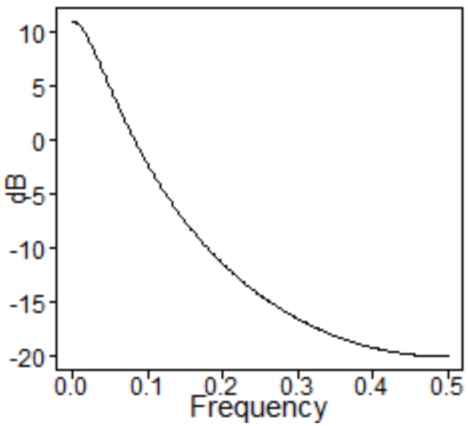
Realization from model above



Autocorrelations



Spectral density



Notice that this model has two positive real roots. We see that the behaviors of all three plots are similar to those for the AR(1) model, $(1 - .8B)X_t$, which had one real root.

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AR(2) | tswge

Two Real Roots

Stationary AR(2) Model with Two Negative Real Roots

$$\begin{aligned}X_t + 1.4X_{t-1} + .48X_{t-2} &= (1 + 1.4B + .48B^2)X_t \\ &= (1 + .8B)(1 + .6B)X_t \\ &= a_t\end{aligned}$$

Notice that this model has two negative real roots. Behaviors of the plots (realization, autocorrelations, and spectral density) are similar to those for AR(1) models with a negative real root.

Try it.

With tswge?

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AR(2) | Complex Conjugate Roots

Case 2: Stationary AR(2) Model with a Pair of Complex Conjugate Roots

Recall the stationary AR(2) model.

c. $X_t - 1.6X_{t-1} + .8X_{t-2} = a_t$

Recall, this model was associated with the characteristic equation: $1 - 1.6z + .8z^2 = 0$ which had the complex conjugate roots $1 \pm .5i$ that are greater than one in absolute value.

Stationary AR(2) Model with a Pair of Complex Conjugate Roots

Key facts

- A stationary AR(2) model, $(1 - \varphi_1 B - \varphi_1 B^2)X_t = a_t$, whose characteristic equation has complex conjugate roots, has an autocorrelation function that has the appearance of a *damped sinusoidal curve.*
- This sinusoidal function has frequency f_0, where

$$f_0 = \frac{1}{2\pi}\cos^{-1}\left(\frac{\varphi_1}{2\sqrt{-\varphi_2}}\right)$$

Stationary AR(2) Model with a Pair of Complex Conjugate Roots

Returning to the AR(2) model

c. $X_t - 1.6X_{t-1} + .8X_{t-2} = a_t$

it is seen that $\varphi_1 = 1.6$ and $\varphi_2 = -.8$

(Again, be careful about the signs)

So, $f_0 = \dfrac{1}{2\pi}\cos^{-1}\left(\dfrac{\varphi_1}{2\sqrt{-\varphi_2}}\right)$

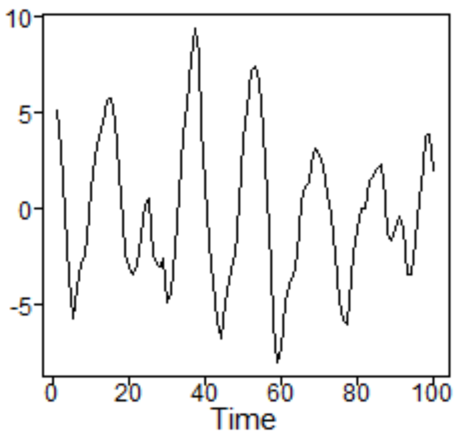
$$= \frac{1}{2\pi}\cos^{-1}\left(\frac{1.6}{2\sqrt{.8}}\right)$$

$$= .0738$$

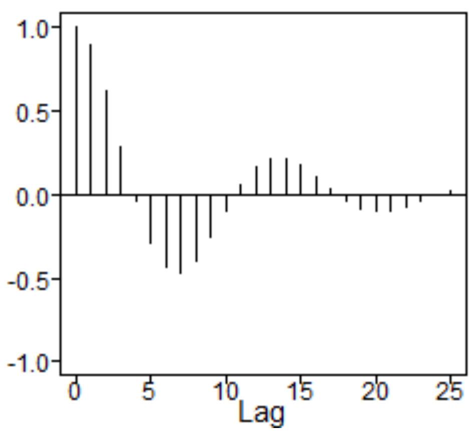
Stationary AR(2) Model with a Pair of Complex Conjugate Roots

c. $X_t - 1.6X_{t-1} + .8X_{t-2} = a_t$

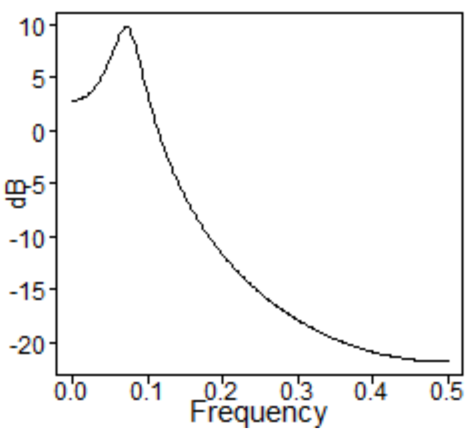
Realization from model above



Autocorrelations



Spectral density



The realization is pseudo-cyclic with about 7 cycles in the series of length 100. That is, the cycle length is about $100/7 \approx 14$.

Autocorrelations have a damped sinusoidal behavior with cycle length about 13 or 14, which is consistent with $f_0 = .0738$.

Spectral density has a peak at about $.07 \approx f_0$.

Stationary AR(2) Model with a Pair of Complex Conjugate Roots

Summarizing: A stationary AR(2) model, $(1-\varphi_1 B-\varphi_1 B^2)X_t=a_t$, whose characteristic equation has complex conjugate roots

- Has realizations that show a *pseudo-cyclic behavior* with cycle length about $1/f_0$, where f_0 is given on previous slides
- The autocorrelation function has the appearance of a *damped sinusoidal* with frequency f_0 (i.e., cycle length $1/f_0$)
- The spectral density has a *peak at about* f_0

