The multiplicity conjecture for Gan-Gross-Prasad pairs in the Special Orthogonal Case

Sam Pease

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1 preliminaries

1.1 General Notation

- F a field, either p-adic (a finite extension of $\mathbb{Q}_p$) or $\mathbb{R}$
- $|.|$ is the normalized absolute value on F. That is for every Haar measure dx on F we have $d(ax) = |a|dx$ for all $a \in F$
- For G a locally compact separable greoup, we will denote by $d_L g$ (resp. $d_R g$) a left (resp. a right) Haar measure on G. If the group is unimodular then we will denote both by dg.
- δ_G will stand for the modular character of G that is defined by $d_l(gg'^{-1} = \delta_G(g')d_L g$ for all $g \in G$

If F is p-adic then:

- $\mathcal{O}_F = \{a \in F : |a| \leq 1\}$ the ring of integers of F
- $\mathfrak{p}_F = \{a \in F : |a| < 1\}$ the maximal ideal of $\mathcal{O}_F$
- q_F the cardinality of the residue field $\mathcal{O}_F \backslash \mathfrak{p}_F$
- ϖ_F the uniformizer of F which generates $\mathfrak{p}_F$
- val_F is the surjective function $F \to \mathbb{Z} \cup \{\infty\}$ such that $\mathrm{val}_F(\varpi_F) = 1$ and $|a| = q_F^{-\mathrm{val}_F(a)}$ for all $a \in F$. Note we can write $a = \varpi^n u$ for some integer n and unit u. Then $\mathrm{val}(a) = n$ and $\mathrm{val}(0) = \infty$.
- If R is a subring of F, and $v \in F$ then $\mathrm{val}_R(v) = -\inf\{k \in \mathbb{Z} \,|\varpi^k v \in R\}$.
- If V is an F vector space, and R is an $\mathcal{O}_F$ lattice of V and $g \in GL_F(V)$ then $\mathrm{val}_R(g) = \inf\{\mathrm{val}_R(gv), v \in R\}$ and $\|g\|_R = |\varpi_F|^{\mathrm{val}_R(g)}$

1.2 Log norm

- For two log norms σ_1, σ_2 on X, we say σ_2 dominates σ_1, or $\sigma_1 \ll \sigma_2$, if there exists $c > 0$ such that $\sigma_1(x) \leq c\sigma_2(x)\,\forall x \in X$
- If $\sigma_1 \ll \sigma_2$ and $\sigma_2 \ll \sigma_1$ then we say σ_1 and σ_2 are equivalent and write $\sigma_1 \sim \sigma_2$
- For a vector space V, $\sigma_V(v) = \log(2 + |v|), v \in V$
- We denote $\sigma := \sigma_V$
- $\sigma(xy) \ll \sigma(x) + \sigma(y) \ll \sigma(x)\sigma(y)\forall(x,y) \in G \times G$
- $\sigma_{T\backslash G}(g) = \inf_{t \in T(F)} \sigma(tg)$
- For every maximal torus $T \subset G$. we have $\sigma(g^{-1}Xg)+\log(2+D^G(X)^{-1}) \sim \sigma_{\mathfrak{g}}(X) + \sigma_{T\backslash G}(g) + \log(2 + D^G(X)^{-1})$

Let $f : X \to Y$ be a morphism of algebraic varieties over F and let σ_X be a log norm on X (but we will only consider its restriction to $X(F)$). Define an abstract log norm $f_*\sigma_X$ on $\mathrm{Im}(X(F) \to Y(F)$ by

$$f_*\sigma_X(y) = \inf_{x \in X(F): f(x)=y} \sigma_X(x)$$

Let σ_Y be a log-norm on Y. We say that f has the *norm descent property* id σ_Y and $f_*\sigma_X$ are equivalent as abstract log-norms on $\mathrm{Im}(X(F) \to Y(F)$. We will need the following facts about the norm descent property. First from Kott3 Prop 18.2

Lemma 1.2.1 *(i) The norm descent property is local on the basis. In other words if $f : X \to Y$ is a morphism of algebraic varieties over F and $(U_i)_{i \in I}$ is a finite covering by Zariski-open subsets of Y defined over F, then f has the norm descent property if and only if each of the $f_i : f^{-1}(U_i) \to U_i, i \in I$ has the norm descent property.*

(ii) If f admites a section, then it has the norm descent property.

and we will also need Kott3 Prop 18.3

Proposition 1.2.2 *Let G be a connected reductive group over F and T an F-subtorus of G. Then the morphism $G \to T\backslash G$ has the norm descent property.*

1.3 Harish-Chandra Schwartz space

We will denote Ξ^G as the Harish-Chandra function. To define it we will define the following. Let P_{min} be a minimal parabolic subgorup of G and let K be a maximal compact subgroup of $G(F)$ which is special in the p-adic case. Then we have $G(F) = P_{min}(F)K$ (Iwasawa decomposition). Consider the (smooth and normalized) induced representation.

$$i^G_{P_{min}}(1)^\infty := \{e \in C^\infty(G(F)) :\, e(pg) = \delta_{min}(p)^{1/2} e(g)\, \forall p \in P_{min}(F), g \in G(F)\}$$

that we equip with the scalar product

$$(e, e') = \int_K e(k)\overline{e'(k)}dk, \quad e, e' \in i^G_{P_{min}}(1)^i nfty$$

Let $e_K \in i^G_{P_{min}}(1)^\infty$ be the unique function such that $e_K(k) = 1$ for all $k \in K$. Then the Harish-Chandra function is defined by

$$\Xi^G(g) = (i^G_{P_{min}}(1)(g)e_K, e_k),\; g \in G(F)$$

This is the action of representaion not the space defined above, make clearer sometime

This definition of Ξ^G depends on various choices, but this doesn't matter as different choices would yield equivalent functions and we will only be using Ξ^G to give estimates. The following give the main properties of Ξ^G that we need.

Proposition 1.3.1 *(i) Set*

$$M^+_{min} = \{m \in M_{min}(F) :\, |\alpha(m)| \leq 1\, \forall \alpha \in R(A_{M_{min}}, P_{min})\}$$

Then, there exists $d > 0$ such that

$$\delta_{P_{min}}(m)^{1/2} \ll \Xi^G(m) \ll \delta_{P_{min}}(m)^{1/2}\sigma(m)^d$$

for all $m \in M^+_{min}$.

(ii) Let $m_{P_{min}} : G(F) \to M_{min}(F)$ be any map such that $g \in m_{P_{min}}(g)U_{min}(F)K$ for all $g \in G(F)$. Then, there exists $d > 0$ such that

$$\Xi^G(g) \ll \delta_{P_{min}}(m_{P_{min}}(g))^{1/2}\sigma(g)^d$$

for all $g \in G(F)$.

(iii) Let $P = MU$ be a parabolic subgroup that contains P_{min}. Let $m_P : G(F) \to M(F)$ be any such map such that $g \in m_P(g)U(F)K$ for all $g \in G(F)$. Then, we have

$$\Xi^G(g) = \int_K \delta_P(m_P(kg))^{1/2}\Xi^M(m_P(kg))dk$$

for all $g \in G(F)$.

(iv) Let $P = MU$ be a parabolic subgroup of G. Then, for all $d > 0$, there exists $d' > 0$ such that

$$\delta_P(m)^{1/2} \int_{U(F)} \Xi^G(mu)\sigma(mu)^{-d'}du \ll \Xi^M(m)\sigma(m)^{-d}$$

for all $m \in M(F)$.

(v) *There exists $d > 0$ such that the integral*

$$\int_{G(F)} \Xi^G(g)^2 \sigma(g)^{-d} dg$$

is convergent.

(vi) *(Doubling principle) We have the equality*

$$\int_K \Xi^G(g_1 k g_2) dk = \Xi^G(g_1)\Xi^G(g_2)$$

for all $g_1, g_2 \in G(F)$.

Proof. sources and stuff todo ☐

1.4 Measures

We will fix a continuous non-trivial additive character $\psi : F \to \mathbb{S}^1$ and equip F with the autodual Haar measure with respect to ψ.

2 Gan-Gross-Prasad Triples

- Let V be a quadratic vector space of dimension n over a local field F. That is, there exists quadratic form $q : V \to F$.
- Let $A \in M_n$ be a symmetric matrix such that $q(v) = v^t A v$.
- Let b be a symmetric bilinear form given by $b(x, y) = x^t A y$
- $O(n) = \{G \in GL_n(F) \big| G^t A G = A\}$
  $= \{G \in GL_n \big| q(Gv) = q(v)\}$
- $SO(n) = \{G \in O(n) \big| \det(G) = 1\}$
- $so(n) = \{g \in M_n \big| g^t A + Ag = 0\}$
- $G = SO(W) \times SO(V)$, $H = SO(W) \ltimes N$
- $D = F z_0$
- $V_0 = W \oplus D$
- $H_0 = SO(W)$ and $G_0 = SO(W) \times SO(V_0)$. We consider H_0 as a subgroup of G_0 via the diagonal embedding $H_0 \hookrightarrow G_0$. The triple $G_0, H_0, 1$ is the GGP triple associated to the admissible pair W, V_0
- T is the subtorus of $SO(V)$ preserving the lines $\langle z_i \rangle$, for $i = \pm 1, \ldots, \pm r$ and acting trivially on V_0. We have $M = T \times G_0$

2.1 Definition of GGP triples

Let (W, V) be a pair of quadratic spaces. We will call (W, V) an *admissible pair* if there exists a quadratic space Z satisfying

- $V \cong W \oplus^{\perp} Z$
- Z is odd dimensional and $SO(Z)$ is quasi-split.

The second condition means that there exists $\nu \in F^{\times}$ and a basis $(z_{-r}, \dots, z_{-1}, z_0, z_1, \dots, z_r)$ of Z such that

2.1.1)

$$b(z_i, z_j) = \nu \delta_{i,-j}$$

for all $i, j \in \{0, \pm 1, \dots, \pm r\}$. Let W, V be an admissible pair. Set $G = SO(W) \times SO(V)$. We will associate a triple (G, H, ξ) where H is an algebraic subgroup of G and $\xi : H(F) \to C^{\times}$ is a continuous character of $H(F)$. This triple is unique up to $G(F)$-conjugacy. Fix an embedding $W \subseteq V$ and set $Z = W^{\perp}$. Also fix $\nu \in F^{\times}$ and a basis $(z_i)_{i=0,\pm 1,\dots,\pm r}$ (where $\dim(Z) = 2r + 1$) of Z satisfying 2.1.1. Denote P_V the stabilizer in $SO(V)$ of the following flag of totally isotropic subspaces of V

$$\langle z_r \rangle \subset \cdots \subset \langle z_r, \dots, z_1 \rangle$$

Then, P_V is a parabolic subgroup of $SO(V)$. We write N as its unipotent radical. Let M_V be the stabilizer in P_V ofthe lines $\langle z_i \rangle$ for $i = \pm 1, \dots, \pm r$. It is a levi component of P_V. Set $P = SO(W) \times P_V$. Then, P is a prabolic subgroup of G with unipotent radical N and $M = SO(W) \times M_V$ is a Levi component of it. We identify $SO(W)$ with its image via the diagonal embedding $SO(W) \hookrightarrow G$, we have $SO(W) \subseteq M$. In particular, conjugation by $SO(W)$ preserves N and we set

$$H = SO(W) \ltimes N$$

Now we will define the character ξ. Define a morphism $\lambda : N \to \mathbb{G}_a$ by

$$\lambda(n) = \sum_{i=0}^{r-1} b(z_{-i-1}.nz_i),\, n \in N$$

λ is $SO(W)$-invariant, so it admits a unique extension, still denoted by λ to a morphism $H \to \mathbb{G}_a$ which is trivial on $SO(W)$. We denote $\lambda_F : H(F) \to F$ induced on the groups of F-points. Recall that we fixed a non-trivial continuous additive character ψ of F. We set

$$\xi(h) = \psi(\lambda_F(h))$$

for all $h \in H(F)$. This gives us the definition of the triple (G, H, ξ). It is easy to check that this definition depends on various choices made only up to $G(F)$-conjugacy. A triple obtained from an admissible pair (W, V) in this is called a *Gan-Gross-Prasad triple* or *GGP triple* for short. We will use the following additional notation:

- $d = \dim(V)$ and $m = \dim(W)$
- $D = Fz_0$
- $V_0 = W \oplus D$
- $H_0 = SO(W)$ and $G_0 = SO(W) \times SO(V_0)$. We consider H_0 as a subgroup of G_0 via the diagonal embedding $H_0 \hookrightarrow G_0$. The triple $(G_0, H_0, 1)$ is the GGP triple associated to the admissible pair W, V_0
- T is the subtorus of $SO(V)$ preserving the lines $\langle z_i \rangle$, for $i = \pm 1, \ldots, \pm r$ and acting trivially on V_0. We have $M = T \times G_0$
- A is the split part of the torus T, it is also the split part of the center of M
- ξ the character of $\mathfrak{h}(F)$, where $\mathfrak{h} = \mathrm{Lie}(H)$, which is trivial on $\mathfrak{u}(W)(F)$ and equal to $\xi \circ \exp$ on $\mathfrak{n}(F)$

Note that when $r = 0$ (that is $Z = D$ is a line), we have $G = G_0$, $H = H_0$, and $\xi = 1$. This is the *codimension one case.*

We will need the following lemma, (1.2 for definition of norm descent property)

Lemma 2.1.2 *(i) The map $G \to H\backslash G$ has the norm descent property.*

(ii) The orbit under M-conjugacy of λ in $(\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$ is a Zariski open subset

Proof. (i) We have a natural identification $H\backslash G = N\backslash SO(V)$, so it is sufficient to prove that $SO(V) \to N\backslash SO(V)$ has the norm descent property. Since this map is $SO(V)$-equivariant for the obvious transitive actions, we only need to show that it admits a section over a nonempty Zariski-open subset. If we denote $\overline{P}_V = M_V\overline{N}$ the parabolic subgroup opposite to P_V with respect to M_V, the multiplication map $N \times M_V \times \overline{N} \to SO(V)$ is an open immersion. The image of that open subset is open in $N\backslash SO(V)$ and the restriction of the projection $SO(V) \to N\backslash SO(V)$ to that open set is $N \times M_V \times \overline{N} \to M_V \times \overline{N}$. This map obviously has a section

(ii) If $r = 0$, *i.e.*, if we are in the codimension one case, we have $\mathfrak{n} = 0$ and the result is trivial. Assume now that $r \geq 1$. It suffices to show that the dimension of the orbit $M \cdot \lambda$ is equal to the dimension of $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$. We can compute

$$\dim(\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]) = 2r$$

and

$$\dim(M) = \frac{m(m-1)}{2} + 2r + \frac{(m+1)m}{2} = m^2 + 2r$$

The stabilizer M_λ of λ is easily seen to be $M_\lambda = Z(G)(SO(W) \times SO(W))$ (where $Z(G)$ denote the center of G). Hence we have

$$\dim(M_\lambda) = m^2$$

and so the dimension of the orbit is

$$\dim(M \cdot \lambda) = \dim(M) - \dim(M_\lambda) = m^2 + 2r - m^2 = 2r$$

which is the same as $\dim(\mathfrak{n} / [\mathfrak{n}, \mathfrak{n}])$.

