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Math 341: Real Analysis
Take-home Final Exam
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1. (15 pts) Let A and B be non-empty subsets of $\mathbb{R}$ with $A \subseteq B$. Suppose that $\inf A$, $\inf B$, and $\sup B$ all exist.

   (a) Prove that $\sup A$ exists.

   Proof. We know A is non-empty by hypothesis, so we only need to show that A is bounded above. Considering any $a \in A$, we see that $a \in B$ as well because $A \subseteq B$. Thus $a \leq \sup B$, and since a was arbitrary we have shown that A is bounded above by $\sup B$. Hence by Axiom III, $\sup A$ exists. □

   (b) Prove that
$$\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B).$$

   Proof. For the first inequality, it suffices to show that $\inf(B)$ is a lower bound for A. So, similar to part (a), we note that for any $a \in A$, a is also in B and therefore $\inf(B) \leq a$. Hence $\inf(B)$ is a lower bound for A and since $\inf(A)$ is the greatest lower bound, we have
$$\inf(B) \leq \inf(A).$$

   The next inequality is quite trivial; since A is non-empty there exists $a \in A$ and by definition of infimum and supremum we have immediately that
$$\inf(A) \leq a \leq \sup(A).$$

   Finally, note that we showed in part (a) that $\sup(B)$ is an upper bound for A. Therefore just as in the infimum case, since $\sup(A)$ is the least upper bound, we have $\sup(A) \leq \sup(B)$. Altogether we have shown that
$$\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B).$$
□

   (c) Show by giving a counterexample that
$$\inf(B) = \inf(A) \qquad \text{and} \qquad \sup(A) = \sup(B)$$
   need not imply that $A = B$.

Counterexample: If we define

$$A = \{0, 1\} \qquad \text{and} \qquad B = [0, 1]$$

we see that

$$\inf(A) = \inf(B) = 0 \qquad \text{and} \qquad \sup(A) = \sup(B) = 1,$$

but of course $A \neq B$.

2. (15 pts) Prove that every non-convergent bounded sequence of real numbers has at least two limit points.

   Proof. Let (s_n) be a non-convergent bounded sequence in $\mathbb{R}$. Because (s_n) is bounded we know from Theorem 3.4.9 that (s_n) has at least one convergent subsequence, and hence at least one limit point x. Consider an arbitrary subsequence (s_{n_k}) of (s_n). This subsequence must also be bounded and thus by the same theorem has a convergent subsequence $(s_{n_{k_i}})$. Well $(s_{n_{k_i}})$ is evidently also a subsequence of (s_n) in its own right, so we see that if x was the only limit point, this would imply $s_{n_{k_i}} \to x$. But we chose an arbitrary subsequence, so we have shown that if x is the only limit point of (s_n) then every subsequence of (s_n) has a subsequence converging to x. By Problem 3.3.6 this would imply $s_n \to x$, which is a contradiction because we are assuming (s_n) is non-convergent. We have shown that (s_n) must have a limit point x and that x cannot be the only limit point, and therefore the only possibility is that (s_n) has at least two limit points. □

3. (20 pts)

   (a) Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = x^3 - 2x^2 + 3x - 6$. Carefully prove that the limit $\lim_{x\to 2} h(x)$ exists. (Your solution should also indicate the limit!)

   Proof. Carefully, and cleverly, we will first prove results about the identity function and constant function. Define the function $i : \mathbb{R} \to \mathbb{R}$ by $i(x) = x$. First note that we have previously shown every $x \in \mathbb{R}$ is a limit point of $\mathbb{R}$, so we know $2 \in \mathbb{R}$ is a limit point of the domain $\mathbb{R}$. To see that $\lim_{x\to 2} i(x) = 2$, let $\epsilon > 0$. Then for $\delta = \epsilon$, if $0 < |x - 2| < \delta$,

   $$|i(x) - 2| = |x - 2| < \delta = \epsilon.$$

   Therefore

   $$\lim_{x\to 2} i(x) = 2.$$

   Next fix $k \in \mathbb{R}$ and define $c_k : \mathbb{R} \to \mathbb{R}$ by $c_k(x) = k$. Let $\epsilon > 0$ and pick $\delta = 64$. Then for $0 < |x - 2| < \delta$, we have

   $$|c_k(x) - k| = |k - k| = 0 < \epsilon.$$

Therefore
$$\lim_{x\to 2} c_k(x) = k.$$
Next recall the function h from above and observe that
$$\begin{aligned}h(x) &= x^3 - 2x^2 + 3x - 6\\ &= (i(x))^3 + c_{-2}(x)(i(x))^2 + c_3(x)i(x) + c_{-6}(x).\end{aligned}$$
Apply Theorem 5.3.1 parts (1) and (3) a total of eight times (if you want me to be *really* careful, we apply part (1) three times and part (3) five times) to obtain
$$\begin{aligned}\lim_{x\to 2} h(x) &= \lim_{x\to 2}\left((i(x))^3 + c_{-2}(x)(i(x))^2 + c_3(x)i(x) + c_{-6}(x)\right)\\ &= (2)^3 + (-2)(2)^2 + (3)(2) + (-6)\\ &= 8 - 8 + 6 - 6\\ &= 0.\end{aligned}$$
Of course applying Theorem 5.3.1 guarantees that the limit does exist, and that it is equal to 0. □