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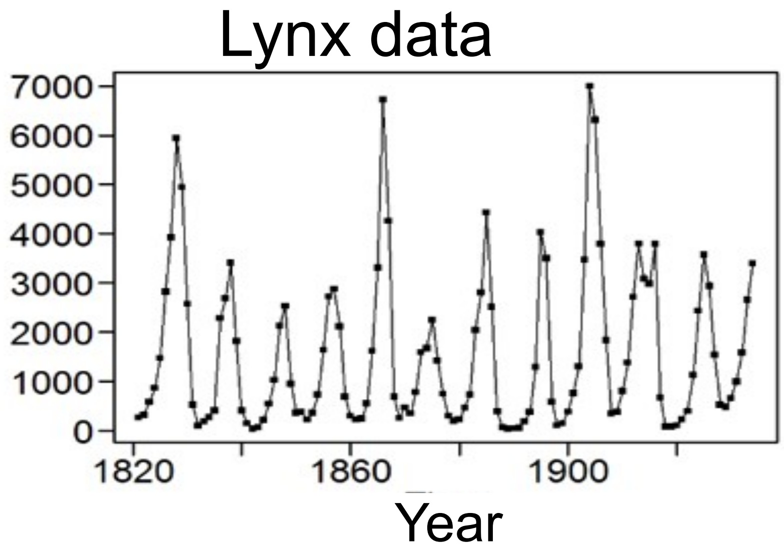
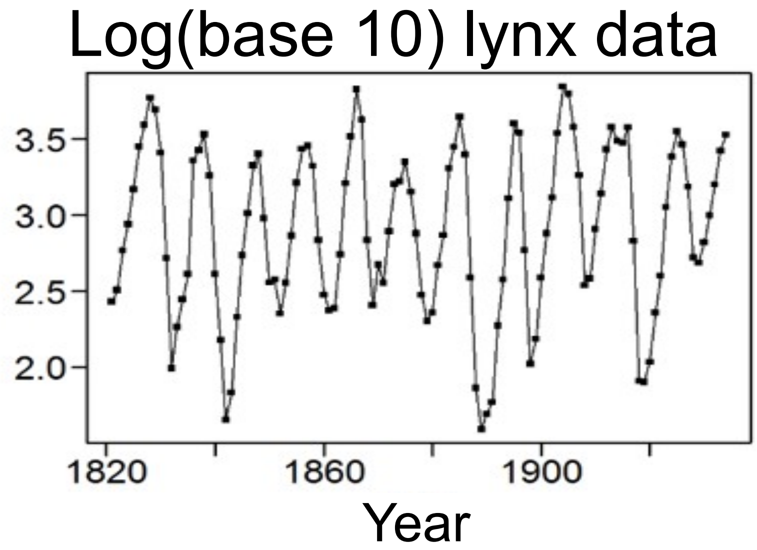
AR(2) | Example

Canadian Lynx Data

Canadian Lynx Data

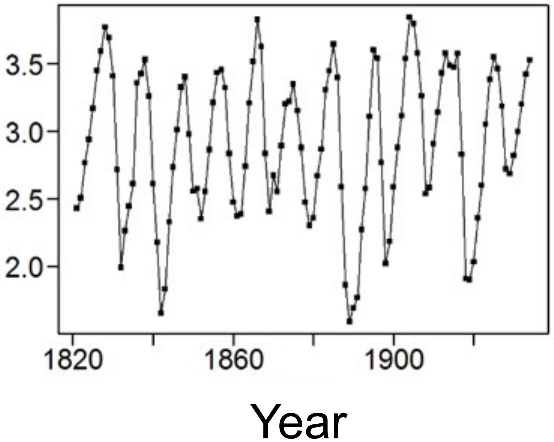




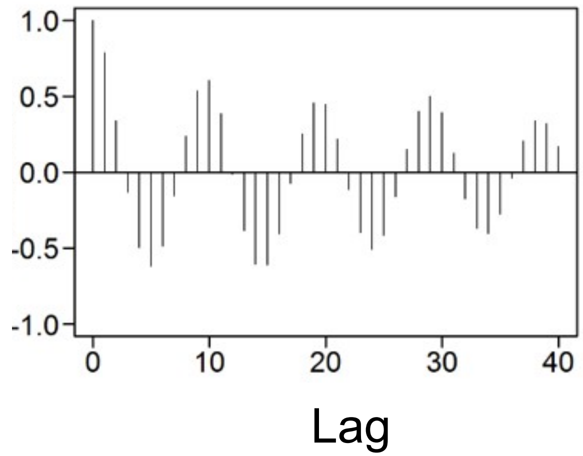
- Classic data set used by time series analysts
- Data are the number of Canadian lynx trapped in the Mackenzie River district of Northwest Canada from 1821–1934 (*above left*)
- Data show a somewhat surprising 10–11 year cycle
- Due to the asymmetric behavior of the cycles, it is common to analyze the “log-lynx” data (*above right*)

Lynx Data

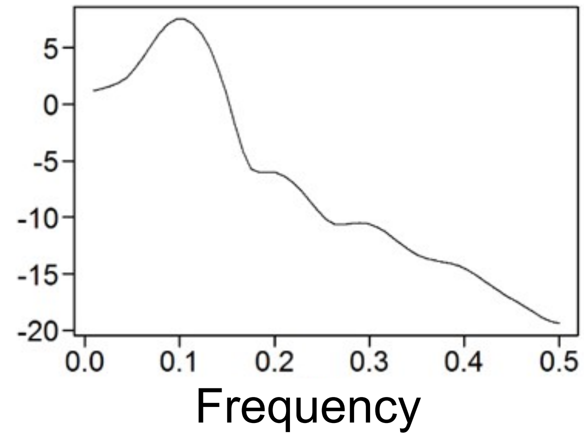
Log lynx data



Autocorrelations



Spectral density



Notes: The above plots are typical of those produced by AR(2) models for which the characteristic equation has complex conjugate roots.

- The log lynx data have a pseudo-cyclic appearance.
- The autocorrelations show damped sinusoidal behavior.
- The spectral density has a peak at some f_0 between 0 and $.5$ (at approximately $.1$ in this case).

Lynx Data

If an AR(2) model is fit to the log lynx data using standard techniques (to be discussed later), we obtain the fitted model

$$(1 - 1.38B + .75B^2)(X_t - 2.9) = a_t$$

Notes about this model:

- The mean of the log lynx data is 2.9
- The characteristic equation $1 - 1.38z + .75z^2 = 0$ has complex conjugate roots greater than 1 in absolute value
- for this model $f_0 = \frac{1}{2\pi}\cos^{-1}\left(\frac{1.38}{2\sqrt{.75}}\right) = .10$

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AR(p) |
Properties and Characteristic Equation

Autoregressive Model of Order *p*: AR(*p*)

We are now ready to define the general AR(*p*) model, where *p* is any positive integer.

$$X_t = \beta + \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \cdots + \varphi_p X_{t-p} + a_t$$

where

$$\beta = (1 - \varphi_1 - \varphi_2 - \cdots - \varphi_p)\mu$$

- "Looks like" a multiple regression model
- Says the value at time t is a linear combination of the p previous values plus a random noise component a_t
- ***Looks "complicated"***

Autoregressive Model of Order *p*: AR(*p*)

Zero mean forms

$$X_t = \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \cdots + \varphi_p X_{t-p} + a_t$$

or

$$X_t - \varphi_1 X_{t-1} - \varphi_2 X_{t-2} - \cdots - \varphi_p X_{t-p} = a_t$$

Operator form: zero mean

$$(1 - \varphi_1 B - \varphi_2 B^2 - \cdots - \varphi_p B^p) X_t = a_t$$

or

$\varphi(B) X_t = a_t$ where $\varphi(B) = 1 - \varphi_1 B - \varphi_2 B^2 - \cdots - \varphi_p B^p$

AR(*p*): characteristic equation

$$\boxed{1 - \varphi_1 z - \varphi_2 z^2 - \cdots - \varphi_p z^p = 0}$$

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AR(p) | Key Result about the Characteristic Equation

Key Result for AR(p)

An AR(p) model is stationary if and only if the roots of the characteristic equation are greater than 1 in absolute value.

