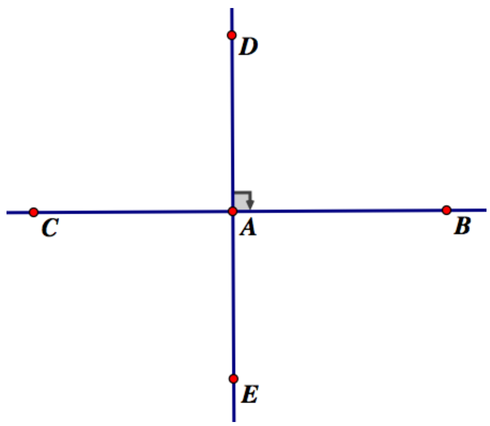
Sam Tay
Professor Schumacher
Math 230
Section 3.5: 1,4,6
Section 3.2: 12(c)
10/02/12

Problem 1: If $l \perp m$, then l and m contain rays that make four different right angles.

Proof. Suppose $l \perp m$. By Definition 3.5.8, there is a point A that lies on both l and m, and $B \in l$, $D \in m$ such that rays $\overrightarrow{AD}$ and $\overrightarrow{AB}$ form a right angle $\angle BAD$. Next let $C \in l$ so that $C * A * B$ and $E \in m$ so that $E * A * D$.



Then of course, since $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are opposite rays, $\angle BAD$ and $\angle DAC$ form a linear pair. By the Linear Pair Theorem,

$$180° = \mu(\angle BAD) + \mu(\angle DAC) = 90° + \mu(\angle DAC),$$

so $\mu(\angle DAC) = 90°$ and thus $\angle DAC$ is a right angle. Next we have opposite rays $\overrightarrow{AE}$ and $\overrightarrow{AD}$, so that $\angle DAC$ and $\angle CAE$ form a linear pair. Again by the Linear Pair Theorem,

$$180° = \mu(\angle DAC) + \mu(\angle CAE) = 90° + \mu(\angle CAE),$$

and similarly $\angle CAE$ is a right angle as well. Finally, the opposite rays $\overrightarrow{AC}$ and $\overrightarrow{AB}$ form the linear pair $\angle CAE$ and $\angle EAB$, so that

$$180° = \mu(\angle CAE) + \mu(\angle EAB) = 90° + \mu(\angle EAB),$$

so we have a fourth right angle $\angle EAB$.

To conclude that these are four distinct right angles, we will show that the four rays $\overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AC}, \overrightarrow{AE}$ are distinct. Our initial assumption is that $\mu(\angle BAD) = 90° \neq 0°$, so $\overrightarrow{AB} \neq \overrightarrow{AD}$. Also since this angle is defined, $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are nonopposite. We defined C such that $\overrightarrow{AC}$ is opposite to $\overrightarrow{AB}$ and thus $\overrightarrow{AC} \neq \overrightarrow{AB}$ and $\overrightarrow{AC} \neq \overrightarrow{AD}$. So the first three rays are distinct. Next, E is defined such that

$\overrightarrow{AE}$ is opposite to $\overrightarrow{AD}$, so $\overrightarrow{AE} \neq \overrightarrow{AD}$. Also $\overrightarrow{AE} \neq \overrightarrow{AC}$, otherwise $\overrightarrow{AE}$ would be opposite to $\overrightarrow{AB}$, implying $\overrightarrow{AD} = \overrightarrow{AB}$, which is not the case. Lastly, $\overrightarrow{AE} \neq \overrightarrow{AB}$, otherwise $\overrightarrow{AB}$ would be opposite to $\overrightarrow{AD}$, which we already said is not the case. Thus these four rays are distinct, and so are the four right angles they form. $\square$

Problem 4: Supplements of congruent angles are congruent.

