Takehome Midterm

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Real Analysis 1

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Problem 1: (15 points)

Let S and T be non-empty subsets of $\mathbb{R}$ that are bounded above. Define a new set

$$S + T := \{s + t : s \in S \text{ and } t \in T\}.$$

Prove that $S+T$ is bounded above and that $\sup(S+T) = \sup(S) + \sup(T)$.

Proof: Let S and T be non-empty subsets of $\mathbb{R}$ that are bounded above and define the set $S+T$ as above. By Axiom III we know there exist least upper bounds for each set S and T, so let $\mathbf{s} = \sup(S)$ and $\mathbf{t} = \sup(T)$.

To show that $\sup(S+T) = \sup(S) + \sup(T) = \mathbf{s} + \mathbf{t}$, we will use Theorem 1.4.4. Let $\epsilon > 0$. Then $\frac{\epsilon}{2} > 0$ as well, and by Theorem 1.4.4 we know that there exist $s \in S$ and $t \in T$ such that

$$|s - \mathbf{s}| < \frac{\epsilon}{2} \qquad \text{and} \qquad |t - \mathbf{t}| < \frac{\epsilon}{2},$$

so that

$$\begin{aligned} |s + t - (\mathbf{s} + \mathbf{t})| &= |s + t - \mathbf{s} - \mathbf{t}| \leq |s - \mathbf{s}| + |t - \mathbf{t}| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore again by Theorem 1.4.4, we have $\sup(S+T) = \mathbf{s} + \mathbf{t} = \sup(S) + \sup(T)$. (Of course, this also proves that $S+T$ is bounded above.) ∎

Problem 2: (18 points) (All roads lead to Rome.) Define the following function $d_{\text{Rome}} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as follows. For $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$,

$$d_{\text{Rome}}(\vec{x}, \vec{y}) := \left\{ \begin{array}{lcl} d(\vec{x}, \vec{y}) & : & \text{if there exists a } t \in \mathbb{R} \text{ such that } \vec{x} = t \cdot \vec{y} \\ d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{y}) & : & \text{otherwise} \end{array} \right\},$$

where $d(\vec{x}, \vec{y}) = d((x_1, x_2), (y_1, y_2)) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$ is the usual (Euclidean) metric on $\mathbb{R}^2$.

(a) Prove that the function d_{Rome} is a metric on $\mathbb{R}^2$.

Proof: We need to show four conditions hold, namely that d_{Rome} is positive, positive definite, symmetric, and satisfies the triangle inequality. Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^2$.

(Positive) We have by definition that

$$d_{Rome}(\vec{x}, \vec{y}) = d(\vec{x}, \vec{y}) \quad \text{or} \quad d_{Rome}(\vec{x}, \vec{y}) = d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{y}),$$

and in either case since d is positive, we have $d_{Rome}(\vec{x}, \vec{y}) \geq 0$. Therefore d_{Rome} is positive as well.

(Positive Definite) ($\Longrightarrow$) If $\vec{x} = \vec{y}$ then there exists $1 \in \mathbb{R}$ such that $\vec{x} = 1(\vec{y})$. Then $d_{Rome}(\vec{x}, \vec{y}) = d(\vec{x}, \vec{y}) = 0$ because d is positive definite.

($\Longleftarrow$) If $d_{Rome}(\vec{x}, \vec{y}) = 0$ then we have two cases.

(a) If $0 = d_{Rome}(\vec{x}, \vec{y}) = d(\vec{x}, \vec{y})$, then since d is positive definite, we have $\vec{x} = \vec{y}$.

(b) If $0 = d_{Rome}(\vec{x}, \vec{y}) = d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{y})$, note that since d is positive, it must be the case that both

$$d(\vec{x}, \vec{0}) = 0 \qquad \text{and} \qquad d(\vec{0}, \vec{y}) = 0.$$

From here, since d is positive definite we conclude that $\vec{x} = \vec{0} = \vec{y}$.

The previous two cases show that $d_{Rome}(\vec{x}, \vec{y}) = 0$ implies $\vec{x} = \vec{y}$. Therefore $d_{Rome}(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$, so d_{Rome} is positive definite.