Note: This is a generalization of the *check for stationarity* given earlier for AR(1) and AR(2) models.

- for AR(1) it's easy (just check to see if $|\varphi_1| < 1$)
- for AR(2) this check is still easy, but you may have to factor the polynomial or use the quadratic formula to find the roots
- for AR(p), $p > 2$ the check is *more complicated*

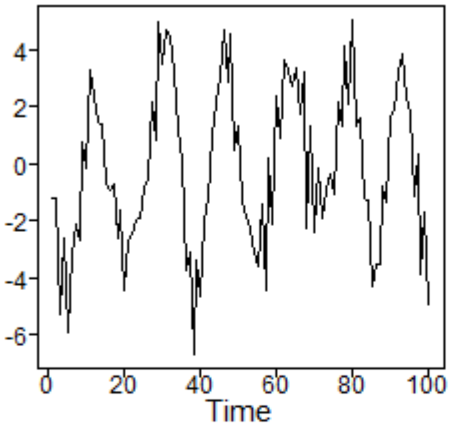
Consider the Following AR(p) Models

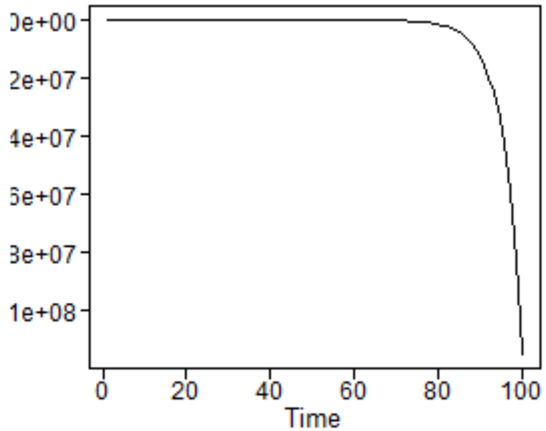
A. $X_t - 1.95\,X_{t-1} + 1.85X_{t-2} - .855\,X_{t-3} = a_t$

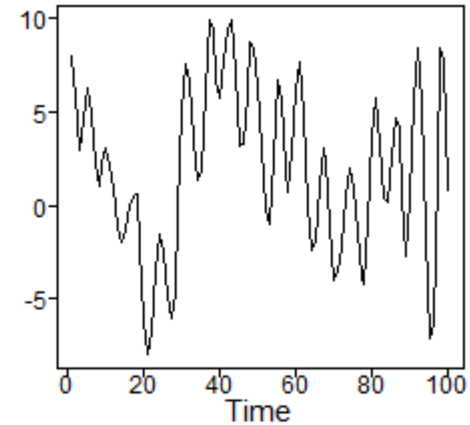
B. $X_t - .2\,X_{t-1} - 1.23\,X_{t-2} + .26\,X_{t-3} + .66\,X_{t-4} = a_t$

C. $X_t - X_{t-1} - .5X_{t-2} + .8\,X_{t-3} - .7X_{t-4} = a_t$

Can you match models with realizations below (shown in random order)?







That's Really a Tough Assignment

I'm guessing you can't match models with realizations?

Good new

I can't either.

Bad news

It seems like AR(p) models may be too complicated (and nearly impossible) to understand.

Good news

There's a simple way to understand these models using a ***key concept*** that we will work towards now and will present in the next section!

Factoring the AR(p)

Note: The roots of a general *p*th order polynomial equation cannot always be found using mathematical formulas (such as the quadratic formula).

- But such polynomial equations can be solved using ***numerical methods.***
- The roots of $\varphi(z) = 1 - \varphi_1 z - \cdots - \varphi_p z^p = 0$ are of ***two types:***
  1. ***Real roots***
  2. ***Complex conjugate pairs*** $(a + bi, a - bi)$
- Correspondingly, $1 - \varphi_1 z - \cdots - \varphi_p z^p$ can ***always*** be factored as a product of:
  - ***First-order (linear)*** factors
  - ***Second-order (quadratic)*** factors

Example 1

AR(3) model

$$1 - 1.95B + 1.85B^2 - .855B^3 = a_t$$

Characteristic equation

$$1 - 1.95z + 1.85z^2 - .855z^3 = 0$$

Factored characteristic equation

Linear factor

$$(1 - .95z)\ (1 - z + .9z^2) = 0$$

Quadratic factor

Note: We often write AR models in factored form (analogous to the characteristic equation factoring).

Factored form of model

$$(1 - .95B)\ (1 - B + .9B^2) = a_t$$

Example 2: Two 4th-Order Polynomials

(a) $1-1.6z+.23z^2+z^3-.576z^4$

(b) $1-2z-1.23z^2+.26z^3+.66z^4$

Factored forms

(a) $(1-.9z)(1-1.5z+.8z^2)(1+.8z)$

Linear factors

Quadratic factor

(b) $(1-1.8z+.95z^2)(1+1.6z+.7z^2)$

No linear factors

Quadratic factors

Key point: ***All*** polynomials can be factored into a product of first and second order factors.

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AR(p) | Factor Tables

Key Concept

Factors like $(1-\alpha_1 B)$ *and* $(1-\alpha_1 B-\alpha_2 B^2)$ serve as building blocks of an AR(*p*) model.

First-order factors, $(1-\alpha_1 B)$

- Associated with real roots
- Contribute AR(1)-type behavior to the AR(p) model
- Are associated with "system frequency" $f_0 = 0$ if α is positive and $f_0 = .5$ if α is negative

Second-order (quadratic) factors, $(1-\alpha_1 B-\alpha_2 B^2)$

- Associated with complex roots
- Contribute cyclic AR(2)-type behavior to the AR(*p*) model, associated with "system frequency"

$$f_0 = \frac{1}{2\pi}\cos^{-1}\left(\frac{\alpha_{1j}}{2\sqrt{-\alpha_{2j}}}\right)$$

Key Concept

Factors like $(1 - \alpha_1 B)$ *and* $(1 - \alpha_1 B - \alpha_2 B^2)$ serve as building blocks of an AR(*p*) model.