(b) Carefully prove that the limit $\lim_{x\to 0} \cos\frac{1}{x}$ fails to exist.

Proof. We need to be quite careful indeed; the problem is actually stated quite ambiguously because it is not clear over which domain we are to consider this function. I think it is safe to assume that we are considering a function $f : \mathbb{R}^* \to \mathbb{R}$ defined by $f(x) = \cos\frac{1}{x}$, where $0 \in \mathbb{R}$ is a limit point of the subset $\mathbb{R}^*$. To show $\lim_{x\to 0} f(x)$ does not exist, we will show that it is not equal to any candidate $L \in \mathbb{R}$. First consider $L = 0$ and pick $\epsilon = \frac{1}{2}$. Let $\delta > 0$ and choose $n \in \mathbb{N}$ such that $n > \frac{1}{\delta}$. Then
$$0 < \left|\frac{1}{2n\pi} - 0\right| < \frac{1}{n} < \delta,$$
yet
$$\left|f\left(\frac{1}{2n\pi}\right) - L\right| = \left|\cos(2n\pi) - 0\right| = |1 - 0| = 1 > \epsilon.$$
This shows that $\lim_{x\to 0} f(x) \neq 0$. Now consider any $L \neq 0$, and pick $\epsilon = |L|$. As before, for any $\delta > 0$ choose $n \in \mathbb{N}$ such that $n > \frac{1}{\delta}$. Then
$$0 < \left|\frac{1}{2n\pi + \frac{\pi}{2}} - 0\right| < \frac{1}{n} < \delta,$$
yet
$$\left|f\left(\frac{1}{2n\pi + \frac{\pi}{2}}\right) - L\right| = \left|\cos\left(2n\pi + \frac{\pi}{2}\right) - L\right| = |0 - L| = |L| = \epsilon.$$

Therefore $\lim_{x\to 0} f(x) \neq L$. So in considering the cases when $L = 0$ and $L \neq 0$, we have shown that the limit of f as x goes to 0 is not L for any L in the codomain $\mathbb{R}$. Therefore the limit does not exist. $\square$

4. (15 pts) Let (X, d) be a metric space. Show that the following statements about X are equivalent.

i. (X, d) is discrete.

ii. All functions on X are continuous. (The range can be *any* metric space!)

Proof. ($\Longrightarrow$) Suppose (X, d) is discrete. Consider any $a \in X$. Since $\{a\}$ is open, there exists $r > 0$ such that $B_r(a) \subseteq \{a\}$, from which it follows that $B_r(a) = \{a\}$. Then obviously $B_r(a)$ does not contain infinitely many points of X, so by the characterization in Theorem 3.5.1.3 we conclude that a is not a limit point of X. Thus for any metric space Y and any function $f : X \to Y$, f is continuous at a by definition. Since a was arbitrary we conclude that all functions on X are continuous.

($\Longleftarrow$) Conversely suppose that all functions on X are continuous. Let $a \in X$ and define the function $f_a : X \to \mathbb{R}$ by

$$f_a(x) = \begin{cases} 64 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

Note that the open interval $(63, 65) = B_1(64)$ is an open subset of $\mathbb{R}$. Since f_a is continuous by hypothesis, Theorem 4.3.5 guarantees that the inverse image of $(63, 65)$ is open in X. By the construction above we have

$$f_a^{-1}\left((63, 65)\right) = \{a\},$$

so the previous sentence implies that $\{a\}$ is open in X. Since a was arbitrary, we conclude X is a discrete metric space. $\square$

5. (15 pts) Let (a_n) and (b_n) be Cauchy sequences in a metric space (X, d). Show that the sequence $(d(a_n, b_n))$ converges. (Note that X need not be complete, but $\mathbb{R}$ is complete!)

