Part 1

Exercise 26.1(a). Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X; suppose that $\mathcal{T} \subset \mathcal{T}'$. What does compactness of X under one of these topologies imply about compactness under the other?

Solution. Compactness in $\mathcal{T}$ does not guarantee compactness in $\mathcal{T}'$; consider the interval $[0, 1]$ closed as a subspace of the usual topology on $\mathbb{R}$, however $[0, 1]$ is obviously not closed in the discrete topology.

On the other hand, compactness in $\mathcal{T}'$ does guarantee compactness in $\mathcal{T}$, since a covering $\mathcal{A}$ of sets open in $\mathcal{T}$ are also open in $\mathcal{T}'$, hence a finite subcover exists in $\mathcal{T}$ as well. $\square$

Exercise 26.1(b). Show that if X is compact Hausdorff under both $\mathcal{T}$ and $\mathcal{T}'$, then either $\mathcal{T}$ and $\mathcal{T}'$ are equal or they are not comparable.

Proof. Suppose for contradiction that one topology strictly contains the other, without loss of generality, $\mathcal{T} \subsetneq \mathcal{T}'$. Let $id : X \to X$ be the identity map from $\mathcal{T}'$ into $\mathcal{T}$. Then id is bijective and for any $U \in \mathcal{T}$, $id^{-1}(U) = U \in \mathcal{T}'$, so id is continuous as well. By Theorem 26.6, id is a homeomorphism. But this implies that for any $V \in \mathcal{T}'$, $id(V) = V \in \mathcal{T}$ as well, contradicting our assumption that $\mathcal{T} \subsetneq \mathcal{T}'$. Thus one topology cannot strictly contain the other, so it must be the case that either $\mathcal{T} = \mathcal{T}'$ or they are not comparable. $\square$

Exercise 26.3. Show that a finite union of compact subspaces of X is compact.

Proof. Let $Y_1, \ldots, Y_n$ be a finite number of compact subspaces of X. Let $\mathcal{A}$ be a covering of the union $\cup_{i=1}^{n} Y_i$ of sets open in X. Then $\mathcal{A}$ obviously covers each subspace Y_i, and by Lemma 26.1 there is a finite subcollection $\mathcal{A}_i \subset \mathcal{A}$ that covers Y_i, for each $i = 1, \ldots, n$. Then $\cup_{i=1}^{n} \mathcal{A}_i$ is a finite subcollection of $\mathcal{A}$ that covers $\cup_{i=1}^{n} Y_i$; again by Lemma 26.1, $\cup_{i=1}^{n} Y_i$ is compact. $\square$

Exercise 26.7. Show that if Y is compact, then the projection $\pi_1 : X \times Y \to X$ is a closed map.

Proof. Let U be closed in $X \times Y$. Then we wish to show that $\pi_1(U)$ is closed in X, so pick $x_0 \in X - \pi_1(U)$; we will find an open set W such that $x_0 \in W \subset X - \pi_1(U)$. Notice that $(X \times Y) - U$ is open and contains $\{x_0\} \times Y$. By the Tube Lemma, there is a neighborhood W of x_0 such that

$$\{x_0\} \times Y \subset W \times Y \subset (X \times Y) - U.$$

Thus, for any $y \in Y$, there is no $x \in W$ such that $x \times y \in U$. Hence $x_0 \in W \subset X - \pi_1(U)$, so $X - \pi_1(U)$ is open and $\pi_1(U)$ is closed. $\square$

Exercise 26.8. Let $f : X \to Y$; let Y be compact Hausdorff. Then f is continuous if and only if the *graph* of f,

$$G_f = \{x \times f(x) : x \in X\},$$

is closed in $X \times Y$.

Proof. ($\Longrightarrow$) Suppose f is continuous. Let $x \times y \in (X \times Y) - G_f$. Then $y \neq f(x)$ and since Y is Hausdorff we can find disjoint open neighborhoods U of y and V of $f(x)$. Then $x \in f^{-1}(V)$ so $x \times y$ is in $f^{-1}(V) \times U$, which is open because f is continuous. Furthermore, for any $z \in f^{-1}(V)$, $f(z) \in V$ and thus $f(z) \notin U$, so $f^{-1}(V) \times U \subset (X \times Y) - G_f$. Therefore G_f is closed.

