Sam Tay Math 601: Homework #8 March 14, 2019

Part 1

Exercise 20.2. Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order is metrizable.

Proof. We define a metric

$$d(x_1 \times x_2, y_1 \times y_2) = \begin{cases} 1 + |x_2 - y_2| & \text{if } x_1 \neq y_1 \\ \min\{|x_2 - y_2|, 1\} & \text{if } x_1 = y_1 \end{cases}$$

and it is tedious but routine to show that d is a metric. Let $\mathcal{T}_d$ denote the topology induced by d and $\mathcal{T}$ be the dictionary order topology.

Let $\mathbf{x} \in \mathbb{R} \times \mathbb{R}$ and $\mathbf{x} \in B(\mathbf{x}, \epsilon)$ for some $\epsilon > 0$. Let $\delta = \min\{\frac{\epsilon}{2}, 1\}$ and

$$B = (x_1 \times (x_2 - \delta), x_1 \times (x_2 + \delta))$$

be a basis element in $\mathcal{T}$. Then $\mathbf{x} \in B$ and for any $\mathbf{y} \in B$, we have $y_1 = x_1$ and

$$d(\mathbf{x}, \mathbf{y}) = \min\{1, |x_2 - y_2|\} \leq \min\{1, \delta\} \leq \delta \leq \frac{\epsilon}{2} < \epsilon$$

so $\mathbf{y} \in B(\mathbf{x}, \epsilon)$ as well. Thus $\mathcal{T}_d \subset \mathcal{T}$.

Next let B be a basis interval in $\mathcal{T}$ with $\mathbf{x} \in B$. Let us denote $B = (a_1 \times a_2, b_1 \times b_2)$.

Case 1: $x_1 = a_1$**.** Let

$$\epsilon = \begin{cases} \min\{\frac{1}{2}, |x_2 - a_2|\} & \text{if } x_2 = b_2 \\ \min\{\frac{1}{2}, |x_2 - a_2|, |x_2 - b_2|\} & \text{if } x_2 \neq b_2 \end{cases}$$

Then $\mathbf{x} \in B(\mathbf{x}, \epsilon)$ and $B(\mathbf{x}, \epsilon) \subset B$.

Case 2: $a_1 < x_1 < b_1$**.** Simply choose $\epsilon = \frac{1}{2}$. Then $\mathbf{x} \in B(\mathbf{x}, \epsilon)$ and $B(\mathbf{x}, \epsilon) \subset B$.

Case 3: $x_1 = b_1$**.** Similar to Case 1, let

$$\epsilon = \begin{cases} \min\{\frac{1}{2}, |x_2 - b_2|\} & \text{if } x_2 = a_2 \\ \min\{\frac{1}{2}, |x_2 - a_2|, |x_2 - b_2|\} & \text{if } x_2 \neq a_2 \end{cases}$$

Then $\mathbf{x} \in B(\mathbf{x}, \epsilon)$ and $B(\mathbf{x}, \epsilon) \subset B$.

In all cases we have shown that $\mathcal{T} \subset \mathcal{T}_d$. We conclude that $\mathcal{T} = \mathcal{T}_d$. $\square$

Exercise 20.4. Consider the product, uniform, and box topologies on $\mathbb{R}_\omega$.

(a) In which topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous?

$$f(t) = (t, 2t, 3t, \ldots)$$
$$g(t) = (t, t, t, \ldots)$$
$$h(t) = \left(t, \frac{1}{2}t, \frac{1}{3}t, \ldots\right)$$

- f is not continuous in box nor uniform, but is continuous in product topology
- g is not continuous in box, but is continuous in uniform and product topologies.
- h is not continuous in box, but is continuous in uniform and product topologies.

