The multiplicity conjecture for Gan-Gross-Prasad pairs in the Special Orthogonal Case

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1 preliminaries

1.1 General Notation

- F a field, either p-adic (a finite extension of $\mathbb{Q}_p$) or $\mathbb{R}$
- $|.|$ is the normalized absolute value on F. That is for every Haar measure dx on F we have $d(ax) = |a|dx$ for all $a \in F$
- For G a locally compact separable greoup, we will denote by $d_L g$ (resp. $d_R g$) a left (resp. a right) Haar measure on G. If the group is unimodular then we will denote both by dg.
- δ_G will stand for the modular character of G that is defined by $d_l(gg'^{-1} = \delta_G(g')d_L g$ for all $g \in G$

If F is p-adic then:

- $\mathcal{O}_F = \{a \in F : |a| \leq 1\}$ the ring of integers of F
- $\mathfrak{p}_F = \{a \in F : |a| < 1\}$ the maximal ideal of $\mathcal{O}_F$
- q_F the cardinality of the residue field $\mathcal{O}_F \backslash \mathfrak{p}_F$
- ϖ_F the uniformizer of F which generates $\mathfrak{p}_F$
- val_F is the surjective function $F \to \mathbb{Z} \cup \{\infty\}$ such that $\mathrm{val}_F(\varpi_F) = 1$ and $|a| = q_F^{-\mathrm{val}_F(a)}$ for all $a \in F$. Note we can write $a = \varpi^n u$ for some integer n and unit u. Then $\mathrm{val}(a) = n$ and $\mathrm{val}(0) = \infty$.
- If R is a subring of F, and $v \in F$ then $\mathrm{val}_R(v) = -\inf\{k \in \mathbb{Z} \,|\varpi^k v \in R\}$.
- If V is an F vector space, and R is an $\mathcal{O}_F$ lattice of V and $g \in GL_F(V)$ then $\mathrm{val}_R(g) = \inf\{\mathrm{val}_R(gv), v \in R\}$ and $\|g\|_R = |\varpi_F|^{\mathrm{val}_R(g)}$

1.2 Log norm

- For two log norms σ_1, σ_2 on X, we say σ_2 dominates σ_1, or $\sigma_1 \ll \sigma_2$, if there exists $c > 0$ such that $\sigma_1(x) \leq c\sigma_2(x) \, \forall x \in X$
- If $\sigma_1 \ll \sigma_2$ and $\sigma_2 \ll \sigma_1$ then we say σ_1 and σ_2 are equivalent and write $\sigma_1 \sim \sigma_2$
- For a vector space V, $\sigma_V(v) = \log(2 + |v|), v \in V$
- We denote $\sigma := \sigma_V$
- $\sigma(xy) \ll \sigma(x) + \sigma(y) \ll \sigma(x)\sigma(y) \forall (x, y) \in G \times G$
- $\sigma_{T\backslash G}(g) = \inf_{t \in T(F)} \sigma(tg)$
- For every maximal torus $T \subset G$. we have $\sigma(g^{-1}Xg) + \log(2 + D^G(X)^{-1}) \sim \sigma_{\mathfrak{g}}(X) + \sigma_{T\backslash G}(g) + \log(2 + D^G(X)^{-1})$

Let $f : X \to Y$ be a morphism of algebraic varieties over F and let σ_X be a log norm on X (but we will only consider its restriction to $X(F)$). Define an abstract log norm $f_*\sigma_X$ on $\mathrm{Im}(X(F) \to Y(F)$ by

$$f_*\sigma_X(y) = \inf_{x \in X(F): f(x) = y} \sigma_X(x)$$

Let σ_Y be a log-norm on Y. We say that f has the *norm descent property* id σ_Y and $f_*\sigma_X$ are equivalent as abstract log-norms on $\mathrm{Im}(X(F) \to Y(F)$. We will need the following facts about the norm descent property. First from Kott3 Prop 18.2

Lemma 1.2.1 *(i) The norm descent property is local on the basis. In other words if $f : X \to Y$ is a morphism of algebraic varieties over F and $(U_i)_{i \in I}$ is a finite covering by Zariski-open subsets of Y defined over F, then f has the norm descent property if and only if each of the $f_i : f^{-1}(U_i) \to U_i, i \in I$ has the norm descent property.*

(ii) If f admites a section, then it has the norm descent property.

and we will also need Kott3 Prop 18.3

Proposition 1.2.2 *Let G be a connected reductive group over F and T an F-subtorus of G. Then the morphism $G \to T\backslash G$ has the norm descent property.*

1.3 Harish-Chandra Schwartz space

We will denote Ξ^G as the Harish-Chandra function. To define it we will define the following. Let P_{min} be a minimal parabolic subgorup of G and let K be a maximal compact subgroup of $G(F)$ which is special in the p-adic case. Then we have $G(F) = P_{min}(F)K$ (Iwasawa decomposition). Consider the (smooth and normalized) induced representation.

$$i^G_{P_{min}}(1)^\infty := \{e \in C^\infty(G(F)) :\, e(pg) = \delta_{min}(p)^{1/2} e(g)\, \forall p \in P_{min}(F), g \in G(F)\}$$

that we equip with the scalar product

$$(e, e') = \int_K e(k)\overline{e'(k)}dk, \quad e, e' \in i^G_{P_{min}}(1)^i nfty$$

Let $e_K \in i^G_{P_{min}}(1)^\infty$ be the unique function such that $e_K(k) = 1$ for all $k \in K$. Then the Harish-Chandra function is defined by

$$\Xi^G(g) = (i^G_{P_{min}}(1)(g)e_K, e_k),\ g \in G(F)$$

This is the action of representaion not the space defined above, make clearer sometime

This definition of Ξ^G depends on various choices, but this doesn't matter as different choices would yield equivalent functions and we will only be using Ξ^G to give estimates. The following give the main properties of Ξ^G that we need.

Proposition 1.3.1 *(i) Set*

$$M^+_{min} = \{m \in M_{min}(F) :\, |\alpha(m)| \leq 1\, \forall \alpha \in R(A_{M_{min}}, P_{min})\}$$

Then, there exists $d > 0$ such that

$$\delta_{P_{min}}(m)^{1/2} \ll \Xi^G(m) \ll \delta_{P_{min}}(m)^{1/2}\sigma(m)^d$$

for all $m \in M^+_{min}$.

(ii) Let $m_{P_{min}} : G(F) \to M_{min}(F)$ be any map such that $g \in m_{P_{min}}(g)U_{min}(F)K$ for all $g \in G(F)$. Then, there exists $d > 0$ such that

$$\Xi^G(g) \ll \delta_{P_{min}}(m_{P_{min}}(g))^{1/2}\sigma(g)^d$$

for all $g \in G(F)$.

(iii) Let $P = MU$ be a parabolic subgroup that contains P_{min}. Let $m_P : G(F) \to M(F)$ be any such map such that $g \in m_P(g)U(F)K$ for all $g \in G(F)$. Then, we have

$$\Xi^G(g) = \int_K \delta_P(m_P(kg))^{1/2}\Xi^M(m_P(kg))dk$$

for all $g \in G(F)$.

(iv) Let $P = MU$ be a parabolic subgroup of G. Then, for all $d > 0$, there exists $d' > 0$ such that

$$\delta_P(m)^{1/2} \int_{U(F)} \Xi^G(mu)\sigma(mu)^{-d'}du \ll \Xi^M(m)\sigma(m)^{-d}$$

for all $m \in M(F)$.

(v) *There exists* $d > 0$ *such that the integral*

$$\int_{G(F)} \Xi^G(g)^2 \sigma(g)^{-d} dg$$

is convergent.

(vi) *(Doubling principle) We have the equality*

$$\int_K \Xi^G(g_1 k g_2) dk = \Xi^G(g_1)\Xi^G(g_2)$$

for all $g_1, g_2 \in G(F)$.

Proof. sources and stuff todo ☐

1.4 Measures

We will fix a continuous non-trivial additive character $\psi : F \to \mathbb{S}^1$ and equip F with the autodual Haar measure with respect to ψ.

2 Gan-Gross-Prasad Triples

- Let V be a quadratic vector space of dimension n over a local field F. That is, there exists quadratic form $q : V \to F$.
- Let $A \in M_n$ be a symmetric matrix such that $q(v) = v^t A v$.
- Let b be a symmetric bilinear form given by $b(x, y) = x^t A y$
- $O(n) = \{G \in GL_n(F) \big| G^t A G = A\}$
  $= \{G \in GL_n \big| q(Gv) = q(v)\}$
- $SO(n) = \{G \in O(n) \big| \det(G) = 1\}$
- $so(n) = \{g \in M_n \big| g^t A + A g = 0\}$
- $G = SO(W) \times SO(V)$, $H = SO(W) \ltimes N$
- $D = F z_0$
- $V_0 = W \oplus D$
- $H_0 = SO(W)$ and $G_0 = SO(W) \times SO(V_0)$. We consider H_0 as a subgroup of G_0 via the diagonal embedding $H_0 \hookrightarrow G_0$. The triple $G_0, H_0, 1$ is the GGP triple associated to the admissible pair W, V_0
- T is the subtorus of $SO(V)$ preserving the lines $\langle z_i \rangle$, for $i = \pm 1, \ldots, \pm r$ and acting trivially on V_0. We have $M = T \times G_0$

2.1 Definition of GGP triples

Let (W, V) be a pair of quadratic spaces. We will call (W, V) an *admissible pair* if there exists a quadratic space Z satisfying

- $V \cong W \oplus^{\perp} Z$
- Z is odd dimensional and $SO(Z)$ is quasi-split.

The second condition means that there exists $\nu \in F^{\times}$ and a basis $(z_{-r}, \dots, z_{-1}, z_0, z_1, \dots, z_r)$ of Z such that

2.1.1)
$$b(z_i, z_j) = \nu\delta_{i,-j}$$

for all $i, j \in \{0, \pm 1, \dots, \pm r\}$. Let W, V be an admissible pair. Set $G = SO(W) \times SO(V)$. We will associate a triple (G, H, ξ) where H is an algebraic subgroup of G and $\xi : H(F) \to C^{\times}$ is a continuous character of $H(F)$. This triple is unique up to $G(F)$-conjugacy. Fix an embedding $W \subseteq V$ and set $Z = W^{\perp}$. Also fix $\nu \in F^{\times}$ and a basis $(z_i)_{i=0,\pm 1,\dots,\pm r}$ (where $\dim(Z) = 2r + 1$) of Z satisfying 2.1.1. Denote P_V the stabilizer in $SO(V)$ of the following flag of totally isotropic subspaces of V

$$\langle z_r \rangle \subset \dots \subset \langle z_r, \dots, z_1 \rangle$$

Then, P_V is a parabolic subgroup of $SO(V)$. We write N as its unipotent radical. Let M_V be the stabilizer in P_V ofthe lines $\langle z_i \rangle$ for $i = \pm 1, \dots, \pm r$. It is a levi component of P_V. Set $P = SO(W) \times P_V$. Then, P is a prabolic subgroup of G with unipotent radical N and $M = SO(W) \times M_V$ is a Levi component of it. We identify $SO(W)$ with its image via the diagonal embedding $SO(W) \hookrightarrow G$, we have $SO(W) \subseteq M$. In particular, conjugation by $SO(W)$ preserves N and we set
$$H = SO(W) \ltimes N$$

Now we will define the character ξ. Define a morphism $\lambda : N \to \mathbb{G}_a$ by

$$\lambda(n) = \sum_{i=0}^{r-1} b(z_{-i-1}.nz_i),\ n \in N$$

λ is $SO(W)$-invariant, so it admits a unique extension, still denoted by λ to a morphism $H \to \mathbb{G}_a$ which is trivial on $SO(W)$. We denote $\lambda_F : H(F) \to F$ induced on the groups of F-points. Recall that we fixed a non-trivial continuous additive character ψ of F. We set

$$\xi(h) = \psi(\lambda_F(h))$$

for all $h \in H(F)$. This gives us the definition of the triple (G, H, ξ). It is easy to check that this definition depends on various choices made only up to $G(F)$-conjugacy. A triple obtained from an admissible pair (W, V) in this is called a *Gan-Gross-Prasad triple* or *GGP triple* for short. We will use the following additional notation:

- $d = \dim(V)$ and $m = \dim(W)$
- $D = Fz_0$
- $V_0 = W \oplus D$
- $H_0 = SO(W)$ and $G_0 = SO(W) \times SO(V_0)$. We consider H_0 as a subgroup of G_0 via the diagonal embedding $H_0 \hookrightarrow G_0$. The triple $(G_0, H_0, 1)$ is the GGP triple associated to the admissible pair W, V_0
- T is the subtorus of $SO(V)$ preserving the lines $\langle z_i \rangle$, for $i = \pm 1, \ldots, \pm r$ and acting trivially on V_0. We have $M = T \times G_0$
- A is the split part of the torus T, it is also the split part of the center of M
- ξ the character of $\mathfrak{h}(F)$, where $\mathfrak{h} = \mathrm{Lie}(H)$, which is trivial on $\mathfrak{u}(W)(F)$ and equal to $\xi \circ \exp$ on $\mathfrak{n}(F)$

Note that when $r = 0$ (that is $Z = D$ is a line), we have $G = G_0$, $H = H_0$, and $\xi = 1$. This is the *codimension one case.*

We will need the following lemma, (1.2 for definition of norm descent property)

Lemma 2.1.2 *(i) The map $G \to H\backslash G$ has the norm descent property.*

(ii) The orbit under M-conjugacy of λ in $(\mathfrak{n}/[\mathfrak{n},\mathfrak{n}])^*$ is a Zariski open subset

Proof. (i) We have a natural identification $H\backslash G = N\backslash SO(V)$, so it is sufficient to prove that $SO(V) \to N\backslash SO(V)$ has the norm descent property. Since this map is $SO(V)$-equivariant for the obvious transitive actions, we only need to show that it admits a section over a nonempty Zariski-open subset. If we denote $\overline{P}_V = M_V\overline{N}$ the parabolic subgroup opposite to P_V with respect to M_V, the multiplication map $N \times M_V \times \overline{N} \to SO(V)$ is an open immersion. The image of that open subset is open in $N\backslash SO(V)$ and the restriction of the projection $SO(V) \to N\backslash SO(V)$ to that open set is $N \times M_V \times \overline{N} \to M_V \times \overline{N}$. This map obviously has a section

(ii) If $r = 0$, *i.e.*, if we are in the codimension one case, we have $\mathfrak{n} = 0$ and the result is trivial. Assume now that $r \geq 1$. It suffices to show that the dimension of the orbit $M \cdot \lambda$ is equal to the dimension of $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$. We can compute

$$\dim(\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]) = 2r$$

and

$$\dim(M) = \frac{m(m-1)}{2} + 2r + \frac{(m+1)m}{2} = m^2 + 2r$$

The stabilizer M_λ of λ is easily seen to be $M_\lambda = Z(G)(SO(W) \times SO(W))$ (where $Z(G)$ denote the center of G). Hence we have

$$\dim(M_\lambda) = m^2$$

and so the dimension of the orbit is

$$\dim(M \cdot \lambda) = \dim(M) - \dim(M_\lambda) = m^2 + 2r - m^2 = 2r$$

which is the same as $\dim(\mathfrak{n} / [\mathfrak{n}, \mathfrak{n}])$.

$\square$

check these dimension in above proof sometime

2.2 The multiplicity of $m(\pi)$

For $\pi \in \mathrm{Temp}(G)$, let us denote $\mathrm{Hom}_H(\pi, \xi)$ for the space of all continuous linear forms $l : \pi^\infty \to \mathbb{C}$ such that

$$l(\pi(h)e) = \xi(h)l(e)$$

for all $e \in \pi^\infty$ and for all $h \in H(F)$. We define the *multiplicity* $m(\pi)$ to be the dimension of that space of linear forms, that is

$$m(\pi) = \dim \mathrm{Hom}_H(\pi, \xi),\ \pi \in \mathrm{Temp}(G)$$

We have the following theorem

Theorem 2.2.1 (6.3.1) *We have*

$$m(\pi) \leq 1$$

for all $\pi \in \mathrm{Temp}(G)$.

need sources

Note that we have

on a second look bar probs means complex conjugation so maybe disregard below

2.2.2)

$$m(\pi) = m(\overline{\pi})$$

for all $\pi \in \mathrm{Temp}(G)$. Indeed, the conjugation map $l \to \bar{l}$ induces an isomorphism

$$\mathrm{Hom}_H(\pi, \xi) \cong \mathrm{Hom}(\overline{\pi}, \overline{\xi})$$

and we can easily check, there exists an element $a \in A(F)$ such that $\xi(aha^{-1}) = \overline{\xi(h)}$ for all $h \in H(F)$, hence the linear map $l \to l \circ \pi(a)$ induces an isomorphism

$$\mathrm{Hom}(\overline{\pi}, \overline{\xi}) \cong \mathrm{Hom}_H(\overline{\pi}, \xi)$$

and 2.2.2 follows.

2.3 $H\backslash G$ is a spherical variety, good parabolic subgroups

A parabolic subgroup $\overline{Q}$ of G is *good* if $H\overline{Q}$ is Zariski-open in G. This is equivalent to $H(F)\overline{Q}(F)$ being open in $G(F)$.

Proposition 2.3.1 (6.4.1) *(i) There exists minimal parabolic subgroups of G that are good and they are all conjugate under $H(F)$. Moreover, if $\overline{P}_{min} = M_{min}\overline{U}_{min}$ is a good minimal parabolic subgroup then $H \cap \overline{U}_{min} = \{1\}$ and the complement of $H(F)\overline{P}_{min}(F))$ in $G(F)$ has null measure;*

(ii) A Parabolic subgroup $\overline{Q}$ of G is good if and only if it contains a good minimal parabolic subgroup.

(iii) Let $\overline{P}_{min} = M_{min}\overline{U}_{min}$ be a good parabolic subgroup and let $A_{min} = A_{M_{min}}$ be the maximal split central subtorus of M_{min}. Set

$$A_{min}^{+} = \{a \in A_{min}(F); |\alpha(a)| \geq 1 \forall \alpha \in R(A_{min}, \overline{P}_{min})\}$$

Then, we have the inequalities

2.3.2) $\sigma(h) + \sigma(a) \ll \sigma(ha)$ for all $a \in A_{min}^{+}$, $h \in H(F)$.

2.3.3) $\sigma(h) \ll \sigma(a^{-1}ha)$ for all $a \in A_{min}^{+}$, $h \in H(F)$.

