Algebraic Topology — Feedback Exercise 5

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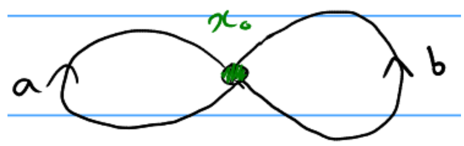
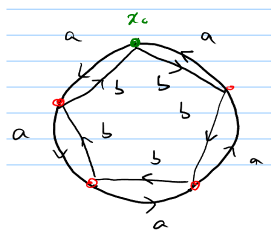
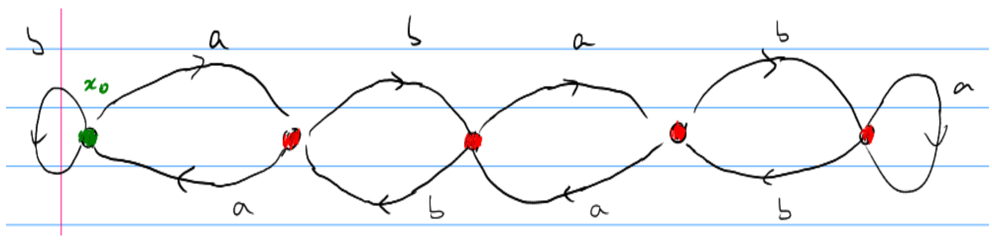
Question (1)

a) For the given case of X, a wedge sum of two circles, we have one vertex. Define X as in the image below:



The vertex, and basepoint, x_0 has an input and an output for directed edges a and b. A five-sheeted covering requires that every point $x \in X$ has 5 elements in the preimage. The two coverings below are 5-sheeted coverings (denoted H, K respectively):





(Apologies, I could not get them to look nice together).

For each vertex, coloured red or green, there is an outgoing a and b edge alongside an ingoing a and b edge. Note that the edges on end vertices of the chain-based covering are both ingoing and outgoing edges.

These are clearly not homeomorphic since a^2 (from X) lifts to a loop with the chain-sheet covering but does not lift to a loop in the circular-sheet covering.

b) The covering H permits a spanning tree which travels anti-clockwise along the a edges. Using the spanning tree, we find the generators for H to be $A := \{ab, a^2ba^{-1}, a^3ba^{-2}, a^4ba^{-3}, a^4b^{-1}, a^5\}$. Then, the elements of A generate a subgroup of $\pi_1(X, x_0)$.

Similarly, for the covering K, a spanning tree which travels along the top edges, alternating a and b, provides the generators $B := \{b, a^2, ab^2a^{-1}, aba^2b^{-1}a^{-1}, abab^2a^{-1}b^{-1}a^{-1}, ababab^{-1}a^{-1}b^{-1}a^{-1}\}$. Hence, we have the subgroup of $\pi_1(X, x_0)$ generated by the elements of B.

Question (2)

a) For the given X, $\Delta_0(X) = \langle t, u, w, v, x, y, z \rangle$, $\Delta_1(X) = \langle a, b, c, d, e, f, g \rangle$ and finally $\Delta_2(X) = 0$, the trivial group.

   We define ∂_1 to be $\partial_1 : \Delta_1(X) \to \Delta_0(X)$. For an edge p, $\partial_1(p) = p_1 - p_0$, where p_0, p_1 are the start and end of the edge respectively.

   Similarly, $\partial_2 : \Delta_2(X) \to \Delta_1(X)$. However, since Δ_2 is the trivial group, ∂_2 is the trivial homomorphism. This gives us the groups $\ker(\partial_1) = \langle d, e, f - g \rangle$ and $\mathrm{im}(\partial_1) = \langle t - w, u - w, z - y, v - w \rangle$.

b) By definition of $H_0(X)$, we have $H_0(X) = \Delta_0(X)/\mathrm{im}(\partial_1) =$ $\langle t, u, w, v, x, y, z : t = w, u = w, v = w, z = y \rangle = \langle w, x, y \rangle \cong \mathbb{Z}^3$.

c) Similar to last question, by definition $H_1(X) = \ker(\partial_1)/\mathrm{im}(\partial_2)$. Since $\Delta_2(X)$ is the trivial group, the homomorphism is the trivial homomorphism, hence $\mathrm{im}(\partial_2) = 0$. Then, $H_1(X) = \ker(\partial_1) = \langle d, e, f - g \rangle \cong \mathbb{Z}^3$.