(Symmetric) If $\vec{x} = \vec{y}$ then we have $1 \in \mathbb{R}$ such that $\vec{x} = 1(\vec{y})$ and $\vec{y} = 1(\vec{x})$. Therefore since d is symmetric we have

$$d_{Rome}(\vec{x}, \vec{y}) = d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x}) = d_{Rome}(\vec{y}, \vec{x}).$$

So suppose $\vec{x} \neq \vec{y}$. We will consider the separate cases as in the piecewise definition of d_{Rome}.

(a) Suppose there exists $t \in \mathbb{R}$ such that $\vec{x} = t\vec{y}$. Then $d_{Rome}(\vec{x}, \vec{y}) = d(\vec{x}, \vec{y})$. Of course we have $\vec{x} = \vec{0}$ or $\vec{x} \neq \vec{0}$, which yields two subcases.

(i) If $\vec{x} \neq \vec{0}$ then it must be the case that $t \neq 0$. Therefore there exists $t^{-1} \in \mathbb{R}$ and $\vec{x} = t\vec{y}$ implies $\vec{y} = t^{-1}\vec{x}$, so

$$d_{Rome}(\vec{y}, \vec{x}) = d(\vec{y}, \vec{x}) = d(\vec{x}, \vec{y}) = d_{Rome}(\vec{x}, \vec{y}),$$

which follows because d is symmetric.

(ii) If $\vec{x} = \vec{0}$ then since $\vec{x} \neq \vec{y}$ we have $\vec{y} \neq 0$. Thus in considering $d_{Rome}(\vec{y}, \vec{x})$, we see that there is no $t' \in \mathbb{R}$ such that $\vec{y} = t'\vec{x}$; we can never multiply $\vec{0}$ by a real number to get a nonzero vector $\vec{y}$. Hence

$$\begin{aligned} d_{Rome}(\vec{y}, \vec{x}) &= d(\vec{y}, \vec{0}) + d(\vec{0}, \vec{x}) \\ &= d(\vec{y}, \vec{x}) + d(\vec{0}, \vec{0}) = d(\vec{x}, \vec{y}) = d_{Rome}(\vec{x}, \vec{y}), \end{aligned}$$

where again the penultimate equality follows because d is both symmetric and positive definite.

(b) Next suppose that there is no $t \in \mathbb{R}$ such that $\vec{x} = t\vec{y}$. Then $d_{Rome}(\vec{x}, \vec{y}) = d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{y})$. In considering $d_{Rome}(\vec{y}, \vec{x})$, we again have two subcases from the piecewise definition of d_{Rome}.

(i) Suppose there is a $t \in \mathbb{R}$ such that $\vec{y} = t\vec{x}$, so that $d_{Rome}(\vec{y}, \vec{x}) = d(\vec{y}, \vec{x})$. If $t \neq 0$ then there would exist $t^{-1} \in \mathbb{R}$ such that $\vec{x} = t^{-1}\vec{y}$, which we are assuming is *not* the case. Hence $t = 0$ so that $\vec{y} = t\vec{x} = 0$ and

$$\begin{aligned} d_{Rome}(\vec{y}, \vec{x}) = d(\vec{y}, \vec{x}) = d(\vec{0}, \vec{x}) &= d(\vec{x}, \vec{0}) \\ &= d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{0}) \\ &= d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{y}) \\ &= d_{Rome}(\vec{x}, \vec{y}). \end{aligned}$$

(ii) If instead there is no $t \in \mathbb{R}$ such that $\vec{y} = t\vec{x}$ then

$$\begin{aligned} d_{Rome}(\vec{y}, \vec{x}) &= d(\vec{y}, \vec{0}) + d(\vec{0}, \vec{x}) \\ &= d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{y}) \\ &= d_{Rome}(\vec{x}, \vec{y}). \end{aligned}$$

Therefore d_{Rome} is symmetric.

(Triangle Inequality) We have two overall cases as given in the piecewise definition of d_{Rome}.

