BELYI STUFF

SAM SCHIAVONE

I. Bielliptic Belyi

Combining my recent interests of Belyi maps and glueing curves along their 2-torsion, I decided to look at which genus 2 Belyi maps we've computed so far are defined on bielliptic curves, i.e., genus 2 hyperelliptic curves X with distinct morphisms $\psi_1 : X \to E_1$ and $\psi_2 : X \to E_2$ where E_1 and E_2 are elliptic curves. You can easily spot at least some of these from their equations: when $y^2 = f(x)$ and f contains only even powers of x. Let's see why in an example: consider

$$X : y^2 = x^6 + 4x^4 + 6x^2 + 3\,,$$

the curve from `6T6-6_6_3.3-a`. Replacing x^2 by x, we obtain the elliptic curve

$$E_1 : y^2 = x^3 + 4x^2 + 6x + 3\,.$$

Homogenizing the equation defining X (gradedly, of course), we have

$$y^2 = x^6 + 4x^4z^2 + 6x^2z^4 + 3z^6$$

and dehomogenizing by setting $x = 1$, we have

$$y^2 = 1 + 4z^2 + 6z^4 + 3z^6$$

find the second elliptic curve

$$E_2 : y^2 = 1 + 4z + 6z^2 + 3z^3\,.$$

So we get the maps

$$\begin{aligned} X &\to E_1 \\ (x, y) &\mapsto (x^2, y) \end{aligned}$$

and

$$\begin{aligned} X &\to E_2 \\ (z, y) &\mapsto (z^2, y)\,. \end{aligned}$$

on the open subsets where these equations hold. (I could work out the maps on the homogenized version, but I'm too lazy.)

Anyway, here's a list of all the genus 2 bielliptic Belyi maps I've found so far.

- `https://beta.lmfdb.org/Belyi/5T4/5/5/5/a`
- `https://beta.lmfdb.org/Belyi/6T1/6/6/3.3/a`
- `https://beta.lmfdb.org/Belyi/6T5/6/6/3.3/a`
- `https://beta.lmfdb.org/Belyi/6T6/6/6/3.3/a`

Date: May 23, 2020.

- https://beta.lmfdb.org/Belyi/7T7/7/4.3/4.3/b

I was hoping to find an example where we have computed a Belyi map on the genus 2 curve and both elliptic curves. Maybe we could find a commutative diagram?

$$\begin{tikzcd}
 & X \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_2"] \arrow[dd, "\varphi"] & \\
E_1 \arrow[dr, "\varphi_1"'] & & E_2 \arrow[dl, "\varphi_2"] \\
 & \mathbb{P}^1 &
\end{tikzcd}$$

I came close, but no cigar so far. Here's the near-example. The map 6T6-6_6_3.3-a (https://beta.lmfdb.org/Belyi/6T6/6/6/3.3/a) is defined on the curve 1728.b.442368.1 (https://beta.lmfdb.org/Genus2Curve/Q/1728/b/442368/1). It is a 2-glueing of the elliptic curves 36.a4 (https://beta.lmfdb.org/EllipticCurve/Q/36/a/4) and 48.a5 (https://beta.lmfdb.org/EllipticCurve/Q/48/a/5). The curve 36.a4 has minimal Weierstrass model

$$E_1 : y^2 = x^3 + 1$$

and we've computed a couple Belyi maps on E_1, namely the Euclidean map 3T1-3_3_3-a (https://beta.lmfdb.org/Belyi/3T1/3/3/3/a) and the hyperbolic map 9T20-9_2.2.2.1.1.1_6.3-a (https://beta.lmfdb.org/Belyi/9T20/9/2.2.2.1.1.1/6.3/a). But despite some near misses, we haven't gotten to a Belyi map on 48.a5. However, while searching for Belyi maps of conductor 48, I did make an interesting observation.

II. Curve with many maps

I noticed that the elliptic curve with label 48.a6

$$E : y^2 = x^3 + 47/768x + 2359/55296$$

admits a bunch of Belyi maps of low degree. First we have one of my favorites, the genus 1 hyperbolic triple of minimal degree 4T5-4_4_3.1-a (https://beta.lmfdb.org/Belyi/4T5/4/4/3.1/a). Today I noticed there are a bunch of degree 8 maps defined on the isomorphic curve

$$E' : y^2 = x^3 + 47/12288x + 2359/3538944$$

(this equation just differs by a factor of 2), namely 8T46-8_8_3.1.1.1.1.1-a (https://beta.lmfdb.org/Belyi/8T46/8/8/3.1.1.1.1.1/a), 8T47-8_2.2.2.2_4.3.1-a (https://beta.lmfdb.org/Belyi/8T47/8/2.2.2.2/4.3.1/a), and 8T47-8_6.2_4.1.1.1.1-a (https://beta.lmfdb.org/Belyi/8T47/8/6.2/4.1.1.1.1/a). Even curiouser, all 3 have nearly the same equation! I wonder if there's some explanation for this. Maybe they arise from composing the degree 4 map with a degree 2 map, or maybe from taking a fiber product?

III. DIFFERENT FIELDS OF DEFINITION FOR CURVE AND MAP

When I was first computing genus 1 Belyi maps, I ran into a problem. I thought that the Belyi map $\varphi : X \to \mathbb{P}^1$ with label `5T3-5_4.1_4.1-a` was defined over $\mathbb{Q}$, as the curve X was. But it turned out that φ is only defined over $\mathbb{Q}(i)$: see `https://beta.lmfdb.org/Belyi/5T3/5/4.1/4.1/a`. I saw another instance of this today: `https://beta.lmfdb.org/Belyi/7T7/7/4.3/4.3/c`.