Measure & Probability Theory — Feedback Exercise 4

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Question (1)

(i) We define $f(x) = \lim_{n\to\infty} f_n(x)$. Assume that DCT is true without an integrable function $g \geq 0$ such that $|f_n| \leq g$ for all n. Then, by the DCT, $\lim_{n\to\infty} \int_X f_n d\mu = \int_X \lim_{n\to\infty} f_n d\mu = \int_X f d\mu$.

We see that f_n is not bounded by an integrable function g, for all $n \in \mathbb{N}$, since for any $x \in \mathbb{R}$, there exist an $n \in \mathbb{N}$ such that $f_n(x) = 1$, so $g(x) \geq 1$ for all $x \in \mathbb{R}$. Given that this would not be a finite integral, g is not an integrable function.

Notice that for all $x \in \mathbb{R}$ and for all $\epsilon > 0$, there is $N = \text{ceil}(x) + 2$ such that for all $n > N$, $|f_n(x) - 0| < \epsilon$. Then, we can see that f_n pointwise converges to 0, hence $f(x) = 0$ for all $x \in \mathbb{R}$.

For each $n \in \mathbb{N}$, f_n is a characteristic function on a measurable set. Hence, we can calculate integral as the simple measurable function $\int_X f_n d\mu = \int_X \chi_{[n,n+1]} d\mu = \mu([n, n+1]) = 1$.

Combining this, we see that $\int_X \lim_{n\to\infty} f_n d\mu = \int_X f d\mu = 0$, while $\lim_{n\to\infty} \int_X f_n d\mu = \lim_{n\to\infty} 1 = 1$. Contradicting part (iii) of the DCT.

(ii) Following from the previous part, we have that $\int_X f_n d\mu = 1$, for all $n \in \mathbb{N}$, while $\int_X f d\mu = 0$. Hence, $\int_X \lim_{n\to\infty} f_n d\mu = \int_X f d\mu < \lim_{n\to\infty} \int_X f_n d\mu$.

Question (2)

(i) Since the Riemann integral coincides with Lebesgue integral, provided that the Riemann integral exists, then $\int_A f(x) d\mu$ is the integral of $\sin(x)$ on the interval $[0, \pi/2]$, which is $-\cos(\pi/2) + \cos(0) = 0 + 1 = 1$.

(ii) Given that the integral over a null-set is a 0 then we can split the integral of this piecewise function. The rationals on the interval A are countable and, hence, are a null-set on the Lebesgue measure.

We write the integral such as $\int_A g(x) d\mu = \int_{A\setminus\mathbb{I}} g(x) d\mu + \int_{A\setminus\mathbb{Q}} g(x) d\mu$. Due to the piecewise definition of the function, this becomes $\int_{A\setminus\mathbb{I}} \sin(x) d\mu + \int_{A\setminus\mathbb{Q}} \cos(x) d\mu = \int_A \cos(x) d\mu$. Then, this integral coincides with the Riemann integral on $[0, \pi/2]$ and $\int_{[0,\pi/2]} \cos(x) = 1$.

(iii) Consider $A = U \cup V$, where $U := \{x \in A : \cos(x) \in \mathbb{I}\}$, $V := \{x \in A : \cos(x) \in \mathbb{Q}\}$. Since U and V are trivially disjoint then $\int_A h(x) d\mu = \int_U h(x) d\mu + \int_V h(x) d\mu$. However, $\cos(x)$ is irrational for all non-zero rational values x, hence $V = \{0, \pi/2\}$ and $\mu(V) = 0$. Consequently, $\int_A h(x) d\mu = \int_U h(x) d\mu$.

By definition of h, then h constrained to U is $\sin^2(x)$, hence $\int_A h(x) = \int_U \sin^2(x) d\mu$. Once more, this is a continuous function which coincides with the Riemann integral and — TO FINISH.