Taylor Series

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For each function, only consider its valid ranges as indicated in the notes when you are computing the Taylor Series expansion.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n a}{n!}(x-a)^n$$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + f'''(a^2) + ...$$

For each Taylor series expansion, we will consider $a = 0$ so that the expressions are easier to simplify... This is a Maclaurin series.

1.

$$f(x) = \frac{1}{(1-x)}$$

$$f'(x) = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{2}{(1-x)^3}$$

$$f'''(x) = \frac{6}{(1-x)^4}$$

$$f''''(x) = \frac{24}{(1-x)^5}$$

Using the equality above, we can arrange the first n primes into a series:

$$\frac{1}{1-x} = \frac{1}{1-a} + \frac{\frac{1}{(1-a)^2}}{1!}(x-a) + \frac{\frac{2}{(1-a)^3}}{2!}(x-a)^2 + \frac{\frac{6}{(1-a)^4}}{3!}(x-a)^3 + \frac{\frac{24}{(1-a)^5}}{4!}(x-a)^4 + ...$$

Substituting 0 for a, and recognizing the equality of the numerator and the factorial in the denominator, we arrive at:

$$1 + \frac{x}{1!} + \frac{2x^2}{2!} + \frac{6x^3}{3!} + \frac{24x^4}{4!} + ...$$

$$f(x) = 1 + x + x^2 + x^3 + ...$$

$$= \sum_{n=0}^{\infty} x^n$$

2.

$$f(x) = e^x$$

Any derivative of e^x is e^x.

$$f^{(n)}(x) = e^x$$

Expansion via the handy equality:

$$e^x = e^a + \frac{e^a}{1!}(x-a) + \frac{e^a}{2!}(x-a)^2 + \frac{e^a}{3!}(x-a)^3 + \frac{e^a}{4!}(x-a)^4 + ...$$

Substituting 0 and simplifying:

$$e^x = e^0 + \frac{e^0}{1!}(x-0) + \frac{e^0}{2!}(x-0)^2 + \frac{e^0}{3!}(x-0)^3 + \frac{e^0}{4!}(x-0)^4 + ...$$

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$
$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

3.

$$f(x) = ln(1+x)$$

$$f'(x) = \frac{1}{1+x}$$
$$f''(x) = \frac{1}{(1+x)^2}$$
$$f'''(x) = \frac{2}{(1+x)^3}$$
$$f''''(x) = \frac{6}{(1+x)^4}$$

$$ln(1+x) =$$
$$ln(1+0) + \frac{\frac{1}{1+0}}{1!}(x-0) - \frac{\frac{1}{(1+0)^2}}{2!}(x-0)^2 + \frac{\frac{2}{(1+0)^3}}{3!}(x-0)^3 - \frac{\frac{6}{(1+0)^4}}{4!}(x-0)^4 + ...$$

$$= 0 + \frac{x}{1!} - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!}$$

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}...$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

Here, we start the series with $n = 1$ because the first term is equal to zero. Also, the coefficient in the sum gives us the sign for each added term in the series.