Sums

ITT9130 Konkreetne Matemaatika

Contents

Next section

Sequences

Definition

A sequence of elements of a set A is any function $f : \mathbb{N} \to A$, where $\mathbb{N}$ is set of natural numbers.

Notations used:

- $f = \{a_n\}$, where $a_n = f(n)$
- $\{a_n\}_{n \in \mathbb{N}}$
- $\{a_n\}$

a_n is called n-th term of a sequence f

Sequences

Definition

A sequence of elements of a set A is any function $f : \mathbb{N} \to A$, where $\mathbb{N}$ is set of natural numbers.

Notations used:

- $f = \{a_n\}$, where $a_n = f(n)$
- $\{a_n\}_{n\in\mathbb{N}}$
- $\{a_n\}$

a_n is called n-th term of a sequence f

Sequences

Definition

A sequence of elements of a set A is any function $f : \mathbb{N} \to A$, where $\mathbb{N}$ is set of natural numbers.

Notations used:

- $f = \{a_n\}$, where $a_n = f(n)$
- $\{a_n\}_{n \in \mathbb{N}}$
- $\{a_n\}$

a_n is called *n*-th term of a sequence f

Sequences

Definition

A sequence of elements of a set A is any function $f : \mathbb{N} \to A$, where $\mathbb{N}$ is set of natural numbers.

Notations used:

- $f = \{a_n\}$, where $a_n = f(n)$
- $\{a_n\}_{n\in\mathbb{N}}$
- $\{a_n\}$

a_n is called *n-th term* of a sequence f

Example

$$a_0 = 0,\ a_1 = \frac{1}{2\cdot 3},\ a_2 = \frac{2}{3\cdot 4},\ a_3 = \frac{3}{4\cdot 5},\cdots$$

or

$$\left\langle 0,\ \frac{1}{6},\ \frac{1}{6},\ \frac{3}{20},\ \frac{2}{15},\cdots,\ \frac{n}{(n+1)(n+2)},\cdots \right\rangle$$

Sequences

Definition

A sequence of elements of a set A is any function $f : \mathbb{N} \to A$, where $\mathbb{N}$ is set of natural numbers.

Notations used:

- $f = \{a_n\}$, where $a_n = f(n)$
- $\{a_n\}_{n \in \mathbb{N}}$
- $\{a_n\}$

a_n is called *n*-th term of a sequence f

Notation

$$f(n) = \frac{n}{(n+1)(n+2)}$$

or

$$a_n = \frac{n}{(n+1)(n+2)}$$

Sets of indexes

- $\mathbb{N}$ – set of indexes of the sequence $f = \{a_n\}_{n\in\mathbb{N}}$
- Any countably infinite set can be used for index. Examples of other frequently used indexes are:
  - $\mathbb{N}^+ = \mathbb{N} - \{0\} \sim \mathbb{N}$
  - $\mathbb{N} - K \sim \mathbb{N}$, where K is any finite subset of $\mathbb{N}$
  - $\mathbb{Z} \sim \mathbb{N}$
  - $\{1, 3, 5, 7, \dots\} = ODD \sim \mathbb{N}$
  - $\{0, 2, 4, 6, \dots\} = EVEN \sim \mathbb{N}$

Sets of indexes

- $\mathbb{N}$ – set of indexes of the sequence $f = \{a_n\}_{n\in\mathbb{N}}$
- Any countably infinite set can be used for index. Examples of other frequently used indexes are:
  - $\mathbb{N}^+ = \mathbb{N} - \{0\} \sim \mathbb{N}$
  - $\mathbb{N} - K \sim \mathbb{N}$, where K is any finite subset of $\mathbb{N}$
  - $\mathbb{Z} \sim \mathbb{N}$
  - $\{1, 3, 5, 7, \ldots\} = ODD \sim \mathbb{N}$
  - $\{0, 2, 4, 6, \ldots\} = EVEN \sim \mathbb{N}$

Sets of indexes

- $\mathbb{N}$ – set of indexes of the sequence $f = \{a_n\}_{n\in\mathbb{N}}$
- Any countably infinite set can be used for index. Examples of other frequently used indexes are:
  - $\mathbb{N}^+ = \mathbb{N} - \{0\} \sim \mathbb{N}$
  - $\mathbb{N} - K \sim \mathbb{N}$, where K is any finite subset of $\mathbb{N}$
  - $\mathbb{Z} \sim \mathbb{N}$
  - $\{1,3,5,7,\dots\} = ODD \sim \mathbb{N}$
  - $\{0,2,4,6,\dots\} = EVEN \sim \mathbb{N}$

$A \sim B$ denotes that sets A and B are of the same cardinality, i.e. $|A| = |B|$.

Sets of indexes

- $\mathbb{N}$ – set of indexes of the sequence $f = \{a_n\}_{n\in\mathbb{N}}$
- Any countably infinite set can be used for index. Examples of other frequently used indexes are:
  - $\mathbb{N}^+ = \mathbb{N} - \{0\} \sim \mathbb{N}$
  - $\mathbb{N} - K \sim \mathbb{N}$, where K is any finite subset of $\mathbb{N}$
  - $\mathbb{Z} \sim \mathbb{N}$
  - $\{1,3,5,7,\dots\} = ODD \sim \mathbb{N}$
  - $\{0,2,4,6,\dots\} = EVEN \sim \mathbb{N}$

Two sets A and B have the same cardinality if there exists a bijection, that is, an injective and surjective function, from A to B.

(See

`http://www.mathsisfun.com/sets/injective-surjective-bijective.html`

for detailed explanation)

Finite sequence

- Finite sequence of elements of a set A is a function $f : K \to A$, where K is set a finite subset of natural numbers

  For example: $f : \{1, 2, 3, 4, \cdots, n\}\, n \to A,\ n \in \mathbb{N}$

  Special case: $n = 0$, i.e. empty sequence: $f(\emptyset) = e$

Domain of the sequence

$$f : T \to A$$

$$a_n = \frac{n}{(n-2)(n-5)}$$

Domain of f is $T = \mathbb{N} - \{2, 5\}$

Next section

Notation

For a finite set $K = \{1, 2, \cdots, m\}$ and a given sequence $f : K \to \mathbb{R}$ with $f(n) = a_n$ we write

$$\sum_{k=1}^{m} a_k = a_1 + a_2 + \cdots + a_m$$

Alternative notations

$$\sum_{k=1}^{m} a_k = \sum_{1 \leqslant k \leqslant m} a_k = \sum_{k \in \{1, \cdots, m\}} a_k = \sum_{K} a_k$$

