

Full course of Statistical Modelling

L3 MIDO

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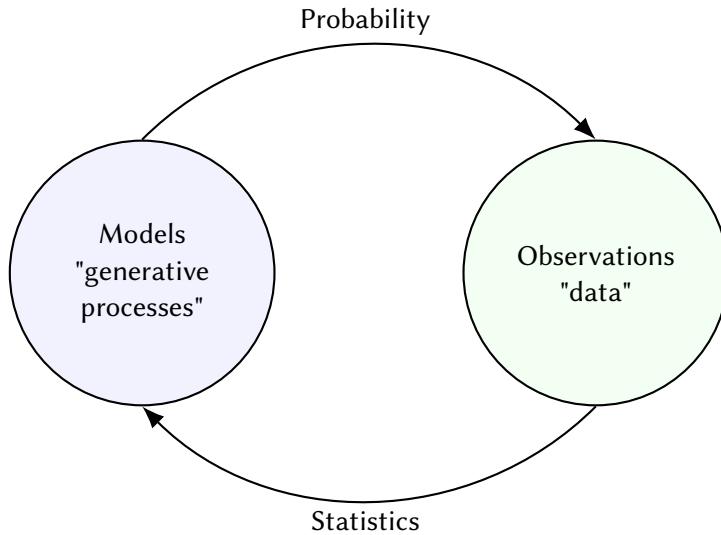
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Chapter 1: Introduction to Statistical Modelling

Introduction Schema



I. Data

1) $n = \# \text{ L3 MIDO students}$

- Were students in Dauphine in L2?
- Data $\in \{0, 1\}^n$:
 - 1: in L2 in Dauphine.
 - 0: not in L2 in Dauphine.
- $x = (0, 0, 1, 1, 0, \dots)$
- $\downarrow \text{sum}(x)$

$\implies x$ is a distribution of n independent experiences of Bernoulli of parameter p .

$$x \sim \text{Ber}(p)^{\otimes n} \quad (\text{i.i.d.})$$

$$\implies \text{sum}(x) \sim \text{Bin}(n, p)$$

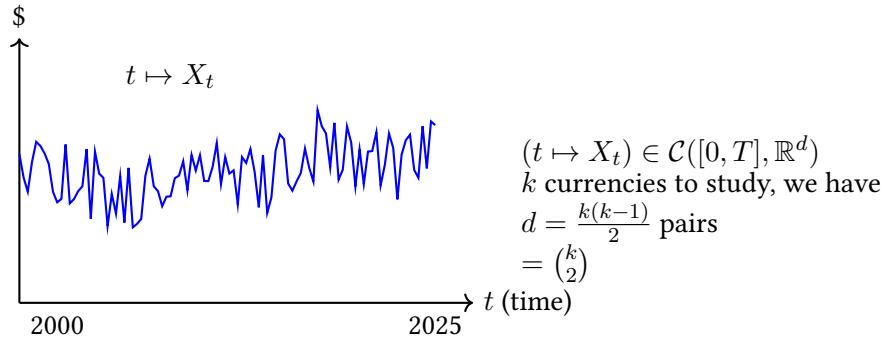
2) INSEE (economy)

Example: Employment rate between 2005 & 2025.

Male ♂	2005	2006	...	2025		Female ♀	2005	...
15-24 yo	α	α	α	α		15-24	α	α
25-49 yo	α	56%	α	α		25-49	α	α
50 +	α		α	α		50 +	α	α

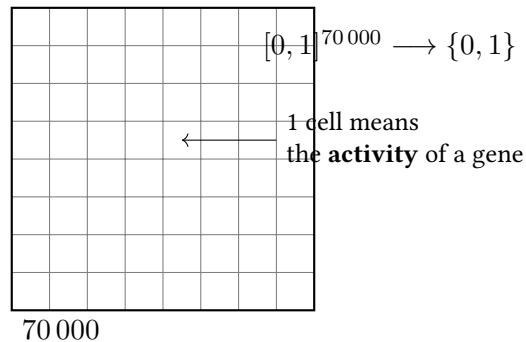
\implies **Tensor:** multiple entries array.

3) Financial data



\implies Brownian motion. \implies Diffusion processes.

4) Microarrays



- Compute a big space into a smaller one: the cells of an individual translated to *healthy* or *not*.
 - $[0, 1]$ measures the activity of one cell. One person has 70 000 cells.

II. Statistical Modelling

1) Stat models with words

To each set of data, we must associate a scientific question/objective.

To answer that question, we must have at our disposition:

- A methodology.
 - A quantitative mathematical (probabilistic) model that accounts for the properties of the data.
 - Well suited methods (mathematical) that combine... and...

2) Abstract (simple) example

We toss a coin 18 times and observe ($n = 18$, $H = 0$, $T = 1$).

$$(0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0)$$

\equiv Data (raw).

- **Statistical model:** We observe $n = 18$ random variables X_i ; that are independent and that have the same distribution.

- $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = \theta$.
- $\text{Ber}(\theta)$ where $\theta \in \Theta = [0, 1]$.
- θ is the **unknown parameter**.

Questions:

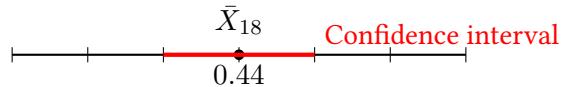
1) What is a good (the best) estimation of θ ?

- A good (?) estimator:

$$\underbrace{\bar{X}_n}_{\substack{\text{notation} \\ \text{of the mean}}} = \frac{1}{n} \sum_{i=1}^n X_i$$

$$= \frac{1}{18} \sum_{i=1}^{18} X_i = \frac{8}{18} = 0.44$$

- What is my accuracy of estimation?



⇒ We want:

1. Small length.
2. Good coverage.

2) Is the coin fair?

For instance, we compare \bar{X}_{18} to 0.5. If $|\bar{X}_{18} - 0.5|$ is small, we accept the idea that the coin is fair, otherwise we reject that hypothesis.

Chapter 2: Models

I. What is a statistical model?

Definition 1: Statistical Model

Let $X = (X_1, \dots, X_n) \in \mathcal{X}^n$ be a vector of n random variables, where each variable $X_i \in \mathcal{X}$ and \mathcal{X} is a measurable space.

A **model** for X is a set \mathcal{P} of probability distributions on \mathcal{X}^n .

Statistical inference will consist in estimating P (the distribution of X) or $F(P)$, with $P \in \mathcal{P}$.

Examples

- i) Let X_1, \dots, X_n be i.i.d. $\mathcal{N}(\mu, \sigma^2)$.

Notation: $X_i \sim_{i.i.d.} \mathcal{N}(\mu, \sigma^2)$ (Gaussian distribution with parameters μ and σ^2). The density is given by: $x \mapsto \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

Here, $\mathcal{X} = \mathbb{R}$. The model is defined as:

$$\mathcal{P} = \{\mathcal{N}(\mu, \sigma^2)^{\otimes n}, (\mu, \sigma^2) \in \mathbb{R} \times (0, +\infty)\}$$

Remark

Notations: If (X_1, X_2) is a random vector of \mathbb{R}^2 , with $X_1 \sim P_1$ and $X_2 \sim P_2$.

(By the way: $X \sim P$ means that P is the distribution of $X \in \mathbb{R}$, meaning: $\forall A \in \mathcal{B}(\mathbb{R})$, $P(A) = \mathbb{P}(X \in A) = \mathbb{P}(X^{-1}(A))$.)

$$(\Omega, \mathcal{F}, \mathbb{P}) \xrightarrow{X} (\mathbb{R}, \mathcal{B}(\mathbb{R}), P)$$

P is defined via: $\forall A \in \mathcal{B}(\mathbb{R})$, $P(A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}(\{\omega \in \Omega, X(\omega) \in A\})$.

If $X_1 \sim P_1$, $X_2 \sim P_2$ and $X_1 \perp\!\!\!\perp X_2$ (X_1 and X_2 are independent):

$$\mathbb{P}(X_1 \in A, X_2 \in B) = \mathbb{P}(X_1 \in A) \cdot \mathbb{P}(X_2 \in B)$$

$\forall A, B \in \mathcal{B}(\mathbb{R})$, the distribution of (X_1, X_2) is defined on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ and is denoted by $P_1 \otimes P_2$.

Computation formula:

$$P_1 \otimes P_2(A \times B) = P_1(A) \cdot P_2(B) = \mathbb{P}(X_1 \in A) \cdot \mathbb{P}(X_2 \in B)$$

In our example, saying $(X_1, \dots, X_n) \sim \mathcal{N}(\mu, \sigma^2)^{\otimes n}$ exactly means:

$$\begin{aligned} \mathbb{P}(X_1 \in A_1, \dots, X_n \in A_n) &= \prod_{i=1}^n \mathbb{P}(X_i \in A_i) \\ &= \prod_{i=1}^n \int_{A_i} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} dx_i \\ &= \int_{A_1} \dots \int_{A_n} \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2\right) dx_1 \dots dx_n \end{aligned}$$

- ii) $X_i \in \{0, 1\}$, $\forall i = 1, \dots, n$, independently with same distribution $\text{Ber}(p)$, $p \in [0, 1]$.

$$\mathcal{P} = \{\text{Ber}(p)^{\otimes n}, p \in [0, 1]\}, \quad \mathcal{X}^n = \{0, 1\}^n$$

Model Comparison and Types

Let us consider models of the form $\mathcal{P} = \{P^{\otimes n}, P \in \mathcal{P}_0\}$.

- Example (1): $\mathcal{P}_0 = \{\mathcal{N}(\mu, \sigma^2), (\mu, \sigma^2) \in \mathbb{R} \times (0, +\infty)\}$ (Parametric).
- Example (2): $\mathcal{P}_0 = \{\text{Ber}(p), p \in [0, 1]\}$.
- Example (3): $\mathcal{P}_0 = \{\text{All distributions on } (\mathbb{R}, \mathcal{B}(\mathbb{R}))\}$ (Huge model).

Which is the best model? Generally, the best model is the one where the set of parameter values is the **smallest** (Parsimony principle).

Remark

In the 3 examples considered above, all observations are **i.i.d.** But, in many situations, this is not the case.

Example of a very standard non-i.i.d. model:

$$X_0 = x_0; \quad X_t = \rho X_{t-1} + \sigma \varepsilon_t, \quad t = 1, \dots, T$$

We observe (X_1, \dots, X_T) . They are not i.i.d. Here $\rho \in \mathbb{R}$, $\sigma > 0$ (Volatility), and $\varepsilon_t \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$. X_t could represent the log price of a financial asset.

In this course, we will mainly focus on the i.i.d. case.

II. Parametric Model

Definition 2: Parametric Model

Let there be data $(X_1, \dots, X_n) \in \mathcal{X}^n$. A model \mathcal{P} for (X_1, \dots, X_n) is called **parametric** if:

$$\mathcal{P} = \{P_\theta, \theta \in \Theta\}$$

with $\Theta \subset \mathbb{R}^d$ for some $d \geq 1$. Here P_θ is a distribution on \mathcal{X}^n .

Definition 3: Nuisance Parameters

Let $X = (X_1, \dots, X_n)$ i.i.d. with model $\mathcal{P} = \{P_\theta, \theta \in \Theta\}$. If $\Theta = (\theta_1, \theta_2)$, and if we only are interested in estimating or predicting θ_1 and we don't care about θ_2 , we say that:

- θ_1 is the **parameter of interest**.
- θ_2 is the **nuisance parameter**.

III. Identifiability

Context: We have a statistical model for $X = (X_1, \dots, X_n)$. $X_i \sim P_\theta$, $\theta \in \Theta \subset \mathbb{R}^d$. So $\mathcal{P} = \{P_\theta^{\otimes n}, \theta \in \Theta\}$.

Aim: Can we learn/estimate θ from X ? This is only possible if we can learn θ from $P_\theta^{\otimes n}$ or P_θ .

Definition 4: Identifiability

A statistical model $\{P_\theta, \theta \in \Theta\}$ is **identifiable** (for θ) if:

$$P_{\theta_1} = P_{\theta_2} \implies \theta_1 = \theta_2$$

(i.e., the map $\theta \mapsto P_\theta$ is injective).

Examples

Example 1: Exponential Distribution

Let $(P_\theta, \theta \in \Theta) = (\mathcal{E}(\theta), \theta > 0)$. P_θ has density $f_\theta(x) = \theta e^{-\theta x} \mathbb{1}_{\{x>0\}}$.

If $P_{\theta_1} = P_{\theta_2}$, then for every $x \in \mathbb{R}$:

$$f_{\theta_1}(x) = f_{\theta_2}(x) \implies \theta_1 e^{-\theta_1 x} = \theta_2 e^{-\theta_2 x} \quad \text{whenever } x > 0$$

Take $x \rightarrow 0$ (or $x = 0$ in the limit), we get $\theta_1 = \theta_2$. Thus, the model is identifiable.

Example 2: Gaussian Distribution

Let $\theta = (\mu, \sigma^2) \in \Theta = \mathbb{R} \times (0, +\infty)$.

$$P_\theta = \mathcal{N}(\mu, \sigma^2)$$

$$f_{\mu, \sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad \forall x \in \mathbb{R}$$

Injectivity? Assume $P_{\theta_1} = P_{\theta_2}$. Then $\forall x \in \mathbb{R}$:

$$\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} = \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(x-\mu_2)^2}{2\sigma_2^2}}$$

Take $x = \mu_1$, then:

$$\frac{1}{\sigma_1} = \frac{1}{\sigma_2} e^{-\frac{(\mu_1-\mu_2)^2}{2\sigma_2^2}}$$

Take $x = \mu_2$, then:

$$\frac{1}{\sigma_1} e^{-\frac{(\mu_2-\mu_1)^2}{2\sigma_1^2}} = \frac{1}{\sigma_2}$$

This implies:

$$\frac{\sigma_2}{\sigma_1} = e^{-\frac{(\mu_1-\mu_2)^2}{2\sigma_2^2}} \leq 1 \quad (\text{because } e^{-\cdot} \leq 1)$$

$$= e^{-\frac{(\mu_1-\mu_2)^2}{2\sigma_1^2}} \geq 1 \quad (\text{because } e^{\cdot} \geq 1)$$

Hence:

$$\frac{\sigma_2}{\sigma_1} = 1 \implies \sigma_1 = \sigma_2$$

And substituting back:

$$e^{-\frac{(\mu_1-\mu_2)^2}{2\sigma^2}} = 1 \implies \frac{(\mu_1 - \mu_2)^2}{2\sigma^2} = 0 \implies \mu_1 = \mu_2$$

Thus $\theta_1 = \theta_2$. The model is identifiable.

Proposition 1: Identifiability via CDF and Density

- If P_θ admits a cumulative distribution function (CDF) F_θ , the model is identifiable $\iff (\forall \theta_1, \theta_2, F_{\theta_1} = F_{\theta_2} \implies \theta_1 = \theta_2)$.
- If P_θ admits a density f_θ , the model is identifiable $\iff (\forall \theta_1, \theta_2, f_{\theta_1} = f_{\theta_2} \implies \theta_1 = \theta_2)$.

Example 3: Non-identifiable Model (Mixture)

Let $Y_1 \sim \text{Ber}(p_1)$, $Y_2 \sim \text{Ber}(p_2)$. $Y_1 \in \{0, 1\}$. $\mathbb{P}(Y_1 = 1) = p_1$.

Let X be defined as:

$$X = \begin{cases} Y_1 & \text{with proba } \pi \\ Y_2 & \text{with proba } 1 - \pi \end{cases}$$

$$\begin{aligned} \mathbb{P}(X = 1) &= \mathbb{P}(X = 1 | X = Y_1)\mathbb{P}(X = Y_1) + \mathbb{P}(X = 1 | X = Y_2)\mathbb{P}(X = Y_2) \\ &= \mathbb{P}(Y_1 = 1) \times \pi + \mathbb{P}(Y_2 = 1) \times (1 - \pi) \\ &= p_1\pi + p_2(1 - \pi) \end{aligned}$$

We have a model with 3 parameters (p_1, p_2, π) . $X \sim \text{Ber}(p)$ with $p = p_1\pi + p_2(1 - \pi)$.

Knowing $\mathbb{P}(X = 1)$ gives limited information on the law of X .

Numeric Examples:

- $\pi = 1/2, p_1 = 0.6, p_2 = 0.2$. $\mathbb{P}(X = 1) = 0.6 \times 0.5 + 0.2 \times 0.5 = 0.4$.
- $\pi = 0.6, p_1 = 1/2, p_2 = 0.25$. $\mathbb{P}(X = 1) = 0.6 \times 0.5 + 0.4 \times 0.25 = 0.3 + 0.1 = 0.4$.

We have different parameters but the same distribution \implies **Not identifiable**.

