Lecture Notes: Statistical Modelling
L3 MIDO

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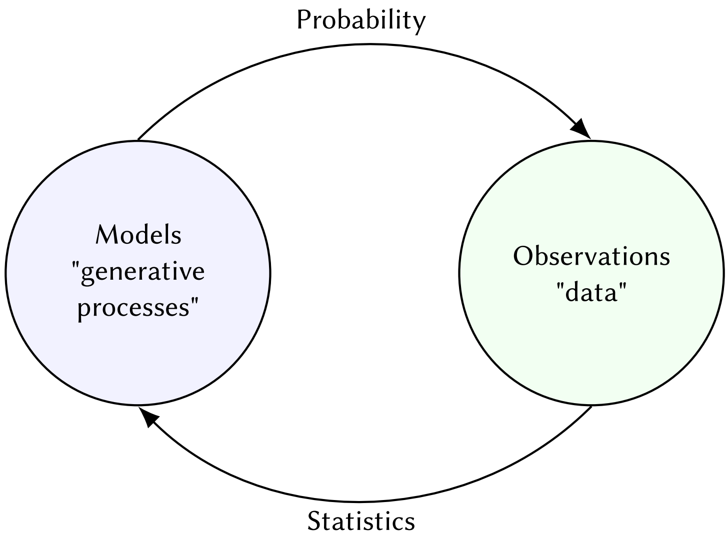
Chapter 1: Introduction to Statistical Modelling

Introduction Schema



I. Data

1) $n = \#$ L3 MIDO students

- Were students in Dauphine in L2?
- Data $\in \{0, 1\}^n$:
  - 1: in L2 in Dauphine.
  - 0: not in L2 in Dauphine.
- $x = (0, 0, 1, 1, 0, \dots)$
- $\downarrow$ sum(x)

$\Longrightarrow$ x is a distribution of n independent experiences of Bernoulli of parameter p.

$$x \sim \text{Ber}(p)^{\otimes n} \quad \text{(i.i.d.)}$$

$$\Longrightarrow \text{sum}(x) \sim \text{Bin}(n, p)$$

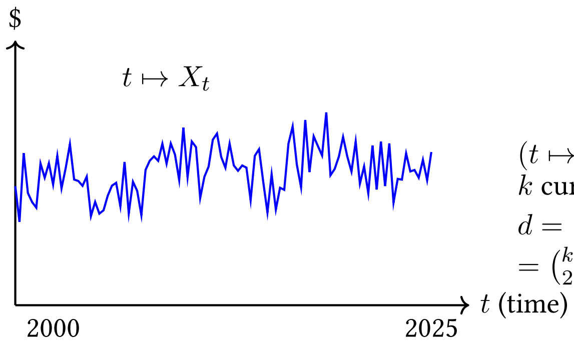
2) INSEE (economy)

Example: Employment rate between 2005 & 2025.

Male ♂	2005	2006	...	2025		**Female** ♀	2005	...
15-24 yo	α	α	α	α		15-24	α	α
25-49 yo	α	56%	α	α		25-49	α	α
50 +	α		α	α		50 +	α	α

$\Longrightarrow$ **Tensor:** multiple entries array.

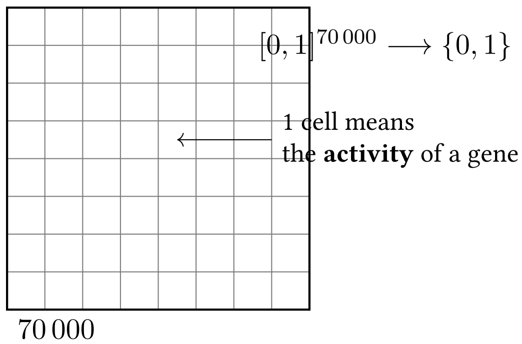
3) Financial data



$(t \mapsto X_t) \in \mathcal{C}([0, T], \mathbb{R}^d)$
k currencies to study, we have
$d = \frac{k(k-1)}{2}$ pairs
$= \binom{k}{2}$

$\implies$ Brownian motion. $\implies$ Diffusion processes.

4) Microarrays



- Compute a big space into a smaller one: the cells of an individual translated to *healthy* or *not*.
- $[0, 1]$ measures the activity of one cell. One person has 70 000 cells.

II. Statistical Modelling

1) Stat models with words

To each set of data, we must associate a scientific question/objective.
To answer that question, we must have at our disposition:

- A methodology.
- A quantitative mathematical (probabilistic) model that accounts for the properties of the data.
- Well suited methods (mathematical) that combine... and...

2) Abstract (simple) example

We toss a coin 18 times and observe ($n = 18$, $H = 0, T = 1$).

$$(0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0)$$

$\equiv$ Data (raw).

- **Statistical model:** We observe $n = 18$ random variables X_i; that are independent and that have the same distribution.

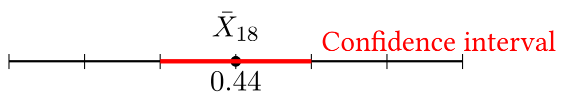
- $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = \theta$.
- $\mathrm{Ber}(\theta)$ where $\theta \in \Theta = [0, 1]$.
- θ is the **unknown parameter**.

Questions:
1) What is a good (the best) estimation of θ?

- A good (?) estimator:

$$\begin{aligned}\underbrace{\bar{X}_n}_{\substack{\text{notation}\\ \text{of the mean}}} &= \frac{1}{n}\sum_{i=1}^{n} X_i \\ &= \frac{1}{18}\sum_{i=1}^{18} X_i = \frac{8}{18} = 0.44\end{aligned}$$

- What is my accuracy of estimation?



$\implies$ We want:

1. Small length.
2. Good coverage.

2) Is the coin fair?
For instance, we compare $\bar{X}_{18}$ to 0.5. If $|\bar{X}_{18} - 0.5|$ is small, we accept the idea that the coin is fair, otherwise we reject that hypothesis.

Chapter 2: Models

I. What is a statistical model?

Definition 1: Statistical Model

Let $X = (X_1, \dots, X_n) \in \mathcal{X}^n$ be a vector of n random variables, where each variable $X_i \in \mathcal{X}$ and $\mathcal{X}$ is a measurable space.
A **model** for X is a set $\mathcal{P}$ of probability distributions on $\mathcal{X}^n$.
Statistical inference will consist in estimating P (the distribution of X) or $F(P)$, with $P \in \mathcal{P}$.

Examples

i) Let $X_1, \dots, X_n$ be i.i.d. $\mathcal{N}(\mu, \sigma^2)$.
Notation: $X_i \underset{i.i.d.}{\sim} \mathcal{N}(\mu, \sigma^2)$ (Gaussian distribution with parameters μ and σ^2). The density is given by: $x \mapsto \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.
Here, $\mathcal{X} = \mathbb{R}$. The model is defined as:

$$\mathcal{P} = \left\{ \mathcal{N}(\mu, \sigma^2)^{\otimes n},\ (\mu, \sigma^2) \in \mathbb{R} \times (0, +\infty) \right\}$$

Remark

Notations: If (X_1, X_2) is a random vector of $\mathbb{R}^2$, with $X_1 \sim P_1$ and $X_2 \sim P_2$.
(By the way: $X \sim P$ means that P is the distribution of $X \in \mathbb{R}$, meaning: $\forall A \in \mathcal{B}(\mathbb{R})$, $P(A) = \mathbb{P}(X \in A) = \mathbb{P}(X^{-1}(A))$.)

$$(\Omega, \mathcal{F}, \mathbb{P}) \xrightarrow{X} (\mathbb{R}, \mathcal{B}(\mathbb{R}), P)$$

P is defined via: $\forall A \in \mathcal{B}(\mathbb{R}), P(A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}(\{\omega \in \Omega, X(\omega) \in A\})$.
If $X_1 \sim P_1$, $X_2 \sim P_2$ and $X_1 \perp\!\!\!\perp X_2$ (X_1 and X_2 are independent):

$$\mathbb{P}(X_1 \in A, X_2 \in B) = \mathbb{P}(X_1 \in A) \cdot \mathbb{P}(X_2 \in B)$$

$\forall A, B \in \mathcal{B}(\mathbb{R})$, the distribution of (X_1, X_2) is defined on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ and is denoted by $P_1 \otimes P_2$.
Computation formula:

$$P_1 \otimes P_2(A \times B) = P_1(A) \cdot P_2(B) = \mathbb{P}(X_1 \in A) \cdot \mathbb{P}(X_2 \in B)$$

In our example, saying $(X_1, \dots, X_n) \sim \mathcal{N}(\mu, \sigma^2)^{\otimes n}$ exactly means:

$$\mathbb{P}(X_1 \in A_1, \dots, X_n \in A_n) = \prod_{i=1}^{n} \mathbb{P}(X_i \in A_i)$$

$$= \prod_{i=1}^{n} \int_{A_i} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} dx_i$$

$$= \int_{A_1} \cdots \int_{A_n} \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right) dx_1 \dots dx_n$$

ii) $X_i \in \{0, 1\}, \forall i = 1, \dots, n$, independently with same distribution $\mathrm{Ber}(p)$, $p \in [0, 1]$.

$$\mathcal{P} = \left\{ \mathrm{Ber}(p)^{\otimes n},\ p \in [0, 1] \right\}, \quad \mathcal{X}^n = \{0, 1\}^n$$

Model Comparison and Types

Let us consider models of the form $\mathcal{P} = \{P^{\otimes n}, P \in \mathcal{P}_0\}$.

- Example (1): $\mathcal{P}_0 = \{\mathcal{N}(\mu, \sigma^2), (\mu, \sigma^2) \in \mathbb{R} \times (0, +\infty)\}$ (Parametric).
- Example (2): $\mathcal{P}_0 = \{\text{Ber}(p), p \in [0, 1]\}$.
- Example (3): $\mathcal{P}_0 = \{\text{All distributions on } (\mathbb{R}, \mathcal{B}(\mathbb{R}))\}$ (Huge model).

Which is the best model? Generally, the best model is the one where the set of parameter values is the **smallest** (Parsimony principle).

Remark

In the 3 examples considered above, all observations are **i.i.d.** But, in many situations, this is not the case.

Example of a very standard non-i.i.d. model:

$$X_0 = x_0; \quad X_t = \rho X_{t-1} + \sigma \varepsilon_t, \quad t = 1, \dots, T$$

We observe $(X_1, \dots, X_T)$. They are not i.i.d. Here $\rho \in \mathbb{R}$, $\sigma > 0$ (Volatility), and $\varepsilon_t \overset{i.i.d}{\sim} \mathcal{N}(0, 1)$. X_t could represent the log price of a financial asset.

In this course, we will mainly focus on the i.i.d. case.

II. Parametric Model

Definition 2: Parametric Model

Let there be data $(X_1, \dots, X_n) \in \mathcal{X}^n$. A model $\mathcal{P}$ for $(X_1, \dots, X_n)$ is called **parametric** if:

$$\mathcal{P} = \{P_\theta, \theta \in \Theta\}$$

with $\Theta \subset \mathbb{R}^d$ for some $d \geq 1$. Here P_θ is a distribution on $\mathcal{X}^n$.

Definition 3: Nuisance Parameters

Let $X = (X_1, \dots, X_n)$ i.i.d. with model $\mathcal{P} = \{P_\theta, \theta \in \Theta\}$. If $\Theta = (\theta_1, \theta_2)$, and if we only are interested in estimating or predicting θ_1 and we don't care about θ_2, we say that:

- θ_1 is the **parameter of interest**.
- θ_2 is the **nuisance parameter**.

III. Identifiability

Context: We have a statistical model for $X = (X_1, \dots, X_n)$. $X_i \sim P_\theta$, $\theta \in \Theta \subset \mathbb{R}^d$. So $\mathcal{P} = \{P_\theta^{\otimes n}, \theta \in \Theta\}$.

Aim: Can we learn/estimate θ from X? This is only possible if we can learn θ from $P_\theta^{\otimes n}$ or P_θ.

Definition 4: Identifiability

A statistical model $\{P_\theta, \theta \in \Theta\}$ is **identifiable** (for θ) if:

$$P_{\theta_1} = P_{\theta_2} \implies \theta_1 = \theta_2$$

(i.e., the map $\theta \mapsto P_\theta$ is injective).

Examples

Example 1: Exponential Distribution

Let $(P_\theta, \theta \in \Theta) = (\mathcal{E}(\theta), \theta > 0)$. P_θ has density $f_\theta(x) = \theta e^{-\theta x} \mathbb{1}_{\{x>0\}}$.
If $P_{\theta_1} = P_{\theta_2}$, then for every $x \in \mathbb{R}$:

$$f_{\theta_1}(x) = f_{\theta_2}(x) \implies \theta_1 e^{-\theta_1 x} = \theta_2 e^{-\theta_2 x} \quad \text{whenever } x > 0$$

Take $x \to 0$ (or $x = 0$ in the limit), we get $\theta_1 = \theta_2$. Thus, the model is identifiable.

Example 2: Gaussian Distribution

Let $\theta = (\mu, \sigma^2) \in \Theta = \mathbb{R} \times (0, +\infty)$.

$$P_\theta = \mathcal{N}(\mu, \sigma^2)$$

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad \forall x \in \mathbb{R}$$

Injectivity? Assume $P_{\theta_1} = P_{\theta_2}$. Then $\forall x \in \mathbb{R}$:

$$\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} = \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(x-\mu_2)^2}{2\sigma_2^2}}$$

Take $x = \mu_1$, then:

$$\frac{1}{\sigma_1} = \frac{1}{\sigma_2} e^{-\frac{(\mu_1-\mu_2)^2}{2\sigma_2^2}}$$

Take $x = \mu_2$, then:

$$\frac{1}{\sigma_1} e^{-\frac{(\mu_2-\mu_1)^2}{2\sigma_1^2}} = \frac{1}{\sigma_2}$$

This implies:

$$\begin{aligned} \frac{\sigma_2}{\sigma_1} &= e^{-\frac{(\mu_1-\mu_2)^2}{2\sigma_2^2}} \leq 1 \quad (\text{because } e^{-\cdot} \leq 1) \\ &= e^{\frac{(\mu_1-\mu_2)^2}{2\sigma_1^2}} \geq 1 \quad (\text{because } e^{\cdot} \geq 1) \end{aligned}$$

Hence:

$$\frac{\sigma_2}{\sigma_1} = 1 \implies \sigma_1 = \sigma_2$$

And substituting back:

$$e^{-\frac{(\mu_1-\mu_2)^2}{2\sigma^2}} = 1 \implies \frac{(\mu_1-\mu_2)^2}{2\sigma^2} = 0 \implies \mu_1 = \mu_2$$

Thus $\theta_1 = \theta_2$. The model is identifiable.

Proposition 1: Identifiability via CDF and Density

- If P_θ admits a cumulative distribution function (CDF) F_θ, the model is identifiable $\iff$ $(\forall\theta_1, \theta_2, F_{\theta_1} = F_{\theta_2} \implies \theta_1 = \theta_2)$.
- If P_θ admits a density f_θ, the model is identifiable $\iff$ $(\forall\theta_1, \theta_2, f_{\theta_1} = f_{\theta_2} \implies \theta_1 = \theta_2)$.