Proof. (i) Set $w_0 = z_0$ and choose a family $(w_1, \dots, w_l)$ of mutually orthogonal vectors in W which is maximal subject to the condition

$$q(w_i) = (-1)^i \nu,\ i = 1, \dots, l$$

Define u_i for $i = 1, \dots, \lceil \frac{l}{2} \rceil$ by

$$u_i = w_{2i-2} + w_{2i-1}$$

and u_i' for $i = 1, \dots, \lfloor \frac{l}{2} \rfloor$ by

$$u_i' = w_{2i-1} + w_{2i}$$

Then the subspaces

$$Z_{V_0} = \langle u_1, \dots, u_{\lceil \frac{l}{2} \rceil} \rangle$$
$$Z_W = \langle u_1', \dots, u'_{\lfloor \frac{l}{2} \rfloor} \rangle$$

are maximal isotropic subspaces of V_0 and W respectively. Let $\overline{P}_{V_0}$ and $\overline{P}_W$ be the stabilizers in $SO(V_0)$ and $SO(W)$ of the totally isotropic flags

$$\langle u_1 \rangle \subseteq \langle u_1, u_2 \rangle \subseteq \dots \subseteq \langle u_1, \dots, u_{\lceil \frac{l}{2} \rceil} \rangle$$

and

$$\langle u_1' \rangle \subseteq \langle u_1', u_2' \rangle \subseteq \dots \subseteq \langle u_1', \dots, u'_{\lfloor \frac{l}{2} \rfloor} \rangle$$

respectively. Then $\overline{P}_{V_0}$ and $\overline{P}_W$ are minimal parabolic subgroups of $SO(V_0)$ and $SO(W)$ respectively. Set

$$\overline{P}_0 = \overline{P}_W \times \overline{P}_{V_0}$$

It is a minimal parabolic subgroup of G_0. Let W_{an} be the orthogonal complement in W of $\langle w_1, \ldots, w_l \rangle$. We claim the following

2.3.4) We have $H_0 \cap \overline{P}_0 = SO(W_{an})$ and $H_0\overline{P}_0$ is Zariski-open in G_0 (i.e., $\overline{P}_0$ is a good parabolic subgroup of G_0).

The second claim follows from the first one by dimension consideration. To prove the first claim let $h_0 \in H_0 \cap \overline{P}_0$. Consider the action of h_0 on V_0. Since h_0 belongs to H_0 it must stabilize $w_0 = z_0$. On the other hand, since h_0 belongs to $\overline{P}_0$ it must stabilize $\langle w_0 + w_1 \rangle$. Because w_0 is orthogonal to w_1, it follows that h_0 fixes w_1. We show similarly that h_0 fixes $w_2, \ldots, w_l$, hence $h_0 \in SO(W_{an})$. This prove 2.3.4.

Let $\overline{P} = M\overline{N}$ be the parabolic subgroup opposite to P with respect to M and set

$$\overline{P}_{min} = \overline{P}_0 T \overline{N}$$

which is a minimal parabolic subgroup of G. We deduce easily from 2.3.4 the following

2.3.5) $\overline{P}_{min}$ is a good parabolic subgroup and we have $\overline{P}_{min} \cap H = SO(W_{an})$.

This proves that there exists minimal parabolic subgroups that are good. Let $\overline{P}'_{min}$ be another good minimal parabolic subgroup. We ill show that $\overline{P}_{min}$ and $\overline{P}'_{min}$ are conjugate under $H(F)$. Let $g \in G(F)$ such that $\overline{P}'_{min} = g\overline{P}_{min}g^{-1}$. Set $\mathcal{U} = H\overline{P}_{min}$ and $\mathcal{Z} = G - \mathcal{U}$. Then $\mathcal{Z}$ is a proper Zariski-closed subset of G which is $H \times \overline{P}_{min}$-invariant (for the left and right multiplication respectively). If $g \in \mathcal{Z}$, then we would have

$$H\overline{P}'_{min} = Hg\overline{P}_{min}g^{-1} \subseteq \mathcal{Z}g^{-1}$$

which is impossible since $\overline{P}'_{min}$ is a good parabolic subgroup. Thus, we have $g \in \mathcal{U} \cap G(F) = \mathcal{U}(F)$. If we can prove that $g \in H(F)\overline{P}_{min}(F)$, then we are done. Hence, it suffices to show that

2.3.6)

$$\mathcal{U}(F) = H(F)\overline{P}_{min}(F)$$

This follows from

2.3.7) The map $H^1(F, H \cap \overline{P}_{min}) \to H^1(F, H)$ is injective.

By 2.3.5 we have $H^1(F, H \cap \overline{P}_{min}) = H^1(F, SO(W_{an}))$. Since $H = SO(W) \rtimes N$ with N unipotent, we also have $H^1(F, H) = H^1(F, SO(W))$. The two sets $H^1(F, SO(W_{an}))$ and $H^1(F, SO(W))$ classify the (isomorphism classes of) quadratic spaces of the same dimension as W_{an} and W respectively. Moreover, the map $H^1(F, SO(W_{an})) \to H^1(F, SO(W))$ we are considering sends W'_{an} to $W'_{an} \oplus W_{an}^{\perp}$, where $W_{an}^{\perp}$ denotes the orthogonal complement of W_{an} in W. By Witt's theorem, this map is injective. This proves 2.3.7 and ends the proof that all good minimal parabolic subgroups are conjugate under $H(F)$.

unsure abt above argument

It only remains to show the last part of (i) which is that $H \cap \overline{U}_{min} = \{1\}$ and the complement of $H(F)\overline{P}_{min}(F)$ in $G(F)$ has null measure for every good minimal parabolic subgroup $\overline{P}_{min} = M_{min}\overline{U}_{min}$. Since we already proved that all good minimal parabolic subgroups are $H(F)$-conjugate, we only need to consider one of them. Let $\overline{P}_{min}$ be the parabolic subgroup tht we constructed above, then the result follows directly from 2.3.5 and 2.3.6.

(ii) Let $\overline{Q}$ be a good parabolic subgroup and choose $P_{min} \subseteq \overline{Q}$ a minimal parabolic subgroup. Set

$$\mathcal{G} := \{g \in G : g^{-1}P_{min}g \text{ is good}\}$$

This is a Zariski-open subset of G since it is the inverse image of the Zariski-open subset $\{\mathcal{V} \in Gr_n(\mathfrak{g}) : \mathcal{V} + \mathfrak{h} = \mathfrak{g}\}$ of the Grassmannian variety $Gr_n(\mathfrak{g})$, where $n = \dim(P_{min})$, by the regular map $g \in G \mapsto g^{-1}\mathfrak{p}_{min}g \in Gr_n(\mathfrak{g})$. Moreover, it is non-empty as (i) tells us there exists good minimal parabolic subgroups. Since $\overline{Q}$ is good, the intersection $\overline{Q}H \cap \mathcal{G}$ is non-empty too. Hence, we may find $\overline{q}_0 \in \overline{Q}$ such that $\overline{q}_0^{-1}P_{min}\overline{q}_0$ is a good parabolic subgroup. This parabolic subgroup is contained in $\overline{Q}$ but it may not be defined over F. Define

$$\mathcal{Q} := \{\overline{q} \in \overline{Q} : \overline{q}^{-1}P_{min}\overline{q} \text{ is good}\}$$

But $\mathcal{Q}$ is a Zariski-open subset of $\overline{Q}$ which is non-empty. Since $\overline{Q}(F)$ is Zariski-dense in $\overline{Q}$, $\mathcal{Q}(F)$ is non-empty. Thus, for all $\overline{q} \in \mathcal{Q}(F)$ the parabolic subgroup $\overline{q}^{-1}P_{min}\overline{q}$ has all the desired properties.

(iii) First we will show that 2.3.2 and 2.3.3 don't depend on the particular pair $(\overline{P}_{min}, M_{min})$ chosen. Let $(\overline{P}'_{min}, M'_{min})$ be another such pair with $\overline{P}'_{min}$ a good prabolic subgroup and M'_{min} is a levi component of it. Then, by (i), there exists $h \in H(F)$ such that $\overline{P}'_{min} = h\overline{P}_{min}h^{-1}$. The inequalities 2.3.2 and 2.3.3 are true for the pair $(\overline{P}_{min}, M_{min})$ if and only if they are true for the pair $(h\overline{P}_{min}h^{-1}, hM_{min}h^{-1}) = (\overline{P}'_{min}, hM_{min}h^{-1})$. So we may assume without loss of generality that $\overline{P}_{min} = \overline{P}'_{min}$ and all that remains to show is that the inequalities do not depend on the choice of M_{min}.

There exists $\overline{u} \in \overline{U}_{min}(F)$ such that $M'_{min} = \overline{u}M_{min}\overline{u}^{-1}$ so we have $A'^{+}_{min} = \overline{u}A^{+}_{min}\overline{u}^{-1}$. By definition of A^{+}_{min}, the sets $\{a^{-1}\overline{u}a \,|\, a \in A^{+}_{min}\}$ and $\{a^{-1}\overline{u}^{-1}a \,|\, a \in A^{+}_{min}$ are bounded. It follows that

$$\sigma(h\overline{u}a\overline{u}^{-1}) \sim \sigma(ha)$$

$$\sigma(\overline{u}a^{-1}\overline{u}^{-1}h\overline{u}a\overline{u}^{-1}) \sim \sigma(a^{-1}ha)$$

for all $a \in A^{+}_{min}$ and all $h \in H(F)$. Thus, 2.3.2 and 2.3.3 are independent of choice of M_{min} and are satisfied for $(\overline{P}_{min}, M_{min})$ if and only if they are satisfied for $(\overline{P}'_{min}, M'_{min})$.

Now we will now reduce the proof of 2.3.2 and 2.3.3 to the codimension one case. Let $\overline{P}_0 = M_0\overline{U}_0$ be a good minimal parabolic subgroup of G_0. Let $A_0 = A_{M_0}$ be the split part of the center of M_0 and let

$$A_0^{+} = \{a_0 \in A_0(F) : |\alpha(a)| \geq 1 \,\forall \alpha \in R(A_0, \overline{P}_0)\}$$

Set $\overline{P}_{min} = \overline{P}_0 T\overline{N}$ and $M_{min} = M_0T$. Then, $\overline{P}_{min}$ is a good parabolic subgroup of G, M_{min} is a Levi component of it, and $A^{+}_{min} \subseteq A(F)A_0^{+}$. We have

$$\sigma(n) + \sigma(a) + \sigma(h_0a_0) \ll \sigma(nh_0aa_0)$$

for all $h = nh_0 \in H(F) = N(F)H_0(F)$ and all $(a, a_0) \in A(F) \times A_0^{+}$. Since $\sigma(aa_0) \sim \sigma(a) + \sigma(a_0)$ for all $(a, a_0) \in A(F) \times A_0^{+}$ and $\sigma(nh_0) \sim \sigma(n) + \sigma(h_0)$ for all $(n, h_0) \in N(F) \times H_0(F)$, we have that 2.3.2 will follow from

2.3.8) $\sigma(h_0) + \sigma(a_0) \ll \sigma(h_0a_0)$, for all $a_0 \in A_0^{+}$ and all $h_0 \in H_0(F)$.

On the other hand, we also have $\sigma(n) \ll \sigma(a^{-1}na)$ for all $a \in A^{+}_{min}$ and all $n \in N(F)$. Thus

$$\begin{aligned}
\sigma(a^{-1}nh_0a) &\gg \sigma(a^{-1}naa^{-1}h_0a) \\
&\gg \sigma(a^{-1}na) + \sigma(a^{-1}h_0a) \\
&\gg \sigma(n) + \sigma(a^{-1}h_0a) \\
&= \sigma(n) + \sigma(a_0^{-1}(aa_0^{-1})^{-1}h_0(aa_0^{-1})a_0) \\
&= \sigma(n) + \sigma(a_0^{-1}h_0a_0)
\end{aligned}$$

for all $h = nh_0 \in H(F) = N(F)H_0(F)$ and all $a \in A^{+}_{min}$ where a_0 denotes the unique element of A_0^{+} such that $aa_0^{-1} \in A(F)$. Hence, 2.3.3 will follow from

2.3.9) $\sigma(h_0) \ll \sigma(a_0^{-1}h_0a_0)$, for all $a_0 \in A_0^+$ and all $h_0 \in H_0(F)$.

What is left now is to show that 2.3.8 and 2.3.9 are true for any pair $(\overline{P}_0, M_0)$. We will choose the following pair.

Introduce a sequence $(w_0, \ldots, w_l)$ and a parabolic subgroup $\overline{P}_0 = \overline{P}_W \times \overline{P}_{V_0}$ of G_0 as in part (i). By 2.3.4, $\overline{P}_0$ is a good parabolic subgroup of G_0. Let M_{V_0} be the Levi component of P_{V_0} that preserve the lines

$$\langle u_1\rangle, \ldots, \langle u_{\lceil \frac{l}{2}\rceil}\rangle \text{ and } \langle u_{-1}\rangle, \ldots, \langle u_{-\lceil \frac{l}{2}\rceil}\rangle$$

where here we have set $u_{-i} = w_{2i-2} - w_{2i-1}$ for $i = 1, \ldots, \lceil \frac{l}{2}\rceil$. Let M_W be the levi component of $\overline{P}_W$ that preserves the lines

$$\langle u'_1\rangle, \ldots, u'_{\lfloor \frac{l}{2}\rfloor}\rangle \text{ and } \langle u'_{-1}\rangle, \ldots, u'_{-\lfloor \frac{l}{2}\rfloor}\rangle$$

where we have set $u'_{-i} = w_{2i-1} - w_{2i}$ for $i = 1, \ldots, \lfloor \frac{l}{2}\rfloor$. Set

$$M_0 = M_W \times M_{V_0}$$

.

This is a Levi component of $\overline{P}_0$ and we will prove 2.3.8 and 2.3.9 for this pair $(\overline{P}_0, M_0)$. We have the decomposition

$$A_0^+ = A_W^+ \times A_{V_0}^+$$

where A_W^+ and $A_{V_0}^+$ are defined in the obvious way. For all $a_{V_0} \in A_{V_0}^+$ (resp. $a_W \in A_W^+$) let us denote the eigenvalues of a_{V_0} (resp. a_W) acting on $u_1, \ldots, u_{\lceil \frac{l}{2}\rceil}$ (resp. on $u'_1, \ldots, u'_{\lfloor \frac{l}{2}\rfloor}$) by $a_{V_0}^1, \ldots, a_{V_0}^{\lceil \frac{l}{2}\rceil}$ (resp. $a_W^1, \ldots, a_W^{\lfloor \frac{l}{2}\rfloor}$). Then, we have

$$|a_{V_0}^1| \geq \cdots \geq |a_{V_0}^{\lceil \frac{l}{2}\rceil}| \geq 1$$
$$|a_W^1| \geq \cdots \geq |a_W^{\lfloor \frac{l}{2}\rfloor}| \geq 1$$

for all $a_{V_0} \in A_{V_0}^+$ and all $a_W^+ \in A_W^+$.

We also have

$$\begin{aligned}
\sigma(h_0) + \sigma(a_0) &\ll \sigma(h_0) + \sigma(h_0a_0) \\
\sigma(h_0) &\ll \sigma_{SO(V_0)}(h_0a_{V_0}) + \sigma_{SO(V_0)}(a_{V_0}) \\
\sigma_{SO(V_0)}(h_0a_{V_0}) &\ll \sigma(h_0a_0)
\end{aligned}$$

for all $a_0 = (a_W, a_{V_0}) \in A_0^+ = A_W^+ \times A_{V_0}^+$ and all $h_0 \in H_0(F)$. So 2.3.8 will follow from

2.3.10) $\sigma_{SO(V_0)}(a_{V_0}) \ll \sigma_{SO(V_0)}(h_0 a_{V_0})$, for all $a_{V_0} \in A^+_{V_0}$ and all $h_0 \in H_0(F)$.

We have

$$\sigma(a_{V_0}) \sim \log(1 + |a^1_{V_0}|)$$

for all $a_{V_0} \in A^+_{V_0}$. Moreover, for all $a_{V_0} \in A^+_{V_0}$ and all $h_0 \in H_0(F)$ we have

$$\begin{aligned} b(h_0 a_{V_0} u_1, w_0) &= a^1_{V_0} b(h_0 u_1, w_0) \\ &= a^1_{V_0}(b(w_0, w_0) + b(h_0 w_1, w_0)) \\ &= a^1_{V_0}(\nu + 0) \\ &= a^1_{V_0}\nu \end{aligned}$$

and we also have $\log(1 + |b(gu_1, w_0)|) \ll \sigma_{SO(V_0)}(g)$ for all $g \in SO(W)$. Thus

$$\sigma_{SO(V_0)}(a_{V_0}) \sim \log(1+|a^1_{V_0}|) \sim \log(1+|b(h_0 a_{V_0} u_1, w_0)|) \ll \sigma_{SO(V_0)}(h_0 a_{V_0})$$

as desired which proves the first inequality.

Now we will prove 2.3.9. As $\sigma(h_0) \sim \max_{v,v'} \log(2+|b(h_0 v, v')|)$ for $v, v' \in V_0$ and $h_0 \in H_0(F)$ it is sufficient to prove the following

2.3.11) For all $v, v' \in V_0$ we have

$$\log(2 + |b(h_0 v, v')|) \ll \sigma(a_0^{-1} h_0 a_0)$$

for all $a_0 \in A^+_0$ and all $h_0 \in H_0(F)$.

By bilinearity and since $b(h_0 v, v') = b(h_0^{-1} v', v)$, it suffices to prove 2.3.11 in the following cases

- $v = w_i$ and $v' = \langle w_i, \dots, w_l\rangle \oplus W_{an}$ for $0 \le i \le l$,
- $v, v' \in W_{an}$

Recall that W_{an} is the orthogonal complement of $\langle w_0, \dots, w_l\rangle$ in V_0.

The proof of 2.3.11 is easy in the second case. As we have

$$\begin{aligned} b(a^{-1}_{V_0} h_0 a_{V_0} v, v') &= b(h_0 a_{V_0} v, a_{V_0} v') \\ &= b(h_0 v, v') \end{aligned}$$

for all $a_{V_0} \in A_{V_0}(F)$, $h_0 \in H_0(F)$, and $v, v' \in W_{an}$. Here $A_{V_0}(F)$ is the split part of the central torus of M_{V_0}.

For the first case we will proceed by induction on i. If $i = 0$ then we have $h_0 w_0 = w_0$ for all $h_0 \in H_0(F)$ so $\sigma(h_0) \sim \sigma(1) \ll \sigma(a_0^{-1} h_0 a_0)$ for all $a_0 \in A_0^+$ and all $h_0 \in H_0(F)$.

Now let $1 \leq i \leq l$ and assume that 2.3.11 is true for $v = w_{i-1}$ and all $v' = \langle w_{i-1}, \dots, w_l \rangle \oplus W_{an}$. We have that $w_i = u_{(i-1)/2} - w_{i-1}$. So

$$\log(2 + |b(h_0 w_i, v')|) \ll \log(2 + |b(h_0 u_{(i-1)/2}, v')|) + \log(2 + |b(h_0 w_{i-1}, v')|)$$

and thus by our induction hypothesis we just need to show that

$$\text{2.3.12)} \quad \log(2 + |b(h_0 u_{(i-1)/2}, v')|) \ll \sigma(a_0^{-1} h_0 a_0)$$

for all $a_0 \in A_0^+$ and all $h_0 \in H_0(F)$

If i is odd then the subspace $\langle w_{i-1}, \dots, w_l \rangle \oplus W_{an} = \langle u_{i-1}, \dots, u_{\lceil \frac{l}{2} \rceil} \rangle \oplus W_{an}$ is preserved by A_{V_0}. By bilinearity we only need to prove 2.3.12 for v' an eigenvector for the action of A_{V_0} on that subspace. For each $a_{V_0} \in A_{V_0}^+$, the eigenvalue of a_{V_0} on v' has an absolute value which is greater than or equal to $|a_{V_0}^{(i-1)/2}|^{-1}$. Hence we have

$$\begin{aligned}
\sigma(a_0^{-1} h_0 a_0) &\gg \log(2 + |b(a_{V_0}^{-1} h_0 a_{V_0} u_{(i-1)/2}, v')|) \\
&\gg \log(2 + |a_{V_0}^{(i-1)/2}||b(h_0 u_{(i-1)/2}, a_{V_0} v')|) \\
&\gg \log(2 + |b(h_0 u_{(i-1)/2}, v')|)
\end{aligned}$$

for all $a_0 = (a_w, a_{V_0}) \in A_0^+ = A_W^+ \times A_{V_0}^+$ and all $h_0 \in H_0(F)$. If i is even, the proof is similar using the action on W rather than on V_0. This gives us the desired inequality.

