

# Lecture Notes: Statistical Modelling

## L3 MIDO

Course by Prof. Judith Rousseau

Transcribed by Ayda Atmani

Compiled by Samuel Lelouch and Arris Bouzouane with Gemini

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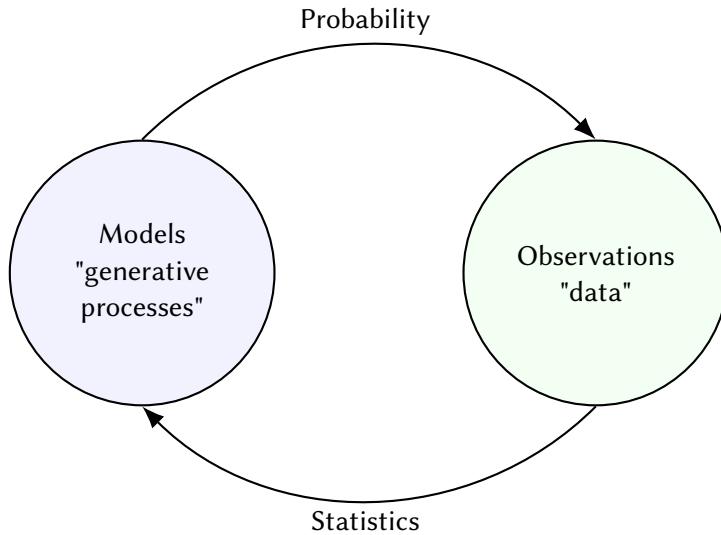
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# Chapter 1: Introduction to Statistical Modelling

## Introduction Schema



## I. Data

### 1) $n = \# \text{ L3 MIDO students}$

- Were students in Dauphine in L2?
- Data  $\in \{0, 1\}^n$ :
  - 1: in L2 in Dauphine.
  - 0: not in L2 in Dauphine.
- $x = (0, 0, 1, 1, 0, \dots)$
- $\downarrow \text{sum}(x)$

$\implies x$  is a distribution of  $n$  independent experiences of Bernoulli of parameter  $p$ .

$$x \sim \text{Ber}(p)^{\otimes n} \quad (\text{i.i.d.})$$

$$\implies \text{sum}(x) \sim \text{Bin}(n, p)$$

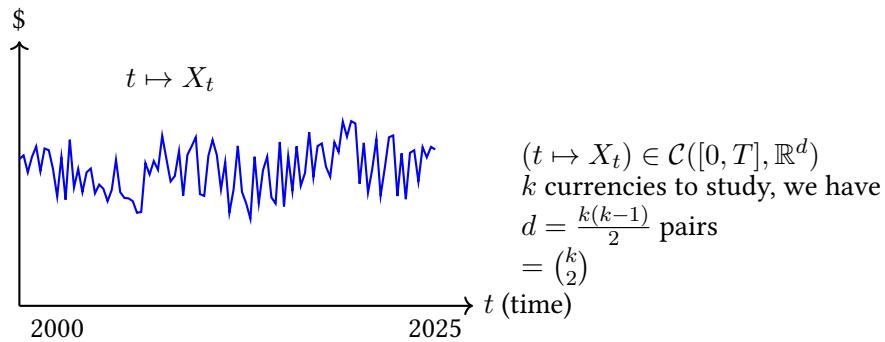
### 2) INSEE (economy)

**Example:** Employment rate between 2005 & 2025.

Male ♂	2005	2006	...	2025		Female ♀	2005	...
15-24 yo	$\alpha$	$\alpha$	$\alpha$	$\alpha$		15-24	$\alpha$	$\alpha$
25-49 yo	$\alpha$	56%	$\alpha$	$\alpha$		25-49	$\alpha$	$\alpha$
50 +	$\alpha$		$\alpha$	$\alpha$		50 +	$\alpha$	$\alpha$

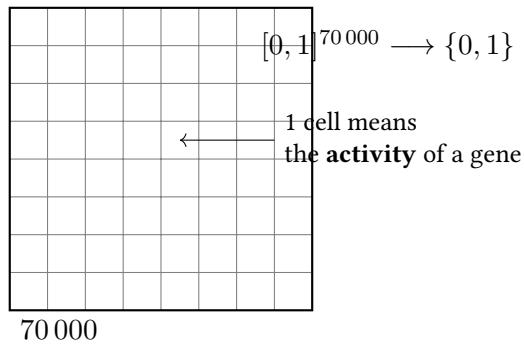
$\implies$  **Tensor:** multiple entries array.

### 3) Financial data



⇒ Brownian motion. ⇒ Diffusion processes.

### 4) Microarrays



- Compute a big space into a smaller one: the cells of an individual translated to *healthy* or *not*.
- $[0, 1]$  measures the activity of one cell. One person has 70 000 cells.

## II. Statistical Modelling

### 1) Stat models with words

To each set of data, we must associate a scientific question/objective.

To answer that question, we must have at our disposition:

- A methodology.
- A quantitative mathematical (probabilistic) model that accounts for the properties of the data.
- Well suited methods (mathematical) that combine... and...

### 2) Abstract (simple) example

We toss a coin 18 times and observe ( $n = 18, H = 0, T = 1$ ).

$$(0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0)$$

≡ Data (raw).

- **Statistical model:** We observe  $n = 18$  random variables  $X_i$ ; that are independent and that have the same distribution.

- $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = \theta$ .
- $\text{Ber}(\theta)$  where  $\theta \in \Theta = [0, 1]$ .
- $\theta$  is the **unknown parameter**.

**Questions:**

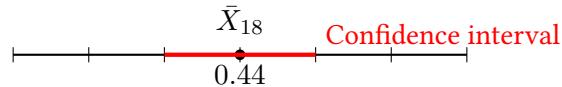
1) What is a good (the best) estimation of  $\theta$ ?

- A good (?) estimator:

$$\underbrace{\bar{X}_n}_{\substack{\text{notation} \\ \text{of the mean}}} = \frac{1}{n} \sum_{i=1}^n X_i$$

$$= \frac{1}{18} \sum_{i=1}^{18} X_i = \frac{8}{18} = 0.44$$

- What is my accuracy of estimation?



$\implies$  We want:

1. Small length.
2. Good coverage.

2) Is the coin fair?

For instance, we compare  $\bar{X}_{18}$  to 0.5. If  $|\bar{X}_{18} - 0.5|$  is small, we accept the idea that the coin is fair, otherwise we reject that hypothesis.

# Chapter 2: Models

## I. What is a statistical model?

### Definition 1: Statistical Model

Let  $X = (X_1, \dots, X_n) \in \mathcal{X}^n$  be a vector of  $n$  random variables, where each variable  $X_i \in \mathcal{X}$  and  $\mathcal{X}$  is a measurable space.

A **model** for  $X$  is a set  $\mathcal{P}$  of probability distributions on  $\mathcal{X}^n$ .

Statistical inference will consist in estimating  $P$  (the distribution of  $X$ ) or  $F(P)$ , with  $P \in \mathcal{P}$ .

### Examples

- i) Let  $X_1, \dots, X_n$  be i.i.d.  $\mathcal{N}(\mu, \sigma^2)$ .

**Notation:**  $X_i \sim_{i.i.d.} \mathcal{N}(\mu, \sigma^2)$  (Gaussian distribution with parameters  $\mu$  and  $\sigma^2$ ). The density is given by:  $x \mapsto \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ .

Here,  $\mathcal{X} = \mathbb{R}$ . The model is defined as:

$$\mathcal{P} = \{\mathcal{N}(\mu, \sigma^2)^{\otimes n}, (\mu, \sigma^2) \in \mathbb{R} \times (0, +\infty)\}$$

### Remark

**Notations:** If  $(X_1, X_2)$  is a random vector of  $\mathbb{R}^2$ , with  $X_1 \sim P_1$  and  $X_2 \sim P_2$ .

(By the way:  $X \sim P$  means that  $P$  is the distribution of  $X \in \mathbb{R}$ , meaning:  $\forall A \in \mathcal{B}(\mathbb{R})$ ,  $P(A) = \mathbb{P}(X \in A) = \mathbb{P}(X^{-1}(A))$ .)

$$(\Omega, \mathcal{F}, \mathbb{P}) \xrightarrow{X} (\mathbb{R}, \mathcal{B}(\mathbb{R}), P)$$

$P$  is defined via:  $\forall A \in \mathcal{B}(\mathbb{R})$ ,  $P(A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}(\{\omega \in \Omega, X(\omega) \in A\})$ .

If  $X_1 \sim P_1$ ,  $X_2 \sim P_2$  and  $X_1 \perp\!\!\!\perp X_2$  ( $X_1$  and  $X_2$  are independent):

$$\mathbb{P}(X_1 \in A, X_2 \in B) = \mathbb{P}(X_1 \in A) \cdot \mathbb{P}(X_2 \in B)$$

$\forall A, B \in \mathcal{B}(\mathbb{R})$ , the distribution of  $(X_1, X_2)$  is defined on  $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$  and is denoted by  $P_1 \otimes P_2$ .

**Computation formula:**

$$P_1 \otimes P_2(A \times B) = P_1(A) \cdot P_2(B) = \mathbb{P}(X_1 \in A) \cdot \mathbb{P}(X_2 \in B)$$

In our example, saying  $(X_1, \dots, X_n) \sim \mathcal{N}(\mu, \sigma^2)^{\otimes n}$  exactly means:

$$\begin{aligned} \mathbb{P}(X_1 \in A_1, \dots, X_n \in A_n) &= \prod_{i=1}^n \mathbb{P}(X_i \in A_i) \\ &= \prod_{i=1}^n \int_{A_i} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} dx_i \\ &= \int_{A_1} \dots \int_{A_n} \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2\right) dx_1 \dots dx_n \end{aligned}$$

- ii)  $X_i \in \{0, 1\}$ ,  $\forall i = 1, \dots, n$ , independently with same distribution  $\text{Ber}(p)$ ,  $p \in [0, 1]$ .

$$\mathcal{P} = \{\text{Ber}(p)^{\otimes n}, p \in [0, 1]\}, \quad \mathcal{X}^n = \{0, 1\}^n$$

## Model Comparison and Types

Let us consider models of the form  $\mathcal{P} = \{P^{\otimes n}, P \in \mathcal{P}_0\}$ .

- Example (1):  $\mathcal{P}_0 = \{\mathcal{N}(\mu, \sigma^2), (\mu, \sigma^2) \in \mathbb{R} \times (0, +\infty)\}$  (Parametric).
- Example (2):  $\mathcal{P}_0 = \{\text{Ber}(p), p \in [0, 1]\}$ .
- Example (3):  $\mathcal{P}_0 = \{\text{All distributions on } (\mathbb{R}, \mathcal{B}(\mathbb{R}))\}$  (Huge model).

**Which is the best model?** Generally, the best model is the one where the set of parameter values is the **smallest** (Parsimony principle).

### Remark

In the 3 examples considered above, all observations are **i.i.d.** But, in many situations, this is not the case.

### Example of a very standard non-i.i.d. model:

$$X_0 = x_0; \quad X_t = \rho X_{t-1} + \sigma \varepsilon_t, \quad t = 1, \dots, T$$

We observe  $(X_1, \dots, X_T)$ . They are not i.i.d. Here  $\rho \in \mathbb{R}$ ,  $\sigma > 0$  (Volatility), and  $\varepsilon_t \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$ .  $X_t$  could represent the log price of a financial asset.

In this course, we will mainly focus on the i.i.d. case.

## II. Parametric Model

### Definition 2: Parametric Model

Let there be data  $(X_1, \dots, X_n) \in \mathcal{X}^n$ . A model  $\mathcal{P}$  for  $(X_1, \dots, X_n)$  is called **parametric** if:

$$\mathcal{P} = \{P_\theta, \theta \in \Theta\}$$

with  $\Theta \subset \mathbb{R}^d$  for some  $d \geq 1$ . Here  $P_\theta$  is a distribution on  $\mathcal{X}^n$ .

### Definition 3: Nuisance Parameters

Let  $X = (X_1, \dots, X_n)$  i.i.d. with model  $\mathcal{P} = \{P_\theta, \theta \in \Theta\}$ . If  $\Theta = (\theta_1, \theta_2)$ , and if we only are interested in estimating or predicting  $\theta_1$  and we don't care about  $\theta_2$ , we say that:

- $\theta_1$  is the **parameter of interest**.
- $\theta_2$  is the **nuisance parameter**.

## III. Identifiability

**Context:** We have a statistical model for  $X = (X_1, \dots, X_n)$ .  $X_i \sim P_\theta$ ,  $\theta \in \Theta \subset \mathbb{R}^d$ . So  $\mathcal{P} = \{P_\theta^{\otimes n}, \theta \in \Theta\}$ .

**Aim:** Can we learn/estimate  $\theta$  from  $X$ ? This is only possible if we can learn  $\theta$  from  $P_\theta^{\otimes n}$  or  $P_\theta$ .

#### Definition 4: Identifiability

A statistical model  $\{P_\theta, \theta \in \Theta\}$  is **identifiable** (for  $\theta$ ) if:

$$P_{\theta_1} = P_{\theta_2} \implies \theta_1 = \theta_2$$

(i.e., the map  $\theta \mapsto P_\theta$  is injective).

#### Examples

##### Example 1: Exponential Distribution

Let  $(P_\theta, \theta \in \Theta) = (\mathcal{E}(\theta), \theta > 0)$ .  $P_\theta$  has density  $f_\theta(x) = \theta e^{-\theta x} \mathbb{1}_{\{x>0\}}$ .

If  $P_{\theta_1} = P_{\theta_2}$ , then for every  $x \in \mathbb{R}$ :

$$f_{\theta_1}(x) = f_{\theta_2}(x) \implies \theta_1 e^{-\theta_1 x} = \theta_2 e^{-\theta_2 x} \quad \text{whenever } x > 0$$

Take  $x \rightarrow 0$  (or  $x = 0$  in the limit), we get  $\theta_1 = \theta_2$ . Thus, the model is identifiable.

##### Example 2: Gaussian Distribution

Let  $\theta = (\mu, \sigma^2) \in \Theta = \mathbb{R} \times (0, +\infty)$ .

$$P_\theta = \mathcal{N}(\mu, \sigma^2)$$

$$f_{\mu, \sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad \forall x \in \mathbb{R}$$

**Injectivity?** Assume  $P_{\theta_1} = P_{\theta_2}$ . Then  $\forall x \in \mathbb{R}$ :

$$\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} = \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(x-\mu_2)^2}{2\sigma_2^2}}$$

Take  $x = \mu_1$ , then:

$$\frac{1}{\sigma_1} = \frac{1}{\sigma_2} e^{-\frac{(\mu_1-\mu_2)^2}{2\sigma_2^2}}$$

Take  $x = \mu_2$ , then:

$$\frac{1}{\sigma_1} e^{-\frac{(\mu_2-\mu_1)^2}{2\sigma_1^2}} = \frac{1}{\sigma_2}$$

This implies:

$$\frac{\sigma_2}{\sigma_1} = e^{-\frac{(\mu_1-\mu_2)^2}{2\sigma_2^2}} \leq 1 \quad (\text{because } e^{-\cdot} \leq 1)$$

$$= e^{-\frac{(\mu_1-\mu_2)^2}{2\sigma_1^2}} \geq 1 \quad (\text{because } e^{\cdot} \geq 1)$$

Hence:

$$\frac{\sigma_2}{\sigma_1} = 1 \implies \sigma_1 = \sigma_2$$

And substituting back:

$$e^{-\frac{(\mu_1-\mu_2)^2}{2\sigma^2}} = 1 \implies \frac{(\mu_1 - \mu_2)^2}{2\sigma^2} = 0 \implies \mu_1 = \mu_2$$

Thus  $\theta_1 = \theta_2$ . The model is identifiable.

### Proposition 1: Identifiability via CDF and Density

- If  $P_\theta$  admits a cumulative distribution function (CDF)  $F_\theta$ , the model is identifiable  $\iff (\forall \theta_1, \theta_2, F_{\theta_1} = F_{\theta_2} \implies \theta_1 = \theta_2)$ .
- If  $P_\theta$  admits a density  $f_\theta$ , the model is identifiable  $\iff (\forall \theta_1, \theta_2, f_{\theta_1} = f_{\theta_2} \implies \theta_1 = \theta_2)$ .

### Example 3: Non-identifiable Model (Mixture)

Let  $Y_1 \sim \text{Ber}(p_1)$ ,  $Y_2 \sim \text{Ber}(p_2)$ .  $Y_1 \in \{0, 1\}$ .  $\mathbb{P}(Y_1 = 1) = p_1$ .

Let  $X$  be defined as:

$$X = \begin{cases} Y_1 & \text{with proba } \pi \\ Y_2 & \text{with proba } 1 - \pi \end{cases}$$

$$\begin{aligned} \mathbb{P}(X = 1) &= \mathbb{P}(X = 1 | X = Y_1)\mathbb{P}(X = Y_1) + \mathbb{P}(X = 1 | X = Y_2)\mathbb{P}(X = Y_2) \\ &= \mathbb{P}(Y_1 = 1) \times \pi + \mathbb{P}(Y_2 = 1) \times (1 - \pi) \\ &= p_1\pi + p_2(1 - \pi) \end{aligned}$$

We have a model with 3 parameters  $(p_1, p_2, \pi)$ .  $X \sim \text{Ber}(p)$  with  $p = p_1\pi + p_2(1 - \pi)$ .