Example 3: Non-identifiable Model (Mixture)

Let $Y_1 \sim \text{Ber}(p_1)$, $Y_2 \sim \text{Ber}(p_2)$. $Y_1 \in \{0, 1\}$. $\mathbb{P}(Y_1 = 1) = p_1$.
Let X be defined as:

$$X = \begin{cases} Y_1 & \text{with proba } \pi \\ Y_2 & \text{with proba } 1 - \pi \end{cases}$$

$$\begin{aligned} \mathbb{P}(X = 1) &= \mathbb{P}(X = 1 | X = Y_1)\mathbb{P}(X = Y_1) + \mathbb{P}(X = 1 | X = Y_2)\mathbb{P}(X = Y_2) \\ &= \mathbb{P}(Y_1 = 1) \times \pi + \mathbb{P}(Y_2 = 1) \times (1 - \pi) \\ &= p_1\pi + p_2(1 - \pi) \end{aligned}$$

We have a model with 3 parameters (p_1, p_2, π). $X \sim \text{Ber}(p)$ with $p = p_1\pi + p_2(1 - \pi)$.
Knowing $\mathbb{P}(X = 1)$ gives limited information on the law of X.
Numeric Examples:

- $\pi = 1/2, p_1 = 0.6, p_2 = 0.2$. $\mathbb{P}(X = 1) = 0.6 \times 0.5 + 0.2 \times 0.5 = 0.4$.
- $\pi = 0.6, p_1 = 1/2, p_2 = 0.25$. $\mathbb{P}(X = 1) = 0.6 \times 0.5 + 0.4 \times 0.25 = 0.3 + 0.1 = 0.4$.

We have different parameters but the same distribution $\implies$ **Not identifiable**.

IV. Empirical Distribution Function

Model: We observe $X_1, \dots, X_n$ i.i.d. P. We set $F(t) =$ Cumulative distribution function of P at t

$$= \mathbb{P}(X_i \leq t), \quad \forall i$$

Remark

Reminder:

- F is non-decreasing.
- F is càdlàg (right-continuous with left limits).
- $\lim_{t\to-\infty} F(t) = 0$ and $\lim_{t\to+\infty} F(t) = 1$.

Definition 5: Empirical CDF

$$\hat{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{X_i \leq t\}}$$

It is a good idea since when $n \to +\infty$: Fix t: $\hat{F}_n(t) \xrightarrow[n\to\infty]{} \mathbb{E}[Y_i]$ in probability (Weak Law of Large

Numbers).

$$\frac{1}{n}\sum_{i=1}^{n} Y_i$$

with $Y_i = \mathbb{1}_{\{X_i \leq t\}}$. The Y_i are i.i.d., $\mathbb{E}[|Y_i|] < +\infty$.

$$\mathbb{E}[Y_1] = \mathbb{P}(X_1 \leq t) = F(t)$$

So, we have: $\hat{F}_n(t) \to F(t)$ in probability.

Examples

1. $X_i = \begin{cases} 1 & \text{if I have an accident on day } i \\ 0 & \text{otherwise} \end{cases}$ $X_i \overset{i.i.d.}{\sim} \text{Ber}(p)$; $p = \mathbb{P}(\text{avoir un accident})$.

   $$\hat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{\mathbb{P}} p \quad \text{(Law of Large Numbers)}$$

   ($\xrightarrow{\mathbb{P}}$ en probabilité).

2. $X_1, \dots, X_n$ i.i.d. $X_i =$ Lifetime of computer i.

   We want to estimate the function $t \mapsto F(t) = \mathbb{P}(X_i \leq t)$. To estimate F, we use at time t, $\hat{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{X_i \leq t\}}$.

Chapter 3: Point estimators and MLE

I. Point Estimators

We consider a parametric model $(X_1, \dots, X_n) \overset{i.i.d.}{\sim} P_\theta$, with $\theta \in \Theta \subset \mathbb{R}^d$.

Goal: Constructions of estimators for θ or for $g(\theta)$ with $g : \Theta \to \mathbb{R}^p$.

Definition 6: Point Estimator

A **point estimator** for θ is a quantity $\hat{\theta}$ which depends **only** on $X_1, \dots, X_n$ (the data).

Examples:

- We observe $X_1, \dots, X_n \overset{i.i.d.}{\sim} \mathrm{Ber}(\theta)$.

$$\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{is a point estimator of } \theta.$$

- We observe $X_1, \dots, X_n \overset{i.i.d.}{\sim} \mathcal{N}(\theta, 1)$.

$$\hat{\theta}_n = \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{is an estimator of } \theta.$$

- We observe $X_1, \dots, X_n \overset{i.i.d.}{\sim} \mathcal{E}(\theta)$.

$$\hat{\theta}_n = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} X_i} = \frac{1}{\bar{X}_n}$$

Recall: If $X \sim \mathcal{E}(\theta)$, $\theta > 0$:

$$E[X] = \frac{1}{\theta}$$

$$\frac{1}{\hat{\theta}_n} \xrightarrow[n\to\infty]{} \frac{1}{\theta}$$

$$\frac{1}{\hat{\theta}_n} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

By the Weak Law of Large Numbers (WLLN), $\hat{\theta}_n \xrightarrow[n\to\infty]{\mathbb{P}} \theta$.

For this last example, if we want to estimate $g(\theta) = \frac{1}{\theta}$, in this case, $\hat{g} = \frac{1}{\hat{\theta}_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is a "good estimator".

II. Quadratic Risk, Bias

Measure of error = Loss function. The most common loss function is the **quadratic loss**:

$$l(\hat{\theta}, \theta) = (\hat{\theta}_n - \theta)^2$$

We cannot compute $l(\hat{\theta}_n, \theta)$ since we do not know θ.

Definition 7: Quadratic Risk (Mean Squared Error)

We call **quadratic risk** (or mean squared error):

$$R(\hat{\theta}_n, \theta) = \mathbb{E}[(\hat{\theta}_n - \theta)^2]$$

Definition 8: Bias

Let $\hat{\theta}_n$ be an estimator of θ in a statistical model $(X_i)_{i \leq n} \overset{i.i.d.}{\sim} P_\theta, \theta \in \Theta$.

(i) We say that $\hat{\theta}_n$ is **unbiased** at θ iff $\mathbb{E}_\theta(\hat{\theta}_n) = \theta$.

(ii) We say that $\hat{\theta}_n$ is **unbiased** over Θ iff it is unbiased at all $\theta \in \Theta$.

The **bias** is defined as:

$$b(\theta) = \mathbb{E}_\theta(\hat{\theta}_n) - \theta$$

Theorem 1: Bias-Variance Decomposition

In a statistical model $(X_i)_{i \leq n}$ i.i.d. $P_\theta, \theta \in \Theta$. If $\hat{\theta}_n$ is an estimator of θ, then:

$$R(\hat{\theta}_n, \theta) = b(\theta)^2 + \mathbb{V}_\theta(\hat{\theta}_n)$$

where $\mathbb{V}_\theta(\hat{\theta}_n)$ is the variance of $\hat{\theta}_n$ assuming that $X_i \overset{i.i.d.}{\sim} P_\theta$.

Proof.

$$R(\hat{\theta}_n, \theta) = \mathbb{E}_\theta[(\hat{\theta}_n - \theta)^2]$$

Add and subtract $\mathbb{E}_\theta(\hat{\theta}_n)$:

$$= \mathbb{E}_\theta[(\hat{\theta}_n - \mathbb{E}_\theta(\hat{\theta}_n) + \mathbb{E}_\theta(\hat{\theta}_n) - \theta)^2]$$

Let $A = \hat{\theta}_n - \mathbb{E}_\theta(\hat{\theta}_n)$ and $B = \mathbb{E}_\theta(\hat{\theta}_n) - \theta = b(\theta)$.

$$= \mathbb{E}_\theta[(A+B)^2] = \mathbb{E}_\theta[A^2 + B^2 + 2AB]$$

$$= \underbrace{\mathbb{E}_\theta(A^2)}_{\mathbb{V}_\theta(\hat{\theta}_n)} + \underbrace{\mathbb{E}_\theta(B^2)}_{b(\theta)^2 \text{ (constant)}} + 2\underbrace{\mathbb{E}_\theta[(\hat{\theta}_n - \mathbb{E}_\theta(\hat{\theta}_n)) \cdot b(\theta)]}_{0}$$

The cross term is zero because:

$$2b(\theta)\mathbb{E}_\theta(\hat{\theta}_n - \mathbb{E}_\theta(\hat{\theta}_n)) = 2b(\theta)(\mathbb{E}_\theta(\hat{\theta}_n) - \mathbb{E}_\theta(\hat{\theta}_n)) = 0$$

Hence:

$$R(\hat{\theta}_n, \theta) = b(\theta)^2 + \mathbb{V}_\theta(\hat{\theta}_n)$$

(Q.E.D) □

Examples of Risk Calculation:

Example 4: Normal Mean

$X_1, \dots, X_n \overset{i.i.d.}{\sim} \mathcal{N}(\theta, 1)$. Estimator $\hat{\theta}_n = \frac{1}{n}\sum X_i = \bar{X}_n$.

$$R(\hat{\theta}_n, \theta) = \mathbb{E}[(\hat{\theta}_n - \theta)^2]$$

$$
\begin{aligned}
&= \mathbb{E}[(\bar{X}_n - \theta)^2] \\
&= \mathrm{Var}(\bar{X}_n) \quad (\text{as } \mathbb{E}[\bar{X}_n] = \theta\text{, it is unbiased}) \\
&= \frac{1}{n}\mathrm{Var}(X_1) = \frac{1}{n}
\end{aligned}
$$

Example 5: Bernoulli Mean

We observe $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \mathrm{Ber}(\theta)$.

$$\hat{\theta}_n = \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$
\begin{aligned}
R(\hat{\theta}_n, \theta) &= \mathbb{E}[(\hat{\theta}_n - \theta)^2] \\
&= \mathbb{E}[(\bar{X}_n - \theta)^2] \\
&= \mathrm{Var}(\bar{X}_n) = \frac{\mathrm{Var}(X_1)}{n} = \frac{\theta(1-\theta)}{n}
\end{aligned}
$$

Remark

The function $x \in [0,1] \mapsto x(1-x)$ attains its maximum at $x = 1/2$.

$$\implies \forall \theta \in [0,1], \quad \theta(1-\theta) \leq \frac{1}{4}$$

$$\implies R(\hat{\theta}_n, \theta) \leq \frac{1}{4n}$$

Reminders on Convergence

Definitions

Definition 9: Convergence in Probability

$$X_n \xrightarrow{\mathbb{P}} X \iff \forall \varepsilon > 0, \mathbb{P}(|X_n - X| > \varepsilon) \xrightarrow[n\to\infty]{} 0.$$

Definition 10: Convergence in L^p

$$X_n \xrightarrow{L^p} X \iff \mathbb{E}(|X_n - X|^p) \xrightarrow[n\to\infty]{} 0.$$

Definition 11: Convergence in Distribution

$$\begin{aligned} X_n \xrightarrow{\mathcal{D}} X & \\ &\iff \forall x \text{ (where } F_X \text{ is continuous)}, F_{X_n}(x) \xrightarrow[n\to\infty]{} F_X(x) \\ &\iff \forall g \text{ continuous and bounded: } \mathbb{E}(g(X_n)) \xrightarrow[n\to\infty]{} \mathbb{E}(g(X)) \end{aligned}$$

Main Theorems

Theorem 2: Law of Large Numbers (Weak)

If $(X_n)_n$ are i.i.d. random variables such that $\mathbb{E}(|X_n|) < \infty$, then:

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{\mathbb{P}} \mathbb{E}(X_1)$$

Theorem 3: Central Limit Theorem (CLT)

If $(X_n)_n$ are i.i.d. such that $\mathbb{E}(X_n^2) < \infty$, then:

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

where $\bar{X}_n = \frac{1}{n}\sum X_i$, $\mu = \mathbb{E}(X_1)$, $\sigma^2 = \mathrm{Var}(X_1)$.

Theorem 4: Continuity Mapping Theorem

i) If $X_n \xrightarrow{\mathbb{P}} X$ and if g is continuous, then $g(X_n) \xrightarrow{\mathbb{P}} g(X)$.

ii) If $X_n \xrightarrow{L^p} X$, $p \geq 1$, then $X_n \xrightarrow{L^q} X$ for all $1 \leq q \leq p$.

iii) If $X_n \xrightarrow{\mathcal{D}} X$, and if g is continuous and bounded, then $g(X_n) \xrightarrow{\mathcal{D}} g(X)$. (Actually holds for any continuous g).

Theorem 5: Delta Method

If $(Y_n)_n$ are like $\sqrt{n}(Y_n - \mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$. If g is $\mathcal{C}^1$ at μ so that $|g'(\mu)| > 0$, then:

$$\sqrt{n}(g(Y_n) - g(\mu)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, g'(\mu)^2\sigma^2)$$

Sketch of proof. Taylor expansion of $g(Y_n)$ around μ:

$$g(Y_n) \approx g(\mu) + g'(\mu)(Y_n - \mu)$$
$$\sqrt{n}(g(Y_n) - g(\mu)) \approx g'(\mu)\sqrt{n}(Y_n - \mu)$$

Since $\sqrt{n}(Y_n - \mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$, the result follows. $\square$

Remark

If $\sqrt{n}(Y_n - \mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$, then $Y_n \xrightarrow{\mathbb{P}} \mu$.

III. Empirical Distribution Function

Definition 12: Dirac Mass

Dirac mass at $a \in \mathbb{R}$ is the distribution defined by:

$$\forall A \subset \mathbb{R}, \quad \delta_{\{a\}}(A) = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{if } a \notin A \end{cases}$$

Let $\mathbb{P}_n = \frac{1}{n} \sum_{i=1}^n \delta_{\{X_i\}}$.

Definition 13: Empirical CDF

F_n = Empirical Cumulative Distribution Function (CDF) for $(X_i)_i$ i.i.d. with distribution P_X.

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{X_i \leq x\}} = \mathbb{P}_n((-\infty, x])$$

Remark: F_n is the CDF of $\mathbb{P}_n$.

Theorem 6: Asymptotic Normality of Empirical CDF

$$\forall x, \quad \sqrt{n}(F_n(x) - F_X(x)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, F_X(x)(1 - F_X(x)))$$

Where F_X is the CDF of P_X.

Proof.

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n \underbrace{\mathbb{1}_{\{X_i \leq x\}}}_{=Y_i}$$

The variables Y_i are i.i.d. Bernoulli variables:

$$Y_i \sim \text{Ber}(p) \quad \text{with } p = \mathbb{P}(X_i \leq x) = F_X(x)$$

$$\mathbb{E}[Y_i^2] < \infty$$

Applying CLT:

$$\sqrt{n}(\bar{Y}_n - \mathbb{E}(Y_1)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \mathrm{Var}(Y_1))$$

Here $\bar{Y}_n = F_n(x)$ and $\mathbb{E}(Y_1) = F_X(x)$.

$$\mathrm{Var}(Y_1) = F_X(x)(1 - F_X(x))$$

(Q.E.D) $\square$

More on Quadratic Risk

Expression of Risk using Density:

$$R(\hat{\theta}_n, \theta) = \mathbb{E}_\theta[(\hat{\theta}_n - \theta)^2]$$

Under the model $X_i \overset{i.i.d.}{\sim} P_\theta$:

$$= \begin{cases} \displaystyle\int (\hat{\theta}_n(x_1, \ldots, x_n) - \theta)^2 \prod_{i=1}^n f_\theta(x_i)\, dx_i & \text{if continuous (density } f_\theta) \\ \displaystyle\sum_{x_1, \ldots, x_n} (\hat{\theta}_n(x_1, \ldots, x_n) - \theta)^2 \prod_{i=1}^n P_\theta(X = x_i) & \text{if discrete observations} \end{cases}$$

Example: Exponential Family

Example 2 (from beginning): $X_i \overset{i.i.d.}{\sim} \mathcal{E}(\theta)$, $\theta > 0$.

$$X_i \text{ are continuous random variables with density } f_\theta(x) = \theta e^{-\theta x} \mathbb{1}_{\{x>0\}}$$

Estimator:

$$\hat{\theta}_n = \frac{1}{\bar{X}_n}; \quad \bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$$

If $X \sim \mathcal{E}(\theta)$, $\mathbb{E}_\theta(X) = \frac{1}{\theta}$. What is $R(\hat{\theta}_n, \theta)$? What is $b(\theta)$?
We need to calculate $\mathbb{E}_\theta(\hat{\theta}_n) = \mathbb{E}_\theta\left(\frac{1}{\bar{X}_n}\right)$.
Recall:

- $\mathbb{E}(h(X)) = \int h(x) f_X(x) dx$ (if continuous).
- We write: $\mathbb{E}(h(X)) = \int h(x) dF_X(x) = \int h(x) dP_X(x)$.
- If $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \Gamma(a, b)$, then $\sum_{i=1}^n X_i \sim \Gamma(na, b)$.