Notation

For a finite set $K = \{1, 2, \cdots, m\}$ and a given sequence $f : K \to \mathbb{R}$ with $f(n) = a_n$ we write

$$\sum_{k=1}^{m} a_k = a_1 + a_2 + \cdots + a_m$$

Alternative notations

$$\sum_{k=1}^{m} a_k = \sum_{1 \leqslant k \leqslant m} a_k = \sum_{k \in \{1, \cdots, m\}} a_k = \sum_{K} a_k$$

Next section

Sums and Recurrences

Computation of any sum

$$S_n = \sum_{k=1}^{n} a_k$$

can be presented in the recursive form:

$$\begin{aligned} S_0 &= a_0 \\ S_n &= S_{n-1} + a_n \end{aligned}$$

$\Rightarrow$ Techniques from CHAPTER ONE can be used for finding closed formulas for evaluating sums.

Recalling repertoire method (presented by Silvio last time)

- Given

$$\begin{aligned} g(0) &= \alpha \\ g(n) &= \Phi(g(n-1)) + \Psi(\beta,\gamma,\ldots) \qquad \text{for } n>0. \end{aligned}$$

  where Φ and Ψ are linear, for example if $g(n) = \lambda_1 g_1(n) + \lambda_2 g_2(n)$ then $\Phi(g(n)) = \lambda_1 \Phi(g_1(n)) + \lambda_2 \Phi(g_2(n))$.

- Closed form is

$$g(n) = \alpha A(n) + \beta B(n) + \gamma C(n) + \cdots \tag{1}$$

- Functions $A(n), B(n), C(n), \ldots$ could be found from the system of equations

$$\begin{aligned} \alpha_1 A(n) + \beta_1 B(n) + \gamma_1 C(n) + \cdots &= g_1(n) \\ \vdots \qquad\qquad &= \vdots \\ \alpha_m A(n) + \beta_m B(n) + \gamma_m C(n) + \cdots &= g_m(n) \end{aligned}$$

  where $\alpha_i, \beta_i, \gamma_i \cdots$ are constants committing (1) and recurrence relationship for the repertoire case $g_i(n)$ and any n.

Recalling repertoire method (presented by Silvio last time)

- Given

$$
\begin{aligned}
g(0) &= \alpha \\
g(n) &= \Phi(g(n-1)) + \Psi(\beta, \gamma, \ldots) \qquad \text{for } n > 0.
\end{aligned}
$$

  where Φ and Ψ are linear, for example if $g(n) = \lambda_1 g_1(n) + \lambda_2 g_2(n)$ then $\Phi(g(n)) = \lambda_1 \Phi(g_1(n)) + \lambda_2 \Phi(g_2(n))$.

- Closed form is

$$
g(n) = \alpha A(n) + \beta B(n) + \gamma C(n) + \cdots \tag{1}
$$

- Functions $A(n), B(n), C(n), \ldots$ could be found from the system of equations

$$
\begin{aligned}
\alpha_1 A(n) + \beta_1 B(n) + \gamma_1 C(n) + \cdots &= g_1(n) \\
\vdots \qquad\qquad &= \vdots \\
\alpha_m A(n) + \beta_m B(n) + \gamma_m C(n) + \cdots &= g_m(n)
\end{aligned}
$$

  where $\alpha_i, \beta_i, \gamma_i \cdots$ are constants committing (1) and recurrence relationship for the repertoire case $g_i(n)$ and any n.

Recalling repertoire method (presented by Silvio last time)

- Given
$$\begin{aligned} g(0) &= \alpha \\ g(n) &= \Phi(g(n-1)) + \Psi(\beta,\gamma,\ldots) \qquad \text{for } n > 0. \end{aligned}$$

  where Φ and Ψ are linear, for example if $g(n) = \lambda_1 g_1(n) + \lambda_2 g_2(n)$ then $\Phi(g(n)) = \lambda_1 \Phi(g_1(n)) + \lambda_2 \Phi(g_2(n))$.
- Closed form is
$$g(n) = \alpha A(n) + \beta B(n) + \gamma C(n) + \cdots \tag{1}$$
- Functions $A(n), B(n), C(n), \ldots$ could be found from the system of equations
$$\begin{aligned} \alpha_1 A(n) + \beta_1 B(n) + \gamma_1 C(n) + \cdots &= g_1(n) \\ \vdots \qquad\qquad &= \vdots \\ \alpha_m A(n) + \beta_m B(n) + \gamma_m C(n) + \cdots &= g_m(n) \end{aligned}$$

  where $\alpha_i, \beta_i, \gamma_i \cdots$ are constants committing (1) and recurrence relationship for the repertoire case $g_i(n)$ and any n.

Next subsection

Example 1: arithmetic sequence

Arithmetic sequence: $a_n = a + bn$

Recurrent equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n$:

$$S_0 = a$$
$$S_n = S_{n-1} + (a + bn)\ , \text{ for } n > 0.$$

Let's find a closed form for a bit more general recurrent equation:

$$R_0 = \alpha$$
$$R_n = R_{n-1} + (\beta + \gamma n)\ , \text{ for } n > 0.$$

Example 1: arithmetic sequence

Arithmetic sequence: $a_n = a + bn$

Recurrent equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n$:

$$\begin{aligned} S_0 &= a \\ S_n &= S_{n-1} + (a + bn) \text{ , for } n > 0. \end{aligned}$$

Let's find a closed form for a bit more general recurrent equation:

$$\begin{aligned} R_0 &= \alpha \\ R_n &= R_{n-1} + (\beta + \gamma n) \text{ , for } n > 0. \end{aligned}$$

Evaluation of terms $R_n = R_{n-1} + (\beta + \gamma n)$

$$
\begin{aligned}
R_0 &= \alpha \\
R_1 &= \alpha + \beta + \gamma \\
R_2 &= \alpha + \beta + \gamma + (\beta + 2\gamma) = \alpha + 2\beta + 3\gamma \\
R_3 &= \alpha + 2\beta + 3\gamma + (\beta + 3\gamma) = \alpha + 3\beta + 6\gamma
\end{aligned}
$$

Observation

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$$

$A(n), B(n), C(n)$ can be evaluated using repertoire method:
we will consider three cases

1. $R_n = 1$ for all n
2. $R_n = n$ for all n
3. $R_n = n^2$ for all n

Evaluation of terms $R_n = R_{n-1} + (\beta + \gamma n)$

$$R_0 = \alpha$$
$$R_1 = \alpha + \beta + \gamma$$
$$R_2 = \alpha + \beta + \gamma + (\beta + 2\gamma) = \alpha + 2\beta + 3\gamma$$
$$R_3 = \alpha + 2\beta + 3\gamma + (\beta + 3\gamma) = \alpha + 3\beta + 6\gamma$$

Observation

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$$

$A(n), B(n), C(n)$ can be evaluated using repertoire method:
we will consider three cases

1. $R_n = 1$ for all n
2. $R_n = n$ for all n
3. $R_n = n^2$ for all n

.