Proof. As hinted above, we will show that $(d(a_n, b_n))$ is a Cauchy sequence and invoke the completeness of $\mathbb{R}$ to complete the proof. Let $\epsilon > 0$. Since (a_n) and (b_n) are Cauchy sequences, we can choose $N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$ so that

$$d(a_n, a_m) < \frac{\epsilon}{2} \quad \text{for all } n, m > N_1$$

$$d(b_n, b_m) < \frac{\epsilon}{2} \quad \text{for all } n, m > N_2.$$

Let $N = \max\{N_1, N_2\}$ and $n, m > N$. Supposing without loss of generality that $d(a_n, b_n) \geq d(a_m, b_m)$,

$$\begin{aligned}
|d(a_n, b_n) - d(a_m, b_m)| &= d(a_n, b_n) - d(a_m, b_m) \\
&\leq d(a_n, a_m) + d(a_m, b_n) - d(a_m, b_m) \\
&\leq d(a_n, a_m) + d(b_n, b_m) + d(b_m, a_m) - d(a_m, b_m) \\
&= d(a_n, a_m) + d(b_n, b_m) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Therefore $(d(a_n, b_n))$ is a Cauchy sequence in $\mathbb{R}$ and since $\mathbb{R}$ is complete, $(d(a_n, b_n))$ converges. □

6. (20 pts)

(a) Let (f_n) be a sequence of real-valued continuous functions on a metric space X. Suppose that (f_n) converges uniformly to a function f. Prove that if all terms (i.e. all functions) of the sequence (f_n) are bounded, then f is bounded.

Proof. Let X be a metric space and $f_n : X \to \mathbb{R}$ be a sequence of bounded functions that converges uniformly to f. To show that f is bounded we need to show that the range $f(X) = \{f(x) : x \in X\}$ is bounded in $\mathbb{R}$, and to do so we will use the characterization given in Corollary 3.1.14. Explicitly, we will show that there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all elements $f(x) \in f(X)$. We will need to use the uniform convergence assumption, so let $\epsilon = 64$. Since (f_n) converges uniformly to f, we can choose $N \in \mathbb{N}$ so that for all $n > N$ and all $x \in X$, $|f_n(x) - f(x)| < 64$. Now fix an $n > N$. Since f_n is bounded, by Corollary 3.1.14 there exists $K \in \mathbb{R}$ such that $|f_n(x)| \leq K$ for all $x \in X$. Then for any $x \in X$,

$$\begin{aligned}
|f(x)| = |f(x) - 0| &\leq |f(x) - f_n(x)| + |f_n(x) - 0| \\
&< 64 + |f_n(x)| \\
&\leq 64 + K.
\end{aligned}$$

Therefore $|f(x)| \leq 64 + K$ for all $x \in X$, so $f(X)$ is bounded in $\mathbb{R}$. By definition, f is bounded. □

(b) Recall that a function $f : X \to Y$ is said to be **bounded** if its range $f(X)$ is a bounded subset of Y. Give an example to show that the pointwise limit of real-valued bounded functions on a metric space X need not be a bounded function. Hint: consider piecewise defined functions on $\mathbb{R}$.

Example: Consider $\mathbb{N}$ under the Euclidean metric and the sequence of functions $f_n : \mathbb{N} \to \mathbb{R}$ defined by

$$f_n(m) = \begin{cases} m & \text{if } m \leq n \\ 0 & \text{if } m > n. \end{cases}$$

Now each f_n is bounded by n:

$$\begin{aligned} f_n(\mathbb{N}) &= \{1, 2, 3, \ldots, n, 0, 0, 0, ...\} \\ &= \{0, 1, 2, \ldots, n\}. \end{aligned}$$

However, it is clear that (f_n) converges pointwise to the function $f(m) = m$. To show this rigorously, let $m \in \mathbb{N}$ (as a point in the domain!) and $\epsilon > 0$. Then for all $n > m$ (as a sequence index!),

$$|f_n(m) - f(m)| = |m - m| = 0 < \epsilon.$$

Therefore $f_n(m) \to f(m)$ at the point m, and since m was arbitrary we have shown that (f_n) converges pointwise to f. But obviously $f(\mathbb{N}) = \mathbb{N}$ is unbounded! Therefore uniform convergence is in fact a necessary condition for part (a).

Of course, using the same idea I could have constructed $f_n : \mathbb{R} \to \mathbb{R}$ as

$$f_n(x) = \begin{cases} |x| & \text{if } x \leq n \\ 0 & \text{if } x > n \end{cases}$$

and gotten the same result. This would have even avoided confusion with regards to choosing $m \in \mathbb{N}$ in the domain and shortly thereafter using $N = m$ for the sequence index that we need in showing a sequence converges. However, I like the way I did it because $\mathbb{N}$ is 1-separated under the Euclidean metric and thus has no limit points. So the sequence functions f_n, while piecewise, are actually continuous.