Part 1

Exercise 18.2. Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

Solution. No. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 64$, which is a constant and hence continuous function. Notice 0 is a limit point of $(0, 1)$, however $f(0) = 64$ is not a limit point of the image $f((0, 1)) = \{64\}$. □

Exercise 18.4. Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by

$$f(x) = x \times y_0 \quad \text{and} \quad g(y) = x_0 \times y$$

are embeddings.

Proof. Let $c_X : X \to Y$ and $c_Y : Y \to X$ be the constant functions $c_X(x) = y_0$ and $c_Y(y) = x_0$. Then let $f' : X \to X \times \{y_0\}$ and $g' : Y \to \{x_0\} \times Y$ be the functions f, g with ranges restricted to their respective images. These restricted functions are obviously bijective, and furthermore

$$f' = i_X \times c_1 \quad \text{and} \quad g' = c_2 \times i_Y.$$

Now we have written f' and g' as products of the constant and identity functions, so by Theorems 18.2 and 18.4 we conclude f' and g' are continuous.

Next we must show $(f')^{-1}$ and $(g')^{-1}$ are continuous. So let $U \subset X$ and $V \subset Y$ be open sets. Then $f'(U) = U \times \{y_0\}$ and $g'(V) = \{x_0\} \times V$, which are each open in the respective topologies $X \times \{y_0\}$ and $\{x_0\} \times Y$. Hence $(f')^{-1}$ and $(g')^{-1}$ are both continuous, so that f' and g' are indeed homeomorphisms. Therefore f and g are embeddings. □

Exercise 18.7a. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is "continuous from the right," that is,

$$\lim_{x \to a^+} f(x) = f(a),$$

for each $a \in \mathbb{R}$. Show that f is continuous when considered as a function from $\mathbb{R}_\ell$ to $\mathbb{R}$.

Proof. I take our hypothesis to mean that "for each $a \in \mathbb{R}$, for all $\epsilon > 0$, there exists $\delta > 0$ such that $x \in [a, a + \delta)$ implies $f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$." To show $f : \mathbb{R}_\ell \to \mathbb{R}$ is continuous, let $a \in \mathbb{R}_\ell$ and V be any neighborhood of $f(a)$. Then there is a basis interval

(ϵ_1, ϵ_2) such that $f(x) \in (\epsilon_1, \epsilon_2) \subset V$. Choosing $\epsilon = \min\{f(a) - \epsilon_1, \epsilon_2 - f(a)\}$, we have $(f(a) - \epsilon, f(a) + \epsilon) \subset V$. Now there must exist $\delta > 0$ such that $x \in [a, a + \delta)$ implies $f(x) \in (f(a) - \epsilon, f(a) + \epsilon) \subset V$. But $[a, a + \delta)$ is open in $\mathbb{R}_\ell$. Thus we have found an open set $[a, a + \delta)$ in X such that $a \in [a, a + \delta) \subset f^{-1}(V)$. Therefore $f : \mathbb{R}_\ell \to \mathbb{R}$ is continuous. □

Exercise 18.8. Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.

(a) Show that the set $\{x : f(x) \leq g(x)\}$ is closed in X.

Proof. Let $A = \{x : f(x) \leq g(x)\}$ and $C = X - A = \{x : f(x) > g(x)\}$. Let $x \in C$. Then $g(x) < f(x)$, and recalling that the ordered set Y is Hausdorff, we know there exist disjoint open sets and hence basis intervals (a, b) and (c, d) such that $g(x) \in (a, b)$ and $f(x) \in (c, d)$. Note that this means

$$a \ < \ g(x) \ < \ b \ \leq \ c \ < \ f(x) \ < \ d.$$

Now since f, g are continuous, $f^{-1}(c, d)$ and $g^{-1}(a, b)$ are open and their intersection $U = f^{-1}(c, d) \cap g^{-1}(a, b)$ is open with $x \in U$. Next we will show that $U \subset C$, so let $y \in U$. Then $f(y) \in (c, d)$ and $g(y) \in (a, b)$, so that $g(y) < b \leq c < f(y)$. Therefore $y \in C$ and $U \subset C$. We have shown that for any $x \in C$ there is an open set U such that $x \in U \subset C$. Thus C is open and A is closed. □

(b) Let $h : X \to Y$ be the function

$$h(x) = \min\{f(x), g(x)\},$$

Show that h is continuous.

Proof. Notice that for each $x \in X$, we have $f(x) \leq g(x)$ or $f(x) \geq g(x)$. Thus, letting $A = \{x : f(x) \leq g(x)\}$ and $B = \{x : f(x) \geq g(x)\}$, we have $X = A \cup B$ and $f(x) = g(x)$ for all $x \in A \cap B$. Now it is clear that

$$h(x) = \min\{f(x), g(x)\} = \begin{cases} f(x) \text{ if } x \in A \\ g(x) \text{ if } x \in B \end{cases}$$

and by the Pasting Lemma, h is continuous. □

Exercise 18.10. Let $f : A \to B$ and $g : C \to D$ be continuous functions. Let us define a map $f \times g : A \times C \to B \times D$ by the equation

$$(f \times g)(a \times c) = f(a) \times g(c).$$

Show that $f \times g$ is continuous.

Proof. Define $f' : A \times C \to B$ and $g' : A \times C \to D$ by $f'(a \times c) = f(a)$ and $g'(a \times c) = g(c)$. Clearly f', g' are continuous because for any open sets $U \subset B$, $V \subset D$, we have $(f')^{-1}(U) = f^{-1}(U) \times C$ and $(g')^{-1}(V) = A \times g^{-1}(V)$, which are both open sets in $A \times C$. Now

$$(f \times g)(a \times c) = f(a) \times g(c) = f'$$

and by Theorem 18.4 we know $f \times g$ is continuous. □

Exercise 18.13. Let $A \subset X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \overline{A} \to Y$, then g is uniquely determined by f.

