Part 1

Exercise 19.2. Let A_α be a subspace of X_α for each $\alpha \in J$. Then ΠA_α is a subspace of ΠX_α if both products are given the box topology or if both products are given the product topology.

Proof. We wish to show that the box, product topologies on ΠA_α are the same as the subspace topologies that the subset ΠA_α inherits from the box, product topologies on ΠX_α. To do so we will use Lemmas 13.2 and 16.1.

Let $\mathcal{B}_b$ denote the usual basis for the box topology on ΠA_α, $\mathcal{C}_b$ denote the usual basis for the box topology on ΠX_α, and $\mathcal{B}_s$ denote the usual basis for the subspace topology that ΠA_α inherits from the box topology on ΠX_α. Notice that

$$\begin{aligned} B \in \mathcal{B}_b \iff B &= \prod_{\alpha \in J} U_\alpha \text{ where } U_\alpha \text{ is open in } A_\alpha \\ &= \prod_{\alpha \in J} A_\alpha \cap V_\alpha \text{ where } V_\alpha \text{ is open in } X_\alpha \\ &= \prod_{\alpha \in J} A_\alpha \bigcap \prod_{\alpha \in J} V_\alpha \\ &= \prod_{\alpha \in J} A_\alpha \bigcap C \text{ for some } C \in \mathcal{C}_b \\ \iff B &\in \mathcal{B}_s \end{aligned}$$

Thus, $\mathcal{B}_b = \mathcal{B}_s$ and they must generate the same topologies. We have shown that the box topology on ΠA_α is the same as its subspace topology inherited from the box topology on ΠX_α.

Let $\mathcal{B}_p$ denote the usual basis for the product topology on ΠA_α, $\mathcal{C}_p$ denote the usual basis for the product topology on ΠX_α, and $\mathcal{B}_s$ denote the usual basis for the subspace topology that ΠA_α inherits from the product topology on ΠX_α. Notice that

$$\begin{aligned}
B \in \mathcal{B}_p \iff B &= \prod_{\alpha \in J} U_\alpha \text{ where } U_\alpha \text{ is open in } A_\alpha, \text{ only finitely many } U_\alpha \neq A_\alpha \\
&= \prod_{\alpha \in J} A_\alpha \cap V_\alpha \text{ where } V_\alpha \text{ is open in } X_\alpha, \text{ only finitely many } A_\alpha \cap V_\alpha \neq A_\alpha \\
&= \prod_{\alpha \in J} A_\alpha \cap V_\alpha \text{ where } V_\alpha \text{ is open in } X_\alpha, \text{ only finitely many } V_\alpha \neq X_\alpha \\
&= \prod_{\alpha \in J} A_\alpha \bigcap \prod_{\alpha \in J} V_\alpha \\
&= \prod_{\alpha \in J} A_\alpha \bigcap C \text{ for some } C \in \mathcal{C}_p \\
\iff B &\in \mathcal{B}_s
\end{aligned}$$

Thus, $\mathcal{B}_b = \mathcal{B}_s$ and they must generate the same topologies. We have shown that the product topology on ΠA_α is the same as its subspace topology inherited from the product topology on ΠX_α.

(I also think it's worth elaborating on the implication that only finitely many α such that $A_\alpha \cap V_\alpha \neq A_\alpha$ leads to only finitely many α such that $V_\alpha \neq X_\alpha$. One might consider the case where we have chosen a basis element such that $A_\alpha \not\subset V_\alpha$for only finitely many $\alpha \in J$, and otherwise $A_\beta \subset V_\beta \subsetneq X_\beta$. In this case, simply notice that such terms $A_\beta \cap V_\beta$ can be replaced with $A_\beta \cap X_\beta$ and the equality still holds.) □

Exercise 19.6. Let $\mathbf{x}_1, \mathbf{x}_2, \dots$ be a sequence of the points of the product space ΠX_α. Show that this sequence converges to the point $\mathbf{x}$ if and only if the sequence $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \dots$ converges to $\pi_\alpha(\mathbf{x})$ for each α. Is this fact true if one uses the box topology instead of the product topology?

Proof. First recall that Munkres defines convergence of a sequence $x_1, x_2, \dots$ to x as the statement that for all neighborhoods U of x, there exists N such that for all $n > N$, $x_n \in U$. Clearly, the statement is equivalent when we replace neighborhood U with a basis element B.