(a) Suppose there exists $t \in \mathbb{R}$ such that $\vec{x} = t\vec{z}$, so that $d_{Rome}(\vec{x}, \vec{z}) = d(\vec{x}, \vec{z})$. Now note that by definition of d_{Rome} one of the following must be true:

(1) $d_{Rome}(\vec{x}, \vec{y}) + d_{Rome}(\vec{y}, \vec{z}) = d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$,

(2) $d_{Rome}(\vec{x}, \vec{y}) + d_{Rome}(\vec{y}, \vec{z}) = d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{0}) + d(\vec{0}, \vec{z})$,

(3) $d_{Rome}(\vec{x}, \vec{y}) + d_{Rome}(\vec{y}, \vec{z}) = d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{y}) + d(\vec{y}, \vec{0}) + d(\vec{0}, \vec{z})$, or

(4) $d_{Rome}(\vec{x}, \vec{y}) + d_{Rome}(\vec{y}, \vec{z}) = d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{y}) + d(\vec{y}, \vec{z})$.

Since d satisfies the triangle inequality we have

$$\begin{aligned} d_{Rome}(\vec{x}, \vec{z}) = d(\vec{x}, \vec{z}) &\leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z}) && (1) \\ &\leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{0}) + d(\vec{0}, \vec{z}) && (2) \\ &\leq d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{y}) + d(\vec{y}, \vec{0}) + d(\vec{0}, \vec{z}) && (3) \end{aligned}$$

Also

$$\begin{aligned} d_{Rome}(\vec{x}, \vec{z}) = d(\vec{x}, \vec{z}) &\leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z}) \\ &\leq d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{y}) + d(\vec{y}, \vec{z}) && (4) \end{aligned}$$

The inequalities (1)-(4) altogether imply that

$$d_{Rome}(\vec{x}, \vec{z}) \leq d_{Rome}(\vec{x}, \vec{y}) + d_{Rome}(\vec{y}, \vec{z}).$$

(b) Next suppose there is no $t \in \mathbb{R}$ such that $\vec{x} = t\vec{z}$. Then $d_{Rome}(\vec{x}, \vec{z}) = d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{z})$. Now we have two subcases.

(i) Suppose first that there does exist $t \in \mathbb{R}$ such that $\vec{x} = t\vec{y}$, so that $d_{Rome}(\vec{x}, \vec{y}) = d(\vec{x}, \vec{y})$. In this case there cannot be any $t' \in \mathbb{R}$ such that $\vec{y} = t'\vec{z}$, otherwise there would be $tt' \in \mathbb{R}$ such that $\vec{x} = t\vec{y} = (tt')\vec{z}$, which we are assuming (in case (b)) is not the case. Thus $d_{Rome}(\vec{y}, \vec{z}) = d(\vec{y}, \vec{0}) + d(\vec{0}, \vec{z})$ so that

$$\begin{aligned} d_{Rome}(\vec{x}, \vec{z}) &= d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{z}) \\ &\leq (d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{0})) + d(\vec{0}, \vec{z}) \\ &= d(\vec{x}, \vec{y}) + (d(\vec{y}, \vec{0}) + d(\vec{0}, \vec{z})) \\ &= d_{Rome}(\vec{x}, \vec{y}) + d_{Rome}(\vec{y}, \vec{z}). \end{aligned}$$

(ii) Finally, suppose that there is no $t \in \mathbb{R}$ such that $\vec{x} = t\vec{y}$. Then $d_{Rome}(\vec{x}, \vec{y}) = d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{y})$ and as earlier, it must be the case that either

(1) $d_{Rome}(\vec{x}, \vec{y}) + d_{Rome}(\vec{y}, \vec{z}) = d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{y}) + d(\vec{y}, \vec{z})$, or
(2) $d_{Rome}(\vec{x}, \vec{y}) + d_{Rome}(\vec{y}, \vec{z}) = d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{y}) + d(\vec{y}, \vec{0}) + d(\vec{0}, \vec{z})$.

We then have

$$\begin{aligned} d_{Rome}(\vec{x}, \vec{z}) &= d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{z}) \\ &\leq d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{y}) + d(\vec{y}, \vec{z}) && (1) \\ &\leq d(\vec{x}, \vec{0}) + d(\vec{0}, \vec{y}) + d(\vec{y}, \vec{0}) + d(\vec{0}, \vec{z}), && (2) \end{aligned}$$

