Measure & Probability Theory — Feedback Exercise 4

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Question (1)

(i) We define $f(x) = \lim_{n\to\infty} f_n(x)$. Assume that DCT is true without an integrable function $g \geq 0$ such that $|f_n| \leq g$ for all n. Then, by the DCT, $\lim_{n\to\infty} \int_X f_n d\mu = \int_X \lim_{n\to\infty} f_n d\mu = \int_X f d\mu$.

We see that f_n is not bounded by an integrable function g, for all $n \in \mathbb{N}$, since for any $x \in \mathbb{R}$, there exist an $n \in \mathbb{N}$ such that $f_n(x) = 1$, so $g(x) \geq 1$ for all $x \in \mathbb{R}$. Given that this would not be a finite integral, g is not an integrable function.

Notice that for all $x \in \mathbb{R}$ and for all $\epsilon > 0$, there is $N = \text{ceil}(x) + 2$ such that for all $n > N$, $|f_n(x) - 0| < \epsilon$. Then, we can see that f_n pointwise converges to 0, hence $f(x) = 0$ for all $x \in \mathbb{R}$.

For each $n \in \mathbb{N}$, f_n is a characteristic function on a measurable set. Hence, we can calculate integral as the simple measurable function $\int_X f_n d\mu = \int_X \chi_{[n,n+1]} d\mu = \mu([n, n+1]) = 1$.

Combining this, we see that $\int_X \lim_{n\to\infty} f_n d\mu = \int_X f d\mu = 0$, while $\lim_{n\to\infty} \int_X f_n d\mu = \lim_{n\to\infty} 1 = 1$. Contradicting part (iii) of the DCT.

(ii) Following from the previous part, we have that $\int_X f_n d\mu = 1$, for all $n \in \mathbb{N}$, while $\int_X f d\mu = 0$. Hence, $\int_X \lim_{n\to\infty} f_n d\mu = \int_X f d\mu < \lim_{n\to\infty} \int_X f_n d\mu$.

Question (2)

(i) Since the Riemann integral coincides with Lebesgue integral, provided that the Riemann integral exists, then $\int_A f(x) d\mu$ is the integral of $\sin(x)$ on the interval $[0, \pi/2]$, which is $-\cos(\pi/2) + \cos(0) = 0 + 1 = 1$.

(ii) Since $\mathbb{Q}$ and $\mathbb{I}$ are disjoint then $\int_A g d\mu = \int_{A\cap\mathbb{Q}} g d\mu + \int_{A\cap\mathbb{I}} g d\mu$. However, $\mathbb{Q}$ is countable and $\mu(\mathbb{Q}) = 0$ which means that $\int_{A\cap Q} g d\mu = 0$.

By definition of g, we have that $\int_{A\cap\mathbb{I}} g d\mu = \int_{A\cap\mathbb{I}} \cos d\mu$. Then $\int_{A\cap\mathbb{I}} \cos(x) d\mu = \int_A \cos(x) d\mu - \int_{A\cap\mathbb{Q}} \cos(x) d\mu$. Once again, $\int_{A\cap\mathbb{I}} \cos(x) d\mu = 0$, so $\int_A g d\mu = \int_A \cos d\mu$. This coincides with the Riemann integral and gives us that $\int_A g d\mu = \sin(\pi/2) - \sin(0) = 1$.

(iii) To solve this question, we discuss when $\cos(x)$ is rational, on the set A. Since $\cos(x)$ is bijective on $[0, \pi/2]$, then it must map countable sets to countable sets. Hence, $\cos(x)$ is rational precisely when x is rational, implying that $U := \{x : \cos(x) \in \mathbb{Q}\}$ is countable.

We redefine the function h as follows:

$$h(x) = \begin{cases} \sin(x) & x \in U \\ \sin^2(x) & x \in A \setminus U \end{cases}$$

Then, we have that $\mu(U) = 0$, since U is countable. So, $\int_A h(x) d\mu = \int_U h(x) d\mu + \int_{A\setminus U} h(x) d\mu = \int_{A\setminus U} h(x) d\mu$. By definition of h, we have that $\int_{A\setminus U} h(x) d\mu = \int_{A\setminus U} \sin^2(x) d\mu$. We expand this into $\int_{A\setminus U} \sin^2(x) d\mu = \int_A \sin^2(x) d\mu - \int_U \sin^2(x) d\mu$. The integral over U is 0 since U is countable and so $\int_A h(x) d\mu = \int_A \sin^2(x) d\mu$. This is a continuous function on a bounded interval A and must coincide with the Riemann integral.

The Riemann integral of $\int_A \sin^2(x) d\mu = \frac{1}{2}(\frac{\pi}{2} - \frac{\sin(\pi)}{2}) = \frac{\pi}{4}$. Hence $\int_A h d\mu = \pi/4$.