AR(*p*) models reflect a mixture of these ***first-*** and ***second-order*** behaviors in the:

- Realizations
- Autocorrelations
- Spectral densities

Factor Table

A tabular presentation of an AR(p) model that:

- Shows the underlying first- and second-order factors
- Provides information about the roots of the characteristic equation to assess whether the model is stationary
- Gives information concerning the underlying frequency domain characteristics
- The format of the ***factor table*** is given on the next slide

Factor Table

Factor	Root	Absolute Reciprocal of root	System frequency (f_0)
First- or second-order factor: $(1 - \alpha_1 B)$ or $(1 - \alpha_1 B - \alpha_2 B^2)$	Root(s) of first- or second-order equations associated with factor: $1 - \alpha_1 z = 0$ $1 - \alpha_1 z - \alpha_2 z^2 = 0$	$\lvert r \rvert^{-1}$, where r is the root in second column	First-order factors: $f_0 = 0$, if $\alpha_1 > 0$ $f_0 = .5$, if $\alpha_1 < 0$ Second-order: $f_0 = \frac{1}{2\pi}\cos^{-1}\left(\frac{\alpha_1}{2\sqrt{-\alpha_2}}\right)$

Note: If $\lvert r \rvert^{-1} < 1$ for all roots, then the process is ***stationary.***

$$\boxed{X_t - 1.95X_{t-1} + 1.85X_{t-2} - .855X_{t-3} = a_t}$$

Is this a stationary process?

What are its characteristics?

The ***factor table*** helps answer these questions.

$$\boxed{X_t - 1.95X_{t-1} + 1.85X_{t-2} - .855X_{t-3} = a_t}$$

Factor Table

Factor	Root	Absolute Reciprocal of root	System frequency (f_0)
$1 - .95B$	1.053	0.95	0.0
$1 - B + .9B^2$	$.556 \pm .896i$	0.95	0.16

What does the factor table tell us?

1. The model is stationary.

- It is not immediately obvious whether $.556 + .896i$ is greater than 1 in absolute value, so we show the absolute value

  (we choose to show absolute value of the reciprocal, $|r|^{-1}$).
- $|r|^{-1} < 1$ for all roots implies stationarity.

$$\boxed{X_t - 1.95X_{t-1} + 1.85X_{t-2} - .855X_{t-3} = a_t}$$

Factor Table

Factor	Root	Absolute Reciprocal of root	System frequency (f_0)
$1 - .95B$	1.053	0.95	0.0
$1 - B + .9B^2$	$.556 \pm .896i$	0.95	0.16

What does the factor table tell us?

2. The model has one positive real root and a pair of complex conjugate roots.

$$\boxed{X_t - 1.95X_{t-1} + 1.85X_{t-2} - .855X_{t-3} = a_t}$$

Factor Table

Factor	Root	Absolute Reciprocal of root	System frequency (f_0)
$1 - .95B$	1.053	0.95	0.0
$1 - B + .9B^2$	$.556 \pm .896i$	0.95	0.16

What does the factor table tell us?

3. The positive real root will be associated with:

- "Wandering" behavior in realizations
- Damped exponential autocorrelations
- A peak at zero in the spectral density

$$X_t - 1.95X_{t-1} + 1.85X_{t-2} - .855X_{t-3} = a_t$$

Factor Table

Factor	Root	Absolute Reciprocal of root	System frequency (f_0)
$1 - .95B$	1.053	0.95	0.0
$1 - B + .9B^2$	$.556 \pm .896i$	0.95	0.16

What does the factor table tell us?

4. The complex conjugate roots will be associated with:

- Pseudo cyclic behavior in the realizations with frequency about $f_0 = .16$ (cycle length $(1/.16 = 6)$)
- Damped sinusoidal autocorrelations with a period (cycle length) of about 6
- A peak at about $.16$ in the spectral density

$$\boxed{X_t - 1.95X_{t-1} + 1.85X_{t-2} - .855X_{t-3} = a_t}$$

Factor Table

Factor	Root	Absolute Reciprocal of root	System frequency (f_0)
$1 - .95B$	1.053	0.95	0.0
$1 - B + .9B^2$	$.556 \pm .896i$	0.95	0.16

What does the factor table tell us?

5. For the above AR(3) model:

- Realizations
- Autocorrelations
- Spectral density

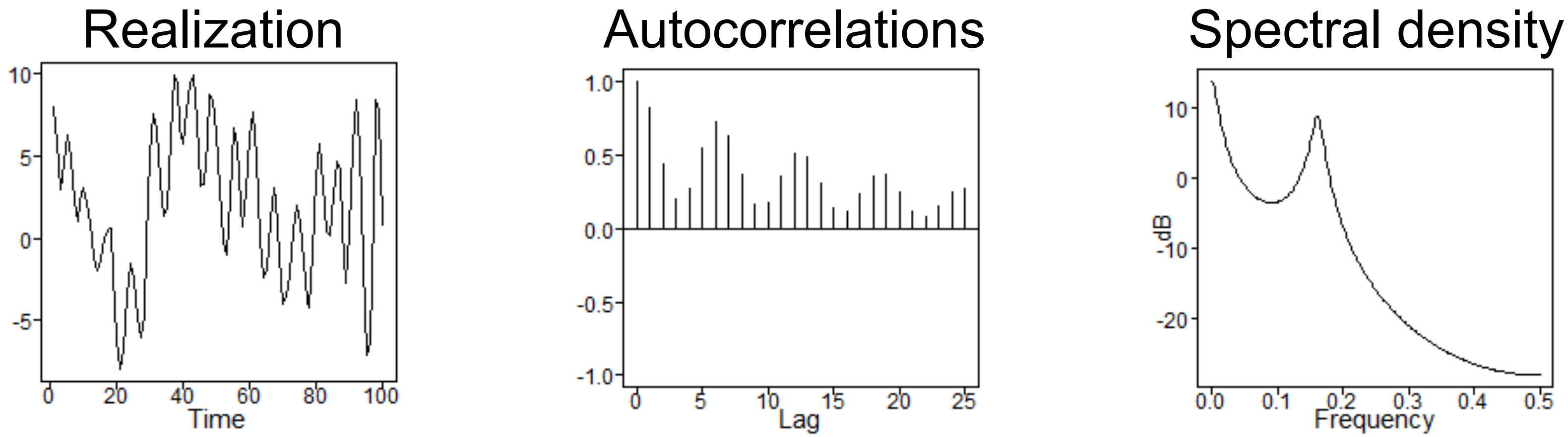
will show a "mixture" of these behaviors

$$X_t - 1.95X_{t-1} + 1.85X_{t-2} - .855X_{t-3} = a_t$$

Factor Table

Factor	Root	Absolute Reciprocal of root	System frequency (f_0)
$1 - .95B$	1.053	0.95	0.0
$1 - B + .9B^2$	$.556 \pm .896i$	0.95	0.16

What does the factor table tell us?