($\Longleftarrow$) Suppose G_f is closed. To show f is continuous, let U be open in Y. Then $X \times U$ is open, so the complement

$$(X \times Y) - (X \times U) = X \times (Y - U)$$

is closed. Then the intersection

$$(X \times (Y - U)) \cap G_f = \{x \times f(x) : f(x) \notin U\}$$

is also closed. Since Y is compact we know from Exercise 26.7 that π_1 is a closed map, so

$$\pi_1(\{x \times f(x) : f(x) \notin U\}) = X - f^{-1}(U)$$

is closed in X. Hence $f^{-1}(U)$ is open and f is continuous.

□

Exercise 27.1. Prove that if X is an ordered set in which every closed interval is compact, then X has the least upper bound property.

Proof. We proceed by contrapositive. Let $A \subset X$ be a nonempty set that is bounded above without a least upper bound. Let B be the (nonempty) set of upper bounds for A. A couple things to note initially: A cannot have a maximum element, as it would be the least upper bound; B cannot have a least element, as it would be the least upper bound; A and B are disjoint, otherwise they would intersect at the least upper bound.

Let $a_0 \in A$ and $b_0 \in B$. Since A has no maximum and B has no minimum, we can find $c \in A$ and $d \in B$ such that $a_0 < c$ and $d < b_0$. We will show that

$$\mathcal{A} = \{(a_0, a) : a \in A \text{ and } a > a_0\} \cup \{(b, b_0) : b \in B \text{ and } b < b_0\}$$

is a covering of $[c, d]$.

Let $x \in [c, d]$. If $x \in B$ then there must exist $b \in B$ with $b < x$ so that $x \in (b, d] \subset (b, b_0)$. If $x \notin B$ then x is not an upper bound for A, so there exists an $a > x$ so that $x \in [c, a) \subset (a_0, a)$. Therefore $\mathcal{A}$ covers $[c, d]$.

Notice $\mathcal{A}$ is a union of two sets which each consist of nested intervals. Any finite subcollection will have the form

$$\mathcal{C} = \bigcup_{i=1}^{n} \{(a_0, a_i)\} \cup \bigcup_{j=1}^{m} \{(b_j, b_0)\}$$

and

$$\bigcup \mathcal{C} = (a_0, a_N) \cup (b_M, b_0)$$

where $a_N = \max\{a_i\}$ and $b_M = \min\{b_j\}$. Then $a_N, b_M \in [c, d]$, but $a_N, b_M \notin \bigcup \mathcal{C}$, as we stated earlier that A and B are disjoint. Thus no finite subcollection of $\mathcal{A}$ can still cover $[c, d]$, and we have found a closed interval that is not compact. □

Part 2

Let C be the Cantor set. Also, give $\{0, 1\}$ the discrete topology and $\{0, 1\}^\omega$ the product topology. Note $\{0, 1\}^\omega$ can be regarded as the space of all sequences of zeroes and ones.

Munkres defines the Cantor set by constructing a sequence of sets A_n each of which is a union of disjoint closed intervals of length $\frac{1}{3^n}$. Define a function $f : C \to \{0, 1\}^\omega$ by saying that the n^{th} term in the sequence $f(x)$ is 0 if for any $k \in \mathbb{Z}$, $x \in \left[\frac{3k}{3^n}, \frac{1+3k}{3^n}\right] \subset A_n$ and the n^{th} term of $f(x)$ is 1 if for any $k \in \mathbb{Z}$, $x \in \left[\frac{2+3k}{3^n}, \frac{3+3k}{3^n}\right] \subset A_n$.