Gamma distribution $\Gamma(a, b)$**:** $\mathcal{C}^0$ distribution on $\mathbb{R}^+$ with density:

$$f_{a,b}(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} \mathbb{1}_{\{x>0\}}$$

Note: $\mathcal{E}(\theta) = \Gamma(1, \theta)$.
$\implies \sum_{i=1}^n X_i \sim \Gamma(n, \theta)$.
Let $Z_n = \sum_{i=1}^n X_i$. Then $\hat{\theta}_n = \frac{n}{Z_n}$.
We want to compute $\mathbb{E}_\theta\left(\frac{n}{Z_n}\right) = n\mathbb{E}_\theta\left(\frac{1}{Z_n}\right)$.

$$\mathbb{E}_\theta\left(\frac{1}{Z_n}\right) = \int_0^{+\infty} \frac{1}{z} \cdot \frac{\theta^n}{\Gamma(n)} z^{n-1} e^{-\theta z} dz$$
$$= \frac{\theta^n}{\Gamma(n)} \int_0^{+\infty} z^{n-2} e^{-\theta z} dz$$

Remark

Let $f_{n-1,\theta}(z) = \frac{\theta^{n-1}}{\Gamma(n-1)} z^{n-2} e^{-\theta z}$ be the density of a $\Gamma(n-1,\theta)$.

$$\implies \int_0^{+\infty} z^{n-2} e^{-\theta z} dz \times \frac{\theta^{n-1}}{\Gamma(n-1)} = 1$$

$$\implies \int_0^{+\infty} z^{n-2} e^{-\theta z} dz = \frac{\Gamma(n-1)}{\theta^{n-1}}$$

Back to the expectation:

$$\mathbb{E}_\theta(\hat{\theta}_n) = \frac{n\theta^n}{\Gamma(n)} \times \frac{\Gamma(n-1)}{\theta^{n-1}}$$

And we have: $\Gamma(x+1) = x\Gamma(x)$ if $x > 0$. So $\Gamma(n) = (n-1)\Gamma(n-1)$.

$$\implies \mathbb{E}_\theta(\hat{\theta}_n) = \frac{n}{n-1}\theta$$

$$\implies b(\theta) = \mathbb{E}_\theta(\hat{\theta}_n) - \theta = \frac{n}{n-1}\theta - \theta = \frac{\theta}{n-1}$$

Now, let's compute the quadratic risk. We need $\mathbb{E}_\theta(\hat{\theta}_n^2) = n^2\mathbb{E}(\frac{1}{Z_n^2})$.

$$\mathbb{E}\left(\frac{1}{Z_n^2}\right) = \int y^{n-3} e^{-\theta y} dy \frac{\theta^n}{\Gamma(n)}$$

Using the same trick (identification with $\Gamma(n-2,\theta)$) for $n > 2$:

$$= \frac{\theta^n}{\Gamma(n)} \cdot \frac{\Gamma(n-2)}{\theta^{n-2}} = \frac{\theta^2\Gamma(n-2)}{(n-1)(n-2)\Gamma(n-2)} = \frac{\theta^2}{(n-1)(n-2)}$$

$$\implies \mathbb{E}_\theta(\hat{\theta}_n^2) = \frac{n^2\theta^2}{(n-1)(n-2)}$$

$$R(\theta,\hat{\theta}_n) = \underbrace{\frac{n^2\theta^2}{(n-1)(n-2)}}_{\mathbb{E}(\hat{\theta}_n^2)} - \underbrace{\left(\frac{n\theta}{n-1}\right)^2}_{\mathbb{E}(\hat{\theta}_n)^2} + \underbrace{\frac{\theta^2}{(n-1)^2}}_{b(\theta)^2}$$

(Using variance decomposition $\mathbb{V}(\hat{\theta}_n) = \mathbb{E}(\hat{\theta}_n^2) - \mathbb{E}(\hat{\theta}_n)^2$). Actually, let's calculate directly:

$$R(\theta,\hat{\theta}_n) = \frac{n^2\theta^2}{(n-1)(n-2)} - 2\theta\frac{n\theta}{n-1} + \theta^2$$

Simplifying (algebra):

$$= \frac{\theta^2 n^2}{n-1}\left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \frac{\theta^2}{(n-1)^2}$$
$$= \frac{\theta^2}{(n-1)^2}\left[\frac{n^2}{n-2} + 1\right]$$

Unbiased estimator

We can construct an unbiased estimator $\tilde{\theta}_n$:

$$\tilde{\theta}_n = \hat{\theta}_n \times \left(\frac{n-1}{n}\right) \implies \mathbb{E}_\theta(\tilde{\theta}_n) = \mathbb{E}_\theta(\hat{\theta}_n) \times \frac{n-1}{n} = \theta.$$

Risk of the unbiased estimator:

$$\begin{aligned} R(\theta, \tilde{\theta}_n) = \mathbb{V}_\theta(\tilde{\theta}_n) &= \left(\frac{n-1}{n}\right)^2 \mathbb{V}_\theta(\hat{\theta}_n) \\ &= \frac{(n-1)^2}{n^2} \times \frac{n^2\theta^2}{(n-1)^2(n-2)} = \frac{\theta^2}{n-2} \end{aligned}$$

We observe that:

$$\forall \theta, R(\theta, \tilde{\theta}_n) < R(\hat{\theta}_n, \theta) \implies \tilde{\theta}_n \text{ is better than } \hat{\theta}_n.$$

Remark

Unbiased estimators are not necessarily better than biased estimators.

Example 6: Empirical Variance

$X_1, \ldots, X_n \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$.

$$S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \quad \text{(Empirical variance)}$$

We know that $S_n^2 = \frac{1}{n}\sum X_i^2 - \bar{X}_n^2$.

$$\mathbb{E}_{\sigma^2}\left(\frac{1}{n}\sum X_i^2\right) = \sigma^2$$

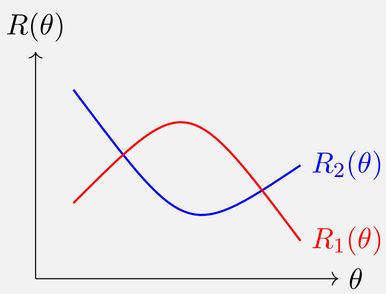
Since $\bar{X}_n \sim \mathcal{N}(0, \frac{\sigma^2}{n})$, $\mathbb{E}_{\sigma^2}(S_n^2) = \sigma^2 - \frac{\sigma^2}{n} = \sigma^2\frac{n-1}{n}$.
Define $\tilde{\sigma}_n^2 = S_n^2 \cdot \frac{n}{n-1} = \frac{1}{n-1}\sum(X_i - \bar{X}_n)^2$.

$$\implies \mathbb{E}_{\sigma^2}(\tilde{\sigma}_n^2) = \sigma^2 \implies \text{Unbiased.}$$

Exercise: Compute $R(\sigma^2, S_n^2)$ and $R(\sigma^2, \tilde{\sigma}_n^2)$ and show that $\forall \sigma > 0$, $R(\sigma^2, S_n^2) < R(\sigma^2, \tilde{\sigma}_n^2)$.

Remark

In most cases, if $\hat{\theta}_{n,1}$ and $\hat{\theta}_{n,2}$ are 2 estimators of θ, then risk functions $R_1(\theta) = R(\theta, \hat{\theta}_{n,1})$ and $R_2(\theta, \hat{\theta}_{n,2})$ cross.



$\implies$ $\hat{\theta}_{n,1}$ and $\hat{\theta}_{n,2}$ cannot be compared uniformly.

Example 7: Normal Mean Estimators

$X_1, \dots, X_n \overset{i.i.d.}{\sim} \mathcal{N}(\theta, 1)$.

$$\hat{\theta}_{n,1} = \bar{X}_n$$

$$\hat{\theta}_{n,2} = \frac{n\bar{X}_n}{n + \tau^2} \quad \text{with } \tau > 0$$

Compute their risks and show that they are not comparable.

Admissibility

Definition 14: Admissibility

An estimator $\hat{\theta}_n$ is **not admissible** iff $\exists \tilde{\theta}_n$, another estimator such that:

- $\forall \theta, R(\theta, \tilde{\theta}_n) \leq R(\theta, \hat{\theta}_n)$
- and $\exists \theta_0, R(\theta_0, \tilde{\theta}_n) < R(\theta_0, \hat{\theta}_n)$.

$\hat{\theta}_n$ is **admissible** iff it is not non-admissible.

IV. Consistency

Example 8: Bernoulli Consistency

$X_1, \dots, X_n \overset{i.i.d.}{\sim} \mathrm{Ber}(p)$. $\theta = p \in]0, 1[$. $\bar{X}_n = \hat{\theta}_n$ is a possible estimator for θ.

$$\bar{X}_n \xrightarrow[n \to \infty]{P_\theta} \theta = \mathbb{E}_\theta(X_1) \quad \text{(LLN)}$$

Question: $R(\theta, \hat{\theta}_n) \xrightarrow[n \to \infty]{} 0, \forall \theta$?

$$R(\theta, \hat{\theta}_n) = \mathbb{E}_\theta[(\bar{X}_n - \theta)^2] = \mathbb{V}_\theta(\bar{X}_n) = \frac{1}{n^2} n \mathbb{V}_\theta(X_1)$$

$$= \frac{\theta(1 - \theta)}{n} \leq \frac{1}{4n}$$

So $\sup_{\theta \in (0,1)} R(\theta, \hat{\theta}_n) \xrightarrow[n \to \infty]{} 0$.

Can I say something like: $\hat{\theta}_{n,1} \leq \theta \leq \hat{\theta}_{n,2}$ with proba ≈ 0.95?

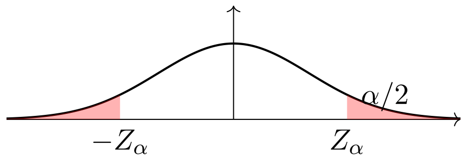
$$\sqrt{n}(\bar{X}_n - \theta) \xrightarrow[n \to \infty]{\mathcal{D}} \mathcal{N}(0, \theta(1 - \theta)) \quad \text{(CLT)}$$

$$\implies \frac{\sqrt{n}(\bar{X}_n - \theta)}{\sqrt{\theta(1 - \theta)}} \xrightarrow[n \to \infty]{\mathcal{D}} \mathcal{N}(0, 1)$$

Since $\bar{X}_n \xrightarrow{\mathbb{P}} \theta$, by Slutsky's lemma (replacing θ with $\bar{X}_n$ in the denominator):

$$\frac{\sqrt{n}(\bar{X}_n - \theta)}{\sqrt{\bar{X}_n(1 - \bar{X}_n)}} \xrightarrow[n \to \infty]{\mathcal{D}} \mathcal{N}(0, 1)$$

Let Z_α be such that $\mathbb{P}(|\mathcal{N}(0,1)| \leq Z_\alpha) = 1-\alpha$. (Usually $1-\alpha = 0.95 \implies Z_\alpha \approx 1.96$).



$$\implies \mathbb{P}_\theta\left(-Z_\alpha \leq \frac{\sqrt{n}(\bar{X}_n - \theta)}{\sqrt{\bar{X}_n(1-\bar{X}_n)}} \leq Z_\alpha\right) \xrightarrow[n\to\infty]{} 1-\alpha$$

$$\implies \mathbb{P}_\theta\left(\underbrace{\bar{X}_n - \frac{\sqrt{\bar{X}_n(1-\bar{X}_n)}Z_\alpha}{\sqrt{n}}}_{\hat{\theta}_{n,1}} \leq \theta \leq \underbrace{\bar{X}_n + \frac{Z_\alpha\sqrt{\bar{X}_n(1-\bar{X}_n)}}{\sqrt{n}}}_{\hat{\theta}_{n,2}}\right) \to 1-\alpha$$

With probability under $P_\theta \approx 1-\alpha$, $\hat{\theta}_{n,1} \leq \theta \leq \hat{\theta}_{n,2}$.

Definition 15: Consistency

1. We say that an estimator $\hat{\theta}_n$ is **consistent in probability** at θ iff $\hat{\theta}_n \xrightarrow[n\to\infty]{P_\theta} \theta$.
2. $\hat{\theta}_n$ is consistent **over** Θ if it is consistent at $\theta, \forall\theta \in \Theta$.
3. $\hat{\theta}_n$ converges in **quadratic mean** at θ iff $R(\hat{\theta}_n, \theta) \xrightarrow[n\to\infty]{} 0$ (Notation: $\hat{\theta}_n \xrightarrow{q.m} \theta$).
4. $\hat{\theta}_n$ converges in quadratic mean over Θ iff $\hat{\theta}_n \xrightarrow{q.m} \theta, \forall\theta \in \Theta$.
5. $\hat{\theta}_n$ is **asymptotically normal** at θ with rate $\frac{1}{\sqrt{n}}$ iff $\exists V > 0$ s.t. $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, V)$.

Estimation of $g(\theta)$

$X_1, \dots, X_n \overset{i.i.d.}{\sim} P_\theta, \theta \in \Theta$.

Examples:

1. $P_\theta \sim \text{Ber}(\theta)$. Interested in estimating $\eta = \log(\frac{\theta}{1-\theta})$.
2. $X_i \overset{i.i.d.}{\sim} \mathcal{N}(\mu, \sigma^2)$, $\theta = (\mu, \sigma^2)$. $g_1(\theta) = \mu$. $g_2(\theta) = \mathbb{P}_\theta(X > 1)$.

If $\hat{\theta}$ is an estimator of θ, then we can use $g(\hat{\theta})$ as an estimator of $g(\theta)$ (one possibility among others).

Theorem 7: Plug-in Consistency

Let $X_1, \dots, X_n \overset{i.i.d.}{\sim} P_\theta, \theta \in \Theta \subset \mathbb{R}^d$. Let $\hat{\theta}_n$ be an estimator of θ and $g: \Theta \to \mathbb{R}^p$. Then:

i) If $\hat{\theta}_n$ is consistent at θ in probability and g is $\mathcal{C}^0$ (continuous), then $\hat{\eta}_n = g(\hat{\theta}_n)$ is consistent in probability at $\eta = g(\theta)$.

ii) If $\hat{\theta}_n \xrightarrow{L^2} \theta$ and g is $\mathcal{C}^0$ and **bounded**, then $g(\hat{\theta}_n) \xrightarrow{L^2(P_\theta)} g(\theta), \forall p \geq 1$.

iii) If $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[n\to\infty]{\mathcal{L}(P_\theta)} \mathcal{N}(0,\sigma^2)$ and g is $\mathcal{C}^1$ with $g'(\theta) \neq 0$ (or $\nabla g(\theta)$ of full rank), then $\sqrt{n}(g(\hat{\theta}_n) - g(\theta)) \xrightarrow{\mathcal{L}(P_\theta)} \mathcal{N}(0, g'(\theta)^2\sigma^2)$.

Remark

$\hat{\theta}_n$ is unbiased $\not\Rightarrow g(\hat{\theta}_n)$ is unbiased.

