Part 1

Exercise 8. Let A, B, A_α denote subsets of a space X. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\supset$ or $\subset$ holds.

(a) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

Proof. Equality does not hold, however $\subset$ does hold. Suppose $x \in \overline{A \cap B}$. Then for all neighborhoods U of x, $U \cap (A \cap B) \neq \varnothing$, which implies that both $U \cap A \neq \varnothing$ and $U \cap B \neq \varnothing$. Therefore $x \in \overline{A}$ and $x \in \overline{B}$. Hence $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. □

(c) $\overline{A - B} = \overline{A} - \overline{B}$.

Proof. Equality does not hold, however $\supset$ does hold. Suppose $x \in \overline{A} - \overline{B}$, so every neighborhood of x intersects A but there exists a neighborhood V of x disjoint from B. Suppose for contradiction that there exists a neighborhood U of x such that $U \cap (A - B) = \varnothing$. Then since $U \cap V \subset U$, we have $(U \cap V) \cap (A - B) = \varnothing$. Notice $U \cap V$ is also a neighborhood of x and since $x \in \overline{A}$, we know $(U \cap V) \cap A \neq \varnothing$, so it must be the case that $(U \cap V) \cap B \neq \varnothing$. However, this implies $V \cap B \neq \varnothing$, which contradicts the assumption that V is disjoint from B. Therefore all neighborhoods U of x intersect $A - B$, so that $x \in \overline{A - B}$. □

Exercise 13. Show that X is Hausdorff if and only if the *diagonal* $\Delta = \{x \times x : x \in X\}$ is closed in $X \times X$.

Proof. ($\Rightarrow$) Suppose X is Hausdorff and let $x \times y$ be in the complement of the diagonal: $x \times y \in X \times X - \Delta$. Then $x \neq y$, so there exist disjoint neighborhoods U of x and V of y. Note that because U and V are disjoint, they share no equal elements, hence $U \times V$ does not intersect Δ. Therefore $x \times y \in U \times V \subset X \times X - \Delta$. Since we have shown that arbitrary elements of the complement of the diagonal are elements of open sets contained within the complement of the diagonal, the complement of the diagonal is open. Hence the diagonal is closed.

($\Leftarrow$) Suppose Δ is closed in $X \times X$. Let x, y be distinct elements of X. Then $x \times y \notin \Delta$, so $x \times y \in X \times X - \Delta$ which is open. Thus there exists some basis element $U \times V$ such that $x \times y \in U \times V \subset X \times X - \Delta$, where U, V are open in X. However, since $U \times V$ is contained in the complement of the diagonal, we see that U and V must be disjoint. We have shown that we can find disjoint neighborhoods around any two distinct elements in X, therefore X is Hausdorff. □

Exercise 16.b. Consider the five topologies on $\mathbb{R}$ given in Exercise 13.7. Which of these topologies satisfy the Hausdorff axiom? the T_1 axiom?

Solution. Recall these are

$$\begin{aligned}
\mathcal{T}_1 &= \text{ the standard topology} \\
\mathcal{T}_2 &= \text{ the topology of } \mathbb{R}_K \\
\mathcal{T}_3 &= \text{ the finite complement topology} \\
\mathcal{T}_4 &= \text{ the upper limit topology, having all sets } (a,b] \text{ as basis} \\
\mathcal{T}_5 &= \text{ the topology having all sets } (-\infty, a) = \{x : x < a\} \text{ as basis.}
\end{aligned}$$

$\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$ satisfy the T_1 axiom and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_4$ satisfy the Hausdorff axiom. □

Exercise 18. Determine the closures of the following subsets of the ordered square:

$$\begin{aligned}
A &= \{(\frac{1}{n}) \times 0 : n \in \mathbb{Z}_+\} \\
B &= \{(1 - \frac{1}{n}) \times \frac{1}{2} : n \in \mathbb{Z}_+\} \\
C &= \{x \times 0 : 0 < x < 1\} \\
D &= \{x \times \frac{1}{2} : 0 < x < 1\} \\
E &= \{\frac{1}{2} \times y : 0 < y < 1\}
\end{aligned}$$

Solution.

$$\begin{aligned}
\overline{A} &= A \cup \{0 \times 1\} \\
\overline{B} &= B \cup \{1 \times 0\} \\
\overline{C} &= \{x \times 1 : 0 \leq x < 1\} \cup \{x \times 0 : 0 < x \leq 1\} \\
\overline{D} &= D \cup \overline{C} \\
\overline{E} &= \left[\frac{1}{2} \times 0, \frac{1}{2} \times 1\right]
\end{aligned}$$

□

Part 2

Let X be a topological space. Let $A \subset X$ be a subset.