$\square$

check these dimension in above proof sometime

2.2 The multiplicity of $m(\pi)$

For $\pi \in \mathrm{Temp}(G)$, let us denote $\mathrm{Hom}_H(\pi, \xi)$ for the space of all continuous linear forms $l : \pi^\infty \to \mathbb{C}$ such that

$$l(\pi(h)e) = \xi(h)l(e)$$

for all $e \in \pi^\infty$ and for all $h \in H(F)$. We define the *multiplicity* $m(\pi)$ to be the dimension of that space of linear forms, that is

$$m(\pi) = \dim \mathrm{Hom}_H(\pi, \xi),\ \pi \in \mathrm{Temp}(G)$$

We have the following theorem

Theorem 2.2.1 (6.3.1) *We have*

$$m(\pi) \leq 1$$

for all $\pi \in \mathrm{Temp}(G)$.

need sources

Note that we have

on a second look bar probs means complex conjugation so maybe disregard below

2.2.2)

$$m(\pi) = m(\overline{\pi})$$

for all $\pi \in \mathrm{Temp}(G)$. Indeed, the conjugation map $l \to \bar{l}$ induces an isomorphism

$$\mathrm{Hom}_H(\pi, \xi) \cong \mathrm{Hom}(\overline{\pi}, \overline{\xi})$$

and we <u>can</u> easily check, there exists an element $a \in A(F)$ such that $\xi(aha^{-1}) = \overline{\xi(h)}$ for all $h \in H(F)$, hence the linear map $l \to l \circ \pi(a)$ induces an isomorphism

$$\mathrm{Hom}(\overline{\pi}, \overline{\xi}) \cong \mathrm{Hom}_H(\overline{\pi}, \xi)$$

and 2.2.2 follows.

2.3 $H\backslash G$ is a spherical variety, good parabolic subgroups

A parabolic subgroup $\overline{Q}$ of G is *good* if $H\overline{Q}$ is Zariski-open in G. This is equivalent to $H(F)\overline{Q}(F)$ being open in $G(F)$.

Proposition 2.3.1 (6.4.1) *(i) There exists minimal parabolic subgroups of G that are good and they are all conjugate under $H(F)$. Moreover, if $\overline{P}_{min} = M_{min}\overline{U}_{min}$ is a good minimal parabolic subgroup then $H \cap \overline{U}_{min} = \{1\}$ and the complement of $H(F)\overline{P}_{min}(F))$ in $G(F)$ has null measure;*

(ii) A Parabolic subgroup $\overline{Q}$ of G is good if and only if it contains a good minimal parabolic subgroup.

(iii) Let $\overline{P}_{min} = M_{min}\overline{U}_{min}$ be a good parabolic subgroup and let $A_{min} = A_{M_{min}}$ be the maximal split central subtorus of M_{min}. Set

$$A^+_{min} = \{a \in A_{min}(F); |\alpha(a)| \geq 1 \forall \alpha \in R(A_{min}, \overline{P}_{min})\}$$

Then, we have the inequalities

2.3.2) $\sigma(h) + \sigma(a) \ll \sigma(ha)$ for all $a \in A^+_{min}$, $h \in H(F)$.

2.3.3) $\sigma(h) \ll \sigma(a^{-1}ha)$ for all $a \in A^+_{min}$, $h \in H(F)$.

Proof. (i) Set $w_0 = z_0$ and choose a family $(w_1, \ldots, w_l)$ of mutually orthogonal vectors in W which is maximal subject to the condition

$$q(w_i) = (-1)^i \nu,\ i = 1, \ldots, l$$

Define u_i for $i = 1, \ldots, \lceil \frac{l}{2} \rceil$ by

$$u_i = w_{2i-2} + w_{2i-1}$$

and u'_i for $i = 1, \ldots, \lfloor \frac{l}{2} \rfloor$ by

$$u'_i = w_{2i-1} + w_{2i}$$

Then the subspaces

$$Z_{V_0} = \langle u_1, \ldots, u_{\lceil \frac{l}{2} \rceil} \rangle$$
$$Z_W = \langle u'_1, \ldots, u'_{\lfloor \frac{l}{2} \rfloor} \rangle$$

are maximal isotropic subspaces of V_0 and W respectively. Let $\overline{P}_{V_0}$ and $\overline{P}_W$ be the stabilizers in $SO(V_0)$ and $SO(W)$ of the totally isotropic flags

$$\langle u_1 \rangle \subseteq \langle u_1, u_2 \rangle \subseteq \cdots \subseteq \langle u_1, \ldots, u_{\lceil \frac{l}{2} \rceil} \rangle$$

and

$$\langle u'_1 \rangle \subseteq \langle u'_1, u'_2 \rangle \subseteq \cdots \subseteq \langle u'_1, \ldots, u'_{\lfloor \frac{l}{2} \rfloor} \rangle$$

respectively. Then $\overline{P}_{V_0}$ and $\overline{P}_W$ are minimal parabolic subgroups of $SO(V_0)$ and $SO(W)$ respectively. Set

$$\overline{P}_0 = \overline{P}_W \times \overline{P}_{V_0}$$

It is a minimal parabolic subgroup of G_0. Let W_{an} be the orthogonal complement in W of $\langle w_1, \ldots, w_l \rangle$. We claim the following

2.3.4) We have $H_0 \cap \overline{P}_0 = SO(W_{an})$ and $H_0\overline{P}_0$ is Zariski-open in G_0 (i.e., $\overline{P}_0$ is a good parabolic subgroup of G_0).

The second claim follows from the first one by dimension consideration. To prove the first claim let $h_0 \in H_0 \cap \overline{P}_0$. Consider the action of h_0 on V_0. Since h_0 belongs to H_0 it must stabilize $w_0 = z_0$. On the other hand, since h_0 belongs to $\overline{P}_0$ it must stabilize $\langle w_0 + w_1 \rangle$. Because w_0 is orthogonal to w_1, it follows that h_0 fixes w_1. We show similarly that h_0 fixes $w_2, \ldots, w_l$, hence $h_0 \in SO(W_{an})$. This prove 2.3.4.

Let $\overline{P} = M\overline{N}$ be the parabolic subgroup opposite to P with respect to M and set

$$\overline{P}_{min} = \overline{P}_0 T \overline{N}$$

which is a minimal parabolic subgroup of G. We deduce easily from 2.3.4 the following

2.3.5) $\overline{P}_{min}$ is a good parabolic subgroup and we have $\overline{P}_{min} \cap H = SO(W_{an})$.

This proves that there exists minimal parabolic subgroups that are good. Let $\overline{P}'_{min}$ be another good minimal parabolic subgroup. We ill show that $\overline{P}_{min}$ and $\overline{P}'_{min}$ are conjugate under $H(F)$. Let $g \in G(F)$ such that $\overline{P}'_{min} = g\overline{P}_{min}g^{-1}$. Set $\mathcal{U} = H\overline{P}_{min}$ and $\mathcal{Z} = G - \mathcal{U}$. Then $\mathcal{Z}$ is a proper Zariski-closed subset of G which is $H \times \overline{P}_{min}$-invariant (for the left and right multiplication respectively). If $g \in \mathcal{Z}$, then we would have

$$H\overline{P}'_{min} = Hg\overline{P}_{min}g^{-1} \subseteq \mathcal{Z}g^{-1}$$

which is impossible since $\overline{P}'_{min}$ is a good parabolic subgroup. Thus, we have $g \in \mathcal{U} \cap G(F) = \mathcal{U}(F)$. If we can prove that $g \in H(F)\overline{P}_{min}(F)$, then we are done. Hence, it suffices to show that

2.3.6)

$$\mathcal{U}(F) = H(F)\overline{P}_{min}(F)$$

This follows from

2.3.7) The map $H^1(F, H \cap \overline{P}_{min}) \to H^1(F, H)$ is injective.

By 2.3.5 we have $H^1(F, H \cap \overline{P}_{min}) = H^1(F, SO(W_{an}))$. Since $H = SO(W) \rtimes N$ with N unipotent, we also have $H^1(F, H) = H^1(F, SO(W))$. The two sets $H^1(F, SO(W_{an}))$ and $H^1(F, SO(W))$ classify the (isomorphism classes of) quadratic spaces of the same dimension as W_{an} and W respectively. Moreover, the map $H^1(F, SO(W_{an})) \to H^1(F, SO(W))$ we are considering sends W'_{an} to $W'_{an} \oplus W_{an}^{\perp}$, where $W_{an}^{\perp}$ denotes the orthogonal complement of W_{an} in W. By Witt's theorem, this map is injective. This proves 2.3.7 and ends the proof that all good minimal parabolic subgroups are conjugate under $H(F)$.

unsure abt above argument

It only remains to show the last part of (i) which is that $H \cap \overline{U}_{min} = \{1\}$ and the complement of $H(F)\overline{P}_{min}(F)$ in $G(F)$ has null measure for every good minimal parabolic subgroup $\overline{P}_{min} = M_{min}\overline{U}_{min}$. Since we already proved that all good minimal parabolic subgroups are $H(F)$-conjugate, we only need to consider one of them. Let $\overline{P}_{min}$ be the parabolic subgroup tht we constructed above, then the result follows directly from 2.3.5 and 2.3.6.

(ii) Let $\overline{Q}$ be a good parabolic subgroup and choose $P_{min} \subseteq \overline{Q}$ a minimal parabolic subgroup. Set

$$\mathcal{G} := \{g \in G : g^{-1}P_{min}g \text{ is good}\}$$

This is a Zariski-open subset of G since it is the inverse image of the Zariski-open subset $\{\mathcal{V} \in Gr_n(\mathfrak{g}) : \mathcal{V} + \mathfrak{h} = \mathfrak{g}\}$ of the Grassmannian variety $Gr_n(\mathfrak{g})$, where $n = \dim(P_{min})$, by the regular map $g \in G \mapsto g^{-1}\mathfrak{p}_{min}g \in Gr_n(\mathfrak{g})$. Moreover, it is non-empty as (i) tells us there exists good minimal parabolic subgroups. Since $\overline{Q}$ is good, the intersection $\overline{Q}H \cap \mathcal{G}$ is non-empty too. Hence, we may find $\overline{q}_0 \in \overline{Q}$ such that $\overline{q}_0^{-1}P_{min}\overline{q}_0$ is a good parabolic subgroup. This parabolic subgroup is contained in $\overline{Q}$ but it may not be defined over F. Define

$$\mathcal{Q} := \{\overline{q} \in \overline{Q} : \overline{q}^{-1}P_{min}\overline{q} \text{ is good}\}$$

But $\mathcal{Q}$ is a Zariski-open subset of $\overline{Q}$ which is non-empty. Since $\overline{Q}(F)$ is Zariski-dense in $\overline{Q}$, $\mathcal{Q}(F)$ is non-empty. Thus, for all $\overline{q} \in \mathcal{Q}(F)$ the parabolic subgroup $\overline{q}^{-1}P_{min}\overline{q}$ has all the desired properties.

(iii) First we will show that 2.3.2 and 2.3.3 don't depend on the particular pair $(\overline{P}_{min}, M_{min})$ chosen. Let $(\overline{P}'_{min}, M'_{min})$ be another such pair with $\overline{P}'_{min}$ a good prabolic subgroup and M'_{min} is a levi component of it. Then, by (i), there exists $h \in H(F)$ such that $\overline{P}'_{min} = h\overline{P}_{min}h^{-1}$. The inequalities 2.3.2 and 2.3.3 are true for the pair $(\overline{P}_{min}, M_{min})$ if and only if they are true for the pair $(h\overline{P}_{min}h^{-1}, hM_{min}h^{-1}) = (\overline{P}'_{min}, hM_{min}h^{-1})$. So we may assume without loss of generality that $\overline{P}_{min} = \overline{P}'_{min}$ and all that remains to show is that the inequalities do not depend on the choice of M_{min}.

There exists $\overline{u} \in \overline{U}_{min}(F)$ such that $M'_{min} = \overline{u}M_{min}\overline{u}^{-1}$ so we have $A'^{+}_{min} = \overline{u}A^{+}_{min}\overline{u}^{-1}$. By definition of A^{+}_{min}, the sets $\{a^{-1}\overline{u}a \,\big|\, a \in A^{+}_{min}\}$ and $\{a^{-1}\overline{u}^{-1}a \,\big|\, a \in A^{+}_{min}$ are bounded. It follows that

$$\sigma(h\overline{u}a\overline{u}^{-1}) \sim \sigma(ha)$$

$$\sigma(\overline{u}a^{-1}\overline{u}^{-1}h\overline{u}a\overline{u}^{-1}) \sim \sigma(a^{-1}ha)$$

for all $a \in A^{+}_{min}$ and all $h \in H(F)$. Thus, 2.3.2 and 2.3.3 are independent of choice of M_{min} and are satisfied for $(\overline{P}_{min}, M_{min})$ if and only if they are satisfied for $(\overline{P}'_{min}, M'_{min})$.

Now we will now reduce the proof of 2.3.2 and 2.3.3 to the codimension one case. Let $\overline{P}_0 = M_0\overline{U}_0$ be a good minimal parabolic subgroup of G_0. Let $A_0 = A_{M_0}$ be the split part of the center of M_0 and let

$$A_0^{+} = \{a_0 \in A_0(F) : |\alpha(a)| \geq 1 \,\forall \alpha \in R(A_0, \overline{P}_0)\}$$

Set $\overline{P}_{min} = \overline{P}_0 T\overline{N}$ and $M_{min} = M_0 T$. Then, $\overline{P}_{min}$ is a good parabolic subgroup of G, M_{min} is a Levi component of it, and $A^{+}_{min} \subseteq A(F)A_0^{+}$. We have

$$\sigma(n) + \sigma(a) + \sigma(h_0 a_0) \ll \sigma(nh_0 a a_0)$$

for all $h = nh_0 \in H(F) = N(F)H_0(F)$ and all $(a, a_0) \in A(F) \times A_0^{+}$. Since $\sigma(aa_0) \sim \sigma(a) + \sigma(a_0)$ for all $(a, a_0) \in A(F) \times A_0^{+}$ and $\sigma(nh_0) \sim \sigma(n) + \sigma(h_0)$ for all $(n, h_0) \in N(F) \times H_0(F)$, we have that 2.3.2 will follow from

2.3.8) $\quad \sigma(h_0) + \sigma(a_0) \ll \sigma(h_0 a_0)$, for all $a_0 \in A_0^{+}$ and all $h_0 \in H_0(F)$.