IV. Empirical Distribution Function

Model: We observe X_1, \dots, X_n i.i.d. P . We set $F(t) =$ Cumulative distribution function of P at t

$$= \mathbb{P}(X_i \leq t), \quad \forall i$$

Remark

Reminder:

- F is non-decreasing.
- F is càdlàg (right-continuous with left limits).
- $\lim_{t \rightarrow -\infty} F(t) = 0$ and $\lim_{t \rightarrow +\infty} F(t) = 1$.

Definition 5: Empirical CDF

$$\hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{X_i \leq t\}}$$

It is a good idea since when $n \rightarrow +\infty$: Fix t : $\hat{F}_n(t) \xrightarrow[n \rightarrow \infty]{\text{in probability}} \mathbb{E}[Y_i]$ in probability (Weak Law of Large

Numbers).

$$\frac{1}{n} \sum_{i=1}^n Y_i$$

with $Y_i = \mathbb{1}_{\{X_i \leq t\}}$. The Y_i are i.i.d., $\mathbb{E}[|Y_i|] < +\infty$.

$$\mathbb{E}[Y_1] = \mathbb{P}(X_1 \leq t) = F(t)$$

So, we have: $\hat{F}_n(t) \rightarrow F(t)$ in probability.

Examples

1. $X_i = \begin{cases} 1 & \text{if I have an accident on day } i \\ 0 & \text{otherwise} \end{cases}$ $X_i \stackrel{i.i.d.}{\sim} \text{Ber}(p)$; $p = \mathbb{P}(\text{avoir un accident})$.

$$\hat{p} = \frac{1}{n} \sum_{i=1}^n X_i \xrightarrow[n \rightarrow \infty]{\mathbb{P}} p \quad (\text{Law of Large Numbers})$$

($\xrightarrow{\mathbb{P}}$ en probabilité).

2. X_1, \dots, X_n i.i.d. $X_i = \text{Lifetime of computer } i$.

We want to estimate the function $t \mapsto F(t) = \mathbb{P}(X_i \leq t)$. To estimate F , we use at time t , $\hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{X_i \leq t\}}$.

Chapter 3: Point estimators and MLE

I. Point Estimators

We consider a parametric model $(X_1, \dots, X_n) \stackrel{i.i.d.}{\sim} P_\theta$, with $\theta \in \Theta \subset \mathbb{R}^d$.

Goal: Constructions of estimators for θ or for $g(\theta)$ with $g : \Theta \rightarrow \mathbb{R}^p$.

Definition 6: Point Estimator

A **point estimator** for θ is a quantity $\hat{\theta}$ which depends **only** on X_1, \dots, X_n (the data).

Examples:

- We observe $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \text{Ber}(\theta)$.

$$\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^n X_i \quad \text{is a point estimator of } \theta.$$

- We observe $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathcal{N}(\theta, 1)$.

$$\hat{\theta}_n = \bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i \quad \text{is an estimator of } \theta.$$

- We observe $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathcal{E}(\theta)$.

$$\hat{\theta}_n = \frac{1}{\frac{1}{n} \sum_{i=1}^n X_i} = \frac{1}{\bar{X}_n}$$

Recall: If $X \sim \mathcal{E}(\theta)$, $\theta > 0$:

$$\begin{aligned} E[X] &= \frac{1}{\theta} \\ \frac{1}{\hat{\theta}_n} &\xrightarrow{n \rightarrow \infty} \frac{1}{\theta} \\ \frac{1}{\hat{\theta}_n} &= \frac{1}{n} \sum_{i=1}^n X_i \end{aligned}$$

By the Weak Law of Large Numbers (WLLN), $\hat{\theta}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \theta$.

For this last example, if we want to estimate $g(\theta) = \frac{1}{\theta}$, in this case, $\hat{g} = \frac{1}{\hat{\theta}_n} = \frac{1}{n} \sum_{i=1}^n X_i$ is a "good estimator".

II. Quadratic Risk, Bias

Measure of error = Loss function. The most common loss function is the **quadratic loss**:

$$l(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$$

We cannot compute $l(\hat{\theta}_n, \theta)$ since we do not know θ .

Definition 7: Quadratic Risk (Mean Squared Error)

We call **quadratic risk** (or mean squared error):

$$R(\hat{\theta}_n, \theta) = \mathbb{E}[(\hat{\theta}_n - \theta)^2]$$

Definition 8: Bias

Let $\hat{\theta}_n$ be an estimator of θ in a statistical model $(X_i)_{i \leq n} \stackrel{i.i.d.}{\sim} P_\theta, \theta \in \Theta$.

- (i) We say that $\hat{\theta}_n$ is **unbiased** at θ iff $\mathbb{E}_\theta(\hat{\theta}_n) = \theta$.
- (ii) We say that $\hat{\theta}_n$ is **unbiased** over Θ iff it is unbiased at all $\theta \in \Theta$.

The **bias** is defined as:

$$b(\theta) = \mathbb{E}_\theta(\hat{\theta}_n) - \theta$$

Theorem 1: Bias-Variance Decomposition

In a statistical model $(X_i)_{i \leq n}$ i.i.d. $P_\theta, \theta \in \Theta$. If $\hat{\theta}_n$ is an estimator of θ , then:

$$R(\hat{\theta}_n, \theta) = b(\theta)^2 + \mathbb{V}_\theta(\hat{\theta}_n)$$

where $\mathbb{V}_\theta(\hat{\theta}_n)$ is the variance of $\hat{\theta}_n$ assuming that $X_i \stackrel{i.i.d.}{\sim} P_\theta$.

Proof.

$$R(\hat{\theta}_n, \theta) = \mathbb{E}_\theta[(\hat{\theta}_n - \theta)^2]$$

Add and subtract $\mathbb{E}_\theta(\hat{\theta}_n)$:

$$= \mathbb{E}_\theta[(\hat{\theta}_n - \mathbb{E}_\theta(\hat{\theta}_n) + \mathbb{E}_\theta(\hat{\theta}_n) - \theta)^2]$$

Let $A = \hat{\theta}_n - \mathbb{E}_\theta(\hat{\theta}_n)$ and $B = \mathbb{E}_\theta(\hat{\theta}_n) - \theta = b(\theta)$.

$$\begin{aligned} &= \mathbb{E}_\theta[(A + B)^2] = \mathbb{E}_\theta[A^2 + B^2 + 2AB] \\ &= \underbrace{\mathbb{E}_\theta(A^2)}_{\mathbb{V}_\theta(\hat{\theta}_n)} + \underbrace{\mathbb{E}_\theta(B^2)}_{b(\theta)^2 \text{ (constant)}} + 2 \underbrace{\mathbb{E}_\theta[(\hat{\theta}_n - \mathbb{E}_\theta(\hat{\theta}_n)) \cdot b(\theta)]}_{0} \end{aligned}$$

The cross term is zero because:

$$2b(\theta)\mathbb{E}_\theta(\hat{\theta}_n - \mathbb{E}_\theta(\hat{\theta}_n)) = 2b(\theta)(\mathbb{E}_\theta(\hat{\theta}_n) - \mathbb{E}_\theta(\hat{\theta}_n)) = 0$$

Hence:

$$R(\hat{\theta}_n, \theta) = b(\theta)^2 + \mathbb{V}_\theta(\hat{\theta}_n)$$

(Q.E.D)

□

Examples of Risk Calculation:

Example 4: Normal Mean

$X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathcal{N}(\theta, 1)$. Estimator $\hat{\theta}_n = \frac{1}{n} \sum X_i = \bar{X}_n$.

$$R(\hat{\theta}_n, \theta) = \mathbb{E}[(\hat{\theta}_n - \theta)^2]$$

$$\begin{aligned}
&= \mathbb{E}[(\bar{X}_n - \theta)^2] \\
&= \text{Var}(\bar{X}_n) \quad (\text{as } \mathbb{E}[\bar{X}_n] = \theta, \text{ it is unbiased}) \\
&= \frac{1}{n} \text{Var}(X_1) = \frac{1}{n}
\end{aligned}$$

Example 5: Bernoulli Mean

We observe $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \text{Ber}(\theta)$.

$$\begin{aligned}
\hat{\theta}_n &= \bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i \\
R(\hat{\theta}_n, \theta) &= \mathbb{E}[(\hat{\theta}_n - \theta)^2] \\
&= \mathbb{E}[(\bar{X}_n - \theta)^2] \\
&= \text{Var}(\bar{X}_n) = \frac{\text{Var}(X_1)}{n} = \frac{\theta(1-\theta)}{n}
\end{aligned}$$

Remark

The function $x \in [0, 1] \mapsto x(1-x)$ attains its maximum at $x = 1/2$.

$$\begin{aligned}
&\implies \forall \theta \in [0, 1], \quad \theta(1-\theta) \leq \frac{1}{4} \\
&\implies R(\hat{\theta}_n, \theta) \leq \frac{1}{4n}
\end{aligned}$$

Reminders on Convergence

Definitions

Definition 9: Convergence in Probability

$$X_n \xrightarrow{\mathbb{P}} X \iff \forall \varepsilon > 0, \mathbb{P}(|X_n - X| > \varepsilon) \xrightarrow{n \rightarrow \infty} 0.$$

Definition 10: Convergence in L^p

$$X_n \xrightarrow{L^p} X \iff \mathbb{E}(|X_n - X|^p) \xrightarrow{n \rightarrow \infty} 0.$$

Definition 11: Convergence in Distribution

$$\begin{aligned} X_n &\xrightarrow{\mathcal{D}} X \\ &\iff \forall x \text{ (where } F_X \text{ is continuous), } F_{X_n}(x) \xrightarrow{n \rightarrow \infty} F_X(x) \\ &\iff \forall g \text{ continuous and bounded: } \mathbb{E}(g(X_n)) \xrightarrow{n \rightarrow \infty} \mathbb{E}(g(X)) \end{aligned}$$

Main Theorems

Theorem 2: Law of Large Numbers (Weak)

If $(X_n)_n$ are i.i.d. random variables such that $\mathbb{E}(|X_n|) < \infty$, then:

$$\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{n \rightarrow \infty} \mathbb{E}(X_1)$$

Theorem 3: Central Limit Theorem (CLT)

If $(X_n)_n$ are i.i.d. such that $\mathbb{E}(X_n^2) < \infty$, then:

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

where $\bar{X}_n = \frac{1}{n} \sum X_i$, $\mu = \mathbb{E}(X_1)$, $\sigma^2 = \text{Var}(X_1)$.

Theorem 4: Continuity Mapping Theorem

- i) If $X_n \xrightarrow{\mathbb{P}} X$ and if g is continuous, then $g(X_n) \xrightarrow{\mathbb{P}} g(X)$.
- ii) If $X_n \xrightarrow{L^p} X$, $p \geq 1$, then $X_n \xrightarrow{L^q} X$ for all $1 \leq q \leq p$.
- iii) If $X_n \xrightarrow{\mathcal{D}} X$, and if g is continuous and bounded, then $g(X_n) \xrightarrow{\mathcal{D}} g(X)$. (Actually holds for any continuous g).

Theorem 5: Delta Method

If $(Y_n)_n$ are like $\sqrt{n}(Y_n - \mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$. If g is C^1 at μ so that $|g'(\mu)| > 0$, then:

$$\sqrt{n}(g(Y_n) - g(\mu)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, g'(\mu)^2 \sigma^2)$$

Sketch of proof. Taylor expansion of $g(Y_n)$ around μ :

$$g(Y_n) \approx g(\mu) + g'(\mu)(Y_n - \mu)$$

$$\sqrt{n}(g(Y_n) - g(\mu)) \approx g'(\mu)\sqrt{n}(Y_n - \mu)$$

Since $\sqrt{n}(Y_n - \mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$, the result follows. \square

Remark

If $\sqrt{n}(Y_n - \mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$, then $Y_n \xrightarrow{\mathbb{P}} \mu$.

III. Empirical Distribution Function

Definition 12: Dirac Mass

Dirac mass at $a \in \mathbb{R}$ is the distribution defined by:

$$\forall A \subset \mathbb{R}, \quad \delta_{\{a\}}(A) = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{if } a \notin A \end{cases}$$

Let $\mathbb{P}_n = \frac{1}{n} \sum_{i=1}^n \delta_{\{X_i\}}$.

Definition 13: Empirical CDF

F_n = Empirical Cumulative Distribution Function (CDF) for $(X_i)_i$ i.i.d. with distribution P_X .

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{X_i \leq x\}} = \mathbb{P}_n((-\infty, x])$$

Remark: F_n is the CDF of \mathbb{P}_n .

Theorem 6: Asymptotic Normality of Empirical CDF

$$\forall x, \quad \sqrt{n}(F_n(x) - F_X(x)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, F_X(x)(1 - F_X(x)))$$

Where F_X is the CDF of P_X .

Proof.

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n \underbrace{\mathbb{1}_{\{X_i \leq x\}}}_{=Y_i}$$

The variables Y_i are i.i.d. Bernoulli variables:

$$Y_i \sim \text{Ber}(p) \quad \text{with } p = \mathbb{P}(X_i \leq x) = F_X(x)$$

$$\mathbb{E}[Y_i^2] < \infty$$

Applying CLT:

$$\sqrt{n}(\bar{Y}_n - \mathbb{E}(Y_1)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \text{Var}(Y_1))$$

Here $\bar{Y}_n = F_n(x)$ and $\mathbb{E}(Y_1) = F_X(x)$.

$$\text{Var}(Y_1) = F_X(x)(1 - F_X(x))$$

(Q.E.D) □

More on Quadratic Risk

Expression of Risk using Density:

$$R(\hat{\theta}_n, \theta) = \mathbb{E}_{\theta}[(\hat{\theta}_n - \theta)^2]$$

Under the model $X_i \stackrel{i.i.d.}{\sim} P_{\theta}$:

$$= \begin{cases} \int (\hat{\theta}_n(x_1, \dots, x_n) - \theta)^2 \prod_{i=1}^n f_{\theta}(x_i) dx_i & \text{if continuous (density } f_{\theta}) \\ \sum_{x_1, \dots, x_n} (\hat{\theta}_n(x_1, \dots, x_n) - \theta)^2 \prod_{i=1}^n P_{\theta}(X = x_i) & \text{if discrete observations} \end{cases}$$

Example: Exponential Family

Example 2 (from beginning): $X_i \stackrel{i.i.d.}{\sim} \mathcal{E}(\theta), \theta > 0$.

X_i are continuous random variables with density $f_{\theta}(x) = \theta e^{-\theta x} \mathbb{1}_{\{x>0\}}$

Estimator:

$$\hat{\theta}_n = \frac{1}{\bar{X}_n}; \quad \bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

If $X \sim \mathcal{E}(\theta)$, $\mathbb{E}_{\theta}(X) = \frac{1}{\theta}$. What is $R(\hat{\theta}_n, \theta)$? What is $b(\theta)$?

We need to calculate $\mathbb{E}_{\theta}(\hat{\theta}_n) = \mathbb{E}_{\theta}\left(\frac{1}{\bar{X}_n}\right)$.

Recall:

- $\mathbb{E}(h(X)) = \int h(x)f_X(x)dx$ (if continuous).
- We write: $\mathbb{E}(h(X)) = \int h(x)dF_X(x) = \int h(x)dP_X(x)$.
- If $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \Gamma(a, b)$, then $\sum_{i=1}^n X_i \sim \Gamma(na, b)$.

Gamma distribution $\Gamma(a, b)$: \mathcal{C}^0 distribution on \mathbb{R}^+ with density:

$$f_{a,b}(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} \mathbb{1}_{\{x>0\}}$$

Note: $\mathcal{E}(\theta) = \Gamma(1, \theta)$.

$$\implies \sum_{i=1}^n X_i \sim \Gamma(n, \theta).$$

Let $Z_n = \sum_{i=1}^n X_i$. Then $\hat{\theta}_n = \frac{n}{Z_n}$.

