Gmail **Sam Schiavone <sam.schiavone@gmail.com>**

Re: degree-7 Belyi maps on genus-3 curves

3 messages

John Voight <jvoight@gmail.com> Tue, Jul 17, 2018 at 9:18 AM
To: Noam Elkies <elkies@math.harvard.edu>
Cc: Michael Musty <michaelmusty@gmail.com>, Samuel Schiavone <sam.schiavone@gmail.com>

Good morning, Noam!

> . . . and come to think of it, before delving into Weil pairings on J_C[7],
> there's the more elementary invariant of whether the monodromy
> generators are all in the same *A_7* conjugacy class of 7-cycles.
> This kind of refinement of your "passport" must arise in some other
> cases; in fact I already find in my files from 1991 a computation of
> the two degree-5 Belyi maps with cycle structures 5, 5, 311,
> which are both defined over Q (Antwerp 75C and 150C) because
> they're distinguished by whether the 5-cycles are conjugate in A_5.

Yes, good point! We'd say that the passport splits up into refined
passports based on the conjugacy classes in G = A_7 (as opposed to
conjugacy classes in S_7). For the conjugacy classes
  C <-> (1, 2, 3, 4, 5, 6, 7),
  D <-> (1, 3, 4, 5, 6, 7, 2)   (= C^-1)
I find that the size 23 passport breaks up into
  8 with C, C, C
  5 with C, C, D
  5 with C, D, C
  0 with D, C, C
  5 with C, D, D
  0 with D, C, D
  0 with D, D, C
  0 with D, D, D
I'm momentarily confused about the evident action of S_3, acting by
automorphisms of PP^1 permuting {0,1,oo}: it must be the case that
there is some stabilizer.

Anyway, the associated characters are rational, so this splits the
passport over QQ and at least we should only have to compute with at
most degree 8 and degree 5 extensions.

> I thought a bit more about the problem of computing
> the missing Belyi covers of degree 7 with full ramification
> at each branch point. If I remember right you were able to
> get it numerically to some accuracy but not enough to surmise
> the underlying algebraic numbers. I don't know in what form
> you have the numerical approximation to the curve (presumably
> a quartic, because I think you might have mentioned if it
> looked hyperelliptic) and the Belyi function; I can ask you
> about this tomorrow.

Yes, that's almost right: we have power series expansions for the 3 differential forms to some precision, but we weren't able to do much with these to our satisfaction. But I'll kick this off again so we have them in hand.

> Meanwhile there may be a nontrivial
> invariant for the Belyi maps with this "passport": if the preimages
> of the branch points are P,P',P" then any two of them differ by
> a 7-torsion point on J(C) that span a 2-dimensional subspace
> of J(C)[7], and I can imagine that this subspace could be
> either isotropic or not -- and if not then we get a choice of
> 7th root of unity for any ordering of P,P',P".
>
> Unfortunately I don't know how to parametrize genus-3 curves C
> with a 7-torsion point of J(C), nor the hyperplane in that
> moduli space where the torsion point is P-P'.
> (for "hyperplane" read "hypersurface" -- it should have
> codimension 1 but I have no reason to expect an identification
> with a P^5 in P^6 . . .)
>
> Unfortunately I don't know how to parametrize genus-3 curves C
> with a 7-torsion point of J(C), nor the hyperplane in that
> moduli space where the torsion point is P-P'.

That's very interesting! But I don't see how to take advantage of it.

> Did you see either this or the J_C[5] invariant in the completely ramified
> quintic Belyi covers by curves of genus 2?

We didn't think about the J_C[5] invariant, but check out
http://beta.lmfdb.org/Belyi/?deg=5&abc_list=%5B5%2C5%2C5%5D&count=50
for the answer to your second question.

There are 3 cyclic passports and one A_5-passport 5T4-[5,5,5]-5-5-5-g2-a
http://beta.lmfdb.org/Belyi/5T4/%5B5%2C5%2C5%5D/5/5/5/g2/a
but it's of size one.

JV

John Voight <jvoight@gmail.com> Tue, Jul 17, 2018 at 12:40 PM
To: Noam Elkies <elkies@math.harvard.edu>
Cc: Michael Musty <michaelmusty@gmail.com>, Samuel Schiavone <sam.schiavone@gmail.com>

I had some partial success this morning: for

```
> sigma;
[
    (1, 2, 3, 4, 5, 6, 7),
    (1, 3, 6, 4, 5, 2, 7),
    (1, 2, 7, 3, 5, 6, 4)
]
```

I compute the curve equation

```
x^3*z - x^2*y^2 - 11*x^2*z^2 + 13*x*y^2*z - 40*x*y*z^2 + 75*x*z^3 - 11*y^4 +
   40*y^3*z - 156*y^2*z^2 + 144*y*z^3 - 297*z^4
```

It's too nice not to be correct! But I still need to compute the
Belyi map, etc.

So maybe this apparently huge genus 3 passport just has lots of curves over QQ!

JV

[Quoted text hidden]

John Voight <jvoight@gmail.com> Wed, Jul 18, 2018 at 9:42 AM

To: Noam Elkies <elkies@math.harvard.edu>
Cc: Michael Musty <michaelmusty@gmail.com>, Samuel Schiavone <sam.schiavone@gmail.com>

Good morning!

For

```
> sigma;
[
   (1, 2, 3, 4, 5, 6, 7),
   (1, 3, 4, 5, 6, 2, 7),
   (1, 2, 7, 5, 3, 6, 4)
]
```

which comes from the refined passport of size 8, I compute that its
field of definition K has minimal polynomial
  x^4 - x^3 - 3*x^2 - x + 1
so maybe this is an 8 = 4 + 4 passport. (The field K has Galois group
dihedral of order 8, and has quadratic subfield of discriminant 21.)

I need more precision to finish the job, so I'll get that running.

JV

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