Knowing  $\mathbb{P}(X = 1)$  gives limited information on the law of  $X$ .

#### Numeric Examples:

- $\pi = 1/2, p_1 = 0.6, p_2 = 0.2$ .  $\mathbb{P}(X = 1) = 0.6 \times 0.5 + 0.2 \times 0.5 = 0.4$ .
- $\pi = 0.6, p_1 = 1/2, p_2 = 0.25$ .  $\mathbb{P}(X = 1) = 0.6 \times 0.5 + 0.4 \times 0.25 = 0.3 + 0.1 = 0.4$ .

We have different parameters but the same distribution  $\implies$  **Not identifiable**.

## IV. Empirical Distribution Function

**Model:** We observe  $X_1, \dots, X_n$  i.i.d.  $P$ . We set  $F(t) =$  Cumulative distribution function of  $P$  at  $t$

$$= \mathbb{P}(X_i \leq t), \quad \forall i$$

#### Remark

#### Reminder:

- $F$  is non-decreasing.
- $F$  is càdlàg (right-continuous with left limits).
- $\lim_{t \rightarrow -\infty} F(t) = 0$  and  $\lim_{t \rightarrow +\infty} F(t) = 1$ .

### Definition 5: Empirical CDF

$$\hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{X_i \leq t\}}$$

It is a good idea since when  $n \rightarrow +\infty$ : Fix  $t$ :  $\hat{F}_n(t) \xrightarrow[n \rightarrow \infty]{\text{in probability}} \mathbb{E}[Y_i]$  in probability (Weak Law of Large

Numbers).

$$\frac{1}{n} \sum_{i=1}^n Y_i$$

with  $Y_i = \mathbb{1}_{\{X_i \leq t\}}$ . The  $Y_i$  are i.i.d.,  $\mathbb{E}[|Y_i|] < +\infty$ .

$$\mathbb{E}[Y_1] = \mathbb{P}(X_1 \leq t) = F(t)$$

So, we have:  $\hat{F}_n(t) \rightarrow F(t)$  in probability.

### Examples

1.  $X_i = \begin{cases} 1 & \text{if I have an accident on day } i \\ 0 & \text{otherwise} \end{cases}$   $X_i \stackrel{i.i.d.}{\sim} \text{Ber}(p)$ ;  $p = \mathbb{P}(\text{avoir un accident})$ .

$$\hat{p} = \frac{1}{n} \sum_{i=1}^n X_i \xrightarrow[n \rightarrow \infty]{\mathbb{P}} p \quad (\text{Law of Large Numbers})$$

( $\xrightarrow{\mathbb{P}}$  en probabilité).

2.  $X_1, \dots, X_n$  i.i.d.  $X_i = \text{Lifetime of computer } i$ .

We want to estimate the function  $t \mapsto F(t) = \mathbb{P}(X_i \leq t)$ . To estimate  $F$ , we use at time  $t$ ,  $\hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{X_i \leq t\}}$ .

# Chapter 3: Point estimators and MLE

## I. Point Estimators

We consider a parametric model  $(X_1, \dots, X_n) \stackrel{i.i.d.}{\sim} P_\theta$ , with  $\theta \in \Theta \subset \mathbb{R}^d$ .

**Goal:** Constructions of estimators for  $\theta$  or for  $g(\theta)$  with  $g : \Theta \rightarrow \mathbb{R}^p$ .

### Definition 6: Point Estimator

A **point estimator** for  $\theta$  is a quantity  $\hat{\theta}$  which depends **only** on  $X_1, \dots, X_n$  (the data).

**Examples:**

- We observe  $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \text{Ber}(\theta)$ .

$$\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^n X_i \quad \text{is a point estimator of } \theta.$$

- We observe  $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathcal{N}(\theta, 1)$ .

$$\hat{\theta}_n = \bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i \quad \text{is an estimator of } \theta.$$

- We observe  $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathcal{E}(\theta)$ .

$$\hat{\theta}_n = \frac{1}{\frac{1}{n} \sum_{i=1}^n X_i} = \frac{1}{\bar{X}_n}$$

Recall: If  $X \sim \mathcal{E}(\theta)$ ,  $\theta > 0$ :

$$\begin{aligned} E[X] &= \frac{1}{\theta} \\ \frac{1}{\hat{\theta}_n} &\xrightarrow{n \rightarrow \infty} \frac{1}{\theta} \\ \frac{1}{\hat{\theta}_n} &= \frac{1}{n} \sum_{i=1}^n X_i \end{aligned}$$

By the Weak Law of Large Numbers (WLLN),  $\hat{\theta}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \theta$ .

For this last example, if we want to estimate  $g(\theta) = \frac{1}{\theta}$ , in this case,  $\hat{g} = \frac{1}{\hat{\theta}_n} = \frac{1}{n} \sum_{i=1}^n X_i$  is a "good estimator".

## II. Quadratic Risk, Bias

**Measure of error = Loss function.** The most common loss function is the **quadratic loss**:

$$l(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$$

We cannot compute  $l(\hat{\theta}_n, \theta)$  since we do not know  $\theta$ .

### Definition 7: Quadratic Risk (Mean Squared Error)

We call **quadratic risk** (or mean squared error):

$$R(\hat{\theta}_n, \theta) = \mathbb{E}[(\hat{\theta}_n - \theta)^2]$$

### Definition 8: Bias

Let  $\hat{\theta}_n$  be an estimator of  $\theta$  in a statistical model  $(X_i)_{i \leq n} \stackrel{i.i.d.}{\sim} P_\theta, \theta \in \Theta$ .

- (i) We say that  $\hat{\theta}_n$  is **unbiased** at  $\theta$  iff  $\mathbb{E}_\theta(\hat{\theta}_n) = \theta$ .
- (ii) We say that  $\hat{\theta}_n$  is **unbiased** over  $\Theta$  iff it is unbiased at all  $\theta \in \Theta$ .

The **bias** is defined as:

$$b(\theta) = \mathbb{E}_\theta(\hat{\theta}_n) - \theta$$

### Theorem 1: Bias-Variance Decomposition

In a statistical model  $(X_i)_{i \leq n}$  i.i.d.  $P_\theta, \theta \in \Theta$ . If  $\hat{\theta}_n$  is an estimator of  $\theta$ , then:

$$R(\hat{\theta}_n, \theta) = b(\theta)^2 + \mathbb{V}_\theta(\hat{\theta}_n)$$

where  $\mathbb{V}_\theta(\hat{\theta}_n)$  is the variance of  $\hat{\theta}_n$  assuming that  $X_i \stackrel{i.i.d.}{\sim} P_\theta$ .

*Proof.*

$$R(\hat{\theta}_n, \theta) = \mathbb{E}_\theta[(\hat{\theta}_n - \theta)^2]$$

Add and subtract  $\mathbb{E}_\theta(\hat{\theta}_n)$ :

$$= \mathbb{E}_\theta[(\hat{\theta}_n - \mathbb{E}_\theta(\hat{\theta}_n) + \mathbb{E}_\theta(\hat{\theta}_n) - \theta)^2]$$

Let  $A = \hat{\theta}_n - \mathbb{E}_\theta(\hat{\theta}_n)$  and  $B = \mathbb{E}_\theta(\hat{\theta}_n) - \theta = b(\theta)$ .

$$\begin{aligned} &= \mathbb{E}_\theta[(A + B)^2] = \mathbb{E}_\theta[A^2 + B^2 + 2AB] \\ &= \underbrace{\mathbb{E}_\theta(A^2)}_{\mathbb{V}_\theta(\hat{\theta}_n)} + \underbrace{\mathbb{E}_\theta(B^2)}_{b(\theta)^2 \text{ (constant)}} + 2 \underbrace{\mathbb{E}_\theta[(\hat{\theta}_n - \mathbb{E}_\theta(\hat{\theta}_n)) \cdot b(\theta)]}_{0} \end{aligned}$$

The cross term is zero because:

$$2b(\theta)\mathbb{E}_\theta(\hat{\theta}_n - \mathbb{E}_\theta(\hat{\theta}_n)) = 2b(\theta)(\mathbb{E}_\theta(\hat{\theta}_n) - \mathbb{E}_\theta(\hat{\theta}_n)) = 0$$

Hence:

$$R(\hat{\theta}_n, \theta) = b(\theta)^2 + \mathbb{V}_\theta(\hat{\theta}_n)$$

(Q.E.D)

□

### Examples of Risk Calculation:

#### Example 4: Normal Mean

$X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathcal{N}(\theta, 1)$ . Estimator  $\hat{\theta}_n = \frac{1}{n} \sum X_i = \bar{X}_n$ .

$$R(\hat{\theta}_n, \theta) = \mathbb{E}[(\hat{\theta}_n - \theta)^2]$$

$$\begin{aligned}
&= \mathbb{E}[(\bar{X}_n - \theta)^2] \\
&= \text{Var}(\bar{X}_n) \quad (\text{as } \mathbb{E}[\bar{X}_n] = \theta, \text{ it is unbiased}) \\
&= \frac{1}{n} \text{Var}(X_1) = \frac{1}{n}
\end{aligned}$$

### Example 5: Bernoulli Mean

We observe  $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \text{Ber}(\theta)$ .

$$\begin{aligned}
\hat{\theta}_n &= \bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i \\
R(\hat{\theta}_n, \theta) &= \mathbb{E}[(\hat{\theta}_n - \theta)^2] \\
&= \mathbb{E}[(\bar{X}_n - \theta)^2] \\
&= \text{Var}(\bar{X}_n) = \frac{\text{Var}(X_1)}{n} = \frac{\theta(1-\theta)}{n}
\end{aligned}$$

### Remark

The function  $x \in [0, 1] \mapsto x(1-x)$  attains its maximum at  $x = 1/2$ .

$$\begin{aligned}
&\implies \forall \theta \in [0, 1], \quad \theta(1-\theta) \leq \frac{1}{4} \\
&\implies R(\hat{\theta}_n, \theta) \leq \frac{1}{4n}
\end{aligned}$$

## Reminders on Convergence

### Definitions

#### Definition 9: Convergence in Probability

$$X_n \xrightarrow{\mathbb{P}} X \iff \forall \varepsilon > 0, \mathbb{P}(|X_n - X| > \varepsilon) \xrightarrow{n \rightarrow \infty} 0.$$

#### Definition 10: Convergence in $L^p$

$$X_n \xrightarrow{L^p} X \iff \mathbb{E}(|X_n - X|^p) \xrightarrow{n \rightarrow \infty} 0.$$

#### Definition 11: Convergence in Distribution

$$\begin{aligned} X_n &\xrightarrow{\mathcal{D}} X \\ &\iff \forall x \text{ (where } F_X \text{ is continuous), } F_{X_n}(x) \xrightarrow{n \rightarrow \infty} F_X(x) \\ &\iff \forall g \text{ continuous and bounded: } \mathbb{E}(g(X_n)) \xrightarrow{n \rightarrow \infty} \mathbb{E}(g(X)) \end{aligned}$$

### Main Theorems

#### Theorem 2: Law of Large Numbers (Weak)

If  $(X_n)_n$  are i.i.d. random variables such that  $\mathbb{E}(|X_n|) < \infty$ , then:

$$\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{n \rightarrow \infty} \mathbb{E}(X_1)$$

#### Theorem 3: Central Limit Theorem (CLT)

If  $(X_n)_n$  are i.i.d. such that  $\mathbb{E}(X_n^2) < \infty$ , then:

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

where  $\bar{X}_n = \frac{1}{n} \sum X_i$ ,  $\mu = \mathbb{E}(X_1)$ ,  $\sigma^2 = \text{Var}(X_1)$ .

#### Theorem 4: Continuity Mapping Theorem

- i) If  $X_n \xrightarrow{\mathbb{P}} X$  and if  $g$  is continuous, then  $g(X_n) \xrightarrow{\mathbb{P}} g(X)$ .
- ii) If  $X_n \xrightarrow{L^p} X$ ,  $p \geq 1$ , then  $X_n \xrightarrow{L^q} X$  for all  $1 \leq q \leq p$ .
- iii) If  $X_n \xrightarrow{\mathcal{D}} X$ , and if  $g$  is continuous and bounded, then  $g(X_n) \xrightarrow{\mathcal{D}} g(X)$ . (Actually holds for any continuous  $g$ ).

### Theorem 5: Delta Method

If  $(Y_n)_n$  are like  $\sqrt{n}(Y_n - \mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$ . If  $g$  is  $C^1$  at  $\mu$  so that  $|g'(\mu)| > 0$ , then:

$$\sqrt{n}(g(Y_n) - g(\mu)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, g'(\mu)^2 \sigma^2)$$

*Sketch of proof.* Taylor expansion of  $g(Y_n)$  around  $\mu$ :

$$g(Y_n) \approx g(\mu) + g'(\mu)(Y_n - \mu)$$

$$\sqrt{n}(g(Y_n) - g(\mu)) \approx g'(\mu)\sqrt{n}(Y_n - \mu)$$

Since  $\sqrt{n}(Y_n - \mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$ , the result follows.  $\square$

### Remark

If  $\sqrt{n}(Y_n - \mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$ , then  $Y_n \xrightarrow{\mathbb{P}} \mu$ .

## III. Empirical Distribution Function

### Definition 12: Dirac Mass

Dirac mass at  $a \in \mathbb{R}$  is the distribution defined by:

$$\forall A \subset \mathbb{R}, \quad \delta_{\{a\}}(A) = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{if } a \notin A \end{cases}$$

Let  $\mathbb{P}_n = \frac{1}{n} \sum_{i=1}^n \delta_{\{X_i\}}$ .

### Definition 13: Empirical CDF

$F_n$  = Empirical Cumulative Distribution Function (CDF) for  $(X_i)_i$  i.i.d. with distribution  $P_X$ .

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{X_i \leq x\}} = \mathbb{P}_n((-\infty, x])$$

**Remark:**  $F_n$  is the CDF of  $\mathbb{P}_n$ .

### Theorem 6: Asymptotic Normality of Empirical CDF

$$\forall x, \quad \sqrt{n}(F_n(x) - F_X(x)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, F_X(x)(1 - F_X(x)))$$

Where  $F_X$  is the CDF of  $P_X$ .

*Proof.*

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n \underbrace{\mathbb{1}_{\{X_i \leq x\}}}_{=Y_i}$$

The variables  $Y_i$  are i.i.d. Bernoulli variables:

$$Y_i \sim \text{Ber}(p) \quad \text{with } p = \mathbb{P}(X_i \leq x) = F_X(x)$$

$$\mathbb{E}[Y_i^2] < \infty$$

Applying CLT:

$$\sqrt{n}(\bar{Y}_n - \mathbb{E}(Y_1)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \text{Var}(Y_1))$$

Here  $\bar{Y}_n = F_n(x)$  and  $\mathbb{E}(Y_1) = F_X(x)$ .

$$\text{Var}(Y_1) = F_X(x)(1 - F_X(x))$$

(Q.E.D) □

## More on Quadratic Risk

**Expression of Risk using Density:**

$$R(\hat{\theta}_n, \theta) = \mathbb{E}_{\theta}[(\hat{\theta}_n - \theta)^2]$$

Under the model  $X_i \stackrel{i.i.d.}{\sim} P_{\theta}$ :

$$= \begin{cases} \int (\hat{\theta}_n(x_1, \dots, x_n) - \theta)^2 \prod_{i=1}^n f_{\theta}(x_i) dx_i & \text{if continuous (density } f_{\theta}) \\ \sum_{x_1, \dots, x_n} (\hat{\theta}_n(x_1, \dots, x_n) - \theta)^2 \prod_{i=1}^n P_{\theta}(X = x_i) & \text{if discrete observations} \end{cases}$$

**Example: Exponential Family**

**Example 2 (from beginning):**  $X_i \stackrel{i.i.d.}{\sim} \mathcal{E}(\theta), \theta > 0$ .

$X_i$  are continuous random variables with density  $f_{\theta}(x) = \theta e^{-\theta x} \mathbb{1}_{\{x>0\}}$

Estimator:

$$\hat{\theta}_n = \frac{1}{\bar{X}_n}; \quad \bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

If  $X \sim \mathcal{E}(\theta)$ ,  $\mathbb{E}_{\theta}(X) = \frac{1}{\theta}$ . What is  $R(\hat{\theta}_n, \theta)$ ? What is  $b(\theta)$ ?

We need to calculate  $\mathbb{E}_{\theta}(\hat{\theta}_n) = \mathbb{E}_{\theta}\left(\frac{1}{\bar{X}_n}\right)$ .

**Recall:**

- $\mathbb{E}(h(X)) = \int h(x)f_X(x)dx$  (if continuous).
- We write:  $\mathbb{E}(h(X)) = \int h(x)dF_X(x) = \int h(x)dP_X(x)$ .
- If  $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \Gamma(a, b)$ , then  $\sum_{i=1}^n X_i \sim \Gamma(na, b)$ .