Proof for (ii). $\forall\varepsilon > 0$,

$$\mathbb{E}_\theta(|g(\hat{\theta}_n) - g(\theta)|^p) = \mathbb{E}_\theta(|g(\hat{\theta}_n) - g(\theta)|^p \mathbb{1}_{|g(\hat{\theta}_n)-g(\theta)|>\varepsilon}) + \mathbb{E}_\theta(|g(\hat{\theta}_n) - g(\theta)|^p \mathbb{1}_{|g(\hat{\theta}_n)-g(\theta)|\leq\varepsilon})$$

The second term is $\leq \varepsilon^p$. The first term:

$$\begin{aligned} &\leq \mathbb{E}_\theta((2||g||_\infty)^p \mathbb{1}_{|g(\hat{\theta}_n)-g(\theta)|>\varepsilon}) \\ &= (2||g||_\infty)^p \mathbb{P}_\theta(|g(\hat{\theta}_n) - g(\theta)| > \varepsilon) \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

Because $g(\hat{\theta}_n) \xrightarrow[n\to\infty]{\mathbb{P}} g(\theta)$ (from point (i), continuous mapping theorem).

Thus $\mathbb{E}_\theta(|g(\hat{\theta}_n) - g(\theta)|^p) \leq \varepsilon^p + 2||g||_\infty^p \times (\cdots \to 0)$. Taking limit $n \to \infty$ then $\varepsilon \to 0$ yields the result. □

The Multivariate Case

$X_i \overset{i.i.d.}{\sim} P_\theta, \theta \in \Theta \subset \mathbb{R}^d$. $\theta = (\theta_1, \ldots, \theta_d)$.

$$\begin{aligned} \hat{\theta}_n \xrightarrow[n\to\infty]{\mathbb{P}} \theta &\iff \forall\varepsilon > 0, \mathbb{P}_\theta(||\hat{\theta}_n - \theta|| > \varepsilon) \xrightarrow[n\to\infty]{} 0 \\ &\iff \forall j \leq d, \hat{\theta}_{n,j} \xrightarrow{\mathbb{P}_\theta} \theta_j \end{aligned}$$

Because:

- If $\forall\varepsilon > 0, \mathbb{P}_\theta(||\hat{\theta}_n - \theta|| > \varepsilon) \to 0$. Let $j \leq d, \forall\varepsilon > 0$, $||\hat{\theta}_n - \theta|| \geq |\hat{\theta}_{n,j} - \theta_j|$.

$$\mathbb{P}_\theta(|\hat{\theta}_{n,j} - \theta_j| > \varepsilon) \leq \mathbb{P}_\theta(||\hat{\theta}_n - \theta|| > \varepsilon) \xrightarrow[n\to\infty]{} 0$$

- **Reverse:** If $\forall j \leq d, \mathbb{P}_\theta(|\hat{\theta}_{n,j} - \theta_j| > \varepsilon) \to 0$.

$$||\hat{\theta}_n - \theta||^2 = \sum_j |\hat{\theta}_{n,j} - \theta_j|^2 \leq d \cdot \max_j |\hat{\theta}_{n,j} - \theta_j|^2$$

$$\begin{aligned} \mathbb{P}_\theta(||\hat{\theta}_n - \theta|| > \varepsilon) &\leq \mathbb{P}_\theta\left(\max_j |\hat{\theta}_{n,j} - \theta_j| > \frac{\varepsilon}{\sqrt{d}}\right) \\ &= \mathbb{P}_\theta\left(\bigcup_{j=1}^d \{|\hat{\theta}_{n,j} - \theta_j| > \frac{\varepsilon}{\sqrt{d}}\}\right) \\ &\leq \sum_{j=1}^d \mathbb{P}_\theta\left(\{|\hat{\theta}_{n,j} - \theta_j| > \frac{\varepsilon}{\sqrt{d}}\}\right) \\ &\xrightarrow[n\to\infty]{} 0 \quad \text{because } d \text{ is fixed.} \end{aligned}$$

$\to$ Similarly, show that $\hat{\theta}_n \xrightarrow{q.m} \theta \iff \forall j, \hat{\theta}_{n,j} \xrightarrow{q.m} \theta_j$.

But, if $\sqrt{n}(\hat{\theta}_{n,j} - \theta_j) \xrightarrow{\mathcal{L}(P_\theta)} \mathcal{N}(0, \sigma_j^2)$, it does **NOT** imply directly the joint normality.

$$\implies \sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}(P_\theta)} \mathcal{N}(\mathbf{0}, \Sigma)$$

(Multivariate Central Limit Theorem required).

Chapter 4: Maximum Likelyhood Estimation

I. The Likelihood

Model: $X_i \overset{i.i.d.}{\sim} P_\theta, \theta \in \Theta$.

Observations: $(x_1, \dots, x_n)$, where x_i is a realisation of X_i.

- If P_θ is **discrete**, then proba mass function of $X_1 \sim P_\theta$:

$$x \in \mathcal{X}, \quad f_\theta(x) = P_\theta(X_1 = x)$$

  ($\rightarrow$ density with respect to counting measure).

- If P_θ is a **continuous** distribution, then it has density f_θ on $\mathbb{R}$ (or $\mathbb{R}^d$) with respect to Lebesgue measure.

$$\int_A f_\theta(x)dx = P_\theta(X \in A)$$

In both cases, there is a density, with respect to either Lebesgue or counting measure, denoted f_θ.

Definition 16: Likelihood

Let $(X_i)_{i=1}^n \overset{i.i.d.}{\sim} P_\theta, \theta \in \Theta$ with $\{P_\theta, \theta \in \Theta\}$ either discrete or continuous. Let f_θ be the density of P_θ (or probability mass function). We call the **likelihood** at $(x_1, \dots, x_n)$ (realisations of $(X_1, \dots, X_n)$) the function:

$$L_n : \Theta \to \mathbb{R}_+$$

$$\theta \mapsto \prod_{i=1}^n f_\theta(x_i)$$

Examples

1. $X_1, \dots, X_n \overset{i.i.d.}{\sim} \mathcal{P}(\lambda), \lambda > 0$ (Poisson). Observations $(x_1, \dots, x_n), \forall x \in \mathbb{N}, f_\lambda(x) = \frac{e^{-\lambda}\lambda^x}{x!}$.

$$\begin{aligned} L_n(\lambda) &= \prod_{i=1}^n f_\lambda(x_i) = \prod_{i=1}^n \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \\ &= \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!} \end{aligned}$$

2. $X_1, \dots, X_n \overset{i.i.d.}{\sim} \mathcal{E}(\theta), \theta > 0$. Observations $(x_1, \dots, x_n), x_i > 0, \forall i$.

$$\forall x > 0, f_\theta(x) = \theta e^{-\theta x}$$

$$\begin{aligned} L_n(\theta) &= \prod_{i=1}^n \theta e^{-\theta x_i} \mathbb{1}_{\{x_i > 0\}} \\ &= \theta^n e^{-\theta \sum_{i=1}^n x_i} \mathbb{1}_{\{\min_i x_i > 0\}} \end{aligned}$$

Remark

In the **discrete** case, $\forall\theta, L_n(\theta) = \prod_{i=1}^n f_\theta(x_i) = \prod_{i=1}^n \mathbb{P}_\theta(X_i = x_i) = \mathbb{P}_\theta(X_1 = x_1, \ldots, X_n = x_n)$.
But, in the **continuous** case, it is different:

$$L_n(\theta) = \prod_{i=1}^n f_\theta(x_i) \neq \mathbb{P}_\theta(X_1 = x_1, \ldots, X_n = x_n) = 0$$

(Probability of exact points in continuous case is 0).

If $L_n(\theta)$ is large, it means that the observations are highly likely for θ. "There is a good fit between $(x_1, \ldots, x_n)$ and P_θ".
Exercise 1: Show that if $P_\theta \sim \mathcal{N}(\mu, \sigma^2)$, $\theta = (\mu, \sigma^2)$, then:

$$L_n(\theta) = \frac{e^{-n\frac{(\bar{x}_n-\mu)^2}{2\sigma^2}}}{(2\pi\sigma^2)^{n/2}} e^{-\sum_{i=1}^n \frac{(x_i-\bar{x}_n)^2}{2\sigma^2}}$$

II. Kullback-Leibler Divergence (K.L)

2nd interpretation: Kullback-Leibler divergence.

Definition 17: Kullback-Leibler Divergence

Let P and Q be 2 probabilities.

- If P, Q are **discrete** on $\mathcal{X}$, then:

$$\begin{aligned} KL(P,Q) &= \sum_{x\in\mathcal{X}} P(x) \ln\left(\frac{P(x)}{Q(x)}\right) \\ &= \mathbb{E}_P\left(\ln\left(\frac{P(X)}{Q(X)}\right)\right) \end{aligned}$$

  (where $P(x) = \mathbb{P}(X = x)$, $Q(x) = \mathbb{Q}(X = x)$).

- If P and Q are **continuous** with density f_P and f_Q (resp), then:

$$\begin{aligned} KL(P,Q) &= \int_{\mathcal{X}} f_P(x) \ln\left(\frac{f_P(x)}{f_Q(x)}\right) dx \\ &= \mathbb{E}_P\left(\ln\left(\frac{f_P(X)}{f_Q(X)}\right)\right) \end{aligned}$$

Proposition 2: Properties of KL

i) $\forall P, Q, \quad KL(P,Q) \geq 0$.

ii) $KL(P,Q) = 0 \iff P = Q$.

But, $KL(P,Q) \neq KL(Q,P)$ (Not symmetric).

Proof. Assume that P, Q are continuous.

i)

$$\begin{aligned}
KL(P,Q) &= \int_{\mathcal{X}} f_P(x) \ln\left(\frac{f_P(x)}{f_Q(x)}\right) dx \\
&= \int_{\mathcal{X}} f_P(x) \left[-\ln\left(\frac{f_Q(x)}{f_P(x)}\right)\right] dx
\end{aligned}$$

By Jensen's inequality:

$$\begin{aligned}
&\geq -\ln\left[\int_{\mathcal{X}} f_P(x)\frac{f_Q(x)}{f_P(x)} dx\right] \\
&= -\ln\left[\int_{\mathcal{X}} f_Q(x) dx\right] = -\ln(1) = 0
\end{aligned}$$

(Because $x \mapsto -\ln(x)$ is strictly convex on $\mathbb{R}$).

ii) Since $-\ln(\cdot)$ is strictly convex:

$$\mathbb{E}_P\left(-\ln(R(X))\right) > -\ln(\mathbb{E}_P(R(X))) \quad \text{with } R(x) = \frac{f_Q(x)}{f_P(x)}$$

if $\forall c, \mathbb{P}(R(X) = c) < 1$ ($R(X)$ is not constant).

$R(x) = \frac{f_Q(x)}{f_P(x)}$ is constant iff $\exists c \neq 0$ so that:

$$f_Q(x) = c f_P(x), \quad \forall x \in \mathcal{X}$$

$$\implies c = 1 \quad (\text{since } \int f_P = \int f_Q = 1)$$

$R(x) = \frac{f_Q(x)}{f_P(x)}$ is constant iff $f_Q = f_P \iff P = Q$. □

Links with the likelihood

$\forall \theta, \theta_0 \in \Theta$:

$$\begin{aligned}
&\ln(L_n(\theta)) - \ln(L_n(\theta_0)) \\
&= \ln\left(\prod_{i=1}^n f_\theta(x_i)\right) - \ln\left(\prod_{i=1}^n f_{\theta_0}(x_i)\right) \\
&= \sum_{i=1}^n \ln(f_\theta(x_i)) - \sum_{i=1}^n \ln(f_{\theta_0}(x_i)) \\
&= \sum_{i=1}^n \ln\left(\frac{f_\theta(x_i)}{f_{\theta_0}(x_i)}\right)
\end{aligned}$$

If $(x_i)_{i=1}^n$ are the realisations of $X_1, \dots, X_n \overset{i.i.d.}{\sim} P_{\theta_0}$. Then when $X_i \overset{i.i.d.}{\sim} P_{\theta_0}$:

$$\frac{1}{n}\sum_{i=1}^n \ln\left(\frac{f_\theta(X_i)}{f_{\theta_0}(X_i)}\right) \underbrace{\text{converges under } P_{\theta_0}}_{n\to\infty} \text{ to } \mathbb{E}_{\theta_0}\left(\ln\left(\frac{f_\theta(X)}{f_{\theta_0}(X)}\right)\right)$$

as soon as $\mathbb{E}_{\theta_0}\left(\left|\ln \frac{f_\theta(X)}{f_{\theta_0}(X)}\right|\right) < \infty$.

$$\implies \text{if } \mathbb{E}_{\theta_0}(\dots) < \infty, \text{ then we have:}$$

$$\begin{aligned}
\frac{1}{n}(\ln(L_n(\theta)) - \ln(L_n(\theta_0))) &\xrightarrow[n\to\infty]{P_{\theta_0}} \mathbb{E}_{\theta_0}\left[\ln\left(\frac{f_\theta(X)}{f_{\theta_0}(X)}\right)\right] \\
&= -KL(P_{\theta_0}, P_\theta)
\end{aligned}$$

- θ_0 minimizes $KL(P_{\theta_0}, P_\theta)$ over θ.
- θ_0 maximizes $-KL(P_{\theta_0}, P_\theta)$ over θ.

We cannot maximize $-KL(P_{\theta_0}, P_\theta)$ because it depends on θ_0 unknown. $\implies$ We maximize:

$$\mathbb{E}_{P_n}\left[-\ln\left(\frac{f_\theta(X)}{f_{\theta_0}(X)}\right)\right] = \frac{1}{n}[\ln(L_n(\theta)) - \ln(L_n(\theta_0))]$$

(Empirical version of $-KL(P_{\theta_0}, P_\theta)$).

Maximizing in θ, $\ln(L_n(\theta)) - \ln(L_n(\theta_0))$ is equivalent to maximizing in θ, $\ln L_n(\theta)$.

Definition 18: Maximum Likelihood Estimator (MLE)

Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} P_\theta$, $\theta \in \Theta$, the model P_θ with density (or probability mass function) f_θ. We call **Maximum Likelihood Estimator** any selection $\hat{\theta}_n$ (when it exists) such that:

$$\hat{\theta}_n \in \underset{\theta \in \Theta}{\operatorname{argmax}}\{\ln(L_n(\theta))\}$$

$$\iff \log(L_n(\hat{\theta}_n)) \geq \log(L_n(\theta)), \forall \theta \in \Theta$$

Examples of Calculations

1) Bernoulli Model

$X_1, \ldots, X_n \overset{i.i.d.}{\sim} \text{Ber}(\theta)$ ($\theta \in]0,1[$).

$$\begin{aligned} L_n(\theta) &= \prod_{i=1}^n P_\theta(X_i = x_i) \\ &= \prod_{i=1}^n \theta^{x_i}(1-\theta)^{1-x_i} \\ &= \theta^{\sum x_i}(1-\theta)^{n-\sum x_i} \end{aligned}$$

Define $l(\theta) = \log(L_n(\theta))$.

$$l(\theta) = \left(\sum_{i=1}^n x_i\right)\log(\theta) + \left(n - \sum_{i=1}^n x_i\right)\log(1-\theta)$$

$\hat{\theta}_n$ maximizes $l(\theta)$ in θ. Let $\bar{x}_n = \frac{1}{n}\sum x_i$.

$$l(\theta) = n\bar{x}_n \log(\theta) + n(1-\bar{x}_n)\log(1-\theta)$$

Derivative:

$$l'(\theta) = \frac{n\bar{x}_n}{\theta} - \frac{n(1-\bar{x}_n)}{1-\theta}$$

Second derivative:

$$l''(\theta) = -\frac{n\bar{x}_n}{\theta^2} - \frac{n(1-\bar{x}_n)}{(1-\theta)^2} < 0$$

So $\theta \mapsto l(\theta)$ is **concave**. $\implies$ $\hat{\theta}_n$ is the solution of $l'(\theta) = 0$.

$$l'(\theta) = 0 \iff \frac{n\bar{x}_n}{\theta} = \frac{n(1-\bar{x}_n)}{1-\theta} \iff \theta = \bar{x}_n$$

$$\implies \hat{\theta}_n = \bar{x}_n \text{ is the MLE (Maximum Likelihood Estimator)}$$

2) Exponential Model

$X_1, \dots, X_n \overset{i.i.d.}{\sim} \mathcal{E}(\theta), \theta > 0.$

$$X \sim \mathcal{E}(\theta) \iff f_\theta(x) = \theta e^{-\theta x} \mathbb{1}_{\{x>0\}}$$

$$L_n(\theta) = \theta^n e^{-n\theta \bar{x}_n} \prod_{i=1}^{n} \mathbb{1}_{\{x_i>0\}}$$

$$l(\theta) = \log(L_n(\theta)) = n\log(\theta) - n\bar{x}_n\theta \quad (\text{if } \forall i, x_i > 0)$$

Exercise: Show that $\theta \mapsto l(\theta)$ is concave and that $\hat{\theta}_n = \frac{1}{\bar{x}_n}$.

