To: Dr. Wu
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Memo: Delta Hedging and Black-Scholes Pricing Models

Introduction

Similar to Beta Hedging, Delta Hedging is a process by which an individual preserves the value of his or her portfolio via purchasing assets that are inversely correlated with market movement. That is, purchasing stock offsets the returns of shorted call options. The amount of aforementioned stock to be purchased is determined by the coefficient delta, which compares the change in the return of a derivative security (usually a call or a put option) to the changes of the underlying asset from which the option is derived. This ratio—sometimes referred to as a hedge ratio—encapsulates the returns of a derivative in terms of the returns of its underlying asset. In practice, a call option with a delta ratio of 0.5 will increase by $0.50 for every $1.00 increase in the stock it is associated with.

Due to the Random Walk theory, the future returns of any security (derivative or otherwise) are unknown. With that said, there are a few processes by which analysts forecast future prices and, therefore, returns. First and foremost, the Black-Scholes (or Black-Scholes Merton) pricing model determines the price of European Call and Put options. Earning a Nobel Prize for their work in 1997, Fischer Black, Myron Scholes, and Robert Merton implemented the Capital Asset Pricing Model (CAPM) in order to determine the correct discount rate to apply to future payoffs.

Similarly, the Binomial Tree model considers price movements in discrete steps, modeled by probabilistic movement derived by the volatility of a security (which is how U, D, and P are derived; see the excel spreadsheet for more information). With regard to this project specifically, we hope to gain an understanding of the nuances of these pricing models by hedging a portfolio of AT&T (T) stock and call options. American and European call options will be the keystone security analyzed in this project Excel modeling software will be implemented to create pricing engines for both of these models, and the results will be compared to real-time results captured by activity on the Chicago Board of Options Exchange (CBOE) virtual trading portfolio.

Findings

While a fairly simple process, delta hedging can be incredibly efficient at preserving the value of a derivative-asset portfolio. Moreover, these portfolios are better managed when valuation of the assets therein is accurate. In this regard, the Black-Scholes model appears to be the more effective at emulating the actual call option prices posted on exchanges. However, as stated below, the Binomial Tree pricing model is versatile, valuing American options as well as European. Analogously, a Binomial Tree can be improved by adding more time steps to the system, as this allows the model to approach a continuous spectrum of price outcomes. Lastly, historical and implied volatility did not appear to have any major differences between each other for the duration of the project. We credit this occurrence to the extremely short hedging period.

Thus, static and dynamic delta hedging appear quite similar in this project, but it is our group's conclusion that dynamic hedging is the most effective method of preserving portfolio value. We posit that had our hedging period been longer—perhaps weeks or months—our static hedging portfolio would have been unable to shelter our portfolio from the vicissitudes of the stock market. Going even further, delta hedging can be quite easy when delta coefficients are constantly provided and change in real time; even so, these coefficients are not as effective without the incorporation of human intuition and outside knowledge of a company. By having both of these sources of knowledge available, hedging becomes markedly more effective. The individual in our group who knew stock markets and macro-economic trends best managed the portfolio that performed the best, and we do not believe this to be a coincidence.

Discussion

III. a. Methods

American and European Options: Before delving further into this delta-hedging project, it is essential to understand the differences between European and American options. Moreover, different pricing models must be applied to these derivatives in order to correctly value them. European Options allow the owner the right to buy or sell a security at a set price some time in the future. The date the investor may exercise these options is a single date at the terminus of the option's maturity. Conversely, American options allow the holder the same rights, but afford them the freedom to exercise the option prior to its expiration date.

Now, the Black-Scholes model can only be implemented on European options (with or without dividends); the assumptions of the Black-Scholes model do not allow early exercise actions to take place. As a result, the Binomial Tree model must be implemented in ordered to price American call options. It is for this reason that our Excel hedging profile was constructed using the Binomial Tree model. The more steps in a binomial tree, the closer its calculations will be to the Black-Scholes model.

Delta Hedging Fundamentals: At a high level, delta hedging offsets a short (or long) options position by buying (or selling) delta shares of the underlying asset. Delta is calculated—for the purposes of this project—as the value change in option payoffs between two scenarios (e.g. stock price increase or decrease) divided by the price change in the asset itself. If done correctly, and adjusted consistently over the course of the hedging period, portfolio value should be preserved against movement in the asset's price (and therefore the call option). Delta hedging typically relates to the Binomial Tree pricing model, as this valuation engine relies on delta to calculate option payoffs.