**Gamma distribution**  $\Gamma(a, b)$ :  $\mathcal{C}^0$  distribution on  $\mathbb{R}^+$  with density:

$$f_{a,b}(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} \mathbb{1}_{\{x>0\}}$$

Note:  $\mathcal{E}(\theta) = \Gamma(1, \theta)$ .

$$\implies \sum_{i=1}^n X_i \sim \Gamma(n, \theta).$$

Let  $Z_n = \sum_{i=1}^n X_i$ . Then  $\hat{\theta}_n = \frac{n}{Z_n}$ .

We want to compute  $\mathbb{E}_{\theta}\left(\frac{n}{Z_n}\right) = n\mathbb{E}_{\theta}\left(\frac{1}{Z_n}\right)$ .

$$\begin{aligned}\mathbb{E}_\theta \left( \frac{1}{Z_n} \right) &= \int_0^{+\infty} \frac{1}{z} \cdot \frac{\theta^n}{\Gamma(n)} z^{n-1} e^{-\theta z} dz \\ &= \frac{\theta^n}{\Gamma(n)} \int_0^{+\infty} z^{n-2} e^{-\theta z} dz\end{aligned}$$

**Remark**

Let  $f_{n-1,\theta}(z) = \frac{\theta^{n-1}}{\Gamma(n-1)} z^{n-2} e^{-\theta z}$  be the density of a  $\Gamma(n-1, \theta)$ .

$$\begin{aligned}\implies \int_0^{+\infty} z^{n-2} e^{-\theta z} dz \times \frac{\theta^{n-1}}{\Gamma(n-1)} &= 1 \\ \implies \int_0^{+\infty} z^{n-2} e^{-\theta z} dz &= \frac{\Gamma(n-1)}{\theta^{n-1}}\end{aligned}$$

Back to the expectation:

$$\mathbb{E}_\theta(\hat{\theta}_n) = \frac{n\theta^n}{\Gamma(n)} \times \frac{\Gamma(n-1)}{\theta^{n-1}}$$

And we have:  $\Gamma(x+1) = x\Gamma(x)$  if  $x > 0$ . So  $\Gamma(n) = (n-1)\Gamma(n-1)$ .

$$\implies \mathbb{E}_\theta(\hat{\theta}_n) = \frac{n}{n-1}\theta$$

$$\implies b(\theta) = \mathbb{E}_\theta(\hat{\theta}_n) - \theta = \frac{n}{n-1}\theta - \theta = \frac{\theta}{n-1}$$

Now, let's compute the quadratic risk. We need  $\mathbb{E}_\theta(\hat{\theta}_n^2) = n^2 \mathbb{E}(\frac{1}{Z_n^2})$ .

$$\mathbb{E} \left( \frac{1}{Z_n^2} \right) = \int y^{n-3} e^{-\theta y} dy \frac{\theta^n}{\Gamma(n)}$$

Using the same trick (identification with  $\Gamma(n-2, \theta)$ ) for  $n > 2$ :

$$\begin{aligned}&= \frac{\theta^n}{\Gamma(n)} \cdot \frac{\Gamma(n-2)}{\theta^{n-2}} = \frac{\theta^2 \Gamma(n-2)}{(n-1)(n-2)\Gamma(n-2)} = \frac{\theta^2}{(n-1)(n-2)} \\ \implies \mathbb{E}_\theta(\hat{\theta}_n^2) &= \frac{n^2 \theta^2}{(n-1)(n-2)} \\ R(\theta, \hat{\theta}_n) &= \underbrace{\frac{n^2 \theta^2}{(n-1)(n-2)}}_{\mathbb{E}(\hat{\theta}_n^2)} - \underbrace{\left( \frac{n\theta}{n-1} \right)^2}_{\mathbb{E}(\hat{\theta}_n)^2} + \underbrace{\frac{\theta^2}{(n-1)^2}}_{b(\theta)^2}\end{aligned}$$

(Using variance decomposition  $\mathbb{V}(\hat{\theta}_n) = \mathbb{E}(\hat{\theta}_n^2) - \mathbb{E}(\hat{\theta}_n)^2$ ). Actually, let's calculate directly:

$$R(\theta, \hat{\theta}_n) = \frac{n^2 \theta^2}{(n-1)(n-2)} - 2\theta \frac{n\theta}{n-1} + \theta^2$$

Simplifying (algebra):

$$\begin{aligned}&= \frac{\theta^2 n^2}{n-1} \left( \frac{1}{n-2} - \frac{1}{n-1} \right) + \frac{\theta^2}{(n-1)^2} \\ &= \frac{\theta^2}{(n-1)^2} \left[ \frac{n^2}{n-2} + 1 \right]\end{aligned}$$

## Unbiased estimator

We can construct an unbiased estimator  $\tilde{\theta}_n$ :

$$\tilde{\theta}_n = \hat{\theta}_n \times \left( \frac{n-1}{n} \right) \implies \mathbb{E}_{\theta}(\tilde{\theta}_n) = \mathbb{E}_{\theta}(\hat{\theta}_n) \times \frac{n-1}{n} = \theta.$$

Risk of the unbiased estimator:

$$\begin{aligned} R(\theta, \tilde{\theta}_n) &= \mathbb{V}_{\theta}(\tilde{\theta}_n) = \left( \frac{n-1}{n} \right)^2 \mathbb{V}_{\theta}(\hat{\theta}_n) \\ &= \frac{(n-1)^2}{n^2} \times \frac{n^2 \theta^2}{(n-1)^2(n-2)} = \frac{\theta^2}{n-2} \end{aligned}$$

We observe that:

$$\forall \theta, R(\theta, \tilde{\theta}_n) < R(\hat{\theta}_n, \theta) \implies \tilde{\theta}_n \text{ is better than } \hat{\theta}_n.$$

### Remark

Unbiased estimators are not necessarily better than biased estimators.

## Example 6: Empirical Variance

$$X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2).$$

$$S_n^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2 \quad (\text{Empirical variance})$$

We know that  $S_n^2 = \frac{1}{n} \sum X_i^2 - \bar{X}_n^2$ .

$$\mathbb{E}_{\sigma^2} \left( \frac{1}{n} \sum X_i^2 \right) = \sigma^2$$

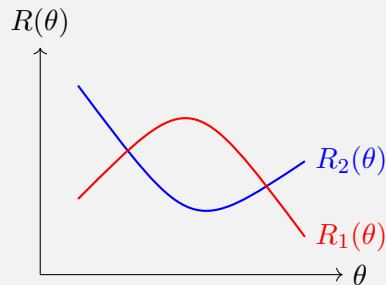
Since  $\bar{X}_n \sim \mathcal{N}(0, \frac{\sigma^2}{n})$ ,  $\mathbb{E}_{\sigma^2}(S_n^2) = \sigma^2 - \frac{\sigma^2}{n} = \sigma^2 \frac{n-1}{n}$ .  
Define  $\tilde{\sigma}_n^2 = S_n^2 \cdot \frac{n}{n-1} = \frac{1}{n-1} \sum (X_i - \bar{X}_n)^2$ .

$$\implies \mathbb{E}_{\sigma^2}(\tilde{\sigma}_n^2) = \sigma^2 \implies \text{Unbiased.}$$

**Exercise:** Compute  $R(\sigma^2, S_n^2)$  and  $R(\sigma^2, \tilde{\sigma}_n^2)$  and show that  $\forall \sigma > 0, R(\sigma^2, S_n^2) < R(\sigma^2, \tilde{\sigma}_n^2)$ .

### Remark

In most cases, if  $\hat{\theta}_{n,1}$  and  $\hat{\theta}_{n,2}$  are 2 estimators of  $\theta$ , then risk functions  $R_1(\theta) = R(\theta, \hat{\theta}_{n,1})$  and  $R_2(\theta, \hat{\theta}_{n,2})$  cross.



$\implies \hat{\theta}_{n,1}$  and  $\hat{\theta}_{n,2}$  cannot be compared uniformly.

### Example 7: Normal Mean Estimators

$X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathcal{N}(\theta, 1)$ .

$$\hat{\theta}_{n,1} = \bar{X}_n$$

$$\hat{\theta}_{n,2} = \frac{n\bar{X}_n}{n + \tau^2} \quad \text{with } \tau > 0$$

Compute their risks and show that they are not comparable.

## Admissibility

### Definition 14: Admissibility

An estimator  $\hat{\theta}_n$  is **not admissible** iff  $\exists \tilde{\theta}_n$ , another estimator such that:

- $\forall \theta, R(\theta, \tilde{\theta}_n) \leq R(\theta, \hat{\theta}_n)$
- and  $\exists \theta_0, R(\theta_0, \tilde{\theta}_n) < R(\theta_0, \hat{\theta}_n)$ .

$\hat{\theta}_n$  is **admissible** iff it is not non-admissible.

## IV. Consistency

### Example 8: Bernoulli Consistency

$X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \text{Ber}(p)$ .  $\theta = p \in ]0, 1[$ .  $\bar{X}_n = \hat{\theta}_n$  is a possible estimator for  $\theta$ .

$$\bar{X}_n \xrightarrow[n \rightarrow \infty]{P_\theta} \theta = \mathbb{E}_\theta(X_1) \quad (\text{LLN})$$

**Question:**  $R(\theta, \hat{\theta}_n) \xrightarrow[n \rightarrow \infty]{} 0, \forall \theta?$

$$\begin{aligned} R(\theta, \hat{\theta}_n) &= \mathbb{E}_\theta[(\bar{X}_n - \theta)^2] = \mathbb{V}_\theta(\bar{X}_n) = \frac{1}{n^2} n \mathbb{V}_\theta(X_1) \\ &= \frac{\theta(1 - \theta)}{n} \leq \frac{1}{4n} \end{aligned}$$

So  $\sup_{\theta \in (0,1)} R(\theta, \hat{\theta}_n) \xrightarrow[n \rightarrow \infty]{} 0$ .

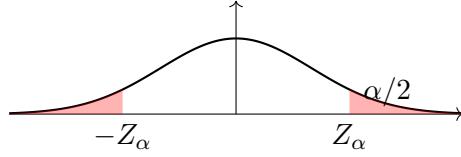
Can I say something like:  $\hat{\theta}_{n,1} \leq \theta \leq \hat{\theta}_{n,2}$  with proba  $\approx 0.95$ ?

$$\begin{aligned} \sqrt{n}(\bar{X}_n - \theta) &\xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \theta(1 - \theta)) \quad (\text{CLT}) \\ \implies \frac{\sqrt{n}(\bar{X}_n - \theta)}{\sqrt{\theta(1 - \theta)}} &\xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1) \end{aligned}$$

Since  $\bar{X}_n \xrightarrow{\mathbb{P}} \theta$ , by Slutsky's lemma (replacing  $\theta$  with  $\bar{X}_n$  in the denominator):

$$\frac{\sqrt{n}(\bar{X}_n - \theta)}{\sqrt{\bar{X}_n(1 - \bar{X}_n)}} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1)$$

Let  $Z_\alpha$  be such that  $\mathbb{P}(|\mathcal{N}(0, 1)| \leq Z_\alpha) = 1 - \alpha$ . (Usually  $1 - \alpha = 0.95 \implies Z_\alpha \approx 1.96$ ).



$$\begin{aligned} &\implies \mathbb{P}_\theta \left( -Z_\alpha \leq \frac{\sqrt{n}(\bar{X}_n - \theta)}{\sqrt{\bar{X}_n(1 - \bar{X}_n)}} \leq Z_\alpha \right) \xrightarrow{n \rightarrow \infty} 1 - \alpha \\ &\implies \mathbb{P}_\theta \left( \underbrace{\bar{X}_n - \frac{\sqrt{\bar{X}_n(1 - \bar{X}_n)}Z_\alpha}{\sqrt{n}}}_{\hat{\theta}_{n,1}} \leq \theta \leq \underbrace{\bar{X}_n + \frac{Z_\alpha\sqrt{\bar{X}_n(1 - \bar{X}_n)}}{\sqrt{n}}}_{\hat{\theta}_{n,2}} \right) \rightarrow 1 - \alpha \end{aligned}$$

With probability under  $P_\theta \approx 1 - \alpha$ ,  $\hat{\theta}_{n,1} \leq \theta \leq \hat{\theta}_{n,2}$ .

### Definition 15: Consistency

1. We say that an estimator  $\hat{\theta}_n$  is **consistent in probability** at  $\theta$  iff  $\hat{\theta}_n \xrightarrow[n \rightarrow \infty]{P_\theta} \theta$ .
2.  $\hat{\theta}_n$  is consistent **over**  $\Theta$  if it is consistent at  $\theta, \forall \theta \in \Theta$ .
3.  $\hat{\theta}_n$  converges in **quadratic mean** at  $\theta$  iff  $R(\hat{\theta}_n, \theta) \xrightarrow[n \rightarrow \infty]{} 0$  (Notation:  $\hat{\theta}_n \xrightarrow{q.m.} \theta$ ).
4.  $\hat{\theta}_n$  converges in quadratic mean over  $\Theta$  iff  $\hat{\theta}_n \xrightarrow{q.m.} \theta, \forall \theta \in \Theta$ .
5.  $\hat{\theta}_n$  is **asymptotically normal** at  $\theta$  with rate  $\frac{1}{\sqrt{n}}$  iff  $\exists V > 0$  s.t.  $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, V)$ .

### Estimation of $g(\theta)$

$X_1, \dots, X_n \stackrel{i.i.d.}{\sim} P_\theta, \theta \in \Theta$ .

**Examples:**

1.  $P_\theta \sim \text{Ber}(\theta)$ . Interested in estimating  $\eta = \log(\frac{\theta}{1-\theta})$ .
  2.  $X_i \stackrel{i.i.d.}{\sim} \mathcal{N}(\mu, \sigma^2), \theta = (\mu, \sigma^2)$ .  $g_1(\theta) = \mu$ .  $g_2(\theta) = \mathbb{P}_\theta(X > 1)$ .
- If  $\hat{\theta}$  is an estimator of  $\theta$ , then we can use  $g(\hat{\theta})$  as an estimator of  $g(\theta)$  (one possibility among others).

### Theorem 7: Plug-in Consistency

Let  $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} P_\theta, \theta \in \Theta \subset \mathbb{R}^d$ . Let  $\hat{\theta}_n$  be an estimator of  $\theta$  and  $g : \Theta \rightarrow \mathbb{R}^p$ . Then:

- i) If  $\hat{\theta}_n$  is consistent at  $\theta$  in probability and  $g$  is  $\mathcal{C}^0$  (continuous), then  $\hat{\eta}_n = g(\hat{\theta}_n)$  is consistent in probability at  $\eta = g(\theta)$ .
- ii) If  $\hat{\theta}_n \xrightarrow{L^2} \theta$  and  $g$  is  $\mathcal{C}^0$  and **bounded**, then  $g(\hat{\theta}_n) \xrightarrow{L^2(P_\theta)} g(\theta), \forall p \geq 1$ .

iii) If  $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[n \rightarrow \infty]{\mathcal{L}(P_\theta)} \mathcal{N}(0, \sigma^2)$  and  $g$  is  $C^1$  with  $g'(\theta) \neq 0$  (or  $\nabla g(\theta)$  of full rank), then  $\sqrt{n}(g(\hat{\theta}_n) - g(\theta)) \xrightarrow{\mathcal{L}(P_\theta)} \mathcal{N}(0, g'(\theta)^2 \sigma^2)$ .

### Remark

$\hat{\theta}_n$  is unbiased  $\not\Rightarrow g(\hat{\theta}_n)$  is unbiased.

*Proof for (ii).*  $\forall \varepsilon > 0$ ,

$$\mathbb{E}_\theta(|g(\hat{\theta}_n) - g(\theta)|^p) = \mathbb{E}_\theta(|g(\hat{\theta}_n) - g(\theta)|^p \mathbb{1}_{|g(\hat{\theta}_n) - g(\theta)| > \varepsilon}) + \mathbb{E}_\theta(|g(\hat{\theta}_n) - g(\theta)|^p \mathbb{1}_{|g(\hat{\theta}_n) - g(\theta)| \leq \varepsilon})$$

The second term is  $\leq \varepsilon^p$ . The first term:

$$\begin{aligned} &\leq \mathbb{E}_\theta((2||g||_\infty)^p \mathbb{1}_{|g(\hat{\theta}_n) - g(\theta)| > \varepsilon}) \\ &= (2||g||_\infty)^p \mathbb{P}_\theta(|g(\hat{\theta}_n) - g(\theta)| > \varepsilon) \xrightarrow[n \rightarrow \infty]{} 0 \end{aligned}$$

Because  $g(\hat{\theta}_n) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} g(\theta)$  (from point (i), continuous mapping theorem).