Repertoire method: case 1

Lemma 1: $A(n) = 1$ for all n

- $1 = R_0 = \alpha$
- From $R_n = R_{n-1} + (\beta + \gamma n)$ follows that $1 = 1 + (\beta + \gamma n)$. This is possible only when $\beta = \gamma = 0$

Hence

$$1 = A(n) \cdot 1 + B(n) \cdot 0 + C(n) \cdot 0$$

$\square$

Repertoire method: case 2

Lemma 2: $B(n) = n$ for all n

- $\alpha = R_0 = 0$
- From $R_n = R_{n-1} + (\beta + \gamma n)$ follows that $n = (n-1) + (\beta + \gamma n)$.
  I.e. $1 = \beta + \gamma n$.
  This gives that $\beta = 1$ and $\gamma = 0$

Hence

$$n = A(n) \cdot 0 + B(n) \cdot 1 + C(n) \cdot 0$$

$\square$

Repertoire method: case 3

Lemma 3: $C(n) = \frac{n^2+n}{2}$ for all n

- $\alpha = R_0 = 0^2 = 0$
- Equation $R_n = R_{n-1} + (\beta + \gamma n)$ can be transformed as
  $n^2 = (n-1)^2 + \beta + \gamma n$
  $n^2 = n^2 - 2n + 1 + \beta + \gamma n$
  $0 = (1+\beta) + n(\gamma - 2)$
  This is valid iff $1 + \beta = 0$ and $\gamma - 2 = 0$

Hence

$$n^2 = A(n) \cdot 0 + B(n) \cdot (-1) + C(n)\gamma \cdot 2$$

Due to Lemma 2 we get

$$n^2 = -n + 2C(n)$$

$\square$

Repertoire method: summing up

According to Lemma 1, 2, 3, we get

1. $R_n = 1$ for all n $\Longrightarrow$ $A(n) = 1$
2. $R_n = n$ for all n $\Longrightarrow$ $B(n) = n$
3. $R_n = n^2$ for all n $\Longrightarrow$ $C(n) = (n^2 + n)/2$

.

Repertoire method: summing up

According to Lemma 1, 2, 3, we get

1. $R_n = 1$ for all n $\Longrightarrow$ $A(n) = 1$
2. $R_n = n$ for all n $\Longrightarrow$ $B(n) = n$
3. $R_n = n^2$ for all n $\Longrightarrow$ $C(n) = (n^2 + n)/2$

.

That means that

$$R_n = \alpha + n\beta + \left(\frac{n^2+n}{2}\right)\gamma$$

Repertoire method: summing up

According to Lemma 1, 2, 3, we get

1. $R_n = 1$ for all n $\implies$ $A(n) = 1$
2. $R_n = n$ for all n $\implies$ $B(n) = n$
3. $R_n = n^2$ for all n $\implies$ $C(n) = (n^2 + n)/2$

.

That means that

$$R_n = \alpha + n\beta + \left(\frac{n^2+n}{2}\right)\gamma$$

The sum for arithmetic sequence we obtain taking $\alpha = \beta = a$ and $\gamma = b$:

$$S_n = \sum_{k=0}^{n}(a+bk) = (n+1)a + \frac{n(n+1)}{2}b$$

Next subsection

Perturbation method

Finding the closed form for $S_n = \sum_{0 \leqslant k \leqslant n} a_k$:

- Rewrite S_{n+1} by splitting off first and last term:

$$S_n + a_{n+1} = a_0 + \sum_{1 \leqslant k \leqslant n+1} a_k =$$

$$= a_0 + \sum_{1 \leqslant k+1 \leqslant n+1} a_{k+1} =$$

$$= a_0 + \sum_{0 \leqslant k \leqslant n} a_{k+1}$$

- Work on last sum and express in terms of S_n.
- Finally, solve for S_n.

Example 2: geometric sequence

Geometric sequence: $a_n = ax^n$

Recurrent equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{0 \leqslant k \leqslant n} ax^k$:

$$S_0 = a$$
$$S_n = S_{n-1} + ax^n \text{ , for } n > 0.$$

Example 2: geometric sequence

Geometric sequence: $a_n = ax^n$

Recurrent equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{0 \leqslant k \leqslant n} ax^k$:

$$S_0 = a$$
$$S_n = S_{n-1} + ax^n \text{ , for } n > 0.$$

- Splitting off the first term gives

$$\begin{aligned} S_n + a_{n+1} = a_0 + \sum_{0 \leqslant k \leqslant n} a_{k+1} &= \\ = a + \sum_{0 \leqslant k \leqslant n} ax^{k+1} &= \\ = a + x \sum_{0 \leqslant k \leqslant n} ax^k &= \\ = a + xS_n & \end{aligned}$$

- Hence, we have the equation

Example 2: geometric sequence

Geometric sequence: $a_n = ax^n$

Recurrent equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{0 \leqslant k \leqslant n} ax^k$:

$$S_0 = a$$
$$S_n = S_{n-1} + ax^n \text{ , for } n > 0.$$

- Hence, we have the equation

$$S_n + ax^{n+1} = a + xS_n$$

- Solution:

$$S_n = \frac{a - ax^{n+1}}{1-x}$$

Example 2: geometric sequence

Geometric sequence: $a_n = ax^n$

Recurrent equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{0 \leqslant k \leqslant n} ax^k$:

$$S_0 = a$$
$$S_n = S_{n-1} + ax^n \text{ , for } n > 0.$$

- Hence, we have the equation

$$S_n + ax^{n+1} = a + xS_n$$

- Solution:

$$S_n = \frac{a - ax^{n+1}}{1 - x}$$

Example 2: geometric sequence

Geometric sequence: $a_n = ax^n$

Recurrent equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{0 \leqslant k \leqslant n} ax^k$:

$$S_0 = a$$
$$S_n = S_{n-1} + ax^n \text{ , for } n > 0.$$

Closed formula for geometric sum:

$$S_n = \frac{a(x^{n+1} - 1)}{x - 1}$$

Next subsection

Example 3: Hanoi sequence

The Tower of Hanoi recurrence:

$$
\begin{aligned}
T_0 &= 0 \\
T_n &= 2T_{n-1} + 1
\end{aligned}
$$

Example 3: Hanoi sequence

The Tower of Hanoi recurrence:

$$T_0 = 0$$
$$T_n = 2T_{n-1} + 1$$

This sequence can be transformed into geometric sum using following manipulations:

- Divide equations by 2^n:

$$\begin{aligned} T_0/2^0 &= 0 \\ T_n/2^n &= T_{n-1}/2^{n-1} + 1/2^n \end{aligned}$$

- Set $S_n = T_n/2^n$ to have:

$$\begin{aligned} S_0 &= 0 \\ S_n &= S_{n-1} + 2^{-n} \end{aligned}$$

(This is geometric sum with the parameters $a = 1$ and $x = 1/2$.)

Example 3: Hanoi sequence

The Tower of Hanoi recurrence:

$$T_0 = 0$$
$$T_n = 2T_{n-1} + 1$$

This sequence can be transformed into geometric sum using following manipulations:

- Divide equations by 2^n:

$$\begin{aligned} T_0/2^0 &= 0 \\ T_n/2^n &= T_{n-1}/2^{n-1} + 1/2^n \end{aligned}$$

- Set $S_n = T_n/2^n$ to have:

$$\begin{aligned} S_0 &= 0 \\ S_n &= S_{n-1} + 2^{-n} \end{aligned}$$

(This is geometric sum with the parameters $a = 1$ and $x = 1/2$.)

Example 3: Hanoi sequence

The Tower of Hanoi recurrence:

$$T_0 = 0$$
$$T_n = 2T_{n-1} + 1$$

Hence,

$$S_n = \frac{0.5(0.5^n - 1)}{0.5 - 1} = 1 - 2^{-n} \qquad (a_0 = 0 \text{ has been left out of the sum})$$

$$T_n = 2^n S_n = 2^n - 1$$

Example 3: Hanoi sequence

The Tower of Hanoi recurrence:

$$T_0 = 0$$
$$T_n = 2T_{n-1} + 1$$

Hence,

$$S_n = \frac{0.5(0.5^n - 1)}{0.5 - 1} = 1 - 2^{-n} \qquad (a_0 = 0 \text{ has been left out of the sum})$$

$$T_n = 2^n S_n = 2^n - 1$$

Just the same result we have proven by means of induction! :))

Next subsection

Linear recurrence in form $a_n T_n = b_n T_{n-1} + c_n$

Here $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are any sequences and initial value T_0 is a constant.

The idea:

Find a summation factor s_n satisfying the property

$$s_n b_n = s_{n-1} a_{n-1} \qquad \text{for any } n$$

If such a factor exists, one can do following transformations:

- $s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n = s_{n-1} a_{n-1} T_{n-1} + s_n c_n$
- Setting $S_n = s_n a_n T_n$, to rewrite the equation as

$$\begin{aligned} S_0 &= s_0 a_0 T_0 \\ S_n &= S_{n-1} + s_n c_n \end{aligned}$$

- Closed formula (!) for solution:

$$T_n = \frac{1}{s_n a_n}(s_0 a_0 T_0 + \sum_{k=1}^{n} s_k c_k) = \frac{1}{s_n a_n}(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k)$$

Linear recurrence in form $a_n T_n = b_n T_{n-1} + c_n$

Here $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are any sequences and initial value T_0 is a constant.

The idea:

Find a summation factor s_n satisfying the property

$$s_n b_n = s_{n-1} a_{n-1} \qquad \text{for any } n$$

If such a factor exists, one can do following transformations:

- $s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n = s_{n-1} a_{n-1} T_{n-1} + s_n c_n$
- Setting $S_n = s_n a_n T_n$, to rewrite the equation as

$$S_0 = s_0 a_0 T_0$$
$$S_n = S_{n-1} + s_n c_n$$

- Closed formula (!) for solution:

$$T_n = \frac{1}{s_n a_n}(s_0 a_0 T_0 + \sum_{k=1}^{n} s_k c_k) = \frac{1}{s_n a_n}(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k)$$

Finding summation factor

Assuming that $b_n \neq 0$ for all n:

- Set $s_0 = 1$
- Compute next elements using the property $s_n b_n = s_{n-1} a_{n-1}$:

$$
\begin{aligned}
s_1 &= \frac{a_0}{b_1} \\
s_2 &= \frac{s_1 a_1}{b_2} = \frac{a_0 a_1}{b_1 b_2} \\
s_3 &= \frac{s_2 a_2}{b_3} = \frac{a_0 a_1 a_2}{b_1 b_2 b_3} \\
&\dots\dots\dots \\
s_n &= \frac{s_{n-1} a_{n-1}}{b_n} = \frac{a_0 a_1 \dots a_{n-1}}{b_1 b_2 \dots b_n}
\end{aligned}
$$

(To be proved by induction!)

Example: application of summation factor

$a_n = c_n = 1$ and $b_n = 2$ gives Hanoi Tower sequence:

- Evaluate summation factor

$$s_n = \frac{s_{n-1}a_{n-1}}{b_n} = \frac{a_0 a_1 \dots a_{n-1}}{b_1 b_2 \dots b_n} = \frac{1}{2^n}$$

- Solution is

$$T_n = \frac{1}{s_n a_n}(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k) = 2^n \sum_{k=1}^{n} \frac{1}{2^k} = 2^n(1-2^{-n}) = 2^n - 1$$

YAE: constant coefficients

Equation $Z_n = aZ_{n-1} + b$

Taking $a_n = 1, b_n = a$ and $c_n = b$:

- Evaluate summation factor

$$s_n = \frac{s_{n-1}a_{n-1}}{b_n} = \frac{a_0 a_1 \ldots a_{n-1}}{b_1 b_2 \ldots b_n} = \frac{1}{a^n}$$