Black-Scholes Pricing Model: The Black-Scholes model was built on a few assumptions, primarily that percent changes in stock price in a short period of time are normally distributed. Furthermore, the model utilizes the Capital Asset Pricing Model to determine the rate at which future option payoffs will be discounted back. Determining this discount rate is particularly difficult, as it involves considering stock price as well as time. The main characteristic that sets the Black-Scholes model apart from the Binomial Tree Model is that the Black Scholes model assumes that the stock and derivative position are riskless for only a short period of time (as opposed to indefinitely). On the whole, it is a more nuanced model; rather than discrete time steps, the Black-Scholes pricing engine considers continuous changes in stock price.

Binomial Tree Pricing Model: By contrast, the Binomial Tree model introduces variation through a series of two-outcome time steps. By calculating the stock-up multiplier (U) and the stock down multiplier (D) through the manipulation of the stock's volatility and the time of the hedging period, one can forecast the future stock price. Almost surely, these stock price forecasts will be incorrect, but as more and more branches are added to a hedging period (with the change in time for each step getting shorter and shorter), the pricing estimates for American Call options become more accurate.

Along with this, due to the nature of Binomial tree pricing models, it is possible to approximate the value of American Call and Put options, a feat unattainable with the Black-Scholes model. By determining individual payoffs at each node of the binomial tree, and then making a conscious decision of whether or not to exercise the option early, one can—depending on the intricacy of the model—value American options quite accurately. Thus, with these two models (and their corresponding output) will be compared and corroborated in order to ensure the efficacy of the delta hedging process overall.

Justification of Inputs: It is central to the efficacy of this project to ensure that the inputs utilized by either pricing model are appropriate and accurate. Dividend yield, stock price, and risk free rate were found by searching data gathered by Yahoo Finance and the U.S. Department of Treasury. These values, based on the ethos of their providers, will be assumed to be correct. For volatility, forty days of stock price movement was analyzed to determine daily volatility (risk), which was then converted to an annual figure. This calculation is assumed to be representative of the current volatility of AT&T. Lastly, the time to maturity was calculated by dividing the life of the call option (in days) by the number of week (trading) days in a year, or 252.

<u>Portfolio Re-allocation and Dynamic Hedging:</u> Along with the pricing models implemented in Excel, our group also utilized the CBOE's virtual trading tool to actually create a portfolio to implement delta hedging in. In order to do this, 10 American Call Options of AT&T (T) (100 shares of stock per option) were sold. In order to counteract the price movements of these derivatives, delta shares of AT&T stock were purchased. Delta values change constantly, and are provided by the CBOE by examining their "find chain" feature. After the initial hedging position is established, the stock price, option price, and delta continue to change constantly during trading hours. In order to maintain portfolio value at $1,000,000, shares of AT&T were purchased or sold such that the new position reflected that of the current delta.

With that said, our group implemented two strategies into our dynamic hedging. While we had the potential to alter our hedging position every day (at most), we decided to only change our hedging position if the portfolio value strayed more than $10 away from the initial portfolio value. While this is an incredibly small change, this allowed for our group to not allow the (very) short-term vicissitudes of the market dramatically alter our hedging position. We felt that being too eager to alter our hedging position may in fact have led to disruptions to initial portfolio value, despite our best intentions. In the same fashion, when our group decided to hedge, we did so proactively. That is, if delta increased consistently, we would purchase more than delta shares of stock in anticipation of further upward price pressure. The same practice was implemented when delta consistently declined. By deliberately overshooting the delta values in our hedging, we were able to anticipate further price changes and preserve our market value more effectively.

<u>Implied Volatility:</u> As witnessed in the "(Q1) Volatility Estimates" tab of our Excel spreadsheet, volatility for a stock can be found by analyzing past price fluctuations over time. These calculated daily volatility values could then be converted into annual percentages and used to price European Call Options via either the Black-Scholes or Binomial Tree models. However, when presented with call option prices, there is no analytical solution to determining the volatility that would lead to such a price. As a result, the Excel solver add-in is utilized in the "(Q3) Implied Volatility" tab and algorithmically determines the volatility for which these price values would be true. Since this volatility value is deduced and not derived from historical data, it is known as implied volatility.

III. b. Analysis

Figure 1.