Thus  $\mathbb{E}_\theta(|g(\hat{\theta}_n) - g(\theta)|^p) \leq \varepsilon^p + 2||g||_\infty^p \times (\dots \rightarrow 0)$ . Taking limit  $n \rightarrow \infty$  then  $\varepsilon \rightarrow 0$  yields the result.  $\square$

## The Multivariate Case

$X_i \stackrel{i.i.d.}{\sim} P_\theta, \theta \in \Theta \subset \mathbb{R}^d. \theta = (\theta_1, \dots, \theta_d)$ .

$$\begin{aligned} \hat{\theta}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \theta &\iff \forall \varepsilon > 0, \mathbb{P}_\theta(||\hat{\theta}_n - \theta|| > \varepsilon) \xrightarrow[n \rightarrow \infty]{} 0 \\ &\iff \forall j \leq d, \hat{\theta}_{n,j} \xrightarrow{\mathbb{P}_\theta} \theta_j \end{aligned}$$

**Because:**

- If  $\forall \varepsilon > 0, \mathbb{P}_\theta(||\hat{\theta}_n - \theta|| > \varepsilon) \rightarrow 0$ . Let  $j \leq d, \forall \varepsilon > 0, ||\hat{\theta}_n - \theta|| \geq |\hat{\theta}_{n,j} - \theta_j|$ .

$$\mathbb{P}_\theta(|\hat{\theta}_{n,j} - \theta_j| > \varepsilon) \leq \mathbb{P}_\theta(||\hat{\theta}_n - \theta|| > \varepsilon) \xrightarrow[n \rightarrow \infty]{} 0$$

- **Reverse:** If  $\forall j \leq d, \mathbb{P}_\theta(|\hat{\theta}_{n,j} - \theta_j| > \varepsilon) \rightarrow 0$ .

$$||\hat{\theta}_n - \theta||^2 = \sum_j |\hat{\theta}_{n,j} - \theta_j|^2 \leq d \cdot \max_j |\hat{\theta}_{n,j} - \theta_j|^2$$

$$\begin{aligned} \mathbb{P}_\theta(||\hat{\theta}_n - \theta|| > \varepsilon) &\leq \mathbb{P}_\theta \left( \max_j |\hat{\theta}_{n,j} - \theta_j| > \frac{\varepsilon}{\sqrt{d}} \right) \\ &= \mathbb{P}_\theta \left( \bigcup_{j=1}^d \{|\hat{\theta}_{n,j} - \theta_j| > \frac{\varepsilon}{\sqrt{d}}\} \right) \\ &\leq \sum_{j=1}^d \mathbb{P}_\theta \left( \{|\hat{\theta}_{n,j} - \theta_j| > \frac{\varepsilon}{\sqrt{d}}\} \right) \\ &\xrightarrow[n \rightarrow \infty]{} 0 \quad \text{because } d \text{ is fixed.} \end{aligned}$$

→ Similarly, show that  $\hat{\theta}_n \xrightarrow{q.m} \theta \iff \forall j, \hat{\theta}_{n,j} \xrightarrow{q.m} \theta_j$ .

**But**, if  $\sqrt{n}(\hat{\theta}_{n,j} - \theta_j) \xrightarrow{\mathcal{L}(P_\theta)} \mathcal{N}(0, \sigma_j^2)$ , it does **NOT** imply directly the joint normality.

$$\implies \sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}(P_\theta)} \mathcal{N}(\mathbf{0}, \Sigma)$$

(Multivariate Central Limit Theorem required).

# Chapter 4: Maximum Likelihood Estimation

## I. The Likelihood

**Model:**  $X_i \stackrel{i.i.d.}{\sim} P_\theta, \theta \in \Theta$ .

**Observations:**  $(x_1, \dots, x_n)$ , where  $x_i$  is a realisation of  $X_i$ .

- If  $P_\theta$  is **discrete**, then proba mass function of  $X_1 \sim P_\theta$ :

$$x \in \mathcal{X}, \quad f_\theta(x) = P_\theta(X_1 = x)$$

(→ density with respect to counting measure).

- If  $P_\theta$  is a **continuous** distribution, then it has density  $f_\theta$  on  $\mathbb{R}$  (or  $\mathbb{R}^d$ ) with respect to Lebesgue measure.

$$\int_A f_\theta(x) dx = P_\theta(X \in A)$$

In both cases, there is a density, with respect to either Lebesgue or counting measure, denoted  $f_\theta$ .

### Definition 16: Likelihood

Let  $(X_i)_{i=1}^n \stackrel{i.i.d.}{\sim} P_\theta, \theta \in \Theta$  with  $\{P_\theta, \theta \in \Theta\}$  either discrete or continuous. Let  $f_\theta$  be the density of  $P_\theta$  (or probability mass function). We call the **likelihood** at  $(x_1, \dots, x_n)$  (realisations of  $(X_1, \dots, X_n)$ ) the function:

$$L_n : \Theta \rightarrow \mathbb{R}_+$$

$$\theta \mapsto \prod_{i=1}^n f_\theta(x_i)$$

### Examples

1.  $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathcal{P}(\lambda), \lambda > 0$  (Poisson). Observations  $(x_1, \dots, x_n), \forall x \in \mathbb{N}, f_\lambda(x) = \frac{e^{-\lambda} \lambda^x}{x!}$ .

$$\begin{aligned} L_n(\lambda) &= \prod_{i=1}^n f_\lambda(x_i) = \prod_{i=1}^n \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} \\ &= \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!} \end{aligned}$$

2.  $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathcal{E}(\theta), \theta > 0$ . Observations  $(x_1, \dots, x_n), x_i > 0, \forall i$ .

$$\forall x > 0, f_\theta(x) = \theta e^{-\theta x}$$

$$\begin{aligned} L_n(\theta) &= \prod_{i=1}^n \theta e^{-\theta x_i} \mathbb{1}_{\{x_i > 0\}} \\ &= \theta^n e^{-\theta \sum_{i=1}^n x_i} \mathbb{1}_{\{\min_i x_i > 0\}} \end{aligned}$$

### Remark

In the **discrete** case,  $\forall \theta, L_n(\theta) = \prod_{i=1}^n f_\theta(x_i) = \prod_{i=1}^n \mathbb{P}_\theta(X_i = x_i) = \mathbb{P}_\theta(X_1 = x_1, \dots, X_n = x_n)$ .

But, in the **continuous** case, it is different:

$$L_n(\theta) = \prod_{i=1}^n f_\theta(x_i) \neq \mathbb{P}_\theta(X_1 = x_1, \dots, X_n = x_n) = 0$$

(Probability of exact points in continuous case is 0).

If  $L_n(\theta)$  is large, it means that the observations are highly likely for  $\theta$ . "There is a good fit between  $(x_1, \dots, x_n)$  and  $P_\theta$ ".

**Exercise 1:** Show that if  $P_\theta \sim \mathcal{N}(\mu, \sigma^2)$ ,  $\theta = (\mu, \sigma^2)$ , then:

$$L_n(\theta) = \frac{e^{-n\frac{(\bar{x}_n - \mu)^2}{2\sigma^2}}}{(2\pi\sigma^2)^{n/2}} e^{-\sum_{i=1}^n \frac{(x_i - \bar{x}_n)^2}{2\sigma^2}}$$

## II. Kullback-Leibler Divergence (K.L)

**2nd interpretation:** Kullback-Leibler divergence.

### Definition 17: Kullback-Leibler Divergence

Let  $P$  and  $Q$  be 2 probabilities.

- If  $P, Q$  are **discrete** on  $\mathcal{X}$ , then:

$$\begin{aligned} KL(P, Q) &= \sum_{x \in \mathcal{X}} P(x) \ln \left( \frac{P(x)}{Q(x)} \right) \\ &= \mathbb{E}_P \left( \ln \left( \frac{P(X)}{Q(X)} \right) \right) \end{aligned}$$

(where  $P(x) = \mathbb{P}(X = x)$ ,  $Q(x) = \mathbb{Q}(X = x)$ ).

- If  $P$  and  $Q$  are **continuous** with density  $f_P$  and  $f_Q$  (resp), then:

$$\begin{aligned} KL(P, Q) &= \int_{\mathcal{X}} f_P(x) \ln \left( \frac{f_P(x)}{f_Q(x)} \right) dx \\ &= \mathbb{E}_P \left( \ln \left( \frac{f_P(X)}{f_Q(X)} \right) \right) \end{aligned}$$

### Proposition 2: Properties of KL

- $\forall P, Q, \quad KL(P, Q) \geq 0$ .
- $KL(P, Q) = 0 \iff P = Q$ .

**But**,  $KL(P, Q) \neq KL(Q, P)$  (Not symmetric).

*Proof.* Assume that  $P, Q$  are continuous.

i)

$$\begin{aligned} KL(P, Q) &= \int_{\mathcal{X}} f_P(x) \ln \left( \frac{f_P(x)}{f_Q(x)} \right) dx \\ &= \int_{\mathcal{X}} f_P(x) \left[ -\ln \left( \frac{f_Q(x)}{f_P(x)} \right) \right] dx \end{aligned}$$

By Jensen's inequality:

$$\begin{aligned} &\geq -\ln \left[ \int_{\mathcal{X}} f_P(x) \frac{f_Q(x)}{f_P(x)} dx \right] \\ &= -\ln \left[ \int_{\mathcal{X}} f_Q(x) dx \right] = -\ln(1) = 0 \end{aligned}$$

(Because  $x \mapsto -\ln(x)$  is strictly convex on  $\mathbb{R}$ ).

ii) Since  $-\ln(\cdot)$  is strictly convex:

$$\mathbb{E}_P (-\ln(R(X))) > -\ln(\mathbb{E}_P(R(X))) \quad \text{with } R(x) = \frac{f_Q(x)}{f_P(x)}$$

if  $\forall c, \mathbb{P}(R(X) = c) < 1$  ( $R(X)$  is not constant).

$R(x) = \frac{f_Q(x)}{f_P(x)}$  is constant iff  $\exists c \neq 0$  so that:

$$f_Q(x) = cf_P(x), \quad \forall x \in \mathcal{X}$$

$$\implies c = 1 \quad (\text{since } \int f_P = \int f_Q = 1)$$

$R(x) = \frac{f_Q(x)}{f_P(x)}$  is constant iff  $f_Q = f_P \iff P = Q$ . □

## Links with the likelihood

$\forall \theta, \theta_0 \in \Theta$ :

$$\begin{aligned} &\ln(L_n(\theta)) - \ln(L_n(\theta_0)) \\ &= \ln \left( \prod_{i=1}^n f_\theta(x_i) \right) - \ln \left( \prod_{i=1}^n f_{\theta_0}(x_i) \right) \\ &= \sum_{i=1}^n \ln(f_\theta(x_i)) - \sum_{i=1}^n \ln(f_{\theta_0}(x_i)) \\ &= \sum_{i=1}^n \ln \left( \frac{f_\theta(x_i)}{f_{\theta_0}(x_i)} \right) \end{aligned}$$

If  $(x_i)_{i=1}^n$  are the realisations of  $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} P_{\theta_0}$ . Then when  $X_i \stackrel{i.i.d.}{\sim} P_{\theta_0}$ :

$$\frac{1}{n} \sum_{i=1}^n \ln \left( \frac{f_\theta(X_i)}{f_{\theta_0}(X_i)} \right) \underbrace{\text{converges under } P_{\theta_0}}_{n \rightarrow \infty} \text{ to } \mathbb{E}_{\theta_0} \left( \ln \left( \frac{f_\theta(X)}{f_{\theta_0}(X)} \right) \right)$$

as soon as  $\mathbb{E}_{\theta_0} \left( \left| \ln \frac{f_\theta(X)}{f_{\theta_0}(X)} \right| \right) < \infty$ .

$\implies$  if  $\mathbb{E}_{\theta_0}(\dots) < \infty$ , then we have:

$$\begin{aligned} \frac{1}{n} (\ln(L_n(\theta)) - \ln(L_n(\theta_0))) &\xrightarrow[n \rightarrow \infty]{P_{\theta_0}} \mathbb{E}_{\theta_0} \left[ \ln \left( \frac{f_\theta(X)}{f_{\theta_0}(X)} \right) \right] \\ &= -KL(P_{\theta_0}, P_\theta) \end{aligned}$$

- $\theta_0$  minimizes  $KL(P_{\theta_0}, P_\theta)$  over  $\theta$ .
- $\theta_0$  maximizes  $-KL(P_{\theta_0}, P_\theta)$  over  $\theta$ .

We cannot maximize  $-KL(P_{\theta_0}, P_\theta)$  because it depends on  $\theta_0$  unknown.  $\implies$  We maximize:

$$\mathbb{E}_{P_n} \left[ -\ln \left( \frac{f_\theta(X)}{f_{\theta_0}(X)} \right) \right] = \frac{1}{n} [\ln(L_n(\theta)) - \ln(L_n(\theta_0))]$$

(Empirical version of  $-KL(P_{\theta_0}, P_\theta)$ ).

Maximizing in  $\theta$ ,  $\ln(L_n(\theta)) - \ln(L_n(\theta_0))$  is equivalent to maximizing in  $\theta$ ,  $\ln L_n(\theta)$ .

### Definition 18: Maximum Likelihood Estimator (MLE)

Let  $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} P_\theta$ ,  $\theta \in \Theta$ , the model  $P_\theta$  with density (or probability mass function)  $f_\theta$ . We call **Maximum Likelihood Estimator** any selection  $\hat{\theta}_n$  (when it exists) such that:

$$\hat{\theta}_n \in \operatorname{argmax}_{\theta \in \Theta} \{\ln(L_n(\theta))\}$$

$$\iff \log(L_n(\hat{\theta}_n)) \geq \log(L_n(\theta)), \forall \theta \in \Theta$$

## Examples of Calculations

### 1) Bernoulli Model

$X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \text{Ber}(\theta)$  ( $\theta \in [0, 1]$ ).

$$\begin{aligned} L_n(\theta) &= \prod_{i=1}^n P_\theta(X_i = x_i) \\ &= \prod_{i=1}^n \theta^{x_i} (1-\theta)^{1-x_i} \\ &= \theta^{\sum x_i} (1-\theta)^{n-\sum x_i} \end{aligned}$$

Define  $l(\theta) = \log(L_n(\theta))$ .

$$l(\theta) = \left( \sum_{i=1}^n x_i \right) \log(\theta) + \left( n - \sum_{i=1}^n x_i \right) \log(1-\theta)$$

$\hat{\theta}_n$  maximizes  $l(\theta)$  in  $\theta$ . Let  $\bar{x}_n = \frac{1}{n} \sum x_i$ .

$$l(\theta) = n\bar{x}_n \log(\theta) + n(1-\bar{x}_n) \log(1-\theta)$$

Derivative:

$$l'(\theta) = \frac{n\bar{x}_n}{\theta} - \frac{n(1-\bar{x}_n)}{1-\theta}$$

Second derivative:

$$l''(\theta) = -\frac{n\bar{x}_n}{\theta^2} - \frac{n(1-\bar{x}_n)}{(1-\theta)^2} < 0$$

So  $\theta \mapsto l(\theta)$  is **concave**.  $\implies \hat{\theta}_n$  is the solution of  $l'(\theta) = 0$ .

$$l'(\theta) = 0 \iff \frac{n\bar{x}_n}{\theta} = \frac{n(1-\bar{x}_n)}{1-\theta} \iff \theta = \bar{x}_n$$

$\implies \hat{\theta}_n = \bar{x}_n$  is the MLE (Maximum Likelihood Estimator)

## 2) Exponential Model

$X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathcal{E}(\theta), \theta > 0.$

$$X \sim \mathcal{E}(\theta) \iff f_\theta(x) = \theta e^{-\theta x} \mathbb{1}_{\{x>0\}}$$

$$L_n(\theta) = \theta^n e^{-n\theta \bar{x}_n} \prod_{i=1}^n \mathbb{1}_{\{x_i>0\}}$$

$$l(\theta) = \log(L_n(\theta)) = n \log(\theta) - n \bar{x}_n \theta \quad (\text{if } \forall i, x_i > 0)$$

**Exercise:** Show that  $\theta \mapsto l(\theta)$  is concave and that  $\hat{\theta}_n = \frac{1}{\bar{x}_n}$ .

### Remark

1. The log likelihood as a function is not necessarily concave, therefore,  $\hat{\theta}_n$  does not always exist and is not always unique.
2. In practice: how do we compute  $\hat{\theta}_n$ ?

$$\hat{\theta}_n \in \operatorname{argmax}\{l(\theta), \theta \in \Theta\}$$

Algorithms such as gradient descent, Newton-Raphson.

### Summary:

- **Model:**  $f_\theta(\cdot), \theta \in \Theta$  density or proba mass function.  $X_i \stackrel{i.i.d.}{\sim} f_\theta$ .
- **Log-likelihood:**  $l(\theta) = \log(L_n(\theta)) = \sum_{i=1}^n \log(f_\theta(x_i))$ .
- **MLE:**  $\hat{\theta}_n$  such that  $l(\hat{\theta}_n) \geq l(\theta), \forall \theta \in \Theta$ .

### Proposition 3: Invariance

If  $\eta = g(\theta)$ , where  $g$  is invertible ( $\iff$  new parametrization).

For instance:  $X_i \stackrel{i.i.d.}{\sim} \text{Ber}(p), \theta = p \in ]0, 1[$ .

$$\eta = \log\left(\frac{p}{1-p}\right) \in \mathbb{R}$$

Let  $\tilde{l}(\eta) = \sum_{i=1}^n \log(f_{g^{-1}(\eta)}(x_i))$ . Let  $l(\theta) = \sum_{i=1}^n \log(f_\theta(x_i))$ .