(b) In which topologies do the following sequences converge?

$$\mathbf{w}_1 = (1, 1, 1, \ldots),\ \mathbf{w}_2 = (0, 2, 2, \ldots)\ \mathbf{w}_3 = (0, 0, 3, \ldots)$$
$$\mathbf{x}_1 = (1, 1, 1, \ldots),\ \mathbf{x}_2 = \left(0, \frac{1}{2}, \frac{1}{2}, \ldots\right),\ \mathbf{x}_3 = \left(0, 0, \frac{1}{3}, \ldots\right)$$
$$\mathbf{y}_1 = (1, 0, 0, \ldots),\ \mathbf{y}_2 = \left(\frac{1}{2}, \frac{1}{2}, 0\right),\ \mathbf{y}_3 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\right)$$
$$\mathbf{z}_1 = (1, 1, 0, \ldots),\ \mathbf{z}_2 = \left(\frac{1}{2}, \frac{1}{2}, 0\right),\ \mathbf{z}_3 = \left(\frac{1}{3}, \frac{1}{3}, 0, \ldots\right)$$

- $\mathbf{w}$ converges in the product topology
- $\mathbf{x}$ converges in the product and uniform topologies
- $\mathbf{y}$ converges in the product and uniform topologies
- $\mathbf{z}$ converges in the product, uniform, and box topologies

Exercise 20.5. Let $\mathbb{R}^\infty \subset \mathbb{R}^\omega$ be sequences that are eventually zero. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the uniform topology?

Solution. Suppose every open ball $B(\mathbf{x}, \epsilon)$ intersects $\mathbb{R}^\infty$ for some $\mathbf{x} \in \mathbb{R}^\omega$. Notice this is equivalent to the statement that for every $\epsilon > 0$ there exists N such that for all $n > N$, $|x_n| < \epsilon$. Therefore the closure of $\mathbb{R}^\infty$ is precisely the set of sequences which converge to 0 in $\mathbb{R}$. $\square$

Part 2

Give $\{0, 1\}$ the discrete topology and then put the product topology on $\{0, 1\}^\omega$. Construct a function $f : \{0, 1\}^\omega \to [0, 1] \subset \mathbb{R}$ defined by $f(a_1, a_2, a_3, \ldots) = \sum_{k=1}^{\infty} \frac{a_k}{2^k}$.

(a) Prove that f is well-defined, i.e. that the series does converge and results in a number in $[0, 1]$. f happens to be surjective, but you don]t have to prove that.

Proof. We know that the geometric series $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ and $\sum_{k=1}^{\infty} 0 = 0$. For any $x \in \{0, 1\}^\omega$, we know that for each term k, $0 \le f(x)_k \le \frac{1}{2^k}$ so $f(x)$ converges as well and $0 \le f(x) \le 1$. □

(b) Show that f is continuous.

Proof. It will make things a bit easier to prove that $g : \{0, 1\}^\omega \to \mathbb{R}$ defined by $g(x) = f(x)$ is continuous, and then recall that restricting the range to the subspace $[0, 1]$ (which contains the image of g) results in f being continuous.

Let U be open in $\mathbb{R}$. If $f^{-1}(U)$ is empty, it's open, so suppose otherwise and let $\mathbf{x} \in f^{-1}(U)$. Since U is open we can find an ϵ small enough so that $0 < \epsilon < 1$ and

$$f(\mathbf{x}) \in (f(\mathbf{x}) - \epsilon, f(\mathbf{x}) + \epsilon) \subset U$$

Now choose n large enough so that $\sum_{k=1}^{n} \frac{1}{2^k} > 1 - \epsilon$. Notice we can find such an n because $0 < 1 - \epsilon < 1$ and as $n \to \infty$, $\sum_{k=1}^{n} \frac{1}{2^k} \to 1$. Consequently,

$$\sum_{k=n+1}^{\infty} \frac{1}{2^k} = 1 - \sum_{k=1}^{n} \frac{1}{2^k} < \epsilon.$$

Now construct the basis element

$$B = \prod_{k=1}^{\infty} \begin{cases} \{x_k\} & \text{if } k \le n \\ \{0, 1\} & \text{if } k > n \end{cases}$$

and notice $\mathbf{x} \in B \subset f^{-1}((f(\mathbf{x}) - \epsilon, f(\mathbf{x}) + \epsilon)) \subset f^{-1}(U)$. Therefore $f^{-1}(U)$ is open and f is continuous. □