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Block II Take-Home
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Problem 1. **(a)** Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = 3n + 5$. We will prove that f is one-to-one, but not onto.
Proof:

To prove that f is one-to-one, let's suppose that for some $z_1, z_2 \in \mathbb{Z}$, $f(z_1) = f(z_2)$. Then

$$3z_1 + 5 = 3z_2 + 5 \iff 3z_1 = 3z_2 \iff z_1 = z_2.$$

We conclude that f is one-to-one.

To show that f is not onto, consider $1 \in \mathbb{Z}$. Suppose there exists $n \in \mathbb{Z}$ such that $f(n) = 3n + 5 = 1$. Then $3n = -4 \iff n = \frac{-4}{3}$. Thus $n \notin \mathbb{Z}$, and we conclude by contradiction that there is no $n \in \mathbb{Z}$ such that $f(n) = 1$. Therefore f is not onto. ∎

(b) Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n(n + 4)$. We will prove that f is one-to-one, but not onto.
Proof:

To prove that f is one-to-one, let's suppose that for some $n, m \in \mathbb{N}$, $f(n) = f(m)$. Then

$$n(n+4) = m(m+4) \iff n^2 + 4n = m^2 + 4m$$

$$\iff n^2 + 4n + 4 = m^2 + 4m + 4 \iff (n+2)^2 = (m+2)^2 \iff n + 2 = \pm(m+2).$$

This last equation is true if and only if $n = m$ or $n = -m - 4$. However, since we know $n, m \in \mathbb{N}$, we disregard the negative solution, and conclude $n = m$. Therefore f is one-to-one.

To prove that f is not onto, consider $1 \in \mathbb{N}$. Suppose there is some $n \in \mathbb{N}$ such that $f(n) = n(n + 4) = 1$. Then

$$n^2 + 4n = 1 \iff n^2 + 4n + 4 = 5$$

$$\iff (n+2)^2 = 5 \iff n + 2 = \pm\sqrt{5}.$$

Once again, we are interested only in the positive solution, such that $n = \sqrt{5} - 2$. Clearly then $n \notin \mathbb{N}$, and we conclude by contradiction that there is no $n \in \mathbb{N}$ such that $f(n) = 1$. Therefore f is not onto. ∎

(c) Let $g : \mathbb{N} \to \mathbb{Q}$ be defined by $g(n) = \frac{n}{n+1}$. We will prove that g is one-to-one, but not onto.
Proof:

To prove that g is one-to-one, let's suppose $g(n) = g(m)$ for some $n, m \in \mathbb{N}$. Then

$$\frac{n}{n+1} = \frac{m}{m+1} \iff \frac{n(m+1)}{(n+1)(m+1)} = \frac{m(n+1)}{(n+1)(m+1)}$$

$$\iff n(m+1) = m(n+1) \iff nm + n = nm + m \iff n = m.$$

Therefore g is one-to-one.

To prove that g is not onto, consider $64 \in \mathbb{Q}$. There is no $n \in \mathbb{N}$ such that $f(n) = \frac{n}{n+1} = 64$, since $\frac{n}{n+1} < 1$ for all $n \in \mathbb{N}$. Therefore g is not onto.
∎

Problem 2. Let $A = \mathbb{Z}^+ \times \mathbb{Z}^+$. Define a relation $\sim$ on A by $(a,b) \sim (c,d)$ if $a^b = c^d$.

(a) We will show that $\sim$ is an equivalence relation on A.
Proof:

To show that $\sim$ is an equivalence relation, we must show that $\sim$ is reflexive, symmetric, and transitive. To prove that $\sim$ is reflexive, consider $(a,b) \in A$. Clearly $a^b = a^b$, so $(a,b) \sim (a,b)$.

To prove that $\sim$ is symmetric, suppose that for some $(a,b), (c,d) \in A$, $(a,b) \sim (c,d)$. Then $a^b = c^d$, so clearly $c^d = a^b$, and thus $(c,d) \sim (a,b)$. Therefore $\sim$ is symmetric.

To prove that $\sim$ is transitive, suppose that for some $(a,b), (c,d), (e,f) \in A$, $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. Then $a^b = c^d$ and $c^d = e^f$, so we must have $a^b = e^f$. Therefore $(a,b) \sim (e,f)$, and we conclude $\sim$ is transitive.

Therefore, $\sim$ is an equivalence relation. ∎

(b) The equivalence class $[(16,1)]$ consists of all $(a,b) \in A$ such that $(16,1) \sim (a,b)$, or equivalently $16^1 = a^b$. We find that the set of ordered pairs of positive integers satisfying this equation is $[(16,1)] = \{(2,4), (4,2), (16,1)\}$. Similarly, the equivalence class $[(3,4)]$ consists of ordered pairs of positive integers (a,b) satisfying $3^4 = 81 = a^b$. We find that this set is $[(3,4)] = \{(81,1), (9,2), (3,4)\}$.

(c) One natural number with many nice properties is 64. We find that the equivalence class $[(64,1)] = \{(64,1), (8,2), (4,3), (2,6)\}$ and thus has exactly four elements.

(d) An equivalence class with infinitely many elements is $[(1,1)] = \{(1,n)\}_{n\in\mathbb{Z}^+}$, since $1^n = 1$ for all $n \in \mathbb{Z}^+$.

Problem 3. (a) The set A has a maximal element k, and minimal elements a, b, c. The greatest element is k, and there is no least element.

