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Math 335
Section 14: 2, 10, 24, 25, 34
1/4/11

I'm going to be a little redundant in answering these problems, because I see a couple of ways to go about them. Let me know if you have any suggestions for how to think about the computational problems. I didn't really have any problem with them, but I don't feel too comfortable with the end of Section 14. It's not any specific part I don't understand, but it's just hard to keep everything in my mind at once (FHT for instance). Again, there are some unstarred problems listed at the end.

Problem 2: Find the order of $(\mathbb{Z}_4 \times \mathbb{Z}_{12})/(\langle 2\rangle \times \langle 2\rangle)$.

Well $\phi : (\mathbb{Z}_4 \times \mathbb{Z}_{12}) \to (\mathbb{Z}_2 \times \mathbb{Z}_2)$ defined by $\phi(x, y) = (x \mod 2, y \mod 2)$ is a homomorphism with $\ker \phi = \langle 2\rangle \times \langle 2\rangle$. Clearly ϕ is onto, so by the Fundamental Homomorphism Theorem,

$$(\mathbb{Z}_4 \times \mathbb{Z}_{12})/(\langle 2\rangle \times \langle 2\rangle) \cong \phi[\mathbb{Z}_4 \times \mathbb{Z}_{12}] = \mathbb{Z}_2 \times \mathbb{Z}_2,$$

which has order 4.

On the other hand, since $|\langle 2\rangle| = 2$ in $\mathbb{Z}_4$ and $|\langle 2\rangle| = 6$ in $\mathbb{Z}_{12}$, we see that the subgroup $\langle 2\rangle \times \langle 2\rangle$ has order $2 \cdot 6 = 12$. Therefore the order of the factor group above, which is the number of cosets of $\langle 2\rangle \times \langle 2\rangle$, is just the index

$$\big(\mathbb{Z}_4 \times \mathbb{Z}_{12} : \langle 2\rangle \times \langle 2\rangle\big) = (4 \cdot 12)/12 = 4.$$

Problem 10: To find the order of $26 + \langle 12\rangle$ in $\mathbb{Z}_{60}/\langle 12\rangle$, we will use the Fundamental Homomorphism Theorem. We construct the homomorphism ϕ of $\mathbb{Z}_{60}$ onto $\mathbb{Z}_{12}$ defined by $\phi(n) = n \mod 12$. The kernel is evidently $\langle 12\rangle$, so

$$\mathbb{Z}_{60}/\langle 12\rangle \cong \phi[\mathbb{Z}_{60}] = \mathbb{Z}_{12}$$

via the group isomorphism μ.[1] Since

$$\mu\big(26 + \langle 12\rangle\big) = \phi(26) = 2$$

has order 6 in $\mathbb{Z}_{12}$, we know that the coset $26 + \langle 12\rangle$ has order 6 in the factor group above.

We can also try to find the least $n \in \mathbb{Z}^+$ such that

$$n(26 + \langle 12\rangle) = n26 + \langle 12\rangle = \langle 12\rangle,$$

[1]This is the function $\mu : G/H \to \phi[G]$ that Fraleigh defines as $\mu(gH) = \phi(g)$ in the Fundamental Homomorphism Theorem.

which occurs when $n26 \in \langle 12 \rangle = \{0, 12, 24, 36, 48\}$, or when $12|n26$. Then the least such positive integer is

$$n = \frac{\operatorname{lcm}(12, 26)}{26} = \frac{156}{26} = 6,$$

as we see that $6(26) = 36 \in \langle 12 \rangle$.

Problem 24: Show that A_n is a normal subgroup of S_n and compute S_n/A_n.

If $n = 1$ then both A_1 and S_1 are trivial, and therefore S_1/A_1 is unquestionably trivial. For $n \geq 2$, Theorem 9.20 states that $A_n \leq S_n$ where $|A_n| = \frac{n!}{2} = \frac{|S_n|}{2}$, and thus $(S_n : A_n) = 2$. From Exercise 10.39 we know A_n is normal. As in Example 13.3, we have a homomorphism $\phi : S_n \to \mathbb{Z}_2$ given by

$$\phi(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ even} \\ 1 & \text{if } \sigma \text{ odd} \end{cases}.$$

Obviously $\ker \phi = A_n$, and from the Fundamental Homomorphism Theorem we have

$$A_n/S_n \cong \phi[S_n] = \mathbb{Z}_2.$$

Problem 25: Show that if $H \leq G$ and left coset multiplication $(aH)(bH) = (ab)H$ is well defined, then $Ha \subseteq aH$.

Proof. Let $x \in Ha$. Then $x = h_1 a$ for some $h_1 \in H$, so $x^{-1} = a^{-1}h_1^{-1} \in a^{-1}H$. Since the cosets partition G, this means $x^{-1}H = a^{-1}H$, so

$$H = eH = (a^{-1}a)H = (a^{-1}H)(aH) = (x^{-1}H)(aH) = (x^{-1}a)H.$$

It must be the case that $x^{-1}a \in H$ and because H is a group, $(x^{-1}a)^{-1} = a^{-1}x = h_2$ for some $h_2 \in H$. Hence $x = ah_2$ and $x \in aH$. $\square$

Problem 34: If G is finite and H is the only subgroup of a given order $m \in \mathbb{Z}^+$, then H is normal.

