Part 1

Exercise 10.2.

(a) Show that in a well-ordered set, every element except the largest (if one exists) has an immediate successor.

Proof. Let X be a well-ordered set and $x \in X$ not be the largest element. Then the set $\{y \in X : y > x\}$ is nonempty and has a smallest element, which is clearly the immediate successor of x. □

(b) Find a set in which every element has an immediate successor that is not well-ordered.

Solution. The integers $\mathbb{Z}$. □

Exercise 10.4.

(a) Let $\mathbb{Z}_-$ denote the set of negative integers in the usual order. Show that a simply ordered set A fails to be well-ordered if and only if it contains a subset having the same order type as $\mathbb{Z}_-$.

Proof. ($\Longrightarrow$) Suppose A has a simple order $<$ that is not well-ordered. Let B be a nonempty subset of A with no least element and pick some $b_1 \in B$. For any $k \geq 1$ choose $b_{k+1} < b_k$, which will always exist since B has no least element. Then $\{b_n\} \subset A$ and $f : \{b_n\} \to \mathbb{Z}_-$ defined by

$$f(b_n) = -n$$

is clearly an order preserving bijection.

($\Longleftarrow$) Suppose A has a subset B with the same order type as $\mathbb{Z}_-$. Then clearly B has no least element because, letting $f : \mathbb{Z}_- \to B$ denote an order preserving bijection, for any $b \in B$ we have some $n \in \mathbb{Z}_-$ such that $f(n) = b$ and some $f(n-1) \in B$ such that $f(n-1) < b$. Therefore A is not well-ordered. □

(b) Show that if A is simply ordered and every countable subset of A is well-ordered, then A is well-ordered.

Proof. Suppose A is not well-ordered. By part (a), there exists a subset B of A that has the same order type as $\mathbb{Z}_-$, so B is a countable subset of A that is not well-ordered. □

Exercise 23.2. Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \varnothing$ for all n. Show that $\bigcup A_n$ is connected.

Proof. Suppose $\bigcup A_n = C \cup D$ is a separation of $\bigcup A_n$. Since each A_n is a subset of the union $\bigcup A_n$, by Lemma 23.2, each A_n lies entirely within C or D. For $n = 1$, without loss of generality suppose A_1 lies entirely within C. Notice that if A_n lies within C for all $n > 1$, then $\bigcup A_n = C$ and $D = \varnothing$, contradicting C, D being a separation. So let m be the least integer such that A_m lies within D. Our hypothesis indicates that there exists $a \in A_{m-1} \cap A_m$, but this is a contradiction, since A_{m-1} lies entirely in C and A_m lies entirely in D, where C and D are disjoint. Therefore $\bigcup A_n$ must be connected. □

Exercise 23.5. A space is *totally disconnected* if its only connected subspaces are one-point sets. Show that if X has the discrete topology, then X is totally disconnected. Does the converse hold?

Proof. Let X have the discrete topology and A be a connected subspace of X. Notice that if A is empty it cannot be connected. So let $x \in A$. If $A - \{x\}$ is nonempty then $\{x\}, A - \{x\}$ forms a separation of A, so it must be the case that $A - \{x\}$ is empty and $A = \{x\}$. Hence all connected subspaces of X are singleton sets.

No, the converse does not hold. The rational numbers as the subspace of the usual topology on $\mathbb{R}$ are totally disconnected but not discrete. □

Exercise 23.8. Determine whether or not $\mathbb{R}^\omega$ is connected in the uniform topology.

Solution. $\mathbb{R}^\omega$ is not connected in the uniform topology, for an argument identical to the one given for the box topology in Example 23.6. Specifically, we separate $\mathbb{R}^\omega$ into bounded and unbounded sequences of real numbers. These are clearly nonempty disjoint sets, and they are open, since for any $\mathbf{a} \in \mathbb{R}^\omega$ the ball

$$B(\mathbf{a}, 1) = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \cdots$$

consists entirely of bounded sequences if $\mathbf{a}$ is bounded and unbounded sequences if $\mathbf{a}$ is unbounded. Therefore $\mathbb{R}^\omega$ is not connected. □

Exercise 24.4. Let X be an ordered set in the order topology. Show that if X is connected, then X is a linear continuum.

Proof. Suppose $x < y$ for some $x, y \in X$. Then $(-\infty, y)$ and (x, ∞) are open, nonempty, and their union is all of X. Thus they cannot be disjoint, so there exists $z \in (-\infty, y) \cap (x, \infty)$ so that $x < z < y$.

Next suppose for contradiction that there is a nonempty subset A of X that is bounded above without a least upper bound. Let B be the (nonempty) set of upper bounds for A.

Notice that if there was an upper bound $a_u \in A$ then for any upper bound $b \in B$, $a_u \leq b$, hence a_u would be a least upper bound of A. Therefore A and B are disjoint nonempty sets. Define the open sets

$$U = \bigcup_{a \in A} (-\infty, a) \qquad V = \bigcup_{b \in B} (b, \infty)$$

Note that V is actually just equal to the set of upper bounds B. This is because there is no least bound, so for any $b \in B$ there will exist another $b_0 \in B$ so that $b \in (b_0, \infty)$. Also, it is worth noting that $A \subset U$ which follows by similar reasoning, since there is no maximum within A.

