Question 4:

Part (B): $\theta = (\lambda_c, \theta_c)$

$$E[\lambda_c | x] = \int_0^\infty \int_0^\infty \lambda_c \, f(\lambda_c, \theta_c | x) \, \pi_{\lambda_c}(\lambda) \, \pi_{\theta_c}(\lambda) \, d\lambda_c \, d\theta_c$$

$$= \int_0^\infty \int_0^\infty \lambda_c \left(\prod_{i=1}^{19} f(x_{1i}, \lambda_c, \theta_c) \cdot \prod_{i=1}^{19} f(x_{2i}, \lambda_c) \right) \pi_{\lambda_c}(\lambda) \, \pi_{\theta_c}(\theta) \, d\lambda_c \, d\theta_c$$

$$= \int_0^\infty \int_0^\infty \lambda_c \cdot e^{-\sum_{i=1}^{19}(\lambda_c+\theta_c)} \cdot \frac{(\lambda_c+\theta_c)^{\sum_{i=1}^{19} x_{1i}}}{\prod_{i=1}^{19} x_{1i}} \cdot e^{-\sum_{i=1}^{19}(\lambda_c)} \cdot \frac{(\lambda_c)^{\sum_{i=1}^{19} x_{2i}}}{\prod_{i=1}^{19} x_{2i}} \cdot \frac{1}{(1+\lambda_c)(1+\theta_c)} \, d\lambda_c \, d\theta_c$$

$$E[\lambda_c | x] = \frac{\int_\lambda \int_\theta \lambda \cdot e^{\sum_{i=1}^{19}(\lambda_c+\theta_c)} \cdot \frac{1}{\pi(x_{1i})} \cdot (\lambda+\theta)^{\sum_{i=1}^{19} x_{1i}} \cdot e^{\sum_{i=1}^{19}(\lambda_c)} \cdot \frac{1}{\pi(x_{2i})} \cdot (\lambda)^{\sum_{i=1}^{19} x_{2i}} \cdot \frac{1}{(1+\lambda)(1+\theta)} \, d\lambda \, d\theta}{\int_\lambda \int_\theta e^{\sum_{i=1}^{19}(\lambda_c+\theta_c)} \cdot \frac{1}{\pi(x_{1i})} \cdot (\lambda+\theta)^{\sum_{i=1}^{19} x_{1i}} \cdot e^{\sum_{i=1}^{19}(\lambda_c)} \cdot \frac{1}{\pi(x_{2i})} \cdot (\lambda)^{\sum_{i=1}^{19} x_{2i}} \cdot \frac{1}{(1+\lambda)(1+\theta)} \, d\lambda \, d\theta}$$

$$= \frac{\int_\lambda \int_\theta \lambda \, e^{\sum_{i=1}^{19}(\lambda_c+\theta_c)} (\lambda+\theta)^{\sum_{i=1}^{19} x_{1i}} \cdot e^{\sum_{i=1}^{19}(\lambda_c)} \cdot (\lambda)^{\sum_{i=1}^{19} x_{2i}} \cdot \frac{1}{(1+\lambda)(1+\theta)} \, d\lambda \, d\theta}{\int_\theta e^{\sum_{i=1}^{19}(\lambda_c+\theta_c)} (\lambda+\theta)^{\sum_{i=1}^{19} x_{1i}} \cdot e^{\sum_{i=1}^{19}(\lambda_c)} \cdot (\lambda)^{\sum_{i=1}^{19} x_{2i}} \cdot \frac{1}{(1+\lambda)(1+\theta)} \, d\lambda \, d\theta}$$

$$= \frac{\int_\lambda \int_\theta \lambda \left(e^{19(\lambda_c+\theta_c)} \cdot (\lambda+\theta)^{\sum_{i=1}^{19} x_{1i}} \right) \left(e^{19\lambda_c} \cdot (\lambda)^{\sum_{i=1}^{19} x_{2i}} \right) \cdot \frac{1}{(1+\lambda)(1+\theta)} \, d\lambda \, d\theta}{\int_\lambda \int_\theta \left(e^{19(\lambda_c+\theta_c)} (\lambda+\theta)^{\sum_{i=1}^{19} x_{1i}} \right) \left(e^{19\lambda_c} (\lambda)^{\sum_{i=1}^{19} x_{2i}} \right) \cdot \frac{1}{(1+\lambda)(1+\theta)} \, d\lambda \, d\theta}$$

$$= \frac{\text{Constant} \int_\lambda \int_\theta h_1(\lambda, \theta) \, g(\lambda, \theta) \, d\lambda \, d\theta}{\text{Constant} \int_\lambda \int_\theta h_2(\lambda, \theta) \, g(\lambda, \theta) \, d\lambda \, d\theta}.$$

where $\begin{cases} h_1(\lambda, \theta) = \lambda \cdot \frac{1}{(1+\lambda)(1+\theta)} \\ h_2(\lambda, \theta) = \frac{1}{(1+\lambda)(1+\theta)} \end{cases}$ and $\lambda \sim \text{Gamma}\left(\sum_{i=1}^{19} x_{2i} + 1, 19\right)$

$\lambda + \theta \sim \text{Gamma}\left(\sum_{i=1}^{19} x_{1i} + 1, 19\right)$

And $\theta \sim \text{Gamma}\left(\sum_{i=1}^{19} x_{1i} + 1, 19\right) - \text{Gamma}\left(\sum_{i=1}^{19} x_{2i} + 1, 19\right)$

$$\therefore E[\lambda_c | x] = \frac{\sum_{k=1}^{n} \lambda_k \frac{1}{(1+\lambda_k)(1+\theta_k)}}{\sum_{k=1}^{n} \frac{1}{(1+\lambda_k)(1+\theta_k)}}$$ ~~and n is the~~

and n is the generate times for λ and θ.

$$E[\theta | x] = \frac{\sum_{k=1}^{n} \theta_k \cdot \frac{1}{(1+\lambda_k)(1+\theta_k)}}{\sum_{k=1}^{n} \frac{1}{(1+\lambda_k)(1+\theta_k)}}$$