Measure & Probability Theory — Feedback Exercise 4

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Question (1)

(i) Part 1

(ii) Part 2

Question (2)

(i) We are back baby!

(ii) Since the Riemann integral coincides with Lebesgue integral, provided that the Riemann integral exists, then $\int_A f(x)d\mu$ is the integral of $\sin(x)$ on the interval $[0, \pi/2]$, which is $-\cos(\pi/2) + \cos(0) = 0 + 1 = 1$.

(iii) Given that the integral over a null-set is a 0 then we can split the integral of this piecewise function. The rationals on the interval A are countable and, hence, are a null-set on the Lebesgue measure.

We write the integral such as $\int_A g(x)d\mu = \int_{A\setminus\mathbb{I}} g(x)d\mu + \int_{A\setminus\mathbb{Q}} g(x)d\mu$. Due to the piecewise definition of the function, this becomes $\int_{A\setminus\mathbb{I}} \sin(x)d\mu + \int_{A\setminus\mathbb{Q}} \cos(x)d\mu = \int_A \cos(x)d\mu$. Then, this integral coincides with the Riemann integral on $[0, \pi/2]$ and $\int_{[0,\pi/2]} \cos(x) = 1$.

(iv) Consider $A = U \cup V$, where $U := \{x \in A : \cos(x) \in \mathbb{I}\}$, $V := \{x \in A : \cos(x) \in \mathbb{Q}\}$. Since U and V are trivially disjoint then $\int_A h(x)d\mu = \int_U h(x)d\mu + \int_V h(x)d\mu$. However, $\cos(x)$ is irrational for all non-zero rational values x, hence $V = \{0, \pi/2\}$ and $\mu(V) = 0$. Consequently, $\int_A h(x)d\mu = \int_U h(x)d\mu$.

By definition of h, then h constrained to U is $\sin^2(x)$, hence $\int_A h(x) = \int_U \sin^2(x)d\mu$. Once more, this is a continuous function which coincides with the Riemann integral and — TO FINISH.