On the other hand, we also have $\sigma(n) \ll \sigma(a^{-1}na)$ for all $a \in A^{+}_{min}$ and all $n \in N(F)$. Thus

$$\begin{aligned}
\sigma(a^{-1}nh_0a) &\gg \sigma(a^{-1}naa^{-1}h_0a) \\
&\gg \sigma(a^{-1}na) + \sigma(a^{-1}h_0a) \\
&\gg \sigma(n) + \sigma(a^{-1}h_0a) \\
&= \sigma(n) + \sigma(a_0^{-1}(aa_0^{-1})^{-1}h_0(aa_0^{-1})a_0) \\
&= \sigma(n) + \sigma(a_0^{-1}h_0a_0)
\end{aligned}$$

for all $h = nh_0 \in H(F) = N(F)H_0(F)$ and all $a \in A^{+}_{min}$ where a_0 denotes the unique element of A_0^{+} such that $aa_0^{-1} \in A(F)$. Hence, 2.3.3 will follow from

2.3.9) $\sigma(h_0) \ll \sigma(a_0^{-1}h_0a_0)$, for all $a_0 \in A_0^+$ and all $h_0 \in H_0(F)$.

What is left now is to show that 2.3.8 and 2.3.9 are true for any pair $(\overline{P}_0, M_0)$. We will choose the following pair.

Introduce a sequence $(w_0, \ldots, w_l)$ and a parabolic subgroup $\overline{P}_0 = \overline{P}_W \times \overline{P}_{V_0}$ of G_0 as in part (i). By 2.3.4, $\overline{P}_0$ is a good parabolic subgroup of G_0. Let M_{V_0} be the Levi component of P_{V_0} that preserve the lines

$$\langle u_1\rangle, \ldots, \langle u_{\lceil \frac{l}{2}\rceil}\rangle \text{ and } \langle u_{-1}\rangle, \ldots, \langle u_{-\lceil \frac{l}{2}\rceil}\rangle$$

where here we have set $u_{-i} = w_{2i-2} - w_{2i-1}$ for $i = 1, \ldots, \lceil \frac{l}{2}\rceil$. Let M_W be the levi component of $\overline{P}_W$ that preserves the lines

$$\langle u'_1\rangle, \ldots, u'_{\lfloor \frac{l}{2}\rfloor}\rangle \text{ and } \langle u'_{-1}\rangle, \ldots, u'_{-\lfloor \frac{l}{2}\rfloor}\rangle$$

where we have set $u'_{-i} = w_{2i-1} - w_{2i}$ for $i = 1, \ldots, \lfloor \frac{l}{2}\rfloor$. Set

$$M_0 = M_W \times M_{V_0}$$

.

This is a Levi component of $\overline{P}_0$ and we will prove 2.3.8 and 2.3.9 for this pair $(\overline{P}_0, M_0)$. We have the decomposition

$$A_0^+ = A_W^+ \times A_{V_0}^+$$

where A_W^+ and $A_{V_0}^+$ are defined in the obvious way. For all $a_{V_0} \in A_{V_0}^+$ (resp. $a_W \in A_W^+$) let us denote the eigenvalues of a_{V_0} (resp. a_W) acting on $u_1, \ldots, u_{\lceil \frac{l}{2}\rceil}$ (resp. on $u'_1, \ldots, u'_{\lfloor \frac{l}{2}\rfloor}$) by $a_{V_0}^1, \ldots, a_{V_0}^{\lceil \frac{l}{2}\rceil}$ (resp. $a_W^1, \ldots, a_W^{\lfloor \frac{l}{2}\rfloor}$). Then, we have

$$|a_{V_0}^1| \geq \cdots \geq |a_{V_0}^{\lceil \frac{l}{2}\rceil}| \geq 1$$
$$|a_W^1| \geq \cdots \geq |a_W^{\lfloor \frac{l}{2}\rfloor}| \geq 1$$

for all $a_{V_0} \in A_{V_0}^+$ and all $a_W^+ \in A_W^+$.

We also have

$$\begin{aligned}
\sigma(h_0) + \sigma(a_0) &\ll \sigma(h_0) + \sigma(h_0a_0) \\
\sigma(h_0) &\ll \sigma_{SO(V_0)}(h_0a_{V_0}) + \sigma_{SO(V_0)}(a_{V_0}) \\
\sigma_{SO(V_0)}(h_0a_{V_0}) &\ll \sigma(h_0a_0)
\end{aligned}$$

for all $a_0 = (a_W, a_{V_0}) \in A_0^+ = A_W^+ \times A_{V_0}^+$ and all $h_0 \in H_0(F)$. So 2.3.8 will follow from

2.3.10) $\sigma_{SO(V_0)}(a_{V_0}) \ll \sigma_{SO(V_0)}(h_0 a_{V_0})$, for all $a_{V_0} \in A_{V_0}^+$ and all $h_0 \in H_0(F)$.

We have

$$\sigma(a_{V_0}) \sim \log(1 + |a_{V_0}^1|)$$

for all $a_{V_0} \in A_{V_0}^+$. Moreover, for all $a_{V_0} \in A_{V_0}^+$ and all $h_0 \in H_0(F)$ we have

$$\begin{aligned} b(h_0 a_{V_0} u_1, w_0) &= a_{V_0}^1 b(h_0 u_1, w_0) \\ &= a_{V_0}^1 (b(w_0, w_0) + b(h_0 w_1, w_0)) \\ &= a_{V_0}^1 (\nu + 0) \\ &= a_{V_0}^1 \nu \end{aligned}$$

and we also have $\log(1 + |b(gu_1, w_0)|) \ll \sigma_{SO(V_0)}(g)$ for all $g \in SO(W)$. Thus

$$\sigma_{SO(V_0)}(a_{V_0}) \sim \log(1+|a_{V_0}^1|) \sim \log(1+|b(h_0 a_{V_0} u_1, w_0)|) \ll \sigma_{SO(V_0)}(h_0 a_{V_0})$$

as desired which proves the first inequality.

Now we will prove 2.3.9. As $\sigma(h_0) \sim \max_{v,v'} \log(2+|b(h_0 v, v')|)$ for $v, v' \in V_0$ and $h_0 \in H_0(F)$ it is sufficient to prove the following

2.3.11) For all $v, v' \in V_0$ we have

$$\log(2 + |b(h_0 v, v')|) \ll \sigma(a_0^{-1} h_0 a_0)$$

for all $a_0 \in A_0^+$ and all $h_0 \in H_0(F)$.

By bilinearity and since $b(h_0 v, v') = b(h_0^{-1} v', v)$, it suffices to prove 2.3.11 in the following cases

- $v = w_i$ and $v' = \langle w_i, \ldots, w_l \rangle \oplus W_{an}$ for $0 \leq i \leq l$,
- $v, v' \in W_{an}$

Recall that W_{an} is the orthogonal complement of $\langle w_0, \ldots, w_l \rangle$ in V_0.

The proof of 2.3.11 is easy in the second case. As we have

$$\begin{aligned} b(a_{V_0}^{-1} h_0 a_{V_0} v, v') &= b(h_0 a_{V_0} v, a_{V_0} v') \\ &= b(h_0 v, v') \end{aligned}$$

for all $a_{V_0} \in A_{V_0}(F)$, $h_0 \in H_0(F)$, and $v, v' \in W_{an}$. Here $A_{V_0}(F)$ is the split part of the central torus of M_{V_0}.

For the first case we will proceed by induction on i. If $i = 0$ then we have $h_0 w_0 = w_0$ for all $h_0 \in H_0(F)$ so $\sigma(h_0) \sim \sigma(1) \ll \sigma(a_0^{-1} h_0 a_0)$ for all $a_0 \in A_0^+$ and all $h_0 \in H_0(F)$.

Now let $1 \leq i \leq l$ and assume that 2.3.11 is true for $v = w_{i-1}$ and all $v' = \langle w_{i-1}, \dots, w_l \rangle \oplus W_{an}$. We have that $w_i = u_{(i-1)/2} - w_{i-1}$. So

$$\log(2+|b(h_0 w_i, v')|) \ll \log(2+|b(h_0 u_{(i-1)/2}, v')|) + \log(2+|b(h_0 w_{i-1}, v')|)$$

and thus by our induction hypothesis we just need to show that

$$2.3.12) \quad \log(2+|b(h_0 u_{(i-1)/2}, v')|) \ll \sigma(a_0^{-1} h_0 a_0)$$

for all $a_0 \in A_0^+$ and all $h_0 \in H_0(F)$

If i is odd then the subspace $\langle w_{i-1}, \dots, w_l \rangle \oplus W_{an} = \langle u_{i-1}, \dots, u_{\lceil \frac{l}{2} \rceil} \rangle \oplus W_{an}$ is preserved by A_{V_0}. By bilinearity we only need to prove 2.3.12 for v' an eigenvector for the action of A_{V_0} on that subspace. For each $a_{V_0} \in A_{V_0}^+$, the eigenvalue of a_{V_0} on v' has an absolute value which is greater than or equal to $|a_{V_0}^{(i-1)/2}|^{-1}$. Hence we have

$$\begin{aligned}
\sigma(a_0^{-1} h_0 a_0) &\gg \log(2+|b(a_{V_0}^{-1} h_0 a_{V_0} u_{(i-1)/2}, v')|) \\
&\gg \log(2+|a_{V_0}^{(i-1)/2}||b(h_0 u_{(i-1)/2}, a_{V_0} v')|) \\
&\gg \log(2+|b(h_0 u_{(i-1)/2}, v')|)
\end{aligned}$$

for all $a_0 = (a_w, a_{V_0}) \in A_0^+ = A_W^+ \times A_{V_0}^+$ and all $h_0 \in H_0(F)$. If i is even, the proof is similar using the action on W rather than on V_0. This gives us the desired inequality.

□

2.4 Some Estimates

Lemma 2.4.1 (6.5.1) *(i) There exists $\epsilon > 0$ such that the integral*

$$\int_{H_0(F)} \Xi^{G_0}(h_0) e^{\epsilon \sigma(h_0)} dh_0$$

is absolutely convergent.

(ii) There exists $d > 0$ such that the integral

$$\int_{H(F)} \Xi^G \sigma(h)^{-d} dh$$

is absolutely convergent

(iii) For all $\delta > 0$ there exists $\epsilon > 0$ such that the integral

$$\int_{H(F)} \Xi^G(h) e^{\epsilon\sigma(h)} (1 + |\lambda(h)|)^{-\delta} dh$$

is absolutely convergent (where $\lambda : H \to \mathbb{G}_a$ is the homomorphism defined in Section 2.1).

Let $\overline{P}_{min} = M_{min}\overline{U}_{min}$ be a good minimal parabolic subgroup of G. We have the following

(iv) For all $\delta > 0$ there exists $\epsilon > 0$ such that the integral

$$I^1_{\epsilon,\delta}(m_{min}) = \int_{H(F)} \Xi^G(hm_{min}) e^{\epsilon\sigma(h)} (1 + |\lambda(h)|)^{-\delta} dh$$

is absolutely convergent for all $m_{min} \in M_{min}(F)$ and there exists $d > 0$ such that

$$I^1_{\epsilon,\delta}(m_{min}) \ll \delta_{\overline{P}_{min}}(m_{min})^{-1/2} \sigma(m_{min})^d$$

for all $m_{min} \in M_{min}(F)$.

(v) Assumes moreover that A is contained in $A_{M_{min}}$. Then, for all $\delta > 0$ there exists $\epsilon > 0$ such that the integral

$$I^2_{\epsilon,\delta}(m_{min}) = \int_{H(f)} \int_{H(F)} \Xi^G(hm_{min}) \Xi^G(h'hm_{min}) e^{\epsilon\sigma(h)} e^{\epsilon\sigma(h')} (1+|\lambda(h')|)^{-\delta} dh' dh$$

is absolutely convergent for all $m_{min} \in M_{min}(F)$ and there exists $d > 0$ such that

$$I^2_{\epsilon,\delta} \ll \delta_{\overline{P}_{min}}(m_{min})^{-1} \sigma(m_{min})^d$$

for all $m_{min} \in M_{min}(F)$.

Proof. (i) This follows from the following fact

2.4.2) There exists $\epsilon' > 0$ such that

$$\Xi^{G_0}(h_0) \ll \Xi^{H_0}(h_0)^2 e^{-\epsilon'\sigma(h_0)}$$

for all $h_0 \in H_0(F)$.

Proof in paper by waldspurger ref from chen later

(ii) Let $d > 0$. By Proposition 1.3.1(iv), if d is sufficiently large, we have

$$\int_{H(F)} \Xi^G(h)\sigma(h)^{-d} dh = \int_{H_0(F)} \int_{N(F)} \Xi^G(h_0 n)\sigma(h_0 n)^{-d} dn dh_0$$
$$\ll \int_{H_0(F)} \Xi^{G_0}(h_0) dh_0$$

Note that $\delta_P(h_0) = 1$ and $\Xi^M(h_0) = \Xi^{G_0}(h_0)$ for all $h_0 \in H_0(F)$ and this last integral is absolutely convergent by (i).

(iii) By (i) and since $\sigma(h_0 n) \ll \sigma(h_0) + \sigma(n)$ for all $h_0 \in H_0(F)$ and all $n \in N(F)$, it suffices to establish

2.4.3) For all $\delta > 0$ and all $\epsilon_0 > 0$, there exists $\epsilon > 0$ such that the integral

$$I^0_{\epsilon,\delta}(h_0) = \int_{N(F)} \Xi^G(nh_0)e^{\epsilon\sigma(n)}(1 + |\lambda(n)|)^{-\delta}dn$$

is absolutely convergent for all $h_0 \in H_0(F)$ and satisfies the inequality

$$I^0_{\epsilon,\delta}(h_0) \ll \Xi^{G_0}(h_0)e^{\epsilon_0\sigma(h_0)}$$

for all $h_0 \in H_0(F)$.