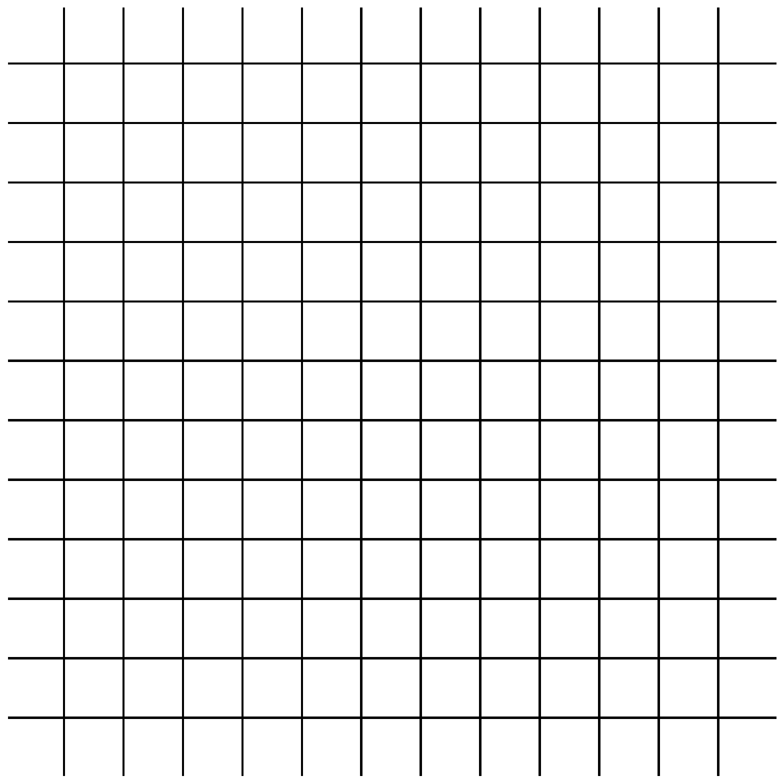
and of course from the reasoning above, the two inequalities (1)-(2) ensure that

$$d_{Rome}(\vec{x}, \vec{z}) \leq d_{Rome}(\vec{x}, \vec{y}) + d_{Rome}(\vec{y}, \vec{z}).$$

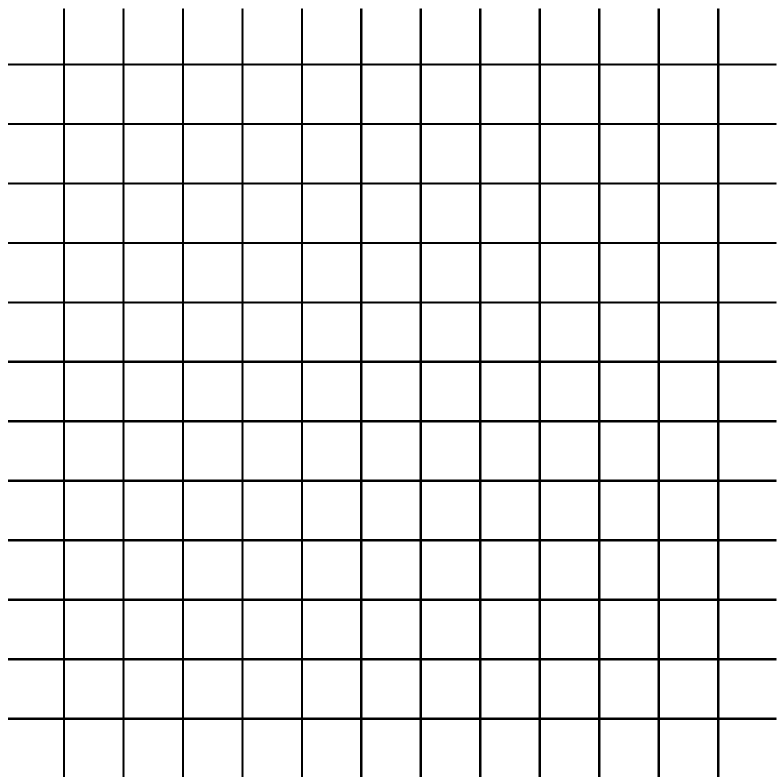
Therefore d_{Rome} satisfies the triangle inequality. We have now shown that d_{Rome} is a metric. ▮

(b) Draw carefully labeled pictures of open balls of radius 3 about the points $(0, 0)$, $(2, 2)$, and $(3, 0)$.