□

2.4 Some Estimates

Lemma 2.4.1 (6.5.1) *(i) There exists $\epsilon > 0$ such that the integral*

$$\int_{H_0(F)} \Xi^{G_0}(h_0) e^{\epsilon \sigma(h_0)} dh_0$$

is absolutely convergent.

(ii) There exists $d > 0$ such that the integral

$$\int_{H(F)} \Xi^G \sigma(h)^{-d} dh$$

is absolutely convergent

(iii) *For all $\delta > 0$ there exists $\epsilon > 0$ such that the integral*

$$\int_{H(F)} \Xi^G(h) e^{\epsilon\sigma(h)} (1 + |\lambda(h)|)^{-\delta} dh$$

is absolutely convergent (where $\lambda : H \to \mathbb{G}_a$ is the homomorphism defined in Section 2.1).

Let $\overline{P}_{min} = M_{min}\overline{U}_{min}$ be a good minimal parabolic subgroup of G. We have the following

(iv) *For all $\delta > 0$ there exists $\epsilon > 0$ such that the integral*

$$I^1_{\epsilon,\delta}(m_{min}) = \int_{H(F)} \Xi^G(hm_{min}) e^{\epsilon\sigma(h)} (1 + |\lambda(h)|)^{-\delta} dh$$

is absolutely convergent for all $m_{min} \in M_{min}(F)$ and there exists $d > 0$ such that

$$I^1_{\epsilon,\delta}(m_{min}) \ll \delta_{\overline{P}_{min}}(m_{min})^{-1/2} \sigma(m_{min})^d$$

for all $m_{min} \in M_{min}(F)$.

(v) *Assumes moreover that A is contained in $A_{M_{min}}$. Then, for all $\delta > 0$ there exists $\epsilon > 0$ such that the integral*

$$I^2_{\epsilon,\delta}(m_{min}) = \int_{H(f)} \int_{H(F)} \Xi^G(hm_{min}) \Xi^G(h'hm_{min}) e^{\epsilon\sigma(h)} e^{\epsilon\sigma(h')} (1+|\lambda(h')|)^{-\delta} dh'dh$$

is absolutely convergent for all $m_{min} \in M_{min}(F)$ and there exists $d > 0$ such that

$$I^2_{\epsilon,\delta} \ll \delta_{\overline{P}_{min}}(m_{min})^{-1} \sigma(m_{min})^d$$

for all $m_{min} \in M_{min}(F)$.

Proof. (i) This follows from the following fact

2.4.2) There exists $\epsilon' > 0$ such that

$$\Xi^{G_0}(h_0) \ll \Xi^{H_0}(h_0)^2 e^{-\epsilon'\sigma(h_0)}$$

for all $h_0 \in H_0(F)$.

Proof in paper by waldspurger ref from chen later

(ii) Let $d > 0$. By Proposition 1.3.1(iv), if d is sufficiently large, we have

$$\int_{H(F)} \Xi^G(h)\sigma(h)^{-d} dh = \int_{H_0(F)} \int_{N(F)} \Xi^G(h_0 n)\sigma(h_0 n)^{-d} dn dh_0$$
$$\ll \int_{H_0(F)} \Xi^{G_0}(h_0) dh_0$$

Note that $\delta_P(h_0) = 1$ and $\Xi^M(h_0) = \Xi^{G_0}(h_0)$ for all $h_0 \in H_0(F)$ and this last integral is absolutely convergent by (i).

(iii) By (i) and since $\sigma(h_0 n) \ll \sigma(h_0) + \sigma(n)$ for all $h_0 \in H_0(F)$ and all $n \in N(F)$, it suffices to establish

2.4.3) For all $\delta > 0$ and all $\epsilon_0 > 0$, there exists $\epsilon > 0$ such that the integral

$$I^0_{\epsilon,\delta}(h_0) = \int_{N(F)} \Xi^G(nh_0) e^{\epsilon\sigma(n)} (1 + |\lambda(n)|)^{-\delta} dn$$

is absolutely convergent for all $h_0 \in H_0(F)$ and satisfies the inequality

$$I^0_{\epsilon,\delta}(h_0) \ll \Xi^{G_0}(h_0) e^{\epsilon_0 \sigma(h_0)}$$

for all $h_0 \in H_0(F)$.

Let $\delta > 0, \epsilon_0 > 0$ and $\epsilon > 0$. We want to prove 2.4.3 holds if ϵ is sufficiently small (compared to δ and ϵ_0). We will introduce an auxiliary parameter $b > 0$ that we will make more precise later. For all $h_0 \in H_0(F)$, we have $I^0_{\epsilon,\delta}(h_0) = I^0_{\epsilon,\delta,\leq b}(h_0) + I^0_{\epsilon,\delta,>b}(h_0)$ where

$$I^0_{\epsilon,\delta,\leq b}(h_0) = \int_{N(F)} \mathbb{1}_{\sigma \leq b}(n) \Xi^G(nh_0) e^{\epsilon\sigma(n)} (1 + |\lambda(n)|)^{-\delta} dn$$

$$I^0_{\epsilon,\delta,>b}(h_0) = \int_{N(F)} \mathbb{1}_{\sigma > b}(n) \Xi^G(nh_0) e^{\epsilon\sigma(n)} (1 + |\lambda(n)|)^{-\delta} dn$$

For all $d > 0$, we have

$$I_{\epsilon,\delta \leq b}(h_0) \leq e^{\epsilon b} b^d \int_{N(F)} \Xi^G(nh_0) \sigma(n)^{-d} dn$$

for all $h_0 \in H_0(F)$ and all $b > 0$. By proposition 1.3.1(iv), we may choose $d > 0$ such that the last integral above is essentially bounded by $\delta_P(h_0)^{1/2} \Xi^M(h_0) = \Xi^{G_0}(h_0)$ for all $h_0 \in H_0(F)$. From here on we will fix such a $d > 0$. Hence, we have

2.4.4)
$$I^0_{\epsilon,\delta,\leq b}(h_0) \ll e^{\epsilon b} b^d \Xi^{G_0}(h_0)$$

for all $h_0 \in H_0(F)$ and all $b > 0$.

There exists $\alpha > 0$ such that $\Xi^G(g_1 g_2) \ll e^{\alpha\sigma(g_2)} \Xi^G(g_1)$ for all $g_1, g_2 \in G(F)$. It follows that

2.4.5)

$$I^0_{\epsilon,\delta,>b}(h_0) \ll e^{\alpha\sigma(h_0) - \sqrt{\epsilon} b} \int_{N(F)} \Xi^G(n) e^{(\epsilon + \sqrt{\epsilon})\sigma(n)} (1 + |\lambda(n)|)^{-\delta} dn$$

for all $h_0 \in H_0(F)$ and all $b > 0$. Assume that 2.4.5 is convergent if ϵ is sufficiently small. Then we can combine 2.4.4 and 2.4.5 and get

$$I^0_{\epsilon,\delta}(h_0) \ll e^{\epsilon b} b^d \Xi^{G_0}(h_0) + e^{\alpha\sigma(h_0) - \sqrt{\epsilon} b}$$

for all $h_0 \in H_0(F)$ and for all $b > 0$. There exists $\beta > 0$ such that $e^{-\beta\sigma(h_0)} \ll \Xi^{G_0}(h_0)$ for all $h_0 \in H_0(F)$. Plugging $b = \frac{\alpha+\beta}{\sqrt{\epsilon}}\sigma(h_0)$ in the last inequality, we obtain

$$I^0_{\epsilon,\delta}(h_0) \ll e^{\sqrt{\epsilon}(\alpha+\beta+1)\sigma(h_0)} \Xi^{G_0}(h_0)$$

for all $h_0 \in H_0(F)$. Hence, for $\epsilon \leq \epsilon_0^2(\alpha+\beta+1)^{-2}$, 2.4.3 holds.

It remains to show that 2.4.5 converges for ϵ sufficiently small. If P is a minimal parabolic subgroup of G then it follows from Corollary B.3.2 in paper (since in this case λ is a generic additive character on N). Assume this is not the case. Then we can find two isotropic vectors $z_{0,+}, z_{0,-} \in V_0$ such that $z_0 = z_{0,+} - z_{0,-}$. We have a decomposition $\lambda = \lambda_+ - \lambda_-$ where

$$\lambda_+(n) = \sum_{i=1}^{r-1} b(z_{-i-1}, nz_i) + b(z_{-1}, nz_{0,+}) \qquad n \in N$$
$$\lambda_-(n) = b(z_{-1}, nz_{0,-}) \qquad n \in N$$

Note that the additive character λ_+ is the restriction to N of a generic additive character of the unipotent radical of a minimal parabolic subgroup contained in P. Hence, same ref as before applies to λ_+. Choose a one-parameter subgroup $a : \mathbb{G}_m \to M$ such that $\lambda_+(a(t)na(t)^{-1}) = t\lambda_+(n)$ and $\lambda_-(a(t)na(t)^{-1}) = t^{-1}\lambda_-(n)$ for all $t \in \mathbb{G}_m$ and $n \in N$ (such a one-parameter subgroup is easily seen to exist). Let $\mathcal{U} \subseteq F^\times$ be a compact neighborhood of 1. Then, for all $\epsilon > 0$, we have

$$\begin{aligned}
&\int_{N(F)} \Xi^G(n) e^{\epsilon\sigma(n)} (1 + |\lambda(n)|)^{-\delta} dn \\
&\ll \int_{N(F)} \Xi^G(n) e^{\epsilon\sigma(n)} (1 + |\lambda(a(t)na(t)^{-1})|)^{-\delta} dn \\
&= \int_{N(F)} \Xi^G(n) e^{\epsilon\sigma(n)} (1 + |t\lambda_+(n) - t^{-1}\lambda_-(n)|)^{-\delta} dn
\end{aligned}$$

for all $t \in \mathcal{U}$. Integrating this inequality over $\mathcal{U}$, we get that for all $\epsilon > 0$, we have

$$\begin{aligned}
&\int_{N(F)} \Xi^G(n) e^{\epsilon\sigma(n)} (1 + |\lambda(n)|)^{-\delta} dn \\
&\qquad \ll \int_{N(F)} \Xi^G(n) e^{\epsilon\sigma(n)} \int_{\mathcal{U}} (1 + |t\lambda_+(n) - t^{-1}\lambda_-(n)|)^{-\delta} dt dn
\end{aligned}$$

By Lemma B.1.1 there exists $\delta' > 0$ depending only on $\delta > 0$ such that the last expression is essentially bounded by

$$\int_{N(F)} \Xi^G(n) e^{\epsilon\sigma(n)} (1 + |\lambda_+(n)|)^{-\delta'} dn$$

Now by Corollary B.3.2, this last integral is convergent if ϵ is sufficiently small.

(iv) Let $\delta > 0$ and $\epsilon > 0$. We want to show that (iv) holds if ϵ is sufficiently small (compared to δ). Since $\Xi^G(g^{-1}) \sim \Xi^G(g), \sigma(g^{-1}) \sim \sigma(g)$ and $\lambda(h^{-1})) = -\lambda(h)$ for all $g \in G(F)$ and all $h \in H(F)$, it is equivalent to show the following

2.4.6) If ϵ is sufficiently small the integral

$$J^1_{\epsilon,\delta}(m_{min}) = \int_{H(F)} \Xi^G(m_{min}h) e^{\epsilon\sigma(h)} (1 + |\lambda(n)|)^{-\delta} dh$$

is absolutely convergent for all $m_{min} \in M_{min}(F)$ and there exists $d > 0$ such that

$$J^1_{\epsilon,\delta}(m_{min}) \ll \delta_{\overline{P}_{min}}(m_{min})^{1/2} \sigma(m_{min})^d$$

for all $m_{min} \in M_{min}(F)$

Let K be a maximal compact subgroup of $G(F)$ that is special in the p-adic case. Fix a map $m_{\overline{P}_{min}} : G(F) \to M_{min}(F)$ such that $g \in m_{\overline{P}_{min}}(g)\overline{U}_{min}(F)K$ for all $g \in G(F)$. By Proposition 1.3.1(ii), there exists $d > 0$ such that we have

$$J^1_{\epsilon,\delta}(m_{min}) \ll \delta_{\overline{P}_{min}}(m_{min})^{1/2} \sigma(m_{min})^d *$$
$$\int_H (F) \delta_{\overline{P}_{min}}(m_{\overline{P}_{min}}(h))^{1/2} \sigma(h)^d e^{\epsilon\sigma(h)} (1 + |\lambda(h)|)^{-\delta} dh$$

for all $m_{min}(F)$. Of course, for any $\epsilon' > \epsilon$ we have $\sigma(h)^d e^{\epsilon\sigma(h)} \ll e^{\epsilon'\sigma(h)}$, for all $h \in H(F)$. Hence, we only need to prove that for ϵ sufficiently small the integral

2.4.7)
$$\int_{H(F)} \delta_{\overline{P}_{min}}(m_{\overline{P}_{min}}(h))^{1/2} e^{\epsilon\sigma(h)} (1 + |\lambda(h)|)^{-\delta} dh$$

is absolutely convergent. Since $\overline{P}_{min}$ is a good parabolic subgroup, we may find compact neighborhoods of the identity $\mathcal{U}_K \subseteq K, \mathcal{U}_H \subseteq H(F)$ and $\mathcal{U}_{\overline{P}} \subseteq \overline{P}_{min}(F)$ such that $\mathcal{U}_K \subseteq \mathcal{U}_{\overline{P}}\mathcal{U}_H$. We have the inequalities

$$e^{\epsilon\sigma(k_H h)} \ll e^{\epsilon\sigma(h)}$$

and

$$(1+|\lambda(k_H h)|)^{-\delta} \ll (1+|\lambda(h)|)^{-\delta}$$

for all $h \in H(F)$ and for all $k_H \in \mathcal{U}_H$. Hence, we have

$$\begin{aligned}
&\int_H (F)\delta \overline{P}_{min}(m_{\overline{P}_{min}}(h))^{1/2} e^{\epsilon\sigma(h)}(1+|\lambda(h)|)^{-\delta} dh \\
\ll\ & \delta_{\overline{P}_{min}}(k_{\overline{P}})^{1/2} \int_{H(F)} \delta_{\overline{P}_{min}}(m_{\overline{P}_{min}}(k_H h))^{1/2} e^{\epsilon\sigma(h)}(1+|\lambda(h)|)^{-\delta} dh \\
=\ & \int_{H(F)} \delta_{\overline{P}_{min}}(m_{\overline{P}_{min}}(k_{\overline{P}} k_H h))^{1/2} e^{\epsilon\sigma(h)}(1+|\lambda(h)|)^{-\delta} dh
\end{aligned}$$

for all $k_H \in \mathcal{U}_H$ and all $k_{\overline{P}} \in \mathcal{U}_{\overline{P}}$. It follows that

$$\begin{aligned}
&\int_{H(F)} \delta_{\overline{P}_{min}}(m_{\overline{P}min}(h))^{1/2} e^{\epsilon\sigma(h)}(1+|\lambda(h)|)^{-\delta} dh \\
\ll\ & \int_{H(F)} \int_{\mathcal{U}_K} \delta_{\overline{P}_{min}}(m_{\overline{P}_{min}}(kh))^{1/2} dk e^{\epsilon\sigma(h)}(1+|\lambda(h)|)^{-\delta} dh \\
\ll\ & \int_{H(F)} \int_K \delta_{\overline{P}_{min}}(m_{\overline{P}_{min}}(kh))^{1/2} dk e^{\epsilon\sigma(h)}(1+|\lambda(h)|)^{-\delta} dh
\end{aligned}$$

By Proposition 1.3.1(iii), the innter integral above is equal to $\Xi^G(h)$ (for a suitable normalization) and the convergence of 2.4.7 for ϵ sufficiently small now follows from (iii).

(v) Let $\delta > 0$ and $\epsilon > 0$. We want to prove that (v) holds if ϵ is sufficiently small (compared to δ). After the variable change $h \mapsto h'h^{-1}$, we are left with proving that for $\epsilon > 0$ sufficiently small the following integral is absolutely convergent

$$\begin{aligned}
&I^3_{\epsilon,\delta}(m_{min}) \\
&= \int_{H(F)} \int_{H(F)} \Xi^G(hm_{min}) \Xi^G(h'm_{min}) e^{\epsilon\sigma(h)} e^{\epsilon\sigma(h')} \\
&\qquad\qquad (1+|\lambda(h')-\lambda(h)|)^{-\delta} dh' dh
\end{aligned}$$

for all $m_{min} \in M_{min}(F)$ and that there exists $d > 0$ such that

2.4.8)
$$I^3_{\epsilon,\delta}(m_{min}) \ll \delta_{\overline{P}_{min}}(m_{min})^{-1} \sigma(m_{min})^d$$

for all $m_{min} \in M_{min}(F)$. Let $a : \mathbb{G}_m \to A$ be a one-parameter subgroup such that $\lambda(a(t)ha(t)^{-1}) = t\lambda(h)$ for all $h \in H$ and for all $t \in \mathbb{G}_m$. Let $\mathcal{U} \subseteq F^\times$ be a compact neighborhood of 1. Since A is in the center of M_{min}, we have the inequality

$$I^3_{\epsilon,\delta}(m_{min})$$

$$\ll \int_{H(F)} \int_{H(F)} \Xi^G(hm_{min})\Xi^G(h'm_{min})e^{\epsilon\sigma(h)}e^{\epsilon\sigma(h')}$$

$$\int_{\mathcal{U}} (1 + |\lambda(a(t)h'a(t)^{-1}) - \lambda(h)|)^{-\delta} dt dh' dh$$

$$= \int_{H(F)} \int_{H(F)} \Xi^G(hm_{min})\Xi^G(h'm_{min})e^{\epsilon\sigma(h)}e^{\epsilon\sigma(h')}$$

$$\int_{\mathcal{U}} (1 + |t\lambda(h') - \lambda(h)|)^{-\delta} dt dh' dh$$

for all $m_{min} \in M_{min}(F)$. By Lemma B.1.1, there exists $\delta' > 0$ depending only on δ such that the last integral above is essentially bounded by

$$\int_{H(F)} \int_{H(F)} \Xi^G(hm_{min})\Xi^G(h'm_{min})e^{\epsilon\sigma(h)}e^{\epsilon\sigma(h')}$$

$$(1 + |\lambda(h')|)^{-\delta'}(1 + |\lambda(h)|)^{-\delta} dh' dh$$

for all $m_{min} \in M_{min}(F)$. This last integral is equal to $I^1_{\epsilon,\delta'}(m_{min})^2$. Hence, the inequality 2.4.8 for ϵ sufficiently small now follows from (iv).