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AR(p) | Factor Table in tswge

Factor Tables in tswge

tswge demo

```
#Factor Table
factor.wge(phi=c(1.95,-1.85,.855))
#
#Plotting a realization along with true
# autocorrelations and spectral density
plotts.true.wge(phi=c(1.95,-1.85,.855))
```

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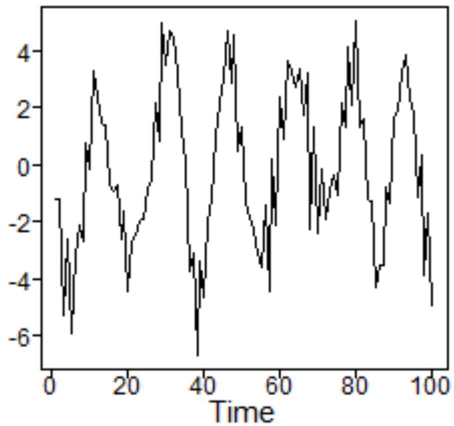
AR(p) | Examples

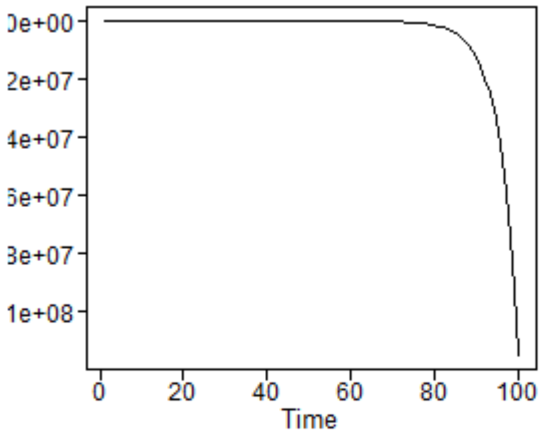
A. $X_t - 1.95X_{t-1} + 1.85X_{t-2} - .855X_{t-3} = a_t$

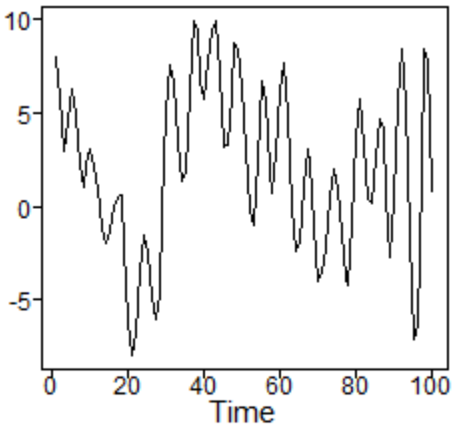
B. $X_t - .2X_{t-1} - 1.23X_{t-2} + .26X_{t-3} + .66X_{t-4} = a_t$

C. $X_t - X_{t-1} - .5X_{t-2} + .8X_{t-3} - .7X_{t-4} = a_t$

Can you match models with realizations below (shown in random order)?







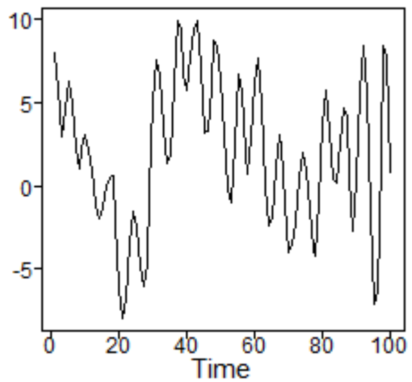
$$X_t - 1.95X_{t-1} + 1.85X_{t-2} - .855X_{t-3} = a_t$$

Factor Table

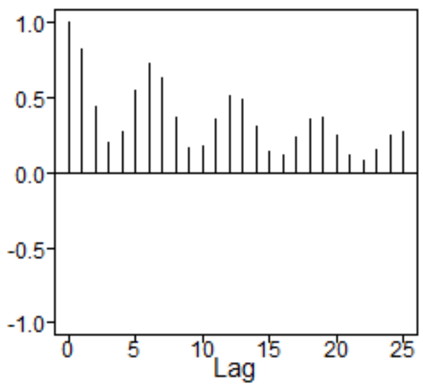
Factor	Root	Absolute Reciprocal of root	System frequency (f_0)
$1 - .95B$	1.053	0.95	0.0
$1 - B + .9B^2$	$.556 \pm .896i$	0.95	0.16

What does the factor table tell us?

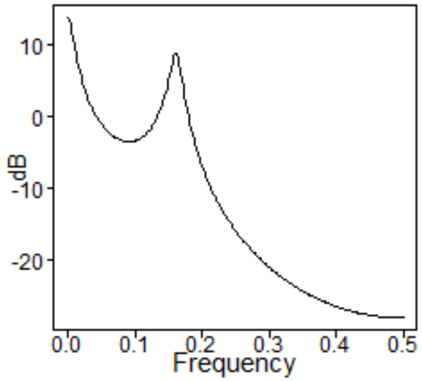
Realization



Autocorrelations



Spectral density



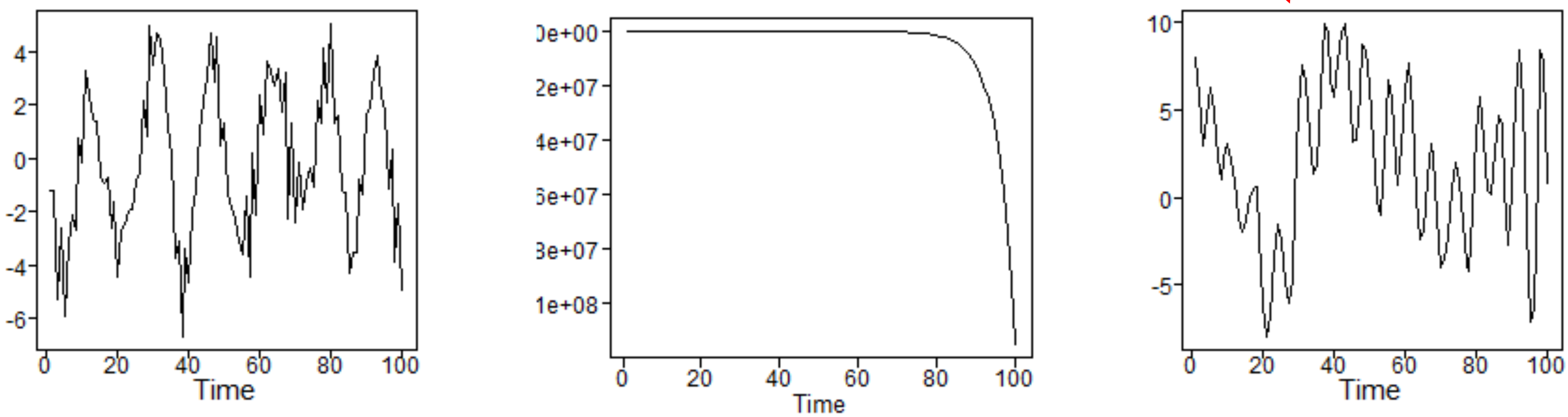
Consider the Following AR(p) Models

A. $X_t - 1.95X_{t-1} + 1.85X_{t-2} - .855X_{t-3} = a_t$

B. $X_t - .2X_{t-1} - 1.23X_{t-2} + .26X_{t-3} + .66X_{t-4} = a_t$

C. $X_t - X_{t-1} - .5X_{t-2} + .8X_{t-3} - .7X_{t-4} = a_t$

Can you match models with realizations below (shown in random order)?

