

Topology: Homework #1

Due on January 23, 2020 at 11:59pm

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Problem 1

Give an appropriate positive constant c such that $f(n) \leq c \cdot g(n)$ for all $n > 1$.

1. $f(n) = n^2 + n + 1, g(n) = 2n^3$
2. $f(n) = n\sqrt{n} + n^2, g(n) = n^2$
3. $f(n) = n^2 - n + 1, g(n) = n^2/2$

Solution

We solve each solution algebraically to determine a possible constant c .

Part One

$$\begin{aligned} n^2 + n + 1 &= \\ &\leq n^2 + n^2 + n^2 \\ &= 3n^2 \\ &\leq c \cdot 2n^3 \end{aligned}$$

Thus a valid c could be when $c = 2$.

Part Two

$$\begin{aligned} n^2 + n\sqrt{n} &= \\ &= n^2 + n^{3/2} \\ &\leq n^2 + n^{4/2} \\ &= n^2 + n^2 \\ &= 2n^2 \\ &\leq c \cdot n^2 \end{aligned}$$

Thus a valid c is $c = 2$.

Part Three

$$\begin{aligned} n^2 - n + 1 &= \\ &\leq n^2 \\ &\leq c \cdot n^2/2 \end{aligned}$$

Thus a valid c is $c = 2$.

Problem 2

Let $\Sigma = \{0, 1\}$. Construct a DFA A that recognizes the language that consists of all binary numbers that can be divided by 5.

Let the state q_k indicate the remainder of k divided by 5. For example, the remainder of 2 would correlate to state q_2 because $7 \bmod 5 = 2$.

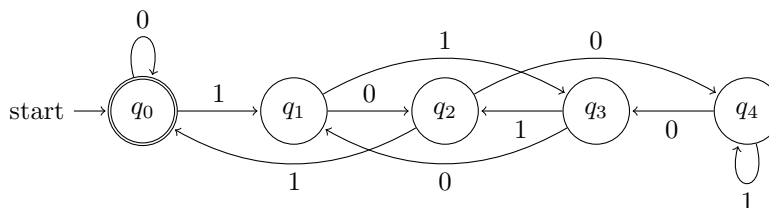


Figure 1: DFA, A , this is really beautiful, ya know?

Justification

Take a given binary number, x . Since there are only two inputs to our state machine, x can either become $x0$ or $x1$. When a 0 comes into the state machine, it is the same as taking the binary number and multiplying it by two. When a 1 comes into the machine, it is the same as multiplying by two and adding one.

Using this knowledge, we can construct a transition table that tell us where to go:

	$x \bmod 5 = 0$	$x \bmod 5 = 1$	$x \bmod 5 = 2$	$x \bmod 5 = 3$	$x \bmod 5 = 4$
$x0$	0	2	4	1	3
$x1$	1	3	0	2	4

Therefore on state q_0 or ($x \bmod 5 = 0$), a transition line should go to state q_0 for the input 0 and a line should go to state q_1 for input 1. Continuing this gives us the Figure 1.

Problem 3

Write part of **Quick-Sort**($list, start, end$)

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1: function QUICK-SORT( $list, start, end$ )
2:   if  $start \geq end$  then
3:     return
4:   end if
5:    $mid \leftarrow \text{PARTITION}(list, start, end)$ 
6:   QUICK-SORT( $list, start, mid - 1$ )
7:   QUICK-SORT( $list, mid + 1, end$ )
8: end function

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Algorithm 1: Start of QuickSort

Problem 4

Suppose we would like to fit a straight line through the origin, i.e., $Y_i = \beta_1 x_i + e_i$ with $i = 1, \dots, n$, $\mathbb{E}[e_i] = 0$, and $\text{Var}[e_i] = \sigma_e^2$ and $\text{Cov}[e_i, e_j] = 0, \forall i \neq j$.

Part A

Find the least squares estimator for $\hat{\beta}_1$ for the slope β_1 .

Solution

To find the least squares estimator, we should minimize our Residual Sum of Squares, RSS:

$$\begin{aligned} RSS &= \sum_{i=1}^n (Y_i - \hat{Y}_i)^2 \\ &= \sum_{i=1}^n (Y_i - \hat{\beta}_1 x_i)^2 \end{aligned}$$

By taking the partial derivative in respect to $\hat{\beta}_1$, we get:

$$\frac{\partial \hat{\beta}_1}{\partial RSS} = -2 \sum_{i=1}^n x_i (Y_i - \hat{\beta}_1 x_i) = 0$$

This gives us:

$$\begin{aligned} \sum_{i=1}^n x_i (Y_i - \hat{\beta}_1 x_i) &= \sum_{i=1}^n x_i Y_i - \sum_{i=1}^n \hat{\beta}_1 x_i^2 \\ &= \sum_{i=1}^n x_i Y_i - \hat{\beta}_1 \sum_{i=1}^n x_i^2 \end{aligned}$$

Solving for $\hat{\beta}_1$ gives the final estimator for β_1 :

$$\hat{\beta}_1 = \frac{\sum x_i Y_i}{\sum x_i^2}$$

Part B

Calculate the bias and the variance for the estimated slope $\hat{\beta}_1$.

Solution

For the bias, we need to calculate the expected value $\mathbb{E}[\hat{\beta}_1]$:

$$\begin{aligned}\mathbb{E}[\hat{\beta}_1] &= \mathbb{E}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\ &= \frac{\sum x_i \mathbb{E}[Y_i]}{\sum x_i^2} \\ &= \frac{\sum x_i (\beta_1 x_i)}{\sum x_i^2} \\ &= \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\ &= \beta_1 \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\ &= \beta_1\end{aligned}$$

Thus since our estimator's expected value is β_1 , we can conclude that the bias of our estimator is 0.

For the variance:

$$\begin{aligned}\text{Var}[\hat{\beta}_1] &= \text{Var}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\ &= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \text{Var}[Y_i] \\ &= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \text{Var}[Y_i] \\ &= \frac{1}{\sum x_i^2} \text{Var}[Y_i] \\ &= \frac{1}{\sum x_i^2} \sigma^2 \\ &= \frac{\sigma^2}{\sum x_i^2}\end{aligned}$$

Problem 5

Prove a polynomial of degree k , $a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0$ is a member of $\Theta(n^k)$ where $a_k \dots a_0$ are nonnegative constants.

Proof. To prove that $a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0$, we must show the following:

$$\exists c_1 \exists c_2 \forall n \geq n_0, c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$

For the first inequality, it is easy to see that it holds because no matter what the constants are, $n^k \leq a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0$ even if $c_1 = 1$ and $n_0 = 1$. This is because $n^k \leq c_1 \cdot a_k n^k$ for any nonnegative constant, c_1 and a_k .

Taking the second inequality, we prove it in the following way. By summation, $\sum_{i=0}^k a_i$ will give us a new constant, A . By taking this value of A , we can then do the following:

$$\begin{aligned} a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0 &= \\ &\leq (a_k + a_{k-1} \dots a_1 + a_0) \cdot n^k \\ &= A \cdot n^k \\ &\leq c_2 \cdot n^k \end{aligned}$$

where $n_0 = 1$ and $c_2 = A$. c_2 is just a constant. Thus the proof is complete. □

Problem 18

Evaluate $\sum_{k=1}^5 k^2$ and $\sum_{k=1}^5 (k-1)^2$.

Problem 19

Find the derivative of $f(x) = x^4 + 3x^2 - 2$

Problem 6

Evaluate the integrals $\int_0^1 (1-x^2) dx$ and $\int_1^\infty \frac{1}{x^2} dx$.