Proof. Let $\angle ABC \cong \angle DEF$, and suppose $\angle XYZ$ is a supplement to $\angle ABC$ and $\angle UVW$ is a supplement to $\angle DEF$. From this assumption of supplements we have

$$\mu(\angle ABC) + \mu(\angle XYZ) = 180^\circ$$

$$\mu(\angle DEF) + \mu(\angle UVW) = 180^\circ,$$

so that

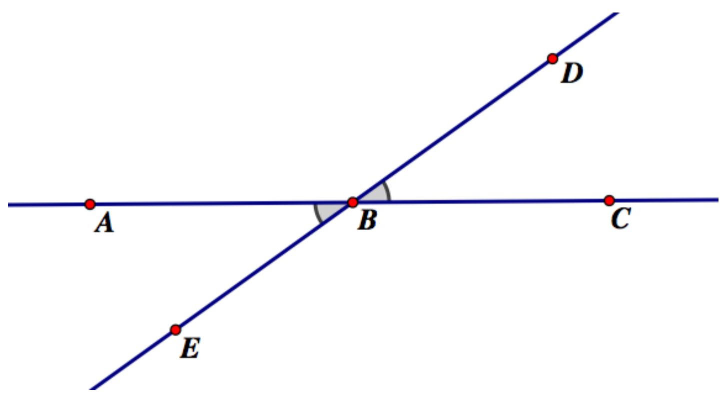
$$\mu(\angle XYZ) = 180^\circ - \mu(\angle ABC)$$

$$\mu(\angle UVW) = 180^\circ - \mu(\angle DEF).$$

However the congruence above implies that $\mu(\angle ABC) = \mu(\angle DEF)$, and it follows immediately that $\mu(\angle XYZ) = \mu(\angle UVW)$, and therefore supplements of congruent angles are also congruent. $\square$

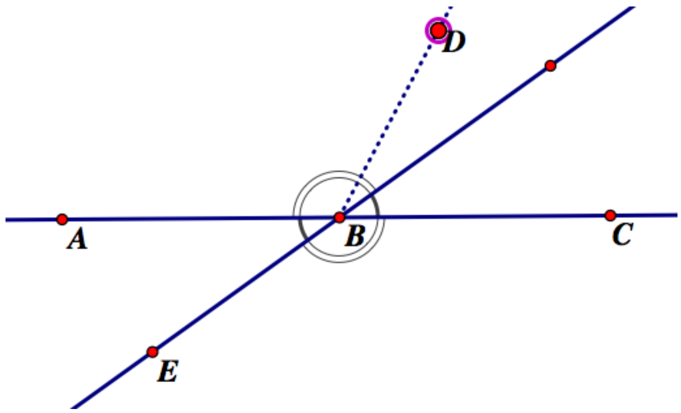
Problem 6: If A, B, C, D, and E are points such that $A * B * C$, D and E are on opposite sides of $\overleftrightarrow{AB}$, and $\angle DBC \cong \angle ABE$, then D, B, and E are collinear.

Proof. As above, suppose A, B, C, D, and E are points such that $A*B*C$, D and E are on opposite sides of $\overleftrightarrow{AB}$, and $\angle DBC \cong \angle ABE$.



We will first show that A and C are on opposite sides of the line $\overleftrightarrow{BE}$. Since E and D are defined to be on opposite sides of $\overleftrightarrow{BA}$, we know that $E \notin \overleftrightarrow{BA}$, where $\overleftrightarrow{BA}=\overleftrightarrow{AC}$, so $\overleftrightarrow{BE}\neq\overleftrightarrow{AC}$. But $\{B\} \subseteq \overleftrightarrow{BE} \cap \overleftrightarrow{AC}$, so these lines are distinct and nonparallel. By Theorem 3.1.7 we know that B is the only point that lies on both $\overleftrightarrow{BE}$ and $\overleftrightarrow{AC}$. Hence $A, C \notin \overleftrightarrow{BE}$ and, since $A * B * C$, we have $\{B\} \subseteq \overleftrightarrow{BE} \cap \overline{AC}$. By Proposition 3.3.4, we know A and C are on opposite sides of $\overleftrightarrow{BE}$ and can define the two distinct half-planes H_A and H_C that are bounded by $\overleftrightarrow{BE}$.

To show that $D \in \overleftrightarrow{BE}$, suppose instead that $D \notin \overleftrightarrow{BE}$. Then we have two cases[1]: either $D \in H_A$ or $D \in H_C$. If $D \in H_A$, then D and A are on the same side of $\overleftrightarrow{BE}$.