Let $\delta > 0, \epsilon_0 > 0$ and $\epsilon > 0$. We want to prove 2.4.3 holds if ϵ is sufficiently small (compared to δ and ϵ_0). We will introduce an auxiliary parameter $b > 0$ that we will make more precise later. For all $h_0 \in H_0(F)$, we have $I^0_{\epsilon,\delta}(h_0) = I^0_{\epsilon,\delta,\leq b}(h_0) + I^0_{\epsilon,\delta,>b}(h_0)$ where

$$I^0_{\epsilon,\delta,\leq b}(h_0) = \int_{N(F)} \mathbb{1}_{\sigma\leq b}(n)\Xi^G(nh_0)e^{\epsilon\sigma(n)}(1 + |\lambda(n)|)^{-\delta}dn$$

$$I^0_{\epsilon,\delta,>b}(h_0) = \int_{N(F)} \mathbb{1}_{\sigma>b}(n)\Xi^G(nh_0)e^{\epsilon\sigma(n)}(1 + |\lambda(n)|)^{-\delta}dn$$

For all $d > 0$, we have

$$I_{\epsilon,\delta\leq b}(h_0) \leq e^{\epsilon b}b^d \int_{N(F)} \Xi^G(nh_0)\sigma(n)^{-d}dn$$

for all $h_0 \in H_0(F)$ and all $b > 0$. By proposition 1.3.1(iv), we may choose $d > 0$ such that the last integral above is essentially bounded by $\delta_P(h_0)^{1/2}\Xi^M(h_0) = \Xi^{G_0}(h_0)$ for all $h_0 \in H_0(F)$. From here on we will fix such a $d > 0$. Hence, we have

2.4.4)
$$I^0_{\epsilon,\delta,\leq b}(h_0) \ll e^{\epsilon b}b^d\Xi^{G_0}(h_0)$$

for all $h_0 \in H_0(F)$ and all $b > 0$.

There exists $\alpha > 0$ such that $\Xi^G(g_1g_2) \ll e^{\alpha\sigma(g_2)}\Xi^G(g_1)$ for all $g_1, g_2 \in G(F)$. It follows that

2.4.5)

$$I^0_{\epsilon,\delta,>b}(h_0) \ll e^{\alpha\sigma(h_0)-\sqrt{\epsilon}b} \int_{N(F)} \Xi^G(n)e^{(\epsilon+\sqrt{\epsilon})\sigma(n)}(1 + |\lambda(n)|)^{-\delta}dn$$

for all $h_0 \in H_0(F)$ and all $b > 0$. Assume that 2.4.5 is convergent if ϵ is sufficiently small. Then we can combine 2.4.4 and 2.4.5 and get

$$I^0_{\epsilon,\delta}(h_0) \ll e^{\epsilon b} b^d \Xi^{G_0}(h_0) + e^{\alpha\sigma(h_0) - \sqrt{\epsilon} b}$$

for all $h_0 \in H_0(F)$ and for all $b > 0$. There exists $\beta > 0$ such that $e^{-\beta\sigma(h_0)} \ll \Xi^{G_0}(h_0)$ for all $h_0 \in H_0(F)$. Plugging $b = \frac{\alpha+\beta}{\sqrt{\epsilon}}\sigma(h_0)$ in the last inequality, we obtain

$$I^0_{\epsilon,\delta}(h_0) \ll e^{\sqrt{\epsilon}(\alpha+\beta+1)\sigma(h_0)} \Xi^{G_0}(h_0)$$

for all $h_0 \in H_0(F)$. Hence, for $\epsilon \leq \epsilon_0^2(\alpha+\beta+1)^{-2}$, 2.4.3 holds.

It remains to show that 2.4.5 converges for ϵ sufficiently small. If P is a minimal parabolic subgroup of G then it follows from Corollary B.3.2 in paper (since in this case λ is a generic additive character on N). Assume this is not the case. Then we can find two isotropic vectors $z_{0,+}, z_{0,-} \in V_0$ such that $z_0 = z_{0,+} - z_{0,-}$. We have a decomposition $\lambda = \lambda_+ - \lambda_-$ where

$$\begin{aligned} \lambda_+(n) &= \sum_{i=1}^{r-1} b(z_{-i-1}, n z_i) + b(z_{-1}, n z_{0,+}) && n \in N \\ \lambda_-(n) &= b(z_{-1}, n z_{0,-}) && n \in N \end{aligned}$$

Note that the additive character λ_+ is the restriction to N of a generic additive character of the unipotent radical of a minimal parabolic subgroup contained in P. Hence, same ref as before applies to λ_+. Choose a one-parameter subgroup $a : \mathbb{G}_m \to M$ such that $\lambda_+(a(t)na(t)^{-1}) = t\lambda_+(n)$ and $\lambda_-(a(t)na(t)^{-1}) = t^{-1}\lambda_-(n)$ for all $t \in \mathbb{G}_m$ and $n \in N$ (such a one-parameter subgroup is easily seen to exist). Let $\mathcal{U} \subseteq F^\times$ be a compact neighborhood of 1. Then, for all $\epsilon > 0$, we have

$$\begin{aligned} &\int_{N(F)} \Xi^G(n) e^{\epsilon\sigma(n)} (1 + |\lambda(n)|)^{-\delta} dn \\ &\ll \int_{N(F)} \Xi^G(n) e^{\epsilon\sigma(n)} (1 + |\lambda(a(t)na(t)^{-1})|)^{-\delta} dn \\ &= \int_{N(F)} \Xi^G(n) e^{\epsilon\sigma(n)} (1 + |t\lambda_+(n) - t^{-1}\lambda_-(n)|)^{-\delta} dn \end{aligned}$$

for all $t \in \mathcal{U}$. Integrating this inequality over $\mathcal{U}$, we get that for all $\epsilon > 0$, we have

$$\begin{aligned} &\int_{N(F)} \Xi^G(n) e^{\epsilon\sigma(n)} (1 + |\lambda(n)|)^{-\delta} dn \\ &\qquad \ll \int_{N(F)} \Xi^G(n) e^{\epsilon\sigma(n)} \int_{\mathcal{U}} (1 + |t\lambda_+(n) - t^{-1}\lambda_-(n)|)^{-\delta} dt dn \end{aligned}$$

By Lemma B.1.1 there exists $\delta' > 0$ depending only on $\delta > 0$ such that the last expression is essentially bounded by

$$\int_{N(F)} \Xi^G(n) e^{\epsilon\sigma(n)} (1 + |\lambda_+(n)|)^{-\delta'} dn$$

Now by Corollary B.3.2, this last integral is convergent if ϵ is sufficiently small.

(iv) Let $\delta > 0$ and $\epsilon > 0$. We want to show that (iv) holds if ϵ is sufficiently small (compared to δ). Since $\Xi^G(g^{-1}) \sim \Xi^G(g), \sigma(g^{-1}) \sim \sigma(g)$ and $\lambda(h^{-1})) = -\lambda(h)$ for all $g \in G(F)$ and all $h \in H(F)$, it is equivalent to show the following

2.4.6) If ϵ is sufficiently small the integral

$$J^1_{\epsilon,\delta}(m_{min}) = \int_{H(F)} \Xi^G(m_{min}h) e^{\epsilon\sigma(h)} (1 + |\lambda(n)|)^{-\delta} dh$$

is absolutely convergent for all $m_{min} \in M_{min}(F)$ and there exists $d > 0$ such that

$$J^1_{\epsilon,\delta}(m_{min}) \ll \delta_{\overline{P}_{min}}(m_{min})^{1/2} \sigma(m_{min})^d$$

for all $m_{min} \in M_{min}(F)$

Let K be a maximal compact subgroup of $G(F)$ that is special in the p-adic case. Fix a map $m_{\overline{P}_{min}} : G(F) \to M_{min}(F)$ such that $g \in m_{\overline{P}_{min}}(g)\overline{U}_{min}(F)K$ for all $g \in G(F)$. By Proposition 1.3.1(ii), there exists $d > 0$ such that we have

$$J^1_{\epsilon,\delta}(m_{min}) \ll \delta_{\overline{P}_{min}}(m_{min})^{1/2} \sigma(m_{min})^d * \int_H (F) \delta_{\overline{P}_{min}}(m_{\overline{P}_{min}}(h))^{1/2} \sigma(h)^d e^{\epsilon\sigma(h)} (1 + |\lambda(h)|)^{-\delta} dh$$

for all $m_{min}(F)$. Of course, for any $\epsilon' > \epsilon$ we have $\sigma(h)^d e^{\epsilon\sigma(h)} \ll e^{\epsilon'\sigma(h)}$, for all $h \in H(F)$. Hence, we only need to prove that for ϵ sufficiently small the integral

2.4.7)

$$\int_{H(F)} \delta_{\overline{P}_{min}}(m_{\overline{P}_{min}}(h))^{1/2} e^{\epsilon\sigma(h)} (1 + |\lambda(h)|)^{-\delta} dh$$

is absolutely convergent. Since $\overline{P}_{min}$ is a good parabolic subgroup, we may find compact neighborhoods of the identity $\mathcal{U}_K \subseteq K, \mathcal{U}_H \subseteq H(F)$ and $\mathcal{U}_{\overline{P}} \subseteq \overline{P}_{min}(F)$ such that $\mathcal{U}_K \subseteq \mathcal{U}_{\overline{P}}\mathcal{U}_H$. We have the inequalities

$$e^{\epsilon\sigma(k_H h)} \ll e^{\epsilon\sigma(h)}$$

and

$$(1+|\lambda(k_H h)|)^{-\delta} \ll (1+|\lambda(h)|)^{-\delta}$$

for all $h \in H(F)$ and for all $k_H \in \mathcal{U}_H$. Hence, we have

$$\begin{aligned}
&\int_H (F)\delta\overline{P}_{min}(m_{\overline{P}_{min}}(h))^{1/2}e^{\epsilon\sigma(h)}(1+|\lambda(h)|)^{-\delta}dh \\
\ll\ & \delta_{\overline{P}_{min}}(k_{\overline{P}})^{1/2}\int_{H(F)} \delta_{\overline{P}_{min}}(m_{\overline{P}_{min}}(k_H h))^{1/2}e^{\epsilon\sigma(h)}(1+|\lambda(h)|)^{-\delta}dh \\
=\ & \int_{H(F)} \delta_{\overline{P}_{min}}(m_{\overline{P}_{min}}(k_{\overline{P}}k_H h))^{1/2}e^{\epsilon\sigma(h)}(1+|\lambda(h)|)^{-\delta}dh
\end{aligned}$$

for all $k_H \in \mathcal{U}_H$ and all $k_{\overline{P}} \in \mathcal{U}_{\overline{P}}$. It follows that

$$\begin{aligned}
&\int_{H(F)} \delta_{\overline{P}_{min}}(m_{\overline{P}min}(h))^{1/2}e^{\epsilon\sigma(h)}(1+|\lambda(h)|)^{-\delta}dh \\
\ll\ & \int_{H(F)}\int_{\mathcal{U}_K} \delta_{\overline{P}_{min}}(m_{\overline{P}_{min}}(kh))^{1/2}dk e^{\epsilon\sigma(h)}(1+|\lambda(h)|)^{-\delta}dh \\
\ll\ & \int_{H(F)}\int_{K} \delta_{\overline{P}_{min}}(m_{\overline{P}_{min}}(kh))^{1/2}dk e^{\epsilon\sigma(h)}(1+|\lambda(h)|)^{-\delta}dh
\end{aligned}$$

By Proposition 1.3.1(iii), the innter integral above is equal to $\Xi^G(h)$ (for a suitable normalization) and the convergence of 2.4.7 for ϵ sufficiently small now follows from (iii).

(v) Let $\delta > 0$ and $\epsilon > 0$. We want to prove that (v) holds if ϵ is sufficiently small (compared to δ). After the variable change $h \mapsto h'h^{-1}$, we are left with proving that for $\epsilon > 0$ sufficiently small the following integral is absolutely convergent

$$\begin{aligned}
&I^3_{\epsilon,\delta}(m_{min}) \\
&= \int_{H(F)}\int_{H(F)} \Xi^G(hm_{min})\Xi^G(h'm_{min})e^{\epsilon\sigma(h)}e^{\epsilon\sigma(h')} \\
&\qquad\qquad (1+|\lambda(h')-\lambda(h)|)^{-\delta}dh'dh
\end{aligned}$$

for all $m_{min} \in M_{min}(F)$ and that there exists $d > 0$ such that

2.4.8)
$$I^3_{\epsilon,\delta}(m_{min}) \ll \delta_{\overline{P}_{min}}(m_{min})^{-1}\sigma(m_{min})^d$$

for all $m_{min} \in M_{min}(F)$. Let $a : \mathbb{G}_m \to A$ be a one-parameter subgroup such that $\lambda(a(t)ha(t)^{-1}) = t\lambda(h)$ for all $h \in H$ and for all $t \in \mathbb{G}_m$. Let $\mathcal{U} \subseteq F^\times$ be a compact neighborhood of 1. Since A is in the center of M_{min}, we have the inequality

$I^3_{\epsilon,\delta}(m_{min})$

$$\ll \int_{H(F)} \int_{H(F)} \Xi^G(hm_{min})\Xi^G(h'm_{min})e^{\epsilon\sigma(h)}e^{\epsilon\sigma(h')}$$
$$\int_{\mathcal{U}} (1 + |\lambda(a(t)h'a(t)^{-1}) - \lambda(h)|)^{-\delta} dtdh'dh$$

$$= \int_{H(F)} \int_{H(F)} \Xi^G(hm_{min})\Xi^G(h'm_{min})e^{\epsilon\sigma(h)}e^{\epsilon\sigma(h')}$$
$$\int_{\mathcal{U}} (1 + |t\lambda(h') - \lambda(h)|)^{-\delta} dtdh'dh$$

for all $m_{min} \in M_{min}(F)$. By Lemma B.1.1, there exists $\delta' > 0$ depending only on δ such that the last integral above is essentially bounded by

$$\int_{H(F)} \int_{H(F)} \Xi^G(hm_{min})\Xi^G(h'm_{min})e^{\epsilon\sigma(h)}e^{\epsilon\sigma(h')}$$
$$(1 + |\lambda(h')|)^{-\delta'}(1 + |\lambda(h)|)^{-\delta} dh'dh$$

for all $m_{min} \in M_{min}(F)$. This last integral is equal to $I^1_{\epsilon,\delta'}(m_{min})^2$. Hence, the inequality 2.4.8 for ϵ sufficiently small now follows from (iv).

□

2.5 Relative Weak Cartan Decomposition

2.5.1 Relative Weak Cartan Decomposition for G_0

First we will address the codimension one case of the triple $(G_0, H_0, 1)$. Of course proposition 2.3.1 applieds to this case, in particular, G_0 admits good minimal parabolic subgroups. Let $\overline{P}_0 = M_0\overline{U}_0$ be such a minimal parabolic subgroup of G_0 and denote $A_0 = A_{M_0}$ the maximal central split subtorus of M_0. Set

$$A_0^+ = \{a \in A_0(F);\ |\alpha(a)| \geq 1\, \forall\alpha \in R(A_0, \overline{P}_0)\}$$

Proposition 2.5.1.1 (6.6.1) *There exists a compact subset $\mathcal{K}_0 \subseteq G_0(F)$ such that*

$$G_0(F) = H_0(F)A_0^+\mathcal{K}_0$$

Proof. First we will prove that the result doesn't depend on the choice of $(\overline{P}_0, M_0)$. Let $(\overline{P}'_0, M'_0)$ be another such pair, i.e. $\overline{P}'_0$ is a good minimal parabolic subgroup of G_0 and M'_0 is a Levi component of it. By 2.3.1(i), there exists $h \in H(F)$ such that $\overline{P}'_0 = h\overline{P}_0h^{-1}$. It is clear that if the proposition is true for the pair $(\overline{P}_0, M_0)$ then it is also true for the pair $(h\overline{P}_0h^{-1}, hM_0h^{-1}) = (\overline{P}'_0, hM_0h^{-1})$. Moreover, there exists a $\overline{p}'_0 \in \overline{P}'_0(F)$ such that $hM_0h^{-1} = \overline{p}'_0M'_0\overline{p}'^{-1}_0$. This means that the theorem being true for the pair $(\overline{P}'_0, hM_0h^{-1})$ implies it is true for the pair $(\overline{P}'_0, M'_0)$ becuase, by definition of A'^+_0, the set

$$\{a'^{-1}_0\overline{p}'_0a'_0,\ a'_0 \in A'^+_0\}$$

is bounded. Thus, it suffices to prove the proposition holds for any particular pair $(\overline{P}_0, M_0)$.