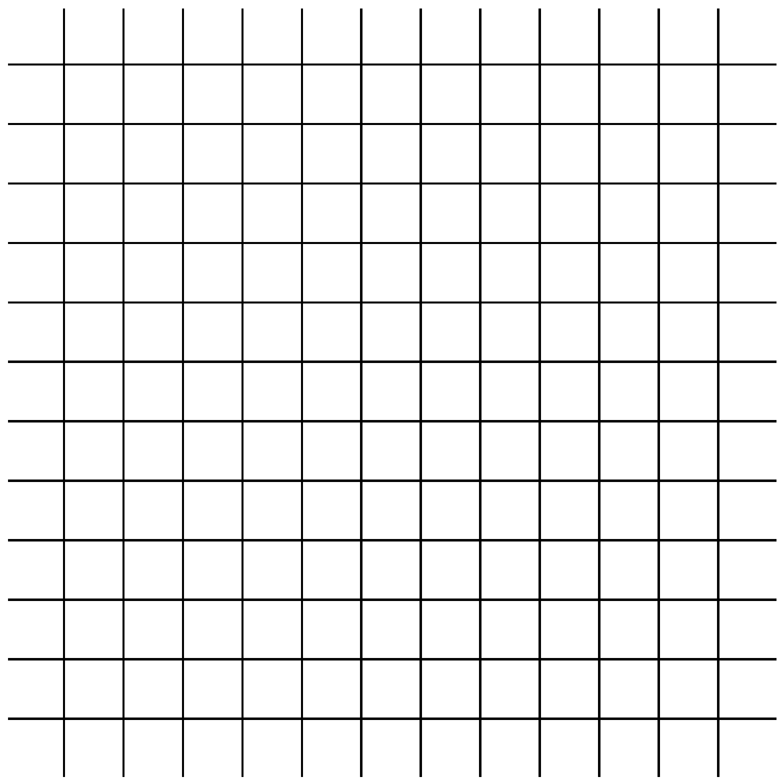
$B_3((0, 0))$



$B_3((2,2))$



$B_3((3,0))$



Problem 3: (15 points) Let X be a metric space. Prove that every set $S \subseteq X$ can be written as the intersection of a collection of open sets in X.

Lemma 1: Let S be a proper subset of X. If $x \notin S$, then there exists an open set U of X such that $S \subseteq U$ and $x \notin U$.

Proof: Let $S \subset X$ and choose $x \in X \setminus S$. We have shown that $\{x\}^{\mathcal{C}}$ is an open set, and clearly for any $s \in S$, we have $s \neq x$ so that $s \in \{x\}^{\mathcal{C}}$. Therefore for any $x \notin S$ there is an open set $\{x\}^{\mathcal{C}}$ such that $S \subseteq \{x\}^{\mathcal{C}}$ and $x \notin \{x\}^{\mathcal{C}}$. ▮

Next we show that every set $S \subseteq X$ is an intersection of a collection of open sets in X.

Proof: First, we have already shown in Exercise 3.1.6 that the improper subset $S = X$ is an open set of the metric space X. So suppose that $S \subset X$ is a proper subset, and let $\hat{S}$ be the intersection of all open sets U containing S. Note that the open set X contains S, so this intersection is in fact nonempty. By construction then, it is evident that $S \subseteq \hat{S}$. If $x \in \hat{S}$, then x is in every open set containing S. Hence by the contrapositive of Lemma 1, $x \in S$. This shows $\hat{S} \subseteq S$, so we conclude that $S = \hat{S}$. ▮

Problem 4: (18 points) Let X be a metric space and S a subset of X. A point $x \in S$ is called an **interior point** of S if there exists an open ball about x that is contained in S. (That is, this open ball is a subset of S.) The set of all interior points of S is called the **interior** of S, and is denoted by $\mathcal{I}nt(S)$.

(a) Give an example of a set $S \subseteq \mathbb{R}^2$ such that $\mathcal{I}nt(S) \neq \emptyset$ and $\mathcal{I}nt(S) \neq S$. Draw a carefully labelled picture of the sets S and $\mathcal{I}nt(S)$.

