Differential Geometry — Feedback Exercise 1

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Question (1)

For the function to be a parameterised curve, we require that that the function is smooth and defined on an open interval. Clearly α is defined on the open interval $(0, 100)$. To determine if α is smooth, we recognise that sin, cos and polynomials are smooth functions as well as the fact that the composition of smooth functions is smooth. Given α is made up of three smooth functions, we know α is also smooth.

For α to be regular, we require that $\dot{\alpha}(s) \neq \mathbf{0}$ for all $s \in (0, 100)$. Deriving $\dot{\alpha}$, we find: $\dot{\alpha} = ((2t-1)\cos(t^2 - t), (1-2t)\sin(t^2 - t), 2t - 1)$. Immediately, we see that for $\dot{\alpha}(\frac{1}{2}) = (0, 0, 0)$, so α is not regular.

Question (2)

Given the components of γ are smooth functions $\sin(s)$ and $\cos(s)$, γ is similarly a smooth function on the open interval $\mathbb{R}$. Furthermore, γ is regular since $\dot{\gamma}$ is solely comprised of similar $-\sin(s)$ and $\cos(s)$ functions which can not be 0 simultaneously, for $s \in \mathbb{R}$. Hence, γ is a regular parameterised curve (RPC).

For the RPC γ, we require that $||\dot{\gamma}|| = 1$ for γ to be unit-speed. Hence, we calculate $\dot{\gamma}(s) = (-\sin(s), \dots, -\sin(s), \cos(s), \dots, \cos(s))$, where there is n-total $-\sin$ and cos components respectively. Then, we calculate the magnitude of $\dot{\gamma}$:

$$
\begin{aligned}
||\dot{\gamma}(s)|| &= \sqrt{\frac{1}{n}(\sin^2(s) + \dots + \sin^2(s) + \cos^2(s) + \dots + \cos^2(s))} \\
||\dot{\gamma}(s)|| &= \sqrt{\frac{1}{n}(n(\sin^2(s) + \cos^2(s))} \\
||\dot{\gamma}(s)|| &= \sqrt{1} \\
||\dot{\gamma}(s)|| &= 1
\end{aligned}
$$

Hence, γ is a unit-speed curve.

Consequently, we calculate the Frenet-Serret frame for $n = 1$. We have $\gamma_1 : \mathbb{R} \to \mathbb{R}^2$, $s \mapsto (\cos(s), \sin(s))$, so $\dot{\gamma_1} = (-\sin(s), \cos(s)) = \mathbf{T}$. Similarly, since γ_1 is unit-speed, we can calculate $\mathbf{N}$, which is simple for curves in $\mathbb{R}^2$. $\mathbf{N} = (-\cos(s), -\sin(s))$. Therefore, the Frenet-Serret frame is $\{\mathbf{T}, \mathbf{N}\}$.

Question (3)

To find that γ is of unit-speed, we calculate the derivative as $\dot{\gamma} = (\frac{1}{2}\sqrt{1+s}, \frac{-1}{2}\sqrt{1-s}, \frac{1}{\sqrt{2}})$. Consequently, we calculate the magnitude as follows:

$$
\begin{aligned}
||\dot{\gamma}(s)|| &= \sqrt{\frac{1}{4}(1+s) + \frac{1}{4}(1-s) + \frac{1}{2}} \\
||\dot{\gamma}(s)|| &= \sqrt{\frac{1}{4} + \frac{s}{4} + \frac{1}{4} - \frac{s}{4} + \frac{1}{2}} \\
||\dot{\gamma}(s)|| &= \sqrt{\frac{1}{2} + \frac{1}{2}} \\
||\dot{\gamma}(s)|| &= \sqrt{1} \\
||\dot{\gamma}(s)|| &= 1
\end{aligned}
$$

Hence, γ is unit-speed.

To calculate the curvature and torsion, we use the respective equations:

$$\kappa = \frac{||\dot{\gamma} \times \ddot{\gamma}||}{||\dot{\gamma}||^3}$$

$$\tau = \frac{\det(\dot{\gamma} \mid \ddot{\gamma} \mid \dddot{\gamma})}{||\dot{\gamma} \times \ddot{\gamma}||^2}$$

As the equations necessitate, we need the second and third derivatives of γ which, respectively, are:

$$\ddot{\gamma} = \left(\frac{1}{4\sqrt{1+s}}, \frac{1}{4\sqrt{1-s}}, 0 \right)$$
$$\dddot{\gamma} = \left(\frac{-1}{8(1+s)^{\frac{-3}{2}}}, \frac{1}{8(1-s)^{\frac{-3}{2}}}, 0 \right)$$

Firstly, note that γ is a unit-speed curve so $||\dot{\gamma}|| = 1$, hence $\kappa = ||\ddot{\gamma}||$. We solve for the curvature, κ, first:

$$\kappa = ||\ddot{\gamma}||$$
$$\kappa = \sqrt{\frac{1}{16(1+s)} + \frac{1}{16(1-s)}}$$
$$\kappa = \frac{1}{\sqrt{16}} \sqrt{\frac{1}{1+s} + \frac{1}{1-s}}$$
$$\kappa = \frac{1}{4} \sqrt{\frac{2}{1-s^2}}$$
$$\kappa = \frac{1}{4} \sqrt{\frac{2}{1-s^2}}$$
$$\kappa = \frac{\sqrt{2}}{4\sqrt{1-s^2}}$$

For τ, it is a longer calculation. We first calculate $\det(\dot{\gamma} \mid \ddot{\gamma} \mid \dddot{\gamma})$:

$$\tau = \det(\dot{\gamma} \mid \ddot{\gamma} \mid \dddot{\gamma})$$
$$\tau = \dot{\gamma} \cdot (\ddot{\gamma} \times \dddot{\gamma})$$
$$\tau = \dot{\gamma} \cdot \left(0, 0, \frac{1}{16(1-s)^{\frac{3}{2}}} \right)$$
$$\tau = \frac{\sqrt{2}}{32(1-s^2)^{\frac{3}{2}}}$$

Similarly, we calculate $||\dot{\gamma} \times \ddot{\gamma}||$, which is just $||\ddot{\gamma}||$ for a unit-speed curve.

$$||\dot{\gamma} \times \ddot{\gamma}|| = ||\, ||\dot{\gamma}||\, ||\ddot{\gamma}||\, \sin(\theta)\mathbf{n}||$$
$$||\dot{\gamma} \times \ddot{\gamma}|| = ||\, 1 \cdot ||\ddot{\gamma}|| \, \cdot 1 \cdot \mathbf{n}||$$
$$||\dot{\gamma} \times \ddot{\gamma}|| = ||\ddot{\gamma}||$$
$$||\dot{\gamma} \times \ddot{\gamma}|| = \frac{\sqrt{2}}{4\sqrt{1-s^2}}$$

Combining these components, we find the torsion, τ:

$$\tau = \frac{\sqrt{2}}{32(1-s^2)^{\frac{3}{2}}} \cdot \frac{16(1-s^2)}{2}$$
$$\tau = \frac{16\sqrt{2}(1-s^2)}{64(1-s^2)^{\frac{3}{2}}}$$
$$\tau = \frac{\sqrt{2}}{4\sqrt{1-s^2}}$$