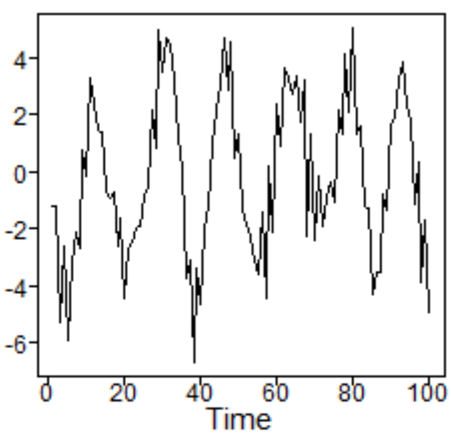


$$\text{(B) } X_t - .2X_{t-1} - 1.23X_{t-2} + .26X_{t-3} + .66X_{t-4} = a_t$$

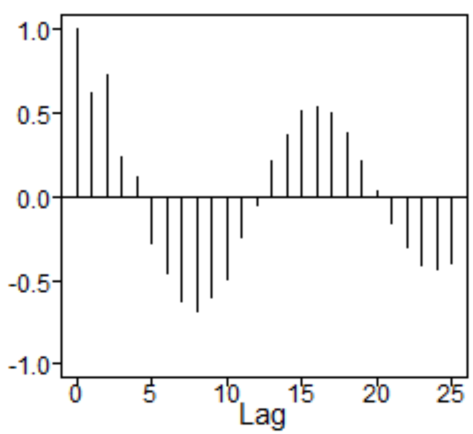
Factor Table

Factor	Root	Absolute Reciprocal of root	System frequency (f_0)
$1 - 1.8B + .95B^2$	$.95 \pm 39i$	0.97	0.06
$1 + 1.6B + .7B^2$	$-1.15 \pm 35i$	0.83	0.45

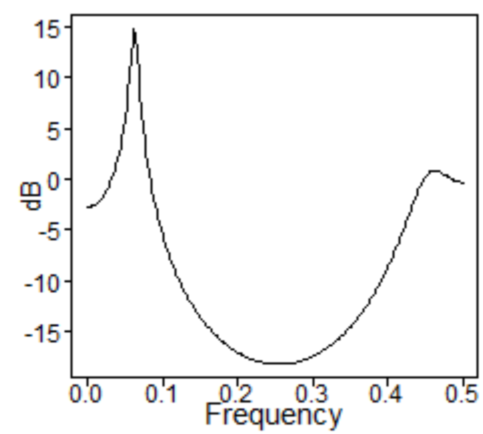
Realization



Autocorrelations



Spectral density



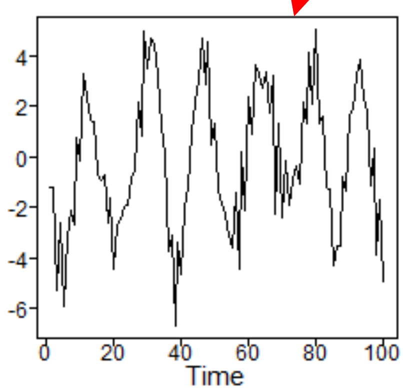
Consider the Following AR(p) Models

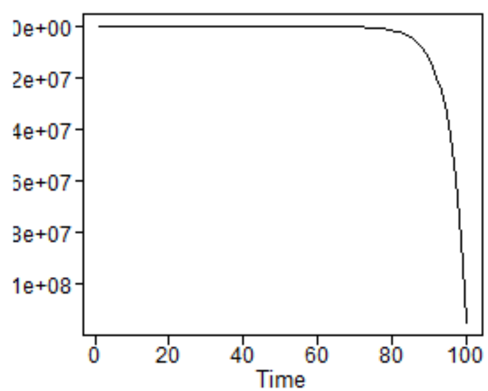
A. $X_t - 1.95X_{t-1} + 1.85X_{t-2} - .855X_{t-3} = a_t$

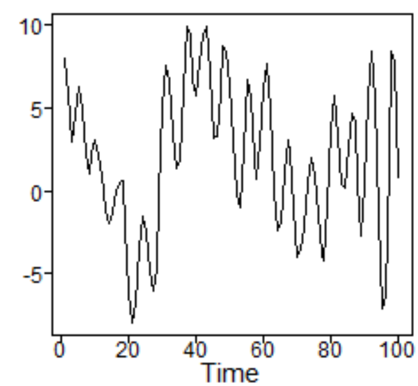
B. $X_t - .2X_{t-1} - 1.23X_{t-2} + .26X_{t-3} + .66X_{t-4} = a_t$

C. $X_t - X_{t-1} + .5X_{t-2} + .8X_{t-3} - .7X_{t-4} = a_t$

Can you match models with realizations below (shown in random order)?







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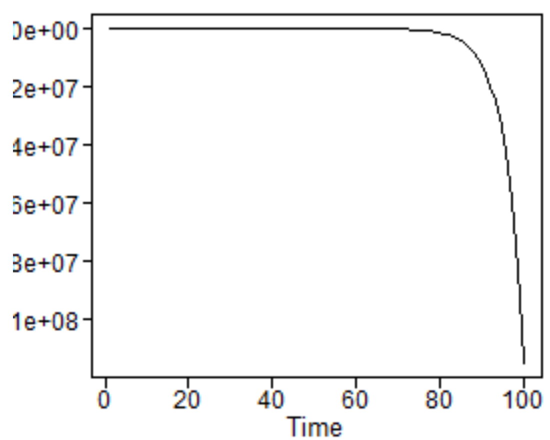
(C) $X_t - X_{t-1} - .5X_{t-2} + .8X_{t-3} - .7X_{t-4} = a_t$

Factor Table

Factor	Root	Absolute Reciprocal of root	System frequency (f_0)
$1 - 1.25B$	$.90$	1.25	0.0
$1 + B$	-1.0	1.00	0.5
$1 - .75B + .56B^2$	$.67 \pm 1.15i$	0.75	0.17