- If  $\hat{\eta}_n$  is the MLE of  $\tilde{l}$  ( $\iff \tilde{l}(\eta) \leq \tilde{l}(\hat{\eta}_n), \forall \eta$ ).
- If  $\hat{\theta}_n$  is the MLE of  $l$  ( $\iff l(\theta) \leq l(\hat{\theta}_n), \forall \theta$ ).

Then:

$$\hat{\eta}_n = g(\hat{\theta}_n)$$

*Proof.* Note:  $\hat{\eta}_n = g(\hat{\theta}_n)$ . We verify that  $\tilde{l}(\hat{\eta}_n) \geq \tilde{l}(\eta), \forall \eta$ , that is, it will imply that  $\hat{\eta}_n$  is an MLE of  $\tilde{l}$ .  
Indeed:

$$\tilde{l}(\eta) = l(g^{-1}(\eta)) \leq l(\hat{\theta}_n), \forall \eta$$

And:

$$\tilde{l}(\hat{\eta}_n) = l(g^{-1}(g(\hat{\theta}_n))) = l(\hat{\theta}_n)$$

$$\implies \forall \eta, \tilde{l}(\eta) \leq \tilde{l}(\hat{\eta}_n). \quad (\text{Q.E.D})$$

□

## Examples

MLE for  $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \text{Ber}(p)$ .

$$l(p) = p^{\sum x_i} (1-p)^{n-\sum x_i}$$

$x_1, \dots, x_n$  are the observations.

$$\hat{p}_n = \frac{1}{n} \sum_{i=1}^n x_i \rightarrow \text{MLE}$$

$$\eta = \log \left( \frac{p}{1-p} \right) \implies \hat{\eta}_n = \log \left( \frac{\hat{p}_n}{1-\hat{p}_n} \right) \text{ is the MLE of } \eta.$$

## Asymptotic Normality of the MLE

Under which conditions on the model is the MLE asymptotically normal?

### Definition 19: Regular Model

The model  $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} f_\theta, \theta \in \Theta \subset \mathbb{R}^k$  (either density or probability mass function) is **regular** on  $\Theta$  iff:

- (i)  $\Theta$  is an open set and the model is identifiable.
- (ii)  $\theta \mapsto \log(f_\theta(x))$  is  $C^2, \forall x \in \mathcal{X}$ .
- (iii)  $\forall \theta \in \Theta, \mathbb{E}_\theta(\|\nabla_\theta \log(f_\theta(X))\|^2) < +\infty$  and  $\exists \delta > 0, \mathbb{E}_\theta(\sup_{|\theta' - \theta| < \delta} \|D_\theta^2 \log(f_{\theta'}(X))\|) < +\infty$ .
- (iv)  $\forall \theta, I(\theta) = \mathbb{E}_\theta(\nabla_\theta \log(f_\theta(X)) \cdot \nabla_\theta \log(f_\theta(X))^T)$  is positive definite.

## Fisher Information

$I(\theta)$  is called the **Fisher Information Matrix**.

**Notations:**  $D^2 \log(f_\theta(x))$  is the matrix:

$$(D^2 \log(f_\theta(x)))_{i,j} = \frac{\partial^2 \log(f_\theta(x))}{\partial \theta_i \partial \theta_j}$$

$$\nabla_\theta \log(f_\theta(x)) = \left( \frac{\partial}{\partial \theta_j} \log(f_\theta(x)) \right)_{j=1,\dots,k}$$

### Lemma 1: Properties of Score and Fisher Information

If the model is regular:

- a)  $\forall \theta \in \Theta, \mathbb{E}_\theta(\nabla_\theta \log(f_\theta(X))) = 0$ .  $S(\theta, x) = \nabla_\theta \log(f_\theta(x))$  is called the **score function**.
- b)  $I(\theta) = -\mathbb{E}_\theta(D^2 \log(f_\theta(X)))$ .

*Proof.* a) The model is regular  $\implies \nabla_\theta \log(f_\theta(X))$  exists and is integrable. And:

$$\nabla_\theta \log(f_\theta(X)) = \frac{\nabla_\theta f_\theta(X)}{f_\theta(X)}$$

$$\mathbb{E}_\theta(\nabla_\theta \log(f_\theta(X))) = \begin{cases} \int_{\mathcal{X}} \frac{\nabla_\theta f_\theta(x)}{f_\theta(x)} f_\theta(x) dx & \text{if } X_i \text{'s are continuous} \\ \sum_{x \in \mathcal{X}} \frac{\nabla_\theta f_\theta(x)}{f_\theta(x)} f_\theta(x) & \text{if } X_i \text{'s are discrete} \end{cases}$$

We have, if the  $X_i$ 's are continuous:

$$\int_{\mathcal{X}} \frac{\partial}{\partial \theta_j} f_\theta(x) dx = \frac{\partial}{\partial \theta_j} \int_{\mathcal{X}} f_\theta(x) dx$$

Because  $\mathbb{E}_\theta(||\nabla_\theta f_\theta(X)||) < +\infty$ . Also  $\forall \theta, \int_{\mathcal{X}} f_\theta(x) dx = 1 \implies \frac{\partial}{\partial \theta_j} (\int f_\theta(x) dx) = 0$ .

$$\text{So } \mathbb{E}_\theta \left( \frac{\partial}{\partial \theta_j} \frac{f_\theta(X)}{f_\theta(X)} \right) = 0.$$

$$\implies \mathbb{E}_\theta(\nabla_\theta \log(f_\theta(X))) = 0$$

If  $f$  is discrete  $\rightarrow$  same argument.

$$\sum_x \frac{\partial}{\partial \theta_j} f_\theta(x) = \frac{\partial}{\partial \theta_j} \sum_x f_\theta(x) = \frac{\partial}{\partial \theta_j}(1) = 0$$

(Q.E.D)

**b)** Show that  $I(\theta) = -\mathbb{E}_\theta(D_\theta^2 \log(f_\theta(X)))$ .

$$\begin{aligned} \frac{\partial^2 \log(f_\theta(x))}{\partial \theta_i \partial \theta_j} &= \frac{\partial}{\partial \theta_j} \left( \frac{\frac{\partial}{\partial \theta_i} f_\theta(x)}{f_\theta(x)} \right) \\ &= \frac{\frac{\partial^2}{\partial \theta_i \partial \theta_j} f_\theta(x)}{f_\theta(x)} - \left( \frac{\frac{\partial}{\partial \theta_i} f_\theta(x)}{f_\theta(x)} \right) \left( \frac{\frac{\partial}{\partial \theta_j} f_\theta(x)}{f_\theta(x)} \right) \end{aligned}$$

Similarly to before:

$$\begin{aligned} \mathbb{E}_\theta \left( \frac{\frac{\partial^2}{\partial \theta_i \partial \theta_j} f_\theta(X)}{f_\theta(X)} \right) &= \int_{\mathcal{X}} \frac{\frac{\partial^2}{\partial \theta_i \partial \theta_j} f_\theta(x)}{f_\theta(x)} f_\theta(x) dx \\ &= \int_{\mathcal{X}} \frac{\partial^2}{\partial \theta_i \partial \theta_j} f_\theta(x) dx = \frac{\partial^2}{\partial \theta_i \partial \theta_j} \int_{\mathcal{X}} f_\theta(x) dx = 0 \end{aligned}$$

$$\begin{aligned} J(\theta)_{i,j} &= -\mathbb{E}_\theta(D_\theta^2 \log(f_\theta(X))) \\ &= -\mathbb{E}_\theta \left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log(f_\theta(X)) \right) \\ &= - \left( -\mathbb{E}_\theta \left( \frac{\frac{\partial f}{\partial \theta_i}}{f} \cdot \frac{\frac{\partial f}{\partial \theta_j}}{f} \right) \right) \\ &= I_{i,j}(\theta) \end{aligned}$$

So  $J(\theta) = I(\theta)$ . (Q.E.D)

□

### Theorem 8: Asymptotic Normality of MLE

Let  $(x_1, \dots, x_n)$  be observations from the model  $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} f_\theta, \theta \in \Theta \subset \mathbb{R}^k$ . Assume the model is **regular**.

(i) **Score Convergence:** For all  $\theta \in \Theta$ , let  $S_n(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \nabla_\theta \log(f_\theta(X_i))$ . Then:

$$S_n(\theta) \xrightarrow[n \rightarrow \infty]{\mathcal{L}_{P_\theta}} \mathcal{N}(0, I(\theta))$$

(ii) **MLE Convergence:** If in addition the MLE  $\hat{\theta}_n$  is **consistent** over  $\Theta$  (in probability), then:

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[n \rightarrow \infty]{\mathcal{L}_{P_\theta}} \mathcal{N}(0, I^{-1}(\theta))$$

$\implies I^{-1}(\theta)$  is the asymptotical variance of  $\sqrt{n}(\hat{\theta}_n - \theta)$ .

### Remark

**Cramer-Rao Lower Bound:** For the variance of a regular estimator is  $\frac{I^{-1}(\theta)}{n}$ .

### Multivariate Gaussians Reminder

$X \sim \mathcal{N}(\mu, \Sigma)$ ,  $X \in \mathbb{R}^k$ ,  $\mu \in \mathbb{R}^k$ .  $\Sigma$  symmetric, positive semi-definite.  
 $\iff$  its density (w.r.t Lebesgue)

$$f_{\mu, \Sigma}(x) = \frac{e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)}}{(2\pi)^{k/2} (\det(\Sigma))^{1/2}}$$

$\iff \forall a \in \mathbb{R}^d, a^T X \sim \mathcal{N}(a^T \mu, a^T \Sigma a)$ .

### Proof of the Theorem

**Objective:** Show that  $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}_{P_\theta}} \mathcal{N}(0, I^{-1}(\theta))$ .

$\hat{\theta}_n$  is defined by  $l(\hat{\theta}_n) \geq l(\theta)$  and verifies  $\nabla_\theta l(\hat{\theta}_n) = 0$ . The rest of the proof is done with  $k = 1$ .  
 $\theta \mapsto l(\theta) = \sum \log(f_\theta(x_i))$  is  $\mathcal{C}^2$ .

Taylor expansion:

$$l'(\theta) = l'(\hat{\theta}_n) + (\theta - \hat{\theta}_n)l''(\tilde{\theta}_n) \quad \text{with } \tilde{\theta}_n \in [\theta, \hat{\theta}_n]$$

$$\implies l'(\theta) = (\theta - \hat{\theta}_n)l''(\tilde{\theta}_n) \quad (\text{since } l'(\hat{\theta}_n) = 0)$$

$$\frac{l''(\tilde{\theta}_n)}{n} = \frac{1}{n} \sum_{i=1}^n \frac{\partial^2}{\partial \theta^2} \log(f_{\tilde{\theta}_n}(x_i))$$

$$\frac{l''(\tilde{\theta}_n) - l''(\theta)}{n} = \Delta_n(\theta)$$

So:

$$l'(\theta) = (\theta - \hat{\theta}_n)(l''(\theta) + \Delta_n(\theta) \times n)$$

And  $\frac{l'(\theta)}{\sqrt{n}} = -S_n(\theta)$ .

$$S_n(\theta) = \sqrt{n}(\theta - \hat{\theta}_n) \left( \frac{l''(\theta)}{n} + \Delta_n(\theta) \right)$$

We examine the terms:

$$\frac{l''(\theta)}{n} = \frac{1}{n} \sum_{i=1}^n D^2 \log(f_\theta(x_i)) \xrightarrow[n \rightarrow \infty]{P_\theta} \mathbb{E}_\theta(D^2 \log(f_\theta(X))) = -I(\theta)$$

$$\implies S_n(\theta) = \sqrt{n}(\hat{\theta}_n - \theta)(-I(\theta) + \Delta_n(\theta) + B_n(\theta))$$

where  $B_n(\theta) = \frac{l''(\theta)}{n} + I(\theta) \xrightarrow{P_\theta} 0$ .

We need to handle  $S_n(\theta)$  and  $\Delta_n(\theta)$ .

**(i) Behavior of  $S_n(\theta)$ :**

$$\forall \theta \in \Theta, S_n(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \nabla_\theta \log(f_\theta(x_i)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, I(\theta))$$

**Proof for (i):**

$$S_n(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^n s_1(\theta, x_i)$$

And  $\forall \theta, (s_1(\theta, x_i))_{i=1,\dots,n}$  are i.i.d. because  $(x_i)$  are i.i.d. And  $\mathbb{E}_\theta(s_1(\theta, X_i)) = 0$  (Lemma). And  $\mathbb{E}_\theta(\|s_1(\theta, X)\|^2) < +\infty$  (Assumption).

By the CLT:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n s_1(\theta, X_i) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \text{Var}_\theta(s_1(\theta, X)))$$

Also  $\text{Var}_\theta(s_1(\theta, X)) = \mathbb{E}_\theta(s_1(\theta, X)s_1(\theta, X)^T) = I(\theta)$ .

**(ii) Behavior of  $\Delta_n(\theta)$ :** We show that  $\Delta_n(\theta) \xrightarrow[n \rightarrow \infty]{P_\theta} 0$ . Because when  $|\hat{\theta}_n - \theta| \leq \delta$ :

$$|\Delta_n(\theta)| \leq \frac{1}{n} \sum \sup_{|\theta' - \theta| \leq \delta} |D_{\theta'}^2 \log(f_{\theta'}(x_i)) - D_\theta^2 \log(f_\theta(x_i))|$$

LLN:

$$\begin{aligned} & \frac{1}{n} \sum \sup_{|\theta' - \theta| \leq \delta} |D_{\theta'}^2 \log(f_{\theta'}(x_i)) - D_\theta^2 \log(f_\theta(x_i))| \\ & \xrightarrow{P_\theta} \mathbb{E}_\theta \left( \sup_{|\theta' - \theta| \leq \delta} |D_{\theta'}^2 \log(f_{\theta'}(x_i)) - D_\theta^2 \log(f_\theta(x_i))| \right) \end{aligned}$$

By the dominated convergence theorem:

$$\mathbb{E}_\theta \left( \sup_{|\theta' - \theta| \leq \delta} |D_{\theta'}^2 \log(f_{\theta'}(x)) - D_\theta^2 \log(f_\theta(x))| \right) \xrightarrow[\delta \rightarrow 0]{} 0$$

$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0, \forall \delta \leq \delta_\varepsilon$ , this expectation is  $< \varepsilon$ .

We know that  $\hat{\theta}_n \xrightarrow{P_\theta} \theta$ .  $\forall \delta > 0, \mathbb{P}_\theta(|\hat{\theta}_n - \theta| > \delta) \xrightarrow{n \rightarrow \infty} 0$ .

Also  $\theta \mapsto D_\theta^2 \log(f_\theta(x))$  is continuous.  $\implies \limsup_{\delta \rightarrow 0} \sup_{|\theta' - \theta| \leq \delta} |D_{\theta'}^2 \log(f_{\theta'}(x)) - D_\theta^2 \log(f_\theta(x))| = 0$  (almost surely). Let  $\delta_p \rightarrow 0, \delta_p \downarrow$ .

$$\begin{aligned} & \sup_{|\theta' - \theta| \leq \delta_p} |D_{\theta'}^2 \log(f_{\theta'}(x)) - D_\theta^2 \log(f_\theta(x))| \\ & \leq \sup_{|\theta' - \theta| \leq \delta_0} |D_{\theta'}^2 \log(f_{\theta'}(x)) - D_\theta^2 \log(f_\theta(x))| \end{aligned}$$

And  $\mathbb{E}_\theta(H(X)) < +\infty$  (by assumption).

So  $\Delta_n(\theta) \xrightarrow{P_\theta} 0$ .

**Conclusion of Proof:**

$$S_n(\theta) = \sqrt{n}(\hat{\theta}_n - \theta)(-I(\theta) + \Delta_n(\theta) + B_n(\theta))$$

$$B_n(\theta) = \frac{l''(\theta)}{n} + I(\theta) \xrightarrow{P_\theta} 0$$

$$\Delta_n(\theta) \xrightarrow{P_\theta} 0$$

$$S_n(\theta) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, I(\theta))$$

$$\implies \sqrt{n}(\hat{\theta}_n - \theta) = \frac{S_n(\theta)}{I(\theta)(1 + \mu_n)} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{I(\theta)}{I^2(\theta)}\right) = \mathcal{N}(0, I^{-1}(\theta))$$

With  $\mu_n \xrightarrow{P_\theta} 0$ .

### III. Exponential Families

#### Definition 20: Exponential Family

Model  $X \in \mathbb{R}^d \sim P_\theta$ ,  $\theta \in \Theta \subset \mathbb{R}^k$  is an **exponential family** iff  $P_\theta$  has density (or proba mass function)  $f_\theta$  and:

- $\exists h : \mathbb{R}^d \rightarrow \mathbb{R}_+$
- $\exists A : \Theta \rightarrow \mathbb{R}^k$
- $\exists S : \mathbb{R}^d \rightarrow \mathbb{R}^k$
- $\exists C : \Theta \rightarrow \mathbb{R}$

Such that:

$$f_\theta(x) = h(x) \exp(A(\theta)^T S(x) - C(\theta))$$

It's a **canonical exponential family** iff  $A(\theta) = \theta$ .