Let $S = \{(x, y) : x^2 + y^2 \leq 1\}$. First to show that $\mathcal{I}nt(S) \neq \emptyset$, note that the point $(0, 0) \in S$ and

$$B_1((0,0)) = \{(x, y) : x^2 + y^2 < 1\} \subseteq S,$$

so $(0, 0) \in \mathcal{I}nt(S)$ and we conclude $\mathcal{I}nt(S) \neq \emptyset$.

To show that $\mathcal{I}nt(S) \neq S$, consider the point $(0,1) \in S$. For any $r > 0$ we see that the point $(0, 1 + \frac{r}{2}) \in B_r((0,1))$ because

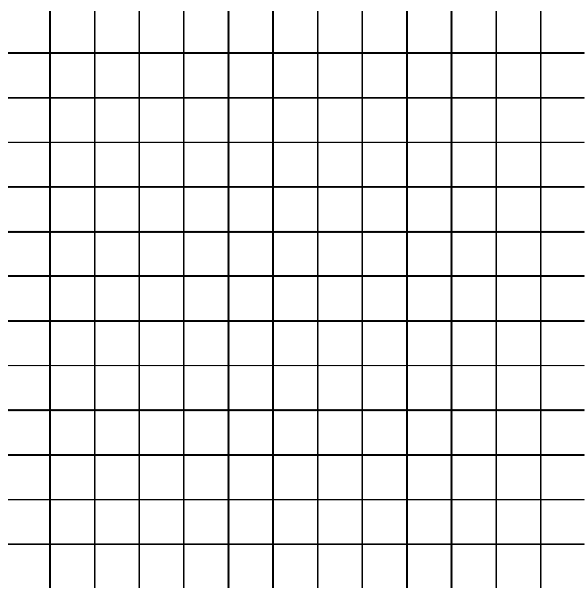
$$d\Big(\big(0,1\big),\big(0,1+\frac{r}{2}\big)\Big) = \sqrt{\Big(1-\big(1+\frac{r}{2}\big)\Big)^2}$$
$$= \sqrt{\frac{r^2}{2^2}} = \frac{|r|}{2} = \frac{r}{2} < r.$$

However since

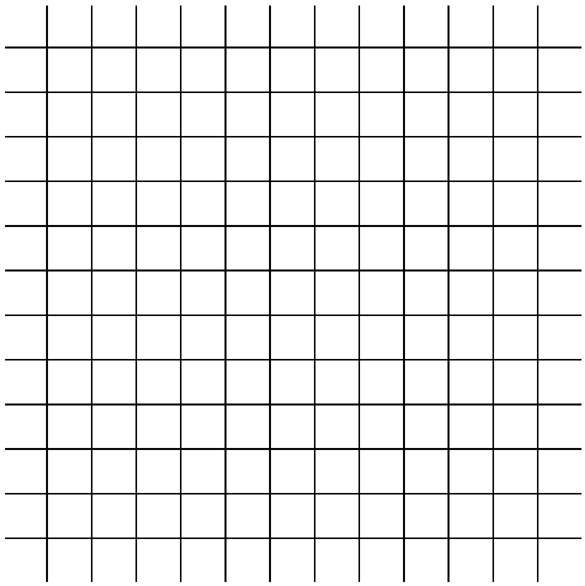
$$(0)^2 + \Big(1+\frac{r}{2}\Big)^2 \;=\; 1 + r + \frac{r^2}{4} \;>\; 1,$$

we have $(0, 1 + \frac{r}{2}) \notin S$. Thus for the point $(0,1) \in S$ we have found that no open ball $B_r((0,1))$ is contained in S, so $(0,1)$ is not an interior point of S. Therefore $\mathcal{I}nt(S) \neq S$.

$$S = \{(x,y) : x^2 + y^2 \leq 1\}$$



$$\mathcal{I}nt(S) = \{(x, y) : x^2 + y^2 < 1\}$$



(b) Prove that $\mathcal{I}nt(S)$ is an open set.

Proof: We will use the characterization in Theorem 3.1.7.2 to prove that $\mathcal{I}nt(S)$ is an open set; specifically we will show that for any $x \in \mathcal{I}nt(S)$, there exists $r > 0$ such that $B_r(x) \subseteq \mathcal{I}nt(S)$.

