Sam Tay Math 601: Homework #4 February 17, 2019

Part 1

Exercise 16.2. If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, what can you say about the corresponding subspace topologies on the subset Y of X?

Solution. We can say that the subspace topology $\mathcal{T}'_Y$ that Y inherits from $\mathcal{T}'$ is finer but possibly not strictly finer than the subspace topology $\mathcal{T}_Y$ that Y inherits from $\mathcal{T}$. Suppose $U \in \mathcal{T}_Y$. Then $U = Y \cap V$ for some $V \in \mathcal{T}$. Since $\mathcal{T} \subset \mathcal{T}'$, we have $V \in \mathcal{T}'$ so that $U = Y \cap V \in \mathcal{T}'_Y$ as well. Therefore $\mathcal{T}_Y \subset \mathcal{T}'_Y$.

For an example where the inherited subspace topologies are equal, consider when $\mathcal{T} = \mathbb{R}, \mathcal{T}' = \mathbb{R}_K$, and $Y = (-2, -1)$. Let $\mathcal{B}_K$ be the standard basis for $\mathbb{R}_K$ and $\mathcal{B}$ be the standard basis for $\mathbb{R}$. Next let $B_K \cap Y$ be any basis element of the inherited subspace topology $\mathcal{T}'_Y$, as given in Lemma 16.1, where $B_K \in \mathcal{B}_K$. Notice that due to our choice of Y there is always an interval $B \in \mathcal{B}$ such that $B_K \cap Y = B \cap Y$. Thus by Lemma 13.3 we have $\mathcal{T}' \subset \mathcal{T}$. So in this case, $\mathcal{T}'$ is not strictly finer than $\mathcal{T}$. $\square$

Exercise 16.8. If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. In each case it is a familiar topology.

Solution. The topology L inherits depends on its slope. When inherited from $\mathbb{R}_\ell \times \mathbb{R}$:

$m = \infty$	$m = 0$	$m > 0$	$m < 0$
$L \longleftrightarrow \mathbb{R}$	$L \longleftrightarrow \mathbb{R}_\ell$	$L \longleftrightarrow \mathbb{R}_\ell$	$L \longleftrightarrow \mathbb{R}_\ell$

When inherited from $\mathbb{R}_\ell \times \mathbb{R}_\ell$:

$m = \infty$	$m = 0$	$m > 0$	$m < 0$
$L \longleftrightarrow \mathbb{R}_\ell$	$L \longleftrightarrow \mathbb{R}_\ell$	$L \longleftrightarrow \mathbb{R}_\ell$	$L \longleftrightarrow \mathcal{P}(\mathbb{R})$

Note that since the lower limit and upper limit topologies are isomorphic, these $\mathbb{R}_\ell$'s could also be considered naturally as $\mathbb{R}_u$'s. For example, it would be reasonable to say that the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ when its slope is positive is isomorphic to $\mathbb{R}_u$ if you consider the order of the points on L to increase as its y-coordinate increases. But the answers above are consistent if you treat the order topology on L as inheriting the dictionary order from $\mathbb{R} \times \mathbb{R}$, that is, defining $(x_1, mx_1 + b) < (x_2, mx_2 + b)$ as long as $x_1 < x_2$ for points on the line $L = \{(x, mx + b) : x \in \mathbb{R}\}$. $\square$

Exercise 16.10. Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology $I \times I$ inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.

Solution. I assume the product topology is referring to the product of each I under the order topology. Let us denote

$$\begin{aligned}
\mathcal{T} &= \text{ the order topology on } I \\
\mathcal{T}_\times &= \text{ the product topology on } I \times I \\
\mathcal{T}_< &= \text{ the dictionary order topology on } I \times I \\
\mathcal{T}_s &= \text{ the subspace topology on } I \times I \\
\mathcal{B} &= \text{ the usual basis for the order topology } \mathcal{T} \\
\mathcal{B}_\times &= \{U \times V : U, V \in \mathcal{T}\} \\
\mathcal{B}_< &= \text{ the usual basis for the dictionary order topology on } I \times I \\
\mathcal{B}_s &= \{B \cap I \times I : B \in \mathcal{C}\} \\
\mathcal{C} &= \text{ the usual basis for the dictionary order topology on } \mathbb{R} \times \mathbb{R}
\end{aligned}$$