Now U, V are nonempty disjoint open sets. Also for any $x \in X$, if x is an upper bound for A then $x \in V$ and if x is not an upper bound for A then there exists $a \in A$ so that $x < a$, hence $x \in U$. Thus $X = U \cup V$ and U, V form a separation of X. By contradiction, it must be the case that X has the least upper bound property. $\square$

Exercise 24.5. Consider the following sets in the dictionary order. Which are linear continua?

(a) $\mathbb{Z}_+ \times [0,1)$
Yes.

(b) $[0,1) \times \mathbb{Z}_+$
No. There is no element between 0×1 and 0×2.

(c) $[0,1) \times [0,1]$
Yes.

(d) $[0,1) \times [0,1)$
No. There is no least upper bound for $A = (-\infty, \frac{1}{2} \times 1)$ but A is nonempty and bounded above by $\frac{3}{4} \times 0$.

Exercise 24.8 (a). Is a product of path-connected spaces necessarily path connected?

Solution. In the product topology, yes. Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of path-connected spaces and give $\prod X_\alpha$ the product topology. Let $\mathbf{x}, \mathbf{y} \in \prod X_\alpha$. Since each X_α is path connected, there exists a continuous map $f_\alpha : [a_\alpha, b_\alpha] \to X_\alpha$ such that $f_\alpha(a_\alpha) = \pi_\alpha(\mathbf{x})$ and $f_\alpha(b_\alpha) = \pi_\alpha(\mathbf{y})$. Define

$$g_\alpha : [0,1] \to [a_\alpha, b_\alpha] \quad \text{by} \quad g(x) = a_\alpha + (b_\alpha - a_\alpha)x$$

which is clearly continuous (when $a_\alpha < b_\alpha$ this acts as expected, when $a_\alpha = b_\alpha$ this is just the constant function which is also continuous). Then define $f : [0,1] \to \prod X_\alpha$ by

$$f(x) = (f_\alpha \circ g_\alpha(x))_{\alpha \in J}$$

which is also continuous by Theorem 19.6. It is also evident that $f(0) = \mathbf{x}$ and $f(1) = \mathbf{y}$. Hence $\prod X_\alpha$ is path connected. $\square$

Part 2

(a) Let X be an uncountable subset of S_Ω. Prove that X and S_Ω have the same order type. Hint: let $f : X \to S_\Omega$ be $f(x) = \min(S_\Omega - f((-\infty, x) \cap X))$.

Proof. First, let us define some notation for sections:

$$X_a = (-\infty, a) \cap X$$

$$S_a = (-\infty, a)$$

so that f defined above can be rewritten as

$$f(x) = \min(S_\Omega - f(X_x))$$

We begin by proving a lemma: images of sections in X are sections in S_Ω. Suppose for contradiction that this is not the case, and let y be the least element of X such that $f(X_y)$ is not a section of S_Ω. Then there exists $a < b$ such that $b \in f(X_y)$ and $a \notin f(X_y)$. Let $x \in X_y$ such that $f(x) = b$, so

$$b = f(x) = \min(S_\Omega - f(X_x)).$$

However, $x < y$ so we know that $f(X_x)$ is a section of S_Ω. Then it is clear that

$$f(X_x) \cup \{\min(S_\Omega - f(X_x))\} = f(X_x) \cup \{b\}$$

is also a section of S_Ω. Since $a < b$, it must be the case that $a \in f(X_x)$, but of course $X_x \subset X_y$ and $f(X_x) \subset f(X_y)$, so this contradicts $a \notin f(X_y)$. We conclude that for all $y \in X$, $f(X_y)$ is a section of S_Ω.

Next we show f is order preserving. Let $x, y \in X$ such that $x < y$. Then $f(X_y)$ is a section of S_Ω which contains $f(x)$. Therefore $S_\Omega - f(X_y)$ contains elements that are all strictly greater than $f(x)$. Hence

$$f(y) = \min(S_\Omega - f(X_y))$$

must be strictly greater than $f(x)$. We conclude f is order preserving and also injective.

Next suppose for contradiction that f is not surjective. Let a be the least element of S_Ω such that $a \notin f(X)$. Then the section $S_a \subset f(X)$. We've already shown that f is injective, and since S_a is countable, it cannot be the case that $f(X) = S_a$ as this would contradict X being uncountable. So $f(X)$ must contain some elements greater than a. Let b be the least element such that $b \in f(X)$ and $a < b$. Let $x \in X$ such that $f(x) = b$. Now there must exist a $y \in X$ such that $x < y$, because $X_x \cup \{x\}$ is countable and X is uncountable. Then $f(X_y)$ is a section containing $f(x)$ and since $a < f(x)$, $f(X_y)$ must also contain a. This contradicts $a \notin f(X)$. We conclude f is surjective.

We have shown f is an order preserving bijection, and therefore X and S_Ω have the same order types.

$\square$

(b) Is it possible for a well-ordered set to have the same cardinality as S_Ω, but not the same order type?

Solution. Yes. Consider the set $A = S_\Omega \cup \{\Omega\}$ from Lemma 10.2. This set has a different order type than S_Ω, as every section of S_Ω is countable, but the section of A by Ω is uncountable. We can construct a bijection fairly easily however: let s denote the immediate successor function and define $f : A \to S_\Omega$ by

$$f(x) = \begin{cases} s(x) & \text{if } x \neq \Omega \\ \min(S_\Omega) & \text{if } x = \Omega. \end{cases}$$

Then f is well defined by Exercise 10.2(a) and clearly a bijection.

$\square$