So let $x \in \mathcal{I}nt(S)$. By definition, there exists $r > 0$ such that $B_r(x) \subseteq S$. However, we must show that $B_r(x) \subseteq \mathcal{I}nt(S)$, which will amount to showing that any y in this open ball is an interior point of S. Let $y \in B_r(x)$ and pick $\epsilon = r - d(x, y)$. Note that ϵ is positive because $y \in B_r(x)$, so $d(x, y) < r$. Now for any $z \in B_\epsilon(y)$ we have

$$\begin{aligned} d(z, x) &\leq d(z, y) + d(x, y) \\ &< \epsilon + d(x, y) \\ &= r - d(x, y) + d(x, y) \\ &= r. \end{aligned}$$

Hence $z \in B_r(x) \subseteq S$. This shows that $B_\epsilon(y) \subseteq S$ and we conclude y is an interior point of S. Since y was arbitrarily chosen from $B_r(x)$, we have

$B_r(x) \subseteq \mathcal{I}nt(S)$. Finally by Theorem 3.1.7.2 we conclude that $\mathcal{I}nt(S)$ is an open set. ▮

(c) Prove that S is open if and only if $S = \mathcal{I}nt(S)$.

Proof: Note that by definition, every interior point of S is in S itself, so we will always have $\mathcal{I}nt(S) \subseteq S$. Thus the proof will be complete if we can show that S is open if and only if $S \subseteq \mathcal{I}nt(S)$. Well by Theorem 3.1.7.2 we know S is open if and only if for every $x \in S$ there exists $r > 0$ such that $B_r(x) \subseteq S$. By definition, this means precisely that every $x \in S$ is an interior point of S, which of course is equivalent to $S \subseteq \mathcal{I}nt(S)$. Thus we conclude that S is open if and only if $S \subseteq \mathcal{I}nt(S)$, and as noted above it follows that S is open if and only if $S = \mathcal{I}nt(S)$. ▮

Problem 5: (15 points) Consider the sequence $(s_n) = \left(\frac{10n-3}{n}\right)_{n=1}^{\infty}$. Does this sequence converge or diverge?

This sequences converges to 10.

Proof: Define (s_n) as above. To show that $s_n \to 10$, let $\epsilon > 0$. Invoking the Archimedean property of $\mathbb{R}$, pick $N \in \mathbb{N}$ so that $N > \frac{3}{\epsilon}$. This implies that $\epsilon > \frac{3}{N}$. We see then for $n > N$,

$$\begin{aligned} d\left(\frac{10n-3}{n},\ 10\right) &= \left|\frac{10n-3}{n} - 10\right| \\ &= \left|\frac{10n-3-10n}{n}\right| \\ &= \left|\frac{-3}{n}\right| \\ &= \frac{3}{n} \ < \ \frac{3}{N} \ < \ \epsilon. \end{aligned}$$

Therefore (s_n) converges to 10. ▮

Problem 6 (a): Construct a sequence of real numbers that has subsequences converging to every rational number.

Proof: We know that $\mathbb{Q}$ is countable from Foundations, so let $b : \mathbb{N} \to \mathbb{Q}$ be a bijection. Then $b(n) = (b_n)$ is a sequence with range $\mathbb{Q}$.

Let $q \in \mathbb{Q}$ and $r > 0$. We proved in class that q is a limit point of $\mathbb{Q}$; we present that argument again for the sake of completeness (and grading). By Theorem 3.5.1.4 it is sufficient to show that for all $r > 0$, $B_r(q)$ contains an element of $\mathbb{Q} \setminus \{q\}$. This will follow from Problem 1.4.8.d: since q and $q - r$ are real numbers and $q - r < q$, we know there exists $p \in \mathbb{Q}$ satisfying

$$q - r < p < q.$$

Thus $p \neq q$ and of course this also implies that

$$q - r < p < q + r.$$

By Theorem 1.3.8.8 this means that

$$|q - p| < r,$$

so $p \in B_r(q)$ where $p \in \mathbb{Q} \setminus \{q\}$. This shows that q is a limit point of $\mathbb{Q}$.

Now from the characterization in Theorem 3.5.1.3 we have that for all $r > 0$, $B_r(q)$ contains infinitely many elements of $\mathbb{Q}$. Recall that the rational numbers are precisely the terms of (b_n) so it must be the case that every open ball $B_r(q)$ about q contains infinitely many terms of the sequence (b_n). Therefore by Problem 3.3.8.a we conclude that (b_n) has a subsequence converging to q. Of course since q was arbitrary, this sequence has subsequences converging to every rational number. ∎

(b) Let $x \in \mathbb{R}$. For the sequence you constructed in part (a), describe how you can find a subsequence that converges to x.