($\Longrightarrow$) Suppose $\mathbf{x}_1, \mathbf{x}_2, \dots$ converges to $\mathbf{x}$ in ΠX_α. Let $\beta \in J$ and U_β be an arbitrary open neighborhood of $\pi_\beta(\mathbf{x})$. We can choose the open set

$$B = \prod_{\alpha \in J} \begin{cases} U_\beta \text{ if } \alpha = \beta \\ X_\alpha \text{ if } \alpha \neq \beta \end{cases}$$

in X_α so that $\mathbf{x} \in B$, and now there exists N such that for all $n > N$, $\mathbf{x}_n \in B$, which also implies that $\pi_\beta(\mathbf{x}_n) \in U_\beta$. Therefore $\pi_\beta(\mathbf{x}_1), \pi_\beta(\mathbf{x}_2), \dots$ converges to $\pi_\beta(\mathbf{x})$ in X_β. Since β was arbitrary, this holds for each $\beta \in J$.

($\Longleftarrow$) The converse is similarly proven. Suppose that $\pi_\beta(\mathbf{x}_1), \pi_\beta(\mathbf{x}_2), \ldots$ converges to $\pi_\beta(\mathbf{x})$ in X_β for each $\beta \in J$. Then let B be a basis element with $\mathbf{x} \in B$. We know that

$$B = \prod_{\alpha \in J} U_\alpha$$

where each U_α is an open set in X_α and only finitely many $U_\alpha \neq X_\alpha$. But for each α there exists N_α such that for all $n > N_\alpha$, $\pi_\alpha(\mathbf{x}_n) \in U_\alpha$. So let

$$N = \max\{N_\alpha : \alpha \in J \text{ and } U_\alpha \neq X_\alpha\}$$

and notice that N exists because there are only finitely many such N_α's. Furthermore, for every $\alpha \in J$, whenever $n > N$ we have $\pi_\alpha(\mathbf{x}_n) \in U_\alpha$; this is trivial when $U_\alpha = X_\alpha$, otherwise this is by construction of N. Therefore for all $n > N$, $\mathbf{x}_n \in B$, so that $\mathbf{x}_1, \mathbf{x}_2, \ldots$ converges to $\mathbf{x}$.

This statement does not hold for the box topology, however, because we are not guaranteed that we can construct the necessary N as above. If there are infinitely many N_α, they may be unbounded. □

Exercise 19.7. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answer.

Solution. In the box topology, $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$. To see why, consider the complement $\mathbb{R}^\omega - \mathbb{R}^\infty$ which consists of all sequences that are *not* eventually zero, in other words for any $i \in \mathbb{Z}_+$ there exists $k > i$ such that $x_k \neq 0$. Let $(x_1, x_2, \ldots) \in \mathbb{R}^\omega - \mathbb{R}^\infty$. Then let

$$B = \prod_{n \in \mathbb{Z}_+} \begin{cases} (0, x_n + 1) & \text{if } x_n > 0 \\ (x_n - 1, 0) & \text{if } x_n < 0 \\ (x_n - 1, x_n + 1) & \text{if } x_n = 0 \end{cases}$$

Now B is a basis element of the box topology and $x \in B \subset \mathbb{R}^\omega - \mathbb{R}^\infty$. Thus $\mathbb{R}^\infty$ is closed.

However, $\mathbb{R}^\infty$ is not closed in the product topology. Because there can only be finitely many proper subsets U_n in the basis element $B = \prod U_n$, there will exist an N such that $U_n = \mathbb{R}$ for all $n > N$. Of course, this makes it difficult to find such a B contained by $\mathbb{R}^\omega - \overline{\mathbb{R}^\infty}$. I conjecture that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$. To this end, let $(x_1, x_2, \ldots) \in \mathbb{R}^\omega$ and $B = \prod U_n$ be a basis element with $x \in B$. As just mentioned, there exists N such that for all $n > N$, $U_n = \mathbb{R}$; hence, for all $n > N$ we have $0 \in U_n$. Thus B contains sequences that are eventually zero, and therefore intersects $\mathbb{R}^\infty$. By Thereom 17.5 we conclude that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$. □

Part 2

Let $\{0,1\}$ have the discrete topology, and let $\{0,1\}^\omega$ have the product topology. Define a function $f : \{0,1\}^\omega \to \mathbb{R}$ by

$$f(\mathbf{x}) = \begin{cases} \frac{1}{\min\{i | x_i = 1\}} & \text{if } \mathbf{x} \neq \mathbf{0} \\ 0 & \text{if } \mathbf{x} = \mathbf{0} \end{cases}$$

Is f continuous? (Notation: $\mathbf{0} = (0,0,0,\ldots)$ and x_i is the i^{th} coordinate of the point $\mathbf{x} = (x_1, x_2, \ldots)$.)

Solution. Yes, f is continuous. Let U be an open set in $\mathbb{R}$ and consider $f^{-1}(U)$. First, if $f^{-1}(U)$ is empty, then trivially it is open, so suppose otherwise and let $\mathbf{x} \in f^{-1}(U)$. We have two cases: $\mathbf{x} \neq \mathbf{0}$ or $\mathbf{x} = \mathbf{0}$.