Next we will consider the archemedian case. Fix a good minimal parabolic subgroup $\overline{P}_0 \subseteq G_0$. By 2.3.1(i) there exists a Levi component M_0 of $\overline{P}_0$ such that $H_0 \cap \overline{P}_0 \subseteq M_0$. By theorem 5.13 of KKSS there exists a compact subset $\mathcal{K}_0 \subseteq G_0(\mathbb{R})$ such that

$$G_0(\mathbb{R}) = H_0(\mathbb{R}) F'' A_Z^- \mathcal{K}_0$$

where A_Z^- is a certain submonoid of $A_0(\mathbb{R})$ (the exponential of the "compression cone" associated to the real spherical variety $Z = H_0(\mathbb{R}) \backslash G_0(\mathbb{R})$, cf. Sections 5.1 of KKSS) and F'' is a subset of $N_{G_0(\mathbb{R})}(H_0)F$, F being any set of representatives for the open $H_0(\mathbb{R}) \times \overline{P}_0(\mathbb{R})$ double cosets in $G_0(\mathbb{R})$. By 2.3.1(i), we can take $F = \{1\}$. Moreover, we can check that $N_{G_0(\mathbb{R})}(H_0) = H_0(\mathbb{R}) Z_{G_0}(\mathbb{R})$. As $Z_{G_0}(\mathbb{R})$ is compact, up to multiplying by $\mathcal{K}_0$ by $Z_{G_0}(\mathbb{R})$, we may also assume that $F'' = \{1\}$. All that remains is to show that $A_Z^- \subseteq A_0^+$ (note that our convention for the positive chamber is the opposite to the notation in KKSS, this is because we are denoting our good parabolic subgroup by $\overline{P}_0$ and not by P_0). But this follows from the fact that the real spherical variety $Z = H_0(\mathbb{R}) \backslash G_0(\mathbb{R})$ is wavefront (cf. definition 6.1 of KKSS noting thet here $\mathfrak{a}_H = 0$. The notion of wavefront spherival variety has been first introduced in CV). To see this, consider the complex homogeneous space $Z_{\mathbb{C}} = H_0(\mathbb{C}) \backslash G_0(\mathbb{C}) \cong GL_{d-1}(\mathbb{C}) \backslash (GL_{d-1}(\mathbb{C}) \times GL_d(\mathbb{C}))$. It is spherical (it follows for example from 2.3.1(i) applied to GGP triples of codimension one with G quasi split) and wavefront by Remark 6.2 of KKSS. But we also know from the characterization of the compression cone given in Lemma 5.9 of KKSS that a real spherical variety is wavefront if its complexification is spherical and wavefront. Thus Z is wavefront and this proves the proposition in the archemedian case.

Now we will consider the p-adic case. To do this we will prove Proposition 11.0.1 of Beu1 in the case of the Special Orthogonal group. But first we will need 2 lemmas from Beu1. In these lemmas we will have $G = SO(V)$, $G_0 = SO(V_0)$ and $H = SO(W)$.

To proceed we will need to following notation. $w_0 = z_0$, and $w_0, \dots, w_l$ is a maximal family of orthogonal vectors in V_0 such that $q(w_i) = (-1)^i \nu$. Let $r_0 = \lfloor \frac{l+1}{2} \rfloor$ and $r_1 = \lfloor \frac{l}{2} \rfloor$. Let $u_{\pm i} = w_{2i-2} \pm w_{2i-1}$ for $i = 1, \dots, r_0$ and $u'_{\pm i} = w_{2i-1} \pm w_{2i}$ for $i = 1, \dots, r_1$. Then $(u_{\pm i})$ and $u'_{\pm i}$ are maximal isotropic subspace of V_0 and W respectively. Let $V_{0,an}$ and W_{an} be the orthogonal complements of $\langle u_1, u_{-1}, \dots u_{r_0}, u_{-r_0} \rangle$ in V_0 and $\langle u'_1, u'_{-1}, \dots u'_{r_1}, u'_{-r_1} \rangle$ in W respectively. Define P_0 and P_H as the parabolic subgroups of G_0 and H which preserve the isotropic flags $\langle u_1 \rangle \subseteq \langle u_1, u_2 \rangle \subseteq \cdots \subseteq \langle u_1, \dots, u_{r_0} \rangle$ and $\langle u'_1 \rangle \subseteq \cdots \subseteq \langle u'_1, \dots, u'_{r_1} \rangle$ respectively.

Let A_0 and A_H be the maximal split central subtorus of G_0 and H which stabilize the lines $\langle u_i \rangle, i = \pm 1, \dots, \pm r_0$ and $\langle u'_i \rangle, i = \pm 1, \dots, \pm r_1$ respectively. Let A_0^+ and A_H^+ be the positive chambers of $A_0(F)$ and $A_H(F)$ respectively. Let $R_0 = \mathcal{O}_F u_1 \oplus \mathcal{O}_F u_{-1} \oplus \ldots \oplus \mathcal{O}_F u_r \oplus \mathcal{O}_F u_{-r} \oplus R_{0,an}$ where $R_{0,an} = \{v \in V_{0,an}; \mathrm{val}_F(q(v)) \geq \mathrm{val}_F(\nu)\}$. If V is an F vector space, and R is an $\mathcal{O}_F$ Lattice of V and $g \in GL_F(V)$ then $\mathrm{val}_R(g) = \inf\{\mathrm{val}_R(gv), v \in R\}$ and

$\|g\|_R = |\varpi_F|^{\mathrm{val}_R(g)}$

Lemma 2.5.1.2 (Beu1 Lemma 11.0.2) *There exists a compact subset $C_1 \subseteq G_0(F)$ such that for all $v, v' \in V_0$ such that $q(v) = q(v') = \nu$ and $\mathrm{val}_{R_0}(v) = \mathrm{val}_{R_0}(v')$, there is a $\gamma \in C_1$ such that $\gamma v = v'$*

Proof. If $l = 0$ then G_0 is compact and then $C_1 = G_0$ satisfies the lemma. Suppose then that $l \geq 1$. Let $K_0 = \mathrm{Stab}_{G_0(F)}(R_0)$ be an open compact subgroup of $G_0(F)$. For all $g \in G_0(F)$, we will note $\|g\| = \|g\|_{R_0}$. The function $g \mapsto \|g\|$ is bounded from below by a positive constant on $G_0(F)$ and is biinvariant by K_0. Let $v \in V_0$. Now we will show

2.5.1.3) There is $k \in K_0$ such that $kv \in \langle u_1, u_{-1}\rangle \oplus V_{0,an}$

We can decompose v as $v = \sum_{i=\pm 1,\ldots,\pm r} \lambda_i u_i + v_{an}$ for $v_{an} \in V_{0,an}$, $\lambda_i \in F$. By multiplying v by an element of the Weyl group of A_0 (identified with a subgroup of K_0), we can assume that $\mathrm{val}(\lambda_1) = \inf_{i=\pm 1,\ldots,\pm r} \mathrm{val}(\lambda_i)$.

Let $k_1 \in K_0$ be the element which sends u_1 to $u_1 - \frac{\lambda_2}{\lambda_1}u_2 - \cdots - \frac{\lambda_r}{\lambda_1}u_r$ and u_{-i} to $u_{-i} + \frac{\lambda_i}{\lambda_1}u_{-1}$ for $i = 2, \ldots, r$ and which acts trivially on $V_{0,an} \oplus \langle u_{-1}, u_2, \ldots u_r\rangle$. We then have that

$$k_1 v = \lambda u_1 + \mu_1 u_{-1} + \lambda_{-2}u_{-2} + \cdots + \lambda_{-r}u_{-r} + v_{an}$$

Let $k_2 \in K_0$ be the element that sends u_1 to $u_1 - \frac{\lambda_{-2}}{\lambda_1}u_{-2} - \cdots - \frac{\lambda_{-r}}{\lambda_1}$ and u_i to $u_i + \frac{\lambda_{-i}}{\lambda_1}u_{-1}$ and acts trivially on $V_{0,an} \oplus \langle u_{-1}, \ldots, u_{-r}$. We then have that $k_2k_1 v \in \langle u_1, u_{-1}\rangle \oplus V_{0,an}$.

And we have

2.5.1.4) For all $v_{an} \in V_{0,an}$, we have

$$R_{0,an} \subseteq \frac{1}{2}\big((R_{0,an} \cap \langle v_{an}\rangle) \oplus (R_{0,an} \cap (v_{an})^{\perp})\big)$$

where $(v_{an})^{\perp}$ is the orthogonal complement of v_{an} in $V_{0,an}$.

To see this, let $v \in R_{0,an}$ and $v = v_1 + v_2$ with $v_1 \in \langle v_{an}\rangle$ and $v_2 \in (v_{an})^{\perp}$. Assume $\mathrm{val}_F(q(v_1) + q(v_2)) > \min(\mathrm{val}_F(q(v_1)), \mathrm{val}_F(q(v_2))) + \mathrm{val}_F(4)$. Then $-\frac{q(v_1)}{q(v_2)} \in 1 + 4\mathfrak{p}_F$. Since $1 + 4\mathfrak{p}_F \subseteq \mathcal{O}_F^{\times,2}$, there exists $\lambda \in \mathcal{O}_F^{\times}$ such that $q(v_1 + \lambda v_2) = 0$, contradicting that $V_{0,an}$ is anisotropic. Consequently,

$$\begin{aligned}
\min(\mathrm{val}_F(q(2v_1)), \mathrm{val}_F(q(2v_2))) &\geq \mathrm{val}_F(q(v_1) + q(v_2)) \\
&= \mathrm{val}_F(q(v_1 + v_2)) \\
&= \mathrm{val}_F(q(v)) \\
&\geq \mathrm{val}_F(\nu)
\end{aligned}$$

Hence $v_1, v_2 \in \frac{1}{2}R_{0,an}$.

Next, we will show

2.5.1.5) There is $g \in G_0(F)$ such that $gv = \lambda_1 u_1 + \lambda_{-1} u_{-1} + v_{an}$ with $v_{an} \in V_{0,an}$, $\mathrm{val}_F(\lambda_1) = \inf(\mathrm{val}_F(\lambda_1), \mathrm{val}_F(\lambda_{-1}), \mathrm{val}_{R_{0,an}}(v_{an}))$ and $\|g\| \leq |2|_F^{-1}$ for all $v \in V_0$.

From 2.5.1.3, we can assume that $v = \lambda_1 u_1 + \lambda_{-1} u_{-1} + v_{an}$ and up to applying the element which exchanges u_1 and u_{-1} and leaves stable the anisotropic part, we can also assume that $\mathrm{val}_F(\lambda_1) \leq \mathrm{val}_F(\lambda_{-1})$. If $\mathrm{val}_F(\lambda_1) \leq \mathrm{val}_{R_{0,an}}(v_{an})$ then we are done. If on the contrary $-d = \mathrm{val}_{R_{0,an}}(v_{an}) < \mathrm{val}_F(\lambda_1)$, we can consider the element $g \in G_0(F)$ which sends u_1 to u_1, v_{an} to $v_{an} + \varpi_F^{-d} u_1$, u_{-1} to $u_{-1} - \frac{2\varpi_F^{-2d}\nu}{q(v_{an})} v_{an} - \frac{\varpi_F^{-d}\nu}{q(v_{an})} u_1$, and which acts trivially on the orthogonal complement of $\langle u_1, u_{-1}, v_{an} \rangle$. Since $\mathrm{val}_F(q(v_{an})) \leq -2d + \mathrm{val}_F(\nu)$, we have

$$g(R_0 \cap \langle u_1, u_{-1}, v_{an} \rangle) = R_0 \cap \langle u_1, u_{-1}, v_{an} \rangle$$

According to 2.5.1.4 we therefore have $\|g\| \leq |2|_F^{-1}$. Still using that $\mathrm{val}_F(q(v_{an})) \leq -2d + \mathrm{val}_F(\nu)$, it is easy to check that gv does in fact satisfy 2.5.1.5.

Next, we will show

2.5.1.6) There exists $g \in G_0(F)$ such that $gv \in \langle u_1, u_{-1} \rangle$ and $\|g\| \leq |\varpi_F|_F^{-1} |2|_F^{-4}$

From 2.5.1.5 we can assume that $v = \lambda_1 u_1 + \lambda_{-1} u_{-1} + v_{an}$ with $v_{an} \in V_{0,an}$ and $\mathrm{val}_F(\lambda_1) = \inf(\mathrm{val}_F(\lambda_1), \mathrm{val}_F(\lambda_{-1}), \mathrm{val}_{R_{0,an}}(v_{an}))$. Let $g \in G_0(F)$ be the element which sends

$$u_{-1} \mapsto u_{-1}$$

$$u_1 \mapsto u_1 - \frac{1}{\lambda_1} v_{an} - \frac{q(v_{an})}{4\lambda_1^2 \nu} u_{-1}$$

$$v_{an} \mapsto v_{an} + \frac{q(v_{an})}{2\lambda_1 \nu} u_{-1}$$

and acts trivially on the orthogonal complement of $\langle u_1, u_{-1}, v_{an} \rangle$. Then we do have $gv \in \langle u_1, u_{-1} \rangle$ and by 2.5.1.4 we have

$$\begin{aligned} \mathrm{val}_{R_0}(g) \leq \inf(& \mathrm{val}_{R_{0,an}}(v_{an}) - \mathrm{val}_F(\lambda_1), \\ & \mathrm{val}_F(q(v_{an})) - \mathrm{val}_F(4\lambda_1^2 \nu), \\ & \mathrm{val}_F(q(v_{an}) - \mathrm{val}_F(2\lambda_1 \nu) - \mathrm{val}_{R_{0,an}}(v_{an}) - \mathrm{val}_F(2), \\ & 0) \end{aligned}$$

Because $\mathrm{val}_F(q(v_{an}) \geq 2\,\mathrm{val}_{R_{0,an}}(v_{an}) + \mathrm{val}_F(\nu) - 1$, we have $\mathrm{val}_{R_0}(g) \geq -1 - 3\,\mathrm{val}_F(2)$.

