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Math 335
Section 15: 10, 12, 16, 33, 40, 41
1/7/11

Well, I'm getting a little worried about this stuff. The winter break is coming to a close and this stuff is starting to seem pretty opaque. I just really want to have some intuition for these computational problems, and in some cases I do, but not with the majority. I have the starred problems all solved, and most of the unstarred. However, I find it very difficult to find homomorphisms with the right kernel for using the Fundamental Homomorphism Theorem. When I can't, I just try to stab in the dark by finding elements (of the factor group) with certain order- although this method can provide a valid proof, it just seems so haphazard. Its not like I can really see why the factor group collapses the way it does, and I feel like I should be able to.

Problem 10: To compute $(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_8)/\langle(0,4,0)\rangle$, we will use the Fundamental Homomorphism Theorem. Consider the homomorphism $\phi : \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_8 \to \mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_8$ defined by

$$\phi(x, y, z) = (x, y \mod 4, z).$$

Then

$$\ker\phi = \{(0, 4n, 0) : n \in \mathbb{Z}\} = \langle(0, 4, 0)\rangle,$$

and since ϕ is onto, we have

$$(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_8)/\langle(0,4,0)\rangle \cong \mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_8.$$

Problem 12: Similarly, to compute $(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z})/\langle(3,3,3)\rangle$, consider the homomorphism $\phi : \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}_3\times\mathbb{Z}\times\mathbb{Z}$ defined by

$$\phi(x, y, z) = (x \mod 3, y - x, z - x).$$

Then

$$\ker\phi = \{(3n, y, z) : n \in \mathbb{Z} \text{ and } 3n = y = z\} = \{(3n, 3n, 3n) : n \in \mathbb{Z}\} = \langle(3,3,3)\rangle,$$

and since ϕ is onto, we have

$$(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z})/\langle(3,3,3)\rangle \cong \mathbb{Z}_3\times\mathbb{Z}\times\mathbb{Z}.$$

Problem 16: The six cyclic subgroups of order 4 of $G = \mathbb{Z}_4\times\mathbb{Z}_4$ are

$$H_1 = \langle(1,0)\rangle, H_2 = \langle(0,1)\rangle, H_3 = \langle(1,2)\rangle, H_4 = \langle(2,1)\rangle, H_5 = \langle(1,3)\rangle, H_6 = \langle(1,1)\rangle.$$

- By Theorem 15.8, $G/H_1 \cong G/H_2 \cong \mathbb{Z}_4$.

- Since $|G/H_3| = |G/H_4| = 4$ and in the respective factor groups, $|(1,1) + H_3| = |(1,1) + H_4| = 4$, we know that these factor groups must be cyclic and isomorphic to $\mathbb{Z}_4$. We know that those cosets have order 4 because $(1,1)^n$ is an ordered pair of equal elements (of $\mathbb{Z}_4$) for all n, and the only element in H_3, H_4 that is an ordered pair of equal elements is the identity $(0,0)$, so the order of the coset is just the order of the element $|(1,1)| = 4$.
- Similarly, since $(1,2)^n \in H_5$ only when $(1,2)^n = (0,0)$, we see that the order of the coset $|(1,2)+H_5|$ is equal to the order of the element $|(1,2)| = 4$. Since this is exactly the order of the factor group, we know that G/H_5 is cyclic and isomorphic to $\mathbb{Z}_4$.
- We see that the homomorphism $\phi : G \to \mathbb{Z}_4$ given by $\phi(x,y) = y - x$ yields $\ker\phi = H_6$, and since ϕ is onto, by the FHT we have $G/H_6 \cong \mathbb{Z}_4$.

The only subgroup of order 4 that is not cyclic is

$$H_7 = \{(0,0), (0,2), (2,0), (2,2)\},$$

and the homomorphism from G onto $\mathbb{Z}_2 \times \mathbb{Z}_2$ given by $\phi(x,y) = (x \bmod 2, y \bmod 2)$ has $\ker\phi = H_7$, so again by the FHT we have $G/H_7 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

The three subgroups of order 2 are

$$H_8 = \langle(0,2)\rangle, H_9 = \langle(2,0)\rangle, H_{10} = \langle(2,2)\rangle.$$

Each of these subgroups form a factor group of order 8 and since each element in G has at most order 4, we know that these factor groups are either isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

- Since $(1,1)^n \in H_8, H_9$ only when $(1,1)^n = (0,0)$, we see that the cosets $(1,1)+H_8$ and $(1,1)+H_9$ have order $|(1,1)| = 4$ in each of their respective factor groups. From above, we must have $G/H_8 \cong G/H_9 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.
- Similarly, $(1,2) + H_{10}$ has order 4, so $G/H_{10} \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.