#### Remark

**Cramer-Rao Lower Bound:** For the variance of a regular estimator, the lower bound is  $\frac{I^{-1}(\theta)}{n}$ . What are regular models? Exponential families are a prime example.

#### Theorem 9: Properties of Exponential Families

Let  $P_\theta$  be an exponential family. If  $f_\theta(x) = h(x) \exp(A(\theta)^T S(x) - C(\theta))$  and if  $\Theta = \{\theta, f_\theta(x)\text{ exists}\}$  is an open set, then:

(i) If  $A(\theta) = \theta$ , then the model is **regular**.

(ii)  $\exp(C(\theta)) = \int_{\mathbb{R}^d} h(x) e^{A(\theta)^T S(x)} dx$  (if continuous) or  $\sum_{x \in \mathcal{X}} h(x) e^{A(\theta)^T S(x)}$  (if discrete).

(iii) If  $A(\theta) = \theta$ , then  $\theta \mapsto C(\theta)$  is  $\mathcal{C}^\infty$  on  $\Theta$  and:

$$\mathbb{E}_\theta(S(X)) = \nabla_\theta C(\theta)$$

$$\mathbb{V}_\theta(S(X)) = \frac{\partial^2 C(\theta)}{\partial \theta^2} = D^2 C(\theta)$$

(iv) If  $A$  is invertible and  $\mathcal{C}^2$ , and  $A^{-1}$  is  $\mathcal{C}^2$ , then the model is regular.

### Remark

" $f_\theta$  exists" means:

- If  $P_\theta$  is continuous:  $\int_{\mathbb{R}^d} e^{A(\theta)^T S(x)} h(x) dx < +\infty$ .
- If  $P_\theta$  is discrete:  $\sum_{x \in \mathcal{X}} e^{A(\theta)^T S(x)} h(x) < +\infty$ .

### Heuristic of the proof (Continuous case)

(ii) We know that  $\int_{\mathbb{R}^d} f_\theta(x) dx = 1$ .

$$\begin{aligned} &\iff \int_{\mathbb{R}^d} h(x) e^{A(\theta)^T S(x) - C(\theta)} dx = 1 \\ &\iff \int_{\mathbb{R}^d} h(x) e^{A(\theta)^T S(x)} dx = e^{C(\theta)} \end{aligned}$$

(iii) If  $A(\theta) = \theta$ .

$$e^{C(\theta)} = \int_{\mathbb{R}^d} e^{\theta^T S(x)} h(x) dx$$

And  $\{\theta, \int_{\mathbb{R}^d} e^{\theta^T S(x)} h(x) dx < +\infty\}$  is open.

Differentiation:

$$\frac{\partial}{\partial \theta} e^{\theta^T S(x)} = S(x) e^{\theta^T S(x)}$$

If  $k = 1$ :

$$\begin{aligned} \frac{d}{d\theta} e^{C(\theta)} &= C'(\theta) e^{C(\theta)} = \int_{\mathbb{R}^d} S(x) e^{\theta^T S(x)} h(x) dx \\ &\implies C'(\theta) = e^{-C(\theta)} \int_{\mathbb{R}^d} S(x) e^{\theta^T S(x)} h(x) dx \\ &= \int_{\mathbb{R}^d} S(x) \underbrace{e^{\theta^T S(x) - C(\theta)} h(x)}_{f_\theta(x)} dx = \mathbb{E}_\theta(S(X)) \end{aligned}$$

Second derivative:

$$\begin{aligned} \frac{d^2}{d\theta^2} e^{C(\theta)} &= C''(\theta) e^{C(\theta)} + (C'(\theta))^2 e^{C(\theta)} = \int_{\mathbb{R}^d} S(x)^2 e^{\theta^T S(x)} h(x) dx \\ &\implies C''(\theta) = - \underbrace{C'(\theta)^2}_{\mathbb{E}_\theta(S(X))^2} + \underbrace{\int_{\mathbb{R}^d} S(x)^2 e^{\theta^T S(x) - C(\theta)} h(x) dx}_{\mathbb{E}_\theta(S(X)^2)} \\ &= \mathbb{E}_\theta(S(X)^2) - \mathbb{E}_\theta(S(X))^2 = \mathbb{V}_\theta(S(X)) \end{aligned}$$

### Remark

Using properties of the log-likelihood:

$$\frac{d^2}{d\theta^2} \log(f_\theta(x)) = -\frac{d^2}{d\theta^2} C(\theta)$$

$$\text{Or } I(\theta) = -\mathbb{E}_\theta \left( \frac{d^2}{d\theta^2} \log f_\theta(X) \right) = \frac{d^2}{d\theta^2} C(\theta).$$

**Justification for differentiation under the integral sign (Proof details):** If  $\theta \in \Theta$ ,  $\theta + \varepsilon \in \Theta$  and  $\theta - \varepsilon \in \Theta$  (since open).

$$\implies \int e^{(\theta+\varepsilon)S(x)} h(x) dx < +\infty \quad \text{and} \quad \int e^{(\theta-\varepsilon)S(x)} h(x) dx < +\infty$$

$$|S(x)|e^{\theta S(x)} \leq e^{\theta S(x)} + \varepsilon |S(x)|e^{\theta S(x)} \dots$$

Wait, actually we use convexity arguments.

$$|S(x)|e^{\theta S(x)} \leq e^{(\theta+\varepsilon)S(x)} + e^{(\theta-\varepsilon)S(x)}$$

This bound (integrable function) allows the use of the Dominated Convergence Theorem.

$$\implies \int |S(x)|e^{\theta S(x)} h(x) dx < +\infty$$

### Proposition 4: Stability of Exponential Families

If  $\{P_\theta, \theta \in \Theta\}$  is an exponential family and  $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} P_\theta$ , then the joint distribution is also an exponential family.

$$f_\theta^{(n)}(x_1, \dots, x_n) = e^{A(\theta)^T \sum_{i=1}^n S(x_i) - nC(\theta)} \prod_{i=1}^n h(x_i)$$

This is an exponential family with:

$$A_n(\theta) = A(\theta), \quad S_n(x_1, \dots, x_n) = \sum_{i=1}^n S(x_i), \quad C_n(\theta) = nC(\theta)$$

*Proof.*

$$\begin{aligned} f_\theta^{(n)}(x_1, \dots, x_n) &= \prod_{i=1}^n f_\theta(x_i) \\ &= \prod_{i=1}^n \left( h(x_i) e^{A(\theta)^T S(x_i) - C(\theta)} \right) \\ &= \left( \prod_{i=1}^n h(x_i) \right) e^{A(\theta)^T \sum S(x_i) - nC(\theta)} \end{aligned}$$

□

### Examples

### Example 9: Binomial Distribution

$X \sim \text{Bin}(n, p)$ ,  $p = \theta$ . Probability mass function:

$$f_p(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x \in \{0, \dots, n\}$$

$$= \underbrace{\binom{n}{x}}_{h(x)} \exp \left( \underbrace{\log \left( \frac{p}{1-p} \right) x}_{A(p)} + \underbrace{n \log(1-p)}_{-C(p)} \right)$$

Here  $S(x) = x$ .

**Canonical Parametrisation:**

$$\eta = \log \left( \frac{p}{1-p} \right) \implies p = \frac{e^\eta}{1+e^\eta}$$

$$\tilde{f}_\eta(x) = \binom{n}{x} \exp(\eta x - n \log(1+e^\eta))$$

With  $C(\eta) = n \log(1+e^\eta)$ .

$$\Theta = \mathbb{R} \text{ (open)}$$

Moments check:

$$\mathbb{E}_\eta(X) = C'(\eta) = n \frac{e^\eta}{1+e^\eta} = np$$

$$\mathbb{V}_\eta(X) = C''(\eta) = n \frac{e^\eta(1+e^\eta) - e^\eta e^\eta}{(1+e^\eta)^2} = n \frac{e^\eta}{(1+e^\eta)^2} = np(1-p)$$

**Remark:** In a canonical exponential family, if  $\Theta$  is an open set, then  $I(\theta) = \frac{d^2 C(\theta)}{d\theta^2}$ .

**Proof:**

$$\begin{aligned} \log(f_\theta(x)) &= \theta^T S(x) - C(\theta) + \log(h(x)) \\ \implies \frac{d}{d\theta} \log(f_\theta(x)) &= S(x) - \nabla C(\theta) \\ \implies \frac{d^2}{d\theta^2} \log(f_\theta(x)) &= -\frac{d^2}{d\theta^2} C(\theta) \\ \implies I(\theta) &= -\mathbb{E}_\theta \left[ \frac{d^2}{d\theta^2} \log(f_\theta(x)) \right] \quad (\text{see Lemma 1}) \\ &= \frac{d^2}{d\theta^2} C(\theta) \end{aligned}$$

### Example 10: Exponential Distribution

$X \sim \mathcal{E}(\theta)$ ,  $\theta > 0$ .

$$f_\theta(x) = \theta e^{-\theta x} \mathbb{1}_{x>0} = \mathbb{1}_{x>0} e^{-\theta x + \log(\theta)}$$

$h(x) = \mathbb{1}_{x>0}$ .  $A(\theta) = \theta$  (or  $-\theta$ ).  $S(x) = -x$ .  $C(\theta) = -\log(\theta)$ . Canonical exponential family.

## IV. Sufficient Condition for Consistency of MLE

### Theorem 10: Consistency of MLE

If  $\Theta$  is compact,

$$\text{If (i) } \sup_{\theta \in \Theta} \left| \frac{\ln(L_n(\theta)) - \ln(L_n(\theta_0))}{n} + KL(\theta_0, \theta) \right| \xrightarrow[n \rightarrow \infty]{P_{\theta_0}} 0$$

$$\text{If (ii) } \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \inf_{|\theta - \theta_0| > \varepsilon} KL(\theta_0, \theta) > \delta.$$

Then the MLE is consistent.

**Where:**

$$KL(\theta_0, \theta) = \mathbb{E}_{\theta_0} [\log f_{\theta_0}(X) - \log f_\theta(X)]$$

(Kullback-Leibler divergence between  $f_{\theta_0}$  and  $f_\theta$ ).

*Proof.*

$$KL(\theta_0, \theta) = \mathbb{E}_{\theta_0} [\log(f_{\theta_0}(X)) - \log(f_\theta(X))]$$

Idea of the proof: Prove that  $\forall \varepsilon > 0, P_{\theta_0} \left[ \sup_{|\theta - \theta_0| > \varepsilon} \ln(\theta) < \sup_{\theta \in \Theta} \ln(\theta) \right] \rightarrow 1$  as  $n \rightarrow \infty$ . (where  $\ln(\theta)$  denotes the log-likelihood).

Because if  $\sup_{|\theta - \theta_0| > \varepsilon} \ln(\theta) < \sup_{\theta \in \Theta} \ln(\theta)$ , then  $|\hat{\theta}_n - \theta_0| \leq \varepsilon$  (since  $\hat{\theta}_n$  maximizes the likelihood).

Show that:

$$\begin{aligned} P_{\theta_0} \left( \sup_{|\theta - \theta_0| > \varepsilon} (\ln(\theta) - \ln(\theta_0)) < 0 \right) &\xrightarrow{n \rightarrow \infty} 1 \\ \sup_{|\theta - \theta_0| > \varepsilon} \left[ \frac{\ln(\theta) - \ln(\theta_0)}{n} + KL(\theta_0, \theta) - KL(\theta_0, \theta_0) \right] \end{aligned}$$

We have:

$$\frac{\ln(\theta) - \ln(\theta_0)}{n} = - \left[ \frac{1}{n} \sum_{i=1}^n \log \left( \frac{f_{\theta_0}(x_i)}{f_\theta(x_i)} \right) \right]$$

By LLN  $\rightarrow -\mathbb{E}_{\theta_0} \left[ \log \left( \frac{f_{\theta_0}(X)}{f_\theta(X)} \right) \right] = -KL(\theta_0, \theta)$ .

By assumption (i):

$$\sup_{|\theta - \theta_0| > \varepsilon} \left| \frac{\ln(\theta) - \ln(\theta_0)}{n} + KL(\theta_0, \theta) \right| \xrightarrow{P_{\theta_0}} 0$$

And using (ii):

$$\sup_{|\theta - \theta_0| > \varepsilon} -KL(\theta_0, \theta) = - \inf_{|\theta - \theta_0| > \varepsilon} KL(\theta_0, \theta) < -\delta$$

So:

$$\sup_{|\theta - \theta_0| > \varepsilon} \frac{\ln(\theta) - \ln(\theta_0)}{n} < -\delta + o(1)$$

Which is  $< 0$  for  $n$  large enough.  $\square$

## V. Using Asymptotic Normality to Compute Confidence Regions

$\theta \in \Theta \subset \mathbb{R}$ . Model  $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} P_\theta$ .  $\hat{\theta}_n$  = MLE.

We know:

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow[n \rightarrow \infty]{\mathcal{L}_{P_{\theta_0}}} \mathcal{N}(0, I^{-1}(\theta_0))$$

We want  $I_n = [\hat{\theta}_{n,1}, \hat{\theta}_{n,2}]$  such that:

$$P_{\theta_0}(\theta_0 \in I_n) \geq 1 - \alpha$$

Then  $I_n$  is a  $(1 - \alpha)$  confidence interval for  $\theta_0$ .

$I_n$  is an **asymptotic**  $(1 - \alpha)$  confidence interval if  $P_{\theta_0}(\theta_0 \in I_n) \xrightarrow{n \rightarrow \infty} 1 - \alpha$ .

**Exercise:** Construct  $I_n$ .

## VI. Delta method and confidence intervals

### VI.1 Delta method

#### Remark

**Reminder:** If  $\sqrt{n}(X_n - \mu) \xrightarrow{\mathcal{L}} \mathcal{N}(0, V_0)$ , with  $X_n \in \mathbb{R}^{d_1}$  and if  $g : \mathbb{R}^{d_1} \rightarrow \mathbb{R}^{d_2}$  is  $\mathcal{C}^1$  and s.t (sub-tangent/standard) ( $d_1 \leq d_2$ ),  $\nabla g(\mu) \in \mathbb{R}^{d_2 \times d_1}$  is of rank  $d_2$

Then:

$$\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \nabla g(\mu)V_0\nabla g(\mu)^\top)$$

*Proof.* Case  $d_1 = d_2 = 1$ .

Taylor expansion:

$$g(X_n) = g(X_n - \mu + \mu) = g(\mu) + (X_n - \mu)g'(\bar{\mu}_n)$$

where  $\bar{\mu}_n \in ]X_n, \mu[$ .

This implies:

$$\begin{aligned} \sqrt{n}(g(X_n) - g(\mu)) &= \sqrt{n}(X_n - \mu)g'(\bar{\mu}_n) \\ &= \sqrt{n}(X_n - \mu)(g'(\mu) + o_p(1)) \end{aligned}$$

We know that  $\sqrt{n}(X_n - \mu)g'(\mu) \xrightarrow{\mathcal{L}} \mathcal{N}(0, g'(\mu)^2 V_0)$ .

And  $\sqrt{n}(X_n - \mu)(g'(\bar{\mu}_n) - g'(\mu)) \xrightarrow{\mathbb{P}} 0$ . □

#### Remark

##### Reminders:

- $X_n = O_p(1) \iff \lim_{C \rightarrow \infty} \limsup_n \mathbb{P}(|X_n| > C) = 0$ .
- $X_n = o_p(1) \iff X_n \xrightarrow{\mathbb{P}} 0$ .
- If  $X_n \xrightarrow{\mathcal{L}} Q$  (where  $Q$  is a probability distribution), then  $X_n = O_p(1)$ .

## Application to the MLE

If  $\hat{\theta}_n$  is the MLE and if  $\hat{\eta}_n = g(\hat{\theta}_n)$  with  $g \in \mathcal{C}^1$ ,  $\hat{\theta}_n \in \mathbb{R}^d$ ,  $g : \mathbb{R}^d \rightarrow \mathbb{R}^k$ .

If  $\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0, I^{-1}(\theta_0))$  with  $\nabla g(\theta_0)$  of rank  $k$ .

Then:

$$\sqrt{n}(\hat{\eta}_n - g(\theta_0)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \nabla g(\theta_0)I^{-1}(\theta_0)\nabla g(\theta_0)^\top)$$

### Example 11: Bernoulli

$X_1, \dots, X_n \stackrel{i.i.d}{\sim} \text{Ber}(p)$  under  $\mathbb{P}_{p_0}$ .

MLE:  $\hat{p}_n = \bar{X}_n$  and  $\sqrt{n}(\hat{p}_n - p_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0, p_0(1 - p_0))$ .

Let  $\eta = \log(\frac{p}{1-p})$  and  $g(p) = \log(\frac{p}{1-p})$ .

$g : ]0, 1[ \rightarrow \mathbb{R}$  is invertible. For  $x \in \mathbb{R}$ ,  $g^{-1}(x) = \frac{e^x}{1+e^x}$ .

Derivative:

$$g'(p) = \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}$$

$\forall p_0 \in ]0, 1[, g'(p_0) \neq 0$ .

$$\sqrt{n}(\hat{\eta}_n - \eta) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{p_0(1-p_0)}\right)$$

Note:  $\frac{1}{p_0(1-p_0)} = \frac{(1+e^{\eta_0})^2}{e^{\eta_0}}$ .