Remark

1. The log likelihood as a function is not necessarily concave, therefore, $\hat{\theta}_n$ does not always exist and is not always unique.
2. In practice: how do we compute $\hat{\theta}_n$?

$$\hat{\theta}_n \in \text{argmax}\{l(\theta), \theta \in \Theta\}$$

Algorithms such as gradient descent, Newton-Raphson.

Summary:

- **Model:** $f_\theta(\cdot), \theta \in \Theta$ density or proba mass function. $X_i \overset{i.i.d.}{\sim} f_\theta$.
- **Log-likelihood:** $l(\theta) = \log(L_n(\theta)) = \sum_{i=1}^n \log(f_\theta(x_i))$.
- **MLE:** $\hat{\theta}_n$ such that $l(\hat{\theta}_n) \geq l(\theta), \forall \theta \in \Theta$.

Proposition 3: Invariance

If $\eta = g(\theta)$, where g is invertible ($\iff$ new parametrization).
For instance: $X_i \overset{i.i.d.}{\sim} \text{Ber}(p), \theta = p \in]0,1[$.

$$\eta = \log\left(\frac{p}{1-p}\right) \in \mathbb{R}$$

Let $\tilde{l}(\eta) = \sum_{i=1}^n \log(f_{g^{-1}(\eta)}(x_i))$. Let $l(\theta) = \sum_{i=1}^n \log(f_\theta(x_i))$.

- If $\hat{\eta}_n$ is the MLE of $\tilde{l}$ ($\iff$ $\tilde{l}(\eta) \leq \tilde{l}(\hat{\eta}_n), \forall \eta$).
- If $\hat{\theta}_n$ is the MLE of l ($\iff$ $l(\theta) \leq l(\hat{\theta}_n), \forall \theta$).

Then:

$$\hat{\eta}_n = g(\hat{\theta}_n)$$

Proof. Note: $\hat{\eta}_n = g(\hat{\theta}_n)$. We verify that $\tilde{l}(\hat{\eta}_n) \geq \tilde{l}(\eta), \forall \eta$, that is, it will imply that $\hat{\eta}_n$ is an MLE of $\tilde{l}$.
Indeed:

$$\tilde{l}(\eta) = l(g^{-1}(\eta)) \leq l(\hat{\theta}_n), \forall \eta$$

And:

$$\tilde{l}(\hat{\eta}_n) = l(g^{-1}(g(\hat{\theta}_n))) = l(\hat{\theta}_n)$$

$$\implies \forall \eta, \tilde{l}(\eta) \leq \tilde{l}(\hat{\eta}_n). \quad \text{(Q.E.D)}$$

□

Examples

MLE for $X_1, \dots, X_n \overset{i.i.d.}{\sim} \text{Ber}(p)$.

$$l(p) = p^{\sum x_i}(1-p)^{n-\sum x_i}$$

$x_1, \dots, x_n$ are the observations.

$$\hat{p}_n = \frac{1}{n}\sum_{i=1}^{n} x_i \to \text{MLE}$$

$$\eta = \log\left(\frac{p}{1-p}\right) \implies \hat{\eta}_n = \log\left(\frac{\hat{p}_n}{1-\hat{p}_n}\right) \text{ is the MLE of } \eta.$$

Asymptotic Normality of the MLE

Under which conditions on the model is the MLE asymptotically normal?

Definition 19: Regular Model

The model $X_1, \dots, X_n \overset{i.i.d.}{\sim} f_\theta, \theta \in \Theta \subset \mathbb{R}^k$ (either density or probability mass function) is **regular** on Θ iff:

(i) Θ is an open set and the model is identifiable.

(ii) $\theta \mapsto \log(f_\theta(x))$ is $\mathcal{C}^2, \forall x \in \mathcal{X}$.

(iii) $\forall \theta \in \Theta, \mathbb{E}_\theta(||\nabla_\theta \log(f_\theta(X))||^2) < +\infty$ and $\exists \delta > 0, \mathbb{E}_\theta(\sup_{|\theta'-\theta|<\delta} ||D^2_\theta \log(f_{\theta'}(X))||) < +\infty$.

(iv) $\forall \theta, I(\theta) = \mathbb{E}_\theta\left(\nabla_\theta \log(f_\theta(X)) \cdot \nabla_\theta \log(f_\theta(X))^T\right)$ is positive definite.

Fisher Information

$I(\theta)$ is called the **Fisher Information Matrix**.
Notations: $D^2 \log(f_\theta(x))$ is the matrix:

$$(D^2 \log(f_\theta(x)))_{i,j} = \frac{\partial^2 \log(f_\theta(x))}{\partial \theta_i \partial \theta_j}$$

$$\nabla_\theta \log(f_\theta(x)) = \left(\frac{\partial}{\partial \theta_j} \log(f_\theta(x))\right)_{j=1,\dots,k}$$

Lemma 1: Properties of Score and Fisher Information

If the model is regular:

a) $\forall \theta \in \Theta, \mathbb{E}_\theta(\nabla_\theta \log(f_\theta(X))) = 0$. $S(\theta, x) = \nabla_\theta \log(f_\theta(x))$ is called the **score function**.

b) $I(\theta) = -\mathbb{E}_\theta(D^2 \log(f_\theta(X)))$.

Proof. **a)** The model is regular $\implies \nabla_\theta \log(f_\theta(X))$ exists and is integrable. And:

$$\nabla_\theta \log(f_\theta(X)) = \frac{\nabla_\theta f_\theta(X)}{f_\theta(X)}$$

$$\mathbb{E}_\theta(\nabla_\theta \log(f_\theta(X))) = \begin{cases} \int_{\mathcal{X}} \frac{\nabla_\theta f_\theta(x)}{f_\theta(x)} f_\theta(x) dx & \text{if } X_i\text{'s are continuous} \\ \sum_{x \in \mathcal{X}} \frac{\nabla_\theta f_\theta(x)}{f_\theta(x)} f_\theta(x) & \text{if } X_i\text{'s are discrete} \end{cases}$$

We have, if the X_i's are continuous:

$$\int_{\mathcal{X}} \frac{\partial}{\partial \theta_j} f_\theta(x) dx = \frac{\partial}{\partial \theta_j} \int_{\mathcal{X}} f_\theta(x) dx$$

Because $\mathbb{E}_\theta(||\nabla_\theta f_\theta(X)||) < +\infty$. Also $\forall \theta, \int_{\mathcal{X}} f_\theta(x) dx = 1 \implies \frac{\partial}{\partial \theta_j}(\int f_\theta(x) dx) = 0$.

So $\mathbb{E}_\theta \left(\frac{\partial}{\partial \theta_j} \frac{f_\theta(X)}{f_\theta(X)} \right) = 0$.

$$\implies \mathbb{E}_\theta(\nabla_\theta \log(f_\theta(X))) = 0$$

If f is discrete $\to$ same argument.

$$\sum_x \frac{\partial}{\partial \theta_j} f_\theta(x) = \frac{\partial}{\partial \theta_j} \sum_x f_\theta(x) = \frac{\partial}{\partial \theta_j}(1) = 0$$

(Q.E.D)

b) Show that $I(\theta) = -\mathbb{E}_\theta(D^2_\theta \log(f_\theta(X)))$.

$$\frac{\partial^2 \log(f_\theta(x))}{\partial \theta_i \partial \theta_j} = \frac{\partial}{\partial \theta_j} \left(\frac{\frac{\partial}{\partial \theta_i} f_\theta(x)}{f_\theta(x)} \right)$$

$$= \frac{\frac{\partial^2}{\partial \theta_i \partial \theta_j} f_\theta(x)}{f_\theta(x)} - \left(\frac{\frac{\partial}{\partial \theta_i} f_\theta(x)}{f_\theta(x)} \right) \left(\frac{\frac{\partial}{\partial \theta_j} f_\theta(x)}{f_\theta(x)} \right)$$

Similarly to before:

$$\mathbb{E}_\theta \left(\frac{\frac{\partial^2}{\partial \theta_i \partial \theta_j} f_\theta(X)}{f_\theta(X)} \right) = \int_{\mathcal{X}} \frac{\frac{\partial^2}{\partial \theta_i \partial \theta_j} f_\theta(x)}{f_\theta(x)} f_\theta(x) dx$$

$$= \int_{\mathcal{X}} \frac{\partial^2}{\partial \theta_i \partial \theta_j} f_\theta(x) dx = \frac{\partial^2}{\partial \theta_i \partial \theta_j} \int_{\mathcal{X}} f_\theta(x) dx = 0$$

$$\begin{aligned} J(\theta)_{i,j} &= -\mathbb{E}_\theta(D^2_\theta \log(f_\theta(X))) \\ &= -\mathbb{E}_\theta \left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \log(f_\theta(X)) \right) \\ &= - \left(-\mathbb{E}_\theta \left(\frac{\frac{\partial f}{\partial \theta_i}}{f} \cdot \frac{\frac{\partial f}{\partial \theta_j}}{f} \right) \right) \\ &= I_{i,j}(\theta) \end{aligned}$$

So $J(\theta) = I(\theta)$. (Q.E.D) $\square$

Theorem 8: Asymptotic Normality of MLE

Let $(x_1, \ldots, x_n)$ be observations from the model $X_1, \ldots, X_n \overset{i.i.d.}{\sim} f_\theta, \theta \in \Theta \subset \mathbb{R}^k$. Assume the model is **regular**.

(i) **Score Convergence:** For all $\theta \in \Theta$, let $S_n(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \nabla_\theta \log(f_\theta(X_i))$. Then:

$$S_n(\theta) \xrightarrow[n \to \infty]{\mathcal{L}_{P_\theta}} \mathcal{N}(0, I(\theta))$$

(ii) **MLE Convergence:** If in addition the MLE $\hat{\theta}_n$ is **consistent** over Θ (in probability), then:

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[n \to \infty]{\mathcal{L}_{P_\theta}} \mathcal{N}(0, I^{-1}(\theta))$$

$\implies I^{-1}(\theta)$ is the asymptotical variance of $\sqrt{n}(\hat{\theta}_n - \theta)$.

Remark

Cramer-Rao Lower Bound: For the variance of a regular estimator is $\frac{I^{-1}(\theta)}{n}$.

Multivariate Gaussians Reminder

$X \sim \mathcal{N}(\mu, \Sigma)$, $X \in \mathbb{R}^k$, $\mu \in \mathbb{R}^k$. Σ symmetric, positive semi-definite.
$\iff$ its density (w.r.t Lebesgue)

$$f_{\mu,\Sigma}(x) = \frac{e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)}}{(2\pi)^{k/2}(\det(\Sigma))^{1/2}}$$

$\iff \forall a \in \mathbb{R}^d, a^T X \sim \mathcal{N}(a^T \mu, a^T \Sigma a)$.

Proof of the Theorem

Objective: Show that $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}_{P_\theta}} \mathcal{N}(0, I^{-1}(\theta))$.

$\hat{\theta}_n$ is defined by $l(\hat{\theta}_n) \geq l(\theta)$ and verifies $\nabla_\theta l(\hat{\theta}_n) = 0$. The rest of the proof is done with $k = 1$. $\theta \mapsto l(\theta) = \sum \log(f_\theta(x_i))$ is $\mathcal{C}^2$.

Taylor expansion:

$$l'(\theta) = l'(\hat{\theta}_n) + (\theta - \hat{\theta}_n) l''(\tilde{\theta}_n) \quad \text{with } \tilde{\theta}_n \in]\theta, \hat{\theta}_n[$$

$$\implies l'(\theta) = (\theta - \hat{\theta}_n) l''(\tilde{\theta}_n) \quad (\text{since } l'(\hat{\theta}_n) = 0)$$

$$\frac{l''(\tilde{\theta}_n)}{n} = \frac{1}{n} \sum_{i=1}^n \frac{\partial^2}{\partial \theta^2} \log(f_{\tilde{\theta}_n}(x_i))$$

$$\frac{l''(\tilde{\theta}_n) - l''(\theta)}{n} = \Delta_n(\theta)$$

So:

$$l'(\theta) = (\theta - \hat{\theta}_n)(l''(\theta) + \Delta_n(\theta) \times n)$$

And $\frac{l'(\theta)}{\sqrt{n}} = -S_n(\theta)$.

$$S_n(\theta) = \sqrt{n}(\theta - \hat{\theta}_n) \left(\frac{l''(\theta)}{n} + \Delta_n(\theta) \right)$$

We examine the terms:

$$\frac{l''(\theta)}{n} = \frac{1}{n}\sum_{i=1}^{n} D^2 \log(f_\theta(x_i)) \xrightarrow[n\to\infty]{P_\theta} \mathbb{E}_\theta(D^2\log(f_\theta(X))) = -I(\theta)$$

$$\implies S_n(\theta) = \sqrt{n}(\hat{\theta}_n - \theta)(-I(\theta) + \Delta_n(\theta) + B_n(\theta))$$

where $B_n(\theta) = \frac{l''(\theta)}{n} + I(\theta) \xrightarrow{P_\theta} 0$.

We need to handle $S_n(\theta)$ and $\Delta_n(\theta)$.

(i) Behavior of $S_n(\theta)$:

$$\forall\theta\in\Theta, S_n(\theta) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \nabla_\theta \log(f_\theta(x_i)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, I(\theta))$$

Proof for (i):

$$S_n(\theta) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} s_1(\theta, x_i)$$

And $\forall\theta, (s_1(\theta, x_i))_{i=1,\ldots,n}$ are i.i.d. because (x_i) are i.i.d. And $\mathbb{E}_\theta(s_1(\theta, X_i)) = 0$ (Lemma). And $\mathbb{E}_\theta(||s_1(\theta, X)||^2) < +\infty$ (Assumption).

By the CLT:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} s_1(\theta, X_i) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, \mathrm{Var}_\theta(s_1(\theta, X)))$$

Also $\mathrm{Var}_\theta(s_1(\theta, X)) = \mathbb{E}_\theta(s_1(\theta, X)s_1(\theta, X)^T) = I(\theta)$.