Now suppose that $q(v) = 2\nu^2$. Then

2.5.1.7) There exists $g \in G_0(F)$ and $\lambda \in \mathcal{O}_F - \{0\}$ such that $gv = \lambda u_1 + \frac{\nu}{2\lambda} u_{-1}$ and $\|g\| \leq |\varpi_F|_F^{-1} |2|_F^{-5}$.

From 2.5.1.6, we know that there is a $g' \in G_0(F)$ such that $g'v = \lambda u_1 + \mu u_{-1}$ and $\|g'\| \leq |\varpi_F|_F^{-1}|2|_F^{-4}$. As we can multiply by an element to swap u_1 and u_{-1} we can assume without loss of generality that $\mathrm{val}(\mu) \geq \mathrm{val}(\lambda)$. Now by the fact that $q(v) = 2\nu^2$ we have that $\mu = \frac{\nu}{2\lambda}$ and the result is trivial

We can now finish the proof of the lemma. Let v and v' be as in the lemma. There are $g_1, g_2 \in G_0(F)$ and $\lambda_1, \lambda_2 \in \mathcal{O}_F - \{0\}$ such that

$$\begin{aligned}
\|g_1\|, \|g_2\| &\leq |\varpi_F|_F^{-1}|2|_F^{-5} \\
g_1 v &= \lambda_1 u_1 + \frac{\nu}{2\lambda_1} u_{-1} \\
g_2 v' &= \lambda_2 u_1 + \frac{\nu}{2\lambda_2} u_{-1}
\end{aligned}$$

There is an integer N_0 such that

$$|\mathrm{val}_{R_0}(gv_0) - \mathrm{val}_{R_0}(v_0)| \leq -\mathrm{val}_{R_0}(g) + N_0$$

for all $g \in G_0(F)$ and for all $v_0 \in V_0$. Since $\mathrm{val}_{R_0}(g_1 v) = -\mathrm{val}_F(\lambda_1) - \mathrm{val}_F(2)$, $\mathrm{val}_{R_0}(g_2 v') = -\mathrm{val}_F(\lambda_2) - \mathrm{val}_F(2)$, and $\mathrm{val}_{R_0}(v) = \mathrm{val}_{R_0}(v')$, we have

$$|\mathrm{val}_F(\lambda_1) - \mathrm{val}_F(\lambda_2)| \leq 2 + 10\,\mathrm{val}_F(2) + 2N_0$$

Let a be the element of $G_0(F)$ which sends u_1 to $\frac{\lambda_2}{\lambda_1}u_1$, and u_{-1} to $\frac{\lambda_1}{\lambda_2}u_{-1}$ while acting trivially on the orthogonal space to $\langle u_1, u_{-1}\rangle$. We then have $ag_1v = g_2v'$. Hence $g_2^{-1}ag_1v = v'$ and $\|g_2^{-1}ag_1\|$ is bounded by a constant, which completes the proof. $\square$

Assume $l \geq 1$. For all $\lambda \in F^\times$ we denote $a(\lambda)$ as the element of $G_0(F)$ which sends u_1 to λu_1 and u_{-1} to $\lambda^{-1}u_{-1}$ and which acts trivially on the orthogonal complement of $\langle u_1, u_{-1}\rangle$. Let $R_{\#,H}$ be an $\mathcal{O}_F$-lattice orthogonal to $\langle w_1, \ldots, w_l\rangle$ in W. Then let $R_1 = \mathcal{O}_F\, w_1 \oplus \cdots \oplus \mathcal{O}_F\, w_l \oplus R_{\#,H}$ and $R_2 = \mathcal{O}_F\, w_2 \oplus \cdots \oplus \mathcal{O}_F\, w_l \oplus R_{\#,H}$

Lemma 2.5.1.8 (Beu1 lema 11.0.3) *Suppose $l \geq 1$. There exists a compact subset C_2 of $G_0(F)$ and a constant $c_0 > 0$ such that for any $h \in H(F)$ and any $\lambda \in \mathcal{O}_F - \{0\}$, there exists $h' \in h\big(a(\lambda)C_2a(\lambda)^{-1} \cap H(F)\big)$ such that*

$$\mathrm{val}_{R_1}(h') \geq \mathrm{val}_{R_1}(h'w_1) - \mathrm{val}_F(\lambda) - c_0$$

Proof. Let $e_1, \ldots, e_t$ be an orthogonal basis of the $\mathcal{O}_F$-module, R_2. There exists a strictly positive integer α such that

- $-\frac{q(e_i)}{q(w_1)} \in \mathfrak{p}_F^{-\alpha}$, for $i = 1, \ldots, t$
- for any integer $k \geq 0$ and for any $x \in 1 + \mathfrak{p}_F^{k+\alpha}$, there exists $y \in 1 + \mathfrak{p}_F^k$ such that $y^2 = x$.

For $i = 1, \ldots, t$ and $\lambda \in \mathcal{O}_F - \{0\}$, we consider the element $\gamma_i(\lambda) \in H(F)$ which acts trivially on $\langle w_1, e_i \rangle^{\perp}$ and which acts in the following way on w_1 and e_i

$$w_1 \mapsto a w_1 + \lambda \varpi_F^{\alpha} e_i$$

$$e_i \mapsto \frac{q(e_i)}{q(w_1)} \lambda \varpi_F^{\alpha} w_1 - a e_i$$

where $a = 1 - \lambda^2 \varpi_F^{\alpha} \frac{q(e_i)}{q(w_1)} \in 1 + \lambda^2 \mathcal{O}_F$. Then $\gamma_i(\lambda)$ stabilizes $\mathcal{O}_F w_1 \oplus R_2$ and maps to a compact set independent of i and λ. Let's check that $a(\lambda)^{-1}\gamma_i(\lambda)a(\lambda)$ also is compact. We have

$$\begin{aligned} a(\lambda)^{-1}\gamma_i(\lambda)a(\lambda)u_{-1} &= \lambda^{-1}a(\lambda)^{-1}\gamma_i(\lambda)u_{-1} \\ &= \lambda^{-1}a(\lambda)^{-1}(w_0 - \gamma_i(\lambda)w_1) \\ &= \lambda^{-1}a(\lambda)^{-1}(u_{-1} + w_1 - \gamma_i(\lambda)w_1) \\ &= u_{-1} + \lambda^{-1}a(\lambda)^{-1}(w_1 - \gamma_i(\lambda)w_1) \end{aligned}$$

but $w_1 - \gamma_i(\lambda)w_1$ is in $\lambda^2 \mathcal{O}_F \oplus R_2$. So $a(\lambda)^{-1}\gamma_i(\lambda)a(\lambda)u_{-1} \in \frac{1}{2}\mathcal{O}_F u_{-1} + R_1$. We can show in the same way that $a(\lambda)^{-1}\gamma_i(\lambda)a(\lambda)u_1$ remains bounded. Finally, since $a(\lambda)^{-1}\gamma_i(\lambda)a(\lambda)e_i = a(\lambda)^{-1}\gamma_i(\lambda)e_i$ and $\gamma_i(\lambda)e_i \in \lambda \mathcal{O}_F w_1 + R_2$, we have $a(\lambda)^{-1}\gamma_i(\lambda)a(\lambda)e_i$ remains bounded.

Let C_2 be a compact subset of $G_0(F)$ which contains all $a(\lambda)^{-1}\gamma_i(\lambda)a(\lambda)$ for all $i = 1, \ldots, t$ and for all $\lambda \in \mathcal{O}_F - \{0\}$. Since the $\gamma_i(\lambda)$ are bounded there exists a constant $c_1 > 0$ such that for all $h \in H(F)$, for all $i \in 1, \ldots, t$ and for all $\lambda \in \mathcal{O}_F$ we have

$$\mathrm{val}_{R_1}(h\gamma_i(\lambda)) \geq \mathrm{val}_{R_1}(h) - c_1$$

We have $\mathrm{val}_{R_1}(h\gamma_i(\lambda)w_1) = \inf(\mathrm{val}_{R_1}(hw_1), \mathrm{val}_F(\lambda) + \alpha + \mathrm{val}_{R_1}(he_i)$. Since $\mathrm{val}_{R_1}(h) = \inf(\mathrm{val}_{R_1}(hw_1), \mathrm{val}_{R_1}(he_1), \ldots, \mathrm{val}_{R_1}(he_t))$, we have

$$\begin{aligned} \inf_{i=1,\ldots,t}(\mathrm{val}_{R_1}(h\gamma_i(\lambda)w_1) &\leq \mathrm{val}_F(\lambda) + \alpha + \mathrm{val}_{R_1}(h) \\ &\leq \mathrm{val}_F(\lambda) + \alpha + c_1 + \inf_{i=1,\ldots,t} \mathrm{val}_{R_1}(h\gamma_i(\lambda)) \end{aligned}$$

To obtain the lemma it is sufficient to take $c_0 = \alpha + c_1$ and for $h \in H(F)$ and $\lambda \in \mathcal{O}_F - \{0\}$ to choose $h' = h\gamma_j(\lambda)$ where j is such that $\inf_{i=1,\ldots,t}(\mathrm{val}_{R_1}(h\gamma_i(\lambda)w_1)) = \mathrm{val}_{R_1}(h\gamma_j(\lambda)w_1)$

□

Now we can prove our desired proposition

Proposition 2.5.1.9 (Beu1 Proposition 11.0.1) *There exist compact subsets $C_0 \subseteq G_0(F)$ and $C_H \subseteq H(F)$ such that*

$$G_0(F) = C_H A_H^+ A_0^+ C_0$$

Proof. We will proceed by induction on l. If $l = 0$, then G_0 is compact and the proposition is true. Now we will assume that $l \geq 1$. 2.5.1.2 gives us compact $C_1 \subseteq G_0(F)$. 2.5.1.8 gives us a compact $C_2 \subseteq G_0(F)$ as well as a constant c_0. Let $C = C_2^{-1} C_1$. Let $g \in G_0(F)$. By 2.5.1.2 there exists $k_1 \in C_1$ and $\lambda \in \mathcal{O}_E - \{0\}$ such that

$$k_1 g^{-1} z_0 = a(\lambda)^{-1} v_0$$

that is $g \in H(F) a(\lambda) k_1$. Using 2.5.1.8 ,we see that there exists $k_2 \in C_2, \lambda \in \mathcal{O}_F - \{0\}$ and $h \in H(F)$ such that

$$g = h a(\lambda) k_2^{-1} k_1$$

and

2.5.1.10) $\quad \mathrm{val}_{R_1}(h) \geq \mathrm{val}_{R_1}(hw_1) - \mathrm{val}_F(\lambda) - c_0$

Set $k = k_2^{-1} k_1 \in C$. Let H' be the special orthogonal group of $W' = \langle w_0, w_1 \rangle^{\perp}$, $P_{H'}$ the parabolic subgroup which preserves the flag $\langle u_2 \rangle \subseteq \cdots \subseteq \langle u_2, \ldots, u_r \rangle$, $A_{H'}$ the maximal torus that preserves the lines $\langle u_i \rangle (i = \pm 2, \ldots, \pm r_0)$, and $A_{H'}^+$ the positive chamber of $A_{H'}(F)$. By our induction hypothesis there exists compact subsets $C_H^{\#} \subseteq H(F)$ and $C_{H'}^{\#} \subseteq H'(F)$ such that $H(F) = C_H^{\#} A_H^+ A_{H'}^+ C_{H'}^{\#}$. Using this decomposition we can write $h = k_1^{\#} a_H a_{H'} k_2^{\#}$. We then have $g = k_1^{\#} a_H a_{H'} a(\lambda) k_2^{\#} k$. We do have that $a_{H'} a(\lambda) \in A_0^+$ but it is not necessarily in the positive chamber of P_0. We will use the inequality 2.5.1.10 to verify that $a_{H'} a(\lambda)$ is in fact in A_0^+ modulo a finite subset.

Let $a_{H',2}$ be the eigenvalue of $a_{H'}$ acting on u_2. We will show $\mathrm{val}_F(a_{H',2}) - \mathrm{val}_F(\lambda)$ is bounded by a constant. There exist positive constants c_1, c_2, c_3 and c_4 such that

- $\mathrm{val}_{R_2}(k^{\#}) \geq -c_1$ for all $k^{\#} \in C_{H'}^{\#}$
- $\mathrm{val}_{R_1}(h^{-1}) \geq \mathrm{val}_{R_1}(h) - c_2$ for all $h \in H(F)$
- $\mathrm{val}_{R_1}(hk^{\#}) \geq \mathrm{val}_{R_1}(h) - c_3$ for all $h \in H(F)$ and $k^{\#} \in C_H^{\#}$
- $\mathrm{val}_{R_1}(hw_1) \geq -\mathrm{val}_{R_1}(a_{H,1}) - c_4$ if $h = k^{\#} a_H h'$ with $k^{\#} \in C_H^{\#}, a_H \in A_H(F)^+$ and $h' \in H'(F)$ and where $a_{H,1}$ is the eigenvalue of a_H acting on u_1'.

We then have

$$\begin{aligned}
-\mathrm{val}_F(a_{H',2}) &= \mathrm{val}_{R_1}(a_{H'}^{-1} w_2) + \mathrm{val}_F(2) \\
&\geq \mathrm{val}_{R_2}((k_2^{\#})^{-1} a_{H'}^{-1} w_2) - c_1 \\
&= \mathrm{val}_{R_1}((k_2^{\#})^{-1} a_{H'}^{-1} u_1; -w_1) - c_1 \\
&= \mathrm{val}_{R_1}(h^{-1} k_1^{\#} a_H u_1' - w_1) - c_1 \\
&\geq \inf(0, \mathrm{val}_{R_1}(h^{-1} k_1^{\#} a_H u_1')) - c_1
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{val}_{R_1}(h_{-1}k_1^{\#}a_H u_1') &= \mathrm{val}_F(a_{H,1}) + \mathrm{val}_{R_1}(h^{-1}k_1^{\#}u_1') \\
&\geq \mathrm{val}_F(a_{H,1}) + \mathrm{val}_{R_1}(h) - c_2 - c_3 \\
&\geq \mathrm{val}_F(a_{H.1}) + \mathrm{val}_{R_1}(hw_1) - \mathrm{val}_F(\lambda) - c_0 - c_2 - c_3 \\
&\geq -\mathrm{val}_F(\lambda) - c_0 - c_2 - c_3 - c_4
\end{aligned}$$

We then deduce $\mathrm{val}_F(a_{H',2}) \geq -\mathrm{val}_F(\lambda) - c_0 - c_1 - c_2 - c_3 - c_4$ which is what we needed.

