The complex response function of the liquid injector is given by the following equation.

$$\Pi_\Sigma = \frac{\dot{m}'_N / \dot{m}_N}{\Delta P'_\Sigma / \Delta P_\Sigma} = \left(\frac{\hat{R}_V}{\hat{r}_{V0}}\right)^2 \frac{\Pi_T \Pi_{VN} \Pi_N}{1 + 2\Pi_T(\Pi_{V2} + \Pi_{V3})}$$

This function quantifies the injector's sensitivity to upstream perturbations, and each component represents a different stage of the injector flow path. The complex response of the tangential inlets to plane waves is defined as follows.

$$\Pi_T = \frac{\dot{m}'_T / \dot{m}_T}{\Delta P'_T / \Delta P_T}$$

The complex response of the vortex chamber to surface waves is characterized by reflections at the boundary between the vortex chamber and nozzle.

$$\Pi_{VN} = \frac{\dot{m}'_{VN} / \dot{m}_{VN}}{\dot{m}'_T / \dot{m}_T}$$

The complex response of the nozzle to surface waves is defined from the proportion of transmitted waves through the reflective boundary.

$$\Pi_N = \frac{\dot{m}'_N / \dot{m}_N}{\dot{m}'_{VN} / \dot{m}_{VN}}$$

The complex response functions of the closed end of the vortex chamber comprise a feedback connection and are associated with surface waves and vorticity waves respectively.

$$\Pi_{V2} = \frac{1}{2} \frac{\Delta P'_{V2} / \Delta P_T}{\dot{m}'_T / \dot{m}_T}$$

$$\Pi_{V3} = \frac{1}{2} \frac{\Delta P'_{V3} / \Delta P_T}{\dot{m}'_T / \dot{m}_T}$$

The injector response function and its component parts are defined as follows. The response of the tangential inlets to pressure oscillations is given by the following equation in terms of the Strouhal number at this section of the injector. In this study, distances are normalized with respect to the nozzle radius and velocities are normalized with respect to the inflow velocity.

$$\Pi_T = \frac{1 - i\mathrm{Sh}_T}{2 + 2\mathrm{Sh}_T^2}$$

$$\mathrm{Sh}_T = \hat{\omega} \frac{\hat{L}_T}{\hat{U}_T}$$

The phase shift for the vortex chamber is defined as follows.

$$\Phi_V = \hat{\omega} \frac{\hat{L}_V + \hat{L}_C}{\hat{c}_V}$$

The liquid wave speed in the uniform section is given by the following equation.

$$\hat{c}_V = \hat{U}_{V,z} + \sqrt{\hat{C}^2 \frac{\hat{R}_V^2 - \hat{r}_V^2}{2\hat{r}_V^4}}$$

The nozzle reflection coefficient is defined by the following expression. This is determined assuming that waves are reflected at an approximated sudden-step discontinuity between the vortex chamber and the nozzle.

$$\Pi_{refl} = 1 - 2\sqrt{\frac{\varphi}{\hat{R}_V^2 - \hat{r}_{V0}^2}}$$

The transfer function of surface waves in the vortex chamber is then defined as follows.

$$\Pi_{V2} = \frac{1}{A\sqrt{2\frac{\left(\hat{R}_V^2 - \hat{r}_{V0}^2\right)}{\hat{R}_N^2}}} \sum_{n=0}^{\infty} \Pi_{refl}^n e^{-2n\Phi_V(i+\nu)}$$

The transfer function of the nozzle entrance is given by this equation. An artificial viscosity coefficient is used to dampen infinite reflections.

$$\Pi_{VN} = \sum_{n=0}^{\infty} \Pi_{refl}^n e^{-\Phi_V(2n+1)(i+\nu)}$$

The Strouhal number at the vortex chamber is given by this equation.

$$\mathrm{Sh}_V = \hat{\omega} \frac{\hat{R}_V}{\hat{U}_{V,z}}$$

A new function and integration constant are introduced for analysing this stage.

$$f(\bar{x}) = \bar{x}\mathrm{Sh}_V \tan\left(\frac{\pi\bar{x}}{2}\right)$$

$$K = 1 - \frac{\hat{r}_{V0}}{\hat{R}_V}$$

The real and imaginary parts of the vorticity wave transfer function are given by the two following equations.

$$\mathrm{Re}(\Pi_{V3}) = 2\int_0^1 \frac{\cos(f(\bar{x}))}{(1-K\bar{x})^3} e^{-\nu f(\bar{x})}\, d\bar{x}$$

$$\mathrm{Im}(\Pi_{V3}) = -2\int_0^1 \frac{\sin(f(\bar{x}))}{(1-K\bar{x})^3} e^{-\nu f(\bar{x})}\, d\bar{x}$$

The phase shift for the nozzle is defined by the following equation.

$$\Phi_N = \hat{\omega}\frac{\hat{L}_N}{\hat{c}_N}$$

The wave speed in the nozzle is given by this expression.

$$\hat{c}_N = \hat{U}_{N,z} + \sqrt{\hat{C}^2 \frac{\hat{R}_N^2 - \hat{r}_N^2}{2\hat{r}_N^4}}$$

The nozzle transfer function is then defined by the following relationship which is based on the proportion of transmitted waves through the reflective boundary.

$$\Pi_N = (1 - \Pi_{refl})e^{-i\Phi_N}$$

The concluding step in this analysis is to combine all transfer functions into the total injector response function.