(ii) Behavior of $\Delta_n(\theta)$: We show that $\Delta_n(\theta) \xrightarrow[n\to\infty]{P_\theta} 0$. Because when $|\hat{\theta}_n - \theta| \le \delta$:

$$|\Delta_n(\theta)| \le \frac{1}{n}\sum \sup_{|\theta'-\theta|\le\delta} |D^2_{\theta'}\log(f_{\theta'}(x_i)) - D^2_\theta \log(f_\theta(x_i))|$$

LLN:

$$\frac{1}{n}\sum \sup_{|\theta'-\theta|\le\delta} |D^2_{\theta'}\log(f_{\theta'}(x_i)) - D^2_\theta \log(f_\theta(x_i))|$$

$$\xrightarrow[P_\theta]{} \mathbb{E}_\theta\left(\sup_{|\theta'-\theta|\le\delta} |D^2_{\theta'}\log(f_{\theta'}(x_i)) - D^2_\theta \log(f_\theta(x))|\right)$$

By the dominated convergence theorem:

$$\mathbb{E}_\theta\left(\sup_{|\theta'-\theta|\le\delta} |D^2_{\theta'}\log(f_{\theta'}(x)) - D^2_\theta \log(f_\theta(x))|\right) \xrightarrow[\delta\to 0]{} 0$$

$\forall\varepsilon > 0, \exists\delta_\varepsilon > 0, \forall\delta \le \delta_\varepsilon$, this expectation is $< \varepsilon$.

We know that $\hat{\theta}_n \xrightarrow{P_\theta} \theta$. $\forall\delta > 0, \mathbb{P}_\theta(|\hat{\theta}_n - \theta| > \delta) \xrightarrow[n\to\infty]{} 0$.

Also $\theta \mapsto D^2_\theta \log(f_\theta(x))$ is continuous. $\implies \limsup_{\delta\to 0}\sup_{|\theta'-\theta|\le\delta} |D^2_{\theta'}\log(f_{\theta'}(x)) - D^2_\theta\log(f_\theta(x))| = 0$ (almost surely). Let $\delta_p \to 0, \delta_p \downarrow$.

$$\sup_{|\theta'-\theta|\le\delta_p} |D^2_{\theta'}\log(f_{\theta'}(x)) - D^2_\theta \log(f_\theta(x))|$$

$$\le \sup_{|\theta'-\theta|\le\delta_0} |D^2_{\theta'}\log(f_{\theta'}(x)) - D^2_\theta \log(f_\theta(x))|$$

And $\mathbb{E}_\theta(H(X)) < +\infty$ (by assumption).

So $\Delta_n(\theta) \xrightarrow{P_\theta} 0$.
Conclusion of Proof:

$$S_n(\theta) = \sqrt{n}(\hat{\theta}_n - \theta)(-I(\theta) + \Delta_n(\theta) + B_n(\theta))$$

$$B_n(\theta) = \frac{l''(\theta)}{n} + I(\theta) \xrightarrow{P_\theta} 0$$

$$\Delta_n(\theta) \xrightarrow{P_\theta} 0$$

$$S_n(\theta) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, I(\theta))$$

$$\implies \sqrt{n}(\hat{\theta}_n - \theta) = \frac{S_n(\theta)}{I(\theta)(1+\mu_n)} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{I(\theta)}{I^2(\theta)}\right) = \mathcal{N}(0, I^{-1}(\theta))$$

With $\mu_n \xrightarrow{P_\theta} 0$.

III. Exponential Families

Definition 20: Exponential Family

Model $X \in \mathbb{R}^d \sim P_\theta$, $\theta \in \Theta \subset \mathbb{R}^k$ is an **exponential family** iff P_θ has density (or proba mass function) f_θ and:

- $\exists h : \mathbb{R}^d \to \mathbb{R}_+$
- $\exists A : \Theta \to \mathbb{R}^k$
- $\exists S : \mathbb{R}^d \to \mathbb{R}^k$
- $\exists C : \Theta \to \mathbb{R}$

Such that:

$$f_\theta(x) = h(x) \exp\left(A(\theta)^T S(x) - C(\theta)\right)$$

It's a **canonical exponential family** iff $A(\theta) = \theta$.

Remark

Cramer-Rao Lower Bound: For the variance of a regular estimator, the lower bound is $\frac{I^{-1}(\theta)}{n}$. What are regular models? Exponential families are a prime example.

Theorem 9: Properties of Exponential Families

Let P_θ be an exponential family. If $f_\theta(x) = h(x)\exp(A(\theta)^T S(x) - C(\theta))$ and if $\Theta = \{\theta, f_\theta(x) \text{ exists}\}$ is an open set, then:

(i) If $A(\theta) = \theta$, then the model is **regular**.

(ii) $\exp(C(\theta)) = \int_{\mathbb{R}^d} h(x) e^{A(\theta)^T S(x)} dx$ (if continuous) or $\sum_{x\in\mathcal{X}} h(x) e^{A(\theta)^T S(x)}$ (if discrete).

(iii) If $A(\theta) = \theta$, then $\theta \mapsto C(\theta)$ is $\mathcal{C}^\infty$ on Θ and:

$$\mathbb{E}_\theta(S(X)) = \nabla_\theta C(\theta)$$

$$\mathbb{V}_\theta(S(X)) = \frac{\partial^2 C(\theta)}{\partial \theta^2} = D^2 C(\theta)$$

(iv) If A is invertible and $\mathcal{C}^2$, and A^{-1} is $\mathcal{C}^2$, then the model is regular.

Remark

"f_θ exists" means:

- If P_θ is continuous: $\int_{\mathbb{R}^d} e^{A(\theta)^T S(x)} h(x) dx < +\infty$.
- If P_θ is discrete: $\sum_{x \in \mathcal{X}} e^{A(\theta)^T S(x)} h(x) < +\infty$.

Heuristic of the proof (Continuous case)

(ii) We know that $\int_{\mathbb{R}^d} f_\theta(x) dx = 1$.

$$\iff \int_{\mathbb{R}^d} h(x) e^{A(\theta)^T S(x) - C(\theta)} dx = 1$$

$$\iff \int_{\mathbb{R}^d} h(x) e^{A(\theta)^T S(x)} dx = e^{C(\theta)}$$

(iii) If $A(\theta) = \theta$.

$$e^{C(\theta)} = \int_{\mathbb{R}^d} e^{\theta^T S(x)} h(x) dx$$

And $\{\theta, \int_{\mathbb{R}^d} e^{\theta^T S(x)} h(x) dx < +\infty\}$ is open.

Differentiation:

$$\frac{\partial}{\partial \theta} e^{\theta^T S(x)} = S(x) e^{\theta^T S(x)}$$

If $k = 1$:

$$\frac{d}{d\theta} e^{C(\theta)} = C'(\theta) e^{C(\theta)} = \int_{\mathbb{R}^d} S(x) e^{\theta S(x)} h(x) dx$$

$$\implies C'(\theta) = e^{-C(\theta)} \int_{\mathbb{R}^d} S(x) e^{\theta S(x)} h(x) dx$$

$$= \int_{\mathbb{R}^d} S(x) \underbrace{e^{\theta S(x) - C(\theta)} h(x)}_{f_\theta(x)} dx = \mathbb{E}_\theta(S(X))$$

Second derivative:

$$\frac{d^2}{d\theta^2} e^{C(\theta)} = C''(\theta) e^{C(\theta)} + (C'(\theta))^2 e^{C(\theta)} = \int_{\mathbb{R}^d} S(x)^2 e^{\theta S(x)} h(x) dx$$

$$\implies C''(\theta) = -\underbrace{C'(\theta)^2}_{\mathbb{E}_\theta(S(X))^2} + \underbrace{\int_{\mathbb{R}^d} S(x)^2 e^{\theta S(x) - C(\theta)} h(x) dx}_{\mathbb{E}_\theta(S(X)^2)}$$

$$= \mathbb{E}_\theta(S(X)^2) - \mathbb{E}_\theta(S(X))^2 = \mathbb{V}_\theta(S(X))$$

Remark

Using properties of the log-likelihood:

$$\frac{d^2}{d\theta^2}\log(f_\theta(x)) = -\frac{d^2}{d\theta^2}C(\theta)$$

Or $I(\theta) = -\mathbb{E}_\theta\left(\frac{d^2}{d\theta^2}\log f_\theta(X)\right) = \frac{d^2}{d\theta^2}C(\theta)$.

Justification for differentiation under the integral sign (Proof details): If $\theta \in \Theta$, $\theta + \varepsilon \in \Theta$ and $\theta - \varepsilon \in \Theta$ (since open).

$$\implies \int e^{(\theta+\varepsilon)S(x)}h(x)dx < +\infty \quad \text{and} \quad \int e^{(\theta-\varepsilon)S(x)}h(x)dx < +\infty$$

$$|S(x)|e^{\theta S(x)} \leq e^{\theta S(x)} + \varepsilon|S(x)|e^{\theta S(x)}\dots$$

Wait, actually we use convexity arguments.

$$|S(x)|e^{\theta S(x)} \leq e^{(\theta+\varepsilon)S(x)} + e^{(\theta-\varepsilon)S(x)}$$

This bound (integrable function) allows the use of the Dominated Convergence Theorem.

$$\implies \int |S(x)|e^{\theta S(x)}h(x)dx < +\infty$$

Proposition 4: Stability of Exponential Families

If $\{P_\theta, \theta \in \Theta\}$ is an exponential family and $X_1, \dots, X_n \overset{i.i.d.}{\sim} P_\theta$, then the joint distribution is also an exponential family.

$$f_\theta^{(n)}(x_1, \dots, x_n) = e^{A(\theta)^T \sum_{i=1}^n S(x_i) - nC(\theta)} \prod_{i=1}^n h(x_i)$$

This is an exponential family with:

$$A_n(\theta) = A(\theta), \quad S_n(x_1, \dots, x_n) = \sum_{i=1}^n S(x_i), \quad C_n(\theta) = nC(\theta)$$

Proof.

$$f_\theta^{(n)}(x_1, \dots, x_n) = \prod_{i=1}^n f_\theta(x_i)$$

$$= \prod_{i=1}^n \left(h(x_i)e^{A(\theta)^T S(x_i) - C(\theta)}\right)$$

$$= \left(\prod_{i=1}^n h(x_i)\right) e^{A(\theta)^T \sum S(x_i) - nC(\theta)}$$

□

Examples

Example 9: Binomial Distribution

$X \sim \text{Bin}(n,p)$, $p = \theta$. Probability mass function:

$$f_p(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x \in \{0, \dots, n\}$$

$$= \underbrace{\binom{n}{x}}_{h(x)} \exp\left(\underbrace{\log\left(\frac{p}{1-p}\right)}_{A(p)} x + \underbrace{n \log(1-p)}_{-C(p)} \right)$$

Here $S(x) = x$.

Canonical Parametrisation:

$$\eta = \log\left(\frac{p}{1-p}\right) \implies p = \frac{e^\eta}{1+e^\eta}$$

$$\tilde{f}_\eta(x) = \binom{n}{x} \exp\left(\eta x - n\log(1+e^\eta)\right)$$

With $C(\eta) = n\log(1+e^\eta)$.

$$\Theta = \mathbb{R} \text{ (open)}$$

Moments check:

$$\mathbb{E}_\eta(X) = C'(\eta) = n\frac{e^\eta}{1+e^\eta} = np$$

$$\mathbb{V}_\eta(X) = C''(\eta) = n\frac{e^\eta(1+e^\eta) - e^\eta e^\eta}{(1+e^\eta)^2} = n\frac{e^\eta}{(1+e^\eta)^2} = np(1-p)$$

Remark: In a canonical exponential family, if Θ is an open set, then $I(\theta) = \frac{d^2 C(\theta)}{d\theta^2}$.

Proof:

$$\log(f_\theta(x)) = \theta^T S(x) - C(\theta) + \log(h(x))$$

$$\implies \frac{d}{d\theta}\log(f_\theta(x)) = S(x) - \nabla C(\theta)$$

$$\implies \frac{d^2}{d\theta^2}\log(f_\theta(x)) = -\frac{d^2}{d\theta^2}C(\theta)$$

$$\implies I(\theta) = -\mathbb{E}_\theta\left[\frac{d^2}{d\theta^2}\log(f_\theta(X))\right] \quad \text{(see Lemma 1)}$$

$$= \frac{d^2}{d\theta^2}C(\theta)$$

Example 10: Exponential Distribution

$X \sim \mathcal{E}(\theta), \theta > 0$.

$$f_\theta(x) = \theta e^{-\theta x}\mathbb{1}_{x>0} = \mathbb{1}_{x>0} e^{-\theta x + \log(\theta)}$$

$h(x) = \mathbb{1}_{x>0}$. $A(\theta) = \theta$ (or $-\theta$). $S(x) = -x$. $C(\theta) = -\log(\theta)$. Canonical exponential family.

IV. Sufficient Condition for Consistency of MLE

Theorem 10: Consistency of MLE

If Θ is compact,
If (i) $\sup_{\theta \in \Theta} \left| \frac{\ln(L_n(\theta)) - \ln(L_n(\theta_0))}{n} + KL(\theta_0, \theta) \right| \xrightarrow[n \to \infty]{P_{\theta_0}} 0$
If (ii) $\forall \varepsilon > 0, \exists \delta > 0$ such that $\inf_{|\theta - \theta_0| > \varepsilon} KL(\theta_0, \theta) > \delta$.

Then the MLE is consistent.

Where:

$$KL(\theta_0, \theta) = \mathbb{E}_{\theta_0}[\log f_{\theta_0}(X) - \log f_\theta(X)]$$

(Kullback-Leibler divergence between f_{θ_0} and f_θ).

Proof.

$$KL(\theta_0, \theta) = \mathbb{E}_{\theta_0}[\log(f_{\theta_0}(X)) - \log(f_\theta(X))]$$

Idea of the proof: Prove that $\forall \varepsilon > 0, P_{\theta_0} \left[\sup_{|\theta - \theta_0| > \varepsilon} \ln(\theta) < \sup_{\theta \in \Theta} \ln(\theta) \right] \to 1$ as $n \to \infty$. (where $\ln(\theta)$ denotes the log-likelihood).

Because if $\sup_{|\theta - \theta_0| > \varepsilon} \ln(\theta) < \sup_{\theta \in \Theta} \ln(\theta)$, then $|\hat{\theta}_n - \theta_0| \leq \varepsilon$ (since $\hat{\theta}_n$ maximizes the likelihood).

Show that:

$$P_{\theta_0} \left(\sup_{|\theta - \theta_0| > \varepsilon} (\ln(\theta) - \ln(\theta_0)) < 0 \right) \xrightarrow{n \to \infty} 1$$

$$\sup_{|\theta - \theta_0| > \varepsilon} \left[\frac{\ln(\theta) - \ln(\theta_0)}{n} + KL(\theta_0, \theta) - KL(\theta_0, \theta) \right]$$

We have:

$$\frac{\ln(\theta) - \ln(\theta_0)}{n} = - \left[\frac{1}{n} \sum_{i=1}^n \log \left(\frac{f_{\theta_0}(x_i)}{f_\theta(x_i)} \right) \right]$$

By LLN $\to -\mathbb{E}_{\theta_0} \left[\log \left(\frac{f_{\theta_0}(X)}{f_\theta(X)} \right) \right] = -KL(\theta_0, \theta)$.

By assumption (i):

$$\sup_{|\theta - \theta_0| > \varepsilon} \left| \frac{\ln(\theta) - \ln(\theta_0)}{n} + KL(\theta_0, \theta) \right| \xrightarrow{P_{\theta_0}} 0$$

And using (ii):

$$\sup_{|\theta - \theta_0| > \varepsilon} -KL(\theta_0, \theta) = - \inf_{|\theta - \theta_0| > \varepsilon} KL(\theta_0, \theta) < -\delta$$

So:

$$\sup_{|\theta - \theta_0| > \varepsilon} \frac{\ln(\theta) - \ln(\theta_0)}{n} < -\delta + o(1)$$

Which is < 0 for n large enough. □

V. Using Asymptotic Normality to Compute Confidence Regions

$\theta \in \Theta \subset \mathbb{R}$. Model $X_1, \ldots, X_n \overset{i.i.d.}{\sim} P_\theta$. $\hat{\theta}_n =$ MLE.

