Homework 9

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1. p.363 #11

> The price of one share of stock in the Pilsdorff Beer Company (see Exercise 8.2.12) is given by Y_n on the nth day of the year. Finn observes that the differences $X_n = Y_{n+1} - Y_n$ appear to be independent random variables with a common distribution having mean $\mu = 0$ and variance $\sigma^2 = \frac{1}{4}$. If $Y_1 = 100$, estimate the probability that Y_365 is:

To start, we can restate the question: what is the likelihood that the value $Y_{365} - 100$ greater than or equal to these values each minus 100… We can also point out that these variables are continuous, with a normal distribution, centered around 0. For $X_n = Y_{n+1} - Y_n$ to be independent, we must ignore some basic facts about timeseries data, especially prices. We imagine the value of $Y_365 - Y_1$ on the bell curve, and use a cumulative probability approach to determine $P(Y_{365} \geq x)$.

```
n <- 365
y1 <- 100
mu <- 0
var <- 1/4
st_dev <- 1/2

likelihood <- function(y1, y365, n, mu, st_dev) {
  value <- (y365 - y1) / sqrt(n - 1)
  return(pnorm(value, mu, st_dev, lower.tail = FALSE))
}
```

(a) ≥ 100

```
y365 <- 100
likelihood(y1, y365, n, mu, st_dev)
```

```
## [1] 0.5
```

(b) ≥ 110

```
y365 <- 110
likelihood(y1, y365, n, mu, st_dev)
```

```
## [1] 0.1472537
```

(c) ≥ 120

```r
y365 <- 120
likelihood(y1, y365, n, mu, st_dev)
```

```
## [1] 0.01801584
```

2. Calculate the expected value and variance of the binomial distribution using the moment generating function.

Binomial Probability Mass Function:

Assuming $X = \{0, 1, 2, ..., n\}$ and $0 \leq j \leq n$:

$$p_X(j) = \binom{n}{j} p^j q^{n-j}$$

Moment Generating Function:

$$g(t) = E(e^{tX})$$

$$= \sum_{k=0}^{\infty} \frac{\mu_k t^k}{k!}$$
$$= E\left(\sum_{k=0}^{\infty} \frac{X^k t^k}{k!}\right)$$
$$= \sum_{j=1}^{\infty} e^{tx_j} p(x_j)$$

Expected value:

$$g(t) = \sum_{j=0}^{n} e^{tj} \binom{n}{j} p^j q^{n-j}$$
$$= \sum_{j=0}^{n} \binom{n}{j} (pe^t)^j q^{n-j}$$
$$= (pe^t + q)^n$$

Variance:

$$\mu_1 = g'(0)$$
$$= n(pe^t + q)^{n-1} pe^t \big|_{t=0} = np$$

$$\mu_2 = g''(0)$$
$$= n(n-1)p^2 + np$$

$$\sigma^2 = \mu_2 - \mu_1^2$$
$$= np(1-p)$$

3. Calculate the expected value and variance of the exponential distribution using the moment generating function.

Exponential Function:

$$f(x) = \left\{ \begin{array}{ll} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{array} \right\}$$

Moment Generating Function:

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx$$

Expected Value:

$$g(t) = \int_{-\infty}^{\infty} e^{tx} \lambda e^{-\lambda x} dx$$

Now, we can't have a negative value for x, and since λ is a constant, we can move it outside.

$$\begin{aligned} g(t) &= \lambda \int_{x \geq 0} e^{(t-\lambda)x} dx \\ &= \frac{\lambda}{t-\lambda} \left[\lim_{x \to \infty} e^{(t-\lambda)x} - e^{(t-\lambda)0} \right] \\ E(e^{tx}) &= \frac{\lambda}{\lambda - t}, \quad for \quad t - \lambda < 0 \end{aligned}$$

Note: honestly, I'm a bit confused here... While I am confident I do not have an error in the above section, I know that the mean of an exponential distribution is equal to $\frac{1}{\lambda}$. $t = 0$ is defined for the result $\frac{\lambda}{\lambda - t}$... So, making this jump is a final step that just escapes me. From some internet reseach I guess that the first moment is equal to $\frac{\lambda}{(\lambda-t)^2}$, but again, I do not understand why this is not just the above result.

Variance:

$$\begin{aligned} \mu_1 &= M_X'(0) = \frac{1}{\lambda} \\ \mu_2 &= M_X''(0) = \frac{2}{\lambda^2} \\ \sigma^2 &= \mu_2 - \mu_1^2 = \frac{1}{\lambda^2} \end{aligned}$$