Part 1

Exercise 5.4 (d). Let $m, n \in \mathbb{Z}_+$ and $X \neq \varnothing$. Find a bijective map $k : X^n \times X^\omega \to X^\omega$.

Solution. Define $k : X^n \times X^\omega \to X^\omega$ by

$$k((z_1, \ldots, z_n) \times (x_1, x_2, \ldots)) = (z_1, \ldots, z_n, x_1, x_2, \ldots)$$

then k is easily seen to be bijective. $\square$

Exercise 7.5. Determine for each of the following sets whether or not it is countable. Justify your answers.

(a) The set A of all functions $f : \{0, 1\} \to \mathbb{Z}_+$.

Solution. A is countable because $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable by Corollary 7.4, and

$$A \;=\; \left\{f : \{0,1\} \to \mathbb{Z}_+\right\} \;=\; \prod_{i \in \{0,1\}} \mathbb{Z}_+ \;\longleftrightarrow\; \mathbb{Z}_+ \times \mathbb{Z}_+.$$

$\square$

(c) The set $C = \bigcup_{n \in \mathbb{Z}_+} B_n$, where B_n is the set of all functions $f : \{1, \ldots, n\} \to \mathbb{Z}_+$.

Solution. C is indeed countable. To see why, first note that each B_n is countable; by the same reasoning in part (a), we see that $B_n \longleftrightarrow Z_+^n$, and by Theorem 7.6, this finite product of countable sets is countable. Now C is a countable union of countable sets, which is itself countable by Theorem 7.5. $\square$

(d) The set D of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$.

Solution. D is uncountable. We can define an injection $i : \{0, 1\}^\omega \to D$ by

$$(i(f))(n) = \begin{cases} f(0) & \text{if } n = 1 \\ f(1) & \text{if } n = 2 \\ 1 & \text{if } n > 2 \end{cases}$$

This shows that $|\{0, 1\}^\omega| \leq |D|$, and $\{0, 1\}^\omega$ is uncountable by Theorem 7.7. $\square$

(f) The set F of all functions $f : \mathbb{Z}_+ \to \{0, 1\}$ that are "eventually zero".

Solution. We can partition $F = \bigcup_{n \in \mathbb{Z}_+} F_n$ where

$$F_n = \{f \in F : n \text{ is the least integer such that } f(m) = 0 \text{ for all } m \geq n\}.$$

Notice that $|F_1| = 1$, $|F_2| = 1$, and for $n > 2$ it is easy to see the bijection between each F_n and

$$\{g : \{1, \ldots, n-2\} \to \{0, 1\}\}$$

which is clearly finite. [1] Thus F is a countable union of finite sets, so F itself is countable. $\square$

(h) The set H of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ that are eventually constant.

Solution. Similar to part (f), for $h \in H$ let n be the least integer such that h is constant for all $m > n$. Partition H into $\bigcup_{n \in \mathbb{Z}_+} H_n$. Then each H_n is bijective to

$$\{g : \{1, \ldots, n\} \to \mathbb{Z}_+\}$$

which of course is the finite product of positive intgers $\mathbb{Z}_+^n$. Thus H is the countable union of countable sets, which is itself countable. $\square$

(j) The set J of all finite subsets of $\mathbb{Z}_+$.

Solution. Let B_n denote the set of all subsets of $\mathbb{Z}_+$ of size n. Then clearly $J = \bigcup_{n \in \{0, \ldots\}} B_n$. But each B_n is finite, so this is a countable union of finite sets. Therefore J is countable. $\square$

Exercise 13.1. Let X be a topological space; let $A \subset X$. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Proof. For each $x \in A$, let U_x denote the open set such that $x \in U_x$ and $U_x \subset A$. We will show that $A = \bigcup_{x \in A} U_x$. Of course, if $a \in A$, then $a \in U_a \subset \bigcup_{x \in A} U_x$. Conversely if $a \in \bigcup_{x \in A} U_x$ then $a \in U_x$ for some $x \in A$, and since $U_x \subset A$, we have $a \in A$. Therefore $A = \bigcup_{x \in A} U_x$. Since A is a union of open sets, A is itself open by definition. $\square$

Exercise 13.4 (a). If $\{\tau_\alpha\}$ is a family of topologies on X, show that $\bigcap \tau_\alpha$ is a topology on X. Is $\bigcup \tau_\alpha$ a topology on X?