Realization



Explosively nonstationary

Consider the Following AR(p) Models

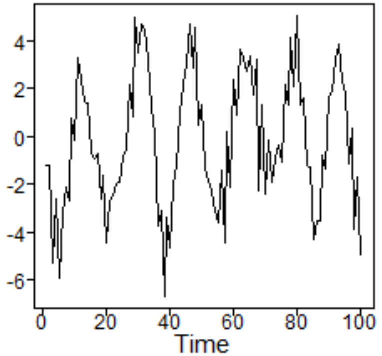
(A) $X_t - 1.95X_{t-1} + 1.85X_{t-2} - .855X_{t-3} = a_t$

(B) $X_t - .2X_{t-1} - 1.23X_{t-2} + .26X_{t-3} + .66X_{t-4} = a_t$

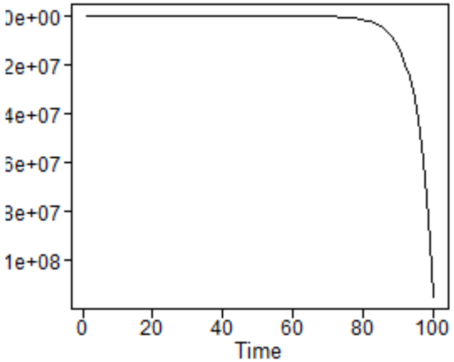
(C) $X_t - X_{t-1} - .5X_{t-2} + .8X_{t-3} - .7X_{t-4} = a_t$

Can you match models with realizations below (shown in random order)?

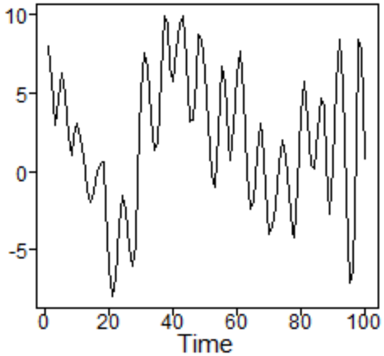
(B)



(C)



(A)



Key point

The coefficients alone are not helpful in understanding the nature of AR(p) models.

- ***Use factor tables.***

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AR(*p*) | Roots and Dominant Behavior

Key Fact

Roots close to the unit circle ***dominate.***

To illustrate this we consider the three models:

Model **A** (in factored form) $$(1\quad .95B)(1\quad B+.9B^2)X_t=a_t$$

Model **A-r** $$(1\quad .95B)(1\quad .76B+.5B^2)X_t=a_t$$

Model **A-c** $$(1\quad .7B)(1\quad B+.9B^2)X_t=a_t$$

To understand these 3 models we examine the ***factor tables*** (of course).

Factor Table

	Factor	Root	Absolute Reciprocal of root	System frequency (f_0)
Model **A**	$1-.95B$	1.053	0.95	0.0
	$1-B+.9B^2$	$.56 \pm .90i$	0.95	0.16
Model **A-r**	$1-.95B$	1.053	0.95	0.0
	$1-.76B+.5B^2$	$.76 \pm 1.19i$	0.70	0.16
Model **A-c**	$1-.7B$	1.43	0.70	0.0
	$1-B+.9B^2$	$.56 \pm .90i$	0.95	0.16

- All three models have a first- and second-order factor
- All three models have system frequencies of 0 and .16
- Model A: roots equally close to the unit circle
- Model A-r: **real** root closer to the unit circle than complex roots
- Model A-c: **complex** root closer to the unit circle than real root

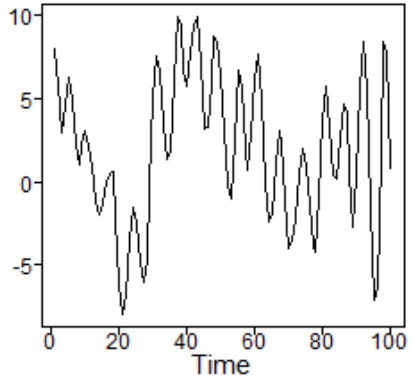
$$X_t - 1.95X_{t-1} + 1.85X_{t-2} - .855X_{t-3} = a_t$$

Factor Table

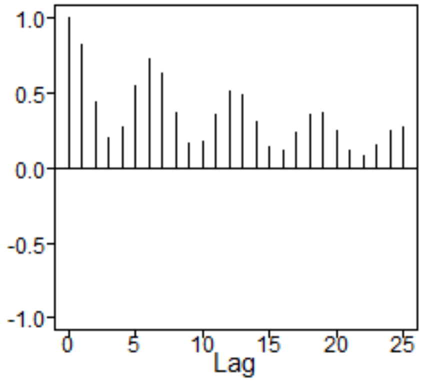
Factor	Root	Absolute Reciprocal of root	System frequency (f_0)
$1 - .95B$	1.053	0.95	0.0
$1 - B + .9B^2$	$.556 \pm .896i$	0.95	0.16

What does the factor table tell us?