- Solution is

$$\begin{aligned} Z_n &= \frac{1}{s_n a_n}\left(s_1 b_1 Z_0 + \sum_{k=1}^{n} s_k c_k\right) = a^n\left(Z_0 + b\sum_{k=1}^{n}\frac{1}{a^k}\right) \\ &= a^n Z_0 + b(1 + a + a^2 + \cdots + a^{n-1}) \\ &= a^n Z_0 + \frac{a^n - 1}{a - 1} b \end{aligned}$$

YAE : check up on results

Equation $Z_n = aZ_{n-1} + b$

$$
\begin{aligned}
Z_n &= aZ_{n-1} + b = \\
&= a^2 Z_{n-2} + ab + b = \\
&= a^3 Z_{n-3} + a^2 b + ab + b = \\
&\quad \ldots\ldots \\
&= a^k Z_{n-k} + (a^{k-1} + a^{k-2} + \ldots + 1)b = \\
&= a^k Z_{n-k} + \frac{a^k - 1}{a - 1} b \qquad \text{(assuming } a \neq 1\text{)}
\end{aligned}
$$

Continuing until $k = n$:

$$
\begin{aligned}
Z_n &= a^n Z_{n-n} + \frac{a^n - 1}{a - 1} b = \\
&= a^n Z_0 + \frac{a^n - 1}{a - 1} b
\end{aligned}
$$

YAE : check up on results

Equation $Z_n = aZ_{n-1} + b$

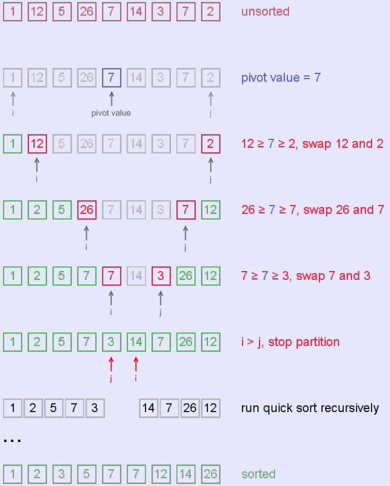
$$
\begin{aligned}
Z_n &= aZ_{n-1} + b = \\
&= a^2 Z_{n-2} + ab + b = \\
&= a^3 Z_{n-3} + a^2 b + ab + b = \\
&\quad \ldots\ldots \\
&= a^k Z_{n-k} + (a^{k-1} + a^{k-2} + \ldots + 1)b = \\
&= a^k Z_{n-k} + \frac{a^k - 1}{a - 1} b \qquad \text{(assuming } a \neq 1\text{)}
\end{aligned}
$$

Continuing until $k = n$:

$$
\begin{aligned}
Z_n &= a^n Z_{n-n} + \frac{a^n - 1}{a - 1} b = \\
&= a^n Z_0 + \frac{a^n - 1}{a - 1} b
\end{aligned}
$$

Efficiency of quick sort

Average number of comparisons: $C_n = n + 1 + \frac{2}{n} \sum_{k=0}^{n-1} C_k$



Efficiency of quick sort (2)

The average number of comparison steps when it is applied to n items

$$C_0 = 0$$

$$C_n = n + 1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k$$

The following transformations reduce this equation

$$nC_n = n^2 + n + 2\sum_{k=0}^{n-2} C_k + 2C_{n-1}$$

Write the last equation for $n-1$ and subtract to eliminate the sum:

$$(n-1)C_{n-1} = (n-1)^2 + (n-1) + 2\sum_{k=0}^{n-2} C_k$$

Efficiency of quick sort (2)

The average number of comparison steps when it is applied to n items

$$C_0 = 0$$

$$C_n = n + 1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k$$

The following transformations reduce this equation

$$nC_n = n^2 + n + 2\sum_{k=0}^{n-2} C_k + 2C_{n-1}$$

Write the last equation for $n-1$ and subtract to eliminate the sum:

$$(n-1)C_{n-1} = (n-1)^2 + (n-1) + 2\sum_{k=0}^{n-2} C_k$$

$$nC_n - (n-1)C_{n-1} = n^2 + n + 2C_{n-1} - (n-1)^2 - (n-1)$$

Efficiency of quick sort (2)

The average number of comparison steps when it is applied to n items

$$C_0 = 0$$

$$C_n = n + 1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k$$

The following transformations reduce this equation

$$nC_n = n^2 + n + 2\sum_{k=0}^{n-2} C_k + 2C_{n-1}$$

Write the last equation for $n-1$ and subtract to eliminate the sum:

$$(n-1)C_{n-1} = (n-1)^2 + (n-1) + 2\sum_{k=0}^{n-2} C_k$$

$$nC_n - nC_{n-1} + C_{n-1} = n^2 + n + 2C_{n-1} - n^2 + 2n - 1 - n + 1$$

Efficiency of quick sort (2)

The average number of comparison steps when it is applied to n items

$$C_0 = 0$$

$$C_n = n + 1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k$$

The following transformations reduce this equation

$$nC_n = n^2 + n + 2\sum_{k=0}^{n-2} C_k + 2C_{n-1}$$

Write the last equation for $n-1$ and subtract to eliminate the sum:

$$(n-1)C_{n-1} = (n-1)^2 + (n-1) + 2\sum_{k=0}^{n-2} C_k$$

$$nC_n - nC_{n-1} = C_{n-1} + 2n$$

Efficiency of quick sort (2)

The average number of comparison steps when it is applied to n items

$$C_0 = 0$$

$$C_n = n + 1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k$$

The following transformations reduce this equation

$$nC_n = n^2 + n + 2\sum_{k=0}^{n-2} C_k + 2C_{n-1}$$

Write the last equation for $n-1$ and subtract to eliminate the sum:

$$(n-1)C_{n-1} = (n-1)^2 + (n-1) + 2\sum_{k=0}^{n-2} C_k$$

$$nC_n = (n+1)C_{n-1} + 2n$$

Efficiency of quick sort (3)

Equation $nC_n = (n+1)C_{n-1} + 2n$

- Assuming $a_n = n, b_n = n+1$ and $c_n = 2n$ evaluate summation factor

$$s_n = \frac{a_1 a_2 \ldots a_{n-1}}{b_2 b_3 \ldots b_n} = \frac{1 \cdot 2 \cdot \ldots \cdot (n-1)}{3 \cdot 4 \cdot \ldots \cdot (n+1)} = \frac{2}{n(n+1)}$$

- Solution is

$$\begin{aligned} C_n &= \frac{1}{s_n a_n}\left(s_1 b_1 C_0 + \sum_{k=1}^{n} s_k c_k\right) \\ &= \frac{n+1}{2}\sum_{k=1}^{n}\frac{4k}{k(k+1)} \\ &= 2(n+1)\sum_{k=1}^{n}\frac{1}{k+1} = 2(n+1)\left(\sum_{k=1}^{n}\frac{1}{k} + \frac{1}{n+1} - 1\right) \\ &= 2(n+1)H_n - 2n \end{aligned}$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ is n-th harmonic number.