### Example 12: Normal Distribution

$X_1, \dots, X_n \stackrel{i.i.d}{\sim} \mathcal{N}(\mu, \sigma^2)$ .  
 $\theta = (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}^{+*}$ .

MLE:  $\hat{\theta}_n = (\bar{X}_n, S_n^2)$  where  $S_n^2 = \frac{1}{n} \sum_{k=1}^n (X_k - \bar{X}_n)^2$ .

Canonical parameters:

$$\begin{aligned} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} &= \frac{e^{-\frac{x^2}{2\sigma^2} - \frac{x\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2} - \frac{\log(\sigma^2)}{2}}}{\sqrt{2\pi}} \\ \implies \eta_1 &= \frac{\mu}{\sigma^2}, \quad \eta_2 = \frac{-1}{2\sigma^2} \end{aligned}$$

$$\implies \sqrt{n}(\hat{\eta}_n - \eta) \rightarrow \mathcal{N}(0, \tilde{V}_0)$$

Let  $g(\eta) = \sigma^2$ . We have  $\sigma^2 = -\frac{1}{2\eta_2}$ .

Gradient:

$$\nabla g(\eta) = \left( \frac{\partial g}{\partial \eta_1}, \frac{\partial g}{\partial \eta_2} \right) = \left( 0, \frac{1}{2\eta_2^2} \right)$$

$$\implies \sqrt{n}(S_n^2 - \sigma^2) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, (0, \frac{1}{2\eta_2^2})\tilde{V}_0\left(\frac{0}{2\eta_2^2}\right)\right)$$

Where  $\tilde{V}_0 = \frac{d^2 c}{d^2 \eta}(\eta_0)$  and  $c(\eta_0) = \frac{\eta_1^2}{8\eta_2} + \frac{1}{2} \log(-\frac{1}{2\eta_2})$ .

$$\implies c'' = \dots$$

## VI.2 Confidence intervals

### Definition 21: Confidence Region

A confidence region of level  $\alpha$  for an estimator of  $g(\theta)$  is any region  $C_\alpha(X_1, \dots, X_n)$  verifying:

$$\forall \theta \in \Theta, \quad \mathbb{P}_\theta[g(\theta) \in C_\alpha(X_1, \dots, X_n)] \geq 1 - \alpha$$

We can use asymptotic normality of an estimator to construct asymptotic confidence regions.

### Example 13: Construction using MLE

$X_1, \dots, X_n \stackrel{i.i.d}{\sim} \mathbb{P}_\theta, \theta \in \Theta \subset \mathbb{R}$ .

If the MLE verifies:

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[\mathbb{P}_\theta]{\mathcal{L}} \mathcal{N}(0, I_\theta^{-1})$$

We look for:

$$C_\alpha(X_1, \dots, X_n) = [\hat{\theta}_{n,1}, \hat{\theta}_{n,2}]$$

s.t.

$$\mathbb{P}_\theta[\theta \in [\hat{\theta}_{n,1}, \hat{\theta}_{n,2}]] \xrightarrow[n \rightarrow \infty]{} 1 - \alpha$$

$$\hat{\theta}_{n,1} \leq \theta \leq \hat{\theta}_{n,2} \iff \sqrt{n}(\hat{\theta}_{n,1} - \hat{\theta}_n) \leq \sqrt{n}(\theta - \hat{\theta}_n) \leq \sqrt{n}(\hat{\theta}_{n,2} - \hat{\theta}_n)$$

But:

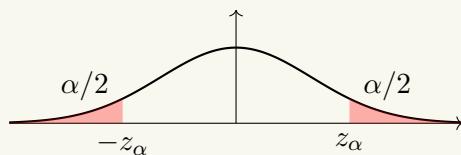
$$\begin{aligned} \sqrt{n}(\theta - \hat{\theta}_n) &\xrightarrow{\mathcal{L}} \mathcal{N}(0, I_\theta^{-1}) \\ &\implies \sqrt{n}I_\theta^{1/2}(\theta - \hat{\theta}_n) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1) \end{aligned}$$

Since  $\hat{\theta}_n \xrightarrow{\mathbb{P}_\theta} \theta \implies I_{\hat{\theta}_n} \xrightarrow{\mathbb{P}_\theta} I_\theta$ .

$$\implies \sqrt{n}I_{\hat{\theta}_n}^{1/2}(\theta - \hat{\theta}_n) \xrightarrow[\mathbb{P}_\theta]{\mathcal{L}} \mathcal{N}(0, 1)$$

If  $z_\alpha$  is the  $1 - \frac{\alpha}{2}$  quantile of a  $\mathcal{N}(0, 1)$ :

$$\iff \Phi(z_\alpha) = 1 - \frac{\alpha}{2}, \quad \Phi = \text{CDF of } \mathcal{N}(0, 1)$$



By symmetry:  $\Phi(-z_\alpha) = \frac{\alpha}{2}$ .

$$\implies \lim_{n \rightarrow \infty} \mathbb{P}_\theta \left( -z_\alpha \leq \sqrt{n}I_{\hat{\theta}_n}^{1/2}(\theta - \hat{\theta}_n) \leq z_\alpha \right) = 1 - \alpha$$

$$\iff \lim_{n \rightarrow \infty} \mathbb{P}_\theta \left( \frac{-z_\alpha}{I_{\hat{\theta}_n}^{1/2}} \leq \sqrt{n}(\theta - \hat{\theta}_n) \leq \frac{z_\alpha}{I_{\hat{\theta}_n}^{1/2}} \right) = 1 - \alpha$$

So if:

$$\hat{\theta}_{n,1} = \hat{\theta}_n - \frac{z_\alpha}{\sqrt{n}\sqrt{I_{\hat{\theta}_n}}} \quad ; \quad \hat{\theta}_{n,2} = \hat{\theta}_n + \frac{z_\alpha}{\sqrt{n}\sqrt{I_{\hat{\theta}_n}}}$$

Then:

$$\mathbb{P}_\theta(\hat{\theta}_{n,1} \leq \theta \leq \hat{\theta}_{n,2}) \xrightarrow{n \rightarrow \infty} 1 - \alpha$$

# Chapter 5: Bayesian statistics

**Model:**  $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \mathbb{P}_\theta$ .  
 $\hat{\theta}_n \rightarrow \mathcal{R}(\theta, \hat{\theta}_n) = \mathbb{E}[(\theta - \hat{\theta}_n)^2] = \text{Quadratic risk.}$

## I. In Bayesian statistics

$\theta$  = unknown  $\rightarrow$  Modelled as a random variable.

### Definition 22: Prior Distribution

We call prior distribution on  $\theta$ , a probability distribution over  $\Theta$ .

**Model:**

$X_1, \dots, X_n | \theta \stackrel{i.i.d.}{\sim} \mathbb{P}_\theta$  : Conditional distribution of  $(X_1, \dots, X_n)$  given  $\theta$ .

**Prior Marginal Probability over  $\Theta$ :**

$\theta \sim \Pi$  Probability on  $\Theta$ .

$\Rightarrow$  **Joint distribution** on  $(X_1, \dots, X_n, \theta)$ .

IF  $\mathbb{P}_\theta$  is continuous, with density  $f_\theta$  (wrt Lebesgue measure).

### Theorem 11: Bayes Theorem

The conditional distribution of  $\theta$  given  $(X_1, \dots, X_n)$  has density wrt  $\nu$ :

$$\Pi_n(\theta | X_1, \dots, X_n) = \frac{\Pi(\theta) \prod_{i=1}^n f_\theta(x_i)}{\int_\Theta \Pi(\theta) \prod_{i=1}^n f_\theta(x_i) d\nu(\theta)}$$

If  $\pi$  is the density of  $\Pi$  wrt  $\nu$ , it is called the **posterior distribution**.

### Definition 23: Bayesian Model

In Bayesian statistics, the Bayesian model is defined by:

- i)  $[X | \theta] \sim \mathbb{P}_\theta$  with likelihood  $L(\theta)$ ,  $\theta \in \Theta$ .
- ii) Prior distribution  $\Pi$  on  $\Theta$ .

Then the inference is made by the **posterior distribution** defined as the conditional distribution of  $\theta$  given  $X$ . If  $\pi$  is the density of  $\Pi$  (wrt a measure  $\nu$ ), then the posterior distribution has density (wrt  $\nu$ ) given by:

$$\Pi(\theta | X) = \frac{L(\theta)\Pi(\theta)}{\int_\Theta L(\theta)\Pi(\theta)d\nu(\theta)}$$

where

$$\int_\Theta L(\theta)\Pi(\theta)d\nu(\theta) = \begin{cases} \int L(\theta)\Pi(\theta)d\nu(\theta) & \text{if } \Pi \text{ continuous} \\ \sum_{\theta \in \Theta} L(\theta)\Pi(\theta) & \text{if } \Pi \text{ discrete} \end{cases}$$

### Remark

- If  $\mathbb{P}_\theta$  is continuous:  
 $L(\theta) = f_\theta(x) = \text{density at the observations when } \theta \text{ is the parameter.}$
- If  $\mathbb{P}_\theta$  is discrete:

$$L(\theta) = f_\theta(x) = \mathbb{P}_\theta(X = x).$$

$$\Pi(\theta|X = x) = \frac{f_\theta(x)\Pi(\theta)}{\int_\Theta f_\theta(x)\Pi(\theta)d\nu(\theta)}$$

### Example 14: Poisson - Gamma

$X = (X_1, \dots, X_n)$  and  $X_i \stackrel{i.i.d.}{\sim} \mathcal{P}(\theta), \theta > 0$ .

**Prior:**  $\theta \sim \Gamma(a, b)$ ,  $\Pi$  is a  $\Gamma(a, b)$ .

$$f_\theta(x) = \prod_{i=1}^n f_\theta(x_i)$$

$$\implies \Pi(\theta|X_1, \dots, X_n) = \frac{\prod_{i=1}^n f_\theta(x_i) \theta^{a-1} e^{-b\theta}}{\int_\Theta \prod_{i=1}^n f_\theta(x_i) \theta^{a-1} e^{-b\theta} d\theta}$$

Normalization constant (denominator) implies:

$$\Pi(\theta|X_1, \dots, X_n) \propto e^{-n\theta} \prod_{i=1}^n \frac{\theta^{x_i}}{x_i!} \times \theta^{a-1} e^{-b\theta}$$

$$\propto e^{-(n+b)\theta} \theta^{a+\sum_{i=1}^n x_i - 1}$$

(Recall:  $f_\theta(x_i) = e^{-\theta} \frac{\theta^{x_i}}{x_i!}$ ).

$\Pi(\cdot)$  is a density  $\implies \frac{f(\theta)}{g(\theta)}$  is a constant implies  $\exists c \neq 0, f(\theta) = cg(\theta)$ .  
 $\exists C > 0$  s.t.

$$\Pi(\theta|X_1, \dots, X_n) = Ce^{-(b+n)\theta} \theta^{\sum_{i=1}^n x_i + a - 1}$$

We recognize the kernel of a Gamma distribution  $\Gamma(\sum_{i=1}^n x_i + a, b + n)$ .

$$\implies \Pi(\theta|X_1, \dots, X_n) = \frac{e^{-(b+n)\theta} \theta^{\sum_{i=1}^n x_i + a - 1} (b + n)^{a + \sum_{i=1}^n x_i}}{\Gamma(a + \sum_{i=1}^n x_i)}$$

Posterior distribution is  $\Gamma(\sum x_i + a, b + n)$ .

Note regarding the constant:

$$g(\theta) = \theta^{b-1} e^{-b\theta} \times c$$

$$\int_{\mathbb{R}^+} g(\theta) d\theta = 1 \iff c = \frac{b^a}{\Gamma(a)}$$

### Example 15: Normal - Normal

**Bayesian model:**  $X_1, \dots, X_n | \mu \stackrel{i.i.d.}{\sim} \mathcal{N}(\mu, 1), \mu \in \mathbb{R}$ .

**Prior:**  $\mu \sim \mathcal{N}(a, b^2), a \in \mathbb{R}, b > 0$ .

Posterior distribution? Continuous with density.

$$\Pi(\mu|X_1, \dots, X_n) \propto \prod_{i=1}^n \frac{e^{-\frac{(x_i - \mu)^2}{2}}}{\sqrt{2\pi}} \frac{e^{-\frac{(\mu - a)^2}{2b^2}}}{\sqrt{2\pi b}}$$

$$\propto \exp\left(-\frac{n}{2}\{\mu^2 + \bar{X}_n^2 - 2\mu\bar{X}_n\} - \frac{1}{2b^2}\{\mu^2 + a^2 - 2a\mu\}\right)$$

Because:

$$\begin{aligned} \sum_{i=1}^n (X_i - \mu)^2 &= \sum_{i=1}^n [(X_i - \bar{X}_n)^2 + (\bar{X}_n - \mu)^2 + 2(\bar{X}_n - \mu)(X_i - \bar{X}_n)] \\ &= n(\bar{X}_n - \mu)^2 + \sum_{i=1}^n (X_i - \bar{X}_n)^2 \\ \Pi(\mu | X_1, \dots, X_n) &\propto e^{-\frac{1}{2}(n+\frac{1}{b^2})\left(\mu - \frac{\frac{a}{b^2} + n\bar{X}_n}{n+\frac{1}{b^2}}\right)^2} \end{aligned}$$

$\implies$  Posterior distribution is:

$$a \mathcal{N}\left(\frac{\frac{a}{b^2} + n\bar{X}_n}{n + \frac{1}{b^2}}, \frac{1}{n + \frac{1}{b^2}}\right)$$

#### Definition 24: Marginal Likelihood

In a Bayesian model  $X|\theta \sim f_\theta(x)$ ,  $\theta \in \Theta$  (Likelihood) and prior  $\theta \sim \Pi$  with density  $\pi$  wrt  $\nu$ .  
The quantity:

$$m(x) = \int_{\Theta} f_\theta(x) \Pi(\theta) d\nu(\theta)$$

is called the **marginal likelihood**.

## II. Bayesian decision theory: Risks

### II.1 Posterior and integrated risks

Recall: Quadratic Risk:

$$\mathcal{R}(\theta, \delta) = \begin{cases} \int (\theta - \delta(x))^2 f_\theta(x) dx \\ \sum_x (\theta - \delta(x))^2 \mathbb{P}_\theta(X = x) \end{cases}$$

#### Definition 25: Loss Function

A loss function is a function:

$$l : \Theta \times \mathcal{D} \rightarrow \mathbb{R}^+$$

where  $\mathcal{D}$  is the set of decisions (i.e., estimators).

#### Example 16: Examples of Loss Functions

1. If  $\mathcal{D} = \Theta$ .

- a) The quadratic loss  $l(\theta, \delta) = (\theta - \delta)^2$ .
- b) The  $L_1$  loss  $l(\theta, \delta) = |\theta - \delta|$ .

2.  $\mathcal{D} = \{0, 1\}$ ,  $\theta \in [0, 1]$ .

Aim is testing if  $\theta > \frac{1}{2}$  (choosing 1) or  $\theta \leq \frac{1}{2}$  (choosing 0).

**0-1 loss function** defined by:

$$l(\theta, \delta) = 1 \quad \text{if } \theta < \frac{1}{2} \text{ and } \delta = 1 \text{ (wrong decision)}$$

$$\quad \quad \quad \text{if } \theta > \frac{1}{2} \text{ and } \delta = 0 \text{ (wrong decision)}$$

$$l(\theta, \delta) = 0 \quad \text{else (correct decision)}$$

### Definition 26: Risks

In a Bayesian model  $X|\theta \sim f_\theta(x)$ ,  $\theta \sim \Pi$  with a loss function  $l : \Theta \times \mathcal{D} \rightarrow \mathbb{R}^+$ . where:

- a) We call the **posterior risk**:

$$l(\Pi, \delta | X) = \int_{\Theta} l(\theta, \delta) \Pi(\theta | X_1, \dots, X_n) d\nu(\theta)$$

- b) We call the **integrated risk**:

$$r(\Pi, \delta) = \begin{cases} \int_x l(\Pi, \delta | X = x) m(x) dx & \text{if } X \text{ continuous} \\ \sum_{x \in \mathcal{X}} l(\Pi, \delta | X = x) m(x) & \text{if } X \text{ discrete} \end{cases}$$

### Example 17: Calculation of Posterior Risk

**1) Bayesian model**,  $X_1, \dots, X_n | \theta \sim \mathcal{P}(\theta)$ ,  $\theta > 0$ .

**Prior**:  $\theta \sim \Gamma(a, b)$ .

**Loss function**: Quadratic.  $\mathcal{D} = \Theta = \mathbb{R}_+$ .

$\forall \theta > 0, \delta > 0$ ,  $l(\theta, \delta) = (\theta - \delta)^2$ .

Posterior risk:

**Posterior distribution**:  $\Gamma(a + \sum_{i=1}^n X_i, b + n)$ .

$$\Pi(\theta | X_1, \dots, X_n) = \frac{(b+n)^{n\bar{X}_n+a} e^{-(b+n)\theta} \theta^{n\bar{X}_n+a-1}}{\Gamma(a+n\bar{X}_n)}$$

With  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ .

$\forall \delta > 0$ :

$$\begin{aligned} l(\Pi, \delta | X_1, \dots, X_n) &= \int_0^\infty (\theta - \delta)^2 \Pi(\theta | X_1, \dots, X_n) d\theta \\ &= \int_0^\infty \theta^2 \Pi(\theta | X_1, \dots, X_n) d\theta + \delta^2 - 2\delta \int_0^\infty \theta \Pi(\theta | X_1, \dots, X_n) d\theta \\ &\implies l(\Pi, \delta | X_1, \dots, X_n) = \frac{a'(a'+1)}{b'^2} + \delta^2 - 2\delta \frac{a'}{b'} \end{aligned}$$

where  $a' = a + n\bar{X}_n$  and  $b' = b + n$ . (Using moments of Gamma distribution).

→ Can be minimized in  $\delta$ .

$\delta(X_1, \dots, X_n)$  = minimizer of  $l(\Pi, \delta | X_1, \dots, X_n)$ .

Show that:

$$\delta(X_1, \dots, X_n) = \frac{a'}{b'} = \frac{a + n\bar{X}_n}{b + n}$$

### Example 18: Example 2

$$\ell : \Theta \times \mathcal{D} \rightarrow \mathbb{R}_+$$

- Posterior risk:

$$\ell(\Pi, \delta | x_1, \dots, x_n) = \int_{\Theta} \ell(\theta, \delta) \Pi(\theta | x_1, \dots, x_n) d\theta$$

$X_1, \dots, X_n \sim \mathcal{P}(\theta) ; \theta \sim \Gamma(a, b)$ .

The posterior is:

$$\Pi(\theta | X_1, \dots, X_n) = \frac{\theta^{a+n\bar{x}_n-1} e^{-(b+n)\theta} (b+n)^{a+n\bar{x}_n}}{\Gamma(a+n\bar{x}_n)}$$

We set the hypotheses:

- $H_0 = \{\theta < 1\}$
- $H_1 = \{\theta \geq 1\}$

Decision rule:

$$\begin{cases} \delta = 1 & \text{if choose } H_0 \\ \delta = 0 & \text{if choose } H_1 \end{cases}$$

Let us use the 0-1 loss:

$$\begin{aligned} \ell(\Pi, \delta | x_1, \dots, x_n) &= \int_0^{+\infty} [\mathbb{1}_{\delta=1} \mathbb{1}_{\theta \geq 1} + \mathbb{1}_{\delta=0} \mathbb{1}_{\theta < 1}] \Pi(\theta | x_1, \dots, x_n) d\theta \\ &= \underbrace{\mathbb{1}_{\delta=1} \int_1^{\infty} \Pi(\theta | x_1, \dots, x_n) d\theta}_{\Pi(\theta > 1 | X)} + \underbrace{\mathbb{1}_{\delta=0} \int_0^1 \Pi(\theta | x_1, \dots, x_n) d\theta}_{\Pi(\theta \leq 1 | X)} \\ &= \mathbb{1}_{\delta=1} \Pi(\theta > 1 | x_1, \dots, x_n) + \mathbb{1}_{\delta=0} \Pi(\theta \leq 1 | x_1, \dots, x_n) \end{aligned}$$

Thus:

$$\ell(\theta, \delta) = \mathbb{1}_{\delta=1} \mathbb{1}_{\theta \geq 1} + \mathbb{1}_{\delta=0} \mathbb{1}_{\theta < 1}$$

### Definition 27: Bayesian Estimator

In a Bayesian model  $X | \theta \sim P_\theta, \theta \in \Theta$ , prior  $\theta \sim \Pi$ .

If  $\ell : \Theta \times \mathcal{D} \rightarrow \mathbb{R}_+$  is a loss function, we define the **Bayes estimators** as (when they exist) the minimizers of  $\ell(\Pi, \delta | X)$  (or posterior risk, denoted  $\ell(\Pi, \delta | X = x)$ ) in  $\delta$ .

### Proposition 5: Property

Bayesian estimators also minimize the integrated Risk.

*Proof.* Bayesian Model  $P_\theta(x), \theta \in \Theta$  and Prior  $\Pi$ . Loss function  $\ell(\theta, \delta), \delta \in \mathcal{D}$ .

Integrated risk  $r(\Pi, \delta) = \int_{\mathcal{X}} \ell(\Pi, \delta | X = x) m(x) dx$ .

Since  $\forall x, \delta^\pi(x)$  verifies:

$$\int_{\Theta} \ell(\Pi, \delta^\pi(x) | X) \leq \int_{\Theta} \ell(\Pi, \delta | X)$$

(Inequality on posterior risk).

$$\begin{aligned} &\implies \int \ell(\Pi, \delta^\pi(x)|X=x)m(x)dx \leq \int \ell(\Pi, \delta(x)|X=x)m(x)dx \\ &\implies r(\Pi, \delta^\Pi) \leq r(\Pi, \delta), \quad \forall \text{ estimator } \delta \text{ when } \delta^\Pi \text{ is a Bayesian estimator.} \end{aligned}$$

□

## Why is it interesting?

### Theorem 12: Statistical Model

$X|\theta \sim P_\theta, \theta \in \Theta$  (Model), Prior  $\theta \sim \pi$ .

- Loss function:  $\ell : \Theta \times \mathcal{D} \rightarrow \mathbb{R}_+$
- The **Frequentist risk** is defined as the function  $R : \Theta \times \mathcal{D} \rightarrow \mathbb{R}_+ ; R(\theta, \delta) = \mathbb{E}_\theta(\ell(\theta, \delta(X)))$ .

$$= \begin{cases} \int_{\mathcal{X}} \ell(\theta, \delta(x))f(x|\theta)dx & \text{if } X \text{ continuous} \\ \sum_{x \in \mathcal{X}} \ell(\theta, \delta(x))f(x|\theta) & \text{if } X \text{ discrete} \end{cases}$$

Then:

$$\int_{\Theta} R(\theta, \delta)\pi(\theta)d\theta = r(\pi, \delta)$$

### Example 19: Quadratic Risk

If  $\ell(\theta, \delta) = (\theta - \delta)^2$ ,  $R(\theta, \delta)$  is the quadratic risk.

*Proof.*

$$r(\Pi, \delta) = \int_{\mathcal{X}} \rho(\Pi, \delta|X=x)m(x)dx$$

$$m(x) = \int_{\Theta} f(x|\theta)\Pi(\theta)d\theta$$

and  $\rho(\Pi, \delta|x=x) = \int_{\Theta} \ell(\theta, \delta)\Pi(\theta|x=x)d\theta$ .

$$\begin{aligned} &\implies r(\pi, \delta) = \int_{\mathcal{X}} \int_{\Theta} \ell(\theta, \delta(x)) \underbrace{\pi(\theta|x=x)m(x)}_{f(x|\theta)\pi(\theta)/m(x) \cdot m(x)} d\theta dx \\ &= \int_{\mathcal{X}} \int_{\Theta} \ell(\theta, \delta(x))f(x|\theta)\pi(\theta)d\theta dx \\ &= \int_{\Theta} \left[ \int_{\mathcal{X}} \ell(\theta, \delta(x))f(x|\theta)dx \right] \pi(\theta)d\theta \\ &= \int_{\Theta} R(\theta, \delta)\pi(\theta)d\theta \end{aligned}$$

□

### Example 20: Example

$X_1, \dots, X_n \sim \mathcal{P}(\theta)$ . Prior  $\Pi(\theta) = \frac{\theta^{a-1} e^{-b\theta} b^a}{\Gamma(a)}$ ,  $\theta > 0$ .  
 $a = 1, b = 1$ :  $\pi(\theta) = e^{-\theta}, \theta > 0$ .

$$\begin{aligned}\Pi(\theta|X_1, \dots, X_n) &= \frac{\theta^{\sum X_i} e^{-(n+1)\theta} (n+1)^{n\bar{X}_n}}{\Gamma(n\bar{X}_n)} \\ &= \frac{\prod_{i=1}^n f(X_i|\theta) \Pi(\theta)}{\int_{\Theta} \prod_{i=1}^n f(X_i|\theta) \Pi(\theta) d\theta}\end{aligned}$$

## II.2 Computation of Bayesian estimators

- Model:  $X|\theta \sim P_{\theta}, \theta \in \Theta$
- Prior:  $\theta \sim \Pi$
- Loss function:  $\ell : \Theta \times \mathcal{D} \rightarrow \mathbb{R}_+$

### a) Quadratic loss

$$\ell(\theta, \delta) = (\theta - \delta)^2, \quad \mathcal{D} = \Theta$$

$$\rho(\Pi, \delta|x = x) = \int_{\Theta} (\theta - \delta)^2 \Pi(\theta|X = x) d\theta$$

is convex in  $\delta$ .

$$\frac{\partial}{\partial \delta} \ell(\Pi, \delta|x = x) = 2 \int_{\Theta} \theta \Pi(\theta|X = x) + 2\delta \underbrace{\int_{\Theta} \Pi(\theta|X = x)}_{=1}$$

Wait, correction from notes:

$$\begin{aligned}&= 2 \int_{\Theta} (\delta - \theta) \Pi(\theta|X = x) d\theta = 2\delta - 2 \int_{\Theta} \theta \Pi(\theta|X = x) d\theta \\ &= 0 \iff \delta(x) = \int_{\Theta} \theta \Pi(\theta|X = x) d\theta = E^{\Pi}(\theta|X = x) = \text{posterior mean.}\end{aligned}$$

### Proposition 6: Posterior Mean

The Bayesian estimator associated to the quadratic loss is the posterior mean.

### b) $L_1$ loss

$$\ell(\theta, \delta) = |\theta - \delta|, \quad \mathcal{D} = \Theta = \mathbb{R}$$

Posterior risk:

$$\ell(\Pi, \delta|X = x) = \int_{\mathbb{R}} |\theta - \delta| \Pi(\theta|X = x) d\theta$$

Convex in  $\delta$  and:

$$\begin{aligned}\rho(\Pi, \delta | X = x) &= \int_{-\infty}^{\delta} (\delta - \theta) \Pi(\theta | X = x) d\theta \\ &\quad + \int_{\delta}^{+\infty} (\theta - \delta) \Pi(\theta | X = x) d\theta\end{aligned}$$

$$\frac{\partial}{\partial \delta} \rho(\Pi, \delta | X = x) = \int_{-\infty}^{\delta} \Pi(\theta | X = x) d\theta - \int_{\delta}^{+\infty} \Pi(\theta | X = x) d\theta$$

(plus boundary terms that cancel out)

$$\begin{aligned}&+ \delta(\Pi(\delta | x = x)) + \dots \\ &- \delta \Pi(\delta | x = x) - \delta \Pi(\delta | X = x)\end{aligned}$$

(Terms from Leibniz rule cancel out)

$$\begin{aligned}&= \Pi(\theta \leq \delta | X = x) - \Pi(\theta > \delta | X = x) \\ &= 0 \iff \delta \text{ is the median of the posterior distribution.}\end{aligned}$$

### III. Confidence intervals in Bayesian analysis: credible regions

Let  $\alpha \in [0, 1]$  be the risk level, and  $\Pi$  be the prior on  $\Theta \subset \mathbb{R}^d$ .

#### Definition 28: Checking Credible Set

An  $\alpha$ -credible set is a random set  $C(\alpha, X_1, \dots, X_n)$  such that:

$$\Pi(\theta \in C(\alpha, X_1, \dots, X_n) | X_1, \dots, X_n) = \int_{C(\alpha, X_1, \dots, X_n)} \pi(\theta | X_1, \dots, X_n) d\theta = 1 - \alpha$$

In particular, if  $\Theta = \mathbb{R}$ , an  $\alpha$ -credible interval is denoted  $[l_\alpha(X_1, \dots, X_n), r_\alpha(X_1, \dots, X_n)]$ .  $l_\alpha$  and  $r_\alpha$  solve (satisfy):

$$\int_{-\infty}^{l_\alpha} \pi(\theta | X_1, \dots, X_n) d\theta = \int_{r_\alpha}^{+\infty} \pi(\theta | X_1, \dots, X_n) d\theta = \frac{\alpha}{2}$$

#### Remark

**Carefull:** A Bayesian credible set  $\neq$  Frequentist confidence set interval.

In Bayesian, we have:

$$\Pi(\theta \in (l_\alpha, r_\alpha) | X_1, \dots, X_n) \geq 1 - \alpha$$

but we do not have:

$$\mathbb{P}_\theta(\theta \in (l_\alpha, r_\alpha)) \geq 1 - \alpha$$

However, if  $f_\theta(x)$  is nice enough, then credible sets can be turned into asymptotically optimal confidence intervals (as  $n \rightarrow \infty$ ).

## IV. Choosing priors

### IV.1 Conjugate priors

Let  $f(x|\theta) \in \mathcal{F}$  be a family of models (Poisson, Gamma, Normal, Ber...).  
Let  $\pi(\theta) \in \tilde{\mathcal{F}}$  be another family.

#### Definition 29: Conjugate Priors

$f(x|\theta) \in \mathcal{F}$  and  $\pi(\theta) \in \tilde{\mathcal{F}}$  are conjugate if:

$$\pi(\theta|X_1, \dots, X_n) \in \tilde{\mathcal{F}}$$

#### Example 21: Uniform / Pareto

Consider the model:

$$f(x|\theta) = \frac{1}{\theta} \mathbb{1}_{[0,\theta]}(x)$$

And the prior (Pareto distribution):

$$\pi(\theta) = \frac{C_{\alpha, \theta_{min}}}{\theta^{\alpha+1}} \mathbb{1}_{\{\theta > \theta_{min}\}}$$

with  $\alpha > 0$  and  $\theta_{min} > 0$ .

#### Posterior derivation:

$$\begin{aligned} \pi(\theta|X_1, \dots, X_n) &\propto L_n(\theta)\pi(\theta) \\ &= \prod_{i=1}^n \frac{1}{\theta} \mathbb{1}_{[0,\theta]}(X_i) \times \frac{1}{\theta^{\alpha+1}} \mathbb{1}_{\{\theta > \theta_{min}\}} \\ &= \frac{1}{\theta^{n+\alpha+1}} \times \mathbb{1}_{\{\theta > \max(X_i)\}} \cap \{\theta > \theta_{min}\} \\ &= \frac{1}{\theta^{n+\alpha+1}} \times \mathbb{1}_{\{\theta > \max(\max(X_i), \theta_{min})\}} \end{aligned}$$

This is a Pareto distribution:  $\text{Pareto}(\alpha + n, \theta_{min} \vee \max_{i=1 \dots n}(X_i))$ .

### IV.2 Flat priors

The prior is to be non-informative.

- e.g.  $\Theta = [a, b]$ ,  $\pi(\theta) = \frac{1}{b-a} \mathbb{1}_{[a,b]}(\theta)$ .
- What if  $\Theta = \mathbb{R}$ ? **Improper prior**.

Take  $\pi(\theta) \propto c$ , then  $\int_{\mathbb{R}} \pi(\theta) d\theta = \infty$ .

But we only need to have:

$$\int_{\Theta} L_n(\theta'|X_1, \dots, X_n) \pi(\theta') d\theta' < +\infty \quad \mathbb{P}_{\theta} - a.s.$$

### Example 22: Normal model with flat prior

Let  $f(x|\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2}$ .

Take  $\pi(\theta) = 1$  on  $\mathbb{R}$ ! (Improper).

**Posterior derivation:**

$$\begin{aligned}\pi(\theta|X_1, \dots, X_n) &\propto L_n(\theta|X_1, \dots, X_n) \\ &\propto e^{-\frac{1}{2} \sum_{i=1}^n (X_i - \theta)^2} \\ &= e^{-\frac{1}{2} \sum_{i=1}^n (\theta - X_i)^2}\end{aligned}$$

Let us expand the sum in the exponential:

$$\begin{aligned}\sum_{i=1}^n (\theta - X_i)^2 &= n\theta^2 - 2\sum_{i=1}^n \theta X_i + \sum_{i=1}^n X_i^2 \\ &= n(\theta^2 - 2\theta \bar{X}_n + \dots) \\ &= n(\theta - \bar{X}_n)^2\end{aligned}$$

(The terms not depending on  $\theta$  are absorbed in the proportionality constant).

Thus:

$$\pi(\theta|X_1, \dots, X_n) \propto e^{-\frac{1}{2} \frac{1}{n} (\theta - \bar{X}_n)^2}$$

We recognize a Normal distribution:

$$\mathcal{N}\left(\bar{X}_n, \frac{1}{n}\right)$$

### IV.3 Jeffrey's prior

**Jeffrey's idea:**

$$\pi(\theta) \propto I(\theta)^{\frac{1}{2}}$$

Where  $I(\theta)$  is the Fisher information:

$$I(\theta) = \mathbb{E}_{\theta} [(\log f_{\theta}(X_1))'^2]$$

**Result:**

If  $\pi(\theta) \propto \sqrt{I(\theta)}$ , then under reparameterisation  $\eta$ , we have  $\tilde{\pi}(\eta) \propto \sqrt{\tilde{I}(\eta)}$ , where  $\tilde{I}$  is the Fisher info on the model parametrisation.