MIT Infinite Series notes

Infinite Series

Improper Integrals (with a finite singularity)

Defin:

$$\int_0^1 f(x)dx = \lim_{a\to 0^+} \int_a^1 f(x)dx$$

The series converges if the limit exists, and diverges if not.

Ex1.

$$\int_0^1 \frac{dx}{\sqrt{x}} = \int_0^1 x^{-\frac{1}{2}} dx$$

$$2x^{\frac{1}{2}}\Big|_0^1 = 2 - 0$$

This one is convergent.

Ex2.

$$\int_0^1 \frac{dx}{x} = lnx|_0^1$$
$$= ln1 - ln0^+ = 0 - (-\inf) = \inf$$

This one diverges.

In general,

$$\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}\Bigg| p \geq 1$$

Contrast:

$$\frac{1}{x^{\frac{1}{2}}} < \frac{1}{x} < \frac{1}{x^2}$$

as $x \to 0^+$, and

$$\frac{1}{x^{\frac{1}{2}}} > \frac{1}{x} > \frac{1}{x^2}$$

as $x \to \inf$

some are divergent........

Series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = 2$$

Geometric Series

$$1 + a + a^2 + a^3 + ... = \frac{1}{1-a}$$
$$|a| < 1, -1 < a < 1$$

Divergences:

$$a = 1, 1 + 1 + 1 + ... = \frac{1}{1-1} = \frac{1}{0}$$

NO no no no no no no no no!

$$a = -1, 1 - 1 + 1 - 1 + ... = \frac{1}{1-(-1)} = \frac{1}{2}$$

$$a = 2, 1 + 2 + 2^2 + 2^3 + ... = \frac{1}{1-2} = -1$$

Notation.

$$S_N = \sum_{n=0}^{N} a_n$$

$$S = \sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} S_N$$

Either the limit exists, the series converges, or it does not exist, and the series does not converge.

Ex3.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sim \int_1^{\infty} \frac{dx}{x^2}$$

Euler computes that the first term here is equal to $\frac{\pi^2}{6}$, the second is equal to 1.

Ex4.

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \sim \int_1 \infty \frac{dx}{x^3}$$

Here, the second term is equal to $\frac{1}{2}$ and the first is some impossible irrational number.

Ex5.

$$\sum_{n=1}^{\infty}\frac{1}{n} \sim \int_1^{\infty}\frac{dx}{x}$$

This diverges!

Upper Riemann Sum –

$$\int_1^N \frac{dx}{x} < 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{N-1} < S_N$$

$$S_N = 1 + \frac{1}{2} + ... + \frac{1}{N-1} + \frac{1}{N}$$

We can prove that these are divergent:

$$\int_1 N\frac{dx}{x} < S_N$$

$$\int_1 N\frac{dx}{x} = lnx|_1^N = (lnN) - 0$$

$$lnN < S_N$$

$$(N \to \infty, S_N \to \infty)$$

We have shown divergence.

Lower Riemann Sum

$$\int_1^N \frac{dx}{x} > \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{N} = S_N - 1$$

$$lnN < S_N < (lnN) + 1$$

Integral Comparison

If $f(x)$ is decreasing and $f(x) > 0$, then:

$$\left|\sum_{n=1}^{\infty} f(n) - \int_1^{\infty} f(x)dx\right| < f(1)$$

and the sum and the integral converge or diverge together.

Limit Comparison

If $f(n) \sim g(n)$, or $\frac{f(n)}{g(n)} \to \infty$ and $g(n) > 0$ (for all positive numbers), then $\sum f(n), \sum g(n)$ both either converge or diverge.

Ex.

$$\sum \frac{1}{\sqrt{n^2+1}} \sim \sum \frac{1}{\sqrt{n^2}} = \sum \frac{1}{n}$$

These diverge together...

Ex.

$$\sum_{2}^{\infty} \frac{1}{\sqrt{n^5-n^2}} \sim \sum \frac{1}{\sqrt{n^5}} = \sum \frac{1}{n^{\frac{5}{2}}}$$

This converges.