We want to compute $\mathbb{E}_{\theta}\left(\frac{n}{Z_n}\right) = n\mathbb{E}_{\theta}\left(\frac{1}{Z_n}\right)$.

$$\begin{aligned}\mathbb{E}_\theta \left(\frac{1}{Z_n} \right) &= \int_0^{+\infty} \frac{1}{z} \cdot \frac{\theta^n}{\Gamma(n)} z^{n-1} e^{-\theta z} dz \\ &= \frac{\theta^n}{\Gamma(n)} \int_0^{+\infty} z^{n-2} e^{-\theta z} dz\end{aligned}$$

Remark

Let $f_{n-1,\theta}(z) = \frac{\theta^{n-1}}{\Gamma(n-1)} z^{n-2} e^{-\theta z}$ be the density of a $\Gamma(n-1, \theta)$.

$$\begin{aligned}\implies \int_0^{+\infty} z^{n-2} e^{-\theta z} dz \times \frac{\theta^{n-1}}{\Gamma(n-1)} &= 1 \\ \implies \int_0^{+\infty} z^{n-2} e^{-\theta z} dz &= \frac{\Gamma(n-1)}{\theta^{n-1}}\end{aligned}$$

Back to the expectation:

$$\mathbb{E}_\theta(\hat{\theta}_n) = \frac{n\theta^n}{\Gamma(n)} \times \frac{\Gamma(n-1)}{\theta^{n-1}}$$

And we have: $\Gamma(x+1) = x\Gamma(x)$ if $x > 0$. So $\Gamma(n) = (n-1)\Gamma(n-1)$.

$$\implies \mathbb{E}_\theta(\hat{\theta}_n) = \frac{n}{n-1}\theta$$

$$\implies b(\theta) = \mathbb{E}_\theta(\hat{\theta}_n) - \theta = \frac{n}{n-1}\theta - \theta = \frac{\theta}{n-1}$$

Now, let's compute the quadratic risk. We need $\mathbb{E}_\theta(\hat{\theta}_n^2) = n^2 \mathbb{E}(\frac{1}{Z_n^2})$.

$$\mathbb{E} \left(\frac{1}{Z_n^2} \right) = \int y^{n-3} e^{-\theta y} dy \frac{\theta^n}{\Gamma(n)}$$

Using the same trick (identification with $\Gamma(n-2, \theta)$) for $n > 2$:

$$\begin{aligned}&= \frac{\theta^n}{\Gamma(n)} \cdot \frac{\Gamma(n-2)}{\theta^{n-2}} = \frac{\theta^2 \Gamma(n-2)}{(n-1)(n-2)\Gamma(n-2)} = \frac{\theta^2}{(n-1)(n-2)} \\ \implies \mathbb{E}_\theta(\hat{\theta}_n^2) &= \frac{n^2 \theta^2}{(n-1)(n-2)} \\ R(\theta, \hat{\theta}_n) &= \underbrace{\frac{n^2 \theta^2}{(n-1)(n-2)}}_{\mathbb{E}(\hat{\theta}_n^2)} - \underbrace{\left(\frac{n\theta}{n-1} \right)^2}_{\mathbb{E}(\hat{\theta}_n)^2} + \underbrace{\frac{\theta^2}{(n-1)^2}}_{b(\theta)^2}\end{aligned}$$

(Using variance decomposition $\mathbb{V}(\hat{\theta}_n) = \mathbb{E}(\hat{\theta}_n^2) - \mathbb{E}(\hat{\theta}_n)^2$). Actually, let's calculate directly:

$$R(\theta, \hat{\theta}_n) = \frac{n^2 \theta^2}{(n-1)(n-2)} - 2\theta \frac{n\theta}{n-1} + \theta^2$$

Simplifying (algebra):

$$\begin{aligned}&= \frac{\theta^2 n^2}{n-1} \left(\frac{1}{n-2} - \frac{1}{n-1} \right) + \frac{\theta^2}{(n-1)^2} \\ &= \frac{\theta^2}{(n-1)^2} \left[\frac{n^2}{n-2} + 1 \right]\end{aligned}$$

Unbiased estimator

We can construct an unbiased estimator $\tilde{\theta}_n$:

$$\tilde{\theta}_n = \hat{\theta}_n \times \left(\frac{n-1}{n} \right) \implies \mathbb{E}_{\theta}(\tilde{\theta}_n) = \mathbb{E}_{\theta}(\hat{\theta}_n) \times \frac{n-1}{n} = \theta.$$

Risk of the unbiased estimator:

$$\begin{aligned} R(\theta, \tilde{\theta}_n) &= \mathbb{V}_{\theta}(\tilde{\theta}_n) = \left(\frac{n-1}{n} \right)^2 \mathbb{V}_{\theta}(\hat{\theta}_n) \\ &= \frac{(n-1)^2}{n^2} \times \frac{n^2 \theta^2}{(n-1)^2(n-2)} = \frac{\theta^2}{n-2} \end{aligned}$$

We observe that:

$$\forall \theta, R(\theta, \tilde{\theta}_n) < R(\hat{\theta}_n, \theta) \implies \tilde{\theta}_n \text{ is better than } \hat{\theta}_n.$$

Remark

Unbiased estimators are not necessarily better than biased estimators.

Example 6: Empirical Variance

$$X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2).$$

$$S_n^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2 \quad (\text{Empirical variance})$$

We know that $S_n^2 = \frac{1}{n} \sum X_i^2 - \bar{X}_n^2$.

$$\mathbb{E}_{\sigma^2} \left(\frac{1}{n} \sum X_i^2 \right) = \sigma^2$$

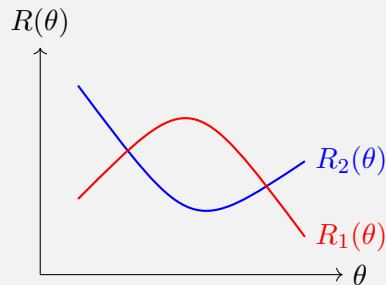
Since $\bar{X}_n \sim \mathcal{N}(0, \frac{\sigma^2}{n})$, $\mathbb{E}_{\sigma^2}(S_n^2) = \sigma^2 - \frac{\sigma^2}{n} = \sigma^2 \frac{n-1}{n}$.
Define $\tilde{\sigma}_n^2 = S_n^2 \cdot \frac{n}{n-1} = \frac{1}{n-1} \sum (X_i - \bar{X}_n)^2$.

$$\implies \mathbb{E}_{\sigma^2}(\tilde{\sigma}_n^2) = \sigma^2 \implies \text{Unbiased.}$$

Exercise: Compute $R(\sigma^2, S_n^2)$ and $R(\sigma^2, \tilde{\sigma}_n^2)$ and show that $\forall \sigma > 0, R(\sigma^2, S_n^2) < R(\sigma^2, \tilde{\sigma}_n^2)$.

Remark

In most cases, if $\hat{\theta}_{n,1}$ and $\hat{\theta}_{n,2}$ are 2 estimators of θ , then risk functions $R_1(\theta) = R(\theta, \hat{\theta}_{n,1})$ and $R_2(\theta, \hat{\theta}_{n,2})$ cross.



$\implies \hat{\theta}_{n,1}$ and $\hat{\theta}_{n,2}$ cannot be compared uniformly.

Example 7: Normal Mean Estimators

$X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathcal{N}(\theta, 1)$.

$$\hat{\theta}_{n,1} = \bar{X}_n$$

$$\hat{\theta}_{n,2} = \frac{n\bar{X}_n}{n + \tau^2} \quad \text{with } \tau > 0$$

Compute their risks and show that they are not comparable.

Admissibility

Definition 14: Admissibility

An estimator $\hat{\theta}_n$ is **not admissible** iff $\exists \tilde{\theta}_n$, another estimator such that:

- $\forall \theta, R(\theta, \tilde{\theta}_n) \leq R(\theta, \hat{\theta}_n)$
- and $\exists \theta_0, R(\theta_0, \tilde{\theta}_n) < R(\theta_0, \hat{\theta}_n)$.

$\hat{\theta}_n$ is **admissible** iff it is not non-admissible.

IV. Consistency

Example 8: Bernoulli Consistency

$X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \text{Ber}(p)$. $\theta = p \in]0, 1[$. $\bar{X}_n = \hat{\theta}_n$ is a possible estimator for θ .

$$\bar{X}_n \xrightarrow[n \rightarrow \infty]{P_\theta} \theta = \mathbb{E}_\theta(X_1) \quad (\text{LLN})$$

Question: $R(\theta, \hat{\theta}_n) \xrightarrow[n \rightarrow \infty]{} 0, \forall \theta?$

$$\begin{aligned} R(\theta, \hat{\theta}_n) &= \mathbb{E}_\theta[(\bar{X}_n - \theta)^2] = \mathbb{V}_\theta(\bar{X}_n) = \frac{1}{n^2} n \mathbb{V}_\theta(X_1) \\ &= \frac{\theta(1 - \theta)}{n} \leq \frac{1}{4n} \end{aligned}$$

So $\sup_{\theta \in (0,1)} R(\theta, \hat{\theta}_n) \xrightarrow[n \rightarrow \infty]{} 0$.

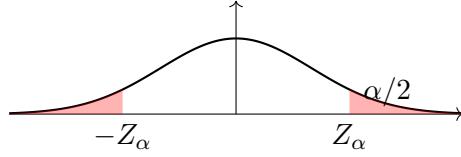
Can I say something like: $\hat{\theta}_{n,1} \leq \theta \leq \hat{\theta}_{n,2}$ with proba ≈ 0.95 ?

$$\begin{aligned} \sqrt{n}(\bar{X}_n - \theta) &\xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \theta(1 - \theta)) \quad (\text{CLT}) \\ \implies \frac{\sqrt{n}(\bar{X}_n - \theta)}{\sqrt{\theta(1 - \theta)}} &\xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1) \end{aligned}$$

Since $\bar{X}_n \xrightarrow{\mathbb{P}} \theta$, by Slutsky's lemma (replacing θ with \bar{X}_n in the denominator):

$$\frac{\sqrt{n}(\bar{X}_n - \theta)}{\sqrt{\bar{X}_n(1 - \bar{X}_n)}} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1)$$

Let Z_α be such that $\mathbb{P}(|\mathcal{N}(0, 1)| \leq Z_\alpha) = 1 - \alpha$. (Usually $1 - \alpha = 0.95 \implies Z_\alpha \approx 1.96$).



$$\begin{aligned} &\implies \mathbb{P}_\theta \left(-Z_\alpha \leq \frac{\sqrt{n}(\bar{X}_n - \theta)}{\sqrt{\bar{X}_n(1 - \bar{X}_n)}} \leq Z_\alpha \right) \xrightarrow{n \rightarrow \infty} 1 - \alpha \\ &\implies \mathbb{P}_\theta \left(\underbrace{\bar{X}_n - \frac{\sqrt{\bar{X}_n(1 - \bar{X}_n)}Z_\alpha}{\sqrt{n}}}_{\hat{\theta}_{n,1}} \leq \theta \leq \underbrace{\bar{X}_n + \frac{Z_\alpha\sqrt{\bar{X}_n(1 - \bar{X}_n)}}{\sqrt{n}}}_{\hat{\theta}_{n,2}} \right) \rightarrow 1 - \alpha \end{aligned}$$

With probability under $P_\theta \approx 1 - \alpha$, $\hat{\theta}_{n,1} \leq \theta \leq \hat{\theta}_{n,2}$.

Definition 15: Consistency

1. We say that an estimator $\hat{\theta}_n$ is **consistent in probability** at θ iff $\hat{\theta}_n \xrightarrow[n \rightarrow \infty]{P_\theta} \theta$.
2. $\hat{\theta}_n$ is consistent **over** Θ if it is consistent at $\theta, \forall \theta \in \Theta$.
3. $\hat{\theta}_n$ converges in **quadratic mean** at θ iff $R(\hat{\theta}_n, \theta) \xrightarrow[n \rightarrow \infty]{} 0$ (Notation: $\hat{\theta}_n \xrightarrow{q.m.} \theta$).
4. $\hat{\theta}_n$ converges in quadratic mean over Θ iff $\hat{\theta}_n \xrightarrow{q.m.} \theta, \forall \theta \in \Theta$.
5. $\hat{\theta}_n$ is **asymptotically normal** at θ with rate $\frac{1}{\sqrt{n}}$ iff $\exists V > 0$ s.t. $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, V)$.

Estimation of $g(\theta)$

$X_1, \dots, X_n \stackrel{i.i.d.}{\sim} P_\theta, \theta \in \Theta$.

Examples:

1. $P_\theta \sim \text{Ber}(\theta)$. Interested in estimating $\eta = \log(\frac{\theta}{1-\theta})$.

2. $X_i \stackrel{i.i.d.}{\sim} \mathcal{N}(\mu, \sigma^2), \theta = (\mu, \sigma^2)$. $g_1(\theta) = \mu$. $g_2(\theta) = \mathbb{P}_\theta(X > 1)$.

If $\hat{\theta}$ is an estimator of θ , then we can use $g(\hat{\theta})$ as an estimator of $g(\theta)$ (one possibility among others).

Theorem 7: Plug-in Consistency

Let $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} P_\theta, \theta \in \Theta \subset \mathbb{R}^d$. Let $\hat{\theta}_n$ be an estimator of θ and $g : \Theta \rightarrow \mathbb{R}^p$. Then:

- i) If $\hat{\theta}_n$ is consistent at θ in probability and g is \mathcal{C}^0 (continuous), then $\hat{\eta}_n = g(\hat{\theta}_n)$ is consistent in probability at $\eta = g(\theta)$.
- ii) If $\hat{\theta}_n \xrightarrow{L^2} \theta$ and g is \mathcal{C}^0 and **bounded**, then $g(\hat{\theta}_n) \xrightarrow{L^2(P_\theta)} g(\theta), \forall p \geq 1$.

iii) If $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[n \rightarrow \infty]{\mathcal{L}(P_\theta)} \mathcal{N}(0, \sigma^2)$ and g is C^1 with $g'(\theta) \neq 0$ (or $\nabla g(\theta)$ of full rank), then $\sqrt{n}(g(\hat{\theta}_n) - g(\theta)) \xrightarrow{\mathcal{L}(P_\theta)} \mathcal{N}(0, g'(\theta)^2 \sigma^2)$.

Remark

$\hat{\theta}_n$ is unbiased $\not\Rightarrow g(\hat{\theta}_n)$ is unbiased.

Proof for (ii). $\forall \varepsilon > 0$,

$$\mathbb{E}_\theta(|g(\hat{\theta}_n) - g(\theta)|^p) = \mathbb{E}_\theta(|g(\hat{\theta}_n) - g(\theta)|^p \mathbb{1}_{|g(\hat{\theta}_n) - g(\theta)| > \varepsilon}) + \mathbb{E}_\theta(|g(\hat{\theta}_n) - g(\theta)|^p \mathbb{1}_{|g(\hat{\theta}_n) - g(\theta)| \leq \varepsilon})$$

The second term is $\leq \varepsilon^p$. The first term:

$$\begin{aligned} &\leq \mathbb{E}_\theta((2||g||_\infty)^p \mathbb{1}_{|g(\hat{\theta}_n) - g(\theta)| > \varepsilon}) \\ &= (2||g||_\infty)^p \mathbb{P}_\theta(|g(\hat{\theta}_n) - g(\theta)| > \varepsilon) \xrightarrow[n \rightarrow \infty]{} 0 \end{aligned}$$

Because $g(\hat{\theta}_n) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} g(\theta)$ (from point (i), continuous mapping theorem).

Thus $\mathbb{E}_\theta(|g(\hat{\theta}_n) - g(\theta)|^p) \leq \varepsilon^p + 2||g||_\infty^p \times (\dots \rightarrow 0)$. Taking limit $n \rightarrow \infty$ then $\varepsilon \rightarrow 0$ yields the result. \square

The Multivariate Case

$X_i \stackrel{i.i.d.}{\sim} P_\theta, \theta \in \Theta \subset \mathbb{R}^d. \theta = (\theta_1, \dots, \theta_d)$.

$$\begin{aligned} \hat{\theta}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \theta &\iff \forall \varepsilon > 0, \mathbb{P}_\theta(||\hat{\theta}_n - \theta|| > \varepsilon) \xrightarrow[n \rightarrow \infty]{} 0 \\ &\iff \forall j \leq d, \hat{\theta}_{n,j} \xrightarrow{\mathbb{P}_\theta} \theta_j \end{aligned}$$

Because:

- If $\forall \varepsilon > 0, \mathbb{P}_\theta(||\hat{\theta}_n - \theta|| > \varepsilon) \rightarrow 0$. Let $j \leq d, \forall \varepsilon > 0, ||\hat{\theta}_n - \theta|| \geq |\hat{\theta}_{n,j} - \theta_j|$.

$$\mathbb{P}_\theta(|\hat{\theta}_{n,j} - \theta_j| > \varepsilon) \leq \mathbb{P}_\theta(||\hat{\theta}_n - \theta|| > \varepsilon) \xrightarrow[n \rightarrow \infty]{} 0$$

- **Reverse:** If $\forall j \leq d, \mathbb{P}_\theta(|\hat{\theta}_{n,j} - \theta_j| > \varepsilon) \rightarrow 0$.

$$||\hat{\theta}_n - \theta||^2 = \sum_j |\hat{\theta}_{n,j} - \theta_j|^2 \leq d \cdot \max_j |\hat{\theta}_{n,j} - \theta_j|^2$$

$$\begin{aligned} \mathbb{P}_\theta(||\hat{\theta}_n - \theta|| > \varepsilon) &\leq \mathbb{P}_\theta \left(\max_j |\hat{\theta}_{n,j} - \theta_j| > \frac{\varepsilon}{\sqrt{d}} \right) \\ &= \mathbb{P}_\theta \left(\bigcup_{j=1}^d \{|\hat{\theta}_{n,j} - \theta_j| > \frac{\varepsilon}{\sqrt{d}}\} \right) \\ &\leq \sum_{j=1}^d \mathbb{P}_\theta \left(\{|\hat{\theta}_{n,j} - \theta_j| > \frac{\varepsilon}{\sqrt{d}}\} \right) \\ &\xrightarrow[n \rightarrow \infty]{} 0 \quad \text{because } d \text{ is fixed.} \end{aligned}$$

→ Similarly, show that $\hat{\theta}_n \xrightarrow{q.m} \theta \iff \forall j, \hat{\theta}_{n,j} \xrightarrow{q.m} \theta_j$.