Efficiency of quick sort (3)

Equation $nC_n = (n+1)C_{n-1} + 2n$

- Assuming $a_n = n, b_n = n+1$ and $c_n = 2n$ evaluate summation factor

$$s_n = \frac{a_1 a_2 \dots a_{n-1}}{b_2 b_3 \dots b_n} = \frac{1 \cdot 2 \cdot \dots \cdot (n-1)}{3 \cdot 4 \cdot \dots \cdot (n+1)} = \frac{2}{n(n+1)}$$

- Solution is

$$\begin{aligned} C_n &= \frac{1}{s_n a_n}\left(s_1 b_1 C_0 + \sum_{k=1}^{n} s_k c_k\right) \\ &= \frac{n+1}{2} \sum_{k=1}^{n} \frac{4k}{k(k+1)} \\ &= 2(n+1) \sum_{k=1}^{n} \frac{1}{k+1} = 2(n+1)\left(\sum_{k=1}^{n} \frac{1}{k} + \frac{1}{n+1} - 1\right) \\ &= 2(n+1)H_n - 2n \end{aligned}$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ is n-th harmonic number.

(k-th harmonic produced by a violin string is the fundamental tone produced by a string that is $1/k$ times as long.)

Efficiency of quick sort (3)

Equation $nC_n = (n+1)C_{n-1} + 2n$

- Assuming $a_n = n, b_n = n+1$ and $c_n = 2n$ evaluate summation factor

$$s_n = \frac{a_1 a_2 \ldots a_{n-1}}{b_2 b_3 \ldots b_n} = \frac{1 \cdot 2 \cdot \ldots \cdot (n-1)}{3 \cdot 4 \cdot \ldots \cdot (n+1)} = \frac{2}{n(n+1)}$$

- Solution is

$$\begin{aligned} C_n &= \frac{1}{s_n a_n}\left(s_1 b_1 C_0 + \sum_{k=1}^{n} s_k c_k\right) \\ &= \frac{n+1}{2}\sum_{k=1}^{n}\frac{4k}{k(k+1)} \\ &= 2(n+1)\sum_{k=1}^{n}\frac{1}{k+1} = 2(n+1)\left(\sum_{k=1}^{n}\frac{1}{k} + \frac{1}{n+1} - 1\right) \\ &= 2(n+1)H_n - 2n \end{aligned}$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \approx \ln n$.

(k-th harmonic produced by a violin string is the fundamental tone produced by a string that is $1/k$ times as long.)

Next section

General Methods: a Review

Example

$$\square_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad \text{for } n \geqslant 0$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
n^2	0	1	4	9	16	25	36	49	64	81	100	121	144
$\square_n$	0	1	5	14	30	55	91	140	204	285	385	506	650

Next subsection

Review: Method 0

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Find solution from a reference books:

- "CRC Standard Mathematical Tables"
- "Valemeid matemaatikast"
- "The On-Line Encyclopedia of Integer Sequences (OEIS)" (`http://oeis.org/`)
- *etc*

Review: Method 0

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Find solution from a reference books:

- "CRC Standard Mathematical Tables"
- "Valemeid matemaatikast"
- "The On-Line Encyclopedia of Integer Sequences (OEIS)" (`http://oeis.org/`)
- *etc*

Review: Method 0

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Find solution from a reference books:

- "CRC Standard Mathematical Tables"
- "Valemeid matemaatikast"
- "The On-Line Encyclopedia of Integer Sequences (OEIS)" (`http://oeis.org/`)
- *etc*

Review: Method 0

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Find solution from a reference books:

- "CRC Standard Mathematical Tables"
- "Valemeid matemaatikast"
- "The On-Line Encyclopedia of Integer Sequences (OEIS)" (`http://oeis.org/`)
- *etc*

Review: Method 0

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad \text{for } n \geqslant 0$

Find solution from a reference books:

- "CRC Standard Mathematical Tables"
- "Valemeid matemaatikast"
- "The On-Line Encyclopedia of Integer Sequences (OEIS)" (`http://oeis.org/`)
- *etc*

Possible answer:

$$\square_n = \frac{n(n+1)(2n+1)}{6} \qquad \text{for } n \geqslant 0$$

Next subsection

Review: Method 1

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad$ for $n \geqslant 0$

Guess the answer, prove it by induction.

Let's compute

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	49	64	81

Review: Method 1

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad \text{for } n \geqslant 0$

Guess the answer, prove it by induction.

Let's compute

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	49	64	81
$\square_n$	0	1	5	14	30	55	91	140	204	285

Review: Method 1

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Guess the answer, prove it by induction.

Let's compute

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	49	64	81
$\Box_n$	0	1	5	14	30	55	91	140	204	285
$\Box_n / n^2$	–	1	1.25	1.56	1.88	2.2	2.53	2.86	3.19	3.52

Review: Method 1

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad \text{for } n \geqslant 0$

Guess the answer, prove it by induction.

Let's compute

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	49	64	81
$\Box_n$	0	1	5	14	30	55	91	140	204	285
$\Box_n/n^2$	–	1	1.25	1.56	1.88	2.2	2.53	2.86	3.19	3.52
$3\Box_n/n^2$	–	3	3.75	4.67	5.63	6.6	7.58	8.57	9.56	10.56

Review: Method 1

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad \text{for } n \geqslant 0$

Guess the answer, prove it by induction.

Let's compute

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	49	64	81
$\Box_n$	0	1	5	14	30	55	91	140	204	285
$\Box_n/n^2$	–	1	1.25	1.56	1.88	2.2	2.53	2.86	3.19	3.52
$3\Box_n/n^2$	–	3	3.75	4.67	5.63	6.6	7.58	8.57	9.56	10.56
$n(n+1)$	0	2	6	12	20	30	42	56	72	90

Review: Method 1

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad$ for $n \geqslant 0$

Guess the answer, prove it by induction.