Since $A * B * C$, the external point D forms the linear pair $\angle ABD$ and $\angle DBC$. By the Linear Pair Theorem, we have

$$\begin{aligned} 180^\circ &= \mu(\angle ABD) + \mu(\angle DBC) \\ &= \mu(\angle ABD) + \mu(\angle ABE), \end{aligned}$$

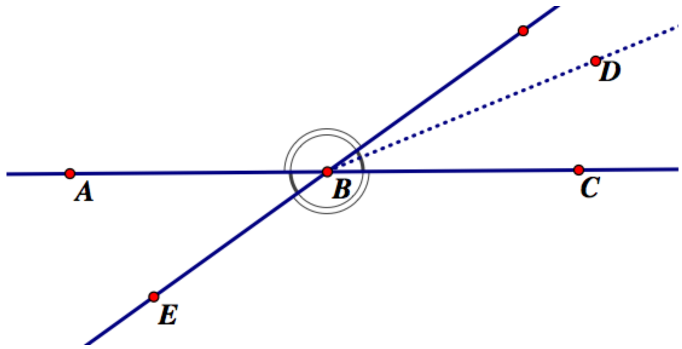
so $\angle ABD$ and $\angle ABE$ are supplementary as well. Note that since $\mu(\angle ABE) < 180^\circ$, we have $\angle ABD \neq 0^\circ$, so that D does not lie on $\overrightarrow{BA}$. Therefore we can apply Theorem 3.4.4 to conclude that either D is in the interior of $\angle EBA$ or A is in the interior of $\angle EBD$. However, D cannot be in the interior of $\angle EBA$, because this would require D and E on the same side of $\overleftrightarrow{AB}$, which contradicts our original hypothesis. So it must be the case that A is in the interior of $\angle EBD$, and the Angle Addition Postulate implies that

$$\begin{aligned} \mu(\angle EBD) &= \mu(\angle EBA) + \mu(\angle ABD) \\ &= \mu(\angle ABE) + \mu(\angle ABD) = 180^\circ. \end{aligned}$$

This of course is a contradiction, as 180° is not in the range of μ. (Our false assumption of $D \in H_A$ implies that rays $\overrightarrow{BE}, \overrightarrow{BD}$ are nonopposite, allowing $\angle EBD$ to be defined.)

Next suppose that $D \in H_C$. Then D and C are on the same side of $\overleftrightarrow{BE}$.

[1] I think I can suppose without loss of generality that $D \in H_A$, but I'll just present both arguments to be on the safe side.



Since $A * B * C$, the external point E forms the linear pair $\angle ABE$ and $\angle EBC$. By the Linear Pair Theorem, we have

$$\begin{aligned} 180° &= \mu(\angle ABE) + \mu(\angle EBC) \\ &= \mu(\angle DBC) + \mu(\angle EBC), \end{aligned}$$

so $\angle DBC$ and $\angle EBC$ are supplementary as well. Note that since $\mu(\angle EBC) < 180°$, we have $\angle DBC \neq 0°$, so that D does not lie on $\overrightarrow{BC}$. Then again by Theorem 3.4.4, we know that either D is in the interior of $\angle EBC$ or C is in the interior of $\angle EBD$. However, since E and D are defined to be on opposite sides of $\overleftrightarrow{AB} = \overleftrightarrow{BC}$, we cannot have D in the interior of $\angle EBC$. Thus it must be the case that C is in the interior of $\angle EBD$. Then by the Angle Addition Postulate,

$$\begin{aligned} \mu(\angle EBD) &= \mu(\angle EBC) + \mu(\angle CBD) \\ &= \mu(\angle EBC) + \mu(\angle DBC) = 180°. \end{aligned}$$

We find the same contradiction in either case, and conclude that D is not in either of the half-planes bounded by $\overleftrightarrow{BE}$. Of course, this means that $D \in \overleftrightarrow{BE}$, so that E, B, and D are collinear. $\square$

Problem 3.2.12(c): If $f : \ell \to \mathbb{R}$ is a coordinate function for a line ℓ and $h : \ell \to \mathbb{R}$ is another coordinate function for ℓ, then there exists a constant c such that either $h(R) = f(R) + c$ or $h(R) = -f(R) + c$.

