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Section 10: 6, 7, 9-11, 25, 28, 29, 34
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Problem 6:
All left cosets of $\{\rho_0, \mu_2\} \leq D_4$ are

$$\{\rho_0, \mu_2\}, \{\rho_1, \delta_2\}, \{\rho_2, \mu_1\}, \{\rho_3, \delta_1\}.$$

Problem 7:
All right cosets of $\{\rho_0, \mu_2\} \leq D_4$ are

$$\{\rho_0, \mu_2\}, \{\rho_1, \delta_1\}, \{\rho_2, \mu_1\}, \{\rho_3, \delta_2\},$$

which are *not* the same as the left cosets because D_4 is nonabelian.

Problem 9:
All left cosets of $\{\rho_0, \rho_2\} \leq D_4$ are

$$\{\rho_0, \rho_2\}, \{\rho_1, \rho_3\}, \{\mu_1, \mu_2\}, \{\delta_1, \delta_2\}.$$

Problem 10:
All right cosets of $\{\rho_0, \rho_2\} \leq D_4$ are

$$\{\rho_0, \rho_2\}, \{\rho_1, \rho_3\}, \{\mu_1, \mu_2\}, \{\delta_1, \delta_2\},$$

which are the same as the left cosets because ρ_0, ρ_2 commute with all elements of D_4.

Problem 11:
We can rearrange Table 8.12 in an order corresponding to the above cosets to find:

	ρ_0	ρ_2	ρ_1	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_0	ρ_0	ρ_2	ρ_1	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_2	ρ_2	ρ_0	ρ_3	ρ_1	μ_2	μ_1	δ_2	δ_1
ρ_1	ρ_1	ρ_3	ρ_2	ρ_0	δ_1	δ_2	μ_2	μ_1
ρ_3	ρ_3	ρ_1	ρ_0	ρ_2	δ_2	δ_1	μ_1	μ_2
μ_1	μ_1	μ_2	δ_2	δ_1	ρ_0	ρ_2	ρ_3	ρ_1
μ_2	μ_2	μ_1	δ_1	δ_2	ρ_2	ρ_0	ρ_1	ρ_3
δ_1	δ_1	δ_2	μ_1	μ_2	ρ_1	ρ_3	ρ_0	ρ_2
δ_2	δ_2	δ_1	μ_2	μ_1	ρ_3	ρ_1	ρ_2	ρ_0

If we assign the letters $A = \{\rho_0, \rho_2\}, B = \{\rho_1, \rho_3\}, C = \{\mu_1, \mu_2\}, D = \{\delta_1, \delta_2\}$, we see that this table becomes

	A	B	C	D
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

from which it is clear that these cosets form a group isomorphic to the Klein-4.

Problem 25:
To see that $|H|\,\big|\,|G|$, simply note that each coset of H has $|H|$ elements and for some r, the left cosets of H partition G into r cells such that $|G| = r|H|$.

Problem 28: Let H be a subgroup of G such that $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$. Then every left coset gH is the same as the right coset Hg.
Proof:
Let $H \leq G$ such that $g^{-1}hg \in H$ for all $h \in H$ and all $g \in G$. Then let $g \in G$ and $x \in gH$. Then $x = gh_1$ for some $h_1 \in H$. Since G is a group, $h_1 = g^{-1}x$. By our previous assumption, since $g^{-1} \in G$ and $h_1 \in H$, we must have that $(g^{-1})^{-1}h_1g^{-1} = gh_1g^{-1} = h_2$ for some $h_2 \in H$. Recalling that $h_1 = g^{-1}x$, we find $h_2 = gg^{-1}xg^{-1} = xg^{-1}$, from which it follows that $x = h_2g$. Therefore $x \in Hg$, so $gH \subseteq Hg$.
Similarly if $x \in Hg$ then $x = h_1g$ for some $h_1 \in H$. Then $h_1 = xg^{-1}$ and again since $g \in G$ we have $g^{-1}h_1g \in H$ where $g^{-1}h_1g = g^{-1}xg^{-1}g = g^{-1}x$. Therefore we have $g^{-1}x = h_2$ for some $h_2 \in H$, and since G is a group, $x = gh_2$. Thus $x \in gH$, so $Hg \subseteq gH$ and we conclude $gH = Hg$. ∎

Problem 29: If H is a subgroup of G such that the partition of G into left cosets of H is the same as the partition into right cosets of H,[1] then $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$.
Proof:
Let G be a group with subgroup H such that $gH = Hg$ for all $g \in G$. Then $x \in Hg$ if and only if $x \in gH$ as well. This means that for any $h_1 \in H$, $x = h_1g$ if and only if $x = gh_2$ for some $h_2 \in H$. Since G is a group, we have $h_2 = g^{-1}x = g^{-1}h_1g$. Therefore for any $g \in G$ and any $h_1 \in H$, we have $g^{-1}h_1g \in H$ as well. ∎

Problem 34: Let G be a group of order pq where p and q are primes. Then every proper subgroup of G is cyclic.
Proof:
Let G be a group of order pq where p and q are primes. For any proper subgroup $H < G$, we must have $1 \leq |H| < G$. We know from the Theorem of Lagrange that $|H|\,\big|\,pq$ and since p, q are primes, it must be the case that $|H| = 1, |H| = p$, or $|H| = q$. Clearly if $|H| = 1$ such that H is trivial, H is cyclic. By Corollary 10.11, in the latter two cases H is cyclic as well. We conclude that all proper subgroups of G are cyclic. ∎

[1]Couldn't we have a situation where this antecedent holds true, where the sets $\{gH : g \in G\} = \{Hg : g \in G\}$ and yet $g_0H \neq Hg_0$ for some $g_0 \in G$?