Let's compute

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	49	64	81
$\Box_n$	0	1	5	14	30	55	91	140	204	285
$\Box_n/n^2$	–	1	1.25	1.56	1.88	2.2	2.53	2.86	3.19	3.52
$3\Box_n/n^2$	–	3	3.75	4.67	5.63	6.6	7.58	8.57	9.56	10.56
$n(n+1)$	0	2	6	12	20	30	42	56	72	90
$3\Box_n/n(n+1)$	–	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5

Review: Method 1

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Guess the answer, prove it by induction.

Let's compute

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	49	64	81
$\square_n$	0	1	5	14	30	55	91	140	204	285
$\square_n/n^2$	–	1	1.25	1.56	1.88	2.2	2.53	2.86	3.19	3.52
$3\square_n/n^2$	–	3	3.75	4.67	5.63	6.6	7.58	8.57	9.56	10.56
$n(n+1)$	0	2	6	12	20	30	42	56	72	90
$3\square_n/n(n+1)$	–	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5

Hypothesis:

$$\frac{3\square_n}{n(n+1)} = n + \frac{1}{2} \qquad \Longrightarrow$$

$$\square_n = \frac{n(n+1/2)(n+1)}{3} = \frac{n(n+1)(2n+1)}{6}$$

Review: Method 1

Proof. $3\Box_n = n(n+\frac{1}{2})(n+1)$

Assume that the formula is true for $n-1$

We know that $\Box_n = \Box_{n-1} + n^2$

We have

$$\begin{aligned} 3\Box_n &= (n-1)(n-\frac{1}{2})n + 3n^2 \\ &= (n^3 - \frac{3}{2}n^2 + \frac{1}{2}n) + 3n^2 \\ &= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \\ &= n(n+\frac{1}{2})(n+1) \end{aligned}$$

Review: Method 1

Proof. $3\square_n = n(n+\frac{1}{2})(n+1)$

Assume that the formula is true for $n-1$

We know that $\square_n = \square_{n-1} + n^2$

We have

$$\begin{aligned} 3\square_n &= (n-1)(n-\frac{1}{2})n + 3n^2 \\ &= (n^3 - \frac{3}{2}n^2 + \frac{1}{2}n) + 3n^2 \\ &= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \\ &= n(n+\frac{1}{2})(n+1) \end{aligned}$$

Review: Method 1

Proof. $3\square_n = n(n+\frac{1}{2})(n+1)$

Assume that the formula is true for $n-1$

We know that $\square_n = \square_{n-1} + n^2$

We have

$$
\begin{aligned}
3\square_n &= (n-1)(n-\frac{1}{2})n + 3n^2 \\
&= (n^3 - \frac{3}{2}n^2 + \frac{1}{2}n) + 3n^2 \\
&= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \\
&= n(n+\frac{1}{2})(n+1)
\end{aligned}
$$

Review: Method 1

Proof. $3\square_n = n(n+\frac{1}{2})(n+1)$

Assume that the formula is true for $n-1$

We know that $\square_n = \square_{n-1} + n^2$

We have

$$\begin{aligned}
3\square_n &= (n-1)(n-\frac{1}{2})n + 3n^2 \\
&= (n^3 - \frac{3}{2}n^2 + \frac{1}{2}n) + 3n^2 \\
&= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \\
&= n(n+\frac{1}{2})(n+1)
\end{aligned}$$

Q.E.D.

Next subsection

Review: Method 2

Example; $\square_n = \sum_{0\leqslant k\leqslant n} k^2$ for $n \geqslant 0$

Perturb the sum.

- Define a sum $\boxdot_n = 0^3 + 1^3 + 2^3 + \ldots + n^3$.
- Then we have

$$\boxdot_n + (n+1)^3 = \sum_{0\leqslant k\leqslant n} (k+1)^3 = \sum_{0\leqslant k\leqslant n} (k^3 + 3k^2 + 3k + 1)$$
$$= \boxdot_n + 3\square_n + 3\frac{(n+1)n}{2} + (n+1).$$

- Cancel $\boxdot_n$ and extract $\square_n$

$$3\square_n = (n+1)^3 - 3(n+1)n/2 - (n+1)$$
$$= (n+1)(n^2 + 2n + 1 - \frac{3}{2}n - 1)$$
$$= (n+1)(n + \frac{1}{2})n.$$

Review: Method 2

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Perturb the sum.

- Define a sum $\cube_n = 0^3 + 1^3 + 2^3 + \ldots + n^3$.
- Then we have

$$\cube_n + (n+1)^3 = \sum_{0 \leqslant k \leqslant n} (k+1)^3 = \sum_{0 \leqslant k \leqslant n} (k^3 + 3k^2 + 3k + 1)$$

$$= \cube_n + 3\square_n + 3\frac{(n+1)n}{2} + (n+1).$$

- Cancel $\cube_n$ and extract $\square_n$

$$3\square_n = (n+1)^3 - 3(n+1)n/2 - (n+1)$$

$$= (n+1)(n^2 + 2n + 1 - \frac{3}{2}n - 1)$$

$$= (n+1)(n + \frac{1}{2})n.$$

Review: Method 2

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Perturb the sum.

- Define a sum $\cube_n = 0^3 + 1^3 + 2^3 + \ldots + n^3$.
- Then we have

$$\cube_n + (n+1)^3 = \sum_{0 \leqslant k \leqslant n} (k+1)^3 = \sum_{0 \leqslant k \leqslant n} (k^3 + 3k^2 + 3k + 1)$$
$$= \cube_n + 3\square_n + 3\frac{(n+1)n}{2} + (n+1).$$

- Cancel $\cube_n$ and extract $\square_n$

$$3\square_n = (n+1)^3 - 3(n+1)n/2 - (n+1)$$
$$= (n+1)(n^2 + 2n + 1 - \frac{3}{2}n - 1)$$
$$= (n+1)(n + \frac{1}{2})n.$$

Review: Method 2

Example; $\square_n = \sum_{0\leqslant k\leqslant n} k^2$ for $n \geqslant 0$

Perturb the sum.

