Topics in Algebra — Feedback Exercise 4

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Question (1)

By Theorem 38.11 in Fraleigh, we know that a subgroup of a free abelian group is also a free abelian group. Furthermore, it has a rank less than or equal to 3.

For example, if G is a free abelian group, then it could be of the form $G \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ - by the Fundamental Theorem of Finitely Generated Abelian Groups. Consequently, we have a rank 3 subgroup: $H \cong \mathbb{Z} \times \mathbb{Z} \times 2\mathbb{Z}$, generated by $\{(1,0,0),(0,1,0),(0,0,2)\}$. We have a rank 2 subgroup, $H \cong \mathbb{Z} \times \mathbb{Z}$, generated by $\{(1,0),(0,1)\}$. We have a rank 1 subgroup, $H \cong \mathbb{Z}$, generated by $\{1\}$. Finally, we have a rank 0 subgroup, the trivial group.

Question (2)

a) Let G be a free group of rank 2, $G = \langle a, b : - \rangle$. Since there is 10 elements in $\mathbb{Z}_{10}$, then there is 10 options for each generator, 100 options total. By theorem 39.12 in Fraleigh, any assignment of generators defines a homomorphism, hence we have 100 homomorphisms.

b) The set of "onto" homomorphisms is a subset of the "into" homomorphisms, so we have at most 100 homomorphisms. For this case, we want to avoid the homomorphisms being assigned into subgroups, in order to be surjective. The subgroup of elements $\{0,2,4,6,8\}$ has 5 elements and consequently contains 25 homomorphisms, as per theorem 39.12. Similarly, the subgroup of elements $\{0,5\}$ has 2 elements and contains 4 homomorphisms. This would be a total of 29 homomorphisms but the homomorphism of sending a and b to 0 is being counted twice. Hence, total "onto" homomorphisms is $100 - (25 + 4 - 1) = 100 - 28 = 72$. There are 72 "onto" homomorphisms from G to $\mathbb{Z}_{10}$.