I argue that $\mathcal{T}_s$ is finer than both $\mathcal{T}_\times$ and $\mathcal{T}_<$, but that $\mathcal{T}_\times$ and $\mathcal{T}_<$ are incomparable with each other. First, I'll show incomparability using Lemma 13.3. Let

$$B_\times = \left(\frac{1}{4}, \frac{1}{2}\right) \times \left[0, \frac{1}{2}\right) \text{ and } x = \frac{1}{3} \times 0.$$

Since $(\frac{1}{4}, \frac{1}{2})$ and $[0, \frac{1}{2})$ are both open in $\mathcal{T}$, we have $B_\times \in \mathcal{B}_\times$. However, notice that $x \in B_\times$ but there is no $B \in \mathcal{B}_<$ such that $x \in B \subset B_\times$. This is easy to see geometrically: any basis element $B \in \mathcal{B}_<$ that contains x will have to spill over and contain an element $a \times 1$ where $a < \frac{1}{3}$, but all the elements of $B_\times$ have y-coordinate less than $\frac{1}{2}$. By Lemma 13.3 we know that $\mathcal{T}_\times \not\subset \mathcal{T}_<$. Next let

$$B_< = \left[0 \times 0, 0 \times \frac{1}{4}\right) \text{ and } x = 0 \times 0.$$

Notice that $B_< \in \mathcal{B}_<$ and $x \in B_<$. However, if we try to find subsets U, V such that $x \in U \times V \subset B_<$, this will force $U = \{0\}$. Of course, $\{0\}$ is not open in $\mathcal{T}$, and therefore there is no basis element $B \in \mathcal{B}_\times$ such that $x \in B \subset B_<$. By Lemma 13.3, $\mathcal{T}_< \not\subset \mathcal{T}_\times$. Therefore $\mathcal{T}_\times$ and $\mathcal{T}_<$ are incomparable.

Next we show that $\mathcal{T}_\times \subset \mathcal{T}_s$; to this end, let $B_\times \in \mathcal{B}_\times$ and $x \in B_\times$. Then $B_\times = U \times V$ for some open sets $U, V \in \mathcal{T}$ and $x = u \times v$ for some $u \in U$ and $v \in V$. Now there exists $B_v \in \mathcal{B}$ such that $v \in B_v \subset V$. We can construct an interval $(a, b) \subset \mathbb{R}$ such that $v \in (a, b) \cap I \subset B_v$; specifically, if $B_v = [0, d)$ then choose $(a, b) = (-1, d)$, if $B_v = (c, d)$ then choose $(a, b) = (c, d)$, and if $B_v = (c, 1]$ then choose $(a, b) = (c, 2)$. Now $v \in (a, b) \cap I \subset B_v$. Finally, let

$$\begin{aligned}
B_s &= (u \times a, u \times b) \cap (I \times I). \\
&= \{u\} \times ((a, b) \cap I) \\
&\subset B_u \times B_v \\
&\subset U \times V = B_\times
\end{aligned}$$

Now it is clear that $B_s \in \mathcal{B}_s$ and $x \in B_s \subset B_\times$. Therefore by Lemma 13.3 $\mathcal{T}_\times \subset \mathcal{T}_s$.

Finally, we'll show that $\mathcal{T}_< \subset \mathcal{T}_s$. Unfortunately, I've only just realized that Lemma 13.3 is a bit of a roundabout way to show containment. Instead, note that if $\mathcal{T}_1, \mathcal{T}_2$ are topologies on a space X and $\mathcal{S}$ is a subbasis for $\mathcal{T}_1$, then $\mathcal{S} \subset \mathcal{T}_2$ implies $\mathcal{T}_1 \subset \mathcal{T}_2$. This is because T_1 is made up of arbitrary unions of finite intersections of elements of $\mathcal{S}$, and $\mathcal{T}_2$ is closed under arbitrary unions and finite intersections; it's rather obvious in hindsight. We know that the open rays $[0 \times 0, a \times b)$ and $(a \times b, 1 \times 1]$ form a subbasis for $\mathcal{T}_<$. It is easy to see that

$$[0 \times 0, a \times b) = (-\infty, a \times b) \ \cap \ I \times I$$
$$[a \times b, 1 \times 1) = (a \times b, \infty) \ \cap \ I \times I$$

It is now clear that these open rays are open in the subspace topology, and as reasoned above, we conclude $\mathcal{T}_< \subset \mathcal{T}_s$.

$\square$

Exercise 17.2. Show that if A is closed in Y and Y is closed in X, then A is closed in X.