$\square$

2.5 Relative Weak Cartan Decomposition

2.5.1 Relative Weak Cartan Decomposition for G_0

First we will address the codimension one case of the triple $(G_0, H_0, 1)$. Of course proposition 2.3.1 applieds to this case, in particular, G_0 admits good minimal parabolic subgroups. Let $\overline{P}_0 = M_0\overline{U}_0$ be such a minimal parabolic subgroup of G_0 and denote $A_0 = A_{M_0}$ the maximal central split subtorus of M_0. Set

$$A_0^+ = \{a \in A_0(F);\, |\alpha(a)| \geq 1\, \forall \alpha \in R(A_0, \overline{P}_0)\}$$

Proposition 2.5.1.1 (6.6.1) *There exists a compact subset $\mathcal{K}_0 \subseteq G_0(F)$ such that*

$$G_0(F) = H_0(F)A_0^+\mathcal{K}_0$$

Proof. First we will prove that the result doesn't depend on the choice of $(\overline{P}_0, M_0)$. Let $(\overline{P}'_0, M'_0)$ be another such pair, i.e. $\overline{P}'_0$ is a good minimal parabolic subgroup of G_0 and M'_0 is a Levi component of it. By 2.3.1(i), there exists $h \in H(F)$ such that $\overline{P}'_0 = h\overline{P}_0h^{-1}$. It is clear that if the proposition is true for the pair $(\overline{P}_0, M_0)$ then it is also true for the pair $(h\overline{P}_0h^{-1}, hM_0h^{-1}) = (\overline{P}'_0, hM_0h^{-1})$. Moreover, there exists a $\overline{p}'_0 \in \overline{P}'_0(F)$ such that $hM_0h^{-1} = \overline{p}'_0M'_0\overline{p}'^{-1}_0$. This means that the theorem being true for the pair $(\overline{P}'_0, hM_0h^{-1})$ implies it is true for the pair $(\overline{P}'_0, M'_0)$ becuase, by definition of A'^+_0, the set

$$\{a'^{-1}_0\overline{p}'_0a'_0,\, a'_0 \in A'^+_0\}$$

is bounded. Thus, it suffices to prove the proposition holds for any particular pair $(\overline{P}_0, M_0)$.

Next we will consider the archemedian case. Fix a good minimal parabolic subgroup $\overline{P}_0 \subseteq G_0$. By 2.3.1(i) there exists a Levi component M_0 of $\overline{P}_0$ such that $H_0 \cap \overline{P}_0 \subseteq M_0$. By theorem 5.13 of KKSS there exists a compact subset $\mathcal{K}_0 \subseteq G_0(\mathbb{R})$ such that

$$G_0(\mathbb{R}) = H_0(\mathbb{R})F'' A_Z^- \mathcal{K}_0$$

where A_Z^- is a certain submonoid of $A_0(\mathbb{R})$ (the exponential of the "compression cone" associated to the real spherical variety $Z = H_0(\mathbb{R})\backslash G_0(\mathbb{R})$, cf. Sections 5.1 of KKSS) and F'' is a subset of $N_{G_0(\mathbb{R})}(H_0)F$, F being any set of representatives for the open $H_0(\mathbb{R}) \times \overline{P}_0(\mathbb{R})$ double cosets in $G_0(\mathbb{R})$. By 2.3.1(i), we can take $F = \{1\}$. Moreover, we can check that $N_{G_0(\mathbb{R})}(H_0) = H_0(\mathbb{R})Z_{G_0}(\mathbb{R})$. As $Z_{G_0}(\mathbb{R})$ is compact, up to multiplying by $\mathcal{K}_0$ by $Z_{G_0}(\mathbb{R})$, we may also assume that $F'' = \{1\}$. All that remains is to show that $A_Z^- \subseteq A_0^+$ (note that our convention for the positive chamber is the opposite to the notation in KKSS, this is because we are denoting our good parabolic subgroup by $\overline{P}_0$ and not by P_0). But this follows from the fact that the real spherical variety $Z = H_0(\mathbb{R})\backslash G_0(\mathbb{R})$ is wavefront (cf. definition 6.1 of KKSS noting thet here $\mathfrak{a}_H = 0$. The notion of wavefront spherival variety has been first introduced in CV). To see this, consider the complex homogeneous space $Z_{\mathbb{C}} = H_0(\mathbb{C})\backslash G_0(\mathbb{C}) \cong GL_{d-1}(\mathbb{C})\backslash(GL_{d-1}(\mathbb{C}) \times GL_d(\mathbb{C}))$. It is spherical (it follows for example from 2.3.1(i) applied to GGP triples of codimension one with G quasi split) and wavefront by Remark 6.2 of KKSS. But we also know from the characterization of the compression cone given in Lemma 5.9 of KKSS that a real spherical variety is wavefront if its complexification is spherical and wavefront. Thus Z is wavefront and this proves the proposition in the archemedian case.

Now we will consider the p-adic case. To do this we will prove Proposition 11.0.1 of Beu1 in the case of the Special Orthogonal group. But first we will need 2 lemmas from Beu1. In these lemmas we will have $G = SO(V)$, $G_0 = SO(V_0)$ and $H = SO(W)$.

To proceed we will need to following notation. $w_0 = z_0$, and $w_0, \dots, w_l$ is a maximal family of orthogonal vectors in V_0 such that $q(w_i) = (-1)^i \nu$. Let $r_0 = \lfloor \frac{l+1}{2} \rfloor$ and $r_1 = \lfloor \frac{l}{2} \rfloor$. Let $u_{\pm i} = w_{2i-2} \pm w_{2i-1}$ for $i = 1, \dots, r_0$ and $u'_{\pm i} = w_{2i-1} \pm w_{2i}$ for $i = 1, \dots, r_1$. Then $(u_{\pm i})$ and $u'_{\pm i}$ are maximal isotropic subspace of V_0 and W respectively. Let $V_{0,an}$ and W_{an} be the orthogonal complements of $\langle u_1, u_{-1}, \dots u_{r_0}, u_{-r_0} \rangle$ in V_0 and $\langle u'_1, u'_{-1}, \dots u'_{r_1}, u'_{-r_1} \rangle$ in W respectively. Define P_0 and P_H as the parabolic subgroups of G_0 and H which preserve the isotropic flags $\langle u_1 \rangle \subseteq \langle u_1, u_2 \rangle \subseteq \cdots \subseteq \langle u_1, \dots, u_{r_0} \rangle$ and $\langle u'_1 \rangle \subseteq \cdots \subseteq \langle u'_1, \dots, u'_{r_1} \rangle$ respectively.

Let A_0 and A_H be the maximal split central subtorus of G_0 and H which stabilize the lines $\langle u_i \rangle, i = \pm 1, \dots, \pm r_0$ and $\langle u'_i \rangle, i = \pm 1, \dots, \pm r_1$ respectively. Let A_0^+ and A_H^+ be the positive chambers of $A_0(F)$ and $A_H(F)$ respectively. Let $R_0 = \mathcal{O}_F u_1 \oplus \mathcal{O}_F u_{-1} \oplus \dots \oplus \mathcal{O}_F u_r \oplus \mathcal{O}_F u_{-r} \oplus R_{0,an}$ where $R_{0,an} = \{v \in V_{0,an}; \mathrm{val}_F(q(v)) \geq \mathrm{val}_F(\nu)\}$. If V is an F vector space, and R is an $\mathcal{O}_F$ Lattice of V and $g \in GL_F(V)$ then $\mathrm{val}_R(g) = \inf\{\mathrm{val}_R(gv), v \in R\}$ and

$\|g\|_R = |\varpi_F|^{\mathrm{val}_R(g)}$

Lemma 2.5.1.2 (Beu1 Lemma 11.0.2) *There exists a compact subset $C_1 \subseteq G_0(F)$ such that for all $v, v' \in V_0$ such that $q(v) = q(v') = \nu$ and $\mathrm{val}_{R_0}(v) = \mathrm{val}_{R_0}(v')$, there is a $\gamma \in C_1$ such that $\gamma v = v'$*

Proof. If $l = 0$ then G_0 is compact and then $C_1 = G_0$ satisfies the lemma. Suppose then that $l \geq 1$. Let $K_0 = \mathrm{Stab}_{G_0(F)}(R_0)$ be an open compact subgroup of $G_0(F)$. For all $g \in G_0(F)$, we will note $\|g\| = \|g\|_{R_0}$. The function $g \mapsto \|g\|$ is bounded from below by a positive constant on $G_0(F)$ and is biinvariant by K_0. Let $v \in V_0$. Now we will show

2.5.1.3) There is $k \in K_0$ such that $kv \in \langle u_1, u_{-1}\rangle \oplus V_{0,an}$

We can decompose v as $v = \sum_{i=\pm 1,\ldots,\pm r} \lambda_i u_i + v_{an}$ for $v_{an} \in V_{0,an}$, $\lambda_i \in F$. By multiplying v by an element of the Weyl group of A_0 (identified with a subgroup of K_0), we can assume that $\mathrm{val}(\lambda_1) = \inf_{i=\pm 1,\ldots,\pm r} \mathrm{val}(\lambda_i)$.

Let $k_1 \in K_0$ be the element which sends u_1 to $u_1 - \frac{\lambda_2}{\lambda_1}u_2 - \cdots - \frac{\lambda_r}{\lambda_1}u_r$ and u_{-i} to $u_{-i} + \frac{\lambda_i}{\lambda_1}u_{-1}$ for $i = 2, \ldots, r$ and which acts trivially on $V_{0,an} \oplus \langle u_{-1}, u_2, \ldots u_r\rangle$. We then have that

$$k_1 v = \lambda u_1 + \mu_1 u_{-1} + \lambda_{-2}u_{-2} + \cdots + \lambda_{-r}u_{-r} + v_{an}$$

Let $k_2 \in K_0$ be the element that sends u_1 to $u_1 - \frac{\lambda_{-2}}{\lambda_1}u_{-2} - \cdots - \frac{\lambda_{-r}}{\lambda_1}$ and u_i to $u_i + \frac{\lambda_{-i}}{\lambda_1}u_{-1}$ and acts trivially on $V_{0,an} \oplus \langle u_{-1}, \ldots, u_{-r}$. We then have that $k_2 k_1 v \in \langle u_1, u_{-1}\rangle \oplus V_{0,an}$.

And we have

2.5.1.4) For all $v_{an} \in V_{0,an}$, we have

$$R_{0,an} \subseteq \frac{1}{2}\big((R_{0,an} \cap \langle v_{an}\rangle) \oplus (R_{0,an} \cap (v_{an})^\perp)\big)$$

where $(v_{an})^\perp$ is the orthogonal complement of v_{an} in $V_{0,an}$.

To see this, let $v \in R_{0,an}$ and $v = v_1 + v_2$ with $v_1 \in \langle v_{an}\rangle$ and $v_2 \in (v_{an})^\perp$. Assume $\mathrm{val}_F(q(v_1) + q(v_2)) > \min(\mathrm{val}_F(q(v_1)), \mathrm{val}_F(q(v_2))) + \mathrm{val}_F(4)$. Then $-\frac{q(v_1)}{q(v_2)} \in 1 + 4\mathfrak{p}_F$. Since $1 + 4\mathfrak{p}_F \subseteq \mathcal{O}_F^{\times,2}$, there exists $\lambda \in \mathcal{O}_F^\times$ such that $q(v_1 + \lambda v_2) = 0$, contradicting that $V_{0,an}$ is anisotropic. Consequently,

$$\begin{aligned}
\min(\mathrm{val}_F(q(2v_1)), \mathrm{val}_F(q(2v_2))) &\geq \mathrm{val}_F(q(v_1) + q(v_2)) \\
&= \mathrm{val}_F(q(v_1 + v_2)) \\
&= \mathrm{val}_F(q(v)) \\
&\geq \mathrm{val}_F(\nu)
\end{aligned}$$

Hence $v_1, v_2 \in \frac{1}{2}R_{0,an}$.

Next, we will show

2.5.1.5) There is $g \in G_0(F)$ such that $gv = \lambda_1 u_1 + \lambda_{-1} u_{-1} + v_{an}$ with $v_{an} \in V_{0,an}$, $\mathrm{val}_F(\lambda_1) = \inf(\mathrm{val}_F(\lambda_1), \mathrm{val}_F(\lambda_{-1}), \mathrm{val}_{R_{0,an}}(v_{an}))$ and $\|g\| \leq |2|_F^{-1}$ for all $v \in V_0$.

From 2.5.1.3, we can assume that $v = \lambda_1 u_1 + \lambda_{-1} u_{-1} + v_{an}$ and up to applying the element which exchanges u_1 and u_{-1} and leaves stable the anisotropic part, we can also assume that $\mathrm{val}_F(\lambda_1) \leq \mathrm{val}_F(\lambda_{-1})$. If $\mathrm{val}_F(\lambda_1) \leq \mathrm{val}_{R_{0,an}}(v_{an})$ then we are done. If on the contrary $-d = \mathrm{val}_{R_{0,an}}(v_{an}) < \mathrm{val}_F(\lambda_1)$, we can consider the element $g \in G_0(F)$ which sends u_1 to u_1, v_{an} to $v_{an} + \varpi_F^{-d} u_1$, u_{-1} to $u_{-1} - \frac{2\varpi_F^{-2d}\nu}{q(v_{an})} v_{an} - \frac{\varpi_F^{-d}\nu}{q(v_{an})} u_1$, and which acts trivially on the orthogonal complement of $\langle u_1, u_{-1}, v_{an} \rangle$. Since $\mathrm{val}_F(q(v_{an})) \leq -2d + \mathrm{val}_F(\nu)$, we have

$$g(R_0 \cap \langle u_1, u_{-1}, v_{an} \rangle) = R_0 \cap \langle u_1, u_{-1}, v_{an} \rangle$$

According to 2.5.1.4 we therefore have $\|g\| \leq |2|_F^{-1}$. Still using that $\mathrm{val}_F(q(v_{an})) \leq -2d + \mathrm{val}_F(\nu)$, it is easy to check that gv does in fact satisfy 2.5.1.5.

Next, we will show

2.5.1.6) There exists $g \in G_0(F)$ such that $gv \in \langle u_1, u_{-1} \rangle$ and $\|g\| \leq |\varpi_F|_F^{-1} |2|_F^{-4}$

From 2.5.1.5 we can assume that $v = \lambda_1 u_1 + \lambda_{-1} u_{-1} + v_{an}$ with $v_{an} \in V_{0,an}$ and $\mathrm{val}_F(\lambda_1) = \inf(\mathrm{val}_F(\lambda_1), \mathrm{val}_F(\lambda_{-1}), \mathrm{val}_{R_{0,an}}(v_{an}))$. Let $g \in G_0(F)$ be the element which sends

$$u_{-1} \mapsto u_{-1}$$

$$u_1 \mapsto u_1 - \frac{1}{\lambda_1} v_{an} - \frac{q(v_{an})}{4\lambda_1^2 \nu} u_{-1}$$

$$v_{an} \mapsto v_{an} + \frac{q(v_{an})}{2\lambda_1 \nu} u_{-1}$$

and acts trivially on the orthogonal complement of $\langle u_1, u_{-1}, v_{an} \rangle$. Then we do have $gv \in \langle u_1, u_{-1} \rangle$ and by 2.5.1.4 we have

$$\begin{aligned} \mathrm{val}_{R_0}(g) \leq \inf(& \mathrm{val}_{R_{0,an}}(v_{an}) - \mathrm{val}_F(\lambda_1), \\ & \mathrm{val}_F(q(v_{an})) - \mathrm{val}_F(4\lambda_1^2 \nu), \\ & \mathrm{val}_F(q(v_{an}) - \mathrm{val}_F(2\lambda_1 \nu) - \mathrm{val}_{R_{0,an}}(v_{an}) - \mathrm{val}_F(2), \\ & 0) \end{aligned}$$

Because $\mathrm{val}_F(q(v_{an}) \geq 2\,\mathrm{val}_{R_{0,an}}(v_{an}) + \mathrm{val}_F(\nu) - 1$, we have $\mathrm{val}_{R_0}(g) \geq -1 - 3\,\mathrm{val}_F(2)$.

Now suppose that $q(v) = 2\nu^2$. Then

2.5.1.7) There exists $g \in G_0(F)$ and $\lambda \in \mathcal{O}_F - \{0\}$ such that $gv = \lambda u_1 + \frac{\nu}{2\lambda} u_{-1}$ and $\|g\| \leq |\varpi_F|_F^{-1} |2|_F^{-5}$.

From 2.5.1.6, we know that there is a $g' \in G_0(F)$ such that $g'v = \lambda u_1 + \mu u_{-1}$ and $\|g'\| \leq |\varpi_F|_F^{-1}|2|_F^{-4}$. As we can multiply by an element to swap u_1 and u_{-1} we can assume without loss of generality that $\mathrm{val}(\mu) \geq \mathrm{val}(\lambda)$. Now by the fact that $q(v) = 2\nu^2$ we have that $\mu = \frac{\nu}{2\lambda}$ and the result is trivial

We can now finish the proof of the lemma. Let v and v' be as in the lemma. There are $g_1, g_2 \in G_0(F)$ and $\lambda_1, \lambda_2 \in \mathcal{O}_F - \{0\}$ such that

$$\begin{aligned} \|g_1\|, \|g_2\| &\leq |\varpi_F|_F^{-1}|2|_F^{-5} \\ g_1 v &= \lambda_1 u_1 + \frac{\nu}{2\lambda_1} u_{-1} \\ g_2 v' &= \lambda_2 u_1 + \frac{\nu}{2\lambda_2} u_{-1} \end{aligned}$$

There is an integer N_0 such that

$$|\mathrm{val}_{R_0}(gv_0) - \mathrm{val}_{R_0}(v_0)| \leq -\mathrm{val}_{R_0}(g) + N_0$$

for all $g \in G_0(F)$ and for all $v_0 \in V_0$. Since $\mathrm{val}_{R_0}(g_1 v) = -\mathrm{val}_F(\lambda_1) - \mathrm{val}_F(2)$, $\mathrm{val}_{R_0}(g_2 v') = -\mathrm{val}_F(\lambda_2) - \mathrm{val}_F(2)$, and $\mathrm{val}_{R_0}(v) = \mathrm{val}_{R_0}(v')$, we have

$$|\mathrm{val}_F(\lambda_1) - \mathrm{val}_F(\lambda_2)| \leq 2 + 10\,\mathrm{val}_F(2) + 2N_0$$

Let a be the element of $G_0(F)$ which sends u_1 to $\frac{\lambda_2}{\lambda_1} u_1$, and u_{-1} to $\frac{\lambda_1}{\lambda_2} u_{-1}$ while acting trivially on the orthogonal space to $\langle u_1, u_{-1} \rangle$. We then have $ag_1 v = g_2 v'$. Hence $g_2^{-1} a g_1 v = v'$ and $\|g_2^{-1} a g_1\|$ is bounded by a constant, which completes the proof. □

Assume $l \geq 1$. For all $\lambda \in F^\times$ we denote $a(\lambda)$ as the element of $G_0(F)$ which sends u_1 to λu_1 and u_{-1} to $\lambda^{-1} u_{-1}$ and which acts trivially on the orthogonal complement of $\langle u_1, u_{-1} \rangle$. Let $R_{\#,H}$ be an $\mathcal{O}_F$-lattice orthogonal to $\langle w_1, \dots, w_l \rangle$ in W. Then let $R_1 = \mathcal{O}_F\, w_1 \oplus \cdots \oplus \mathcal{O}_F\, w_l \oplus R_{\#,H}$ and $R_2 = \mathcal{O}_F\, w_2 \oplus \cdots \oplus \mathcal{O}_F\, w_l \oplus R_{\#,H}$

Lemma 2.5.1.8 (Beu1 lema 11.0.3) *Suppose $l \geq 1$. There exists a compact subset C_2 of $G_0(F)$ and a constant $c_0 > 0$ such that for any $h \in H(F)$ and any $\lambda \in \mathcal{O}_F - \{0\}$, there exists $h' \in h\big(a(\lambda) C_2 a(\lambda)^{-1} \cap H(F)\big)$ such that*

$$\mathrm{val}_{R_1}(h') \geq \mathrm{val}_{R_1}(h' w_1) - \mathrm{val}_F(\lambda) - c_0$$

Proof. Let $e_1, \dots, e_t$ be an orthogonal basis of the $\mathcal{O}_F$-module, R_2. There exists a strictly positive integer α such that

- $-\frac{q(e_i)}{q(w_1)} \in \mathfrak{p}_F^{-\alpha}$, for $i = 1, \dots, t$
- for any integer $k \geq 0$ and for any $x \in 1 + \mathfrak{p}_F^{k+\alpha}$, there exists $y \in 1 + \mathfrak{p}_F^{k}$ such that $y^2 = x$.