But, if $\sqrt{n}(\hat{\theta}_{n,j} - \theta_j) \xrightarrow{\mathcal{L}(P_\theta)} \mathcal{N}(0, \sigma_j^2)$, it does **NOT** imply directly the joint normality.

$$\implies \sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}(P_\theta)} \mathcal{N}(\mathbf{0}, \Sigma)$$

(Multivariate Central Limit Theorem required).

Chapter 4: Maximum Likelihood Estimation

I. The Likelihood

Model: $X_i \stackrel{i.i.d.}{\sim} P_\theta, \theta \in \Theta$.

Observations: (x_1, \dots, x_n) , where x_i is a realisation of X_i .

- If P_θ is **discrete**, then proba mass function of $X_1 \sim P_\theta$:

$$x \in \mathcal{X}, \quad f_\theta(x) = P_\theta(X_1 = x)$$

(→ density with respect to counting measure).

- If P_θ is a **continuous** distribution, then it has density f_θ on \mathbb{R} (or \mathbb{R}^d) with respect to Lebesgue measure.

$$\int_A f_\theta(x) dx = P_\theta(X \in A)$$

In both cases, there is a density, with respect to either Lebesgue or counting measure, denoted f_θ .

Definition 16: Likelihood

Let $(X_i)_{i=1}^n \stackrel{i.i.d.}{\sim} P_\theta, \theta \in \Theta$ with $\{P_\theta, \theta \in \Theta\}$ either discrete or continuous. Let f_θ be the density of P_θ (or probability mass function). We call the **likelihood** at (x_1, \dots, x_n) (realisations of (X_1, \dots, X_n)) the function:

$$L_n : \Theta \rightarrow \mathbb{R}_+$$

$$\theta \mapsto \prod_{i=1}^n f_\theta(x_i)$$

Examples

1. $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathcal{P}(\lambda), \lambda > 0$ (Poisson). Observations $(x_1, \dots, x_n), \forall x \in \mathbb{N}, f_\lambda(x) = \frac{e^{-\lambda} \lambda^x}{x!}$.

$$\begin{aligned} L_n(\lambda) &= \prod_{i=1}^n f_\lambda(x_i) = \prod_{i=1}^n \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} \\ &= \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!} \end{aligned}$$

2. $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathcal{E}(\theta), \theta > 0$. Observations $(x_1, \dots, x_n), x_i > 0, \forall i$.

$$\forall x > 0, f_\theta(x) = \theta e^{-\theta x}$$

$$\begin{aligned} L_n(\theta) &= \prod_{i=1}^n \theta e^{-\theta x_i} \mathbb{1}_{\{x_i > 0\}} \\ &= \theta^n e^{-\theta \sum_{i=1}^n x_i} \mathbb{1}_{\{\min_i x_i > 0\}} \end{aligned}$$

Remark

In the **discrete** case, $\forall \theta, L_n(\theta) = \prod_{i=1}^n f_\theta(x_i) = \prod_{i=1}^n \mathbb{P}_\theta(X_i = x_i) = \mathbb{P}_\theta(X_1 = x_1, \dots, X_n = x_n)$.

But, in the **continuous** case, it is different:

$$L_n(\theta) = \prod_{i=1}^n f_\theta(x_i) \neq \mathbb{P}_\theta(X_1 = x_1, \dots, X_n = x_n) = 0$$

(Probability of exact points in continuous case is 0).

If $L_n(\theta)$ is large, it means that the observations are highly likely for θ . "There is a good fit between (x_1, \dots, x_n) and P_θ ".

Exercise 1: Show that if $P_\theta \sim \mathcal{N}(\mu, \sigma^2)$, $\theta = (\mu, \sigma^2)$, then:

$$L_n(\theta) = \frac{e^{-n\frac{(\bar{x}_n - \mu)^2}{2\sigma^2}}}{(2\pi\sigma^2)^{n/2}} e^{-\sum_{i=1}^n \frac{(x_i - \bar{x}_n)^2}{2\sigma^2}}$$

II. Kullback-Leibler Divergence (K.L)

2nd interpretation: Kullback-Leibler divergence.

Definition 17: Kullback-Leibler Divergence

Let P and Q be 2 probabilities.

- If P, Q are **discrete** on \mathcal{X} , then:

$$\begin{aligned} KL(P, Q) &= \sum_{x \in \mathcal{X}} P(x) \ln \left(\frac{P(x)}{Q(x)} \right) \\ &= \mathbb{E}_P \left(\ln \left(\frac{P(X)}{Q(X)} \right) \right) \end{aligned}$$

(where $P(x) = \mathbb{P}(X = x)$, $Q(x) = \mathbb{Q}(X = x)$).

- If P and Q are **continuous** with density f_P and f_Q (resp), then:

$$\begin{aligned} KL(P, Q) &= \int_{\mathcal{X}} f_P(x) \ln \left(\frac{f_P(x)}{f_Q(x)} \right) dx \\ &= \mathbb{E}_P \left(\ln \left(\frac{f_P(X)}{f_Q(X)} \right) \right) \end{aligned}$$

Proposition 2: Properties of KL

- $\forall P, Q, \quad KL(P, Q) \geq 0$.
- $KL(P, Q) = 0 \iff P = Q$.

But, $KL(P, Q) \neq KL(Q, P)$ (Not symmetric).

Proof. Assume that P, Q are continuous.

i)

$$\begin{aligned} KL(P, Q) &= \int_{\mathcal{X}} f_P(x) \ln \left(\frac{f_P(x)}{f_Q(x)} \right) dx \\ &= \int_{\mathcal{X}} f_P(x) \left[-\ln \left(\frac{f_Q(x)}{f_P(x)} \right) \right] dx \end{aligned}$$

By Jensen's inequality:

$$\begin{aligned} &\geq -\ln \left[\int_{\mathcal{X}} f_P(x) \frac{f_Q(x)}{f_P(x)} dx \right] \\ &= -\ln \left[\int_{\mathcal{X}} f_Q(x) dx \right] = -\ln(1) = 0 \end{aligned}$$

(Because $x \mapsto -\ln(x)$ is strictly convex on \mathbb{R}).

ii) Since $-\ln(\cdot)$ is strictly convex:

$$\mathbb{E}_P (-\ln(R(X))) > -\ln(\mathbb{E}_P(R(X))) \quad \text{with } R(x) = \frac{f_Q(x)}{f_P(x)}$$

if $\forall c, \mathbb{P}(R(X) = c) < 1$ ($R(X)$ is not constant).

$R(x) = \frac{f_Q(x)}{f_P(x)}$ is constant iff $\exists c \neq 0$ so that:

$$f_Q(x) = cf_P(x), \quad \forall x \in \mathcal{X}$$

$$\implies c = 1 \quad (\text{since } \int f_P = \int f_Q = 1)$$

$R(x) = \frac{f_Q(x)}{f_P(x)}$ is constant iff $f_Q = f_P \iff P = Q$. □

Links with the likelihood

$\forall \theta, \theta_0 \in \Theta$:

$$\begin{aligned} &\ln(L_n(\theta)) - \ln(L_n(\theta_0)) \\ &= \ln \left(\prod_{i=1}^n f_\theta(x_i) \right) - \ln \left(\prod_{i=1}^n f_{\theta_0}(x_i) \right) \\ &= \sum_{i=1}^n \ln(f_\theta(x_i)) - \sum_{i=1}^n \ln(f_{\theta_0}(x_i)) \\ &= \sum_{i=1}^n \ln \left(\frac{f_\theta(x_i)}{f_{\theta_0}(x_i)} \right) \end{aligned}$$

If $(x_i)_{i=1}^n$ are the realisations of $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} P_{\theta_0}$. Then when $X_i \stackrel{i.i.d.}{\sim} P_{\theta_0}$:

$$\frac{1}{n} \sum_{i=1}^n \ln \left(\frac{f_\theta(X_i)}{f_{\theta_0}(X_i)} \right) \underbrace{\text{converges under } P_{\theta_0}}_{n \rightarrow \infty} \text{ to } \mathbb{E}_{\theta_0} \left(\ln \left(\frac{f_\theta(X)}{f_{\theta_0}(X)} \right) \right)$$

as soon as $\mathbb{E}_{\theta_0} \left(\left| \ln \frac{f_\theta(X)}{f_{\theta_0}(X)} \right| \right) < \infty$.

\implies if $\mathbb{E}_{\theta_0}(\dots) < \infty$, then we have:

$$\begin{aligned} \frac{1}{n} (\ln(L_n(\theta)) - \ln(L_n(\theta_0))) &\xrightarrow[n \rightarrow \infty]{P_{\theta_0}} \mathbb{E}_{\theta_0} \left[\ln \left(\frac{f_\theta(X)}{f_{\theta_0}(X)} \right) \right] \\ &= -KL(P_{\theta_0}, P_\theta) \end{aligned}$$

- θ_0 minimizes $KL(P_{\theta_0}, P_\theta)$ over θ .
- θ_0 maximizes $-KL(P_{\theta_0}, P_\theta)$ over θ .

We cannot maximize $-KL(P_{\theta_0}, P_\theta)$ because it depends on θ_0 unknown. \implies We maximize:

$$\mathbb{E}_{P_n} \left[-\ln \left(\frac{f_\theta(X)}{f_{\theta_0}(X)} \right) \right] = \frac{1}{n} [\ln(L_n(\theta)) - \ln(L_n(\theta_0))]$$

(Empirical version of $-KL(P_{\theta_0}, P_\theta)$).

Maximizing in θ , $\ln(L_n(\theta)) - \ln(L_n(\theta_0))$ is equivalent to maximizing in θ , $\ln L_n(\theta)$.

Definition 18: Maximum Likelihood Estimator (MLE)

Let $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} P_\theta$, $\theta \in \Theta$, the model P_θ with density (or probability mass function) f_θ . We call **Maximum Likelihood Estimator** any selection $\hat{\theta}_n$ (when it exists) such that:

$$\hat{\theta}_n \in \operatorname{argmax}_{\theta \in \Theta} \{\ln(L_n(\theta))\}$$

$$\iff \log(L_n(\hat{\theta}_n)) \geq \log(L_n(\theta)), \forall \theta \in \Theta$$

Examples of Calculations

1) Bernoulli Model

$X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \text{Ber}(\theta)$ ($\theta \in [0, 1]$).

$$\begin{aligned} L_n(\theta) &= \prod_{i=1}^n P_\theta(X_i = x_i) \\ &= \prod_{i=1}^n \theta^{x_i} (1-\theta)^{1-x_i} \\ &= \theta^{\sum x_i} (1-\theta)^{n-\sum x_i} \end{aligned}$$

Define $l(\theta) = \log(L_n(\theta))$.

$$l(\theta) = \left(\sum_{i=1}^n x_i \right) \log(\theta) + \left(n - \sum_{i=1}^n x_i \right) \log(1-\theta)$$

$\hat{\theta}_n$ maximizes $l(\theta)$ in θ . Let $\bar{x}_n = \frac{1}{n} \sum x_i$.

$$l(\theta) = n\bar{x}_n \log(\theta) + n(1-\bar{x}_n) \log(1-\theta)$$

Derivative:

$$l'(\theta) = \frac{n\bar{x}_n}{\theta} - \frac{n(1-\bar{x}_n)}{1-\theta}$$

Second derivative:

$$l''(\theta) = -\frac{n\bar{x}_n}{\theta^2} - \frac{n(1-\bar{x}_n)}{(1-\theta)^2} < 0$$

So $\theta \mapsto l(\theta)$ is **concave**. $\implies \hat{\theta}_n$ is the solution of $l'(\theta) = 0$.

$$l'(\theta) = 0 \iff \frac{n\bar{x}_n}{\theta} = \frac{n(1-\bar{x}_n)}{1-\theta} \iff \theta = \bar{x}_n$$

$\implies \hat{\theta}_n = \bar{x}_n$ is the MLE (Maximum Likelihood Estimator)

2) Exponential Model

$X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathcal{E}(\theta), \theta > 0.$

$$X \sim \mathcal{E}(\theta) \iff f_\theta(x) = \theta e^{-\theta x} \mathbb{1}_{\{x>0\}}$$

$$L_n(\theta) = \theta^n e^{-n\theta \bar{x}_n} \prod_{i=1}^n \mathbb{1}_{\{x_i>0\}}$$

$$l(\theta) = \log(L_n(\theta)) = n \log(\theta) - n \bar{x}_n \theta \quad (\text{if } \forall i, x_i > 0)$$

Exercise: Show that $\theta \mapsto l(\theta)$ is concave and that $\hat{\theta}_n = \frac{1}{\bar{x}_n}$.

Remark

1. The log likelihood as a function is not necessarily concave, therefore, $\hat{\theta}_n$ does not always exist and is not always unique.
2. In practice: how do we compute $\hat{\theta}_n$?

$$\hat{\theta}_n \in \operatorname{argmax}\{l(\theta), \theta \in \Theta\}$$

Algorithms such as gradient descent, Newton-Raphson.

Summary:

- **Model:** $f_\theta(\cdot), \theta \in \Theta$ density or proba mass function. $X_i \stackrel{i.i.d.}{\sim} f_\theta$.
- **Log-likelihood:** $l(\theta) = \log(L_n(\theta)) = \sum_{i=1}^n \log(f_\theta(x_i))$.
- **MLE:** $\hat{\theta}_n$ such that $l(\hat{\theta}_n) \geq l(\theta), \forall \theta \in \Theta$.

Proposition 3: Invariance

If $\eta = g(\theta)$, where g is invertible (\iff new parametrization).

For instance: $X_i \stackrel{i.i.d.}{\sim} \text{Ber}(p), \theta = p \in]0, 1[$.

$$\eta = \log\left(\frac{p}{1-p}\right) \in \mathbb{R}$$

Let $\tilde{l}(\eta) = \sum_{i=1}^n \log(f_{g^{-1}(\eta)}(x_i))$. Let $l(\theta) = \sum_{i=1}^n \log(f_\theta(x_i))$.

- If $\hat{\eta}_n$ is the MLE of \tilde{l} ($\iff \tilde{l}(\eta) \leq \tilde{l}(\hat{\eta}_n), \forall \eta$).
- If $\hat{\theta}_n$ is the MLE of l ($\iff l(\theta) \leq l(\hat{\theta}_n), \forall \theta$).

Then:

$$\hat{\eta}_n = g(\hat{\theta}_n)$$

Proof. Note: $\hat{\eta}_n = g(\hat{\theta}_n)$. We verify that $\tilde{l}(\hat{\eta}_n) \geq \tilde{l}(\eta), \forall \eta$, that is, it will imply that $\hat{\eta}_n$ is an MLE of \tilde{l} .
Indeed:

$$\tilde{l}(\eta) = l(g^{-1}(\eta)) \leq l(\hat{\theta}_n), \forall \eta$$

And:

$$\tilde{l}(\hat{\eta}_n) = l(g^{-1}(g(\hat{\theta}_n))) = l(\hat{\theta}_n)$$

$$\implies \forall \eta, \tilde{l}(\eta) \leq \tilde{l}(\hat{\eta}_n). \quad (\text{Q.E.D})$$

□

Examples

MLE for $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \text{Ber}(p)$.

$$l(p) = p^{\sum x_i} (1-p)^{n-\sum x_i}$$

x_1, \dots, x_n are the observations.

$$\hat{p}_n = \frac{1}{n} \sum_{i=1}^n x_i \rightarrow \text{MLE}$$

$$\eta = \log \left(\frac{p}{1-p} \right) \implies \hat{\eta}_n = \log \left(\frac{\hat{p}_n}{1-\hat{p}_n} \right) \text{ is the MLE of } \eta.$$

Asymptotic Normality of the MLE

Under which conditions on the model is the MLE asymptotically normal?

Definition 19: Regular Model

The model $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} f_\theta, \theta \in \Theta \subset \mathbb{R}^k$ (either density or probability mass function) is **regular** on Θ iff:

- (i) Θ is an open set and the model is identifiable.
- (ii) $\theta \mapsto \log(f_\theta(x))$ is $C^2, \forall x \in \mathcal{X}$.
- (iii) $\forall \theta \in \Theta, \mathbb{E}_\theta(\|\nabla_\theta \log(f_\theta(X))\|^2) < +\infty$ and $\exists \delta > 0, \mathbb{E}_\theta(\sup_{|\theta' - \theta| < \delta} \|D_\theta^2 \log(f_{\theta'}(X))\|) < +\infty$.
- (iv) $\forall \theta, I(\theta) = \mathbb{E}_\theta(\nabla_\theta \log(f_\theta(X)) \cdot \nabla_\theta \log(f_\theta(X))^T)$ is positive definite.