Problem 33: Theorem 15.18 states that M is a maximal normal subgroup of G if and only if G/M is simple. The forward direction is true because for any nontrivial proper normal subgroup $H \lhd G/M$, $\gamma^{-1}[H]$ is a normal subgroup of G by Theorem 15.16 where $M \lhd \gamma^{-1}[H] \lhd G$, such that M is not maximal. Similarly, if M is not maximal then there is a subgroup N such that $M \lhd N \lhd G$, where $\gamma[N] \unlhd G/M$ and

$$\gamma[N] \neq G/M \quad \text{and} \quad \gamma[N] \neq \{M\},$$

such that G/M is not simple.

Problem 40: Let N and H be subgroups of G where N is normal. Then the set

$$HN = \{hn : h \in H, n \in N\}$$

is also a subgroup of G, where HN is the smallest subgroup containing both N and H.

Proof. To see that $HN \leq G$, let $x_1 = h_1 n_1, x_2 = h_2 n_2 \in HN$. Then

$$x_1 x_2 = h_1 n_1 h_2 n_2$$

where $n_1 h_2 \in N h_2$, and since $N h_2 = h_2 N$ we must have $n_1 h_2 = h_2 n_0$ for some $n_0 \in N$. Thus

$$x_1 x_2 = h_1 (n_1 h_2) n_2 = h_1 (h_2 n_0) n_2$$

where $h_1 h_2 \in H$ and $n_0 n_2 \in N$. So $x_1 x_2 \in HN$ and therefore HN is closed. Also note that since $H, N \leq G$, we have the identity $e \in H$ and $e \in N$ so that $ee = e \in HN$. Finally, for $x = hn \in HN$, we know $x^{-1} = n^{-1} h^{-1}$ where $n^{-1} h^{-1} \in N h^{-1}$. Since $N h^{-1} = h^{-1} N$, there must exist $n_0 \in N$ such that

$$x^{-1} = n^{-1} h^{-1} = h^{-1} n_0 \in HN,$$

so each element has an inverse in HN. Therefore HN is a subgroup of G.

Next, suppose that $K \leq G$ such that $H \cup N \subseteq K$. Let $x \in HN$ so that $x = hn$ for $h \in H$ and $n \in N$. Then since $h, n \in H \cup N \subseteq K$ and K is a closed subgroup, we have $hn = x \in K$. So $HN \subseteq K$ and this holds for all K containing both H and K. We conclude that HN is the smallest such subgroup. □

Problem 41: Let N and M be normal subgroups of G. Then NM is also a normal subgroup of G.

Proof. We know from Problem 40 that $NM \leq G$; to show that $NM \trianglelefteq G$, suppose $g \in G$ and $x = nm \in NM$. By Theorem 14.13, $gng^{-1} \in N$ and $gmg^{-1} \in M$ so we have

$$gng^{-1} gmg^{-1} = gnmg^{-1} = gxg^{-1} \in NH.$$

Again by Theorem 14.13, NM is normal.

□

Questions from the Unstarred Problems (and some neither starred nor unstarred)

Problem 1: By Theorem 15.8, $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(0,1)\rangle \cong \mathbb{Z}_2$

Problem 2: To compute $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(0,2)\rangle$, we first see that the factor $\mathbb{Z}_2$ is left alone (identity still 0) and $\mathbb{Z}_4$ is collapsed by a subgroup of order 2 (identity goes to both 0 and 2), so we expect the factor group to be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The homomorphism $\phi(x, y) = (x, y \bmod 2)$ onto our expected group confirms our suspicions, as $\ker \phi = \langle(0,2)\rangle$.

Problem 3: To compute $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(1,2)\rangle$, where

$$\langle(1,2)\rangle = \{(0,0), (1,2)\},$$

first note that the factor group must have order 4. We also find that the coset $(1,1)+\langle(1,2)\rangle$ has order 4, so the factor group is cyclic and isomorphic to $\mathbb{Z}_4$. I suppose the homomorphism $\phi : \mathbb{Z}_2 \times \mathbb{Z}_4 \to \mathbb{Z}_4$ defined by $\phi(x, y) = y - 2x$ could also help.

Problem 4: To compute $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1,2)\rangle$, where

$$\langle(1,2)\rangle = \{(0,0),(1,2),(2,4),(3,6)\},$$

first note that the factor group must have order 8. We also find that the coset $(1,1)+\langle(1,2)\rangle$ has order 8, so the factor group is cyclic and isomorphic to $\mathbb{Z}_8$. I suppose the homomorphism $\phi : \mathbb{Z}_4 \times \mathbb{Z}_8 \to \mathbb{Z}_8$ defined by $\phi(x, y) = y - 2x$ could also help.