We know:

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow[n \to \infty]{\mathcal{L}_{P_{\theta_0}}} \mathcal{N}(0, I^{-1}(\theta_0))$$

We want $I_n = [\hat{\theta}_{n,1}, \hat{\theta}_{n,2}]$ such that:

$$P_{\theta_0}(\theta_0 \in I_n) \geq 1 - \alpha$$

Then I_n is a $(1-\alpha)$ confidence interval for θ_0.

I_n is an **asymptotic** $(1-\alpha)$ confidence interval if $P_{\theta_0}(\theta_0 \in I_n) \xrightarrow[n\to\infty]{} 1-\alpha$.

Exercise: Construct I_n.

VI. Delta method and confidence intervals

VI.1 Delta method

Remark

Reminder: If $\sqrt{n}(X_n - \mu) \xrightarrow{\mathcal{L}} \mathcal{N}(0, V_0)$, with $X_n \in \mathbb{R}^{d_1}$ and if $g : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is $\mathcal{C}^1$ and s.t (sub-tangent/standard) $(d_1 \leq d_2)$, $\nabla g(\mu) \in \mathbb{R}^{d_2 \times d_1}$ is of rank d_2

Then:

$$\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \nabla g(\mu) V_0 \nabla g(\mu)^\top)$$

Proof. Case $d_1 = d_2 = 1$.

Taylor expansion:

$$g(X_n) = g(X_n - \mu + \mu) = g(\mu) + (X_n - \mu) g'(\bar{\mu}_n)$$

where $\bar{\mu}_n \in]X_n, \mu[$.

This implies:

$$\begin{aligned}\sqrt{n}(g(X_n) - g(\mu)) &= \sqrt{n}(X_n - \mu) g'(\bar{\mu}_n) \\ &= \sqrt{n}(X_n - \mu)(g'(\mu) + o_p(1))\end{aligned}$$

We know that $\sqrt{n}(X_n - \mu) g'(\mu) \xrightarrow{\mathcal{L}} \mathcal{N}(0, g'(\mu)^2 V_0)$.

And $\sqrt{n}(X_n - \mu)(g'(\bar{\mu}_n) - g'(\mu)) \xrightarrow{\mathbb{P}} 0$. □

Remark

Reminders:

- $X_n = O_p(1) \iff \lim_{C\to\infty} \limsup_n \mathbb{P}(|X_n| > C) = 0$.
- $X_n = o_p(1) \iff X_n \xrightarrow{\mathbb{P}} 0$.
- If $X_n \xrightarrow{\mathcal{L}} Q$ (where Q is a probability distribution), then $X_n = O_p(1)$.

Application to the MLE

If $\hat{\theta}_n$ is the MLE and if $\hat{\eta}_n = g(\hat{\theta}_n)$ with $g \in \mathcal{C}^1$, $\hat{\theta}_n \in \mathbb{R}^d$, $g : \mathbb{R}^d \to \mathbb{R}^k$.

If $\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow[\mathbb{P}_{\theta_0}]{\mathcal{L}} \mathcal{N}(0, I^{-1}(\theta_0))$ with $\nabla g(\theta_0)$ of rank k.

Then:

$$\sqrt{n}(\hat{\eta}_n - g(\theta_0)) \xrightarrow[\mathbb{P}_{\theta_0}]{\mathcal{L}} \mathcal{N}(0, \nabla g(\theta_0) I^{-1}(\theta_0) \nabla g(\theta_0)^\top)$$

Example 11: Bernoulli

$X_1, ..., X_n \overset{i.i.d}{\sim} \text{Ber}(p)$ under $\mathbb{P}_{p_0}$.
MLE: $\hat{p}_n = \bar{X}_n$ and $\sqrt{n}(\hat{p}_n - p_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0, p_0(1-p_0))$.
Let $\eta = \log(\frac{p}{1-p})$ and $g(p) = \log(\frac{p}{1-p})$.
$g :]0,1[\to \mathbb{R}$ is invertible. For $x \in \mathbb{R}$, $g^{-1}(x) = \frac{e^x}{1+e^x}$.

Derivative:

$$g'(p) = \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}$$

$\forall p_0 \in]0,1[, g'(p_0) \neq 0$.

$$\sqrt{n}(\hat{\eta}_n - \eta) \xrightarrow[\mathbb{P}_{p_0}]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{p_0(1-p_0)}\right)$$

Note: $\frac{1}{p_0(1-p_0)} = \frac{(1+e^{\eta_0})^2}{e^{\eta_0}}$.

Example 12: Normal Distribution

$X_1, ..., X_n \overset{i.i.d}{\sim} \mathcal{N}(\mu, \sigma^2)$.
$\theta = (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}^{+*}$.

MLE: $\hat{\theta}_n = (\bar{X}_n, S_n^2)$ where $S_n^2 = \frac{1}{n}\sum_{k=1}^n (X_k - \bar{X}_n)^2$.
Canonical parameters:

$$\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} = \frac{e^{-\frac{x^2}{2\sigma^2} - \frac{x\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2} - \frac{\log(\sigma^2)}{2}}}{\sqrt{2\pi}}$$

$$\implies \eta_1 = \frac{\mu}{\sigma^2}, \quad \eta_2 = \frac{-1}{2\sigma^2}$$

$$\implies \sqrt{n}(\hat{\eta}_n - \eta) \to \mathcal{N}(0, \tilde{V}_0)$$

Let $g(\eta) = \sigma^2$. We have $\sigma^2 = -\frac{1}{2\eta_2}$.
Gradient:

$$\nabla g(\eta) = \left(\frac{\partial g}{\partial \eta_1}, \frac{\partial g}{\partial \eta_2}\right) = \left(0, \frac{1}{2\eta_2^2}\right)$$

$$\implies \sqrt{n}(S_n^2 - \sigma^2) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, (0, \frac{1}{2\eta_2^2})\tilde{V}_0 \begin{pmatrix} 0 \\ \frac{1}{2\eta_2^2} \end{pmatrix}\right)$$

Where $\tilde{V}_0 = \frac{d^2 c}{d^2 \eta}(\eta_0)$ and $c(\eta_0) = \frac{\eta_1^2}{8\eta_2} + \frac{1}{2}\log(-\frac{1}{2\eta_2})$.

$$\implies c'' = \dots$$

VI.2 Confidence intervals

Definition 21: Confidence Region

A confidence region of level α for an estimator of $g(\theta)$ is any region $C_\alpha(X_1, ..., X_n)$ verifying:

$$\forall \theta \in \Theta, \quad \mathbb{P}_\theta[g(\theta) \in C_\alpha(X_1, ..., X_n)] \geq 1 - \alpha$$

We can use asymptotic normality of an estimator to construct asymptotic confidence regions.

Example 13: Construction using MLE

$X_1, ..., X_n \overset{i.i.d}{\sim} \mathbb{P}_\theta, \theta \in \Theta \subset \mathbb{R}$.
If the MLE verifies:

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[\mathbb{P}_\theta]{\mathcal{L}} \mathcal{N}(0, I_\theta^{-1})$$

We look for:

$$C_\alpha(X_1, ..., X_n) = [\hat{\theta}_{n,1}, \hat{\theta}_{n,2}]$$

s.t.

$$\mathbb{P}_\theta[\theta \in [\hat{\theta}_{n,1}, \hat{\theta}_{n,2}]] \xrightarrow[n\to\infty]{} 1 - \alpha$$

$$\hat{\theta}_{n,1} \leq \theta \leq \hat{\theta}_{n,2} \iff \sqrt{n}(\hat{\theta}_{n,1} - \hat{\theta}_n) \leq \sqrt{n}(\theta - \hat{\theta}_n) \leq \sqrt{n}(\hat{\theta}_{n,2} - \hat{\theta}_n)$$

But:

$$\sqrt{n}(\theta - \hat{\theta}_n) \xrightarrow{\mathcal{L}} \mathcal{N}(0, I_\theta^{-1})$$

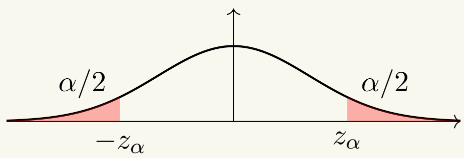
$$\implies \sqrt{n} I_\theta^{1/2}(\theta - \hat{\theta}_n) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1)$$

Since $\hat{\theta}_n \xrightarrow{\mathbb{P}_\theta} \theta \implies I_{\hat{\theta}_n} \xrightarrow{\mathbb{P}_\theta} I_\theta$.

$$\implies \sqrt{n} I_{\hat{\theta}_n}^{1/2}(\theta - \hat{\theta}_n) \xrightarrow[\mathbb{P}_\theta]{\mathcal{L}} \mathcal{N}(0, 1)$$

If z_α is the $1 - \frac{\alpha}{2}$ quantile of a $\mathcal{N}(0, 1)$:

$$\iff \Phi(z_\alpha) = 1 - \frac{\alpha}{2}, \quad \Phi = \text{CDF of } \mathcal{N}(0, 1)$$



By symmetry: $\Phi(-z_\alpha) = \frac{\alpha}{2}$.

$$\implies \lim_{n\to\infty} \mathbb{P}_\theta \left(-z_\alpha \leq \sqrt{n} I_{\hat{\theta}_n}^{1/2}(\theta - \hat{\theta}_n) \leq z_\alpha\right) = 1 - \alpha$$

$$\iff \lim_{n\to\infty} \mathbb{P}_\theta \left(\frac{-z_\alpha}{I_{\hat{\theta}_n}^{1/2}} \leq \sqrt{n}(\theta - \hat{\theta}_n) \leq \frac{z_\alpha}{I_{\hat{\theta}_n}^{1/2}}\right) = 1 - \alpha$$

So if:

$$\hat{\theta}_{n,1} = \hat{\theta}_n - \frac{z_\alpha}{\sqrt{n}\sqrt{I_{\hat{\theta}_n}}} \quad ; \quad \hat{\theta}_{n,2} = \hat{\theta}_n + \frac{z_\alpha}{\sqrt{I_{\hat{\theta}_n}}\sqrt{n}}$$

Then:

$$\mathbb{P}_\theta(\hat{\theta}_{n,1} \leq \theta \leq \hat{\theta}_{n,2}) \xrightarrow[n\to\infty]{} 1 - \alpha$$

Chapter 5: Bayesian statistics

Model: $X_1, ..., X_n \overset{i.i.d}{\sim} \mathbb{P}_\theta$.
$\hat{\theta}_n \to \mathcal{R}(\theta, \hat{\theta}_n) = \mathbb{E}[(\theta - \hat{\theta}_n)^2] =$ Quadratic risk.

I. In Bayesian statistics

$\theta =$ unknown $\to$ Modelled as a random variable.

Definition 22: Prior Distribution

We call prior distribution on θ, a probability distribution over Θ.

Model:
$X_1, ..., X_n | \theta \overset{i.i.d}{\sim} \mathbb{P}_\theta$: Conditional distribution of $(X_1, ..., X_n)$ given θ.
Prior Marginal Probability over Θ:
$\theta \sim \Pi$ Probability on Θ.
$\implies$ **Joint distribution** on $(X_1, ..., X_n, \theta)$.
IF $\mathbb{P}_\theta$ is continuous, with density f_θ (wrt Lebesgue measure).

Theorem 11: Bayes Theorem

The conditional distribution of θ given $(X_1, ..., X_n)$ has density wrt ν:

$$\Pi_n(\theta | X_1, ..., X_n) = \frac{\Pi(\theta) \prod_{i=1}^n f_\theta(x_i)}{\int_\Theta \Pi(\theta) \prod_{i=1}^n f_\theta(x_i) d\nu(\theta)}$$

If π is the density of Π wrt ν, it is called the **posterior distribution**.

Definition 23: Bayesian Model

In Bayesian statistics, the Bayesian model is defined by:

i) $[X|\theta] \sim \mathbb{P}_\theta$ with likelihood $L(\theta)$, $\theta \in \Theta$.

ii) Prior distribution Π on Θ.

Then the inference is made by the **posterior distribution** defined as the conditional distribution of θ given X. If π is the density of Π (wrt a measure ν), then the posterior distribution has density (wrt ν) given by:

$$\Pi(\theta | X) = \frac{L(\theta)\Pi(\theta)}{\int_\Theta L(\theta)\Pi(\theta) d\nu(\theta)}$$

where

$$\int_\Theta L(\theta)\Pi(\theta) d\nu(\theta) = \begin{cases} \int L(\theta)\Pi(\theta) d\nu(\theta) & \text{if } \Pi \text{ continuous} \\ \sum_{\theta \in \Theta} L(\theta)\Pi(\theta) & \text{if } \Pi \text{ discrete} \end{cases}$$

Remark

- If $\mathbb{P}_\theta$ is continuous:
  $L(\theta) = f_\theta(x) =$ density at the observations when θ is the parameter.
- If $\mathbb{P}_\theta$ is discrete:

$L(\theta) = f_\theta(x) = \mathbb{P}_\theta(X = x)$.

$$\Pi(\theta|X = x) = \frac{f_\theta(x)\Pi(\theta)}{\int_\Theta f_\theta(x)\Pi(\theta)d\nu(\theta)}$$

Example 14: Poisson - Gamma

$X = (X_1, ..., X_n)$ and $X_i \overset{i.i.d}{\sim} \mathcal{P}(\theta), \theta > 0$.
Prior: $\theta \sim \Gamma(a, b)$, Π is a $\Gamma(a, b)$.

$$f_\theta(x) = \prod_{i=1}^{n} f_\theta(x_i)$$

$$\implies \Pi(\theta|X_1, ..., X_n) = \frac{\prod_{i=1}^{n} f_\theta(x_i)\theta^{a-1}e^{-b\theta}}{\int_\Theta \prod_{i=1}^{n} f_\theta(x_i)\theta^{a-1}e^{-b\theta}d\theta}$$

Normalization constant (denominator) implies:

$$\Pi(\theta|X_1, ..., X_n) \propto e^{-n\theta}\prod_{i=1}^{n}\frac{\theta^{x_i}}{x_i!} \times \theta^{a-1}e^{-b\theta}$$

$$\propto e^{-(n+b)\theta}\theta^{a+\sum_{i=1}^{n} x_i - 1}$$

(Recall: $f_\theta(x_i) = e^{-\theta}\frac{\theta^{x_i}}{x_i!}$).