Fisher Information

$I(\theta)$ is called the **Fisher Information Matrix**.

Notations: $D^2 \log(f_\theta(x))$ is the matrix:

$$(D^2 \log(f_\theta(x)))_{i,j} = \frac{\partial^2 \log(f_\theta(x))}{\partial \theta_i \partial \theta_j}$$

$$\nabla_\theta \log(f_\theta(x)) = \left(\frac{\partial}{\partial \theta_j} \log(f_\theta(x)) \right)_{j=1, \dots, k}$$

Lemma 1: Properties of Score and Fisher Information

If the model is regular:

- a) $\forall \theta \in \Theta, \mathbb{E}_\theta(\nabla_\theta \log(f_\theta(X))) = 0$. $S(\theta, x) = \nabla_\theta \log(f_\theta(x))$ is called the **score function**.
- b) $I(\theta) = -\mathbb{E}_\theta(D^2 \log(f_\theta(X)))$.

Proof. a) The model is regular $\implies \nabla_\theta \log(f_\theta(X))$ exists and is integrable. And:

$$\nabla_\theta \log(f_\theta(X)) = \frac{\nabla_\theta f_\theta(X)}{f_\theta(X)}$$

$$\mathbb{E}_\theta(\nabla_\theta \log(f_\theta(X))) = \begin{cases} \int_{\mathcal{X}} \frac{\nabla_\theta f_\theta(x)}{f_\theta(x)} f_\theta(x) dx & \text{if } X_i \text{'s are continuous} \\ \sum_{x \in \mathcal{X}} \frac{\nabla_\theta f_\theta(x)}{f_\theta(x)} f_\theta(x) & \text{if } X_i \text{'s are discrete} \end{cases}$$

We have, if the X_i 's are continuous:

$$\int_{\mathcal{X}} \frac{\partial}{\partial \theta_j} f_\theta(x) dx = \frac{\partial}{\partial \theta_j} \int_{\mathcal{X}} f_\theta(x) dx$$

Because $\mathbb{E}_\theta(||\nabla_\theta f_\theta(X)||) < +\infty$. Also $\forall \theta, \int_{\mathcal{X}} f_\theta(x) dx = 1 \implies \frac{\partial}{\partial \theta_j} (\int f_\theta(x) dx) = 0$.

$$\text{So } \mathbb{E}_\theta \left(\frac{\partial}{\partial \theta_j} \frac{f_\theta(X)}{f_\theta(X)} \right) = 0.$$

$$\implies \mathbb{E}_\theta(\nabla_\theta \log(f_\theta(X))) = 0$$

If f is discrete \rightarrow same argument.

$$\sum_x \frac{\partial}{\partial \theta_j} f_\theta(x) = \frac{\partial}{\partial \theta_j} \sum_x f_\theta(x) = \frac{\partial}{\partial \theta_j}(1) = 0$$

(Q.E.D)

b) Show that $I(\theta) = -\mathbb{E}_\theta(D_\theta^2 \log(f_\theta(X)))$.

$$\begin{aligned} \frac{\partial^2 \log(f_\theta(x))}{\partial \theta_i \partial \theta_j} &= \frac{\partial}{\partial \theta_j} \left(\frac{\frac{\partial}{\partial \theta_i} f_\theta(x)}{f_\theta(x)} \right) \\ &= \frac{\frac{\partial^2}{\partial \theta_i \partial \theta_j} f_\theta(x)}{f_\theta(x)} - \left(\frac{\frac{\partial}{\partial \theta_i} f_\theta(x)}{f_\theta(x)} \right) \left(\frac{\frac{\partial}{\partial \theta_j} f_\theta(x)}{f_\theta(x)} \right) \end{aligned}$$

Similarly to before:

$$\begin{aligned} \mathbb{E}_\theta \left(\frac{\frac{\partial^2}{\partial \theta_i \partial \theta_j} f_\theta(X)}{f_\theta(X)} \right) &= \int_{\mathcal{X}} \frac{\frac{\partial^2}{\partial \theta_i \partial \theta_j} f_\theta(x)}{f_\theta(x)} f_\theta(x) dx \\ &= \int_{\mathcal{X}} \frac{\partial^2}{\partial \theta_i \partial \theta_j} f_\theta(x) dx = \frac{\partial^2}{\partial \theta_i \partial \theta_j} \int_{\mathcal{X}} f_\theta(x) dx = 0 \end{aligned}$$

$$\begin{aligned} J(\theta)_{i,j} &= -\mathbb{E}_\theta(D_\theta^2 \log(f_\theta(X))) \\ &= -\mathbb{E}_\theta \left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \log(f_\theta(X)) \right) \\ &= - \left(-\mathbb{E}_\theta \left(\frac{\frac{\partial f}{\partial \theta_i}}{f} \cdot \frac{\frac{\partial f}{\partial \theta_j}}{f} \right) \right) \\ &= I_{i,j}(\theta) \end{aligned}$$

So $J(\theta) = I(\theta)$. (Q.E.D)

□

Theorem 8: Asymptotic Normality of MLE

Let (x_1, \dots, x_n) be observations from the model $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} f_\theta, \theta \in \Theta \subset \mathbb{R}^k$. Assume the model is **regular**.

(i) **Score Convergence:** For all $\theta \in \Theta$, let $S_n(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \nabla_\theta \log(f_\theta(X_i))$. Then:

$$S_n(\theta) \xrightarrow[n \rightarrow \infty]{\mathcal{L}_{P_\theta}} \mathcal{N}(0, I(\theta))$$

(ii) **MLE Convergence:** If in addition the MLE $\hat{\theta}_n$ is **consistent** over Θ (in probability), then:

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[n \rightarrow \infty]{\mathcal{L}_{P_\theta}} \mathcal{N}(0, I^{-1}(\theta))$$

$\implies I^{-1}(\theta)$ is the asymptotical variance of $\sqrt{n}(\hat{\theta}_n - \theta)$.

Remark

Cramer-Rao Lower Bound: For the variance of a regular estimator is $\frac{I^{-1}(\theta)}{n}$.

Multivariate Gaussians Reminder

$X \sim \mathcal{N}(\mu, \Sigma)$, $X \in \mathbb{R}^k$, $\mu \in \mathbb{R}^k$. Σ symmetric, positive semi-definite.
 \iff its density (w.r.t Lebesgue)

$$f_{\mu, \Sigma}(x) = \frac{e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)}}{(2\pi)^{k/2} (\det(\Sigma))^{1/2}}$$

$\iff \forall a \in \mathbb{R}^d, a^T X \sim \mathcal{N}(a^T \mu, a^T \Sigma a)$.

Proof of the Theorem

Objective: Show that $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}_{P_\theta}} \mathcal{N}(0, I^{-1}(\theta))$.

$\hat{\theta}_n$ is defined by $l(\hat{\theta}_n) \geq l(\theta)$ and verifies $\nabla_\theta l(\hat{\theta}_n) = 0$. The rest of the proof is done with $k = 1$.
 $\theta \mapsto l(\theta) = \sum \log(f_\theta(x_i))$ is \mathcal{C}^2 .

Taylor expansion:

$$l'(\theta) = l'(\hat{\theta}_n) + (\theta - \hat{\theta}_n)l''(\tilde{\theta}_n) \quad \text{with } \tilde{\theta}_n \in [\theta, \hat{\theta}_n]$$

$$\implies l'(\theta) = (\theta - \hat{\theta}_n)l''(\tilde{\theta}_n) \quad (\text{since } l'(\hat{\theta}_n) = 0)$$

$$\frac{l''(\tilde{\theta}_n)}{n} = \frac{1}{n} \sum_{i=1}^n \frac{\partial^2}{\partial \theta^2} \log(f_{\tilde{\theta}_n}(x_i))$$

$$\frac{l''(\tilde{\theta}_n) - l''(\theta)}{n} = \Delta_n(\theta)$$

So:

$$l'(\theta) = (\theta - \hat{\theta}_n)(l''(\theta) + \Delta_n(\theta) \times n)$$

And $\frac{l'(\theta)}{\sqrt{n}} = -S_n(\theta)$.

$$S_n(\theta) = \sqrt{n}(\theta - \hat{\theta}_n) \left(\frac{l''(\theta)}{n} + \Delta_n(\theta) \right)$$

We examine the terms:

$$\frac{l''(\theta)}{n} = \frac{1}{n} \sum_{i=1}^n D^2 \log(f_\theta(x_i)) \xrightarrow[n \rightarrow \infty]{P_\theta} \mathbb{E}_\theta(D^2 \log(f_\theta(X))) = -I(\theta)$$

$$\implies S_n(\theta) = \sqrt{n}(\hat{\theta}_n - \theta)(-I(\theta) + \Delta_n(\theta) + B_n(\theta))$$

where $B_n(\theta) = \frac{l''(\theta)}{n} + I(\theta) \xrightarrow{P_\theta} 0$.

We need to handle $S_n(\theta)$ and $\Delta_n(\theta)$.

(i) Behavior of $S_n(\theta)$:

$$\forall \theta \in \Theta, S_n(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \nabla_\theta \log(f_\theta(x_i)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, I(\theta))$$

Proof for (i):

$$S_n(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^n s_1(\theta, x_i)$$

And $\forall \theta, (s_1(\theta, x_i))_{i=1,\dots,n}$ are i.i.d. because (x_i) are i.i.d. And $\mathbb{E}_\theta(s_1(\theta, X_i)) = 0$ (Lemma). And $\mathbb{E}_\theta(\|s_1(\theta, X)\|^2) < +\infty$ (Assumption).

By the CLT:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n s_1(\theta, X_i) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \text{Var}_\theta(s_1(\theta, X)))$$

Also $\text{Var}_\theta(s_1(\theta, X)) = \mathbb{E}_\theta(s_1(\theta, X)s_1(\theta, X)^T) = I(\theta)$.

(ii) Behavior of $\Delta_n(\theta)$: We show that $\Delta_n(\theta) \xrightarrow[n \rightarrow \infty]{P_\theta} 0$. Because when $|\hat{\theta}_n - \theta| \leq \delta$:

$$|\Delta_n(\theta)| \leq \frac{1}{n} \sum \sup_{|\theta' - \theta| \leq \delta} |D_{\theta'}^2 \log(f_{\theta'}(x_i)) - D_\theta^2 \log(f_\theta(x_i))|$$

LLN:

$$\begin{aligned} & \frac{1}{n} \sum \sup_{|\theta' - \theta| \leq \delta} |D_{\theta'}^2 \log(f_{\theta'}(x_i)) - D_\theta^2 \log(f_\theta(x_i))| \\ & \xrightarrow{P_\theta} \mathbb{E}_\theta \left(\sup_{|\theta' - \theta| \leq \delta} |D_{\theta'}^2 \log(f_{\theta'}(x_i)) - D_\theta^2 \log(f_\theta(x_i))| \right) \end{aligned}$$

By the dominated convergence theorem:

$$\mathbb{E}_\theta \left(\sup_{|\theta' - \theta| \leq \delta} |D_{\theta'}^2 \log(f_{\theta'}(x)) - D_\theta^2 \log(f_\theta(x))| \right) \xrightarrow[\delta \rightarrow 0]{} 0$$

$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0, \forall \delta \leq \delta_\varepsilon$, this expectation is $< \varepsilon$.

We know that $\hat{\theta}_n \xrightarrow{P_\theta} \theta$. $\forall \delta > 0, \mathbb{P}_\theta(|\hat{\theta}_n - \theta| > \delta) \xrightarrow{n \rightarrow \infty} 0$.

Also $\theta \mapsto D_\theta^2 \log(f_\theta(x))$ is continuous. $\implies \limsup_{\delta \rightarrow 0} \sup_{|\theta' - \theta| \leq \delta} |D_{\theta'}^2 \log(f_{\theta'}(x)) - D_\theta^2 \log(f_\theta(x))| = 0$ (almost surely). Let $\delta_p \rightarrow 0, \delta_p \downarrow$.

$$\begin{aligned} & \sup_{|\theta' - \theta| \leq \delta_p} |D_{\theta'}^2 \log(f_{\theta'}(x)) - D_\theta^2 \log(f_\theta(x))| \\ & \leq \sup_{|\theta' - \theta| \leq \delta_0} |D_{\theta'}^2 \log(f_{\theta'}(x)) - D_\theta^2 \log(f_\theta(x))| \end{aligned}$$

And $\mathbb{E}_\theta(H(X)) < +\infty$ (by assumption).

So $\Delta_n(\theta) \xrightarrow{P_\theta} 0$.

Conclusion of Proof:

$$S_n(\theta) = \sqrt{n}(\hat{\theta}_n - \theta)(-I(\theta) + \Delta_n(\theta) + B_n(\theta))$$

$$B_n(\theta) = \frac{l''(\theta)}{n} + I(\theta) \xrightarrow{P_\theta} 0$$

$$\Delta_n(\theta) \xrightarrow{P_\theta} 0$$

$$S_n(\theta) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, I(\theta))$$

$$\implies \sqrt{n}(\hat{\theta}_n - \theta) = \frac{S_n(\theta)}{I(\theta)(1 + \mu_n)} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{I(\theta)}{I^2(\theta)}\right) = \mathcal{N}(0, I^{-1}(\theta))$$

With $\mu_n \xrightarrow{P_\theta} 0$.

III. Exponential Families

Definition 20: Exponential Family

Model $X \in \mathbb{R}^d \sim P_\theta$, $\theta \in \Theta \subset \mathbb{R}^k$ is an **exponential family** iff P_θ has density (or proba mass function) f_θ and:

- $\exists h : \mathbb{R}^d \rightarrow \mathbb{R}_+$
- $\exists A : \Theta \rightarrow \mathbb{R}^k$
- $\exists S : \mathbb{R}^d \rightarrow \mathbb{R}^k$
- $\exists C : \Theta \rightarrow \mathbb{R}$

Such that:

$$f_\theta(x) = h(x) \exp(A(\theta)^T S(x) - C(\theta))$$

It's a **canonical exponential family** iff $A(\theta) = \theta$.

Remark

Cramer-Rao Lower Bound: For the variance of a regular estimator, the lower bound is $\frac{I^{-1}(\theta)}{n}$. What are regular models? Exponential families are a prime example.

Theorem 9: Properties of Exponential Families

Let P_θ be an exponential family. If $f_\theta(x) = h(x) \exp(A(\theta)^T S(x) - C(\theta))$ and if $\Theta = \{\theta, f_\theta(x)\text{ exists}\}$ is an open set, then:

(i) If $A(\theta) = \theta$, then the model is **regular**.

(ii) $\exp(C(\theta)) = \int_{\mathbb{R}^d} h(x) e^{A(\theta)^T S(x)} dx$ (if continuous) or $\sum_{x \in \mathcal{X}} h(x) e^{A(\theta)^T S(x)}$ (if discrete).

(iii) If $A(\theta) = \theta$, then $\theta \mapsto C(\theta)$ is \mathcal{C}^∞ on Θ and:

$$\mathbb{E}_\theta(S(X)) = \nabla_\theta C(\theta)$$

$$\mathbb{V}_\theta(S(X)) = \frac{\partial^2 C(\theta)}{\partial \theta^2} = D^2 C(\theta)$$

(iv) If A is invertible and \mathcal{C}^2 , and A^{-1} is \mathcal{C}^2 , then the model is regular.

Remark

" f_θ exists" means:

- If P_θ is continuous: $\int_{\mathbb{R}^d} e^{A(\theta)^T S(x)} h(x) dx < +\infty$.
- If P_θ is discrete: $\sum_{x \in \mathcal{X}} e^{A(\theta)^T S(x)} h(x) < +\infty$.

Heuristic of the proof (Continuous case)

(ii) We know that $\int_{\mathbb{R}^d} f_\theta(x) dx = 1$.

$$\begin{aligned} &\iff \int_{\mathbb{R}^d} h(x) e^{A(\theta)^T S(x) - C(\theta)} dx = 1 \\ &\iff \int_{\mathbb{R}^d} h(x) e^{A(\theta)^T S(x)} dx = e^{C(\theta)} \end{aligned}$$

(iii) If $A(\theta) = \theta$.

$$e^{C(\theta)} = \int_{\mathbb{R}^d} e^{\theta^T S(x)} h(x) dx$$

And $\{\theta, \int_{\mathbb{R}^d} e^{\theta^T S(x)} h(x) dx < +\infty\}$ is open.