Proof. Let A be closed in Y and Y be closed in X. Then by Theorem 17.2 $A = Y \cap C$ for some closed set C in X. Notice that A is an intersection of two closed sets in X, so A is also closed in X. $\square$

Exercise 17.3. Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Proof. Suppose A is closed in X and B is closed in Y, which means that $X - A$ is open in X and $Y - B$ is open in Y. Then $(X - A) \times Y$ and $X \times (Y - B)$ are both open in $X \times Y$, so their union

$$(X - A) \times Y \ \cup \ X \times (Y - B) \ = \ X \times Y - A \times B$$

is open in $X \times Y$. Therefore $A \times B$ is closed in $X \times Y$. $\square$

Exercise 17.4. Show that if U is open in X and A is closed in X, then $U - A$ is open in X, and $A - U$ is closed in X.

Proof. Suppose U is open in X and A is closed in X. Then $X - U$ is closed and $X - A$ is open. Notice that $U - A = U \cap (X - A)$ is an intersection of open sets and $A - U = A \cap (X - U)$ is an intersection of closed sets, so $U - A$ is open and $A - U$ is closed. $\square$

Exercise 17.6. Let A, B, and A_α denote subsets of a space X. Prove the following:

(a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

Proof. Suppose $A \subset B$ and let $x \in \overline{A}$. By Theorem 17.5, all neighborhoods of x intersect A, and since $A \subset B$, these neighborhoods intersect B as well. By the same theorem, $x \in \overline{B}$. Thus $\overline{A} \subset \overline{B}$. □

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. ($\subset$) Suppose $x \notin \overline{A} \cup \overline{B}$. Then

$$x \in X - (\overline{A} \cup \overline{B}) = (X - \overline{A}) \cap (X - \overline{B}),$$

where $(X - \overline{A}) \cap (X - \overline{B})$ is an open set containing x disjoint from $A \cup B$. Thus by Theorem 17.5, $x \notin \overline{A \cup B}$.

($\supset$) Suppose $x \in \overline{A} \cup \overline{B}$. Then all neighborhoods of x intersect A or all neighborhoods of x intersect B. Then of course, all neighborhoods of x intersect $A \cup B$, so that $x \in \overline{A \cup B}$. □

(c) $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$; give an example where equality fails.

Proof. Let $x \in \bigcup \overline{A}_\alpha$, so that $x \in \overline{A}_\alpha$ for some α. Then all neighborhoods of x intersect A_α, and hence intersect $\bigcup A_\alpha$. Thus $x \in \overline{\bigcup A_\alpha}$ and therefore $\bigcup \overline{A}_\alpha \subset \overline{\bigcup A_\alpha}$.

For an example where equality fails, consider the collection of singleton sets $\{\frac{1}{n}\}_{n \in \mathbb{Z}_+}$ in the context of the order topology on $\mathbb{R}$. For each n, $\overline{\{\frac{1}{n}\}} = \{\frac{1}{n}\}$ since singleton sets are closed, so $\bigcup \overline{\{\frac{1}{n}\}} = \bigcup \{\frac{1}{n}\}$. However as shown in class, $\overline{\bigcup \{\frac{1}{n}\}} = \{0\} \cup \bigcup \{\frac{1}{n}\}$. □

Part 2

Show that if X is an ordered set for which the order topology is the discrete topology, then every element of X except the largest one (if there is a largest one) has an immediate successor. Through a practically identical proof, it's also possible to show that every element except the smallest one (if there is a smallest one) has an immediate predecessor.

Proof. We proceed by contrapositive. Suppose there exists $x \in X$ such that x is not the largest element and x does not have an immediate successor. We will show that (x, ∞) is not closed. Let U be any neighborhood of x. Then there is a basis interval B such that $x \in B \subset U$. We need to find an element c in the intersection of U and (x, ∞). There are a few cases for the shape of B:

Case 1: $B = (a, b)$**.** In this case, $x < b$ and since x does not have an immediate successor, there must exist c such that $x < c < b$. Now $c \in (a, b) \cap (x, \infty)$.

Case 2: $B = [a, b)$**.** Just as in Case 1, since $x < b$ with no immediate successor, we choose c such that $x < c < b$. Now $c \in [a, b) \cap (x, \infty)$.

Case 3: $B = (a, b]$**.** In this case, recall x is assumed not the largest element, so $x < b$ and we can choose $c = b$. Then $c \in (a, b] = (a, b] \cap (x, \infty)$.

In all cases we have shown that B (and hence U) intersects (x, ∞) at a point $c \neq x$. This shows that x is a limit point of (x, ∞), however clearly $x \notin (x, \infty)$. By Corollary 17.7, (x, ∞) is not closed, so this is not the discrete topology. $\square$