- Define a sum $\boxdot_n = 0^3 + 1^3 + 2^3 + \ldots + n^3$.
- Then we have

$$\boxdot_n + (n+1)^3 = \sum_{0\leqslant k\leqslant n} (k+1)^3 = \sum_{0\leqslant k\leqslant n} (k^3 + 3k^2 + 3k + 1)$$
$$= \boxdot_n + 3\square_n + 3\frac{(n+1)n}{2} + (n+1).$$

- Cancel $\boxdot_n$ and extract $\square_n$

$$3\square_n = (n+1)^3 - 3(n+1)n/2 - (n+1)$$
$$= (n+1)(n^2 + 2n + 1 - \frac{3}{2}n - 1)$$
$$= (n+1)(n + \frac{1}{2})n.$$

Next subsection

Review: Method 3

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Build a repertoire.

Recurrence:

$$R_0 = 0$$
$$R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$$

We look a solution in the form $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

Review: Method 3

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Build a repertoire.

Recurrence:

$$R_0 = 0$$
$$R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$$

We look a solution in the form $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

Case: $R_n = n$

- Equation: $n = n - 1 + \alpha + \beta n + \gamma n^2$
- That is $\alpha = 1; \beta = \gamma = 0$,
- and the solution has a form $n = A(n)$.

Review: Method 3

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Build a repertoire.

Recurrence:

$$R_0 = 0$$

$$R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$$

We look a solution in the form $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

Case: $R_n = n^2$

- Equation: $n^2 = (n-1)^2 + \alpha + \beta n + \gamma n^2$
- or: $0 = (\alpha + 1) + (\beta - 2)n + \gamma n^2$
- That is $\alpha = -1; \beta = 2; \gamma = 0$,
- and hence, the solution has a form $n^2 = -A(n) + 2B(n) = -n + 2B(n)$.
- That gives $B(n) = \frac{n^2+n}{2}$.

Review: Method 3

Example; $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Build a repertoire.

Recurrence:

$$R_0 = 0$$

$$R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$$

We look a solution in the form $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

Case: $R_n = n^3$

- Equation: $n^3 = (n-1)^3 + \alpha + \beta n + \gamma n^2 = n^3 - 3n^2 + 3n - 1 + \alpha + \beta n + \gamma n^2$
- or: $0 = (\alpha - 1) + (\beta + 3)n + (\gamma - 3)n^2$
- That is $\alpha = 1; \beta = -3; \gamma = 3$,
- and hence, the solution has a form $n^3 = A(n) - 3B(n) + 3C(n) = n - 3\frac{n^2+n}{2} + 3C(n)$.
- That gives $6C(n) = 2n^3 - 2n + 3n^2 + 3n = 2n^3 + 3n^2 + n = n(2n+1)(n+1)$.

Review: Method 3

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Build a repertoire.

Recurrence:

$$R_0 = 0$$

$$R_n = R_{n-1} + \alpha + \beta n + \gamma n^2$$

We look a solution in the form $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

To resume

- $R_n = \Box_n$ iff $\alpha = \beta = 0; \gamma = 1$
- The solution is

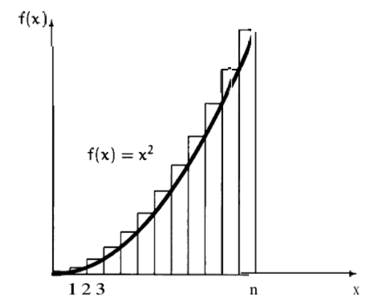
$$\Box_n = \frac{n(n+1)(2n+1)}{6}$$

Next subsection

Review: Method 4

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Replace sums by integrals.



$$\int_0^n x^2 \mathrm{d}x = \frac{n^3}{3} \tag{2}$$

$$\Box_n = \int_0^n x^2 \mathrm{d}x + E_n \tag{3}$$

$$E_n = \sum_{k=1}^{n} \left(k^2 - \int_{k-1}^{k} x^2 \mathrm{d}x \right) \tag{4}$$

Review: Method 4

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad$ for $n \geqslant 0$

Replace sums by integrals.

Evaluate (4):

$$
\begin{aligned}
E_n &= \sum_{k=1}^{n} \left(k^2 - \int_{k-1}^{k} x^2 \mathrm{d}x \right) \\
&= \sum_{k=1}^{n} \left(k^2 - \frac{k^3 - (k-1)^3}{3} \right) \\
&= \sum_{k=1}^{n} \left(k - \frac{1}{3} \right) \\
&= \frac{(n+1)n}{2} - \frac{n}{3} = \frac{3n^2 + n}{6}.
\end{aligned}
$$

Finally, from (3) and (2) we get :

$$\Box_n = \frac{n^3}{3} + \frac{3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

Next subsection

Review: Method 5

Example; $\Box_n = \sum_{0\leqslant k\leqslant n} k^2$ for $n \geqslant 0$

Expand and Contract.

$$\begin{aligned}
\Box_n &= \sum_{1\leqslant k\leqslant n} k^2 = \sum_{1\leqslant j\leqslant k\leqslant n} k \\
&= \sum_{1\leqslant j\leqslant n} \sum_{j\leqslant k\leqslant n} k \\
&= \sum_{1\leqslant j\leqslant n} \frac{(j+n)(n-j+1)}{2} \\
&= \frac{1}{2} \sum_{1\leqslant j\leqslant n} \left(j - j^2 + n(n+1)\right) \\
&= \frac{1}{4} n(n+1) - \frac{1}{2}\Box_n + \frac{1}{2} n^2(n+1)
\end{aligned}$$

Hence
$$\frac{3}{2}\Box_n = \frac{n+1}{4} \cdot \left(2n^2 + n\right)$$

Review: Method 5

Example; $\Box_n = \sum_{0\leqslant k\leqslant n} k^2$ for $n \geqslant 0$

Expand and Contract.

$$\begin{aligned}
\Box_n &= \sum_{1\leqslant k\leqslant n} k^2 = \sum_{1\leqslant j\leqslant k\leqslant n} k \\
&= \sum_{1\leqslant j\leqslant n} \sum_{j\leqslant k\leqslant n} k \\
&= \sum_{1\leqslant j\leqslant n} \frac{(j+n)(n-j+1)}{2} \\
&= \frac{1}{2} \sum_{1\leqslant j\leqslant n} \left(j - j^2 + n(n+1)\right) \\
&= \frac{1}{4} n(n+1) - \frac{1}{2}\Box_n + \frac{1}{2} n^2(n+1)
\end{aligned}$$

Hence

$$\frac{3}{2}\Box_n = \frac{n+1}{4} \cdot \left(2n^2 + n\right)$$

Next subsection

Review: Other methods

Example; $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2 \qquad \text{for } n \geqslant 0$

- Finite calculus
- Generating functions