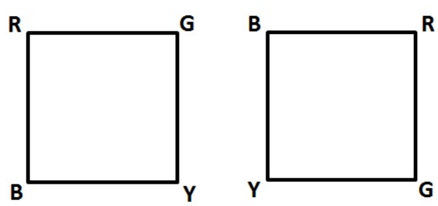
A Small Advertisement for Group Actions
November 2011

Professor Milnikel correctly asserted in class on Wednesday that several important theorems in the field of abstract algebra follow from the underlying theory provided by group actions. In addition to being useful for topics such as the Sylow theorems, Burnside's Lemma, and many counting and coloring problems, group actions provide a natural framework with which to describe and study certain types of sets and objects. Some examples are given below.

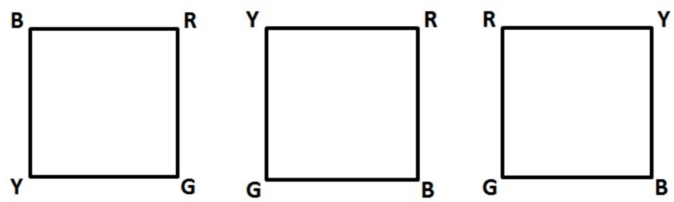
1. Let's begin by considering colorings of the corners of a square. Suppose we decide to color the corners of a square with the colors red (R), yellow (Y), green (G), and blue (B), where each color is used exactly once. Then we know that there are 4!=24 distinct colorings.

   Suppose now that we think of the square as a solid in space – we're allowed to pick it up, move it around (possibly rotating and/or reflecting it in some way), and then replace it in its original position. Given this perspective, we should consider the two colorings shown below to be essentially the same because we can get from one ot the other by rotating our square.

   We see that in this example we have the dihedral group of order 8 acting on the set of oriented colorings of the square (on the set of 24 colorings we discussed above). Suppose we wish to determine how many truly different colorings of the square there are, where we understand 'truly' to mean that two colorings are equivalent if one can be obtained from another by a movement of the square. Then we are actually trying to count the number of distinct orbits under the group action of D_4 on the set of oriented colorings! This is one of the great advantages of defining group actions, and orbits in particular, – they allow us to consider certain subsets to be equivalent based on the structure of the group of our choosing. In this case, we can quickly answer the question of how many truly distinct colorings exist.

We know that the equivalence classes defined by the orbits of the colorings partition the set of oriented colorings. Also, if we start with any element of our set of oriented colorings, applying each element of D_4 results in a new coloring, so the size of each orbit is 8. Therefore, we have (# orbits) $\times 8 = 24$, so there are 3 equivalence classes. In other words, the number of truly different colorings of the square is 3. They are shown below.



The three squares above are representative elements of the three equivalence classes. Based on the theory that we've developed using group actions, we know that if we start with one of the colorings above, there is no movement of the square that will result in one of the other two colorings. This is a much better technique for proving this fact than showing it by brute force because it's quicker and it generalizes nicely to much larger settings.

The moral of the story in this example is this: If X is a G-set, the structure of G determines the structure and meaning of the orbits. By choosing to act by certain groups or subgroups, we gain the ability to efficiently count collections that have desired properties.

2. We have just seen that the equivalence classes defined by orbits can be very useful for counting problems. They can also be useful for the study of certain types of data sets. Suppose that you collect a sequence of observations of an object that exhibits periodic behavior. Each observation gives you a vector of observed values, and therefore the sequence of observations can be collected in a matrix where the columns are ordered by the time of collection. Let D_1 and D_2 be two collections of observations of object A. Let D_3 be a collection of observations of object B. We should expect the matrices for D_1, D_2, and D_3 to be distinct, but we would like to consider D_1 and D_2 to be equivalent since they contain the same information (provided we have observed the object long enough to capture an entire period's worth of information). We can use group actions to create this equivalence.

Let G be the group of cyclic permutations on $\{1, 2, \ldots, n\}$, which is a subgroup of S_n. Let M be a matrix of size $m \times n$ and let $\sigma \in S_n$. Then we define a group action of S_n on the set of matrices of size $m \times n$ by defining $\sigma * M$ to be the matrix obtained by permuting the columns of M according to the permutation σ. (Equivalently, the action of an element of S_n is to right-multiply by a permutation matrix.) The action of the subgroup G is therefore a cyclic permutation of the columns. Hence under this action, the data sets D_1 and D_2 are members of the same equivalence class, while both D_1 and D_2 can be recognized as distinct from the observations collected in D_3.

3. Let's move up a dimension and look at a similar example. Let $E(3)$ denote the Euclidean group, whose elements are the translations, rotations, and reflections in 3-dimensional Euclidean space (i.e. the collection of isometries, or distance-preserving functions, on the space $\mathbb{R}^3$). Under the operation of composition, $E(3)$, and in general $E(n)$, is a group.