(a) Show that $\mathrm{Bd}\,(\mathrm{Bd}\, A) \subset \mathrm{Bd}\, A$.

Proof. Notice that

$$\mathrm{Bd}\,(\mathrm{Bd}\,A) \;=\; \overline{\mathrm{Bd}\,A} \cap \overline{X - \mathrm{Bd}\,A} \;\subset\; \overline{\mathrm{Bd}\,A} \;=\; \mathrm{Bd}\,A.$$

$\square$

(b) Find an example in which $\mathrm{Bd}\,(\mathrm{Bd}\,A) \neq \mathrm{Bd}\,A$.

Solution. When $A = \mathbb{Q} \subset \mathbb{R}$, we have $\mathrm{Bd}\,\mathbb{Q} = \mathbb{R}$ and $\mathrm{Bd}\,(\mathrm{Bd}\,\mathbb{Q}) = \mathrm{Bd}\,\mathbb{R} = \varnothing$. $\square$

(c) Show that $\mathrm{Bd}\,(\mathrm{Bd}\,(\mathrm{Bd}\,A)) = \mathrm{Bd}\,(\mathrm{Bd}\,A)$.

Proof. First note that considering $\mathrm{Bd}\,A$ as a subset of X, part (a) immediately tells us that $\mathrm{Bd}\,(\mathrm{Bd}\,(\mathrm{Bd}\,A)) \subset \mathrm{Bd}\,(\mathrm{Bd}\,A)$. For the reverse direction, suppose $x \in \mathrm{Bd}\,(\mathrm{Bd}\,A)$. To show that $x \in \mathrm{Bd}\,(\mathrm{Bd}\,(\mathrm{Bd}\,A))$, we will simply rule out the alternative possibilities that $x \in \mathrm{Int}\,(\mathrm{Bd}\,(\mathrm{Bd}\,A))$ or $x \in \mathrm{Ext}\,(\mathrm{Bd}\,(\mathrm{Bd}\,A))$.

So suppose for contradiction that $x \in \mathrm{Int}\,(\mathrm{Bd}\,(\mathrm{Bd}\,A))$. Then there exists a neighborhood U of x such that $U \subset \mathrm{Bd}\,(\mathrm{Bd}\,A)$. Since

$$\mathrm{Bd}\,(\mathrm{Bd}\,A) \;=\; \overline{\mathrm{Bd}\,A} \cap \overline{X - \mathrm{Bd}\,A} \;=\; \mathrm{Bd}\,A \cap \overline{X - \mathrm{Bd}\,A} \;\subset\; \mathrm{Bd}\,A,$$

we see that $U \subset \mathrm{Bd}\,A$. Thus $x \in \mathrm{Int}\,(\mathrm{Bd}\,A)$, which contradicts our assumption that $x \in \mathrm{Bd}\,(\mathrm{Bd}\,A)$.

Next suppose for contradiction that $x \in \mathrm{Ext}\,(\mathrm{Bd}\,(\mathrm{Bd}\,A))$. Then there exists a neighborhood U of x such that $U \subset X - \mathrm{Bd}\,(\mathrm{Bd}\,A)$. But then $x \in X - \mathrm{Bd}\,(\mathrm{Bd}\,A)$ which contradicts our assumption that $x \in \mathrm{Bd}\,(\mathrm{Bd}\,A)$.

Therefore the only possibility is that $x \in \mathrm{Bd}\,(\mathrm{Bd}\,(\mathrm{Bd}\,A))$, and we conclude that $\mathrm{Bd}\,(\mathrm{Bd}\,(\mathrm{Bd}\,A)) = \mathrm{Bd}\,(\mathrm{Bd}\,A)$. $\square$