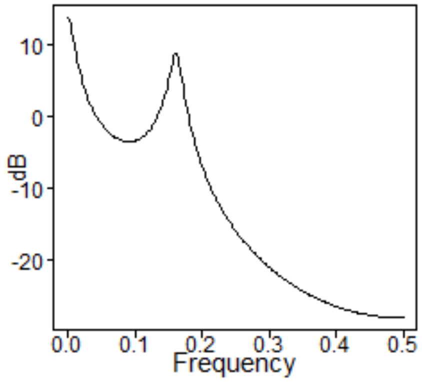
Realization



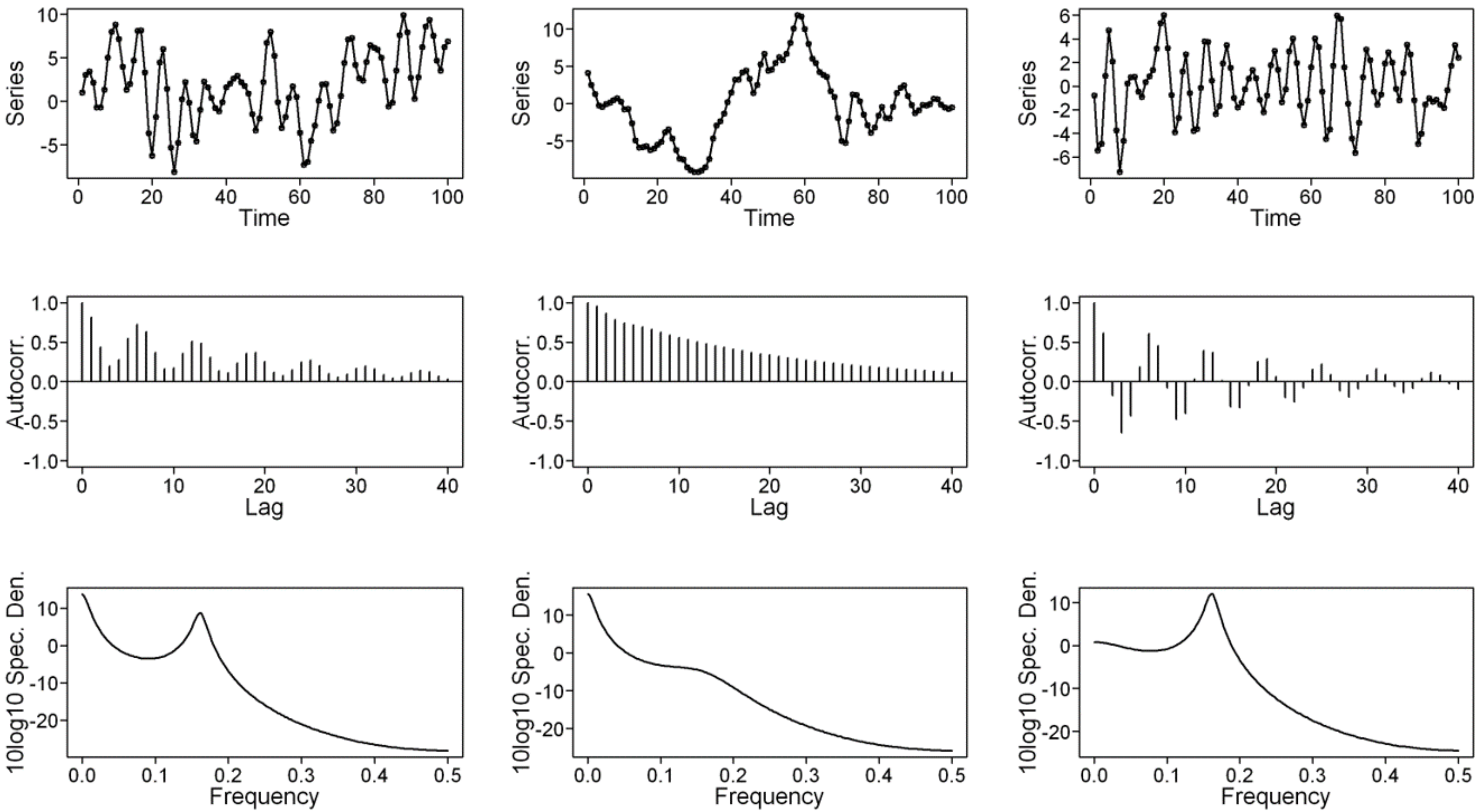
Autocorrelations



Spectral density



Model A　　Model A-r　　Model A-c



Notes

- When roots were equally close to the unit circle, we saw both first- and second-order behaviors.
- When the positive real root was closest to the unit circle, the first-order behavior dominated.
  - We barely see any evidence of a second-order factor.
- When complex roots are closer to the unit circle, second-order behavior dominates.
  - We barely see any evidence of a first-order factor.

Takeaway

Roots closest to the unit circle dominate the behavior of autocorrelation functions, realizations, and spectral densities.

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Review

AR(1) and AR(2)

AR(1) Models: Review

Realizations are wandering or oscillating depending on whether the root of the characteristic equation is positive or negative, respectively.

Autocorrelations are damped exponentials or damped oscillating exponentials depending on whether the root of the characteristic equation is positive or negative, respectively.

Spectral densities have a peak at $f = 0$ or $f = .5$ depending on whether the root of the characteristic equation is positive or negative, respectively.

AR(2) Models: Review

Real roots

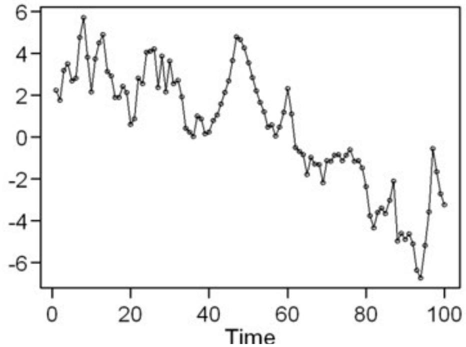
- If *both are positive,* then behavior is similar to that of an AR(1) with a positive real root.
- If *both are negative,* then behavior is similar to that of an AR(1) with a negative real root.
- If *both positive **and** negative* real roots, then both behaviors mentioned above tend to be visible.

Complex conjugate roots

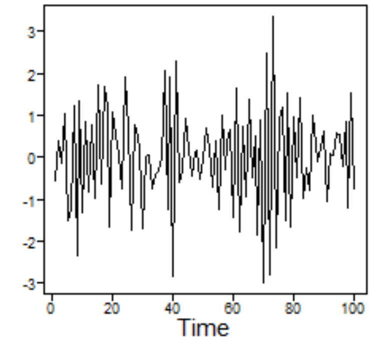
- Realizations are *pseudo-cyclic* with cycle length about $1/f_0$.
- Autocorrelations are *damped sinusoids,* both with cycle length about $1/f_0$.
- Spectral density has a *peak at about* f_0.

AR(1) behaviors

Positive root

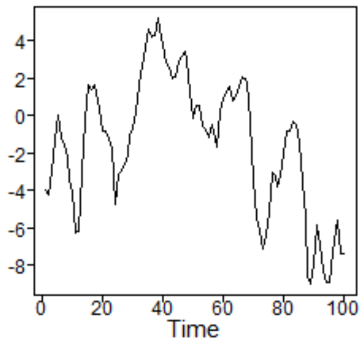


Negative root

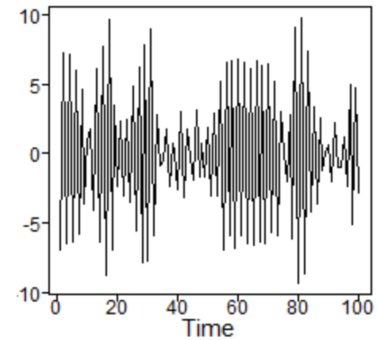


AR(2) behaviors

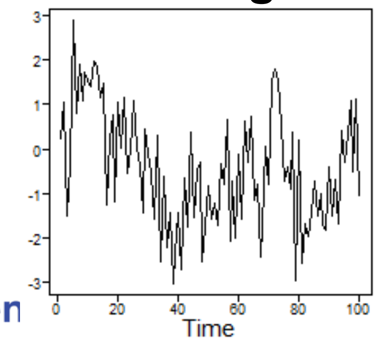
2 positive roots



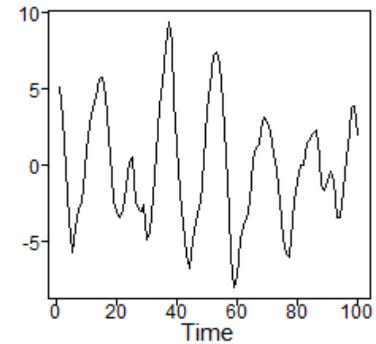
2 negative roots



Positive and negative roots



Complex conjugate roots



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