Proof. As above, suppose that G is finite and H is the only subgroup of order m. Since $i_g : G \to G$ is a homomorphism, it follows from Theorem 13.12.3 that $i_g[H]$ is a subgroup of G. Noting that i_g is one-to-one and clearly $i_g : H \to i_g[H]$ is onto, $i_g[H]$ also has order m and is thus equal to H for all $g \in G$. Therefore H is invariant and by Theorem 14.13, H is normal. $\square$

Questions from the Unstarred Problems

Problem 1: Find the order of $\mathbb{Z}_6/\langle 3\rangle$.

First note that the homomorphism $\phi : \mathbb{Z}_6 \to \mathbb{Z}_3$ given by $\phi(n) = n \bmod 3$ has kernel $\{0, 3\} = \langle 3\rangle$. Since ϕ is onto, by the Fundamental Homomorphism Theorem we have

$$\mathbb{Z}_6/\langle 3\rangle \cong \phi[\mathbb{Z}_6] = \mathbb{Z}_3,$$

and therefore this factor group has order 3.

On the other hand, since $|\langle 3\rangle| = 2$, each coset has order 2 and the number of distinct cosets is exactly the index $(\mathbb{Z}_6 : \langle 3\rangle) = 6/2 = 3$.

Problem 13: Find the order of $(3, 1) + \langle(0, 2)\rangle$ in $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(0, 2)\rangle$.

We need the least $n \in \mathbb{Z}^+$ such that $n(3, 1) \in \langle(0, 2)\rangle$. We see that $3n \equiv_4 0$ when $4|n$ and $1n \equiv_2 0$ when $2|n$, so the order $n = 4$ and

$$4[(3, 1) + \langle(0, 2)\rangle] = (0, 4) + \langle(0, 2)\rangle = \langle(0, 2)\rangle.$$

Problem 14:[2] Find the order of $(3, 3) + \langle(1, 2)\rangle$ in $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1, 2)\rangle$.

First note that since $|\mathbb{Z}_4 \times \mathbb{Z}_8| = 4 \cdot 8 = 32$ and $|\langle(1, 2)\rangle| = 4$, the factor group has order

$$\big|(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1, 2)\rangle\big| = \big((\mathbb{Z}_4 \times \mathbb{Z}_8) : \langle(1, 2)\rangle\big) = 32/4 = 8.$$

By Theorem 10.12, we know that the order of the coset $(3, 3) + \langle(1, 2)\rangle$ is a divisor of 8. We can check in increasing order to find that

$$\begin{aligned} 1(3, 3) &= (3, 3) \notin \langle(1, 2)\rangle \\ 2(3, 3) &= (2, 6) \notin \langle(1, 2)\rangle \\ 4(3, 3) &= (0, 4) \notin \langle(1, 2)\rangle, \end{aligned}$$

and from here we can conclude by process of elimination that the order is 8.

Problem 15: Find the order of $(2, 0) + \langle(4, 4)\rangle$ in $(\mathbb{Z}_6 \times \mathbb{Z}_8)/\langle(4, 4)\rangle$.

Noting that $(2, 0) \in \langle(4, 4)\rangle$, we see that the order is 1.

Problem 21(a): The teacher expects to find nonsense because although it is not incorrect to call a and b elements of G/H, it will be much harder to write clearly with those definitions. Instead, since the elements of G/H are the cosets of H, it is better to write $aH, bH \in G/H$ where $a, b \in G$.

(b) If H is a normal subgroup of an abelian group G, then G/H is abelian.

[2] Do you know a faster why to solve these than tediously computing and checking $(aH)^n$ for all divisors $n|(G : H)$ to find the order of aH? I like trying to find homomorphisms with kernel H, but it is a bit difficult sometimes- is that what I should be trying to do?

Proof. By Theorem 14.9, there exists the homomorphism $\gamma : G \to G/H$ defined by $\gamma(x) = xH$. This homomorphism is certainly onto, and by Example 13.2, G/H is also abelian. To reassure ourselves, let $aH, bH \in G/H$. Then

$$(aH)(bH) = (ab)H = (ba)H = (bH)(aH).$$

$\square$

Problem 23:

(a) True

(b) True

(c) True

(d) True

(e) True

(f) False: $\mathbb{Z}$ is torsion free and the factor group $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$, which is not torsion free.

(g) True

(h) False: For any nonabelian group G, the improper yet normal subgroup forms the factor group G/G which is trivially abelian.

(i) True

(j) False: $\{nr : r \in \mathbb{R}\} = \mathbb{R}$ for all n. Thus $\mathbb{R}/n\mathbb{R}$ is cyclic but of order 1 for all n.

Problem 38: Let $h = \{h \in G : hgh^{-1} = g \text{ for all } x \in G\}$. For $h_1, h_2 \in H$,

$$(h_1h_2)g(h_1h_2)^{-1} = (h_1h_2)g(h_2^{-1}h_1^{-1}) = h_1(h_2gh_2^{-1})h_1^{-1} = h_1gh_1^{-1} = g,$$

so H is closed. Clearly $e \in H$ and for any $h \in H$, $h^{-1}xh = (hx^{-1}h^{-1})^{-1} = (x^{-1})^{-1} = x$, so $h^{-1} \in H$. We have shown that H is a subgroup; to see normality, let $g \in G$ and $h \in H$. Then

$$\begin{aligned}
ghg^{-1} &= (hgh^{-1})h(hgh^{-1})^{-1} \\
&= (hgh^{-1})h(h^{-1}g^{-1}h) \\
&= hg(h^{-1}h)h^{-1}g^{-1}h \\
&= hgh^{-1}g^{-1}h = (hgh^{-1})g^{-1}h \\
&= gg^{-1}h = h
\end{aligned}$$

and thus $ghg^{-1} \in H$. By Theorem 14.13, H is normal.