Proof. Since each τ_α is a topology, $\varnothing, X \in \tau_\alpha$ for all α in the collection, hence

$$\varnothing, X \in \bigcap \tau_\alpha.$$

[1] Our definition implies that for $f \in F_n$, $f(n) = 0$ and $f(n-1) = 1$ are fixed; this is why F_n is isomorphic to the functions with domain $\{1, \ldots, n-2\}$.

Next let $U_1, \ldots, U_n \in \bigcap \tau_\alpha$. Since each τ_α is a topology, the finite intersection $\bigcap_{i=1}^n U_i \in \tau_\alpha$ for all α in the collection, hence

$$\bigcap_{i=1}^n U_i \in \bigcap \tau_\alpha.$$

Similarly for any arbitrary collection of open sets $U_\beta \in \bigcap \tau_\alpha$, since each τ_α is a topology, the union $\bigcup U_\beta \in \tau_\alpha$ for all α in the collection, hence

$$\bigcup U_\beta \in \bigcap \tau_\alpha.$$

Therefore $\bigcap \tau_\alpha$ is a topology.

On the other hand $\bigcup \tau_\alpha$ is not necessarily a topology. One such counterexample is encountered in the very next problem, since the smallest topology containing τ_1 and τ_2 is not their union. Another example is to consider the union of the lower limit and upper limit topologies on $\mathbb{R}$: the intervals $[0, 1)$ and $(0, 1]$ are in the union of these topologies, however the union of the intervals $[0, 1]$ is not in the union of these topologies. $\square$

Exercise 13.4 (c). If $X = \{a, b, c\}$ let

$$\tau_1 = \{\varnothing, X, \{a\}, \{a, b\}\}$$

$$\tau_2 = \{\varnothing, X, \{a\}, \{b, c\}\}.$$

Find the smallest topology containing τ_1, τ_2, and the largest topology contained in τ_1, τ_2.

Solution. To find the smallest topology containing both τ_1 and τ_2, we'll start with $\tau_1 \cup \tau_2$ and ensure it is closed under finite intersections and arbitrary unions:

$$\{\varnothing, X, \{a\}, \{a, b\}, \{b, c\}, \{b\}\}$$

Of course, by Exercise 13.4(a), the largest topology contained in both τ_1 and τ_2 is simply their intersection:

$$\{\varnothing, X, \{a\}\}$$

$\square$

Part 2

(a) Produce an example of a topology on $\mathbb{R}$ which has only finitely many open sets.

Solution.

$$\{\varnothing, \mathbb{R}, \{0\}\}$$

$\square$

(b) Produce an example of a topology on $\mathbb{R}$ which has a countably infinite number of open sets.

Solution.
$$\{\varnothing, \mathbb{R}\} \cup \{(-n, n) : n \in \mathbb{Z}_+\}$$

$\square$

(c) Produce an example of a topology on $\mathbb{R}$ which has an uncountable number of open sets.

Solution. I could choose $\mathcal{P}(\mathbb{R})$ but that's boring. Instead consider the topology generated by the basis
$$\mathcal{B} = \{\varnothing, \mathbb{R}\} \cup \{n : n \in \mathbb{Z}_+\}.$$
It's easy to verify that $\mathcal{B}$ is a basis and since the topology it generates is closed under unions, it is also easy to see that $\mathcal{P}(\mathbb{Z}_+) \subset \tau$, hence τ is uncountably infinite. $\square$

(d) Produce an example of a topology on $\mathbb{R}$ which has a cardinality strictly greater than that of $\mathbb{R}$. Prove that the cardinality really is greater.

Solution. The obvious answer is the discrete topology $\mathcal{P}(\mathbb{R})$. To show that the cardinality is strictly greater, we will use the definition[2] that, for nonempty sets A and B, $|A| < |B|$ if there exists an injection from A to B, but no bijection from A to B.

We have an easy injection $f : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ defined by
$$f(x) = \{x\}$$
however there can be no such bijection, because we proved in class that for any set A, there is no surjection from A to $\mathcal{P}(A)$. Therefore $\mathcal{P}(\mathbb{R})$ has cardinality strictly greater than the cardinality of $\mathbb{R}$. $\square$

[2] I don't believe we have defined this in class, nor is there a specific definition in the book. This is a reasonable definition however.