Measure & Probability Theory — Feedback Exercise 2

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Question (1)

(i) We write $A_1 = [\frac{2}{10}, \frac{3}{10})$. Similarly, we write A_2 as a long chain of unions: $A_2 = [\frac{2}{100}, \frac{3}{100}) \cup [\frac{12}{100}, \frac{13}{100}) \cup [\frac{22}{100}, \frac{23}{100}) \cup [\frac{32}{100}, \frac{33}{100}) \cup [\frac{42}{100}, \frac{43}{100}) \cup [\frac{52}{100}, \frac{53}{100}) \cup [\frac{62}{100}, \frac{63}{100}) \cup [\frac{72}{100}, \frac{73}{100}) \cup [\frac{82}{100}, \frac{83}{100}) \cup [\frac{92}{100}, \frac{93}{100})$. The, admittedly long, chain can be simplified into $\bigcup\limits_{i \in I} [\frac{2+10i}{100}, \frac{3+10i}{100})$, where $I = \{0, 1, \dots, 9\}$.

This generates the intuition to find the general form for the set A_n. The set A_n can be written as $A_n = \bigcup\limits_{i \in I_n} [\frac{2+10i}{10^n}, \frac{3+10i}{10^n})$, where $I_n = \{i \in \mathbb{Z} : i < 10^{n-1}\}$.

Consider the Borel σ-algebra generated by $\mathcal{B}(\mathbb{R}) = \mathcal{A}(\{[r, \infty) : r \in \mathbb{R}\})$. Let $p \in \mathbb{N}$ and $i \in I_p$, defined previously. Then, by definition of $\mathcal{B}(\mathbb{R})$, $[\frac{2+10i}{10^p}, \infty)$ and $[\frac{3+10i}{10^p}, \infty)$ are in $\mathcal{B}(\mathbb{R})$. Furthermore, $\mathcal{B}(\mathbb{R})$ is a σ-algebra so $(-\infty, \frac{3+10i}{10^p}) \in \mathcal{B}(\mathbb{R})$ and $[\frac{2+10i}{10^p}, \infty) \cap (-\infty, \frac{3+10i}{10^p}) = [\frac{2+10i}{10^p}, \frac{3+10i}{10^p}) \in \mathcal{B}(\mathbb{R})$.

Since p and i were arbitrary, A_n is a union of Borel sets and consequently a Borel set.

(ii) The set A can be written simply as a countable union $A = \bigcup\limits_{n \in \mathbb{N}} A_n$. Then A is a countable union of Borel sets and, therefore, a Borel set.

Question (2)

(i) The outer measure $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is defined
$\mu^*(A) = \inf\{\sum_{j=0}^{\infty} \mu(E_j) : E_1, E_2, \dots \in \mathcal{P}(X) \text{ s.t. } A \subseteq \bigcup_{j=0}^{\infty} E_j\}$, for $A \in \mathcal{P}_f(X)$.

For a countable set $A \in \mathcal{P}_f(X)$, $A = \bigcup_{a \in A}\{a\}$. Then, using subadditivity, we see that
$\mu^*(\bigcup_{a \in A}\{a\}) \leq \sum_{a \in A} \mu^*(\{a\}) = \sum_{a \in A} \mu(\{a\})$. Hence, $\mu^*(A) \leq \sum_{a \in A} \mu(\{a\})$.

Furthermore, by the definition of μ^*, we see that $\mu^*(A) \geq \sum_{a \in A} \mu(\{a\})$. Hence, $\mu^*(A) = \sum_{a \in A} \mu(\{a\})$, as required.

(ii) Let $A \in \mathcal{P}(X)$, then A is countable since X is countable. Let S be any subset of X. Then
$\mu^*(S \cap A) + \mu^*(S \cap A^c) = \sum_{a \in (S \cap A)} \mu(\{a\}) + \sum_{a \in (S \cap A^c)} \mu(\{a\}) = \sum_{a \in S} \mu(\{a\}) = \mu^*(S)$. Hence $A \in \mathcal{M}_{\mu^*}$ and since A was arbitrary then $\mathcal{P}(X) \subseteq \mathcal{M}_{\mu^*}$.

However, $\mathcal{M}_{\mu^*} \subseteq \mathcal{P}(X)$ by definition of $\mathcal{P}(X)$, therefore $\mathcal{M}_{\mu^*} = \mathcal{P}(X)$.

(iii) To show that μ^* is a measure on $\mathcal{P}(X)$, we see that $\mu^*(\emptyset) = \sum_{a \in \emptyset} \mu(\{a\}) = 0$, as required. We also see that, for any $A \in \mathcal{P}(X)$, since $\mu^*(A)$ is the sum of μ over the singletons of A and μ is a measure, $\mu^*(A) \geq 0$.

Let $A_1, \dots \in \mathcal{P}(X)$ such that the sets are pairwise disjoint then $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{P}(X)$, by definition of the power set. Let $B = \bigcup_{i \in \mathbb{N}} A_i$. Then $\mu^*(B) = \sum_{b \in B} \mu(\{b\}) = \sum_{i \in \mathbb{N}}(\sum_{a \in A_i} \mu(\{a\})) = \sum_{i \in \mathbb{N}} \mu^*(A_i)$, as required. Hence, μ^* is a measure on $\mathcal{P}(X)$.

Question (3)

i) Let $N \subseteq X$ be a null-set and $A \subseteq N$. Then, by Lemma 2.15, $\mu^*(A) \leq \mu^*(N) = 0$, hence $\mu^*(A) = 0$ and A is also a null-set.

ii) Let $A \subset X$ such that A is a null-set. Then let S be any subset of X. Note that $(S \cap A) \subseteq A$ so $\mu^*(S \cap A) = 0$. Similarly, since $\mu^*(A) = 0$, we can see that $\mu^*(S \setminus A) = \mu^*(S) - \mu^*(A) = \mu^*(S)$.

Hence, $\mu^*(S \cap A) + \mu^*(S \cap A^c) = \mu^*(S)$, satisfying Carathéodory's condition.

iii) Let A be any countable set, such that $A = \{a_1, a_2, \dots\}$. By the definition of the outer measure, λ^*, we know that $\lambda^*(A) = \inf\{\sum_{k=1}^{\infty} \lambda(I_k) : A \subseteq \bigcup_{k=1}^{\infty} I_k\}$. Let $\epsilon > 0$, for a given $a_j \in A$, let $I_j = (a_j - \frac{\epsilon}{2^j}, a_j + \frac{\epsilon}{2^j})$. Since A is countable then the union $\bigcup_{j=1}^{\infty} I_j$ must be countable and $A \subseteq \bigcup_{j=1}^{\infty} I_j$.

For any given $k \in \mathbb{N}$, $\lambda(I_k) = \frac{2\epsilon}{2^k}$. Hence $\sum_{k=1}^{\infty} \lambda(I_k) = \sum_{k=1}^{\infty} \frac{2\epsilon}{2^k} = 2\epsilon \sum_{k=1}^{\infty} 2^{-k} = 2\epsilon$. The outer measure λ^* is defined as the infimum so $\lambda^*(A) \leq 2\epsilon$ but ϵ is arbitrary, so $\lambda^*(A) \leq 0$. Therefore, $\lambda^*(A) = 0$ and A is a null-set.