Proof. Let $g, h : \overline{A} \to Y$ be continuous functions and let x be a limit point of A such that $g(x) \neq h(x)$. Since Y is Hausdorff, there exist disjoint neighborhoods U of $g(x)$ and V of $h(x)$. Then $x \in g^{-1}(U) \cap h^{-1}(V)$ and since g, h are continuous, this is an open set. Since x is a limit point, there exists some $y \in A$ such that $y \in g^{-1}(U) \cap h^{-1}(V)$. But this means that $g(y) \in U$ and $h(y) \in V$, and since U, V are disjoint sets, we have $g(y) \neq h(y)$.

We have shown by contrapositive that if $h(x) = g(x)$ for all $x \in A$, then $h(x) = g(x)$ for all $x \in \overline{A}$. □

Part 2

A *variety* in $\mathbb{R}^n$ is the set of common zeroes of one or more n-variable polynomials. For example, in $\mathbb{R}^2$, the unit circle is a variety because it's the set of zeroes of the polynomial $x^2 + y^2 - 1$. The set $\{2 \times 3, (-2) \times (-5)\}$ is also a variety as it's the set of common zeroes of the polynomials $x^2 - 4$ and $xy + x - 8$. Algebraic geometry is the study of varieties.

Algebraic geometers often use a non-Hausdorff topology called the Zariski topology. The Zariski topology on $\mathbb{R}^n$ is defined by saying that a set is closed if and only if it is a variety, and thus that a set is open if and only if it is the complement of a variety. It's possible to prove that this really is a topology, but the proof is mostly abstract algebra.

(a) Show that on $\mathbb{R}^n$, the Zariski topology is:

- the finite complement topology if $n = 1$.
- strictly finer than the finite complement topology if $n > 1$.

Proof. Let Z be the Zariski topology on $\mathbb{R}^n$ and $\mathcal{T}$ be the finite complement topology on $\mathbb{R}^n$. First we will show that Z is finer than $\mathcal{T}$. So let $U \in \mathcal{T}$. Then $\mathbb{R}^n - U$ is either $\mathbb{R}^n$ or a finite set $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$. Of course if $\mathbb{R}^n - U$ is the entire space, this is precisely the variety corresponding to the single zero polynomial. So suppose the latter, and let us denote

$$\mathbf{x}_1 = (x_{1,1}, x_{1,2}, \dots, x_{1,n})$$
$$\vdots$$
$$\mathbf{x}_m = (x_{m,1}, x_{m,2}, \dots, x_{m,n})$$

Then it should be clear that $\{\mathbf{x}_1, \dots \mathbf{x}_m\}$ is precisely the variety corresponding to the polynomials

$$(x_1 - x_{1,1})(x_1 - x_{2,1}) \cdots (x_1 - x_{m,1})$$
$$\vdots$$
$$(x_n - x_{1,n})(x_n - x_{2,n}) \cdots (x_n - x_{m,1})$$

where, the x_i's are the polynomial placeholders corresponding to the ith coordinate, for each $i = 1, \dots n$. Thus $\mathbb{R}^n - U$ is closed in Z, so $U \in Z$ and Z is finer than $\mathcal{T}$.

Next we'll argue that when $n = 1$, $Z = \mathcal{T}$. In this case, it is easy to see that every polynomial over $\mathbb{R}$ is either zero or of finite dimension, and so the set of common zeros of one or more polynomials is an intersection of sets that are either $\mathbb{R}$ or finite. Of course, such an intersection is also either $\mathbb{R}$ or finite, and hence must be closed in the finite complement topology. Thus when $n = 1$, $Z = \mathcal{T}$.

Lastly, we must show that when $n > 1$, Z is strictly finer than $\mathcal{T}$. Consider the polynomial $x_1 - 1$ over $\mathbb{R}^n$. Clearly the variety is the set $\{(1, x_2, \dots, x_n) : x_2, \dots, x_n \in \mathbb{R}\}$, which is neither finite nor $\mathbb{R}^n$, so its complement is open in Z but not in $\mathcal{T}$. □

(b) Prove that if $p : \mathbb{R}^n \to \mathbb{R}$ is an n-variable polynomial, and both spaces are given the Zariski topology, then p is continuous.

Proof. Let $A \subset \mathbb{R}$ be closed in the Zariski topology. Then as shown above, A is either $\mathbb{R}$ or a finite set $\{c_1, \dots, c_m\}$. If $A = \mathbb{R}$ then obviously $p^{-1}(A) = \mathbb{R}^n$ which is also closed, so suppose the latter. Then consider a "vector" in $p^{-1}(A)$. These are precisely the zeros of the polynomials $p - c_1, \dots, p - c_m$. Thus $p^{-1}(A)$ is the variety corresponding to their product, the single polynomial $(p - c_1) \cdots (p - c_m)$, and is therefore closed. Thus p is continuous. □

(c) What is the closure of the set $\{\frac{1}{n} : n \in \mathbb{Z}_+\} \times \{0\}$ in the Zariski topology on $\mathbb{R}^2$? Which polynomials correspond to the closure?

Solution. I think the closure is $\mathbb{R} \times \{0\}$, but I'm not positive and incapable of a proof. My reasoning is simply that, if there are an infinite number of first coordinate x's that lead to polynomial zeros, we would need something more than 0 in the second coordinate y to "balance" them out.

If I am correct, then the polynomials corresponding to this variety are polynomials where each term is a positive multiple of y. For example, y, xy, x^2y, $xy^2 + y$, etc. □