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Final Exam
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Problem 1

If I don't dive to the bottom or if I come back up at some point,
then the river isn't whiskey or I'm not a duck.
Since I don't play jack o diamonds or I don't trust my luck,
the river is whiskey or I'm a duck.

Problem 2 Let A, B, and C be sets. Then $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
Proof:

Let A, B, and C be sets and let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B \cup C$, which means $y \in B$ or $y \in C$. If $y \in B$, then $(x, y) \in A \times B$, so certainly $(x, y) \in (A \times B) \cup (A \times C)$. Similarly if $y \in C$, then $(x, y) \in A \times C$, so certainly $(x, y) \in (A \times B) \cup (A \times C)$. Therefore $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Let $(x, y) \in (A \times B) \cup (A \times C)$. Then $(x, y) \in (A \times B)$ or $x \in (A \times C)$. If $(x, y) \in A \times B$, then $x \in A$ and $y \in B$, so certainly $y \in B \cup C$. Then $(x, y) \in A \times (B \cup C)$. If $(x, y) \in A \times C$, then $x \in A$ and $y \in C$, so certainly $y \in B \cup C$. Then $(x, y) \in A \times (B \cup C)$. We've shown that in either case $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

Therefore $A \times (B \cup C) = (A \times B) \cup (A \times C)$. ∎

Problem 3 (a) Let A be a set. Then the relation $\sim$ on A is an equivalence relation if and only if $\sim$ is reflexive and circular.
Proof:

Suppose $\sim$ is an equivalence relation on A. Then $\sim$ is reflexive, symmetric, and transitive. Suppose $x \sim y$ and $y \sim z$ for some $x, y, z \in A$. By transitivity, $x \sim z$, and then by symmetry, $z \sim x$. Thus $\sim$ is circular and $\sim$ is reflexive.

Suppose $\sim$ is reflexive and circular, and suppose further that $x \sim y$. Then since $\sim$ is reflexive, $y \sim y$, and since $\sim$ is circular, $y \sim x$. Thus $\sim$ is symmetric. Now suppose $x \sim y$ and $y \sim z$. Then since $\sim$ is circular, $z \sim x$, and we've already proved $\sim$ is symmetric, so $x \sim z$. We have now shown that $\sim$ is transitive, symmetric, and reflexive, and conclude that $\sim$ is an equivalence relation.

Therefore $\sim$ is an equivalence relation if and only if $\sim$ is reflexive and circular. ∎

(b) Consider the set $A = \{a, b, c\}$ and the relation $\sim$ on A where

$$\sim = \{(a, b), (b, c), (c, a)\}.$$

We see that $\sim$ is circular, but clearly not an equivalence relation, as $\sim$ is not reflexive.

Problem 4 Define $g : \mathbb{Z} \to \mathbb{N} \cup \{0\}$ by $g(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x - 1 & \text{if } x < 0 \end{cases}$. Then g is a bijection.
Proof:

To prove that g is one-to-one we suppose that for some $x, y \in \mathbb{Z}$, $g(x) = g(y)$. By definition of g, there are three cases: both $x, y \geq 0$, or $x \geq 0$ and $y < 0$, or both $x, y < 0$ (the case when $x < 0$ and $y \geq 0$ is identical to $x \geq 0$ and $y < 0$). We'll consider each case separately.

Suppose $x, y \geq 0$. Then $2x = 2y$ by definition of g, from which it follows immediately that $x = y$. Next suppose $x \geq 0$ and $y < 0$. Then $2x = -2y - 1$. We see that

$$-2y - 1 = -2y - 2 + 1 = 2(-y - 1) + 1.$$

Since $-y - 1$ is an integer, $g(y)$ is odd. However we have claimed that $g(x) = 2x = g(y)$, and since x is an integer, $g(x)$ is even. By Exercise 1.14.1, no number can be both even and odd, so we conclude by contradiction that this case is impossible. Finally, suppose $x, y < 0$. Then $-2x - 1 = -2y - 1$, so $-2x = -2y$, from which it follows that $x = y$. In all possible cases we have shown that $g(x) = g(y)$ implies $x = y$, and conclude g is one-to-one.

To prove that g is onto, let $n \in \mathbb{N} \cup \{0\}$. Then n is even or n is odd. If n is even, $n = 2x$ for some nonnegative integer x. Then $g(x) = 2x = n$. If n is odd, $n = 2x + 1$ for some nonnegative integer x. Then $n = 2x + 2 - 1$. Equivalently, we can let $y = -x$ which implies $2y = -2x$, such that $n = -2y + 2 - 1 = -2(y - 1) - 1$. Since x is nonnegative, $y - 1 = -x - 1$ is negative. Thus by definition of g, $g(y - 1) = -2(y - 1) - 1 = n$. Therefore g is onto.

We conclude that g is a bijection. ∎

Problem 5 If A is a denumerable set, then $A \cup \{x\}$ is denumerable as well.
Proof:

Suppose A is a denumerable set. Then we know there exists a bijection $g : N \to A$, and we can let $g(n)$ be denoted by a_n for $n \in \mathbb{N}$. Since g is onto, all elements of A can be expressed as a_n for some $n \in \mathbb{N}$. Since g is one-to-one, these will be distinct labels for each of the elements. Thus, we can label the elements of A as $a_1, a_2, \ldots, a_n, \ldots$, for $n \in \mathbb{N}$. If $x \in A$, then $A \cup \{x\} = A$ and is clearly denumerable. So let's consider the more interesting case when $x \notin A$. Let $f : A \cup \{x\} \to \mathbb{N}$ be defined by

$$f(a) = \begin{cases} 1 & \text{if } a = x \\ 2 & \text{if } a = a_1 \\ 3 & \text{if } a = a_2 \\ \vdots & \\ n+1 & \text{if } a = a_n \text{ for } n \in \mathbb{N} \end{cases}$$

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To see that f is one-to-one, suppose $f(a) = f(b)$, for some $a, b \in A \cup \{x\}$. Then since f maps to the naturals, $f(a) = f(b) = n$ for some $n \in \mathbb{N}$. If $n = 1$, then by definition of f, $a = x$ and $b = x$, so $a = b$. If $n > 1$, then $a = a_{n-1}$ and $b = a_{n-1}$, so $a = b$. Therefore f is one-to-one.