Black-Scholes Pricing Engine, First Day

Inputs		Outputs	
Intial Stock Price (S_0)	$ 38.74	d_1	-0.110487
Dividend Rate (q)	4.95%	d_2	-0.227033
Risk Free Rate (r)	0.53%	$N(d_1)$	0.456011522
Volatility (σ)	21.51%	$N(d_2)$	0.410198927
Time Period (T) From First Day	0.293651		
Lowest OTM Strike Price (K)	$ 39.00		

Option Premiums

First Day	
Call Option Premium (c)	**$ 1.43**
Put Option Premium (p)	$ 2.12
Last Day	
Call Option Premium (c)	**$ 1.19**
Put Option Premium (p)	$ 2.42

Black-Scholes Pricing Engine, Last Day

Inputs		Outputs	
Intial Stock Price (S_0)	$ 39.18	d_1	-0.235319
Dividend Rate (q)	4.95%	d_2	-0.347859
Risk Free Rate (r)	0.53%	$N(d_1)$	0.406980492
Volatility (σ)	21.51%	$N(d_2)$	0.363972988
Time Period (T) From Last Day	0.27381		
Lowest OTM Strike Price (K)	$ 40.00		

Actual Call Option Prices (CBOE):

First Day: $ 1.03

Last Day: $ 1.15

Figure 2.

Inputs		
Binomial Tree Delta		0.49
Hedging Period (T)		5 Days
Initial Portfolio Value		$ 1,000,000.00
First Day of Trading		
Stock Price	$	38.78
Call Option Price	$	1.03
Last Day of Trading		
Stock Price	$	39.18
Call Option Price	$	1.15

Step	0	1	2	3	4	5	
						$ 50.49	
					$ 47.93	$ 11.49	
					$ 8.93	$ 45.50	1.00
				$ 45.50	1.00	$ 6.50	
				$ 6.50		$ 45.50	
					$ 43.19	$ 6.50	
					$ 4.19	$ 40.99	1.00
			$ 43.19	1.00		$ 1.99	
			$ 4.19			$ 45.50	
					$ 43.19	$ 6.50	
					$ 4.19	$ 40.99	1.00
				$ 40.99	0.98	$ 1.99	
				$ 1.99		$ 40.99	
					$ 38.91	$ 1.99	
					$ -	$ 36.93	0.49

Static Hedging Portfolio	Purchase Amount	Intial Price	Value	Final Price	Final Value	Final Portfolio Value
AT&T (T) Call Options	-10	$ 1.03	$ (1,030.00)	$ 1.15	$ (1,150.00)	
AT&T (T) Stock	490.85	$ 38.78	$19,035.10	$ 39.18	19231.4371	
Totals	N/A	N/A	$18,005.10	N/A	$18,081.44	$ 1,000,076.34

Figure 3.

Account Activity					
Date	Action	Qty	Symbol/Description	Price	Net Amount
05/12/2016	Buy	88	T - AT&T	$39.58	($3,491.99)
05/11/2016	Buy	6	T - AT&T	$39.23	($244.33)
05/09/2016	Buy	22	T - AT&T	$39.07	($868.49)
05/06/2016	Buy	16	T - AT&T	$38.94	($631.99)
05/05/2016	Buy	498	T - AT&T	$38.76	($19,311.43)
05/05/2016	Sell To Open	10	T Aug16 39 Call	$0.98	$964.74
Total					($23,583.49)

Day	1	2	3	4	5
Stock Price	$38.78	$38.57	$39.01	38.92	$39.18
Call Option Price	$1.03	$0.90	$1.11	$1.02	$1.15
Delta	0.498	0.5135	0.5270	0.5105	0.5789
Shares Purchased	498	16	22	6	0*
Options Purchased	-10	0	0	0	0
Implied Volatility (CBOE)	12.7%	13.3%	14.2%	11.6%	12.4%

*The purchase of 88 stocks as seen in the top of *Figure 3* was outside of the hedging period and is excluded from this analysis.

Black-Scholes Pricing Model Discussion: The initial calculations of the call option price were not markedly different from those depicted by the CBOE (see *Figure 1*); all the same, it is important to characterize the differences between these pricing models. First and foremost, it goes without saying that the volatility coefficient found by the CBOE is certainly more accurate than our own. Our volatility coefficient of 21.51% is much higher than the implied volatility of the CBOE, which initially stood at 12.7%. This difference is likely due to the fact that we completed this project shortly after AT&T released its Q1 earnings, and the price fluctuations we integrated into our volatility generator may have been inflated.