For $i = 1, \ldots, t$ and $\lambda \in \mathcal{O}_F - \{0\}$, we consider the element $\gamma_i(\lambda) \in H(F)$ which acts trivially on $\langle w_1, e_i \rangle^{\perp}$ and which acts in the following way on w_1 and e_i

$$w_1 \mapsto a w_1 + \lambda \varpi_F^{\alpha} e_i$$

$$e_i \mapsto \frac{q(e_i)}{q(w_1)} \lambda \varpi_F^{\alpha} w_1 - a e_i$$

where $a = 1 - \lambda^2 \varpi_F^{\alpha} \frac{q(e_i)}{q(w_1)} \in 1 + \lambda^2 \mathcal{O}_F$. Then $\gamma_i(\lambda)$ stabilizes $\mathcal{O}_F w_1 \oplus R_2$ and maps to a compact set independent of i and λ. Let's check that $a(\lambda)^{-1}\gamma_i(\lambda)a(\lambda)$ also is compact. We have

$$\begin{aligned} a(\lambda)^{-1}\gamma_i(\lambda)a(\lambda)u_{-1} &= \lambda^{-1}a(\lambda)^{-1}\gamma_i(\lambda)u_{-1} \\ &= \lambda^{-1}a(\lambda)^{-1}(w_0 - \gamma_i(\lambda)w_1) \\ &= \lambda^{-1}a(\lambda)^{-1}(u_{-1} + w_1 - \gamma_i(\lambda)w_1) \\ &= u_{-1} + \lambda^{-1}a(\lambda)^{-1}(w_1 - \gamma_i(\lambda)w_1) \end{aligned}$$

but $w_1 - \gamma_i(\lambda)w_1$ is in $\lambda^2 \mathcal{O}_F \oplus R_2$. So $a(\lambda)^{-1}\gamma_i(\lambda)a(\lambda)u_{-1} \in \frac{1}{2}\mathcal{O}_F u_{-1} + R_1$. We can show in the same way that $a(\lambda)^{-1}\gamma_i(\lambda)a(\lambda)u_1$ remains bounded. Finally, since $a(\lambda)^{-1}\gamma_i(\lambda)a(\lambda)e_i = a(\lambda)^{-1}\gamma_i(\lambda)e_i$ and $\gamma_i(\lambda)e_i \in \lambda \mathcal{O}_F w_1 + R_2$, we have $a(\lambda)^{-1}\gamma_i(\lambda)a(\lambda)e_i$ remains bounded.

Let C_2 be a compact subset of $G_0(F)$ which contains all $a(\lambda)^{-1}\gamma_i(\lambda)a(\lambda)$ for all $i = 1, \ldots, t$ and for all $\lambda \in \mathcal{O}_F - \{0\}$. Since the $\gamma_i(\lambda)$ are bounded there exists a constant $c_1 > 0$ such that for all $h \in H(F)$, for all $i \in 1, \ldots, t$ and for all $\lambda \in \mathcal{O}_F$ we have

$$\mathrm{val}_{R_1}(h\gamma_i(\lambda)) \geq \mathrm{val}_{R_1}(h) - c_1$$

We have $\mathrm{val}_{R_1}(h\gamma_i(\lambda)w_1) = \inf(\mathrm{val}_{R_1}(hw_1), \mathrm{val}_F(\lambda) + \alpha + \mathrm{val}_{R_1}(he_i)$. Since $\mathrm{val}_{R_1}(h) = \inf(\mathrm{val}_{R_1}(hw_1), \mathrm{val}_{R_1}(he_1), \ldots, \mathrm{val}_{R_1}(he_t))$, we have

$$\begin{aligned} \inf_{i=1,\ldots,t}(\mathrm{val}_{R_1}(h\gamma_i(\lambda)w_1) &\leq \mathrm{val}_F(\lambda) + \alpha + \mathrm{val}_{R_1}(h) \\ &\leq \mathrm{val}_F(\lambda) + \alpha + c_1 + \inf_{i=1,\ldots,t} \mathrm{val}_{R_1}(h\gamma_i(\lambda)) \end{aligned}$$

To obtain the lemma it is sufficient to take $c_0 = \alpha + c_1$ and for $h \in H(F)$ and $\lambda \in \mathcal{O}_F - \{0\}$ to choose $h' = h\gamma_j(\lambda)$ where j is such that $\inf_{i=1,\ldots,t}(\mathrm{val}_{R_1}(h\gamma_i(\lambda)w_1)) = \mathrm{val}_{R_1}(h\gamma_j(\lambda)w_1)$

□

Now we can prove our desired proposition

Proposition 2.5.1.9 (Beu1 Proposition 11.0.1) *There exist compact subsets $C_0 \subseteq G_0(F)$ and $C_H \subseteq H(F)$ such that*

$$G_0(F) = C_H A_H^+ A_0^+ C_0$$

Proof. We will proceed by induction on l. If $l = 0$, then G_0 is compact and the proposition is true. Now we will assume that $l \geq 1$. 2.5.1.2 gives us compact $C_1 \subseteq G_0(F)$. 2.5.1.8 gives us a compact $C_2 \subseteq G_0(F)$ as well as a constant c_0. Let $C = C_2^{-1}C_1$. Let $g \in G_0(F)$. By 2.5.1.2 there exists $k_1 \in C_1$ and $\lambda \in \mathcal{O}_E - \{0\}$ such that

$$k_1 g^{-1} z_0 = a(\lambda)^{-1} v_0$$

that is $g \in H(F)a(\lambda)k_1$. Using 2.5.1.8 ,we see that there exists $k_2 \in C_2, \lambda \in \mathcal{O}_F - \{0\}$ and $h \in H(F)$ such that

$$g = ha(\lambda)k_2^{-1}k_1$$

and

2.5.1.10) $\quad \mathrm{val}_{R_1}(h) \geq \mathrm{val}_{R_1}(hw_1) - \mathrm{val}_F(\lambda) - c_0$

Set $k = k_2^{-1}k_1 \in C$. Let H' be the special orthogonal group of $W' = \langle w_0, w_1\rangle^{\perp}$, $P_{H'}$ the parabolic subgroup which preserves the flag $\langle u_2\rangle \subseteq \cdots \subseteq \langle u_2, \ldots, u_r\rangle$, $A_{H'}$ the maximal torus that preserves the lines $\langle u_i\rangle (i = \pm 2, \ldots, \pm r_0)$, and $A_{H'}^+$ the positive chamber of $A_{H'}(F)$. By our induction hypothesis there exists compact subsets $C_H^{\#} \subseteq H(F)$ and $C_{H'}^{\#} \subseteq H'(F)$ such that $H(F) = C_H^{\#} A_H^+ A_{H'}^+ C_{H'}^{\#}$. Using this decomposition we can write $h = k_1^{\#} a_H a_{H'} k_2^{\#}$. We then have $g = k_1^{\#} a_H a_{H'} a(\lambda) k_2^{\#} k$. We do have that $a_{H'}a(\lambda) \in A_0^+$ but it is not necessarily in the positive chamber of P_0. We will use the inequality 2.5.1.10 to verify that $a_{H'}a(\lambda)$ is in fact in A_0^+ modulo a finite subset.

Let $a_{H',2}$ be the eigenvalue of $a_{H'}$ acting on u_2. We will show $\mathrm{val}_F(a_{H',2}) - \mathrm{val}_F(\lambda)$ is bounded by a constant. There exist positive constants c_1, c_2, c_3 and c_4 such that

- $\mathrm{val}_{R_2}(k^{\#}) \geq -c_1$ for all $k^{\#} \in C_{H'}^{\#}$
- $\mathrm{val}_{R_1}(h^{-1}) \geq \mathrm{val}_{R_1}(h) - c_2$ for all $h \in H(F)$
- $\mathrm{val}_{R_1}(hk^{\#}) \geq \mathrm{val}_{R_1}(h) - c_3$ for all $h \in H(F)$ and $k^{\#} \in C_H^{\#}$
- $\mathrm{val}_{R_1}(hw_1) \geq -\mathrm{val}_{R_1}(a_{H,1}) - c_4$ if $h = k^{\#} a_H h'$ with $k^{\#} \in C_H^{\#}, a_H \in A_H(F)^+$ and $h' \in H'(F)$ and where $a_{H,1}$ is the eigenvalue of a_H acting on u_1'.

We then have

$$\begin{aligned}
-\mathrm{val}_F(a_{H',2}) &= \mathrm{val}_{R_1}(a_{H'}^{-1}w_2) + \mathrm{val}_F(2) \\
&\geq \mathrm{val}_{R_2}((k_2^{\#})^{-1}a_{H'}^{-1}w_2) - c_1 \\
&= \mathrm{val}_{R_1}((k_2^{\#})^{-1}a_{H'}^{-1}u_1; -w_1) - c_1 \\
&= \mathrm{val}_{R_1}(h^{-1}k_1^{\#}a_H u_1' - w_1) - c_1 \\
&\geq \inf(0, \mathrm{val}_{R_1}(h^{-1}k_1^{\#}a_H u_1')) - c_1
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{val}_{R_1}(h_{-1}k_1^{\#}a_Hu_1') &= \mathrm{val}_F(a_{H,1}) + \mathrm{val}_{R_1}(h^{-1}k_1^{\#}u_1') \\
&\geq \mathrm{val}_F(a_{H,1}) + \mathrm{val}_{R_1}(h) - c_2 - c_3 \\
&\geq \mathrm{val}_F(a_{H.1}) + \mathrm{val}_{R_1}(hw_1) - \mathrm{val}_F(\lambda) - c_0 - c_2 - c_3 \\
&\geq -\mathrm{val}_F(\lambda) - c_0 - c_2 - c_3 - c_4
\end{aligned}$$

We then deduce $\mathrm{val}_F(a_{H',2}) \geq -\mathrm{val}_F(\lambda) - c_0 - c_1 - c_2 - c_3 - c_4$ which is what we needed.

□

Now that we have proved the proposition 2.5.1.9 that there exists compact $C_0 \subseteq SO(V_0)(F)$ and $C_H \subseteq SO(W)(F)$ such that $SO(V_0)(F) = C_H A_H^+ A_0^+ C_0$ we will now prove proposition 2.5.1.1 which is that for the GGP pair $G_0 = SO(V_0) \times SO(W)$ and $H = SO(W)$ there exists a compact subset $\mathcal{K}_0 \subseteq G_0(F)$ such that $G_0(F) = H_0(F)A_0^+\mathcal{K}$.

To see this, let $(g,h) \in G_0 = SO(V_0) \times SO(W)$. Then

$$\begin{aligned}
(g,h) &= (h,h)(g',1) && \\
&= (h,h)(k_Ha_H^+a_G^+k_G,1) && \text{By 2.5.1.9} \\
&= (hk_Ha_H^+a_G^+k_G,h) && \\
&= (h'a_G^+k_G, h'(a_H^+)^{-1}k_H^{-1}) && \text{for } h' = hk_ha_h^+ \\
&= (h',h')(a_G^+,(a_H^+)^{-1})(k_G,k_H^{-1}) && \in H(F)A_G^+\mathcal{K}
\end{aligned}$$

which completes the proof.

□

2.5.2 Relative Weak Cartan decomposition for G

Let the quadruple $(\overline{P}_0, M_0A_0, A_0^+)$ be as in the previous section. Denote $\overline{P} = M\overline{N}$ the parabolic subgroup opposite to P with respect to M and define the following subgroups of G:

$$A_{min} = A_0A \subseteq M_{min} = M_0T \subseteq \overline{P}_{min} = \overline{P}_0T\overline{N}$$

Then, $\overline{P}_{min}$ is a parabolic subgroup, M_{min} is a Levi component of it and A_{min} is the maximal split central subtorus of M_{min}. Moreover, it is easy to see that $\overline{P}_{min}$ is a good parabolic subgroup of G. Set

$$A_{min}^+ = \{a \in A_{min}(F) : |\alpha(a)| \geq 1 \forall \alpha \in R(A_{min}, \overline{P}_{min})\}$$

We will denote P_{min} for the parabolic subgroup opposite to $\overline{P}_{min}$ with respect to M_{min}. We have $P_{min} \subseteq P$. Let Δ be the set of simple roots of A_{min} in P_{min} and $\Delta_P = \Delta \cap R(A_{min}, N)$ be the subset of simple roots appearing in $\mathfrak{n} = \mathrm{Lie}(N)$. For $\alpha \in \Delta_P$, we will denote by $\mathfrak{n}_\alpha$ the corresponding root subspace. Recall als the we have defined in Section 2.1 a character ξ of $\mathfrak{n}(F)$.

Lemma 2.5.2.1 (6.6.2) *We have the following*

(i)

$$A_{min}^{+} = \{a \in A_0^{+}A(F) : |\alpha(a)| \leq 1 \forall \alpha \in \Delta_P\}$$

(ii) There exists a compact subset $\mathcal{K}_G \subseteq G(F)$ *such that*

$$G(F) = H(F)A_0^{+}A(F)\mathcal{K}_G$$

(iii) For all $\alpha \in \Delta_P$*, the restriction of* ξ *to* $\mathfrak{n}_\alpha(F)$ *is nontrivial.*

Proof. (i) is obvious,

(ii) Let K be a maximal compact subgroup of $G(F)$ which is special in the p-adic case. Then we have Iwasawa decomposition

2.5.2.2)

$$G(F) = P(F)K = N(F)G_0(F)T(F)K$$

Since $A = A_T$ is the maximal split subtorus of T, there exists a compact subset $\mathcal{K}_T \subseteq T(F)$ such that

2.5.2.3)

$$T(F) = A(F)\mathcal{K}_T$$

Also by proposition 2.5.1.1, we know there exists a compact subset $\mathcal{K}_0 \subseteq G_0(F)$ such that

2.5.2.4)

$$G_0(F) = H_0(F)A_0^{+}\mathcal{K}_0$$

Combining 2.5.2.2, 2.5.2.3, and 2.5.2.4, and since A and G_0 centralize each other, we get

$$G(F) = H(F)A_0^{+}A(F)\mathcal{K}_G$$

where $\mathcal{K}_G = \mathcal{K}_0\mathcal{K}_T\mathcal{K}$.

(iii) Let $\alpha \in \Delta_P$ and assume, for a contradiction, that ξ is trivial when restricted to $\mathfrak{n}_\alpha(F)$. Recall that ξ is the composition $\xi = \psi \circ \lambda_F$ where λ is an algebraic additive character $\mathfrak{n} \to \mathbb{G}_a$. Since $\mathfrak{n}_\alpha$ is a linera subspace of $\mathfrak{n}$, the condition that ξ is trivial on $\mathfrak{n}_\alpha(F)$ is equivalent to λ being trivial on $\mathfrak{n}_\alpha$. Since λ is invariant by H_0 conjugation and $\mathfrak{n}_\alpha$ is invariant by both T-conjugation and $\overline{P}_0$-conjugation, it follows that λ is trivial on $m\,\mathfrak{n}_\alpha\,m^{-1}$ for all $m \in H_0\overline{P}_0T$. But $\overline{P}_0$ being a good parabolic subgroup of G_0 means $H_0\overline{P}_0T$ is Zariski-dense in $M = G_0T$. Hence, λ is trivial on $m\,\mathfrak{n}_\alpha\,m^{-1}$ for all $m \in M$. This is a contradiction of Lemma 2.1.2(ii) since $\mathfrak{n}_\alpha$ is not included in $[\mathfrak{n}, \mathfrak{n}]$.

□

2.6 The Function $\Xi^{H\backslash G}$

Let $C \subseteq G(F)$ be a compact subset with nonempty interior. We define a function $\Xi_C^{H\backslash G}$ on $H(F)\backslash G(F)$ by

$$\Xi_C^{H\backslash G}(x) = \mathrm{vol}_{H\backslash G}(xC)^{-1/2}$$

for all $x \in H(F)\backslash G(F)$. It is not hard to see that if $C' \subseteq G(F)$ is another compact subset with nonempty interior, we have

$$\Xi_C^{H\backslash G}(x) \sim \Xi_{C'}^{H\backslash G}(x)$$

for all $x \in H(F)\backslash G(F)$. From now on we will assume an implicitly fixed compact subset with nonempty interior $C \subseteq G(F)$ and we will set

$$\Xi^{H\backslash G}(x) = \Xi_C^{H\backslash G}(x)$$

for all $x \in H(F)\backslash G(F)$. The precise choice of C wont matter because the function $\Xi^{H\backslash G}$ will only be used for the purpose of estimates.

Proposition 2.6.1 (6.7.1) *(i) For every compact subset $\mathcal{K} \subseteq G(F)$, we have the following equivalences of functions*

(a)

$$\Xi^{H\backslash G}(xk) \sim \Xi^{H\backslash G}(x)$$

(b)

$$\sigma_{H\backslash G}(xk) \sim \sigma^{H\backslash G}(x)$$

for all $x \in H(F)\backslash G(F)$ and all $k \in \mathcal{K}$.

(ii) Let $\overline{P}_0 = M_0\overline{U}_0 \subseteq G_0$ be a good minimal parabolic subgroup of G_0 and $A_0 = A_{M_0}$ be the split part of the center of M_0. Set

$$A_0^+ = \{a_0 \in A_0(F) : |\alpha(a_0)| \geq 1 \, \forall \alpha \in R(A_0, \overline{P}_0\}$$

Then there exists a positive constant $d > 0$ such that

(a)

$$\Xi^{G_0}(a_0)\delta_P(a)^{1/2}\sigma(a_0)^{-d} \ll \Xi^{H\backslash G}(aa_0) \ll \Xi^{G_0}(a_0)\delta_P(a)^{1/2}$$

(b)

$$\sigma_{H\backslash G}(aa_0) \sim \sigma_G(aa_0)$$

for all $a_0 \in A_0^+$ and all $a \in A(F)$.

