Sam Schiavone <sam.schiavone@gmail.com>

Fiber product

2 messages

John Voight <jvoight@gmail.com> Wed, Jul 18, 2018 at 3:58 PM
To: Samuel Schiavone <sam.schiavone@gmail.com>
Cc: Michael Musty <michaelmusty@gmail.com>, Noam Elkies <elkies@math.harvard.edu>

Hi Sam,

Sorry, I need to expunge this from my brain so I'll stop thinking about it. (I hope you're having fun on your bike ride!)

If we start with a Belyi map

```
 X
  | phi
  v
 PP^1
```

and we want to pullback under a map

```
 PP^1 -f-> PP^1
```

to get a new Belyi map

```
 X'
 | f
 v
PP^1
```

then we can ask given phi, what f's give us a new Belyi map. (It's possible that we end up with a disconnected curve X', but this only happens when the function fields have a common subfield.)

We can do this the hard way, computing tensor products of coordinate rings. But it's just a local question, so we should zoom in to a ramification point; and we can even argue analytically (or, if you like, this is also something you can do in algebraic geometry by working with formal schemes, etc.).

So let's turn this into a diagram of groups

```
 Theta -> Gamma
   |         |
   v         v
 Delta' -> Delta
```

which are the uniformizing orbifold groups (so RHS PP^1 = HH/Delta, etc.). We want the LHS vertical map to be a Belyi map, so that's why I wrote Delta'.

So that means that it is necessary and sufficient to have a pullback diagram when there is an inclusion of one triangle group in another!

But these have been completely classified: see (2.8) and (2.9) of
https://www.math.dartmouth.edu/~jvoight/articles/triangle-071518.pdf
The infinite families in (2.9) each have nice pictures: for example,

Delta(a,a,a) <= Delta(3,3,a)
corresponds to what's called a barycentric subdivision (subdivide the equilateral triangle into 3 triangles in the middle). Usually there isn't much to do, but (7,7,7) is a special case: the full diagram of inclusions in this case is in the attached.

In any case, I think this gives a really nice answer! For each Belyi map we have, looking at (a,b,c) we have a finite list of container triangle groups for which we can pull back that give us new Belyi maps. We need to do a little bit more work to get the new partitions from the old ones, but that's just a little bookkeeping.

JV

Takeuchi - Commensurability Classes of Arithmetic Triangle Groups.pdf
467K

Noam D. Elkies <elkies@math.harvard.edu> Wed, Jul 18, 2018 at 10:52 PM
To: sam.schiavone@gmail.com, jvoight@gmail.com
Cc: michaelmusty@gmail.com

John Voight <jvoight@gmail.com> wrote:

> Sorry, I need to expunge this from my brain so I'll stop thinking
> about it [...]

Thanks for writing this up. It arrived while I was composing the previous e-mail.

While the plane quartic over Q turns out to come from a Belyi cover of genus 0, not 1 as I assumed, one can still apply the construction to some of the genus-1 covers. Here we need 7,331,331:

http://beta.lmfdb.org/Belyi/?deg=7&abc_list=[7,3,3]&g=1

and find two matches. One has Galois group 7T3 (solvable 7:3), and the curve is defined over Q and has j-invariant -3375 and conductor 49, so is one of the the CM curves of discriminant -7 and minimal conductor; I guess that this one gives rise to the Belyi map from the Klein quartic that you mentioned. The other has Galois group 7T5 (the 168-element group); it is an elliptic curve over Q(sqrt(-7)) that has integral but non-CM j-invariant -2848 - 1488*nu (where nu = quadgen(-7) = (1+sqrt(-7))/2). It somehow got represented by a really complicated equation with a4 and a6 having spurious factors of p^4, p^6 where p = 17-12*nu has norm 373, in addition to a bunch of 7's in the numerator and 2's and 3's in the denominator. More interesting -- though I do not have an explanation for this either -- is that while the curve cannot be defined over Q, it's a Q-curve, since the j-invariant is 2-isogenous to its algebraic conjugate.

At any rate, starting from any one of these we can get a degree-7 Belyi map from a genus-3 curve by forming a fiber product with

a cyclic 3:1 cover of the elliptic curve branched at two of the
three preimages of {\infty,0,1} that aren't already of index 3.
There are several choices in each case because cyclic 3:1 covers of
an elliptic curve branched above P,Q are in bijection with points
(P-Q)/3. (Usually some or all of these will be Galois conjugates,
and there will be fewer choices if P-Q is 2-torsion, which may
happen here.) This should account for most if not all of the
remaining maps from your count of 23.

Thanks too for the scan of Takeuchi's paper. This data should go in
the LMFDBelyi too, right? About 20 years ago, when I was starting to
compute Shimura curves, I transcribed the tables, with some additional
annotations. Along the way I found a missing inclusion in the
"Diagrams of inclusions" in section 4 (starting on page 209,
which is p.5 of 7 in the scan): in class III (quaternion algebra
over Q(2^(1/2)) ramified at an infinite prime and at the prime above 2),
the group (2,8,8) contains (4,8,8) with index 2 (ramified at the
e=2 elliptic point and one of the e=8 points); thus the edge from
(2,4,8) to (4,8,8) labeled "4" should be replaced by an edge from
(2,8,8) down to (4,8,8) labeled "2".

NDE