In part (a) we restricted ourselves to considering $q \in \mathbb{Q}$ as a limit point of $\mathbb{Q}$, but really the proof does not require $q \in \mathbb{Q}$; in fact every real number $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$. To be sure, let's run through the argument once more.

This time, let $x \in \mathbb{R}$ and $r > 0$. As before, it is sufficient to show that $B_r(x)$ contains an element of $\mathbb{Q} \setminus \{x\}$. Again using Problem 1.4.8.d, since x and $x - r$ are real numbers and $x - r < x$, we know there exists $p \in \mathbb{Q}$ such that

$$x - r < p < x,$$

and the same reasoning reasoning above allows us to conclude that $p \in B_r(x)$ where $p \in \mathbb{Q} \setminus \{x\}$. This shows that x is a limit point of $\mathbb{Q}$.

Following the same argument above, by Theorem 3.5.1.3 we know that every open ball $B_r(x)$ about x contains infinitely many rational numbers. Since the rational numbers are precisely the terms of the sequence (b_n), we infer that any ball $B_r(x)$ about x must contain infinitely many terms of the sequence (b_n). Again by Problem 3.3.8.a we conclude that there is some subsequence (b_{n_k}) converging to x, for any real number x.

This shows that such a sequence exists. For the skeptics (such as a professor grading this paper), I will describe how one might find such a sequence. Recall from above that since any $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$, by Theorem 3.5.1.3 we know that the intersection

$$B_r(x) \cap \mathbb{Q}$$

is infinite for any $r > 0$. To construct the subsequence converging to x, let $r_k = \frac{1}{k}$ and follow the outline below:

1. From above we can pick a rational number q_1 from $B_{r_1}(x)$. Since $b(n)$ is onto $\mathbb{Q}$, we know $q_1 = b_j$ for some $j \in \mathbb{N}$. Let $n_1 = j$ so that

$$b_{n_1} = b_j \qquad \text{and} \qquad b_{n_1} \in B_{\frac{1}{1}}(x).$$

2. From above there are infinitely many rational numbers in the ball $B_{r_2}(x)$. Therefore there are infinitely many terms of the sequence (b_n) in $B_{r_2}(x)$. Thus there *must* be some $b_m \in B_{r_2}(x)$ such that $m > n_1$. To see this, note that if there were no such $m > n_1$, that is if all terms $b_m \in B_{r_2}(x)$ were such that $m \leq n_1$, then this ball would only have at most n_1 terms of the sequence (b_i), which of course means that this ball would have at most n_1 rational numbers. Hence the set

$$B_{r_2}(x) \cap \mathbb{Q}$$

would be finite, contradicting that x is a limit point of $\mathbb{Q}$! Thus there must be some $b_m \in B_{r_2}(x)$ such that $m > n_1$. We let $n_2 = m$ so that

$$n_1 < n_2 \qquad \text{and} \qquad b_{n_2} \in B_{\frac{1}{2}}(x).$$

$$\vdots$$

k. Just as in step 2, there are infinitely many rational numbers in the ball $B_{r_k}(x)$, and thus infinitely many terms of the sequence (b_n). By the same reasoning above, there must be some $b_m \in B_{r_k}(x)$ where $m > n_{k-1}$. So we let $n_k = m$ so that

$$n_{k-1} < n_k \qquad \text{and} \qquad b_{n_k} \in B_{\frac{1}{k}}(x).$$

It is clear from construction that the sequence (n_k) is a strictly increasing sequence of natural numbers, which means that (b_{n_k}) is indeed a subsequence of (b_n). Also by construction, for all $k \in \mathbb{N}$ we have

$$b_{n_k} \in B_{\frac{1}{k}}(x),$$

so by Problem 3.3.1[1] we conclude that

$$b_{n_k} \to x.$$

[1] Of course, this is relying on the fact (discussed in gross detail in class) that the subsequence (b_{n_k}) is also a sequence; namely a function of k with domain $\mathbb{N}$.