$\Pi(\cdot)$ is a density $\implies \frac{f(\theta)}{g(\theta)}$ is a constant implies $\exists c \neq 0, f(\theta) = cg(\theta)$.
$\exists C > 0$ s.t.

$$\Pi(\theta|X_1, ..., X_n) = Ce^{-(b+n)\theta}\theta^{\sum_{i=1}^{n} x_i + a - 1}$$

We recognize the kernel of a Gamma distribution $\Gamma(\sum_{i=1}^{n} x_i + a, b + n)$.

$$\implies \Pi(\theta|X_1, ..., X_n) = \frac{e^{-(b+n)\theta}\theta^{\sum_{i=1}^{n} x_i + a - 1}(b + n)^{a+\sum x_i}}{\Gamma(a + \sum_{i=1}^{n} x_i)}$$

Posterior distribution is $\Gamma(\sum x_i + a, b + n)$.
Note regarding the constant:

$$g(\theta) = \theta^{b-1}e^{-b\theta} \times c$$

$$\int_{\mathbb{R}^+} g(\theta)d\theta = 1 \iff c = \frac{b^a}{\Gamma(a)}$$

Example 15: Normal - Normal

Bayesian model: $X_1, ..., X_n|\mu \overset{i.i.d}{\sim} \mathcal{N}(\mu, 1), \mu \in \mathbb{R}$.
Prior: $\mu \sim \mathcal{N}(a, b^2), a \in \mathbb{R}, b > 0$.
Posterior distribution? Continuous with density.

$$\Pi(\mu|X_1, ..., X_n) \propto \prod_{i=1}^{n}\frac{e^{-\frac{(x_i-\mu)^2}{2}}}{\sqrt{2\pi}}\frac{e^{-\frac{(\mu-a)^2}{2b^2}}}{\sqrt{2\pi}b}$$

$$\propto \exp\left(-\frac{n}{2}\{\mu^2+\bar{X}_n^2-2\mu\bar{X}_n\}-\frac{1}{2b^2}\{\mu^2+a^2-2a\mu\}\right)$$

Because:

$$\sum_{i=1}^n (X_i-\mu)^2 = \sum_{i=1}^n [(X_i-\bar{X}_n)^2+(\bar{X}_n-\mu)^2+2(\bar{X}_n-\mu)(X_i-\bar{X}_n)]$$

$$= n(\bar{X}_n-\mu)^2+\sum_{i=1}^n (X_i-\bar{X}_n)^2$$

$$\Pi(\mu|X_1,...,X_n) \propto e^{-\frac{1}{2}(n+\frac{1}{b^2})\left(\mu-\frac{\frac{a}{b^2}+n\bar{X}_n}{n+\frac{1}{b^2}}\right)^2}$$

$\implies$ Posterior distribution is:

$$\text{a } \mathcal{N}\left(\frac{\frac{a}{b^2}+n\bar{X}_n}{n+\frac{1}{b^2}}, \frac{1}{n+\frac{1}{b^2}}\right)$$

Definition 24: Marginal Likelihood

In a Bayesian model $X|\theta \sim f_\theta(x)$, $\theta \in \Theta$ (Likelihood) and prior $\theta \sim \Pi$ with density π wrt ν. The quantity:

$$m(x) = \int_\Theta f_\theta(x)\Pi(\theta)d\nu(\theta)$$

is called the **marginal likelihood**.

II. Bayesian decision theory: Risks

II.1 Posterior and integrated risks

Recall: Quadratic Risk:

$$\mathcal{R}(\theta,\delta) = \begin{cases} \int (\theta-\delta(x))^2 f_\theta(x)dx \\ \sum_x (\theta-\delta(x))^2\mathbb{P}_\theta(X=x) \end{cases}$$

Definition 25: Loss Function

A loss function is a function:

$$l : \Theta \times \mathcal{D} \to \mathbb{R}^+$$

where $\mathcal{D}$ is the set of decisions (i.e., estimators).

Example 16: Examples of Loss Functions

1. If $\mathcal{D} = \Theta$.
   a) The quadratic loss $l(\theta,\delta) = (\theta-\delta)^2$.
   b) The L_1 loss $l(\theta,\delta) = |\theta-\delta|$.
2. $\mathcal{D} = \{0,1\}, \theta \in [0,1]$.

Aim is testing if $\theta > \frac{1}{2}$ (choosing 1) or $\theta \leq \frac{1}{2}$ (choosing 0).
0-1 loss function defined by:

$$
\begin{aligned}
l(\theta, \delta) = 1 \quad & \text{if } \theta < \frac{1}{2} \text{ and } \delta = 1 \text{ (wrong decision)} \\
& \text{if } \theta > \frac{1}{2} \text{ and } \delta = 0 \text{ (wrong decision)} \\
l(\theta, \delta) = 0 \quad & \text{else (correct decision)}
\end{aligned}
$$

Definition 26: Risks

In a Bayesian model $X|\theta \sim f_\theta(x)$, $\theta \sim \Pi$ with a loss function $l : \Theta \times \mathcal{D} \to \mathbb{R}^+$. where:

a) We call the **posterior risk**:

$$l(\Pi, \delta|X) = \int_\Theta l(\theta, \delta)\Pi(\theta|X_1, ..., X_n)d\nu(\theta)$$

b) We call the **integrated risk**:

$$r(\Pi, \delta) = \begin{cases} \int_x l(\Pi, \delta|X = x)m(x)dx & \text{if } X \text{ continuous} \\ \sum_{x \in \mathcal{X}} l(\Pi, \delta|X = x)m(x) & \text{if } X \text{ discrete} \end{cases}$$

Example 17: Calculation of Posterior Risk

1) Bayesian model, $X_1, ..., X_n|\theta \sim \mathcal{P}(\theta), \theta > 0$.
Prior: $\theta \sim \Gamma(a, b)$.
Loss function: Quadratic. $\mathcal{D} = \Theta = \mathbb{R}_+$.
$\forall \theta > 0, \delta > 0, \quad l(\theta, \delta) = (\theta - \delta)^2$.
Posterior risk:
Posterior distribution: $\Gamma(a + \sum_{i=1}^n X_i, b + n)$.

$$\Pi(\theta|X_1, ..., X_n) = \frac{(b + n)^{n\bar{X}_n + a} e^{-(b+n)\theta} \theta^{n\bar{X}_n + a - 1}}{\Gamma(a + n\bar{X}_n)}$$

With $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$.
$\forall \delta > 0$:

$$
\begin{aligned}
l(\Pi, \delta|X_1, ..., X_n) &= \int_0^\infty (\theta - \delta)^2 \Pi(\theta|X_1, ..., X_n)d\theta \\
&= \int_0^\infty \theta^2 \Pi(\theta|X_1, ..., X_n)d\theta + \delta^2 - 2\delta \int_0^\infty \theta \Pi(\theta|X_1, ..., X_n)d\theta \\
\implies l(\Pi, \delta|X_1, ..., X_n) &= \frac{a'(a' + 1)}{b'^2} + \delta^2 - 2\delta \frac{a'}{b'}
\end{aligned}
$$

where $a' = a + n\bar{X}_n$ and $b' = b + n$. (Using moments of Gamma distribution).

$\to$ Can be minimized in δ.
$\delta(X_1, ..., X_n) =$ minimizer of $l(\Pi, \delta|X_1, ..., X_n)$.
Show that:

$$\delta(X_1, ..., X_n) = \frac{a'}{b'} = \frac{a + n\bar{X}_n}{b + n}$$

Example 18: Example 2

$\ell : \Theta \times \mathcal{D} \to \mathbb{R}_+$

- Posterior risk:

$$\ell(\Pi, \delta | x_1, \dots, x_n) = \int_\Theta \ell(\theta, \delta) \Pi(\theta | x_1, \dots, x_n) d\theta$$

$X_1, \dots, X_n \sim \mathcal{P}(\theta)$; $\theta \sim \Gamma(a, b)$.
The posterior is:

$$\Pi(\theta | X_1, \dots, X_n) = \frac{\theta^{a + n\bar{x}_n - 1} e^{-(b+n)\theta} (b+n)^{a + n\bar{x}_n}}{\Gamma(a + n\bar{x}_n)}$$

We set the hypotheses:

- $H_0 = \{\theta < 1\}$
- $H_1 = \{\theta \geq 1\}$

Decision rule:

$$\begin{cases} \delta = 1 & \text{if choose } H_0 \\ \delta = 0 & \text{if choose } H_1 \end{cases}$$

Let us use the 0-1 loss:

$$\begin{aligned}
\ell(\Pi, \delta | x_1, \dots, x_n) &= \int_0^{+\infty} \left[\mathbb{1}_{\delta=1} \mathbb{1}_{\theta \geq 1} + \mathbb{1}_{\delta=0} \mathbb{1}_{\theta<1} \right] \Pi(\theta | x_1, \dots, x_n) d\theta \\
&= \mathbb{1}_{\delta=1} \underbrace{\int_1^{\infty} \Pi(\theta | x_1, \dots, x_n) d\theta}_{\Pi(\theta > 1 | X)} + \mathbb{1}_{\delta=0} \underbrace{\int_0^1 \Pi(\theta | x_1, \dots, x_n) d\theta}_{\Pi(\theta \leq 1 | X)} \\
&= \mathbb{1}_{\delta=1} \Pi(\theta > 1 | x_1, \dots, x_n) + \mathbb{1}_{\delta=0} \Pi(\theta \leq 1 | x_1, \dots, x_n)
\end{aligned}$$

Thus:

$$\ell(\theta, \delta) = \mathbb{1}_{\delta=1} \mathbb{1}_{\theta \geq 1} + \mathbb{1}_{\delta=0} \mathbb{1}_{\theta<1}$$

Definition 27: Bayesian Estimator

In a Bayesian model $X | \theta \sim P_\theta, \theta \in \Theta$, prior $\theta \sim \Pi$.
If $\ell : \Theta \times \mathcal{D} \to \mathbb{R}_+$ is a loss function, we define the **Bayes estimators** as (when they exist) the minimizers of $\ell(\Pi, \delta | X)$ (or posterior risk, denoted $\ell(\Pi, \delta | X = x)$) in δ.

Proposition 5: Property

Bayesian estimators also minimize the integrated Risk.

Proof. Bayesian Model $P_\theta(x), \theta \in \Theta$ and Prior Π. Loss function $\ell(\theta, \delta), \delta \in \mathcal{D}$.
Integrated risk $r(\Pi, \delta) = \int_{\mathcal{X}} \ell(\Pi, \delta | X = x) m(x) dx$.
Since $\forall x$, $\delta^\pi(x)$ verifies:

$$\int_\Theta \ell(\Pi, \delta^\pi(x) | X) \leq \int_\Theta \ell(\Pi, \delta | X)$$

(Inequality on posterior risk).

$$\implies \int \ell(\Pi, \delta^\pi(x)|X=x)m(x)dx \leq \int \ell(\Pi, \delta(x)|X=x)m(x)dx$$

$$\implies r(\Pi, \delta^\Pi) \leq r(\Pi, \delta), \quad \forall \text{ estimator } \delta \text{ when } \delta^\Pi \text{ is a Bayesian estimator.}$$

□

Why is it interesting?

Theorem 12: Statistical Model

$X|\theta \sim P_\theta$, $\theta \in \Theta$ (Model), Prior $\theta \sim \pi$.

- Loss function: $\ell : \Theta \times \mathcal{D} \to \mathbb{R}_+$
- The **Frequentist risk** is defined as the function $R : \Theta \times \mathcal{D} \to \mathbb{R}_+ ; R(\theta, \delta) = \mathbb{E}_\theta(\ell(\theta, \delta(X)))$.

$$= \begin{cases} \displaystyle\int_{\mathcal{X}} \ell(\theta, \delta(x))f(x|\theta)dx & \text{if } X \text{ continuous} \\ \displaystyle\sum_{x\in\mathcal{X}} \ell(\theta, \delta(x))f(x|\theta) & \text{if } X \text{ discrete} \end{cases}$$

Then:

$$\int_\Theta R(\theta, \delta)\pi(\theta)d\theta = r(\pi, \delta)$$

Example 19: Quadratic Risk

If $\ell(\theta, \delta) = (\theta - \delta)^2$, $R(\theta, \delta)$ is the quadratic risk.

Proof.

$$r(\Pi, \delta) = \int_{\mathcal{X}} \rho(\Pi, \delta|X=x)m(x)dx$$

$$m(x) = \int_\Theta f(x|\theta)\Pi(\theta)d\theta$$

and $\rho(\Pi, \delta|x=x) = \int_\Theta \ell(\theta, \delta)\Pi(\theta|x=x)d\theta$.

$$\begin{aligned}
\implies r(\pi, \delta) &= \int_{\mathcal{X}} \int_\Theta \ell(\theta, \delta(x)) \underbrace{\pi(\theta|x=x)m(x)}_{f(x|\theta)\pi(\theta)/m(x)\cdot m(x)} d\theta dx \\
&= \int_{\mathcal{X}} \int_\Theta \ell(\theta, \delta(x))f(x|\theta)\pi(\theta)d\theta dx \\
&= \int_\Theta \left[\int_{\mathcal{X}} \ell(\theta, \delta(x))f(x|\theta)dx\right] \pi(\theta)d\theta \\
&= \int_\Theta R(\theta, \delta)\pi(\theta)d\theta
\end{aligned}$$

□

Example 20: Example

$X_1, \dots, X_n \sim \mathcal{P}(\theta)$. Prior $\Pi(\theta) = \frac{\theta^{a-1}e^{-b\theta}b^a}{\Gamma(a)}, \quad \theta > 0.$
$\underline{a = 1, b = 1:}$ $\pi(\theta) = e^{-\theta}, \theta > 0.$

$$\begin{aligned}\Pi(\theta|X_1, \dots, X_n) &= \frac{\theta^{\sum X_i} e^{-(n+1)\theta}(n+1)^{n\bar{X}_n}}{\Gamma(n\bar{X}_n)} \\ &= \frac{\prod_{i=1}^n f(X_i|\theta)\Pi(\theta)}{\int_\Theta \prod_{i=1}^n f(X_i|\theta)\Pi(\theta)d\theta}\end{aligned}$$

II.2 Computation of Bayesian estimators

- Model: $X|\theta \sim P_\theta, \theta \in \Theta$
- Prior: $\theta \sim \Pi$
- Loss function: $\ell : \Theta \times \mathcal{D} \to \mathbb{R}_+$

a) Quadratic loss

$$\ell(\theta, \delta) = (\theta - \delta)^2, \quad \mathcal{D} = \Theta$$

$$\rho(\Pi, \delta|x = x) = \int_\Theta (\theta - \delta)^2 \Pi(\theta|X = x)d\theta$$

is convex in δ.

$$\frac{\partial}{\partial \delta}\ell(\Pi, \delta|x = x) = 2\int_\Theta \theta\Pi(\theta|X = x) + 2\delta \underbrace{\int_\Theta \Pi(\theta|x = x)}_{=1}$$

Wait, correction from notes:

$$= 2\int_\Theta (\delta - \theta)\Pi(\theta|X = x)d\theta = 2\delta - 2\int_\Theta \theta\Pi(\theta|X = x)d\theta$$

$$= 0 \iff \delta(x) = \int_\Theta \theta\Pi(\theta|X = x)d\theta = E^\Pi(\theta|X = x) = \text{posterior mean.}$$

Proposition 6: Posterior Mean

The Bayesian estimator associated to the quadratic loss is the posterior mean.

b) L_1 loss

$$\ell(\theta, \delta) = |\theta - \delta|, \quad \mathcal{D} = \Theta = \mathbb{R}$$

Posterior risk:

$$\ell(\Pi, \delta|X = x) = \int_\mathbb{R} |\theta - \delta|\Pi(\theta|x = x)d\theta$$

Convex in δ and:

$$\rho(\Pi, \delta|X = x) = \int_{-\infty}^{\delta} (\delta - \theta)\Pi(\theta|X = x)d\theta + \int_{\delta}^{+\infty} (\theta - \delta)\Pi(\theta|X = x)d\theta$$

$$\frac{\partial}{\partial\delta}\rho(\Pi, \delta|X = x) = \int_{-\infty}^{\delta} \Pi(\theta|X = x)d\theta - \int_{\delta}^{+\infty} \Pi(\theta|X = x)d\theta$$

(plus boundary terms that cancel out)

$$+\delta(\Pi(\delta|x = x)) + \dots$$

$$-\delta\Pi(\delta|x = x) - \delta\Pi(\delta|X = x)$$

(Terms from Leibniz rule cancel out)

$$= \Pi(\theta \leq \delta|X = x) - \Pi(\theta > \delta|X = x)$$

$= 0 \iff \delta$ is the median of the posterior distribution.