To see that f is onto, consider any $n \in \mathbb{N}$. If $n = 1$, then $f(x) = n$. If $n > 1$, then $f(a_{n-1}) = n$. Thus f is onto.

We conclude that $f : A \cup \{x\} \to \mathbb{N}$ is a bijection, and thus $A \cup \{x\}$ is a denumerable set. ∎

Problem 6(a) The set $\mathcal{M} = \{3x + 5y : x \text{ and } y \text{ are nonnegative integers}\}$ contains all natural numbers greater than 7.
Proof:

We proceed by induction on n where $n \in \mathbb{N}$ such that $n > 7$. We will prove three base cases for $n \in \{8, 9, 10\}$ to make the inductive step simpler. If $n = 8$ then $n = 3(1) + 5(1)$, where 1 is a nonnegative integer. If $n = 9$ then $n = 3(3) + 5(0)$, where 3 and 0 are nonnegative integers. If $n = 10$ then $n = 3(0) + 5(2)$, where 0 and 2 are nonnegative integers. Thus for the base cases $n \in \{8, 9, 10\}$, $n \in \mathcal{M}$.

Now assume that some $k \in \mathbb{N}$ such that $k > 7$ satisfies $k \in \mathcal{M}$. Then $k = 3x + 5y$ for some nonnegative integers x and y. We see then that $k + 3 = 3x + 3 + 5y = 3(x + 1) + 5y$, where $x + 1$ and y are nonnegative integers. Therefore $k + 3 \in \mathcal{M}$.

As we showed in class, the integers can be partitioned into the sets $Z_0 = \{3n : n \in \mathbb{Z}\}$, $Z_1 = \{3n + 1 : n \in \mathbb{Z}\}$, and $Z_2 = \{3n + 2 : n \in \mathbb{Z}\}$. Clearly this partition will still exhaust a subset of the integers (namely the natural numbers), and we have effectively proved by induction within each of these sets that for all natural numbers $n \in Z_i$ such that n is greater than 7, $n \in \mathcal{M}$. Therefore, since $\mathbb{N} \subseteq \bigcup_{i \in \{1,2,3\}} Z_i$, we have proved that $n \in \mathcal{M}$ for all natural numbers n such that $n > 7$. (This is just a complicated way of saying that we have proved induction on every third integer, with three consecutive base cases, so $8, 11, 14, \ldots, 9, 12, 15, \ldots, 10, 13, 16, \ldots$, captures the natural numbers greater than 7.)

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Problem 7 (a) Let A be the set of all finite sequences in $S = \{a, b\}$. Then A is denumerable.
Proof:

We begin by partitioning A into subsets A_k where $s \in A_k$ if s has length k. This partition is rather obvious, but for a quick proof we'll just show that $\sim$ is an equivalence relation on A, where $s \sim t$ holds if and only if the sequence s is the same length as the sequence t. Let s, t, v be sequences in A. Clearly s has the same length as itself, so $\sim$ is reflexive. If s has the same length as t, then t has the same length as s, so $\sim$ is symmetric. If s has the same length as t and t has the same length as v, then s has the same length as v, so $\sim$ is transitive. Thus $\sim$ is an equivalence relation, and the equivalence classes yield the partition $A = \bigcup_{k \in \mathbb{N}} A_k$.

Now consider a finite sequence of length k in S. There are two possibilities for each of the k terms, which means that there are 2^k distinct possible orderings. Thus each A_k contains exactly 2^k elements, so we know that there exist bijections $f_k : A_k \to \{1, \ldots, 2^k\}$. Thus by definition, A_k is finite. Since A is the denumerable union of A_k's, where the A_k's are finite and nonempty, we have by Theorem 7.3.10(2) that A is denumerable. (As I pointed out in class, this also relies on the fact that each A_k is a distinct set, i.e. no trivial union of equal sets to make A a finite set.) ∎

(b) Let B be the set of all infinite sequences $a_1 a_2 a_3 \cdots$ such that each $a_i \in \{a, b\}$. Then B is uncountable.
Proof:

Suppose there is a well defined function $g : \mathbb{N} \to B$. We will show that g cannot be onto to prove that B is uncountable. Let $a_{1n} a_{2n} a_{3n} \cdots$ denote the sequence $g(n)$ for any natural number n. Now define a sequence

$$s = s_1 s_2 s_3 \cdots \text{ where } s_i = \begin{cases} a & \text{if } a_{ii} = b \\ b & \text{if } a_{ii} = a \end{cases}.$$

Now suppose for contradiction that there exists some $n \in \mathbb{N}$ such that $g(n) = s$. Then $a_{1n} a_{2n} \cdots = s_1 s_2 \cdots$, or equivalently $a_{in} = s_i$ for all $i \in \mathbb{N}$. However, we explicitly defined s

such that $s_n = a$ if $a_{nn} = b$ and $s_n = b$ if $a_{nn} = a$. Thus $s_n \neq a_{nn}$, so by contradiction we conclude $s \neq g(n)$ for all $n \in \mathbb{N}$, which means that $s \notin Ran(g)$. Therefore, g is not onto and B is uncountable. ∎