MIT Taylor Series notes

Taylor Series

Stacking blocks.... start from the top:

The block is length 2, so that the leftmost part of the second block is its midpoint, 1. The third block is now at 3/2.

```
series1 <- c(0, 1, 3/2)
```

C_n is the center of mass of the block tower... For N blocks, the new block $N+1$ has a center at C_n+1. Now, we can calculate the center of mass (for just the x coordinate).

$$C_N = \frac{NC_N + C_N + 1}{N+1} = \frac{(N+1)C_N + 1}{N+1}$$

$$C_{N+1} = C_N + \frac{1}{n+a}$$

$$C_1 = 1$$

$$C_2 = 1 + \frac{1}{2}$$

$$C_3 = 1 + \frac{1}{2} + \frac{1}{3}$$

$$C_N = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{N}$$

$$C_N = S_N$$

$$lnN < S_N < (lnN) + 1$$

As $N \to \inf$, $lnN \to \inf$, and $S_N \to \inf$.

We can make a tower with an arbitrary width, we can determine exactly how many blocks are needed.

$$C_{N+1} = \frac{NC_N + 1(C_N + 1)}{N+1} = \frac{(N+1)C_N + 1}{N+1}$$

$$C_{N+1} = C_N + \frac{1}{N+1}$$

$$C_1 = 1$$

$$C_2 = 1 + \frac{1}{2}$$

$$C_3 = C_2 + \frac{1}{3} = 1 + \frac{1}{2} + \frac{1}{3}$$

$$C_N = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + ... + \frac{1}{N}$$

$$C_N = S_N$$

$$lnN < S_N < (lnN) + 1$$

As $N \to \infty$, $lnN \to \infty$ and $S_N \to \infty$.

$$N = e^2 4$$

$$(3cm)e^2 4 \approx 8 \times 10^8 m$$

twice the distance to the moon

Power Series

$$1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$$

suppose:

$$1 + x + x^2 + ... = S$$

multiply by x:

$$x + x^2 + x^3 + ... = Sx$$

subtract one from the other:

$$1 = S - Sx = S(1-x)$$

$$\frac{1}{1-x} = S$$

This reasoning is basically correct, but is incomplete because it requires that S exists...

e.g.

$$x = 1, 1 + 1 + 1 + ... = S$$

$$1 + 1 + 1 = S \times 1$$

$$\infty - \infty = \infty - \infty$$

General Power Series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + ...$$
$$= \sum_{n=0}^{\infty} a_n x^n$$

$$|x| < R \text{radius of convergence}$$
$$-R < x < R$$

where the series converges...

$$|x| > R, \sum a_n x^n \text{diverges}$$

(and $|x| = R very delicate)

What does matter:

$|a_n x^n| \to 0$ exponentially fast for $|x| < R$

$$|a_n x^n| \nrightarrow 0$$

for $|x| > R$

Series are flexible enough to represent all the functions we know... In this form, they are computationally available.

$$1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$$

Rules for convergent power series: Just like polynomials.

$$f(x) + g(x), f(x)g(x),$$
$$f(g(x)), f(x)/g(x)$$

the last two are interesting because we differentiate and integrate them:

$$\frac{d}{dx} f(x), \int f(x)dx$$

$$\frac{d}{dx}(a_0 + a_1x + a_2x^2 + a_3x^3 + ...) = a1 + 2a_2x + 3a_3x^2 + ...$$

$$\int (a_0 + a_1x + a_2x^2 + ...)dx = c + a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + ...$$

Taylor's Formula

$$f(x) = \sum \frac{f^{(n)}(0)}{n!} x^n$$

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + ...$$
$$f''(x) = 2a_2 + 3 \cdot 2a_3x + ...$$
$$f'''(x) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4x + ...$$

$$f'''(0) = 3 \cdot 2a_3$$
$$\frac{f'''(0)}{3 \cdot 2 \cdot 1} = a_3$$

In general:

$$a_n = \frac{f^{(n)}(0)}{n!}$$

$$n! = n \cdot (n-1) \cdot (n-2)...1$$
$$0! = 1$$

$$f(x) = e^x, f'(x) = e^x, f''(x) = e^x$$
$$...$$
$$f^{(n)}(x) = e^x \Big|_{x=0} = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...$$

$$sinx \approx x$$
$$cosx \approx 1 - \frac{x^2}{2}$$

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$
$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$