(iii) There exists $d > 0$ such that the integral

$$\int_{H(F)\backslash G(F)} \Xi^{H\backslash G}(x)^2 \sigma_{H\backslash G}(x)^{-d} dx$$

is absolutely convergent

(iv) For all $d > 0$, there exists $d' > 0$ such that

$$\int_{H(F)\backslash G(F)} \mathbb{1}_{\sigma_{H\backslash G} \leq c}(x) \Xi^{H\backslash G}(x)^2 \sigma_{H\backslash G}(x)^d dx \ll c^{d'}$$

for all $c \geq 1$.

(v) There exists $d > 0$ and $d' > 0$ such that

$$\int_{H(F)} \Xi^G(x^{-1}hx)\sigma_G(x^{-1}hx)^{-d} \ll \Xi^{H\backslash G}(x)^2 \sigma_{H\backslash G}(x)^{d'}$$

for all $x \in H(F)\backslash G(F)$.

(vi) For all $d > 0$, there exists $d' > 0$ such that

$$\int_{H(F)} \Xi^G(hx)\sigma(hx)^{-d'} dh \ll \Xi^{H\backslash G}(x)\sigma_{H\backslash G}(x)^{-d}$$

for all $x \in H(F)\backslash G(F)$.

(vii) Let $\delta > 0$ and $d > 0$. Then, the integral $I_{\delta,d}(c,x) =$

$$\int_{H(F)} \int_{H(F)} \mathbb{1}_{\sigma \geq c}(h')\Xi^G(hx)\Xi^G(h'hx)\sigma(hx)^d \sigma(h'hx)^d (1+|\lambda(h')|)^{-\delta} dh' dh$$

is absolutely convergent for all $x \in H(F)\backslash G(F)$ and all $c \geq 1$. Moreover, there exist $\epsilon > 0$ and $d' > 0$ such that

$$I_{\delta,d}(c,x) \ll \Xi^{H\backslash G}(x)^2 \sigma_{H\backslash G}(x)^{d'} e^{-\epsilon c}$$

for all $x \in H(F)\backslash G(F)$ and all $c \geq 1/$

Proof. Same as ref □

2.7 Parabolic degenerations

Let $\overline{Q} = LU_{\overline{Q}}$ be a good parabolic subgroup of G (ie that $H\overline{Q}$ is Zariski open in G, see section 2.3). Let $\overline{P}_{min} = M_{min}\overline{U}_{min} \subseteq \overline{Q}$ be a good minimal parabolic subroup of G (Proposition 2.3.2(ii)) with the Levi component chosen so that $M_{min} \subseteq L$. Let $A_{min} = A_{M_{min}}$ be the maximal central split torus of M_{min} and set

$$A^+_{min} = \{a \in A_{min}(F) : |\alpha(a)| \geq 1 \forall \in R(A_{min}, \overline{P}_{min})\}$$

Let $H_{\overline{Q}} = H \cap \overline{Q}$ and H_L be the image of $H_{\overline{Q}}$ by the natural projection $H_{\overline{Q}} \twoheadrightarrow L$. Let $Q = LU_Q$ be the parabolic subgroup opposite to $\overline{Q}$ with respect to L. We define $H^Q = H_L \ltimes U_Q$.

Proposition 2.7.1 (6.8.1) *(i) $H_{\overline{Q}} \cap U_{\overline{Q}} = \{1\}$ so that the natural projection $H_{\overline{Q}} \to H_L$ is an isomorphism.*

(ii) $\delta_{\overline{Q}}(h_{\overline{Q}}) = \delta_{H_{\overline{Q}}}(h_{\overline{Q}})$ *and* $\delta_{\overline{Q}}(h_L) = \delta_{H_L}(h_L)$ *for all* $h_{\overline{Q}} \in H_{\overline{Q}}(F)$ *and all* $h_L \in H_L(F)$. *In particular,the group* $H^Q(F)$ *is unimodular.*

Fix a left Haar measure $d_L h_L$ *on* $H_L(F)$ *and a Haar measure* dh^Q *on* $H^Q(F)$.

(iii) *There exists* $d > 0$ *such that the integral*

$$\int_{H_L(F)} \Xi^L(h_L)\sigma(h_L)^{-d}\delta_{H_L}(h_L)^{1/2} d_L h_L$$

converges. Moreover, in the codimension one case (that is $G = G_0$ *and* $H = H_0$, *the integral*

$$\int_{H_L(F)} \Xi^L(h_L)\sigma(h_L)^{d}\delta_{H_L}(h_L)^{1/2} d_L h_L$$

is convergent for all $d > 0$.

(iv) *There exists* $d > 0$ *such that the integral*

$$\int_{H^Q(F)} \Xi^G(h^Q)\sigma(h^Q)^{-d} dh^Q$$

converges.

(v) *We have* $\sigma(h^Q) \ll \sigma(a^{-1}h^Q a)$ *for all* $a \in A^+_{min}$ *and all* $h^Q \in H^Q(F)$.

(vi) *There exists* $d > 0$ *and* $d' > 0$ *such that*

$$\int_{H^Q(F)} \Xi^G(a^{-1}h^Q a_\sigma(a^{-1}h^Q a)^{-d} dh^Q \ll H^{H\backslash G}(a)^2 \sigma_{H\backslash G}(a)^{d'}$$

for all $a \in A^+_{min}$.

Proof. (i) The follows directly from Proposition 2.3.2(i)

(ii) For $h_{\overline{Q}} \in H_{\overline{Q}}(F)$ which maps to $h_L \in H_L(F)$ via the isomorphism $H_{\overline{Q}} \simeq H_L$, we have $\delta_{\overline{Q}}(h_{\overline{Q}}) = \delta_{\overline{Q}}(h_L)$ and $\delta_{H_{\overline{Q}}}(h_{\overline{Q}}) = \delta_{H_L}(h_L)$. Thus, it suffices to show that $\delta_{\overline{Q}}(h_{\overline{Q}}) = \delta_{H_{\overline{Q}}}(h_{\overline{Q}})$ or equivalently

2.7.2)
$$\det\left(\mathrm{Ad}(h_{\overline{Q}})\Big|_{\overline{\mathfrak{q}}/\mathfrak{h}_{\overline{Q}}}\right) = 1$$

for all $h_{\overline{Q}} \in H_{\overline{Q}}(F)$. We have $\overline{\mathfrak{q}}+\mathfrak{h} = \mathfrak{g}$ (because $\overline{Q}$ is a good parabolic subgroup) and $\mathfrak{h}_{\overline{Q}} = \mathfrak{h} \cap \overline{\mathfrak{q}}$, hence the inclusion $\overline{\mathfrak{q}} \subseteq \mathfrak{g}$ induces an isomorphism $\overline{\mathfrak{q}}/\mathfrak{h}_{\overline{Q}} \simeq \mathfrak{g}/\mathfrak{h}$ from which it follows that

$$\begin{aligned}\det\left(\mathrm{Ad}(h_{\overline{Q}})\Big|_{\overline{\mathfrak{q}}/\mathfrak{h}_{\overline{Q}}}\right) &= \det\left(\mathrm{Ad}(h_{\overline{Q}})\Big|_{\mathfrak{g}/\mathfrak{h}}\right) \\ &= \det\left(\mathrm{Ad}(h_{\overline{Q}})\Big|_{\mathfrak{g}}\right)\det\left(\mathrm{Ad}(h_{\overline{Q}})\Big|_{\mathfrak{h}}\right)^{-1}\end{aligned}$$

forall $h_{\overline{Q}} \in H_{\overline{Q}}(F)$. But since G and H are unimodular groups, we have that $\det\left(\mathrm{Ad}(h_{\overline{Q}})\Big|_{\mathfrak{g}}\right) = \det\left(\mathrm{Ad}(h_{\overline{Q}})\Big|_{\mathfrak{h}}\right)^{-1} = 1$ for all $h_{\overline{Q}} \in H_{\overline{Q}}$ and 2.7.2 follows.

(iii) Let K be a maximal compact subgroup of $G(F)$ which is special in good position with respect to L in the p-adic case. Set $K_L = K \cap L(F)$ (a maximal compact subgroup of $L(F)$ which is special in the p-adic case), $\tau = i^L_{P_{min}\cap L}(1)$ and $\pi = i^G_{\overline{Q}}(\tau)$. We will denote by $(-,-)$ and $(-,-)_\tau$ invariant scalar products on π and τ respectively. Let $e_K \in \pi^\infty$ and $e_{K_L} \in \tau^\infty$ be the unique K-fixed and K_L-fixed vectors respectively. Note that we have $e_K(k) = e_{K_L}$ for all $k \in K$. We may assume that the functions ξ^G and Ξ^L are given by

2.7.3)
$$\Xi^G(g) = (\pi(g)e_K, e_K)\,,\; g \in G(F)$$

2.7.4)
$$\Xi^L(l) = (\tau(l)e_{K_L}, e_{K_L})_\tau\,,\; l \in L(F)$$

(note that by the process of induction by stages, we have a natural isomorphism $\pi \simeq i^G_{P_{min}}(1)$). If we choose Haar measures suitably, 2.7.3 gives

$$\Xi^G(g) = \int_{\overline{Q}(F)\backslash G(F)} (e_K(g'g), e_K(g'))_\tau dg'$$

for all $g \in G(F)$. Since $\overline{Q}$ is a good parabolic subgroup, by Proposition 2.3.2(i) (and since $g \mapsto g^{-1}$ is an automorphism of G) the subset $H_{\overline{Q}}(F)\backslash H(F) \subseteq \overline{Q}(F)\backslash G(F)$ has a complement which is negligible. Hence, by (ii), if we choose Haar measures compatibly, we have

□

3 Explicit tempered intertwinings

We will keep the notation from the previous chapter. Given a tempered representation π of $G(F)$, this chapter studies a certain explicit $(H,\xi)\times(H,\xi)$-equivariant sesquiliniear form $\mathcal{L}_\pi$ on (the space of) π, the main result being that $\mathcal{L}_\pi$ is nonzero if and only if the multiplicity $m(\pi)$ is nonzero (Theorem 3.2.2). This will be used in the proof of the spectral side of our local trace formula (Theorem **??**). The sesquilinear form $\mathcal{L}_\pi$ is introduced in ...

3.1 The ξ-integral

For all $f \in \mathcal{C}(G(F))$, the integral

$$\int_{H(F)} f(h)\xi(h)dh$$

is absolutely convergent by Lemma 2.4.1(ii). I presume bc f is bounded by Ξ and ξ is bounded by σ^{-d} Moreover, by lemma 2.4.1(ii) again, this defines a continuous linear form on $\mathcal{C}(G(F))$. Recall that $\mathcal{C}(G(F))$ is a dense subspace of the weak Harish-Chandra Schwartz space $\mathcal{C}^w(G(F))$ (by 1.3.1). have not defined yet.

Proposition 3.1.1 (7.1.1) *The linear form*

$$\mathcal{C}(G(F)) \ni f \mapsto \int_{H(F)} f(h)\xi(h)dh$$

extends continuously to $\mathcal{C}^w(G(G))$.

Proof. Let us fix a one-parameter subgroup $a : \mathbb{G}_m \to A$ such that $\lambda(a(t)ha(t)^{-1}) = t\lambda(h)$ for all $t \in \mathbb{G}_m$ and all $h \in H$ (recall that $\lambda : H \to \mathbb{G}_a$ is the algebraic character such that $\xi = \psi \circ \lambda_F$), such a one-parameter subgroup is easy to construct. We shall now divide the proof into the p-adic and archemedian cases.

- If F is a p-adic field, then we may fix a compact open subgroup $K \subseteq G(F)$ and prove that the linear form

  $$f \in \mathcal{C}_K(G(F)) \mapsto \int_{H(F)} f(h)\xi(h)dh$$

  extends continuously to $C^w_K(G(F))$. Set $K_a = a^{-1}(K \cap A(F)) \subseteq F^\times$. how to define a^{-1} Then for all $f \in \mathcal{C}_K(G(F))$, we have

  3.1.2)

  $$\begin{aligned}\int_{H(F)} f(h)\xi(h)dh &= \mathrm{meas}(K_a)^{-1}\int_{K_a}\int_{H(F)} f(a(t)^{-1}ha(t))\xi(h)dhd^\times t\\ &= \mathrm{meas}(K_a)^{-1}\int_{H(F)} f(h)\int_{K_a}\xi(a(t)ha(t)^{-1})d^\times tdh\\ &= \mathrm{meas}(K_a)^{-1}\int_{H(F)} f(h)\int_{K_a}\psi(t\lambda(h))|t|^{-1}dtdh\end{aligned}$$

  The function $F \ni x \mapsto \int_{K_a}\psi(tx)|t|^{-1}dt$ is the Fourier transform of the function $|\bullet|^{-1}\mathbb{1}_{K_a} \in C^\infty_C(F)$ hence it belongs to $C^\infty_c(F)$. Now by lemma 2.4.1(iii), the last integral of 3.1.2 is absolutely convergent for all $f \in C^w_K(G(F))$ and defines a continuous linear form on that space. This is the extension we were looking for.

- Now assume that $F = \mathbb{R}$. Let us denote Ad the adjoint action of $G(F)$ on $C^w(G(F))$ i.e., one has

$$(\mathrm{Ad}(g)f)(x) = f(g^{-1}xg),\ f \in C^w(G(F)),\ g, x \in G(F)$$

Set $\mathrm{Ad}_a = \mathrm{Ad} \circ a$. Then Ad_a is a smooth representation sanity check, does it matter that this is in center? of $F^\times$ on $C^w(G(F))$ and hence induces an action, also denoted by Ad_a, of $\mathcal{U}$ $(\mathrm{Lie}(F^\times))$ on $\mathcal{C}^w(G(F))$. Set $\Delta = 1 - (t\frac{d}{dt})^2 \in \mathcal{U}(\mathrm{Lie}(F^\times))$. By elliptic regularity (2.1.2), for every integer $m \geq 1$, there exists functions $\varphi_1 \in C_c^{2m-2}(F^\times)$ and $\varphi_2 \in C_c^\infty(F^\times)$ such that

$$\varphi_1 * \Delta^m + \varphi_2 = \delta_1$$

Hence, we have the equality

$$\mathrm{Ad}_a(\varphi_1)\,\mathrm{Ad}_a(\Delta^m) + \mathrm{Ad}_a(\varphi_2) = Id$$

It follows that for all $f \in \mathcal{C}(G(F))$, we have

3.1.3)

$$\begin{aligned}
\int_{H(F)} f(h)\xi(h)dh &= \int_{H(F)} (\mathrm{Ad}_a(\varphi_1)\,\mathrm{Ad}_a(\Delta^m)f)(h)\xi(h)dh \\
&+ \int_{H(F)} (\mathrm{Ad}_a(\varphi_2)f)(h)\xi(h)dh \\
&= \int_{H(F)} (\mathrm{Ad}_a(\Delta^m)f)(h) \int_{F^\times} \varphi_1(t)\xi(a(t)ha(t)^{-1})\delta_P(a(t))d^\times t dh \\
&+ \int_{H(F)} f(h) \int_{F^\times} \varphi_2(t)\xi(a(t)ha(t)^{-1})\delta_P(a(t))d^\times t dh \\
&= \int_{H(F)} (\mathrm{Ad}_a(\Delta^m)f)(h) \int_{F} \varphi_1(t)\delta_P(a(t))|t|^{-1}\psi(t\lambda(h))dtdh \\
&+ \int_{H(F)} f(h) \int_{F} \varphi_2(t)\delta_P(a(t))|t|^{-1}\psi(t\lambda(h))dtdh
\end{aligned}$$

Consider the functions $f_i : x \mapsto \int_F \varphi_i(t)\delta_P(a(t))|t|^{-1}\psi(tx)dt$, $i = 1, 2$, $x \in F$. These are the Fourier transforms of the functions $t \mapsto \varphi_i(t)\delta_H(a(t))|t|^{-1}$, $i = 1, 2$, which both belong to $C_c^{2m-2}(F)$. Hence, f_1 and f_2 are both essentially bounded by $(1 + |x|)^{-2m+2}$. Now, by Lemma 2.4.1(iii), if $m \geq 2$ the two integrals in the last term of 3.1.3 are absolutely convergent for all $f \in C^w(G(F))$ and define on the space continuous linear forms. The extension we were looking for is just the sum of these two integrals.

$\square$

The continuous linear form on $C^w(G(F))$ whose existence is proved by the proposition above will be called the ξ-*integral* on $H(F)$ and will be denoted by

$$C^w(G(F)) \ni f \mapsto \int_{H(F)}^* f(h)\xi(h)dh$$

or

$$C^w(G(F)) \ni f \mapsto \mathcal{P}_{H,\xi}(f)$$

We now note the following properties of the ξ-integral.

Lemma 3.1.4 *(i) For all $f \in \mathcal{C}^w(G(F))$ and all $h_0, h_1 \in H(F)$, we have*

$$\mathcal{P}_{H,\xi}(L(h_0))R(h_1)f) = \xi(h_0)\xi(h_1)^{-1}\mathcal{P}_{H,\xi}(f)$$

(ii) Let $a : \mathbb{G} \to A$ be a one-parameter subgroup such that $\lambda(a(t)ha(t)^{-1}) = t\lambda(h)$ for all $t \in \mathbb{G}_m$ and all $h \in H$. Denote by Ad_a the representation of $F^\times$ on $C^w(G(F))$ given by $\mathrm{Ad}_a(t) = L(a(t))R(a(t))$ for all $t \in F^\times$. Let $\varphi \in C_c^\infty(F^\times)$. Set $\varphi'(t) = |t|^{-1}\delta_P(a(t))\varphi(t)$ for all $t \in F^\times$ and denote $\widehat{\varphi}'$ its Fourier transform, that is

$$\widehat{\varphi}'(x) = \int_F \varphi'(t)\psi(tx)dt, \; x \in F$$

Then, we have

$$\mathcal{P}_{H,\xi}(\mathrm{Ad}_a(\varphi)f) = \int_{H(F)} f(h)\widehat{\varphi}'(\lambda(h))dh$$

for all $f \in \mathcal{C}^w(G(F))$, where the second integral is absolutely convergent.