Differentiation:

$$\frac{\partial}{\partial \theta} e^{\theta^T S(x)} = S(x) e^{\theta^T S(x)}$$

If $k = 1$:

$$\begin{aligned} \frac{d}{d\theta} e^{C(\theta)} &= C'(\theta) e^{C(\theta)} = \int_{\mathbb{R}^d} S(x) e^{\theta^T S(x)} h(x) dx \\ &\implies C'(\theta) = e^{-C(\theta)} \int_{\mathbb{R}^d} S(x) e^{\theta^T S(x)} h(x) dx \\ &= \int_{\mathbb{R}^d} S(x) \underbrace{e^{\theta^T S(x) - C(\theta)} h(x)}_{f_\theta(x)} dx = \mathbb{E}_\theta(S(X)) \end{aligned}$$

Second derivative:

$$\begin{aligned} \frac{d^2}{d\theta^2} e^{C(\theta)} &= C''(\theta) e^{C(\theta)} + (C'(\theta))^2 e^{C(\theta)} = \int_{\mathbb{R}^d} S(x)^2 e^{\theta^T S(x)} h(x) dx \\ &\implies C''(\theta) = - \underbrace{C'(\theta)^2}_{\mathbb{E}_\theta(S(X))^2} + \underbrace{\int_{\mathbb{R}^d} S(x)^2 e^{\theta^T S(x) - C(\theta)} h(x) dx}_{\mathbb{E}_\theta(S(X)^2)} \\ &= \mathbb{E}_\theta(S(X)^2) - \mathbb{E}_\theta(S(X))^2 = \mathbb{V}_\theta(S(X)) \end{aligned}$$

Remark

Using properties of the log-likelihood:

$$\frac{d^2}{d\theta^2} \log(f_\theta(x)) = -\frac{d^2}{d\theta^2} C(\theta)$$

$$\text{Or } I(\theta) = -\mathbb{E}_\theta \left(\frac{d^2}{d\theta^2} \log f_\theta(X) \right) = \frac{d^2}{d\theta^2} C(\theta).$$

Justification for differentiation under the integral sign (Proof details): If $\theta \in \Theta$, $\theta + \varepsilon \in \Theta$ and $\theta - \varepsilon \in \Theta$ (since open).

$$\implies \int e^{(\theta+\varepsilon)S(x)} h(x) dx < +\infty \quad \text{and} \quad \int e^{(\theta-\varepsilon)S(x)} h(x) dx < +\infty$$

$$|S(x)|e^{\theta S(x)} \leq e^{\theta S(x)} + \varepsilon |S(x)|e^{\theta S(x)} \dots$$

Wait, actually we use convexity arguments.

$$|S(x)|e^{\theta S(x)} \leq e^{(\theta+\varepsilon)S(x)} + e^{(\theta-\varepsilon)S(x)}$$

This bound (integrable function) allows the use of the Dominated Convergence Theorem.

$$\implies \int |S(x)|e^{\theta S(x)} h(x) dx < +\infty$$

Proposition 4: Stability of Exponential Families

If $\{P_\theta, \theta \in \Theta\}$ is an exponential family and $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} P_\theta$, then the joint distribution is also an exponential family.

$$f_\theta^{(n)}(x_1, \dots, x_n) = e^{A(\theta)^T \sum_{i=1}^n S(x_i) - nC(\theta)} \prod_{i=1}^n h(x_i)$$

This is an exponential family with:

$$A_n(\theta) = A(\theta), \quad S_n(x_1, \dots, x_n) = \sum_{i=1}^n S(x_i), \quad C_n(\theta) = nC(\theta)$$

Proof.

$$\begin{aligned} f_\theta^{(n)}(x_1, \dots, x_n) &= \prod_{i=1}^n f_\theta(x_i) \\ &= \prod_{i=1}^n \left(h(x_i) e^{A(\theta)^T S(x_i) - C(\theta)} \right) \\ &= \left(\prod_{i=1}^n h(x_i) \right) e^{A(\theta)^T \sum S(x_i) - nC(\theta)} \end{aligned}$$

□

Examples

Example 9: Binomial Distribution

$X \sim \text{Bin}(n, p)$, $p = \theta$. Probability mass function:

$$f_p(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x \in \{0, \dots, n\}$$

$$= \underbrace{\binom{n}{x}}_{h(x)} \exp \left(\underbrace{\log \left(\frac{p}{1-p} \right) x}_{A(p)} + \underbrace{n \log(1-p)}_{-C(p)} \right)$$

Here $S(x) = x$.

Canonical Parametrisation:

$$\eta = \log \left(\frac{p}{1-p} \right) \implies p = \frac{e^\eta}{1+e^\eta}$$

$$\tilde{f}_\eta(x) = \binom{n}{x} \exp(\eta x - n \log(1+e^\eta))$$

With $C(\eta) = n \log(1+e^\eta)$.

$$\Theta = \mathbb{R} \text{ (open)}$$

Moments check:

$$\mathbb{E}_\eta(X) = C'(\eta) = n \frac{e^\eta}{1+e^\eta} = np$$

$$\mathbb{V}_\eta(X) = C''(\eta) = n \frac{e^\eta(1+e^\eta) - e^\eta e^\eta}{(1+e^\eta)^2} = n \frac{e^\eta}{(1+e^\eta)^2} = np(1-p)$$

Remark: In a canonical exponential family, if Θ is an open set, then $I(\theta) = \frac{d^2 C(\theta)}{d\theta^2}$.

Proof:

$$\begin{aligned} \log(f_\theta(x)) &= \theta^T S(x) - C(\theta) + \log(h(x)) \\ \implies \frac{d}{d\theta} \log(f_\theta(x)) &= S(x) - \nabla C(\theta) \\ \implies \frac{d^2}{d\theta^2} \log(f_\theta(x)) &= -\frac{d^2}{d\theta^2} C(\theta) \\ \implies I(\theta) &= -\mathbb{E}_\theta \left[\frac{d^2}{d\theta^2} \log(f_\theta(x)) \right] \quad (\text{see Lemma 1}) \\ &= \frac{d^2}{d\theta^2} C(\theta) \end{aligned}$$

Example 10: Exponential Distribution

$X \sim \mathcal{E}(\theta)$, $\theta > 0$.

$$f_\theta(x) = \theta e^{-\theta x} \mathbb{1}_{x>0} = \mathbb{1}_{x>0} e^{-\theta x + \log(\theta)}$$

$h(x) = \mathbb{1}_{x>0}$. $A(\theta) = \theta$ (or $-\theta$). $S(x) = -x$. $C(\theta) = -\log(\theta)$. Canonical exponential family.

IV. Sufficient Condition for Consistency of MLE

Theorem 10: Consistency of MLE

If Θ is compact,

$$\text{If (i) } \sup_{\theta \in \Theta} \left| \frac{\ln(L_n(\theta)) - \ln(L_n(\theta_0))}{n} + KL(\theta_0, \theta) \right| \xrightarrow[n \rightarrow \infty]{P_{\theta_0}} 0$$

$$\text{If (ii) } \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \inf_{|\theta - \theta_0| > \varepsilon} KL(\theta_0, \theta) > \delta.$$

Then the MLE is consistent.

Where:

$$KL(\theta_0, \theta) = \mathbb{E}_{\theta_0} [\log f_{\theta_0}(X) - \log f_\theta(X)]$$

(Kullback-Leibler divergence between f_{θ_0} and f_θ).

Proof.

$$KL(\theta_0, \theta) = \mathbb{E}_{\theta_0} [\log(f_{\theta_0}(X)) - \log(f_\theta(X))]$$

Idea of the proof: Prove that $\forall \varepsilon > 0, P_{\theta_0} \left[\sup_{|\theta - \theta_0| > \varepsilon} \ln(\theta) < \sup_{\theta \in \Theta} \ln(\theta) \right] \rightarrow 1$ as $n \rightarrow \infty$. (where $\ln(\theta)$ denotes the log-likelihood).

Because if $\sup_{|\theta - \theta_0| > \varepsilon} \ln(\theta) < \sup_{\theta \in \Theta} \ln(\theta)$, then $|\hat{\theta}_n - \theta_0| \leq \varepsilon$ (since $\hat{\theta}_n$ maximizes the likelihood).

Show that:

$$\begin{aligned} P_{\theta_0} \left(\sup_{|\theta - \theta_0| > \varepsilon} (\ln(\theta) - \ln(\theta_0)) < 0 \right) &\xrightarrow{n \rightarrow \infty} 1 \\ \sup_{|\theta - \theta_0| > \varepsilon} \left[\frac{\ln(\theta) - \ln(\theta_0)}{n} + KL(\theta_0, \theta) - KL(\theta_0, \theta_0) \right] \end{aligned}$$

We have:

$$\frac{\ln(\theta) - \ln(\theta_0)}{n} = - \left[\frac{1}{n} \sum_{i=1}^n \log \left(\frac{f_{\theta_0}(x_i)}{f_\theta(x_i)} \right) \right]$$

By LLN $\rightarrow -\mathbb{E}_{\theta_0} \left[\log \left(\frac{f_{\theta_0}(X)}{f_\theta(X)} \right) \right] = -KL(\theta_0, \theta)$.

By assumption (i):

$$\sup_{|\theta - \theta_0| > \varepsilon} \left| \frac{\ln(\theta) - \ln(\theta_0)}{n} + KL(\theta_0, \theta) \right| \xrightarrow{P_{\theta_0}} 0$$

And using (ii):

$$\sup_{|\theta - \theta_0| > \varepsilon} -KL(\theta_0, \theta) = - \inf_{|\theta - \theta_0| > \varepsilon} KL(\theta_0, \theta) < -\delta$$

So:

$$\sup_{|\theta - \theta_0| > \varepsilon} \frac{\ln(\theta) - \ln(\theta_0)}{n} < -\delta + o(1)$$

Which is < 0 for n large enough. \square

V. Using Asymptotic Normality to Compute Confidence Regions

$\theta \in \Theta \subset \mathbb{R}$. Model $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} P_\theta$. $\hat{\theta}_n$ = MLE.

We know:

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow[n \rightarrow \infty]{\mathcal{L}_{P_{\theta_0}}} \mathcal{N}(0, I^{-1}(\theta_0))$$

We want $I_n = [\hat{\theta}_{n,1}, \hat{\theta}_{n,2}]$ such that:

$$P_{\theta_0}(\theta_0 \in I_n) \geq 1 - \alpha$$

Then I_n is a $(1 - \alpha)$ confidence interval for θ_0 .

I_n is an **asymptotic** $(1 - \alpha)$ confidence interval if $P_{\theta_0}(\theta_0 \in I_n) \xrightarrow{n \rightarrow \infty} 1 - \alpha$.

Exercise: Construct I_n .

VI. Delta method and confidence intervals

VI.1 Delta method

Remark

Reminder: If $\sqrt{n}(X_n - \mu) \xrightarrow{\mathcal{L}} \mathcal{N}(0, V_0)$, with $X_n \in \mathbb{R}^{d_1}$ and if $g : \mathbb{R}^{d_1} \rightarrow \mathbb{R}^{d_2}$ is \mathcal{C}^1 and s.t (sub-tangent/standard) ($d_1 \leq d_2$), $\nabla g(\mu) \in \mathbb{R}^{d_2 \times d_1}$ is of rank d_2

Then:

$$\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \nabla g(\mu)V_0\nabla g(\mu)^\top)$$

Proof. Case $d_1 = d_2 = 1$.

Taylor expansion:

$$g(X_n) = g(X_n - \mu + \mu) = g(\mu) + (X_n - \mu)g'(\bar{\mu}_n)$$

where $\bar{\mu}_n \in]X_n, \mu[$.

This implies:

$$\begin{aligned} \sqrt{n}(g(X_n) - g(\mu)) &= \sqrt{n}(X_n - \mu)g'(\bar{\mu}_n) \\ &= \sqrt{n}(X_n - \mu)(g'(\mu) + o_p(1)) \end{aligned}$$

We know that $\sqrt{n}(X_n - \mu)g'(\mu) \xrightarrow{\mathcal{L}} \mathcal{N}(0, g'(\mu)^2 V_0)$.

And $\sqrt{n}(X_n - \mu)(g'(\bar{\mu}_n) - g'(\mu)) \xrightarrow{\mathbb{P}} 0$. □

Remark

Reminders:

- $X_n = O_p(1) \iff \lim_{C \rightarrow \infty} \limsup_n \mathbb{P}(|X_n| > C) = 0$.
- $X_n = o_p(1) \iff X_n \xrightarrow{\mathbb{P}} 0$.
- If $X_n \xrightarrow{\mathcal{L}} Q$ (where Q is a probability distribution), then $X_n = O_p(1)$.

Application to the MLE

If $\hat{\theta}_n$ is the MLE and if $\hat{\eta}_n = g(\hat{\theta}_n)$ with $g \in \mathcal{C}^1$, $\hat{\theta}_n \in \mathbb{R}^d$, $g : \mathbb{R}^d \rightarrow \mathbb{R}^k$.

If $\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0, I^{-1}(\theta_0))$ with $\nabla g(\theta_0)$ of rank k .

Then:

$$\sqrt{n}(\hat{\eta}_n - g(\theta_0)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \nabla g(\theta_0)I^{-1}(\theta_0)\nabla g(\theta_0)^\top)$$

Example 11: Bernoulli

$X_1, \dots, X_n \stackrel{i.i.d}{\sim} \text{Ber}(p)$ under \mathbb{P}_{p_0} .

MLE: $\hat{p}_n = \bar{X}_n$ and $\sqrt{n}(\hat{p}_n - p_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0, p_0(1 - p_0))$.

Let $\eta = \log(\frac{p}{1-p})$ and $g(p) = \log(\frac{p}{1-p})$.

$g :]0, 1[\rightarrow \mathbb{R}$ is invertible. For $x \in \mathbb{R}$, $g^{-1}(x) = \frac{e^x}{1+e^x}$.

Derivative:

$$g'(p) = \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}$$

$\forall p_0 \in]0, 1[, g'(p_0) \neq 0$.

$$\sqrt{n}(\hat{\eta}_n - \eta) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{p_0(1-p_0)}\right)$$

Note: $\frac{1}{p_0(1-p_0)} = \frac{(1+e^{\eta_0})^2}{e^{\eta_0}}$.

Example 12: Normal Distribution

$X_1, \dots, X_n \stackrel{i.i.d}{\sim} \mathcal{N}(\mu, \sigma^2)$.
 $\theta = (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}^{+*}$.

MLE: $\hat{\theta}_n = (\bar{X}_n, S_n^2)$ where $S_n^2 = \frac{1}{n} \sum_{k=1}^n (X_k - \bar{X}_n)^2$.

Canonical parameters:

$$\begin{aligned} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} &= \frac{e^{-\frac{x^2}{2\sigma^2} - \frac{x\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2} - \frac{\log(\sigma^2)}{2}}}{\sqrt{2\pi}} \\ \implies \eta_1 &= \frac{\mu}{\sigma^2}, \quad \eta_2 = \frac{-1}{2\sigma^2} \end{aligned}$$

$$\implies \sqrt{n}(\hat{\eta}_n - \eta) \rightarrow \mathcal{N}(0, \tilde{V}_0)$$

Let $g(\eta) = \sigma^2$. We have $\sigma^2 = -\frac{1}{2\eta_2}$.

Gradient:

$$\nabla g(\eta) = \left(\frac{\partial g}{\partial \eta_1}, \frac{\partial g}{\partial \eta_2} \right) = \left(0, \frac{1}{2\eta_2^2} \right)$$

$$\implies \sqrt{n}(S_n^2 - \sigma^2) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, (0, \frac{1}{2\eta_2^2})\tilde{V}_0\left(\frac{0}{2\eta_2^2}\right)\right)$$

Where $\tilde{V}_0 = \frac{d^2 c}{d^2 \eta}(\eta_0)$ and $c(\eta_0) = \frac{\eta_1^2}{8\eta_2} + \frac{1}{2} \log(-\frac{1}{2\eta_2})$.

$$\implies c'' = \dots$$

VI.2 Confidence intervals

Definition 21: Confidence Region

A confidence region of level α for an estimator of $g(\theta)$ is any region $C_\alpha(X_1, \dots, X_n)$ verifying:

$$\forall \theta \in \Theta, \quad \mathbb{P}_\theta[g(\theta) \in C_\alpha(X_1, \dots, X_n)] \geq 1 - \alpha$$

We can use asymptotic normality of an estimator to construct asymptotic confidence regions.

Example 13: Construction using MLE

$X_1, \dots, X_n \stackrel{i.i.d}{\sim} \mathbb{P}_\theta, \theta \in \Theta \subset \mathbb{R}$.

