Topics in Algebra — Feedback Exercise 5

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Question (1)

Let G be a nonabelian group of order 14.

a) By prime decomposition series, $14 = 7 \cdot 2$. Hence, by Sylow I, there exist a subgroup of H and K of order 7 and 2 respectively. Since H has prime order (7), then it must be cyclic, let $H = \langle a \mid a^7 \rangle$.

   Define $n = |\{P : P \text{ is a Sylow 7-subgroup}\}|$. By Sylow III, $n \equiv 1 (\mod 7)$ and $n \mid 14$. The only overlapping element for n from these conditions is 1, hence $n = 1$. As there is only one subgroup of order 7, it must be normal.

b) Let $b \in G$ and $b \notin H$, then b must have order $1, 2, 7$ or 14 by Lagrange's. Given $b \notin H$, then b cannot have order 7, since H is the only order 7 subgroup in G. Similarly, it cannot be trivial, since the trivial element is in H.

   Since G is nonabelian, it cannot be cyclic so b cannot have order 14. Hence, it must have order 2. This means that $b^2 = 1$. Denote $K = \langle b \mid b^2 \rangle$. Since $H \cap K = \{1\}$ and H is normal in G, we have that $H \times K$ is a subgroup of G order 14, which must be G, hence $H \times K \cong G$. Consequently, as both H and K are cyclic, $\{a, b\}$ are a generating set of G. Furthermore, since G is nonabelian, we know that the generators of H and K must not commute, so $ba \neq ab$.

c) Since H is normal, $bab^{-1} \in H$, for all $b \in G, a \in H$. So, $bab^{-1} = a^r$, for $a^r \in H$, which implies $ba = a^r b$. If $r = 1$, then $ba = ab$, against our hypothesis, so $r \neq 1$. Similarly, if $r = 7$, then $bab^{-1} = 1$ which suggests that a is trivial, however a is order 7 so that is not possible. Hence $2 \leq r \leq 6$.

d) By Exercise 40.13b in Fraleigh, a group presentation of the form $\langle a, b \mid a^m = 1, b^n = 1, ba = a^r b \rangle$ is a group of order mn iff $r^n \equiv 1 \pmod{m}$. Since $G = \langle a, b \mid a^7 = 1, b^2 = 1, ba = a^r b \rangle$ and $|G| = 14$, we know that $r^2 \equiv 1 \pmod 7$. Given that $2 \leq r \leq 6$, we can compute possiblities for r. Through computation, $2^2 \pmod 7 = 4$, $3^2 \pmod 7 = 2$, $4^2 \pmod 7 = 2$, $5^2 \pmod 7 = 4$, $6^2 \pmod 7 = 1$. Hence $r = 6$.

   Consequently, as G is an abstract nonabelian group of order 14, we see that all nonabelian groups of order 14 are isomorphic to the group presentation: $\langle a, b \mid a^7 = 1, b^2 = 1, ba = a^6 b \rangle$

Question (2)

From question 1, we know that the isomorphism type of nonabelian groups is given by the presentation above. We only must consider abelian groups.

If G is an abelian group of order 14, then by the same techniques of Sylow theorems above, there is a unique Sylow 7-subgroup, H, and at least one Sylow 2-subgroup, denoted K. Define $H = \langle a \mid a^7 = 1 \rangle$ and $K = \langle b \mid b^2 = 1 \rangle$. Then $H \cap K = \{1\}$. Since H is normal in G, then $H \times K \cong G$. However, since G is abelian, $ab = ba$ which means that $G = \langle a, b \mid a^7 = 1, b^2 = 1, ab = ba \rangle$. By the theorem of Finitely Generated Abelian Groups, $G \cong \mathbb{Z}_7 \times \mathbb{Z}_2$, so $G \cong \mathbb{Z}_{14}$.
Hence, the isomorphism types of groups of order 14 are $\mathbb{Z}_{14}$ and $\langle a, b \mid a^7 = 1, b^2 = 1, ba = a^6 b \rangle$.