   Consider the set S of vertices of a tetrahedron. We can define a group action of $E(3)$ on S: if $g \in E(3)$ and $s \in S$, then $g * s$ is simply the image of the point s after it has been moved by g (rotated, reflected, or translated as appropriate). The identity element in $E(3)$ is the isometry which leaves every point fixed, so certainly $e * s = s$. If $g, h \in E(3)$ and $s \in S$, then we get that $g * (h * s) = (gh) * s$ essentially by definition of isometry.

   Note that more generlly, we are defining an action of $E(3)$ on the entire solid that is the tetrahedron. However, such an action is completely determined by its action on the vertices so we can simplify the problem by defining the action on the vertices.

   You may notice that there is a subset of $E(3)$ that fixes S. For example, if we fix the tetrahedron to have base in the xy-plane and place its top vertex on the z- axis, then a rotation by $\frac{\pi}{2}$ radians about the z-axis would move the vertices themselves but leave the tetrahedron itself unchanged. We say that a G-set X is **invariant** under the action of G if $GX := \{g * x \mid g \in G, x \in X\} = X$. A natural question to ask is can we find subgroups of $E(3)$ under which S is invariant? We've already found one element of $E(3)$ that fixes S; a quick count reveals that there are 12 such rigid motions that fix the tetrahedron. That is, there is a subgroup $G \leq E(3)$ of order 12 for which $GS = S$. If we allow not only rigid motion isometries but reflections as well, then we find that there is a group of order 24 under which S is invariant. These two groups are the rotational symmetry (or orientation-preserving) group and the symmetry group of the tetrahedron and they are important for a variety of reasons.

In general, the size and structure of the rotational symmetry group and symmetry group of a solid, which are defined in terms of group actions, contain valuable information about the structure of the solid itself. They are useful in fields such as chemistry because they carry information about molecular structure and therefore also help us to understand chemicals and chemical derivatives.

Previously, I mentioned that we could quickly count and determine that the size of the rotational symmetry group is 12. While a brute force count is relatively quick in this case, it is not a desirable method in general and, here again, group actions provide us with a convenient method of counting. It can be shown (and should seem somewhat intuitive) that $|G| = |G_x||Gx|$. In other words, if we pick a vertex, say T at the top of the tetrahedron, we can count the size of G by counting all the ways in which G acts on the tetrahedron while fixing T (which is $|G_x|$) and multiply that by the number of places to which G can send T (which is $|Gx|$). This is called the orbit-stabilizer theorem. If we apply the theorem to our tetrahedron, we find that there are three ways in which G fixes T and there are a total of four possible images of T under an action by G. So there are 3 * 4=12 elements in the rotational symmetry group.

4. As a final example, consider the set of all possible images of an object. We can apply the ideas in the previous examples to this set. If we define a group action using the special orthogonal group $SO(2)$, then our set of equivalence classes will correspond to those images that look the same up to rotation. (For those of you that haven't met this very important group yet, the special orthogonal group is the group of matrices with determinant 1. Every $n \times n$ matrix with determinant one corresponds to a rotation in $\mathbb{R}^n$ and is therefore an isometry, so we can view $SO(2)$ as a subgroup of $E(2)$.) If we define a group action using the special Euclidean group $SE(2)$, then our equivalence classes will partition the

set of images into those that are the same up to a rotation, translation, or combination of the two. Finally, if we define a group action using the Euclidean group $E(2)$, then orbits will consist of those images that are the same up to rotation, translation, and reflection. This use of group actions can be useful if, say, it shows up in both your masters thesis and your disseration.

You might also observe that this framework is useful for people who work on things like registration problems. Imagine using a hand-held video camera to capture a short film. That film is actually just a collection of images that are strung together in succession. Since humans are not very good at keeping the camera steady, the images will most likely consist of images that are slightly translated in varying amounts and directions from a central location. The problem of translating those images to be centered on the same spot so that the film looks steady is called a registration problem. We recognize that each of the images in the set are in an equivalence class defined by a group action of $SE(2)$. The challenge is to choose the 'best' representative from each class so that the resulting film is improved in quality.

One final note. We say that a group action is **transitive** if for every pair $x, y \in S$, there exists an element $g \in G$ such that $g * x = y$. In other words, there is only one orbit. So if we have a set of images under the action of $SO(2)$ and the action is transitive, then we know that our set is actually just a set of rotations of one image. We also know that it is a reasonable use of time to search for an element of $SO(2)$ that sends a given image to another image because we know that such an element of $SO(2)$ exists. This is useful linformation if your data set is numerical – because data is not exact, we typically can't find an element g that sends x precisely to y, but if we know that such a g exists in theory, then we can approximate it.

There is a nice discussion of group actions in 'Introduction to Applied Algebraic Systems' by Norman Reilly if you want to read more on the subject. The example of coloring the corners of the square is presented along with some applications to chemistry and k-colorings.