If the MLE verifies:

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[\mathbb{P}_\theta]{\mathcal{L}} \mathcal{N}(0, I_\theta^{-1})$$

We look for:

$$C_\alpha(X_1, \dots, X_n) = [\hat{\theta}_{n,1}, \hat{\theta}_{n,2}]$$

s.t.

$$\mathbb{P}_\theta[\theta \in [\hat{\theta}_{n,1}, \hat{\theta}_{n,2}]] \xrightarrow[n \rightarrow \infty]{} 1 - \alpha$$

$$\hat{\theta}_{n,1} \leq \theta \leq \hat{\theta}_{n,2} \iff \sqrt{n}(\hat{\theta}_{n,1} - \hat{\theta}_n) \leq \sqrt{n}(\theta - \hat{\theta}_n) \leq \sqrt{n}(\hat{\theta}_{n,2} - \hat{\theta}_n)$$

But:

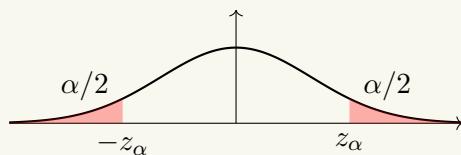
$$\begin{aligned} \sqrt{n}(\theta - \hat{\theta}_n) &\xrightarrow{\mathcal{L}} \mathcal{N}(0, I_\theta^{-1}) \\ &\implies \sqrt{n}I_\theta^{1/2}(\theta - \hat{\theta}_n) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1) \end{aligned}$$

Since $\hat{\theta}_n \xrightarrow{\mathbb{P}_\theta} \theta \implies I_{\hat{\theta}_n} \xrightarrow{\mathbb{P}_\theta} I_\theta$.

$$\implies \sqrt{n}I_{\hat{\theta}_n}^{1/2}(\theta - \hat{\theta}_n) \xrightarrow[\mathbb{P}_\theta]{\mathcal{L}} \mathcal{N}(0, 1)$$

If z_α is the $1 - \frac{\alpha}{2}$ quantile of a $\mathcal{N}(0, 1)$:

$$\iff \Phi(z_\alpha) = 1 - \frac{\alpha}{2}, \quad \Phi = \text{CDF of } \mathcal{N}(0, 1)$$



By symmetry: $\Phi(-z_\alpha) = \frac{\alpha}{2}$.

$$\implies \lim_{n \rightarrow \infty} \mathbb{P}_\theta \left(-z_\alpha \leq \sqrt{n}I_{\hat{\theta}_n}^{1/2}(\theta - \hat{\theta}_n) \leq z_\alpha \right) = 1 - \alpha$$

$$\iff \lim_{n \rightarrow \infty} \mathbb{P}_\theta \left(\frac{-z_\alpha}{I_{\hat{\theta}_n}^{1/2}} \leq \sqrt{n}(\theta - \hat{\theta}_n) \leq \frac{z_\alpha}{I_{\hat{\theta}_n}^{1/2}} \right) = 1 - \alpha$$

So if:

$$\hat{\theta}_{n,1} = \hat{\theta}_n - \frac{z_\alpha}{\sqrt{n}\sqrt{I_{\hat{\theta}_n}}} \quad ; \quad \hat{\theta}_{n,2} = \hat{\theta}_n + \frac{z_\alpha}{\sqrt{n}\sqrt{I_{\hat{\theta}_n}}}$$

Then:

$$\mathbb{P}_\theta(\hat{\theta}_{n,1} \leq \theta \leq \hat{\theta}_{n,2}) \xrightarrow{n \rightarrow \infty} 1 - \alpha$$

Chapter 5: Bayesian statistics

Model: $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathbb{P}_\theta$.

Estimator: $\hat{\theta}_n$.

Risk: $\mathcal{R}(\theta, \hat{\theta}_n) = \mathbb{E}[(\theta - \hat{\theta}_n)^2] = \text{Quadratic risk.}$

I. In Bayesian statistics

The parameter θ is considered **unknown** → It is modelled as a random variable.

Definition 22: Prior Distribution

We call **prior distribution** on θ , a probability distribution Π over Θ .

Model:

$X_1, \dots, X_n | \theta \stackrel{i.i.d.}{\sim} \mathbb{P}_\theta$: Conditional distribution of (X_1, \dots, X_n) given θ .

Prior Marginal Probability over Θ :

$\theta \sim \Pi$ (Probability on Θ).

⇒ This defines a **Joint distribution** on $(X_1, \dots, X_n, \theta)$.

Assume \mathbb{P}_θ is continuous, with density f_θ (with respect to Lebesgue measure).

Theorem 11: Bayes Theorem

The conditional distribution of θ given (X_1, \dots, X_n) has a density with respect to the measure ν (the reference measure of the prior):

$$\pi(\theta | X_1, \dots, X_n) = \frac{\pi(\theta) \prod_{i=1}^n f_\theta(x_i)}{\int_{\Theta} \pi(\theta) \prod_{i=1}^n f_\theta(x_i) d\nu(\theta)}$$

Where $\pi(\theta)$ is the density of the prior Π with respect to ν . This conditional density is called the **posterior distribution** (or posterior density).

Definition 23: Bayesian Model

In Bayesian statistics, the Bayesian model is defined by:

- i) The conditional law $[X | \theta] \sim \mathbb{P}_\theta$ with likelihood $L(\theta)$, for $\theta \in \Theta$.
- ii) The Prior distribution Π on Θ .

Then the inference is made by the **posterior distribution** defined as the conditional distribution of θ given X . If π is the density of Π (wrt a measure ν), then the posterior distribution has a density $\pi(\theta | X)$ (wrt ν) given by:

$$\pi(\theta | X) = \frac{L(\theta)\pi(\theta)}{\int_{\Theta} L(\theta)\pi(\theta)d\nu(\theta)}$$

where the normalization constant is:

$$\int_{\Theta} L(\theta)\pi(\theta)d\nu(\theta) = \begin{cases} \int L(\theta)\pi(\theta)d\nu(\theta) & \text{if } \Pi \text{ is continuous} \\ \sum_{\theta \in \Theta} L(\theta)\pi(\theta) & \text{if } \Pi \text{ is discrete} \end{cases}$$

Remark

- If \mathbb{P}_θ is continuous:
 $L(\theta) = f_\theta(x)$ = density evaluated at the observations when θ is the parameter.
- If \mathbb{P}_θ is discrete:
 $L(\theta) = f_\theta(x) = \mathbb{P}_\theta(X = x)$.

The posterior density becomes:

$$\pi(\theta|X = x) = \frac{f_\theta(x)\pi(\theta)}{\int_{\Theta} f_\theta(x)\pi(\theta)d\nu(\theta)}$$

Example 14: Poisson - Gamma

Let $X = (X_1, \dots, X_n)$ where $X_i \stackrel{i.i.d.}{\sim} \mathcal{P}(\theta)$, with $\theta > 0$.

Prior: $\theta \sim \Gamma(a, b)$. Here Π is a Gamma distribution $\Gamma(a, b)$.

The density of the prior is $\pi(\theta) \propto \theta^{a-1} e^{-b\theta}$.

Likelihood:

$$f_\theta(x) = \prod_{i=1}^n f_\theta(x_i)$$

Posterior density:

$$\Rightarrow \pi(\theta|X_1, \dots, X_n) = \frac{(\prod_{i=1}^n f_\theta(x_i)) \theta^{a-1} e^{-b\theta}}{\int_{\Theta} (\prod_{i=1}^n f_\theta(x_i)) \theta^{a-1} e^{-b\theta} d\theta}$$

Looking at the numerator (and ignoring constants independent of θ):

$$\begin{aligned} \pi(\theta|X_1, \dots, X_n) &\propto e^{-n\theta} \prod_{i=1}^n \frac{\theta^{x_i}}{x_i!} \times \theta^{a-1} e^{-b\theta} \\ &\propto e^{-(n+b)\theta} \theta^{a+\sum_{i=1}^n x_i - 1} \end{aligned}$$

(Recall: for Poisson, $f_\theta(x_i) = e^{-\theta} \frac{\theta^{x_i}}{x_i!}$).

Since $\pi(\cdot|X)$ is a density, if $\frac{f(\theta)}{g(\theta)}$ is a constant (wrt θ), it implies $\exists c \neq 0, f(\theta) = cg(\theta)$.
Thus $\exists C > 0$ such that:

$$\pi(\theta|X_1, \dots, X_n) = Ce^{-(b+n)\theta} \theta^{\sum_{i=1}^n x_i + a - 1}$$

We recognize the kernel of a Gamma distribution $\Gamma(\sum_{i=1}^n x_i + a, b + n)$.

$$\Rightarrow \pi(\theta|X_1, \dots, X_n) = \frac{e^{-(b+n)\theta} \theta^{\sum_{i=1}^n x_i + a - 1} (b + n)^{a + \sum x_i}}{\Gamma(a + \sum_{i=1}^n x_i)}$$

Conclusion: The **Posterior distribution** is $\Gamma(\sum x_i + a, b + n)$.

Note regarding the constant calculation: If $g(\theta) = \theta^{a-1} e^{-b\theta}$, to make it a density we multiply by c :

$$\int_{\mathbb{R}^+} c \cdot g(\theta) d\theta = 1 \iff c = \frac{b^a}{\Gamma(a)}$$

Example 15: Normal - Normal

Bayesian model: $X_1, \dots, X_n | \mu \stackrel{i.i.d.}{\sim} \mathcal{N}(\mu, 1)$, with $\mu \in \mathbb{R}$.

Prior: $\mu \sim \mathcal{N}(a, b^2)$, with $a \in \mathbb{R}, b > 0$.

What is the Posterior distribution? It is continuous with density $\pi(\mu|X)$.

$$\begin{aligned}\pi(\mu|X_1, \dots, X_n) &\propto \left(\prod_{i=1}^n \frac{e^{-\frac{(x_i-\mu)^2}{2}}}{\sqrt{2\pi}} \right) \frac{e^{-\frac{(\mu-a)^2}{2b^2}}}{\sqrt{2\pi}b} \\ &\propto \exp \left(-\frac{n}{2} \{ \mu^2 + \bar{X}_n^2 - 2\mu\bar{X}_n \} - \frac{1}{2b^2} \{ \mu^2 + a^2 - 2a\mu \} \right)\end{aligned}$$

We used the decomposition of the sum of squares:

$$\begin{aligned}\sum_{i=1}^n (X_i - \mu)^2 &= \sum_{i=1}^n [(X_i - \bar{X}_n)^2 + (\bar{X}_n - \mu)^2 + 2(\bar{X}_n - \mu)(X_i - \bar{X}_n)] \\ &= n(\bar{X}_n - \mu)^2 + \sum_{i=1}^n (X_i - \bar{X}_n)^2\end{aligned}$$

(Note: The cross-term sums to zero).

Regrouping terms in μ :

$$\pi(\mu|X_1, \dots, X_n) \propto \exp \left(-\frac{1}{2} \left[\mu^2 \left(n + \frac{1}{b^2} \right) - 2\mu \left(n\bar{X}_n + \frac{a}{b^2} \right) + \text{const} \right] \right)$$

Completing the square:

$$\pi(\mu|X_1, \dots, X_n) \propto e^{-\frac{1}{2} \left(n + \frac{1}{b^2} \right) \left(\mu - \frac{\frac{a}{b^2} + n\bar{X}_n}{n + \frac{1}{b^2}} \right)^2}$$

⇒ The Posterior distribution is:

$$\mathcal{N} \left(\frac{\frac{a}{b^2} + n\bar{X}_n}{n + \frac{1}{b^2}}, \frac{1}{n + \frac{1}{b^2}} \right)$$

Definition 24: Marginal Likelihood

In a Bayesian model $X|\theta \sim f_\theta(x)$, $\theta \in \Theta$ (Likelihood) and prior $\theta \sim \Pi$ with density π wrt ν .

The quantity:

$$m(x) = \int_{\Theta} f_\theta(x) \pi(\theta) d\nu(\theta)$$

is called the **marginal likelihood**.

II. Bayesian decision theory: Risks

II.1 Posterior and integrated risks

Recall: Quadratic Risk:

$$\mathcal{R}(\theta, \delta) = \begin{cases} \int (\theta - \delta(x))^2 f_\theta(x) dx \\ \sum_x (\theta - \delta(x))^2 \mathbb{P}_\theta(X = x) \end{cases}$$

Definition 25: Loss Function

A loss function is a function:

$$l : \Theta \times \mathcal{D} \rightarrow \mathbb{R}^+$$

where \mathcal{D} is the set of decisions (i.e., estimators).

Example 16: Examples of Loss Functions

1. If $\mathcal{D} = \Theta$.
 - a) The quadratic loss $l(\theta, \delta) = (\theta - \delta)^2$.
 - b) The L_1 loss $l(\theta, \delta) = |\theta - \delta|$.

2. $\mathcal{D} = \{0, 1\}, \theta \in [0, 1]$.
Aim is testing if $\theta > \frac{1}{2}$ (choosing 1) or $\theta \leq \frac{1}{2}$ (choosing 0).

0-1 loss function defined by:

$$l(\theta, \delta) = \begin{cases} 1 & \text{if } \theta < \frac{1}{2} \text{ and } \delta = 1 \text{ (wrong decision)} \\ & \text{if } \theta > \frac{1}{2} \text{ and } \delta = 0 \text{ (wrong decision)} \\ 0 & \text{else (correct decision)} \end{cases}$$

Definition 26: Risks

In a Bayesian model $X|\theta \sim f_\theta(x), \theta \sim \Pi$ (with density π) and a loss function $l : \Theta \times \mathcal{D} \rightarrow \mathbb{R}^+$.

- a) We call the **posterior risk**:

$$l(\Pi, \delta | X) = \int_{\Theta} l(\theta, \delta) \pi(\theta | X_1, \dots, X_n) d\nu(\theta)$$

- b) We call the **integrated risk**:

$$r(\Pi, \delta) = \begin{cases} \int_x l(\Pi, \delta | X = x) m(x) dx & \text{if } X \text{ is continuous} \\ \sum_{x \in \mathcal{X}} l(\Pi, \delta | X = x) m(x) & \text{if } X \text{ is discrete} \end{cases}$$

Example 17: Calculation of Posterior Risk

1) Bayesian model, $X_1, \dots, X_n | \theta \sim \mathcal{P}(\theta), \theta > 0$.

Prior: $\theta \sim \Gamma(a, b)$.

Loss function: Quadratic. $\mathcal{D} = \Theta = \mathbb{R}_+$.

$$\forall \theta > 0, \delta > 0, \quad l(\theta, \delta) = (\theta - \delta)^2.$$

Posterior risk:

Posterior distribution: $\Gamma(a + \sum_{i=1}^n X_i, b + n)$. Its density is:

$$\pi(\theta | X_1, \dots, X_n) = \frac{(b+n)^{n\bar{X}_n+a} e^{-(b+n)\theta} \theta^{n\bar{X}_n+a-1}}{\Gamma(a+n\bar{X}_n)}$$

With $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$.

$\forall \delta > 0$:

$$\begin{aligned} l(\Pi, \delta | X_1, \dots, X_n) &= \int_0^\infty (\theta - \delta)^2 \pi(\theta | X_1, \dots, X_n) d\theta \\ &= \int_0^\infty \theta^2 \pi(\theta | X_1, \dots, X_n) d\theta + \delta^2 - 2\delta \int_0^\infty \theta \pi(\theta | X_1, \dots, X_n) d\theta \end{aligned}$$

Using the moments of the Gamma distribution ($E[Y] = \alpha/\beta, Var(Y) = \alpha/\beta^2, E[Y^2] = Var(Y) + E[Y]^2$):

$$\implies l(\Pi, \delta | X_1, \dots, X_n) = \frac{a'(a'+1)}{b'^2} + \delta^2 - 2\delta \frac{a'}{b'}$$

where $a' = a + n\bar{X}_n$ and $b' = b + n$.

→ This can be minimized in δ .

$$\delta(X_1, \dots, X_n) = \text{minimizer of } l(\Pi, \delta | X_1, \dots, X_n)$$

Differentiating with respect to δ and setting to 0, we can show that:

$$\delta(X_1, \dots, X_n) = \frac{a'}{b'} = \frac{a + n\bar{X}_n}{b + n}$$

(which is the Posterior Mean).

Example 18: Hypothesis Testing (0-1 Loss)

Let $\ell : \Theta \times \mathcal{D} \rightarrow \mathbb{R}_+$ be the loss function.

- Posterior risk:

$$\ell(\Pi, \delta | x_1, \dots, x_n) = \int_\Theta \ell(\theta, \delta) \pi(\theta | x_1, \dots, x_n) d\theta$$

Consider the model: $X_1, \dots, X_n \sim \mathcal{P}(\theta)$ with prior $\theta \sim \Gamma(a, b)$.