Proof. In both (i) and (ii), both side of the equality to be proved are continuous in $f \in C^w(G(F))$ (for (ii) this follows from Lemma 2.4.1(iii)). Hence it is sufficient to check the relations for $f \in \mathcal{C}(G(F))$ where by straightforward variable changes we can pass from the left hand side to the right hand side. $\square$

3.2 Definition of $\mathcal{L}_\pi$

Let π be a tempered representation of $G(F)$. For all $T \in \mathrm{End}(\pi)^\infty$, the function

$$G(F) \ni g \mapsto \mathrm{Tr}(\pi(g^{-1})T)$$

belongs to the weak Harish-Chandra Schwartz space $\mathcal{C}^w(G(F))$ by 2.2.4. We can thus define a linear form $\mathcal{L}_\pi : \mathrm{End}(\pi)^\infty \to \mathbb{C}$ by setting

$$\mathcal{L}_\pi(T) = \int_{H(F)}^* \mathrm{Tr}(\pi(h^{-1})T)\xi(h)dh, \; T \in \mathrm{End}(\pi)^\infty$$

By Lemma 3.1.3(i), we have

$$\mathcal{L}_\pi(\pi(h)T\pi(h')) = \xi(h)\xi(h')\,\mathcal{L}_\pi(T)$$

for all $h, h' \in H(F)$ and $T \in \mathrm{End}(\pi)^\infty$. By 2.2.5, the map which associates to $T \in \mathrm{End}(\pi)^\infty$ the function

$$g \mapsto \mathrm{Tr}(\pi(g^{-1})T)$$

in $\mathcal{C}^w$ in $C^w(G(F))$ is continuous. Since the ξ-integral is a continuous linear form on $\mathcal{C}^w(G(F))$, it follows that the linear form $\mathcal{L}_\pi$ is continuous.

Recall that we have a continuous $G(F) \times G(F)$-equivariant embedding with dense image $\pi^\infty \otimes \overline{\pi^\infty} \hookrightarrow \mathrm{End}(\pi)^\infty$, $e \otimes e' \mapsto T_{e,e'}$ (which is an isomorphism in the p-adic case). Where $T_{e,e'}(e_0) = (e_0, e')e$ for all $e_0 \in \pi$. In any case, $\mathrm{End}(\pi)^\infty$ is naturally isomorphic to the completed projective tensor product $\pi^\infty \widehat{\otimes}_p \overline{\pi^\infty}$. Thus we may identify $\mathcal{L}_\pi$ with the continuous sesquilinear form on π^∞ given by

$$\mathcal{L}_\pi(e, e') := \mathcal{L}_\pi(T_{e,e'})$$

for all $e, e' \in \pi^\infty$. Expanding definitions, we have

$$\mathcal{L}_\pi(e, e') = \int_{H(F)}^* (e, \pi(h)e')\xi(h)dh$$

for all $e, e' \in \pi^\infty$. Fixing $e' \in \pi^\infty$, we see that the map $\pi^\infty \ni e \mapsto \mathcal{L}_\pi(e, e')$ belongs to $\mathrm{Hom}_H(\pi^\infty, \xi)$. By the density of $\pi^\infty \otimes \overline{\pi}^\infty$ in $\mathrm{End}(\pi)^\infty$, it follows that

$$\mathcal{L}_\pi \neq 0 \implies m(\pi) \neq 0$$

The purpose of this chapter is to prove the converse direction. Namely, we will show

Theorem 3.2.1 (7.2.1) *For all* $\pi \in \mathrm{Temp}(G)$*, we have*

$$\mathcal{L}_\pi \neq 0 \iff m(\pi) \neq 0$$

credits...

Next, to end this section, we will give some properties of $\mathcal{L}_\pi$. First, since $\mathcal{L}_\pi$ is a continuous sesquillinear form on π^∞, it defines a continuous linear map

$$L_\pi : \pi^\infty \to \overline{\pi^{-\infty}}$$

$$e \mapsto \mathcal{L}_\pi(e, \bullet)$$

where $\overline{\pi^{-\infty}}$ is the topological conjugate-dual of π^∞ endowed with the strong topology. This operator L_π has its image included in $\overline{\pi^{-\infty}}^{H,\xi} = \mathrm{Hom}_H(\overline{\pi^\infty}, \xi)$. By Theorem 2.2.2, this subspace is finite-dimensional and even of dimension less than or equal to 1 if π is irreducible. Let $T \in \mathrm{End}(\pi)^\infty$. Recall that it extends uniquely to a continuous operator $T : \overline{\pi^{-\infty}} \to \pi^\infty$. Thus, we can form the two compositions

$$TL_\pi : \pi^\infty \to \pi^\infty$$

$$L_\pi T : \overline{\pi^{-\infty}} \to \overline{\pi^{-\infty}}$$

which are both finite-rank operators. In particular, their traces are well-defined and we have

3.2.2)

$$\mathrm{Tr}(TL_\pi) = \mathrm{Tr}(L_\pi T) = \mathcal{L}_\pi(T)$$

Lemma 3.2.3 *(i) The maps*

$$\mathcal{X}_{\mathrm{temp}}(G) \ni \pi \mapsto L_\pi \in \mathrm{Hom}(\pi^\infty, \overline{\pi^{-\infty}})$$

$$\mathcal{X}_{\mathrm{temp}}(G) \ni \pi \mapsto \mathcal{L}_\pi \in \mathrm{End}(\pi)^{-\infty}$$

are smooth in the following sense: For every parabolic subgroup $Q = LU_Q$ of G, for all $\sigma \in \Pi_2(L)$ and for every maximal compact subgroup K of $G(F)$, which is special in the p-adic case, the maps

$$iA_L^* \ni \lambda \mapsto \mathcal{L}_{\pi_\lambda} \in \mathrm{End}(\pi_\lambda)^{-\infty} \simeq \mathrm{End}(\pi_K)^{-\infty}$$

$$iA_L^* \ni \lambda \mapsto L_{\pi_\lambda} \in \mathrm{Hom}(\pi_\lambda^\infty, \overline{\pi_\lambda^{-\infty}}) \simeq \mathrm{Hom}(\pi_K^\infty, \overline{\pi_K^{-\infty}})$$

are smooth, where we have set $\pi_\lambda = i_Q^G(\sigma_\lambda)$ and $\pi_K = I_{Q\cap K}^K(\sigma)$.

(ii) Let π be in $\mathrm{Temp}(G)$ or $\mathcal{X}_{\mathrm{temp}}(G)$. Then for all $S, T \in \mathrm{End}(\pi)^\infty$, we have $SL_\pi \in \mathrm{End}(\pi)^\infty$ and

$$\mathcal{L}_\pi(S)\,\mathcal{L}_\pi(T) = \mathcal{L}_\pi(SL_\pi T)$$

(iii) Let $S, T \in \mathcal{C}(\mathcal{X}_{\mathrm{temp}}(G), \mathcal{E}(G))$. Then, the section $\pi \in \mathrm{Temp}(G) \mapsto S_\pi L_\pi T_\pi \in \mathrm{End}(\pi)^\infty$ belongs to $C^\infty(\mathcal{X}_{\mathrm{temp}}(G), \mathcal{E}(G))$.

(iv) Let $f \in \mathcal{C}(G(F))$ and assume that its Fourier transform $\mathcal{X}_{\mathrm{temp}}(G) \ni \pi \mapsto \pi(f)$ is compactly supported (this condition is automatically satisfied when F is p-adic). Then, we have the equality

$$\int_{H(F)} f(h)\xi(h)dh = \int_{\mathcal{X}_{\mathrm{temp}}(G)} \mathcal{L}_\pi(\pi(f))\mu(\pi)d\pi$$

both integrals being absolutely convergent.

(v) Let $f, f' \in \mathcal{C}(G(F))$ and assume that the Fourier transform of f is compactly supported. Then we have the equality

$$\int_{\mathcal{X}_{\mathrm{temp}}(G)} \mathcal{L}_\pi(\pi(f))\overline{\mathcal{L}_\pi(\pi(\overline{f'}))}\mu(\pi)d\pi = \int_{H(F)}\int_{H(F)}\int_{G(F)} f(hgh')f'(g)dg\xi(h')dh'\xi(h)dh$$

where the first integral is absolutely convergent and the second integral is convergent in that order but not necessarily as a triple integral.

Proof. (i) Let $Q = LU_Q$, $\sigma \in \Pi_2^\infty(L)$ and K be as in the statement. Recall that our convention is to equip all the spaces that appear in the statement with the strong topology.

We have $\mathrm{End}(\pi_K)^\infty \simeq \pi_K^\infty \hat{\otimes}_p \overline{\pi_k^\infty}$. Hence, the space $\mathrm{End}(\pi_K)^{-\infty}$ may be identified with the space of continuous sesquilinear forms on π_K^∞ and we get a natural continuous embedding

$$\mathrm{End}(\pi_K)^{-\infty} \hookrightarrow \mathrm{Hom}(\pi_K^\infty, \overline{\pi_K^{-\infty}})$$

The image of $\mathcal{L}_{\pi_\lambda}$ by this map is L_{π_λ}, for all $\lambda \in i\mathcal{A}_L^*$. Consequently, it suffices to prove the smoothness of the map $\lambda \mapsto \mathcal{L}_{\pi_\lambda}$. By proposition A.3.1(iv), this is equivalent to the smoothness of

$$\lambda \mapsto \mathcal{L}_{\pi_\lambda}(T)$$

for all $T \in \mathrm{End}(\pi_K)^\infty$. Because the ξ-integral is a continuous liner form on $\mathcal{C}^w(G(F))$, the smoothness of this last map follows from Lemma 2.3.1(ii)

(ii) The two inclusions $\mathrm{End}(\pi)^\infty \subset \mathrm{Hom}(\overline{\pi^{-\infty}}, \pi)$ and $\mathrm{End}(\pi)^\infty \subset \mathrm{Hom}(\pi, \pi^\infty)$ are continuous. I follows that the bilinear map

$$\mathrm{End}(\pi)^\infty \times \mathrm{End}(\pi)^\infty \to \mathrm{End}(\pi)$$

$$(S,T) \mapsto SL_\pi T$$

is separately continuous. For all $S,T \in \mathrm{End}(\pi)^\infty$, the maps $G(F) \ni g \mapsto \pi(g)S \in \mathrm{End}(\pi)^\infty$ and $G(F) \ni g \mapsto T\pi(g) \in \mathrm{End}(\pi)^\infty$ are smooth. Hence, by Proposition A.3.1(v) in the real case, we have $SL_\pi T \in \mathrm{End}(\pi)^\infty$ for all $S,T \in \mathrm{End}(\pi)^\infty$. We now prove the equality

$$\mathcal{L}_\pi(S)\,\mathcal{L}_\pi(T) = \mathcal{L}_\pi(SL_\pi T)$$

for all $S,T \in \mathrm{End}(\pi)^\infty$. Assume first that $\pi \in \mathrm{Temp}(G)$. Then, this follows directly from 3.2.2 since the operators $L_\pi S, L_\pi T : \overline{\pi^{-\infty}} \to \overline{\pi^{-\infty}}$ have their images contained in the same line (which is $\mathrm{Hom}_H(\overline{\pi^\infty}, \xi)$). Assume now that $\pi \in \mathcal{X}_{\mathrm{temp}}(G)$. We may then find a parabolic subgroup $Q = LU_Q$ of G and a square-integrable representation $\sigma \in \Pi_2(L)$ such that $\pi = i_P^G(\sigma)$. Let K be a maximal compact subgroup of $G(F)$ which is special in the p-adic case and set $\pi_K = i_{Q\cap K}^K(\sigma)$ and $\pi_\lambda = i_P^G(\sigma_\lambda)$ for all $\lambda \in i\mathcal{A}_L^*$. Then, we have isomorphisms $\mathrm{End}(\pi_\lambda)^\infty \simeq \mathrm{End}(\pi_K)^\infty$ for all $\lambda \in i\mathcal{A}_L^*$. Let $S,T \in \mathrm{End}(\pi)^\infty$ and identify them to their images in $\mathrm{End}(\pi_K)^\infty$ by the previous isomorphism. For λ in a dense subset of $i\mathcal{A}_L^*$, the representation π_λ is irreducible. Hence, for every such $\lambda \in i\mathcal{A}_L^*$ we have

$$\mathcal{L}_{\pi_\lambda}(S)\,\mathcal{L}_{\pi_\lambda}(T) = \mathcal{L}_{\pi_\lambda}(SL_{\pi_\lambda}T)$$

By (i), the left hand side of this equality is continuous with respect to $\lambda \in i\mathcal{A}_L^*$. To deduce the equality at $\lambda = 0$ (what we want), it thus suffices to show that the function

$$i\mathcal{A}_L^* \ni \lambda \mapsto \mathcal{L}_{\pi_\lambda}(SL_{\pi_\lambda}T)$$

is continuous. We will, in fact, prove that it is a smooth function. By (i) and proposition A.3.1, it suffices to show that for all $\lambda \in i\mathcal{A}_L^*$, the trilinear map

3.2.4)

$$\operatorname{End}(\pi_\lambda)^\infty \times \operatorname{Hom}(\pi_\lambda^\infty, \overline{\pi_\lambda^{-\infty}}) \times \operatorname{End}(\pi_\lambda)^\infty \to \operatorname{End}(\pi_\lambda)^\infty$$

$$(S, L, T) \mapsto SLT$$

is separately continuous. As the inclusion $\operatorname{End}(\pi_\lambda)^\infty \subset \operatorname{Hom}(\overline{\pi_\lambda^{-\infty}}, \pi_\lambda)$ and $\operatorname{End}(\pi_\lambda)^\infty \subseteq \operatorname{Hom}(\pi_\lambda, \pi_\lambda^\infty)$ are continuous, the trilinear map

$$\operatorname{End}(\pi_\lambda)^\infty \times \operatorname{Hom}(\pi_\lambda^\infty, \overline{\pi_\lambda^{-\infty}}) \times \operatorname{End}(\pi_\lambda)^\infty \to \operatorname{End}(\pi_\lambda)$$

$$(S, L, T) \mapsto SLT$$

is separately continuous for all $\lambda \in i\mathcal{A}_L^*$. By definition of the topology on $\operatorname{End}(\pi_\lambda)^\infty$, this immediately implies that 3.2.4 is separately continuous for all $\lambda \in i\mathcal{A}_L^*$.

(iii) This is a direct consequence of (i) and of the fact that the trilinear map 3.2.4 is separately continuous.

(iv) Let $f \in \mathcal{C}(G(F))$. The left hand side of (iv) is absolutely convergent by Lemma 2.4.1(ii). By Lemma 2.3.1(ii), the map

$$\mathcal{X}_{\text{temp}}(G) \ni \pi \mapsto \varphi(f, \pi) \in \mathcal{C}^w(G(F))$$

where $\varphi(f, \pi)(g) = \operatorname{Tr}(\pi(G^{-1})\pi(f))$, is continuous. By the hypothesis made on f, this map is also compactly supported. It follows that the function $\mathcal{X}_{\text{temp}}(G) \ni \pi \mapsto \mu(\pi)\varphi(f, \pi) \in \mathcal{C}^w(G(F))$ is absolutely integrable. Hence, the function

$$\mathcal{X}_{\text{temp}}(G) \ni \pi \mapsto \mu(\pi)\,\mathcal{L}_\pi(\pi(f)) = \mu(\pi)\,\mathcal{P}_{H,\xi}(\varphi(f, \pi))$$

where $\mathcal{P}_{H,\xi} : \mathcal{C}^w(G(F)) \to \mathbb{C}$ denotes the ξ-integral, is also absolutely integrable, proving the convergence of the right hand side of (iv). We also have the equality

$$f = \int_{\mathcal{X}_{\text{temp}}(G)} \varphi(f, \pi)\mu(\pi)d\pi$$

in $\mathcal{C}^w(G(F))$(or its completion). By the Harish-Chandra Plancherel formula both sides are equal after applying the evaluation map g for all $g \in G(F)$. It follows that

$$\mathcal{P}_{H,\xi}(f) = \int_{\mathcal{X}_{\text{temp}}(G)} \mathcal{P}_{H,\xi}(\varphi, f))\mu(\pi)d\pi$$

which is exactly the content of (iv).

(v) The right hand side of (v) may be rewritten as

3.2.5)
$$\int_{H(F)}\int_{H(F)}(f'^{\vee} * L(h^{-1})f)(h')\xi(h')dh'\xi(h)dh$$

where $f'^{\vee}(g) = f'(g^{-1})$. The Fourier transform of $f'^{\vee} * L(h^{-1})f$ is given by
$$\mathcal{X}_{\text{temp}}(G) \ni \pi \mapsto \pi(f'^{\vee} * L(h^{-1})f) = \pi(f'^{\vee})\pi(h^{-1})\pi(f)$$
In particular, it is compactly supported. Applying (iv) to $f'^{\vee} * L(h^{-1})f$, we deduce that the integral

$$\int_{H(F)}(f'^{\vee} * L(h^{-1})f)(h)'\xi(h')dh'$$

is absolutely convergent and is equal to

$$\int_{\mathcal{X}_{\text{temp}}(G)}\mathcal{L}_{\pi}(\pi(f'^{\vee})\pi(h^{-1})\pi(f))\mu(\pi)d\pi$$

By 3.2.2, this last integral is equal to

$$\int_{\mathcal{X}_{\text{temp}}(G)}\operatorname{Tr}(\pi(h^{-1})\pi(f)L_{\pi}\pi(f'^{\vee}))\mu(\pi)d\pi$$

By (iii), the section $\mathcal{X}_{\text{temp}}(G) \mapsto \pi(f)L_{\pi}\pi(f'^{\vee}) \in \operatorname{End}(\pi)^{\infty}$ is smooth. Moreover, it is compactly supported and so it belongs to $\mathcal{C}(\mathcal{X}_{\text{temp}}(G), \mathcal{E}(G))$. By the matricial Paley-Wiener theorem (Theorem 2.6.1), it is the Fourier transform of a Harish-Chandra Schwartz function. Applying (iv) to this function, we see that the exterior integral of 3.2.5 is absolutely convergent and tha the whole expression is equal to the absolutely convergent integral

$$\int_{\mathcal{X}_{\text{temp}}(G)}\mathcal{L}_{\pi}(\pi(f)L_{\pi}\pi(f'^{\vee}))\mu(\pi)d\pi$$

which by (ii) is equal to

$$\int_{\mathcal{X}_{\text{temp}}(G)}\mathcal{L}_{\pi}(\pi(f))\,\mathcal{L}_{\pi}(\pi(f'^{\vee}))\mu(\pi)d\pi$$

The point (v) now follows from this and the equality

$$\mathcal{L}_{\pi}(\pi(f'^{\vee})) = \overline{\mathcal{L}_{\pi}(\pi(\overline{f'}))}, \quad \pi \in \mathcal{X}_{\text{temp}}(G)$$

$\square$

3.3 Asymptotics of tempered intertwinings

Lemma 3.3.1 *[7.3.1]*

(i) Let π be a tempered representation of $G(F)$ and $l \in \mathrm{Hom}_H(\pi^\infty, \xi)$ be a continuous (H,ξ)-equivarient linear form. Then, there exist $d > 0$ and a continuous semi-norm ν_d on π^∞ such that

$$|l(\pi(x)e)| \leq \nu_d(e)\Xi^{H\backslash G}(x)\sigma_{H\backslash G}(x)^d$$

for all $e \in \pi^\infty$ and all $x \in H(F)\backslash G(F)$

(ii) For all $d > 0$, there exists $d' > 0$ and a continuous semi-norm $\nu_{d,d'}$ on $\mathcal{C}_d^w(G(F))$ such that

$$|\mathcal{P}_{H,\xi}(R(x)L(y)\varphi)| \leq \nu_{d,d'}(\varphi)\Xi^{H\backslash G}(x)\Xi^{H\backslash G}(y)\sigma_{H\backslash G}(x)^{d'}\sigma_{H\backslash G}(y)^{d'}$$

for all $\sigma \in \mathcal{C}_d^w(G(F))$ and all $x, y \in H(F)\backslash G(F)$.