If $\mathbf{x} \neq \mathbf{0}$, then $f(\mathbf{x}) = \frac{1}{n}$ for some $n \in \mathbb{Z}_+$, where n is the first nonzero coordinate of $\mathbf{x}$. Let B be the basis element of $\{0,1\}^\omega$ such that

$$\pi_k(B) = \begin{cases} \{0\} & \text{if } k < n \\ \{1\} & \text{if } k = n \\ \{0,1\} & \text{if } k > n \end{cases}$$

Now $\mathbf{x} \in B = f^{-1}(\{f(\mathbf{x})\}) \subset f^{-1}(U)$.

If $\mathbf{x} = \mathbf{0}$, then $f(\mathbf{x}) = 0 \in U$. Let I be a basis interval such that $0 \in I \subset U$, and notice that there must be (infinitely many) $\frac{1}{n} \in I$. Choose the least such $n \in \mathbb{Z}_+$ and note that $\frac{1}{k} \in I$ for all $k \geq n$. Let B be the basis element of $\{0,1\}^\omega$ such that

$$\pi_k(B) = \begin{cases} \{0\} & \text{if } k < n \\ \{0,1\} & \text{if } k \geq n \end{cases}$$

Now $\mathbf{x} \in B = f^{-1}(I) \subset f^{-1}(U)$.

We have shown in all cases that for any $\mathbf{x} \in f^{-1}(U)$ we can find a basis element B such that $\mathbf{x} \in B \subset f^{-1}(U)$. Therefore $f^{-1}(U)$ is open and f is continuous. $\square$

Part 3

Let X be a topological space. Let J and K be index sets.

(a) For $j \in J$, let $\pi_j : X^j \to X$ be the j^{th} projection function. For $k \in K$, let $\phi_k : (X^J)^K \to X^J$ be the kth projection function. Define $F : (X^J)^K \to X^{J \times K}$ to be $F(\mathbf{y}) = (\pi_j(\phi_k(\mathbf{y})))_{j \times k \in J \times K}$. Show that F is a homeomorphism.

Proof. First we show that F is a bijection. Suppose $\mathbf{x} \neq \mathbf{y}$. Then for some $k \in K$, $\phi_k(\mathbf{x}) \neq \phi_k(\mathbf{y})$. Thus in the kth coordinate there exists some $j \in J$ such that $\pi_j(\phi_k(\mathbf{x})) \neq \pi_j(\phi_k(\mathbf{y}))$. This means that $F(\mathbf{x})$ and $F(\mathbf{y})$ differ in the $j \times k$ coordinate. Hence F is injective.

To show F is surjective, let $\mathbf{x} \in X^{J\times K}$ and consider $\mathbf{x}$ as the function $\mathbf{x} : J \times K \to X$. Now let $\mathbf{y} \in (X^J)^K$ be the function $\mathbf{y} : K \to X^J$ defined by $\mathbf{y}(k) : J \to X$ defined by

$$\mathbf{y}(k)(j) = \mathbf{x}(j \times k).$$

Then for all $j \times k \in J \times K$,

$$\begin{aligned} F(\mathbf{y})(j \times k) &= \pi_j(\phi_k(\mathbf{y})) \\ &= \pi_j(\mathbf{y}(k)) \\ &= \mathbf{y}(k)(j) \\ &= \mathbf{x}(j \times k) \end{aligned}$$

Hence $\mathbf{x} = F(\mathbf{y})$ and F is surjective.

Next we must show F is continuous. So let $\mathbf{x} \in (X^J)^K$ and V be a neighborhood of $f(\mathbf{x})$. We will find a neighborhood B of $\mathbf{x}$ such that $f(B) \subset V$. Since V is open there exists a basis element $\prod U_{j\times k}$ where only finitely many $U_{j\times k} \neq X$ such that

$$f(\mathbf{x}) \in \prod_{j\times k \in J\times K} U_{j\times k} \subset V$$

Define the set $B \subset (X^J)^K$ such that

$$\phi_k(B) = V \subset X^J \text{ such that } \pi_j(V) = U_{j\times k}$$

Notice that since there are only finitely many $j \times k$ such that $U_{j\times k} \neq X$, there are only finitely many j such that $\pi_j(V) \neq X$, and thus each V is open in X^J. Also, since there are only finitely many $j \times k$ such that $U_{j\times k} \neq X$, there are only finitely many k such that $\phi_k(B) \neq X^J$. Therefore B is open in $(X^J)^K$, and we know $\mathbf{x} \in B$ by construction (and the fact that f is injective). Also for any $\mathbf{y} \in B$,

$$F(\mathbf{y}) = \pi_j(\phi_k(\mathbf{y}))_{j\times k \in J\times K}$$

but for any $j \times k \in J \times K$ we have

$$\pi_j(\phi_k(\mathbf{y})) \in \pi_j(\phi_k(B)) = U_{j\times k}$$

and hence $F(\mathbf{y}) \in \prod U_{j\times k} \subset V$. This shows that $F(B) \subset V$, so we conclude that F is continuous.