□

Now that we have proved the proposition 2.5.1.9 that there exists compact $C_0 \subseteq SO(V_0)(F)$ and $C_H \subseteq SO(W)(F)$ such that $SO(V_0)(F) = C_H A_H^+ A_0^+ C_0$ we will now prove proposition 2.5.1.1 which is that for the GGP pair $G_0 = SO(V_0) \times SO(W)$ and $H = SO(W)$ there exists a compact subset $\mathcal{K}_0 \subseteq G_0(F)$ such that $G_0(F) = H_0(F)A_0^+\mathcal{K}$.

To see this, let $(g,h) \in G_0 = SO(V_0) \times SO(W)$. Then

$$\begin{aligned}
(g,h) &= (h,h)(g',1) \\
&= (h,h)(k_H a_H^+ a_G^+ k_G, 1) && \text{By 2.5.1.9} \\
&= (hk_H a_H^+ a_G^+ k_G, h) \\
&= (h' a_G^+ k_G, h'(a_H^+)^{-1} k_H^{-1}) && \text{for } h' = hk_h a_h^+ \\
&= (h', h')(a_G^+, (a_H^+)^{-1})(k_G, k_H^{-1}) && \in H(F)A_G^+\mathcal{K}
\end{aligned}$$

which completes the proof.

□

2.5.2 Relative Weak Cartan decomposition for G

Let the quadruple $(\overline{P}_0, M_0A_0, A_0^+)$ be as in the previous section. Denote $\overline{P} = M\overline{N}$ the parabolic subgroup opposite to P with respect to M and define the following subgroups of G:

$$A_{min} = A_0 A \subseteq M_{min} = M_0 T \subseteq \overline{P}_{min} = \overline{P}_0 T \overline{N}$$

Then, $\overline{P}_{min}$ is a parabolic subgroup, M_{min} is a Levi component of it and A_{min} is the maximal split central subtorus of M_{min}. Moreover, it is easy to see that $\overline{P}_{min}$ is a good parabolic subgroup of G. Set

$$A_{min}^+ = \{a \in A_{min}(F) : |\alpha(a)| \geq 1 \forall \alpha \in R(A_{min}, \overline{P}_{min})\}$$

We will denote P_{min} for the parabolic subgroup opposite to $\overline{P}_{min}$ with respect to M_{min}. We have $P_{min} \subseteq P$. Let Δ be the set of simple roots of A_{min} in P_{min} and $\Delta_P = \Delta \cap R(A_{min}, N)$ be the subset of simple roots appearing in $\mathfrak{n} = \mathrm{Lie}(N)$. For $\alpha \in \Delta_P$, we will denote by $\mathfrak{n}_\alpha$ the corresponding root subspace. Recall als the we have defined in Section 2.1 a character ξ of $\mathfrak{n}(F)$.

Lemma 2.5.2.1 (6.6.2) *We have the following*

(i)
$$A^+_{min} = \{a \in A^+_0 A(F) : |\alpha(a)| \le 1 \forall \alpha \in \Delta_P\}$$

(ii) There exists a compact subset $\mathcal{K}_G \subseteq G(F)$ such that
$$G(F) = H(F)A^+_0 A(F)\mathcal{K}_G$$

(iii) For all $\alpha \in \Delta_P$, the restriction of ξ to $\mathfrak{n}_\alpha(F)$ is nontrivial.

Proof. (i) is obvious,

(ii) Let K be a maximal compact subgroup of $G(F)$ which is special in the p-adic case. Then we have Iwasawa decomposition

2.5.2.2)
$$G(F) = P(F)K = N(F)G_0(F)T(F)K$$

Since $A = A_T$ is the maximal split subtorus of T, there exists a compact subset $\mathcal{K}_T \subseteq T(F)$ such that

2.5.2.3)
$$T(F) = A(F)\mathcal{K}_T$$

Also by proposition 2.5.1.1, we know there exists a compact subset $\mathcal{K}_0 \subseteq G_0(F)$ such that

2.5.2.4)
$$G_0(F) = H_0(F)A^+_0\mathcal{K}_0$$

Combining 2.5.2.2, 2.5.2.3, and 2.5.2.4, and since A and G_0 centralize each other, we get
$$G(F) = H(F)A^+_0 A(F)\mathcal{K}_G$$
where $\mathcal{K}_G = \mathcal{K}_0\mathcal{K}_T\mathcal{K}$.

(iii) Let $\alpha \in \Delta_P$ and assume, for a contradiction, that ξ is trivial when restricted to $\mathfrak{n}_\alpha(F)$. Recall that ξ is the composition $\xi = \psi \circ \lambda_F$ where λ is an algebraic additive character $\mathfrak{n} \to \mathbb{G}_a$. Since $\mathfrak{n}_\alpha$ is a linera subspace of $\mathfrak{n}$, the condition that ξ is trivial on $\mathfrak{n}_\alpha(F)$ is equivalent to λ being trivial on $\mathfrak{n}_\alpha$. Since λ is invariant by H_0 conjugation and $\mathfrak{n}_\alpha$ is invariant by both T-conjugation and $\overline{P}_0$-conjugation, it follows that λ is trivial on $m\,\mathfrak{n}_\alpha\,m^{-1}$ for all $m \in H_0\overline{P}_0T$. But $\overline{P}_0$ being a good parabolic subgroup of G_0 means $H_0\overline{P}_0T$ is Zariski-dense in $M = G_0T$. Hence, λ is trivial on $m\,\mathfrak{n}_\alpha\,m^{-1}$ for all $m \in M$. This is a contradiction of Lemma 2.1.2(ii) since $\mathfrak{n}_\alpha$ is not included in $[\mathfrak{n}, \mathfrak{n}]$.

□

2.6 The Function $\Xi^{H\backslash G}$

Let $C \subseteq G(F)$ be a compact subset with nonempty interior. We define a function $\Xi_C^{H\backslash G}$ on $H(F)\backslash G(F)$ by

$$\Xi_C^{H\backslash G}(x) = \mathrm{vol}_{H\backslash G}(xC)^{-1/2}$$

for all $x \in H(F)\backslash G(F)$. It is not hard to see that if $C' \subseteq G(F)$ is another compact subset with nonempty interior, we have

$$\Xi_C^{H\backslash G}(x) \sim \Xi_{C'}^{H\backslash G}(x)$$

for all $x \in H(F)\backslash G(F)$. From now on we will assume an implicitly fixed compact subset with nonempty interior $C \subseteq G(F)$ and we will set

$$\Xi^{H\backslash G}(x) = \Xi_C^{H\backslash G}(x)$$

for all $x \in H(F)\backslash G(F)$. The precise choice of C wont matter because the function $\Xi^{H\backslash G}$ will only be used for the purpose of estimates.

Proposition 2.6.1 (6.7.1) *(i) For every compact subset $\mathcal{K} \subseteq G(F)$, we have the following equivalences of functions*

(a)

$$\Xi^{H\backslash G}(xk) \sim \Xi^{H\backslash G}(x)$$

(b)

$$\sigma_{H\backslash G}(xk) \sim \sigma^{H\backslash G}(x)$$

for all $x \in H(F)\backslash G(F)$ and all $k \in \mathcal{K}$.

(ii) Let $\overline{P}_0 = M_0\overline{U}_0 \subseteq G_0$ be a good minimal parabolic subgroup of G_0 and $A_0 = A_{M_0}$ be the split part of the center of M_0. Set

$$A_0^+ = \{a_0 \in A_0(F) : |\alpha(a_0)| \geq 1 \, \forall \alpha \in R(A_0, \overline{P}_0\}$$

Then there exists a positive constant $d > 0$ such that

(a)

$$\Xi^{G_0}(a_0)\delta_P(a)^{1/2}\sigma(a_0)^{-d} \ll \Xi^{H\backslash G}(aa_0) \ll \Xi^{G_0}(a_0)\delta_P(a)^{1/2}$$

(b)

$$\sigma_{H\backslash G}(aa_0) \sim \sigma_G(aa_0)$$

for all $a_0 \in A_0^+$ and all $a \in A(F)$.

(iii) There exists $d > 0$ such that the integral

$$\int_{H(F)\backslash G(F)} \Xi^{H\backslash G}(x)^2 \sigma_{H\backslash G}(x)^{-d} dx$$

is absolutely convergent

(iv) *For all $d > 0$, there exists $d' > 0$ such that*

$$\int_{H(F)\backslash G(F)} \mathbb{1}_{\sigma_{H\backslash G}\leq c}(x)\Xi^{H\backslash G}(x)^2\sigma_{H\backslash G}(x)^d dx \ll c^{d'}$$

for all $c \geq 1$.

(v) *There exists $d > 0$ and $d' > 0$ such that*

$$\int_{H(F)} \Xi^G(x^{-1}hx)\sigma_G(x^{-1}hx)^{-d} \ll \Xi^{H\backslash G}(x)^2\sigma_{H\backslash G}(x)^{d'}$$

for all $x \in H(F)\backslash G(F)$.

(vi) *For all $d > 0$, there exists $d' > 0$ such that*

$$\int_{H(F)} \Xi^G(hx)\sigma(hx)^{-d'}dh \ll \Xi^{H\backslash G}(x)\sigma_{H\backslash G}(x)^{-d}$$

for all $x \in H(F)\backslash G(F)$.

(vii) *Let $\delta > 0$ and $d > 0$. Then, the integral $I_{\delta,d}(c,x) =$*

$$\int_{H(F)}\int_{H(F)} \mathbb{1}_{\sigma\geq c}(h')\Xi^G(hx)\Xi^G(h'hx)\sigma(hx)^d\sigma(h'hx)^d(1+|\lambda(h')|)^{-\delta}dh'dh$$

is absolutely convergent for all $x \in H(F)\backslash G(F)$ and all $c \geq 1$. Moreover, there exist $\epsilon > 0$ and $d' > 0$ such that

$$I_{\delta,d}(c,x) \ll \Xi^{H\backslash G}(x)^2\sigma_{H\backslash G}(x)^{d'}e^{-\epsilon c}$$

for all $x \in H(F)\backslash G(F)$ and all $c \geq 1/$

Proof. Same as ref □

2.7 Parabolic degenerations

Let $\overline{Q} = LU_{\overline{Q}}$ be a good parabolic subgroup of G (ie that $H\overline{Q}$ is Zariski open in G, see section 2.3). Let $\overline{P}_{min} = M_{min}\overline{U}_{min} \subseteq \overline{Q}$ be a good minimal parabolic subroup of G (Proposition 2.3.2(ii)) with the Levi component chosen so that $M_{min} \subseteq L$. Let $A_{min} = A_{M_{min}}$ be the maximal central split torus of M_{min} and set

$$A^+_{min} = \{a \in A_{min}(F) : |\alpha(a)| \geq 1 \forall \in R(A_{min}, \overline{P}_{min})\}$$

Let $H_{\overline{Q}} = H \cap \overline{Q}$ and H_L be the image of $H_{\overline{Q}}$ by the natural projection $H_{\overline{Q}} \twoheadrightarrow L$. Let $Q = LU_Q$ be the parabolic subgroup opposite to $\overline{Q}$ with respect to L. We define $H^Q = H_L \ltimes U_Q$.

Proposition 2.7.1 (6.8.1) *(i)* *$H_{\overline{Q}} \cap U_{\overline{Q}} = \{1\}$ so that the natural projection $H_{\overline{Q}} \to H_L$ is an isomorphism.*

(ii) $\delta_{\overline{Q}}(h_{\overline{Q}}) = \delta_{H_{\overline{Q}}}(h_{\overline{Q}})$ *and* $\delta_{\overline{Q}}(h_L) = \delta_{H_L}(h_L)$ *for all* $h_{\overline{Q}} \in H_{\overline{Q}}(F)$ *and all* $h_L \in H_L(F)$. *In particular,the group* $H^Q(F)$ *is unimodular.*

Fix a left Haar measure $d_L h_L$ *on* $H_L(F)$ *and a Haar measure* dh^Q *on* $H^Q(F)$.

(iii) There exists $d > 0$ *such that the integral*

$$\int_{H_L(F)} \Xi^L(h_L)\sigma(h_L)^{-d}\delta_{H_L}(h_L)^{1/2}d_L h_L$$

converges. Moreover, in the codimension one case (that is $G = G_0$ *and* $H = H_0$, *the integral*

$$\int_{H_L(F)} \Xi^L(h_L)\sigma(h_L)^{d}\delta_{H_L}(h_L)^{1/2}d_L h_L$$

is convergent for all $d > 0$.

(iv) There exists $d > 0$ *such that the integral*

$$\int_{H^Q(F)} \Xi^G(h^Q)\sigma(h^Q)^{-d}dh^Q$$

converges.

(v) We have $\sigma(h^Q) \ll \sigma(a^{-1}h^Q a)$ *for all* $a \in A^+_{min}$ *and all* $h^Q \in H^Q(F)$.

(vi) There exists $d > 0$ *and* $d' > 0$ *such that*

$$\int_{H^Q(F)} \Xi^G(a^{-1}h^Q a_\sigma(a^{-1}h^Q a)^{-d}dh^Q \ll H^{H\backslash G}(a)^2\sigma_{H\backslash G}(a)^{d'}$$

for all $a \in A^+_{min}$.

Proof. (i) The follows directly from Proposition 2.3.2(i)

(ii) For $h_{\overline{Q}} \in H_{\overline{Q}}(F)$ which maps to $h_L \in H_L(F)$ via the isomorphism $H_{\overline{Q}} \simeq H_L$, we have $\delta_{\overline{Q}}(h_{\overline{Q}}) = \delta_{\overline{Q}}(h_L)$ and $\delta_{H_{\overline{Q}}}(h_{\overline{Q}}) = \delta_{H_L}(h_L)$. Thus, it suffices to show that $\delta_{\overline{Q}}(h_{\overline{Q}}) = \delta_{H_{\overline{Q}}}(h_{\overline{Q}})$ or equivalently

2.7.2)
$$\det\left(\mathrm{Ad}(h_{\overline{Q}})\Big|_{\overline{\mathfrak{q}}/\mathfrak{h}_{\overline{Q}}}\right) = 1$$

for all $h_{\overline{Q}} \in H_{\overline{Q}}(F)$. We have $\overline{\mathfrak{q}}+\mathfrak{h} = \mathfrak{g}$ (because $\overline{Q}$ is a good parabolic subgroup) and $\mathfrak{h}_{\overline{Q}} = \mathfrak{h}\cap\overline{\mathfrak{q}}$, hence the inclusion $\overline{\mathfrak{q}} \subseteq \mathfrak{g}$ induces an isomorphism $\overline{\mathfrak{q}}/\,\mathfrak{h}_{\overline{Q}} \simeq \mathfrak{g}/\,\mathfrak{h}$ from which it follows that

$$\begin{aligned}\det\left(\mathrm{Ad}(h_{\overline{Q}})\Big|_{\overline{\mathfrak{q}}/\mathfrak{h}_{\overline{Q}}}\right) &= \det\left(\mathrm{Ad}(h_{\overline{Q}})\Big|_{\mathfrak{g}/\,\mathfrak{h}}\right) \\ &= \det\left(\mathrm{Ad}(h_{\overline{Q}})\Big|_{\mathfrak{g}}\right)\det\left(\mathrm{Ad}(h_{\overline{Q}})\Big|_{\mathfrak{h}}\right)^{-1}\end{aligned}$$

forall $h_{\overline{Q}} \in H_{\overline{Q}}(F)$. But since G and H are unimodular groups, we have that $\det\left(\mathrm{Ad}(h_{\overline{Q}})\Big|_{\mathfrak{g}}\right) = \det\left(\mathrm{Ad}(h_{\overline{Q}})\Big|_{\mathfrak{h}}\right)^{-1} = 1$ for all $h_{\overline{Q}} \in H_{\overline{Q}}$ and 2.7.2 follows.