The way we construct the Cantor set is by removing the middle third of an interval, then removing the middle third of the remaining left and right intervals, and so on. What f does is record, at each step of the iteration, whether x was in an interval just left of or just right of a middle third that was removed. For example, for $x = \frac{7}{27}$, when we remove the middle third from $[0, 1]$, $\frac{7}{27}$ lies in the left third $[0, \frac{1}{3}]$, and then when we remove the middle thid from $[0, \frac{1}{3}]$, $\frac{7}{27}$ lies in the right third $[\frac{2}{9}, \frac{1}{3}]$, and then when we remove the middle third from $[\frac{2}{9}, \frac{1}{3}]$, $\frac{7}{27}$ lies in the left third $[\frac{2}{9}, \frac{7}{27}]$, and then after that, it's always in the right third. Thus, "directions" for finding $\frac{7}{27}$ within the Cantor set are left, right, left, right, right, right, $\ldots$, which gets represented as $f\left(\frac{7}{27}\right) = (0, 1, 0, 1, 1, 1, \ldots)$.

(a) Show that C is compact Hausdorff.

Proof. Recall that closed subspaces of compact spaces are compact, and subspaces of Hausdorff spaces are Hausdorff; since $C \subset [0, 1]$ and $[0, 1]$ is compact Hausdorff, it suffices to show that C is closed. But C is defined as an intersection $\bigcap_{n \in \mathbb{Z}_+} A_n$ where each A_n is a finite union of closed intervals; hence C is closed. Therefore C is compact Hausdorff. □

(b) Show that f is bijective.

Proof. First we'll show that f is surjective. Let $\mathbf{x} \in \{0,1\}^\omega$. Define

$$B_1 = \begin{cases} \left[0, \frac{1}{3}\right] & \text{if } x_1 = 0 \\ \left[\frac{2}{3}, 1\right] & \text{if } x_1 = 1 \end{cases}$$

$$B_n = B_{n-1} \cap \begin{cases} \bigcup_{k=0}^{\infty} \left[\frac{3k}{3^n}, \frac{1+3k}{3^n}\right] & \text{if } x_n = 0 \\ \bigcup_{k=0}^{\infty} \left[\frac{2+3k}{3^n}, \frac{3+3k}{3^n}\right] & \text{if } x_n = 1 \end{cases}$$

where each B_i is defined as interval subsets of C. By construction $\mathcal{B} = \{B_n\}_{n=1}^{\infty}$ has the finite intersection property, and since C is compact, we know that $B = \bigcap_{n=1}^{\infty} B_n$ is nonempty and for $b \in B$, $f(b) = \mathbf{x}$. Therefore f is surjective.

Next let $a, b \in C$ such that $a \neq b$. Let n be the least positive integer so that $\frac{1}{3^n} < |a-b|$ Then a, b end up in different thirds at the n^{th} iteration, such that $\pi_n(f(a)) \neq \pi_n(f(b))$, hence $f(a) \neq f(b)$. Therefore f is injective. □

(c) Show that f is continuous.

Proof. Let $c \in C$ and V be a basis neighborhood of $f(c)$. Then $V = \prod_{n \in \mathbb{Z}_+} V_n$ where only finitely many $V_n \neq \{0,1\}$. So there exists N such that $V_n = \{0,1\}$ for all $n > N$. Then we can define

$$U = \prod_{n \in \mathbb{Z}_+} \begin{cases} \{\pi_n(f(c))\} & \text{if } n \leq N \\ \{0,1\} & \text{if } n > N \end{cases}$$

so that $f(c) \in U \subset V$. We want to find a neighborhood of c whose image is contained in U; this amounts to making sure that the neighborhood agrees with c on the directions of the first N iterations of the recursive Cantor definition. We can do this by choosing a suitably small open ball around c, namely $B(c, \frac{1}{3^N})$. For any $x \in B(c, \frac{1}{3^N})$ the distance between x and c is less than $\frac{1}{3^N}$ so for $n \leq N$, x and c fall into the same third of the interval of length $\frac{1}{3^n}$. Thus $f\big(B(c, \frac{1}{3^N})\big) \subset U \subset V$, and f is continuous. □

(d) Show that f is a homeomorphism.

Proof. Since $\{0,1\}$ is discrete, it is Hausdorff, and we know that $\{0,1\}^\omega$ is Hausdorff as well by Theorem 19.4. We showed that C is compact in part (a) and that f is bijective and continuous in parts (b) and (c). By Theorem 26.6, f is a homeomorphism. □