With that said, the price calculations found by the Black-Scholes formula were quite similar to those of the Binomial Tree. Binomial tree call option prices were consistently higher than the Black-Scholes estimations, which may be due in part to the European-American Option differences in these two models. All the same, the extremely short hedging time period and short life of the options themselves should have mitigated most of this difference. Indeed, it appears to have done so; the Black Scholes initial call option premium estimation determined a value of $1.43, while the Binomial Tree model found a value of $1.89.

Lastly, the CBOE appeared to most closely resemble the price estimates of the Black Scholes model. While the initial call premium estimate of $1.43 is rather far from the CBOE call premium of $1.03, the Black Scholes model found a value of $1.19 for the final day of the hedging period, only four cents away from the CBOE call premium of $1.15.

Binomial Tree Pricing Model Discussion: The Binomial Tree model works similarly to that of the Black Scholes model, aside from the fact that it works in discrete time steps. The greater the number of time steps for a hedging period, the higher likelihood that the call option estimates will more closely resemble that of the continuous Black Scholes pricing engine. When observing a portion of the Binomial Tree model showcased in *Figure 2,* one can see that black values represent potential stock price, blue text represents call option payoff, and red represents the calculated delta for a given branch of the tree.

In order to calculate the stock prices for the Binomial Tree, volatility determines the multiplier of stock price increase, given the stock goes up or down (see *"(Q2) Binomial Tree"* for more information). Moreover, the probability of each scenario was also calculated by implementing the volatility value found previously. It goes without saying that both of these models rely heavily on the calculated volatility coefficient, but our group found the Binomial Model to be the most sensitive out of the two.

Implied Volatility Comparisons: Over the course of the hedging period, implied volatility (as provided by the CBOE) was recorded. In turn, the other data provided by the CBOE and Yahoo Finance was compiled into an implied volatility calculator (via the Solver Add-In in Excel). These two values were consistently close to one another, but the Excel calculated implied volatilities were unswervingly higher than those of the CBOE. This increase cannot be attributed to historical volatility calculations in Excel, as the Implied Volatility-pricing engine relies on all inputs EXCEPT the volatility value (as that is what is solved for).

Thus, our group determined a few reasons for the slight increase in our implied volatility calculations compared to those of the CBOE. The most obvious difference (and almost surely correct) is the probability that the CBOE utilizes a more nuanced method of determining implied volatility. With much more processing power and information available to them, the CBOE has much more freedom to implement alternative calculation techniques. Beyond this notion, however, is the possibility that the stock price distribution utilized by the CBOE is not completely normal. Our group implemented a standard normal distribution with Excel function norm.s.dist; perhaps this distribution varies slightly from the one implemented by the CBOE, and therefore is causing some or all of the disparity witnessed in the implied volatility values.

Samuel's Virtual Account Overview

As of 5/5/2016 2:36:41 PM ET.

Print Auto Refresh

Account Balances	More Detail
Account Value	**$999,935.75**
Current Position Value	$18,282.44
Money Markets & Cash[1]	$981,653.31
Futures Cash	$0.00
Stock Buying Power	$1,971,559.10
Option Buying Power	$985,779.55
Futures Buying Power	$985,779.55

Account Positions — More Detail

Symbol	Description	Qty	Price	Value	Action
T	AT&T	498	**$38.78**	**$19,312.44**	Trade | Chain
T Aug16 39 Call	AT&T	-10	**$1.03**	**($1,030.00)**	Trade | Chain

Open Orders — All Orders

Order	Symbol	Description	Action	Qty	Type
You have no open orders					

Samuel's Virtual Account Overview

As of 5/11/2016 9:53:02 AM ET.

Print Auto Refresh

Account Balances	More Detail
Account Value	**$999,993.16**
Current Position Value	$19,840.33
Money Markets & Cash[1]	$980,152.83
Futures Cash	$0.00
Stock Buying Power	$1,970,473.21
Option Buying Power	$985,236.61
Futures Buying Power	$985,236.61

Account Positions — More Detail

Symbol	Description	Qty	Price	Value	Action
T	AT&T	536	**$39.273**	**$21,050.33**	Trade | Chain
T Aug16 39 Call	AT&T	-10	**$1.21**	**($1,210.00)**	Trade | Chain

Open Orders — All Orders

Order	Symbol	Description	Action	Qty	Type
You have no open orders					

Samuel's Virtual Account Overview

As of 5/12/2016 9:33:58 AM ET.