Proof. Since f is a coordinate function there exist $P, Q \in \ell$ such that $f(P) = 0$ and $f(Q) = 1$. We know P and Q are distinct because f is well-defined, and since h is injective, the law of trichotomy in $\mathbb{R}$ gives two possibilities: $h(P) < h(Q)$ or $h(Q) < h(P)$. Let $c = h(P)$. If $h(P) < h(Q)$, we will show that $h(R) = f(R) + c$. First we verify this for the points P and Q. We have

$$h(P) = 0 + h(P) = f(P) + c,$$

and since coordinate functions preserve distance, we find

$$\begin{aligned} QP &= |f(Q) - f(P)| = |f(Q) - 0| = f(Q) \\ &= |h(Q) - h(P)| = h(Q) - h(P) = h(Q) - c. \end{aligned}$$

The penultimate equality follows because $h(P) < h(Q)$. Of course from these equations, we have $h(Q) = f(Q) + c$. Next, suppose $R * P * Q$. Then by Theorem 3.2.17, we have $f(R) < f(P) < f(Q)$ and $h(R) < h(P) < h(Q)$ and therefore

$$\begin{aligned} RP &= |f(R) - f(P)| = f(P) - f(R) = -f(R) \\ &= |h(R) - h(P)| = h(P) - h(R) = c - h(R). \end{aligned}$$

Therefore $h(R) = f(R) + c$. Next we suppose $P * R * Q$. Then again by Theorem 3.2.17, we have $f(P) < f(R) < f(Q)$ and $h(P) < h(R) < h(Q)$. Then

$$\begin{aligned} RP &= |f(R) - f(P)| = f(R) - f(P) = f(R) \\ &= |h(R) - h(P)| = h(R) - h(P) = h(R) - c. \end{aligned}$$

Again we find $h(R) = f(R) + c$. Finally, suppose $P * Q * R$, so that $f(P) < f(Q) < f(R)$ and $h(P) < f(Q) < f(R)$. Again we have

$$\begin{aligned} RP &= |f(R) - f(P)| = f(R) - f(P) = f(R) \\ &= |h(R) - h(P)| = h(R) - h(P) = h(R) - c, \end{aligned}$$

identical to the previous case. We have shown now that if $h(P) < h(Q)$, then for any $R \in \ell$, $h(R) = f(R) + c$.

Next suppose $h(Q) < h(P)$, in which case we will show that $h(R) = -f(R) + c$. First we verify this for the points P and Q. We have

$$h(P) = -0 + h(P) = -f(P) + c,$$

and

$$\begin{aligned} QP &= |f(Q) - f(P)| = |f(Q) - 0| = f(Q) \\ &= |h(Q) - h(P)| = h(P) - h(Q) = -h(Q) + c, \end{aligned}$$

from which it follows that $h(Q) = -f(Q) + c$. Next suppose $R * P * Q$, so that $f(R) < f(P) < f(Q)$ and $h(Q) < h(P) < h(R)$. Then

$$\begin{aligned} RP &= |f(R) - f(P)| = f(P) - f(R) = -f(R) \\ &= |h(R) - h(P)| = h(R) - h(P) = h(R) - c, \end{aligned}$$

so $h(R) = -f(R) + c$. If $P * R * Q$, then $f(P) < f(R) < f(Q)$ and $h(Q) < h(R) < h(P)$, and

$$\begin{aligned} RP &= |f(R) - f(P)| = f(R) - f(P) = f(R) \\ &= |h(R) - h(P)| = h(P) - h(R) = -h(R) + c. \end{aligned}$$

Again it follows that $h(R) = -f(R) + c$. Lastly, if $P * Q * R$, then $f(P) < f(Q) < f(R)$ and $h(R) < h(Q) < h(P)$, and

$$\begin{aligned} RP &= |f(R) - f(P)| = f(R) - f(P) = f(R) \\ &= |h(R) - h(P)| = h(P) - h(R) = -h(R) + c. \end{aligned}$$

Again we find $h(R) = -f(R) + c$. This shows that if $h(Q) < h(P)$, then for any $R \in \ell$, we have $h(R) = -f(R) + c$. We conclude that for any two coordinate functions f, h for a line ℓ, there is a constant c such that $h(R) = f(R) + c$ or $h(R) = -f(R) + c$.

$\square$