$$\Pi_\Sigma = \frac{\dot{m}_N'/\dot{m}_N}{\Delta P_\Sigma'/\Delta P_\Sigma} = \left(\frac{\hat{R}_V}{\hat{r}_{V0}}\right)^2 \frac{\Pi_T \Pi_{VN} \Pi_N}{1 + 2\Pi_T(\Pi_{V2} + \Pi_{V3})}$$

The filling efficiency of the swirl injector is defined as the ratio of the area filled by fluid to the total area.

$$\varphi = 1 - \frac{r_N^2}{R_N^2} = 1 - \hat{r}_N^2$$

The geometric constant is another important parameter which relates injector geometries. This is defined as follows.

$$A \equiv \frac{R_N(R_V - R_T)}{nR_T^2} = \frac{(1-\varphi)\sqrt{2}}{\varphi\sqrt{\varphi}}$$

The filling efficiency can then be used to calculate the mass flow coefficient of the injector.

$$\mu = \sqrt{\frac{\varphi^3}{2-\varphi}}$$

The hydraulic mass flow coefficient is used to account for additional flow losses in the injector. This is modelled by a hydraulic loss coefficient.

$$\mu_h = \sqrt{\frac{\varphi^3\eta^2}{(2-\varphi)\eta^2 + \xi A^2 \varphi^3}}$$

The ideal total velocity magnitude in the injector can be determined from potential flow. This is given by the following equation.

$$U_\Sigma = \sqrt{\frac{2\Delta P_\Sigma}{\rho}}$$

The nozzle radius can then be determined from these parameters for a given filling efficiency.

$$R_N = \sqrt{\frac{\dot{m}_\Sigma}{\pi\mu_h\sqrt{2\rho\Delta P_\Sigma}}}$$

The coefficient of nozzle opening is defined as follows for the injector.

$$(R_V - R_T) = \eta R_N$$

From the definition of the geometric constant and coefficient of nozzle opening, the remaining geometries can be calculated.

$$R_T = \sqrt{\frac{\eta R_N^2}{nA}}$$

$$R_V = \eta R_N + R_T$$

The real internal velocity magnitude of the injector is also lowered due to hydraulic losses.

$$U_{\Sigma,h} = U_\Sigma\sqrt{1 - \xi\frac{\mu_h^2 A^2}{\eta^2}}$$

The pressure drop across the tangential inlets can be expressed as a function of the total pressure drop across the injector. This is valid only for coefficient of nozzle opening values greater than one.

$$\Delta P_T = \Delta P_\Sigma \frac{\hat{r}_{V0}^2}{\eta^2}$$

The ideal inflow velocity to the vortex chamber, which is equivalent to the velocity in the tangential inlets, is then given by this equation.

$$U_{in} = \sqrt{\frac{2\Delta P_T}{\rho}}$$

The spray half angle downstream of the injector exit can then be found from the following relationship. The velocity components vary within the liquid film, so there is some uncertainty in this value.

$$\tan \alpha_E = \frac{U_{E,\theta}}{U_{E,z}} = \frac{\mu A}{\sqrt{1-\mu^2 A^2}}$$

Practically, the coefficient of nozzle opening is chosen based on structural requirements and dynamic stability considerations. The filling efficiency value can then be chosen based on a desired spray angle and the predicted performance of the injector. The continuity and momentum equations for the swirl injector are used to determine the film radius in the vortex chamber. Conservation of angular momentum yields the following relationship.

$$U_{j,\theta,h} = U_{\Sigma,h} \frac{\hat{r}_{V0}}{\hat{r}_j}$$

The axial velocity component can be found using the following equation.

$$U_{j,z,h} = U_{\Sigma,h} \sqrt{1 - \frac{\hat{r}_{V0}^2}{\hat{r}_j^2}}$$

This is equivalent to the flow continuity equation, which can be expressed as follows.

$$U_{j,z,h}\left(\hat{R}_j^2 - \hat{r}_j^2\right) = U_{N,z,h}\left(\hat{R}_N^2 - \hat{r}_N^2\right)$$

The corresponding dimensionless film radius at a given injector stage is given by the following equation.

$$\hat{r}_j = \sqrt{1-\varphi_j}$$

Where the effective filling efficiencies at different stages are defined as follows.

$$\varphi_N = \varphi$$

$$\varphi_{V0} = \frac{3\varphi - 2\varphi^2}{2-\varphi}$$

The resonant modes of the combustion chamber are calculated using the following equation related to the chamber geometry, the local speed of sound, and roots of the Bessel function of the first kind describing the system.

$$f_{m,n,q} = \frac{a}{2}\sqrt{\left(\frac{\alpha_{mn}}{R}\right)^2 + \left(\frac{q}{L}\right)^2}$$

Here m, n, and q are wavenumbers (zero or integer) which correspond to radial, tangential, and longitudinal oscillations.

$$\begin{aligned} P(r,\theta,z,t) \\ = \sum_{m,n,q} \left[J_n\left(\frac{\pi\alpha_{mn}r}{R}\right)\cos\left(\frac{\pi q z}{L}\right)\right] [M\cos(n\theta + \omega t \\ - \delta_1) + N\cos(n\theta - \omega t - \delta_2)] \end{aligned}$$

By evaluating the resonant frequencies of the combustion chamber, the injectors can be designed to limit the potential for acoustic coupling in the system. This calculation can then be performed across the entire throttling range of the engine to access whether any unstable sections exist. The low frequency shift of the injectors during throttling can also be used to approximate when the engine may become susceptible to low-frequency instability.