The posterior density is (as seen previously):

$$\pi(\theta | X_1, \dots, X_n) = \frac{\theta^{a+n\bar{X}_n-1} e^{-(b+n)\theta} (b+n)^{a+n\bar{X}_n}}{\Gamma(a+n\bar{X}_n)}$$

We set the hypotheses:

- $H_0 = \{\theta < 1\}$
- $H_1 = \{\theta \geq 1\}$

Decision rule δ :

$$\begin{cases} \delta = 1 & \text{if we choose } H_0 \\ \delta = 0 & \text{if we choose } H_1 \end{cases}$$

Let us use the 0-1 loss:

$$\ell(\theta, \delta) = \mathbb{1}_{\delta=1} \mathbb{1}_{\theta \geq 1} + \mathbb{1}_{\delta=0} \mathbb{1}_{\theta < 1}$$

The posterior risk becomes:

$$\begin{aligned} \ell(\Pi, \delta | x_1, \dots, x_n) &= \int_0^{+\infty} [\mathbb{1}_{\delta=1} \mathbb{1}_{\theta \geq 1} + \mathbb{1}_{\delta=0} \mathbb{1}_{\theta < 1}] \pi(\theta | x_1, \dots, x_n) d\theta \\ &= \underbrace{\mathbb{1}_{\delta=1} \int_1^{\infty} \pi(\theta | x_1, \dots, x_n) d\theta}_{\Pi(\theta \geq 1 | X)} + \underbrace{\mathbb{1}_{\delta=0} \int_0^1 \pi(\theta | x_1, \dots, x_n) d\theta}_{\Pi(\theta < 1 | X)} \\ &= \mathbb{1}_{\delta=1} \Pi(\theta \geq 1 | x_1, \dots, x_n) + \mathbb{1}_{\delta=0} \Pi(\theta < 1 | x_1, \dots, x_n) \end{aligned}$$

To minimize this risk, we choose $\delta = 1$ (choosing H_0) if $\Pi(\theta \geq 1 | X) < \Pi(\theta < 1 | X)$, i.e., if the posterior probability of H_0 is greater than 0.5.

Definition 27: Bayesian Estimator

Consider a Bayesian model $X | \theta \sim P_\theta$, $\theta \in \Theta$, with prior $\theta \sim \Pi$.

If $\ell : \Theta \times \mathcal{D} \rightarrow \mathbb{R}_+$ is a loss function, we define the **Bayes estimators** δ^Π as (when they exist) the minimizers of the posterior risk $\ell(\Pi, \delta | X)$ in δ .

Proposition 5: Property

Bayesian estimators also minimize the integrated Risk.

Proof. Consider the Bayesian Model $P_\theta(x)$, $\theta \in \Theta$ and Prior Π . Let $\ell(\theta, \delta)$ be the loss function with $\delta \in \mathcal{D}$.

The integrated risk is defined as $r(\Pi, \delta) = \int_{\mathcal{X}} \ell(\Pi, \delta | X = x) m(x) dx$.

Since for all x , the Bayes estimator $\delta^\Pi(x)$ verifies by definition:

$$\ell(\Pi, \delta^\Pi(x) | X = x) \leq \ell(\Pi, \delta | X = x) \quad \forall \delta \in \mathcal{D}$$

(Inequality on posterior risk).

Integrating this inequality with respect to the marginal density $m(x)$ (which is non-negative):

$$\begin{aligned} \implies \int_{\mathcal{X}} \ell(\Pi, \delta^\Pi(x) | X = x) m(x) dx &\leq \int_{\mathcal{X}} \ell(\Pi, \delta(x) | X = x) m(x) dx \\ \implies r(\Pi, \delta^\Pi) &\leq r(\Pi, \delta) \end{aligned}$$

This holds for any estimator δ . Thus, δ^Π minimizes the integrated risk. \square

Why is it interesting?

Theorem 12: Statistical Model

Let $X|\theta \sim P_\theta$, $\theta \in \Theta$ be the Model, and Prior $\theta \sim \Pi$ with density π .

- Loss function: $\ell : \Theta \times \mathcal{D} \rightarrow \mathbb{R}_+$
- The **Frequentist risk** is defined as the function $R : \Theta \times \mathcal{D} \rightarrow \mathbb{R}_+$ given by $R(\theta, \delta) = \mathbb{E}_\theta[\ell(\theta, \delta(X))]$.

$$= \begin{cases} \int_{\mathcal{X}} \ell(\theta, \delta(x)) f(x|\theta) dx & \text{if } X \text{ continuous} \\ \sum_{x \in \mathcal{X}} \ell(\theta, \delta(x)) f(x|\theta) & \text{if } X \text{ discrete} \end{cases}$$

Then:

$$\int_{\Theta} R(\theta, \delta) \pi(\theta) d\theta = r(\Pi, \delta)$$

The integrated risk is the average of the frequentist risk over the prior.

Example 19: Quadratic Risk

If $\ell(\theta, \delta) = (\theta - \delta)^2$, then $R(\theta, \delta) = \mathbb{E}_\theta[(\theta - \delta(X))^2]$ is the classical quadratic risk (MSE).

Proof of the Theorem. Recall that:

$$r(\Pi, \delta) = \int_{\mathcal{X}} \ell(\Pi, \delta|X=x) m(x) dx$$

Where the marginal density is $m(x) = \int_{\Theta} f(x|\theta) \pi(\theta) d\theta$.

And the posterior risk is $\ell(\Pi, \delta|X=x) = \int_{\Theta} \ell(\theta, \delta) \pi(\theta|X=x) d\theta$.

Substituting the expression of the posterior density $\pi(\theta|X=x) = \frac{f(x|\theta)\pi(\theta)}{m(x)}$:

$$\begin{aligned} r(\Pi, \delta) &= \int_{\mathcal{X}} \left[\int_{\Theta} \ell(\theta, \delta(x)) \underbrace{\pi(\theta|X=x)m(x)}_{=f(x|\theta)\pi(\theta)} d\theta \right] dx \\ &= \int_{\mathcal{X}} \int_{\Theta} \ell(\theta, \delta(x)) f(x|\theta) \pi(\theta) d\theta dx \end{aligned}$$

By Fubini's theorem (assuming integrability), we swap the integrals:

$$\begin{aligned} &= \int_{\Theta} \left[\int_{\mathcal{X}} \ell(\theta, \delta(x)) f(x|\theta) dx \right] \pi(\theta) d\theta \\ &= \int_{\Theta} R(\theta, \delta) \pi(\theta) d\theta \end{aligned}$$

□

Example 20: Poisson - Gamma Calculation

$X_1, \dots, X_n \sim \mathcal{P}(\theta)$. Prior density $\pi(\theta) = \frac{\theta^{a-1} e^{-b\theta} b^a}{\Gamma(a)}$, $\theta > 0$.

Case $a = 1, b = 1$: $\pi(\theta) = e^{-\theta} \mathbb{1}_{\theta>0}$.

The posterior density is:

$$\pi(\theta|X_1, \dots, X_n) = \frac{\theta^{\sum X_i} e^{-(n+1)\theta} (n+1)^{n\bar{X}_n + 1}}{\Gamma(n\bar{X}_n + 1)}$$

This comes from the general formula:

$$\pi(\theta|X) = \frac{\prod_{i=1}^n f(X_i|\theta) \pi(\theta)}{\int_{\Theta} \prod_{i=1}^n f(X_i|\theta) \pi(\theta) d\theta}$$

III. Computation of Bayesian estimators

- Model: $X|\theta \sim P_\theta, \theta \in \Theta$
- Prior: $\theta \sim \Pi$
- Loss function: $\ell : \Theta \times \mathcal{D} \rightarrow \mathbb{R}_+$

III.1 Quadratic loss

$$\ell(\theta, \delta) = (\theta - \delta)^2, \quad \mathcal{D} = \Theta$$

The posterior risk is:

$$\ell(\Pi, \delta|X = x) = \int_{\Theta} (\theta - \delta)^2 \pi(\theta|X = x) d\theta$$

This function is convex in δ . To find the minimum, we differentiate with respect to δ :

$$\begin{aligned} \frac{\partial}{\partial \delta} \ell(\Pi, \delta|X = x) &= \int_{\Theta} \frac{\partial}{\partial \delta} (\theta - \delta)^2 \pi(\theta|X = x) d\theta \\ &= \int_{\Theta} -2(\theta - \delta) \pi(\theta|X = x) d\theta \\ &= 2\delta \underbrace{\int_{\Theta} \pi(\theta|X = x) d\theta}_{=1} - 2 \int_{\Theta} \theta \pi(\theta|X = x) d\theta \\ &= 2\delta - 2\mathbb{E}^{\Pi}[\theta|X = x] \end{aligned}$$

Setting the derivative to 0:

$$2\delta - 2\mathbb{E}^{\Pi}[\theta|X = x] = 0 \iff \delta(x) = \mathbb{E}^{\Pi}[\theta|X = x]$$

Proposition 6: Posterior Mean

The Bayesian estimator associated to the quadratic loss is the **posterior mean**.

III.2 L_1 loss

$$\ell(\theta, \delta) = |\theta - \delta|, \quad \mathcal{D} = \Theta = \mathbb{R}$$

Posterior risk:

$$\ell(\Pi, \delta | X = x) = \int_{\mathbb{R}} |\theta - \delta| \pi(\theta | X = x) d\theta$$

This function is convex in δ . We can split the integral:

$$\ell(\Pi, \delta | X = x) = \int_{-\infty}^{\delta} (\delta - \theta) \pi(\theta | X = x) d\theta + \int_{\delta}^{+\infty} (\theta - \delta) \pi(\theta | X = x) d\theta$$

Differentiating with respect to δ (using Leibniz integral rule):

$$\begin{aligned} \frac{\partial}{\partial \delta} \ell(\Pi, \delta | X = x) &= \int_{-\infty}^{\delta} \pi(\theta | X = x) d\theta + [\delta - \delta] \pi(\delta | x) \\ &\quad - \int_{\delta}^{+\infty} \pi(\theta | X = x) d\theta - [\delta - \delta] \pi(\delta | x) \\ &= \int_{-\infty}^{\delta} \pi(\theta | X = x) d\theta - \int_{\delta}^{+\infty} \pi(\theta | X = x) d\theta \\ &= \Pi(\theta \leq \delta | X = x) - \Pi(\theta > \delta | X = x) \end{aligned}$$

Setting to 0:

$$\Pi(\theta \leq \delta | X = x) = \Pi(\theta > \delta | X = x)$$

Since the sum of these probabilities is 1, this implies:

$$\Pi(\theta \leq \delta | X = x) = \frac{1}{2}$$

Proposition 7: Posterior Median

The Bayesian estimator associated to the L_1 loss is the **median of the posterior distribution**.

IV. Confidence intervals in Bayesian analysis: credible regions

Let $\alpha \in [0, 1]$ be the risk level, and Π be the prior distribution on $\Theta \subset \mathbb{R}^d$.

Definition 28: Credible Set

An α -credible set is a random set $C(\alpha, X_1, \dots, X_n)$ such that:

$$\Pi(\theta \in C(\alpha, X_1, \dots, X_n) | X_1, \dots, X_n) = \int_{C(\alpha, X_1, \dots, X_n)} \pi(\theta | X_1, \dots, X_n) d\theta = 1 - \alpha$$

In particular, if $\Theta = \mathbb{R}$, an α -credible interval is denoted $[l_\alpha(X_1, \dots, X_n), r_\alpha(X_1, \dots, X_n)]$. l_α and r_α solve (satisfy):

$$\int_{-\infty}^{l_\alpha} \pi(\theta | X_1, \dots, X_n) d\theta = \int_{r_\alpha}^{+\infty} \pi(\theta | X_1, \dots, X_n) d\theta = \frac{\alpha}{2}$$

Remark

Careful: A Bayesian credible set \neq Frequentist confidence set interval.

In Bayesian analysis, we have:

$$\Pi(\theta \in (l_\alpha, r_\alpha) | X_1, \dots, X_n) \geq 1 - \alpha$$

but we do not have:

$$\mathbb{P}_\theta(\theta \in (l_\alpha, r_\alpha)) \geq 1 - \alpha$$

However, if $f_\theta(x)$ is nice enough (regularity conditions), then credible sets can be turned into asymptotically optimal confidence intervals (as $n \rightarrow \infty$).

V. Choosing priors

V.1 Conjugate priors

Let $f(x|\theta) \in \mathcal{F}$ be a family of models (Poisson, Gamma, Normal, Bernoulli...).
Let $\pi(\theta) \in \tilde{\mathcal{F}}$ be another family of distributions.

Definition 29: Conjugate Priors

The families $f(x|\theta) \in \mathcal{F}$ and $\pi(\theta) \in \tilde{\mathcal{F}}$ are conjugate if:

$$\pi(\theta | X_1, \dots, X_n) \in \tilde{\mathcal{F}}$$

(The posterior distribution belongs to the same family as the prior).

Example 21: Uniform / Pareto

Consider the model:

$$f(x|\theta) = \frac{1}{\theta} \mathbb{1}_{[0,\theta]}(x)$$

And the prior (Pareto distribution):

$$\pi(\theta) = \frac{C_{\alpha, \theta_{min}}}{\theta^{\alpha+1}} \mathbb{1}_{\{\theta > \theta_{min}\}}$$

with $\alpha > 0$ and $\theta_{min} > 0$.

Posterior derivation:

$$\begin{aligned} \pi(\theta | X_1, \dots, X_n) &\propto L_n(\theta) \pi(\theta) \\ &= \prod_{i=1}^n \frac{1}{\theta} \mathbb{1}_{[0,\theta]}(X_i) \times \frac{1}{\theta^{\alpha+1}} \mathbb{1}_{\{\theta > \theta_{min}\}} \\ &= \frac{1}{\theta^n} \mathbb{1}_{\{\theta \geq \max(X_i)\}} \times \frac{1}{\theta^{\alpha+1}} \mathbb{1}_{\{\theta > \theta_{min}\}} \\ &= \frac{1}{\theta^{n+\alpha+1}} \times \mathbb{1}_{\{\theta > \max(\max(X_i), \theta_{min})\}} \end{aligned}$$

This is a Pareto distribution: $\text{Pareto}(\alpha + n, \theta_{min} \vee \max_{i=1 \dots n}(X_i))$.

V.2 Flat priors

The goal is to choose a prior that is non-informative.

- e.g. If $\Theta = [a, b]$, $\pi(\theta) = \frac{1}{b-a} \mathbb{1}_{[a,b]}(\theta)$.
- What if $\Theta = \mathbb{R}$? We use an **Improper prior**.

Take $\pi(\theta) \propto c$, then $\int_{\mathbb{R}} \pi(\theta) d\theta = \infty$.

However, this is allowed as long as the posterior is proper:

$$\int_{\Theta} L_n(\theta' | X_1, \dots, X_n) \pi(\theta') d\theta' < +\infty \quad \mathbb{P}_{\theta} - a.s.$$

Example 22: Normal model with flat prior

Let $f(x|\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2}$.

Take $\pi(\theta) = 1$ on \mathbb{R} ! (Improper).

Posterior derivation:

$$\begin{aligned} \pi(\theta | X_1, \dots, X_n) &\propto L_n(\theta | X_1, \dots, X_n) \times 1 \\ &\propto e^{-\frac{1}{2} \sum_{i=1}^n (X_i - \theta)^2} \\ &= e^{-\frac{1}{2} \sum_{i=1}^n (\theta - X_i)^2} \end{aligned}$$

Let us expand the sum in the exponential:

$$\begin{aligned} \sum_{i=1}^n (\theta - X_i)^2 &= n\theta^2 - 2 \sum_{i=1}^n \theta X_i + \sum_{i=1}^n X_i^2 \\ &= n(\theta^2 - 2\theta \bar{X}_n + \dots) \\ &= n(\theta - \bar{X}_n)^2 + \text{const} \end{aligned}$$

(The terms not depending on θ are absorbed in the proportionality constant).

Thus:

$$\pi(\theta | X_1, \dots, X_n) \propto e^{-\frac{1}{2} \frac{1}{n} (\theta - \bar{X}_n)^2}$$

We recognize a Normal distribution:

$$\mathcal{N}\left(\bar{X}_n, \frac{1}{n}\right)$$

[Image of standard normal distribution curve]

V.3 Jeffrey's prior

Jeffrey's idea: Choose a prior proportional to the square root of the Fisher information.

$$\pi(\theta) \propto I(\theta)^{\frac{1}{2}}$$

Where $I(\theta)$ is the Fisher information:

$$I(\theta) = \mathbb{E}_{\theta} \left[\left(\frac{\partial}{\partial \theta} \log f_{\theta}(X_1) \right)^2 \right]$$

Result:

If $\pi(\theta) \propto \sqrt{I(\theta)}$, then under any reparameterisation $\eta = g(\theta)$, the corresponding prior $\tilde{\pi}(\eta) \propto \sqrt{\tilde{I}(\eta)}$ is consistent, where \tilde{I} is the Fisher information for the model parametrized by η .