BELYI STUFF

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I. Bielliptic Belyi

Combining my recent interests of Belyi maps and glueing curves along their 2-torsion, I decided to look at which genus 2 Belyi maps we've computed so far are defined on bielliptic curves, i.e., genus 2 hyperelliptic curves X with distinct morphisms $\psi_1 : X \to E_1$ and $\psi_2 : X \to E_2$ where E_1 and E_2 are elliptic curves. You can easily spot at least some of these from their equations: when $y^2 = f(x)$ and f contains only even powers of x. Let's see why in an example: consider

$$X : y^2 = x^6 + 4x^4 + 6x^2 + 3\,,$$

the curve from `6T6-6_6_3.3-a`. Replacing x^2 by x, we obtain the elliptic curve

$$E_1 : y^2 = x^3 + 4x^2 + 6x + 3\,.$$

Homogenizing the equation defining X (gradedly, of course), we have

$$y^2 = x^6 + 4x^4z^2 + 6x^2z^4 + 3z^6$$

and dehomogenizing by setting $x = 1$, we have

$$y^2 = 1 + 4z^2 + 6z^4 + 3z^6$$

find the second elliptic curve

$$E_2 : y^2 = 1 + 4z + 6z^2 + 3z^3\,.$$

So we get the maps

$$\begin{aligned} X &\to E_1 \\ (x, y) &\mapsto (x^2, y) \end{aligned}$$

and

$$\begin{aligned} X &\to E_2 \\ (z, y) &\mapsto (z^2, y)\,. \end{aligned}$$

on the open subsets where these equations hold. (I could work out the maps on the homogenized version, but I'm too lazy.)

Anyway, here's a list of all the genus 2 bielliptic Belyi maps I've found so far.

- `https://beta.lmfdb.org/Belyi/5T4/5/5/5/a`
- `https://beta.lmfdb.org/Belyi/6T1/6/6/3.3/a`
- `https://beta.lmfdb.org/Belyi/6T5/6/6/3.3/a`
- `https://beta.lmfdb.org/Belyi/6T6/6/6/3.3/a`

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- `https://beta.lmfdb.org/Belyi/7T7/7/4.3/4.3/b`

I was hoping to find an example where we have computed a Belyi map on the genus 2 curve and both elliptic curves. Maybe we could find a commutative diagram?

$$\begin{tikzcd}
& X \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_2"] \arrow[dd, "\varphi"] & \\
E_1 \arrow[dr, "\varphi_1"'] & & E_2 \arrow[dl, "\varphi_2"] \\
& \mathbb{P}^1 &
\end{tikzcd}$$

I came close, but no cigar so far. Here's the near-example. The map `6T6-6_6_3.3-a` (`https://beta.lmfdb.org/Belyi/6T6/6/6/3.3/a`) is defined on the curve `1728.b.442368.1` (`https://beta.lmfdb.org/Genus2Curve/Q/1728/b/442368/1`). It is a 2-glueing of the elliptic curves `36.a4` (`https://beta.lmfdb.org/EllipticCurve/Q/36/a/4`) and `48.a5` (`https://beta.lmfdb.org/EllipticCurve/Q/48/a/5`). The curve `36.a4` has minimal Weierstrass model

$$E_1 : y^2 = x^3 + 1$$

and we've computed a couple Belyi maps on E_1, namely the Euclidean map `3T1-3_3_3-a` (`https://beta.lmfdb.org/Belyi/3T1/3/3/3/a`) and the hyperbolic map `9T20-9_2.2.2.1.1.1_6.3-a` (`https://beta.lmfdb.org/Belyi/9T20/9/2.2.2.1.1.1/6.3/a`). But despite some near misses, we haven't gotten to a Belyi map on `48.a5`. However, while searching for Belyi maps of conductor 48, I did make an interesting observation.

II. Curve with many maps

I noticed that the elliptic curve with label `48.a6`

$$E : y^2 = x^3 + 47/768x + 2359/55296$$

admits a bunch of Belyi maps of low degree. First we have one of my favorites, the genus 1 hyperbolic triple of minimal degree `4T5-4_4_3.1-a` (`https://beta.lmfdb.org/Belyi/4T5/4/4/3.1/a`). Today I noticed there are a bunch of degree 8 maps defined on the isomorphic curve

$$E' : y^2 = x^3 + 47/12288x + 2359/3538944$$

(this equation just differs by a factor of 2), namely `8T46-8_8_3.1.1.1.1.1-a` (`https://beta.lmfdb.org/Belyi/8T46/8/8/3.1.1.1.1.1/a`), `8T47-8_2.2.2.2_4.3.1-a` (`https://beta.lmfdb.org/Belyi/8T47/8/2.2.2.2/4.3.1/a`), and `8T47-8_6.2_4.1.1.1.1-a` (`https://beta.lmfdb.org/Belyi/8T47/8/6.2/4.1.1.1.1/a`). Even curiouser, all 3 have nearly the same equation! I wonder if there's some explanation for this. Maybe they arise from composing the degree 4 map with a degree 2 map, or maybe from taking a fiber product?