(b) Consider the subset $\{a, d\} \subseteq A$. An upper bound x of this set satisfies that $x \geq a$ and $x \geq d$. From the diagram we see the set of upper bounds is $U = \{f, g, i, k\}$. However, a least upper bound y must satisfy that $y \in U$ and $y \leq x$ for all $x \in U$. We see that $i, k > f$, so we know by antisymmetry that the elements i, k cannot be least upper bounds. This leaves elements f, g as possibilities, but since f is not related to g, we cannot claim that $f \leq g$ or that $g \leq f$. We conclude that $\{a, d\}$ has no least upper bound.

Problem 4. (a) Let A be a partially ordered set. Suppose that $X \subseteq Y \subseteq A$, and that $glb(X), lub(X), glb(Y)$, and $lub(Y)$ all exist. Then $glb(Y) \leq glb(X) \leq lub(X) \leq lub(Y)$.
Proof:
First we'll show that $glb(Y) \leq glb(X)$. Let $y_0 = glb(Y)$. Then $y_0 \in A$ such that $y_0 \leq y$ for all $y \in Y$. Since $X \subseteq Y$, $y_0 \leq x$ for all $x \in X$, which means y_0 is a lower bound for X. Then by definition, $glb(X) \geq y_0$, and we conclude $glb(Y) \leq glb(X)$.
Next we'll show that $glb(X) \leq lub(X)$. Let $x \in X$. By definition, $glb(X) \leq x$, and $x \leq lub(X)$. By transitivity, $glb(X) \leq lub(X)$.
Finally, we'll show that $lub(X) \leq lub(Y)$. Let $y_0 = lub(Y)$. Then $y_0 \geq y$ for all $y \in Y$. Since $X \subseteq Y$, $y_0 \geq x$ for all $x \in X$, which means y_0 is an upper bound for X. Then by definition, $lub(X) \leq y_0$, and we conclude $lub(X) \leq lub(Y)$.
Therefore, by transitivity of the partial order, we have $glb(Y) \leq glb(X) \leq lub(X) \leq lub(Y)$. ∎

(b) Consider the subsets $(0,1), [0,1] \subseteq \mathbb{R}$. We see that $(0,1) \subset [0,1]$, yet $glb((0,1)) = glb([0,1]) = 0$, and $lub((0,1)) = lub([0,1]) = 1$.

Problem 5. Let $f : A \to B$ be a function. Let X, Y be subsets of A and U, V be subsets of B.

(a) $f^{-1}(U) \setminus f^{-1}(V) = f^{-1}(U \setminus V)$.
Proof:
Let $x \in f^{-1}(U) \setminus f^{-1}(V)$. Then $x \in f^{-1}(U)$ but $x \notin f^{-1}(V)$. This means that $f(x) \in U$ and $f(x) \notin V$, from which it follows that $f(x) \in U \setminus V$. Therefore, $x \in f^{-1}(U \setminus V)$.

Now let $x \in f^{-1}(U \setminus V)$. Then $f(x) \in U \setminus V$, which means $f(x) \in U$ but $f(x) \notin V$. Then $x \in f^{-1}(U)$ but $x \notin f^{-1}(V)$. Therefore, $x \in f^{-1}(U) \setminus f^{-1}(V)$. We conclude that $f^{-1}(U) \setminus f^{-1}(V) = f^{-1}(U \setminus V)$. ∎

(b) $f(X) \setminus f(Y) \subseteq f(X \setminus Y)$.
Proof:

Let $z \in f(X) \setminus f(Y)$. Then $z \in f(X)$ but $z \notin f(Y)$. Then there is some $x \in X$ such that $f(x) = z$, but for all $y \in Y$, $f(y) \neq z$. Since $f(x) = z$, we know $x \notin Y$. Thus, $x \in X \setminus Y$ such that $f(x) = z$, and we conclude $z \in f(X \setminus Y)$. ∎

(c) $f(X) \setminus f(Y) = f(X \setminus Y)$ for all subsets X, Y of A if and only if f is one-to-one.
Proof:

We will prove the forward implication by contrapositive, so suppose f is not one-to-one. Then there exist $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$, yet $a_1 \neq a_2$. Let $X = \{a_1\}$ and $Y = \{a_2\}$ such that $X \setminus Y = \{a_1\}$. We know that $f(a_1) \in f(\{a_1\})$, so $f(a_1) \in f(X)$ and $f(a_1) \in f(X \setminus Y)$. However, $f(a_1) = f(a_2)$, and since $f(a_2) \in f(\{a_2\}) = f(Y)$, we must have $f(a_1) \in f(Y)$. Thus, $f(a_1) \in f(X \setminus Y)$ but $f(a_1) \notin f(X) \setminus f(Y)$. We conclude that if f is not one-to-one, then there exist subsets X, Y of A such that $f(X) \setminus f(Y) \neq f(X \setminus Y)$.

Now suppose that f is one-to-one. We know from **5 (b)** that $f(X) \setminus f(Y) \subseteq f(X \setminus Y)$, since this holds for all functions $f : A \to B$ and subsets X, Y of A. So, to prove the other containment, suppose $z \in f(X \setminus Y)$. Then there is some $x \in X \setminus Y$ such that $f(x) = z$. Since $x \in X$, we know that $z \in f(X)$. Suppose that $z \in f(Y)$. Then there exists $y \in Y$ such that $f(y) = z$. Then $f(y) = f(x)$, and since f is one-to-one, $y = x$. This implies $x \in Y$, but we have already said that $x \in X \setminus Y$. Therefore, by contradiction, we know that $z \notin f(Y)$. We have shown that $z \in f(X)$ and $z \notin f(Y)$, so by defintion, $z \in f(X) \setminus f(Y)$. Therefore $f(X \setminus Y) \subseteq f(X) \setminus f(Y)$.

We have proven that $f(X) \setminus f(Y) = f(X \setminus Y)$ for all subsets X, Y of A if and only if f is one-to-one. ∎