Print Auto Refresh

Account Balances	More Detail
Account Value	**$999,994.06**
Current Position Value	$20,085.56
Money Markets & Cash[1]	$979,908.50
Futures Cash	$0.00
Stock Buying Power	$1,970,403.64
Option Buying Power	$985,201.82
Futures Buying Power	$985,201.82

Account Positions — More Detail

Symbol	Description	Qty	Price	Value	Action
T	AT&T	542	**$39.18**	**$21,235.56**	Trade | Chain
T Aug16 39 Call	AT&T	-10	**$1.15**	**($1,150.00)**	Trade | Chain

Open Orders — All Orders

Order	Symbol	Description	Action	Qty	Type
You have no open orders					

Static Hedging Analysis: In order to benchmark the success of the dynamic hedging portfolio implemented with the CBOE virtual trading tool, a static hedging portfolio was set up. By applying the delta coefficient found by the Binomial Tree pricing model, an initial hedging position was created (showcased in *Figure 2.*). Holding to the definition of static hedging, no adjustments to the original hedging position were made. In turn, the results of the hedging profile will be used to reference the relative success of the dynamic hedging portfolio. Likely due to the extremely short hedging period, the static hedging portfolio performed rather well. The final value differed only $76.54 from the original investment, given a total investment of nearly $20,000. With that said, the dynamic hedging section of this memo will elaborate on further strategies to facilitate a successful hedging portfolio.

Dynamic Hedging Analysis: The CBOE dynamic hedging portfolio of this project, time-lined in *Figure 3* carries with it a few distinctions that are different from that of the static hedging portfolio. First and foremost, transaction costs were ignored in both models, but only the dynamic hedging portfolio implemented a strategy to compensate for the value lost via these expenses. In order to do this, the amount of stocks purchased to hedge the short options position was slightly different than that of the given delta value. As a result, if the delta value depicted upward pressure consistently (for the majority of a day), the number of stocks purchased exceeded the delta value by 1% to 3%. Some of this variation was exacted with the discretion of the investor; some of the dynamic hedging alterations were reactionary in that an investor might intuitively change his or her position and only marginally regard the stated delta value.

As one can see from the dynamic hedging final position (bottom of *Figure 4*), this strategy was incredibly successful; the final position of our hedging portfolio differed from the initial portfolio value of $1 million (not shown) by only $5.94. Indeed, it appears that at times human intuition, along with the necessary data, may be more effective than following a cookbook hedging strategy.

III. c. Limitations

Several shortcomings of the models utilized in this project may have contributed to the results obtained. First and foremost, all inputs for the Black-Scholes and Binomial Tree pricing models had to be imputed manually (specifically, stock price). If our model had been more longitudinal, and incorporated more data collected in real time, the option premiums calculated in excel would have been even closer to those posted by the CBOE.

Secondly, it appears that the more steps a binomial tree has, the more accurate its pricing predictions become. However, due to limitations with the input data as well as the relative inflexibility of the model itself limited the efficacy of the model. For longer time periods, it would be imperative to create a binomial tree using different software with many more (automatically adjusting nodes). All the same, the model incorporated in this project performs remarkably well considering the circumstances.

IV. Conclusions

As was the case with the beta-hedging project, this analysis sought to understand the methods by which a portfolio's value could be preserved best. With delta hedging, the portfolio in question was comprised of an asset and a derivative thereof. By approximating the potential payoffs of the option as well as the price movement of the asset, the quantity of stocks in the portfolio could be fixed in a manner such that the portfolio value remained constant (or nearly constant) over time.

Extending off of the findings section, the process found to best preserve portfolio value was to dynamically hedge the portfolio using real-time data along with intuition and understanding of macroeconomic market trends. The Black-Scholes model also appeared to unequivocally be the best pricing model out of the two employed. Thus, to maximize the probability of successful delta hedging, be thorough and meticulous when developing volatility values, implement the Black-Scholes pricing engine whenever possible over the Binomial Tree model, and also exercise thoughtful judgment when re-balancing a portfolio, even if that judgment contradicts slightly with provided data.

V. References

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