Measure & Probability Theory — Feedback Exercise 3

Samuel Jackson — 2520998j

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Question (1)

For a given measure space $(X, \mathcal{A}, \mu)$, let $f : X \to \mathbb{R}$ be measurable. Consider $\mathcal{C} = \{B \in \mathcal{B}(\mathbb{R}) : f^{-1}(B) \in \mathcal{A}\}$, so $C \subseteq B$. Since $\mathcal{A}$ is a σ-algebra then $X, \emptyset \in \mathcal{A}$. Furthermore, we see that $f^{-1}(\mathbb{R}) = X$ and $f^{-1}(\emptyset) = \emptyset$ which means that $\mathbb{R}$ and $\emptyset$ are in $\mathcal{C}$, since $\mathbb{R}$ and $\emptyset$ are part of the Borel σ-algebra.

Suppose that $B \in \mathcal{C}$ then $f^{-1}(B) \in \mathcal{A}$. Then, we view $B^c = \mathbb{R} \setminus B$ and see that $f^{-1}(B^c) = f^{-1}(\mathbb{R} \setminus B) = f^{-1}(\mathbb{R}) \setminus f^{-1}(B)$. Since $\mathcal{A}$ is a σ-algebra and $f^{-1}(R)$ and $f^{-1}(B)$ are in $\mathcal{A}$ then the set difference must be in $\mathcal{A}$ so $f^{-1}(B^c) \in \mathcal{A}$ and $B^c \in \mathcal{C}$.

Let $B_1, B_2, ... \in \mathcal{C}$ then since B_i are Borel sets, for all $i \in \mathbb{N}$, then $\bigcup_{i=0}^{\infty} B_i \in \mathcal{B}(\mathbb{R})$. Furthermore, since $\mathcal{A}$ is a σ-algebra, $f^{-1}(B_1) \cup f^{-1}(B_2) \cup ... = \bigcup_{i=0}^{\infty} f^{-1}(B_i) = f^{-1}(\bigcup_{i=0}^{\infty} B_i) \in \mathcal{A}$. Hence, $\bigcup_{i=0}^{\infty} B_i \in \mathcal{C}$. Therefore, C is a σ-algebra.

By definition of the Borel σ-algebra, $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{C}$. However, given $\mathcal{C}$ was defined as a subset of $\mathcal{B}(\mathbb{R})$ then $\mathcal{C} = \mathcal{B}(\mathbb{R})$. This means that all sets $B \in \mathcal{B}(\mathbb{R})$ satisfy the condition that $f^{-1}(B) \in \mathcal{A}$.

Question (2)

Let $(X, \mathcal{A}, \mu)$ be a measure space. Consider $x \in A$, then $h^{-1}(x) = f^{-1}(x)$, by the definition of h. Similarly, consider $x \in A^c$, then $h^{-1}(x) = g^{-1}(x)$. Consequently, this generalises to the cases that $A \cap h^{-1}(a, \infty] = A \cap f^{-1}(a, \infty]$ and $A^c \cap h^{-1}(a, \infty] = A^c \cap g^{-1}(a, \infty]$. Since f and g are measurable then $f^{-1}(a, \infty]$ and $g^{-1}(a, \infty]$ are in $\mathcal{A}$ for all $a \in \mathbb{R}$. Furthermore, since the measurable sets form a σ-algebra then the intersection of measurable sets is measurable, hence $A \cap f^{-1}(a, \infty]$ and $A^c \cap g^{-1}(a, \infty]$ are measurable sets, consequently they are in $\mathcal{A}$.

Finally, we can view $h^{-1}(a, \infty] = (A \cap h^{-1}(a, \infty]) \cup (A^c \cap h^{-1}(a, \infty]))$ and given that both sets of the union are in $\mathcal{A}$ then $h^{-1}(a, \infty]$ is also in $\mathcal{A}$, as required.