III. Different fields of definition for curve and map

When I was first computing genus 1 Belyi maps, I ran into a problem. I thought that the Belyi map $\varphi : X \to \mathbb{P}^1$ with label 5T3-5_4.1_4.1-a was defined over $\mathbb{Q}$, as the curve X was. But it turned out that φ is only defined over $\mathbb{Q}(i)$: see https://beta.lmfdb.org/Belyi/5T3/5/4.1/4.1/a. I saw another instance of this today: https://beta.lmfdb.org/Belyi/7T7/7/4.3/4.3/c.

IV. Imprimitive Belyi maps

A while ago, Ciaran Schembri was trying to compute some Belyi maps related to his work on quaternionic Shimura curves. He was able to use our code to compute some of them, but had to use the NumericalKernel algorithm as our Newton code unsurprisingly crapped out. As usual, it is the "extra" point that caused trouble, either due to insufficient precision or maybe a bug. I should figure out which it is since it really might be a stupid bug that's holding things up.

Anyway, we could tell that one of his Belyi map, of degree 12, despite being defined on a genus 1 curve, was really the pullback by the x-coordinate projection of a genus 0 Belyi map. It was obvious because there was no y in the answer once we had the equation. I used Nils Bruin and Jeroen's code to compute the monodromy group of the associated degree 6 genus 0 map, and then looked up the equation in our database—it was this one (https://beta.lmfdb.org/Belyi/6T7/4.2/3.3/2.2.1.1/a). Ciaran later did some group theory trickery modding out by centralizers to give a group theoretic way of figuring out the monodromy group for the genus 0 map.

Later, I asked John about this, and he said that all this happened because the monodromy group of the degree 12 map was an imprimitive permutation group. He was right: the monodromy group of the degree 12 map was imprimitive, which I checked this in Magma using IsPrimitive. You can even compute a maximal G-invariant partition, which allows you to compute the "primitivication", if that's a thing. In this case, it turned out that the maximal partition consisted of 3 blocks of size 4. This means that the primitivication is a subgroup of S_3, and actually is S_3 in this case. Which means that even the degree 6 thing I found is imprimitive! They all come from the Belyi map https://beta.lmfdb.org/Belyi/3T2/3/2.1/2.1/a. Here's the code:

```
>     [Sym(12)|
>         (1, 8)(4, 11)(5, 10)(6, 12)(7, 9),
>         (1, 6, 3, 4)(2, 5, 8, 7)(9, 10, 12, 11),
>         (1, 5, 9, 8, 4, 12)(2, 7, 11, 3, 6, 10)
> ];
[
(1, 8)(4, 11)(5, 10)(6, 12)(7, 9),
(1, 6, 3, 4)(2, 5, 8, 7)(9, 10, 12, 11),
(1, 5, 9, 8, 4, 12)(2, 7, 11, 3, 6, 10)
]
> sigma := $1;
> G := sub< Sym(12) | sigma>;
> G;
Permutation group G acting on a set of cardinality 12
```

```
(1, 8)(4, 11)(5, 10)(6, 12)(7, 9)
(1, 6, 3, 4)(2, 5, 8, 7)(9, 10, 12, 11)
(1, 5, 9, 8, 4, 12)(2, 7, 11, 3, 6, 10)
> TransitiveGroupDescription(G);
S_4(12d)x2
> IsPrimitive(G);
false
> MaximalPartition(G);
GSet{@
{ 1, 2, 3, 8 },
{ 4, 5, 6, 7 },
{ 9, 10, 11, 12 }
@}
> BlocksAction(G, $1);
Mapping from: GrpPerm: G to GrpPerm: $, Degree 3
> Image($1);
Permutation group acting on a set of cardinality 3
(2, 3)
(1, 2)
(1, 2, 3)
> $1 eq Sym(3);
true
```

So I guess this is kind of a way of “factoring” Belyi maps: given a monodromy group, we can find its primitivication, compute that smaller degree Belyi map, and then try to lift back up to the original map. I think this is basically what Mike was doing in his thesis: starting with $\mathbb{Z}/2\mathbb{Z}$ and then lifting to 2-groups. So I guess our general strategy should probably be

(1) Compute all primitive Belyi maps
(2) Compute all possible “lifts” of these primitive maps