Next we must show F^{-1} is continuous, so let $y \in X^{J\times K}$ and V be a neighborhood of $F^{-1}(\mathbf{y})$. Next we need to find a neighborhood U of $\mathbf{y}$ such that $F^{-1}(U) \subset V$. Choose a basis element B such that $F^{-1}(\mathbf{y}) \in B \subset V$. Define U such that

$$U = \prod_{j\times k} \pi_j(\phi_k(B)).$$

Because B is a basis element, there are only finitely many k such that $\phi_k(B) \neq X^J$ and of these ks only finitely many j such that $\pi_j(\phi_k(B)) \neq X$, so there are only finitely many $j \times k$ such that $\pi_j(\phi_k(B)) \neq X$. So U is open, $\mathbf{y} \in U$, and for any $\mathbf{x} \in F^{-1}(U)$, we have $\mathbf{x} \in B$ by construction. Therefore F^{-1} is continuous.

We conclude that F is a homeomorphism. $\square$

(b) Suppose that there's a bijection between J and K. Show that X^J and X^K are homeomorphic.

Proof. Let $f : K \to J$ be a bijection. Simply define $g : X^J \to X^K$ by

$$g(\mathbf{x}) = \mathbf{y} \text{ such that } \pi_k(\mathbf{y}) = \pi_{f(k)}(\mathbf{x}) \text{ for all } k \in K.$$

Then g is routinely shown to be a homeomorphism.

Note we can also consider $x : J \to X$ mapped to $g(x) : K \to X$ defined as the composition $g(x) = x \circ f$. $\square$

(c) Show that $(X^\omega)^\omega$ is homeomorphic to X^ω.

Proof. Recall that $\omega = \mathbb{Z}_+$ and $\mathbb{Z}_+$ is isomorphic to $\mathbb{Z}_+ \times \mathbb{Z}_+$. By part (a) there exists a homeomorphism

$$f : (X^\omega)^\omega \to X^{\omega\times\omega}$$

and by part (b) there exists a homeomorphism

$$g : X^{\omega\times\omega} \to X^\omega.$$

Hence $g \circ f$ is a homeomorphism from $(X^\omega)^\omega$ to X^ω.

$\square$

(d) Show that if there is a continuous surjective function from X^ω to some space Z, then there's also a continuous surjective function from X^ω to Z^ω. (Hint: $X^\omega \to (X^\omega)^\omega \to Z^\omega$)

Proof. Let $f : X^\omega \to Z$ be a continuous surjective function. Define $g : (X^\omega)^\omega \to Z^\omega$ to map each projection according to f, that is, such that

$$\pi_n(g(\mathbf{x})) = f(\pi_n(\mathbf{x})) \text{ for all } n \in \mathbb{Z}_+$$

Next let $h : X^\omega \to (X^\omega)^\omega$ be the homeomorphism guaranteed by part (c). Then $F : X^\omega \to Z^\omega$ defined by $F = g \circ h$ is a continuous surjection.

To prove this, we need only show that g is both continuous and surjective. First let $\mathbf{z} \in Z^\omega$. Then let $\mathbf{x} \in (X^\omega)^\omega$ such that for each $n \in \mathbb{Z}_+$, $\pi_n(\mathbf{x})$ is such that $f(\pi_n(\mathbf{x})) = \pi_n(\mathbf{z})$, which exists because f is surjective. Then clearly $g(\mathbf{x}) = \mathbf{z}$, so g is surjective.

Next let $\mathbf{x} \in (X^\omega)^\omega$ and V be a neighborhood of $g(\mathbf{x})$, and choose a basis element B such that $g(\mathbf{x}) \in B \subset V$. Then

$$B = \prod_{n \in \mathbb{Z}_+} U_n \text{ only finitely many } U_n \neq Z$$

Next construct

$$C = \prod_{n \in \mathbb{Z}_+} f^{-1}(U_n).$$

Clearly only finitely many $f^{-1}(U_n) \neq X^\omega$, so C is open, $\mathbf{x} \in C$, and $g(C) \subset B \subset V$. Thus g is continuous.

Since g, h are both continuous and surjective, F is also continuous and surjective.

$\square$