(iii) Let K be a maximal compact subgroup of $G(F)$ which is special in good position with respect to L in the p-adic case. Set $K_L = K \cap L(F)$ (a maximal compact subgroup of $L(F)$ which is special in the p-adic case), $\tau = i_{P_{min}\cap L}^{L}(1)$ and $\pi = i_{\overline{Q}}^{G}(\tau)$. We will denote by $(-,-)$ and $(-,-)_\tau$ invariant scalar products on π and τ respectively. Let $e_K \in \pi^\infty$ and $e_{K_L} \in \tau^\infty$ be the unique K-fixed and K_L-fixed vectors respectively. Note that we have $e_K(k) = e_{K_L}$ for all $k \in K$. We may assume that the functions ξ^G and Ξ^L are given by

2.7.3)
$$\Xi^G(g) = (\pi(g)e_K, e_K)\,,\; g \in G(F)$$

2.7.4)
$$\Xi^L(l) = (\tau(l)e_{K_L}, e_{K_L})_\tau\,,\; l \in L(F)$$

(note that by the process of induction by stages, we have a natural isomorphism $\pi \simeq i_{P_{min}}^{G}(1)$). If we choose Haar measures suitably, 2.7.3 gives

$$\Xi^G(g) = \int_{\overline{Q}(F)\backslash G(F)} (e_K(g'g), e_K(g'))_\tau dg'$$

for all $g \in G(F)$. Since $\overline{Q}$ is a good parabolic subgroup, by Proposition 2.3.2(i) (and since $g \mapsto g^{-1}$ is an automorphism of G) the subset $H_{\overline{Q}}(F)\backslash H(F) \subseteq \overline{Q}(F)\backslash G(F)$ has a complement which is negligible. Hence, by (ii), if we choose Haar measures compatibly, we have

□

3 Explicit tempered intertwinings

We will keep the notation from the previous chapter. Given a tempered representation π of $G(F)$, this chapter studies a certain explicit $(H,\xi)\times(H,\xi)$-equivariant sesquiliniear form $\mathcal{L}_\pi$ on (the space of) π, the main result being that $\mathcal{L}_\pi$ is nonzero if and only if the multiplicity $m(\pi)$ is nonzero (Theorem **??**). This will be used in the proof of the spectral side of our local trace formula (Theorem **??**). The sesquilinear form $\mathcal{L}_\pi$ is introduced in ...

3.1 The ξ-integral

For all $f \in \mathcal{C}(G(F))$, the integral

$$\int_{H(F)} f(h)\xi(h)dh$$

is absolutely convergent by Lemma 2.4.1(ii). I presume bc f is bounded by Ξ and ξ is bounded by σ^{-d} Moreover, by lemma 2.4.1(ii) again, this defines a continuous linear form on $\mathcal{C}(G(F))$. Recall that $\mathcal{C}(G(F))$ is a dense subspace of the weak Harish-Chandra Schwartz space $\mathcal{C}^w(G(F))$ (by 1.3.1). have not defined yet.

Proposition 3.1.1 (7.1.1) *The linear form*

$$\mathcal{C}(G(F)) \ni f \mapsto \int_{H(F)} f(h)\xi(h)dh$$

extends continuously to $\mathcal{C}^w(G(G))$.

Proof. Let us fix a one-parameter subgroup $a : \mathbb{G}_m \to A$ such that $\lambda(a(t)ha(t)^{-1}) = t\lambda(h)$ for all $t \in \mathbb{G}_m$ and all $h \in H$ (recall that $\lambda : H \to \mathbb{G}_a$ is the algebraic character such that $\xi = \psi \circ \lambda_F$), such a one-parameter subgroup is easy to construct. We shall now divide the proof into the p-adic and archemedian cases.

- If F is a p-adic field, then we may fix a compact open subgroup $K \subseteq G(F)$ and prove that the linear form

  $$f \in \mathcal{C}_K(G(F)) \mapsto \int_{H(F)} f(h)\xi(h)dh$$

  extends continuously to $C^w_K(G(F))$. Set $K_a = a^{-1}(K \cap A(F)) \subseteq F^\times$. how to define a^{-1} Then for all $f \in \mathcal{C}_K(G(F))$, we have

  3.1.2)

  $$\begin{aligned}\int_{H(F)} f(h)\xi(h)dh &= \mathrm{meas}(K_a)^{-1}\int_{K_a}\int_{H(F)} f(a(t)^{-1}ha(t))\xi(h)dhd^\times t\\ &= \mathrm{meas}(K_a)^{-1}\int_{H(F)} f(h)\int_{K_a}\xi(a(t)ha(t)^{-1})d^\times tdh\\ &= \mathrm{meas}(K_a)^{-1}\int_{H(F)} f(h)\int_{K_a}\psi(t\lambda(h))|t|^{-1}dtdh\end{aligned}$$

  The function $F \ni x \mapsto \int_{K_a}\psi(tx)|t|^{-1}dt$ is the Fourier transform of the function $|\bullet|^{-1}\mathbb{1}_{K_a} \in C^\infty_C(F)$ hence it belongs to $C^\infty_c(F)$. Now by lemma 2.4.1(iii), the last integral of 3.1.2 is absolutely convergent for all $f \in C^w_K(G(F))$ and defines a continuous linear form on that space. This is the extension we were looking for.

- Now assume that $F = \mathbb{R}$. Let us denote Ad the adjoint action of $G(F)$ on $C^w(G(F))$ i.e., one has

$$(\mathrm{Ad}(g)f)(x) = f(g^{-1}xg),\ f \in C^w(G(F)),\ g, x \in G(F)$$

Set $\mathrm{Ad}_a = \mathrm{Ad} \circ a$. Then Ad_a is a smooth representation sanity check, does it matter that this is in center? of $F^\times$ on $C^w(G(F))$ and hence induces an action, also denoted by Ad_a, of $\mathcal{U}$ (Lie$(F^\times)$) on $\mathcal{C}^w(G(F))$. Set $\Delta = 1 - (t\frac{d}{dt})^2 \in \mathcal{U}(\mathrm{Lie}(F^\times))$. By elliptic regularity (2.1.2), for every integer $m \geq 1$, there exists functions $\varphi_1 \in C_c^{2m-2}(F^\times)$ and $\varphi_2 \in C_c^\infty(F^\times)$ such that

$$\varphi_1 * \Delta^m + \varphi_2 = \delta_1$$

Hence, we have the equality

$$\mathrm{Ad}_a(\varphi_1)\,\mathrm{Ad}_a(\Delta^m) + \mathrm{Ad}_a(\varphi_2) = Id$$

It follows that for all $f \in \mathcal{C}(G(F))$, we have

3.1.3)

$$\begin{aligned}
\int_{H(F)} f(h)\xi(h)dh &= \int_{H(F)} (\mathrm{Ad}_a(\varphi_1)\,\mathrm{Ad}_a(\Delta^m)f)(h)\xi(h)dh \\
&+ \int_{H(F)} (\mathrm{Ad}_a(\varphi_2)f)(h)\xi(h)dh \\
&= \int_{H(F)} (\mathrm{Ad}_a(\Delta^m)f)(h) \int_{F^\times} \varphi_1(t)\xi(a(t)ha(t)^{-1})\delta_P(a(t))d^\times t dh \\
&+ \int_{H(F)} f(h) \int_{F^\times} \varphi_2(t)\xi(a(t)ha(t)^{-1})\delta_P(a(t))d^\times t dh \\
&= \int_{H(F)} (\mathrm{Ad}_a(\Delta^m)f)(h) \int_F \varphi_1(t)\delta_P(a(t))|t|^{-1}\psi(t\lambda(h))dtdh \\
&+ \int_{H(F)} f(h) \int_F \varphi_2(t)\delta_P(a(t))|t|^{-1}\psi(t\lambda(h))dtdh
\end{aligned}$$

Consider the functions $f_i : x \mapsto \int_F \varphi_i(t)\delta_P(a(t))|t|^{-1}\psi(tx)dt$, $i = 1, 2$, $x \in F$. These are the Fourier transforms of the functions $t \mapsto \varphi_i(t)\delta_H(a(t))|t|^{-1}$, $i = 1, 2$, which both belong to $C_c^{2m-2}(F)$. Hence, f_1 and f_2 are both essentially bounded by $(1 + |x|)^{-2m+2}$. Now, by Lemma 2.4.1(iii), if $m \geq 2$ the two integrals in the last term of 3.1.3 are absolutely convergent for all $f \in C^w(G(F))$ and define on the space continuous linear forms. The extension we were looking for is just the sum of these two integrals.

□

The continuous linear form on $C^w(G(F))$ whose existence is proved by the proposition above will be called the ξ-*integral* on $H(F)$ and will be denoted by

$$C^w(G(F)) \ni f \mapsto \int_{H(F)}^* f(h)\xi(h)dh$$

or

$$C^w(G(F)) \ni f \mapsto \mathcal{P}_{H,\xi}(f)$$

We now note the following properties of the ξ-integral.

Lemma 3.1.4 *(i) For all $f \in \mathcal{C}^w(G(F))$ and all $h_0, h_1 \in H(F)$, we have*

$$\mathcal{P}_{H,\xi}(L(h_0))R(h_1)f) = \xi(h_0)\xi(h_1)^{-1}\mathcal{P}_{H,\xi}(f)$$

(ii) Let $a : \mathbb{G} \to A$ be a one-parameter subgroup such that $\lambda(a(t)ha(t)^{-1}) = t\lambda(h)$ for all $t \in \mathbb{G}_m$ and all $h \in H$. Denote by Ad_a the representation of $F^\times$ on $C^w(G(F))$ given by $\mathrm{Ad}_a(t) = L(a(t))R(a(t))$ for all $t \in F^\times$. Let $\varphi \in C_c^\infty(F^\times)$. Set $\varphi'(t) = |t|^{-1}\delta_P(a(t))\varphi(t)$ for all $t \in F^\times$ and denote $\widehat{\varphi}'$ its Fourier transform, that is

$$\widehat{\varphi}'(x) = \int_F \varphi'(t)\psi(tx)dt, \; x \in F$$

Then, we have

$$\mathcal{P}_{H,\xi}(\mathrm{Ad}_a(\varphi)f) = \int_{H(F)} f(h)\widehat{\varphi}'(\lambda(h))dh$$

for all $f \in \mathcal{C}^w(G(F))$, where the second integral is absolutely convergent.

Proof. In both (i) and (ii), both side of the equality to be proved are continuous in $f \in C^w(G(F))$ (for (ii) this follows from Lemma 2.4.1(iii)). Hence it is sufficient to check the relations for $f \in \mathcal{C}(G(F))$ where by straightforward variable changes we can pass from the left hand side to the right hand side. $\square$

3.2 Definition of $\mathcal{L}_\pi$

Let π be a tempered representation of $G(F)$. For all $T \in \mathrm{End}(\pi)^\infty$, the function

$$G(F) \ni g \mapsto \mathrm{Tr}(\pi(g^{-1})T)$$

belongs to the weak Harish-Chandra Schwartz space $\mathcal{C}^w(G(F))$ by 2.2.4. We can thus define a linear form $\mathcal{L}_\pi : \mathrm{End}(\pi)^\infty \to \mathbb{C}$ by setting

$$\mathcal{L}_\pi(T) = \int_{H(F)}^* \mathrm{Tr}(\pi(h^{-1})T)\xi(h)dh, \; T \in \mathrm{End}(\pi)^\infty$$

By Lemma 3.1.3(i), we have

$$\mathcal{L}_\pi(\pi(h)T\pi(h')) = \xi(h)\xi(h')\,\mathcal{L}_\pi(T)$$

for all $h, h' \in H(F)$ and $T \in \mathrm{End}(\pi)^\infty$. By 2.2.5, the map which associates to $T \in \mathrm{End}(\pi)^\infty$ the function

$$g \mapsto \mathrm{Tr}(\pi(g^{-1})T)$$

in $\mathcal{C}^w$ in $C^w(G(F))$ is continuous. Since the ξ-integral is a continuous linear form on $\mathcal{C}^w(G(F))$, it follows that the linear form $\mathcal{L}_\pi$ is continuous.

Recall that we have a continuous $G(F) \times G(F)$-equivariant embedding with dense image $\pi^\infty \otimes \overline{\pi^\infty} \hookrightarrow \mathrm{End}(\pi)^\infty$, $e \otimes e' \mapsto T_{e,e'}$ (which is an isomorphism in the p-adic case). Where $T_{e,e'}(e_0) = (e_0, e')e$ for all $e_0 \in \pi$. In any case, $\mathrm{End}(\pi)^\infty$ is naturally isomorphic to the completed projective tensor product $\pi^\infty \widehat{\otimes}_p \overline{\pi^\infty}$. Thus we may identify $\mathcal{L}_\pi$ with the continuous sesquilinear not bilinear right? form on π^∞ given by

$$\mathcal{L}_\pi(e, e') := \mathcal{L}_\pi(T_{e,e'})$$

for all $e, e' \in \pi^\infty$. Expanding definitions, we have

$$\mathcal{L}_\pi(e, e') = \int_{H(F)}^{*} (e, \pi(h)e')\xi(h)dh$$

how?? for all $e, e' \in \pi^\infty$. Fixing $e' \in \pi^\infty$, we see that the map $\pi^\infty \ni e \mapsto \mathcal{L}_\pi(e, e')$ belongs to $\mathrm{Hom}_H(\pi^\infty, \xi)$. By the density of $\pi^\infty \otimes \overline{\pi}^\infty$ in $\mathrm{End}(\pi)^\infty$, it follows that

$$\mathcal{L}_\pi \neq 0 \implies m(\pi) \neq 0$$

The purpose of this chapter is to prove the converse direction. Namely, we will show

Theorem 3.2.1 (7.2.1) *For all $\pi \in \mathrm{Temp}(G)$, we have*

$$\mathcal{L}_\pi \neq 0 \iff m(\pi) \neq 0$$