Proof. (i) TBD

□

3.4 Explicit intertwinings and parabolic induction

Let $Q = LU_Q$ be a parabolic subgroup of G. Because $G = SO(W) \times SO(V)$, we have decompositions

3.4.1)
$$Q = Q_W \times Q_V \text{ and } L = L_W \times L_V$$

where Q_W and Q_V are parabolic subgroups of $SO(W)$ and $SO(V)$ respectively and L_W, L_V are Levi components of those. By the explicit description of parabolic subgroups of unitary groups (ref), we have

3.4.2)
$$L_W = GL_F(Z_{1,W}) \times \cdots \times GL_F(Z_{a,W}) \times SO(\widetilde{W})$$

3.4.3)
$$L_V = GL_F(Z_{1,V}) \times \cdots \times GL_F(Z_{b,V}) \times SO(\widetilde{V})$$

where $Z_{i,W}$, $1 \leq i \leq a$ (respectively $Z_{i,V}$, $1 \leq i \leq b$) are totally isotropic subspaces of W (respectively V) and $\widetilde{W}$ (respectively $\widetilde{V}$) is a non-degenerate subspace of W (respectively V). Let $\widetilde{G} = SO(\widetilde{W}) \times SO(\widetilde{V})$. The pair $(\widetilde{V}, \widetilde{W})$ is easily seen to be admissible up to permutation, hence it defines a GGP triple $(\widetilde{G}, \widetilde{H}, \widetilde{\xi})$ well-definedup to $(\widetilde{G}(F)$-conjugation. For all tempered representations $\widetilde{\sigma}$ of $\widetilde{G}(F)$, we may define as in Section 3.2 a continuous linear form $\mathcal{L}_{\widetilde{\sigma}} : \mathrm{End}(\widetilde{\sigma})^\infty \to \mathbb{C}$.

Let σ be a tempered representation of $L(F)$ which decomposes according to the decompositions 3.4.1, 3.4.2, and 3.4.3 as a tensor product

3.4.4)
$$\sigma = \sigma_W \boxtimes \sigma_V$$

3.4.5)
$$\sigma_W = \sigma_{1,W} \boxtimes \cdots \boxtimes \sigma_{a,W} \boxtimes \widetilde{\sigma}_W$$

3.4.6)
$$\sigma_V = \sigma_{1,V} \boxtimes \cdots \boxtimes \sigma_{b,V} \boxtimes \widetilde{\sigma}_V$$

where $\sigma_{i,W} \in \mathrm{Temp}(GL_F(Z_{i,W}))$ for $1 \leq i \leq a$, $\sigma_{i,V} \in \mathrm{Temp}(GL_F(Z_{i,V}))$ for $1 \leq i \leq b$, $\widetilde{\sigma}_W$ is a tempered representation of $SO(\widetilde{W})(F)$, and $\widetilde{\sigma}_V$ is a tempered representation of $SO(\widetilde{V})(F)$. Let us set $\widetilde{\sigma} = \widetilde{\sigma}_W \boxtimes \widetilde{\sigma}_V$. It is a tempered representation of $\widetilde{G}(F)$. Finally let us set $\pi = i_Q^G(\sigma)$, $\pi_W = i_{Q_W}^{SO(W)}(\sigma_W)$ and $\pi_V = i_{Q_V}^{SO(V)}(\sigma_V)$ for the parabolic inductions of σ, σ_W, and σ_V respectively. We have $\pi = \pi_W \boxtimes \pi_V$

Proposition 3.4.7 (7.4.1) *With the notation above, we have*

$$\mathcal{L}_\pi \neq 0 \iff \mathcal{L}_{\widetilde{\sigma}} \neq 0$$

Proof. We will use the following notation. If X is an F-vector space of finite dimension, $Q_X = L_X U_X$ is a parabolic subgroup of $GL_F(X)$ with

$$L_X = GL_F(X_1) \times \cdots \times GL_F(X_c)$$

and we have tempered representations $\sigma_{i,X}$ of $GL_F(X_i)$ for $1 \leq i \leq c$, then we will denote by
$$\sigma_{1,X} \times \cdots \times \sigma_{c,X}$$
the induced representation $i_{Q_X}^{GL_F(X)}(\sigma_{1,X} \boxtimes \cdots \boxtimes \sigma_{c,X})$. Note that if all the $\sigma_{i,X}$ $1 \leq i \leq c$, are irreducible so is $\sigma_{1,X} \times \cdots \times \sigma_{c,X}$. Similarly, if X is a quadratic space, $Q_X = L_X U_X$ is a parabolic subgroup of $SO(X)$ with

$$L_X = GL_F(Z_{1,X}) \times \cdots \times GL_F(Z_{d,X}) \times SO(\widetilde{X})$$

and we have tempered representations $\sigma_{i,X}$ of $GL_F(Z_{i,X})$ for $1 \leq i \leq d$ and a tempered representation $\widetilde{\sigma}_X$ of $SO(\widetilde{(X)})(F)$, then we will denote by

$$\sigma_{1,X} \times \cdots \times \sigma_{d,X} \times \widetilde{\sigma}_X$$

the induced representation $i_{Q_X}^{U(X)}(\sigma_{1,X} \boxtimes \cdots \boxtimes \sigma_{d,X} \boxtimes \widetilde{\sigma}_X)$. In particular, with these notations we have

$$\pi_W = \sigma_{1,W} \times \cdots \times \sigma_{a,W} \times \widetilde{\sigma}_W$$

and
$$\pi_V = \sigma_{1,V} \times \cdots \times \sigma_{b,V} \times \widetilde{\sigma}_V$$

By the process of induction by stages, we also have $\pi_W = \sigma'_W \times \widetilde{\sigma}_W$ and $\pi_V = \sigma'_V \times \widetilde{\sigma}_V$ where

$$\sigma'_W = \sigma_{1,W} \times \cdots \times \sigma_{a,W}$$

$$\sigma'_V = \sigma_{1,V} \times \cdots \times \sigma_{b,V}$$

These are tempered irreducible representations of general linear groups. Hence, the statement immediately reduces to the case where $a \leq 1$ and $b \leq 1$.

First, we will treat the codimension one case where $G = G_0$, $H = H_0$ and $(a,b) = (0,1)$. Then $Q = SO(W) \times Q_V$ where Q_V is a maximal proper parabolic subgroup of G. Up to conjugating Q, we may assume that it is a good parabolic subgroup (Section 2.3). Then to fit with the general notation of Chapter 2, we will change our notation and denote Q by $\overline{Q}$ and Q_V by $\overline{Q_V}$. Set $H_{\overline{Q}} = H \cap \overline{Q}$. Clearly, we have a natural embedding $H_{\overline{Q}} \hookrightarrow L$.

We will assume, without loss of generality, that the invariant scalar product on π is given by

$$(e, e') = \int_{\overline{Q}(F)\backslash G(F)} (e(g), e'(g))dg, \quad e, e' \in \pi$$

where the scalar product in the integral is the scalar product on σ. Since we are in the codeimension one case, the integral defining $\mathcal{L}_\pi$ is absolutely convergent and we have

3.4.8)
$$\mathcal{L}_\pi(e, e') = \int_{H(F)} \int_{\overline{Q}(F)\backslash G(F)} (e(g), e'(gh))dgdh$$

for all $e, e' \in \pi^\infty$. Let us show

3.4.9) The expression 3.4.8 is absolutely convergent for all $e.e' \in \pi^\infty$

Let $e, e' \in \pi^\infty$ and choose a maximal compact subgroup K if $G(F)$ which is special in the p-adic case. Then for a suitable choice of Haar measure on K, we have

$$\int_{\overline{Q}(F)\backslash G(F)} |(e(g), e'(gh))|dg = \int_K |(e(k), e'(kh))|dk$$
$$= \int_K \delta_{\overline{Q}}(l_{\overline{Q}}(kh))^{1/2} |(e(k), \sigma(l_{\overline{Q}}(kh))e'(k_{\overline{Q}}(kh)))|dk$$

for all $h \in H(F)$. Here, as usual $l_{\overline{Q}} : G(F) \to L(F)$ and $k_{\overline{Q}} : G(F) \to K$ are maps such that $l_{\overline{Q}}(g)^{-1} g k_{\overline{Q}}(g)^{-1} \in U_{\overline{Q}}(F)$ (the unipotent radical of $\overline{Q}(F)$) for all $g \in G(F)$. Since σ is tempered and the maps $K \ni k \mapsto e(k) \in \sigma$, $K \ni k \mapsto e'(k) \in \sigma$ have bounded image, it follows that

$$\int_{\overline{Q}(F)\backslash G(F)} |(e(g), e'(gh))|dg \ll \int_K \delta_{\overline{Q}}(l_{\overline{Q}}(kg))^{1/2} \Xi^L(l_{\overline{Q}}(kh))dk = \Xi^G(h)$$

for all $h \in H(F)$, where the last equality is Proposition 1.3.1(iii). 3.4.9 now follows from Lemma 2.4.1(i).

Since $\overline{Q}$ is a good parabolic subgroup, by Proposition 2.3.2(i) the quotient $H_{\overline{Q}}(F)\backslash H(F)$ has negligible complement in $\overline{Q}(F)\backslash G(F)$. Hence, if we choose Haar measures compatibly, we have

$$\int_{\overline{Q}(F)\backslash G(F)} \varphi(g)dg = \int_{H_{\overline{Q}}(F)\backslash H(F)} \varphi(h)dh$$

for all $\varphi \in L^1(\overline{Q}(F)\backslash G(F), \delta_{\overline{Q}})$. Thus, 3.4.8 becomes

$$\mathcal{L}_\pi(e, e') = \int_{H(F)} \int_{H_{\overline{Q}}\backslash H(F)} (e(h'), e'(h'h))dh'dh$$

for all $e, e' \in \pi^\infty$, the double integral being absolutely convergent by 3.4.9. Switching the two integrals, we get

3.4.10)
$$\mathcal{L}_\pi(e, e') = \int_{(H_{\overline{Q}}(F)\backslash H(F))^2} \mathcal{L}_\sigma(e(h), e'(h'))dhdh'$$

for all $e, e' \in \pi^\infty$, where we have set

$$\mathcal{L}_\sigma(v, v') = \int_{H_{\overline{Q}}(F)} (v, \sigma(h_{\overline{Q}})v')\delta_{H_{\overline{Q}}}(h_{\overline{Q}})^{1/2} d_L h_{\overline{Q}}$$

for all $v, v' \in \sigma^\infty$. The $\delta_{H_{\overline{Q}}}$ in the above integral instead of $\delta_{\overline{Q}}$ follows from Proposition 2.7.2(ii). Set

$$\mathcal{L}_\sigma(T_\sigma) = \int_{H_{\overline{Q}}(F)} \mathrm{Tr}(\sigma(h_{\overline{Q}}^{-1})T_\sigma)\delta_{H_{\overline{Q}}}(h_{\overline{Q}})^{1/2} d_L h_{\overline{Q}}$$

for all $T_\sigma \in \mathrm{End}(\sigma)^\infty$. We have

3.4.11) The integral defining $\mathcal{L}_\sigma$ is absolutely convergent and $\mathcal{L}_\sigma$ is a continuous linear form on $\mathrm{End}(\sigma)^\infty$.

This follows from Proposition 2.7.2(iii) as σ is tempered. We now prove the following

3.4.12)
$$\mathcal{L}_\pi \neq 0 \iff \mathcal{L}_\sigma \neq 0$$

By 3.4.10 and the density of $\pi^\infty \otimes \overline{\pi^\infty}$ in $\mathrm{End}(\pi)^\infty$, we see that if $\mathcal{L}_\pi$ is nonzero then $\mathcal{L}_\sigma$ is nonzero. Now we'll prove the converse. The analytic fibration $H(F) \to H_{\overline{Q}}(F)\backslash H(F)$ is locally trivial. Let $s : \mathcal{U} \to H(F)$ be an analytic sections over an open subset $\mathcal{U}$ of $H_{\overline{Q}}(F)\backslash H(F)$. For $\varphi \in C_c^\infty(\mathcal{U}, \sigma^\infty)$ a smooth compactly supported function from $\mathcal{U}$ to σ^∞, the following assignment

$$e_\varphi(g) = \begin{cases} \delta_{\overline{Q}}(l)^{1/2}\delta(l)\varphi(h) \text{ if } g = lus(h) \text{ with } l \in L(F), u \in U_{\overline{Q}}(F), h \in \mathcal{U} \\ 0 \end{cases}$$

defines an element of π^∞. By 3.4.10, we have

$$\mathcal{L}_\pi(e_\varphi, e_{\varphi'}) = \int_{(H_{\overline{Q}}(F)\backslash H(F))^2} \mathcal{L}_\sigma(\varphi(h), \varphi'(h'))dhdh'$$

for all $\varphi, \varphi' \in C_c^\infty(\mathcal{U}, \sigma^\infty)$. Now, assume that $\mathcal{L}_\sigma$ is nonzero. Because $\sigma^\infty \otimes \overline{\sigma^\infty}$ is dense in $\mathrm{End}(\sigma)^\infty$, there exists $v_0, v_0' \in \sigma^\infty$ such that $\mathcal{L}_\sigma(v_0, v_0') \neq 0$. Setting $\varphi(h) = f(h)v_0$ and $\varphi'(h) = \overline{f(h)}v_0'$ where $f \in C_c^\infty(\mathcal{U})$ is nonzero in the formula above, we get $\mathcal{L}_\pi(e_\varphi, e_{\varphi'}) \neq 0$ hence $\mathcal{L}_\pi$ doesn't vanish. This proves 3.4.12.

By 3.4.12, we are now reduces to proving

3.4.13)

$$\mathcal{L}_\sigma \neq 0 \iff \mathcal{L}_{\widetilde{\sigma}} \neq 0$$

In order to prove 3.4.13, we will need a precise description of $H_{\overline{Q}}$ and of the embedding $H_{\overline{Q}} \hookrightarrow L$. Since $\overline{Q}_V$ is a maximal proper parabolic subgroup of $SO(V)$ it is the stabilizer of a totally isotropic subspace Z' of V. The quotient $\overline{Q}\backslash G$ classifies the totally isotropic subspaces of V of the same dimension as Z'. The action of $H = SO(W)$ on that space has two orbits: one is open and corresponds to the subspaces Z'' such that $\dim(Z''\cap W) = \dim(Z') - 1$, the other is closed and corresponds to the subspaces Z'' such that $\dim(Z''\cap W) = \dim(Z')$. Since we are assuming that $\overline{Q}$ is a good parabolic subgroup, Z' is in the open orbit. Hence $Z'_W = Z' \cap W$ is a hyperplane in Z'. Moreover, we have

$$L_V = GL_F(Z') \times SO(\widetilde{V})$$

where $\widetilde{V}$ is a non-degenerate subspace of V orthogonal to Z'. Since Z'_W is a hyperplane in Z', up to $\overline{Q}(F)$-conjugation we have $\widetilde{V} \subseteq W$ and so we may as well assume that it is the case. Note that we have a natural identification $H_{\overline{Q}} = SO(W) \cap \overline{Q}_V$. The exact sequence

$$1 \to U_{\overline{Q}} \to \overline{Q}_V \to GL_F(Z') \times SO(\widetilde{V}) \to 1$$

induces an exact sequence

3.4.14)

$$1 \to U^\natural_{\overline{Q}} \to H_{\overline{Q}} \to P_{Z'} \times SO(\widetilde{V}) \to 1$$

where $U^\natural_{\overline{Q}} = SO(W) \cap U_{\overline{Q}}$ and $P_{Z'}$ is the mirabolic subgroup of elements $g \in GL_F(Z')$ preserving the hyperplane Z'_W and acting trivially on the quotient $Z'\backslash Z'_W$. On the other had we have

$$L = SO(W) \times GL_F(Z') \times SO(\widetilde{V})$$

and the embedding $H_{\overline{Q}} \hookrightarrow L$ is the product of the three following maps

- The natural inclusion $H_{\overline{Q}} \subseteq SO(W)$

- The projection $H_{\overline{Q}} \twoheadrightarrow P_{Z'}$ followed by the natural inclusion $P_{Z'} \subseteq GL_F(Z')$
- The projection $H_{\overline{Q}} \twoheadrightarrow SO(\widetilde{V})$

Let $\widetilde{D}$ be a line such that $(Z'_W)^\perp \cap W = Z'_W \oplus (\widetilde{D} \oplus^\perp \widetilde{V})$. Then, the unipotent group $U^\natural_{\overline{Q}}$ may be denoted as the subgroup of $SO(W)$ stabilizing the subspace $\widetilde{D} \oplus Z'_W$. Fix a basis $z'_1, \ldots, z'_l$ of Z'_W and let $B_{Z'}$ be the Borel subgroup of $GL_F(Z')$ fixing the complete flag

$$\langle z'_1 \rangle \subsetneq \langle z'_1, z'_2 \rangle \subsetneq \cdots \subsetneq \langle z'_1, \ldots, z'_l \rangle = Z'_W \subsetneq Z'$$

and denote $Z_{Z'}$ its unipotent radical. Let $\widetilde{\xi}$ be a generic character of $N_{Z'}(F)$. Let us denote $\widetilde{N}$ and $\widetilde{H}$ the inverse images in $H_{\overline{Q}}$ of the subgroups $N_{Z'}$ and $N_{Z'} \times SO(\widetilde{V})$ of $P_{Z'} \times SO(\widetilde{V})$ by the last map of 3.4.14. Recall that $\widetilde{G} = SO(\widetilde{V}) \times SO(W)$ this not tilde bc a=0 right?. We have a natural diagonal embedding $\widetilde{H} \hookrightarrow \widetilde{G}$ (that is: the product of the natural projection $\widetilde{H} \twoheadrightarrow SO(\widetilde{V})$ and the natural inclusion $\widetilde{H} \hookrightarrow SO(W)$) and if we extend $\widetilde{\xi}$ to a character of $\widetilde{H}(F)$ through the projection $\widetilde{H}(F) \twoheadrightarrow N_{Z'}(F)$, then the triple $(\widetilde{G}, \widetilde{H}, \widetilde{\xi}$ is a GGP triple corresponding to the pair of quadratic spaces $(\widetilde{V}, \widetilde{W})$. We can of course use this triple to define $\mathcal{L}_{\widetilde{\sigma}}$. We will assume this is the case.

The representation σ decomposes as $\sigma = \sigma_{1,V} \boxtimes \widetilde{\sigma}$ where $\sigma_{1,V}$ is a tempered irreducible representation of $GL_F(Z')$. The subspace $\sigma_{1,V}^\infty \otimes \widetilde{\sigma}^\infty$ is dense in σ^∞. Hence $\mathcal{L}_\sigma$ is nonzero if and only if there exist vectors $\widetilde{v}, \widetilde{v}' \in \widetilde{\sigma}^\infty$ and $w, w' \in \sigma_{1,V}^\infty$ such that

$$\mathcal{L}_\sigma(w \otimes \widetilde{v}, w' \otimes \widetilde{v}') \neq 0$$

□