Problem 5: The answer to this one is in the back of the book, but it's still driving me nuts! We are computing the factor group

$$G/H = (\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1,2,4)\rangle$$

where

$$\langle(1,2,4)\rangle = \{(0,0,0),(1,2,4),(2,0,0),(3,2,4)\}.$$

Clearly collapsing this subgroup to the identity does not allow for the factors $\mathbb{Z}_4, \mathbb{Z}_4, \mathbb{Z}_8$ to collapse separately. It would be nice to create a homomorphism where we insure that (x, y, z) goes to the identity if $y = 2x$ and $z = 2y = 4x$, but since we are working in modular arithmetic, this does not seem possible. This factor group has order 32; I'm guessing Fraleigh doesn't expect me to analyze each element to figure it out. Noting that all $(x, y, z)^n \in \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_8$ will go to $(0,0,0)$ at $n = 8$, this is the maximum order of any coset in the factor group. We see that the coset $(1,1,1)+\langle(1,2,4)\rangle$ has order 8, which allows us to conclude that this factor group is isomorphic to either $\mathbb{Z}_8 \times \mathbb{Z}_4$ or $\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. I was thinking of coming up with a certain number of elements of a certain order, but there's got to be a better way to solve this.

Problem 6: The factor group $(\mathbb{Z} \times \mathbb{Z})/\langle(0,1)\rangle \cong \mathbb{Z}$ can be computed with a direct application of Theorem 15.8. The homomorphism to consider is $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $\phi(x, y) = x$, which has kernel $\langle(0,1)\rangle$.

Problem 7: I believe the factor group $(\mathbb{Z}\times\mathbb{Z})/\langle(1,2)\rangle$ can be visualized similar to Example 15.12, but representatives must be taken off of the y-axis. The homomorphism to consider is $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $\phi(x, y) = y - 2x$, which has kernel $\langle(1,2)\rangle$.

Problem 8: For the factor group $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})/\langle(1,1,1)\rangle$, we construct $\phi : \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ by $\phi(x, y, z) = (y-x, z-x)$. Then $\phi(x, y, z) = (0,0)$ exactly when $x = y = z$, which are all of the elements in $\langle(1,1,1)\rangle$. Therefore the factor group is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Problem 9: For the factor group $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_4)/\langle(3,0,0)\rangle$, we construct $\phi : \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_4 \to \mathbb{Z}_3\times\mathbb{Z}\times\mathbb{Z}_4$ by $\phi(x, y, z) = (x \bmod 3, y, z)$. Then $\ker\phi = \langle(3,0,0)\rangle$. Therefore the factor group is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}_4$.

Problem 11: For the factor group $(\mathbb{Z} \times \mathbb{Z})/\langle(2,2)\rangle$, we recall in $\mod \langle(1,1)\rangle$, we found the factor group isomorphic to just $\mathbb{Z}$. This is because $\langle(1,1)\rangle \cong \mathbb{Z}$. But this time we are dividing out by $\langle(2,2)\rangle = 2\langle(1,1)\rangle \cong 2\mathbb{Z}$, which is like "half" of $\mathbb{Z}$. As one might expect, defining the homomorphism $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_2 \times \mathbb{Z}$ by $\phi(x, y) = (x \mod 2, y - x)$ yields $\ker \phi = \langle(2,2)\rangle$. Therefore the factor group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}$.

Problem 19:

(a) True?

(b) False

(c) Well $(\frac{1}{2} + \mathbb{Z}) + (\frac{1}{2} + \mathbb{Z}) = \mathbb{Z}$, so false.

(d) True, $(\frac{1}{n} + \mathbb{Z})$.

(e) False

(f) True

(g) False, $C \leq H$.

(h) False, could be $\{e\}$.

(i) True

(g) False, Theorem 15.15 states that A_5 is simple and clearly $5!/2$ is not prime.

Problem 20: Let $K \leq F$ where K consists of all constant functions. Find $H \leq F$ such that $H \cong F/K$.

Well the cosets are just $f + K = \{f + C : C \in \mathbb{R}\}$, which contain all functions that are f just off by a constant. We can choose as representatives those functions that pass through the origin. We have the subgroup $H = \{f \in F : f(0) = 0\}$. In this way, we'd define a homomorphism $\phi : F \to H$ by $\phi(f) = f - f(0)$. (It's very easy to show this is a homomorphism.) We see that $\phi(f) = 0$ if $f(x) = f(0)$ for all x, which are all constant functions! Thus $F/K \cong H$.

Problem 26: Define $\zeta_n = \cos(2\pi/n) + i\sin(2\pi/n)$ for $n \in \mathbb{Z}^+$. Consider $U/\langle\zeta_n\rangle$. We see that mapping $\langle\zeta_n\rangle$ to the identity is to create a modular addition, similar to $\mathbb{Z}/n\mathbb{Z}$. We see that each coset $e^x + \langle\zeta_n\rangle$ has a representative $e^0 \leq e^x < e^{\frac{2\pi}{n}}$, or $0 \leq x < \frac{2\pi}{n}$. Thus

$$U/\langle\zeta_n\rangle \cong \mathbb{R}_{\frac{2\pi}{n}}.$$

Problem 30: The center of a simple

(a) abelian group is all of the group.

(b) nonabelian group must be the trivial subgroup.