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Math 335
Section 11: 2, 8, 16, 25, 29, 50, 51
12/23/11

Problem 2:
We have

$$\mathbb{Z}_3 \times \mathbb{Z}_4 = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,0),(2,1),(2,2),(2,3)\}$$

and this group is cyclic because $\gcd(3,4) = 1$ and we see that $\langle(1,1)\rangle = \mathbb{Z}_3 \times \mathbb{Z}_4$.

Problem 8:
The cyclic subgroups of $\mathbb{Z}_6 \times \mathbb{Z}_8$ are generated by some (a_1, a_2) where $a_1 \in \mathbb{Z}_6$ and $a_2 \in \mathbb{Z}_8$. The order $|(a_1, a_2)| = \operatorname{lcm}(r_1, r_2)$ where $r_1 = |a_1|$ in $\mathbb{Z}_6$ and $r_2 = |a_2|$ in $\mathbb{Z}_8$. By the Theorem of Lagrange, we know that the orders $r_1|6$ and $r_2|8$. Clearly then, their least common multiple is maximized when $r_1 = 6$ and $r_2 = 8$, which includes all factors of 6 and 8. Thus the largest order of an element in $\mathbb{Z}_6 \times \mathbb{Z}_8$ is $\operatorname{lcm}(6,8) = 24$.

With similar reasoning, the largest order of $\mathbb{Z}_{12} \times \mathbb{Z}_{15}$ is $\operatorname{lcm}(12,15) = 60$.

Problem 16:
Yes, the groups $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\mathbb{Z}_4 \times \mathbb{Z}_6$ are isomorphic. From Theorem 11.5 we know that since $\gcd(2,3) = 1$ and $\gcd(3,4) = 1$,

$$\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \qquad \text{and} \qquad \mathbb{Z}_{12} \cong \mathbb{Z}_3 \times \mathbb{Z}_4,$$

so we have

$$\mathbb{Z}_4 \times \mathbb{Z}_6 \;\cong\; \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \;\cong\; \mathbb{Z}_2 \times \mathbb{Z}_{12}.$$

Problem 25:
Note that $1089 = 3^2 \cdot 11^2$. Keeping Theorem 11.5 in mind, we see that there are two ways to combine each factor 3 and 11 such that there are $2 \cdot 2 = 4$ abelian groups of order 1089, up to isomorphism. The possible abelian groups are

1. $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{11} \times \mathbb{Z}_{11}$
2. $\mathbb{Z}_9 \times \mathbb{Z}_{11} \times \mathbb{Z}_{11}$
3. $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{121}$
4. $\mathbb{Z}_9 \times \mathbb{Z}_{121}$

Problem 29 (a) :
Let p be a prime number. By Theorem 11.5, we can infer that the number of abelian groups, up to isomorphism, of order p^n is exactly the number of partitions $p(n)$ of n elements. This is summarized in the table below.

n	2	3	4	5	6	7	8
number of groups	2	3	5	7	11	15	22

(b) Referring to the table above, if p, q, and r are distinct primes then we can combine the factors such that if a group G has order $p^n q^m r^l$, there are exactly $p(n)p(m)p(l)$ possible structures up to isomorphism.

(i) If an abelian group has order $p^3 q^4 r^7$, then there are $3 \cdot 5 \cdot 15 = 225$ possibilities.

(ii) If an abelian group has order $(qr)^7 = q^7 r^7$, then there are $15 \cdot 15 = 225$ possibilities.

(iii) If an abelian group has order $q^5 r^4 q^3 = q^8 r^4$, then there are $22 \cdot 5 = 110$ possibilities.

Problem 50:
Let H and K be groups and let $G = H \times K$. Let e_H, e_K be the identities of H, K respectively, and let us denote the subgroups $H \times \{e_K\} \leq G$ and $\{e_H\} \times K \leq G$ by H_G and K_G respectively.

(a) Every element of G is of the form hk for some $h \in H_G$ and $k \in K_G$.

Proof. Suppose $x \in G$. Then $x \in H \times K$ which means $x = (h', k')$ for $h' \in H$ and $k' \in K$. Clearly since $h' = h'e_H$ and $k' = e_K k'$, we have $x = (h'e_H, e_K k') = (h', e_K)(e_H, k')$ where $(h', e_K) \in H_G$ and $(e_H, k') \in K_G$. Since x was arbitrary, we conclude that all elements of G are of the form hk for some $h \in H_G$ and $k \in K_G$. $\square$

(b) $hk = kh$ for all $h \in H_G$ and $k \in K_G$.

Proof. Let $h \in H_G$ and $k \in K_G$. Then $h = (h', e_K)$ and $k = (e_H, k')$ for some $h' \in H$ and $k' \in K$. Since the identity element of a group commutes with all elements of the group,

$$hk = (h', e_K)(e_H, k') = (h'e_H, e_K k') = (e_H h', k' e_K) = (e_H, k')(h', e_K) = kh.$$

$\square$

(c) $H_G \cap K_G = \{(e_H, e_K)\}$.

Proof. Suppose that $x \in H_G \cap K_G$. Then $x \in H_G$ and $x \in K_G$, which means $x = (h, e_K)$ and $x = (e_H, k)$ for some $h \in H$ and $k \in K$. It follows immediately that $h = e_H$ and $k = e_K$, such that $x = (e_H, e_K)$. Therefore this is the only element in the intersection, such that $H_G \cap K_G = \{(e_H, e_K)\}$. $\square$

Problem 51:
Suppose H and K are subgroups of G satisfying the properties **(a)**, **(b)**, and **(c)** above. Then for every $g \in G$, the expression $g = hk$ for $h \in H$ and $k \in K$ is unique. Further, if we rename g as (h, k) then G is isomorphic to $H \times K$.

Proof. First, we know that if property **(a)** is satisfied then every element $g \in G$ is of the form hk for some $h \in H$ and $k \in K$. Note that the identity is expressed uniquely in this form: if we write $e = hk$, it follows that $h = k^{-1}$. Then since K is a group, it must be the case that $h \in K$. However we know that $H \cap K = \{e\}$ from property **(c)**, so $h = e$ and therefore $k = e$ as well. Now to see that the rest of the elements in g are expressed uniquely as hk, suppose g is expressed as both $h_1k_1 = h_2k_2$ for $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Then

$$e = gg^{-1} = (h_1k_1)(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1},$$

and from property **(b)** we have that

$$h_1k_1k_2^{-1}h_2^{-1} = h_1k_1h_2^{-1}k_2^{-1} = h_1h_2^{-1}k_1k_2^{-1}.$$

Now since H and K are closed, $h_1h_2^{-1} \in H$ and $k_1k_2^{-1} \in K$, so again the identity is being expressed as the product of an element of H and of K. As shown above, it must be the case that $h_1h_2^{-1} = e$ and $k_1k_2^{-1} = e$. Cancellation on the right shows that $h_1 = h_2$ and $k_1 = k_2$, so the expression $g = hk$ is unique for all $g \in G$.

Now we will show that G is isomorphic to $H \times K$. Since each $g \in G$ can be expressed uniquely as $g = hk$, let us define the function $\varphi : G \to H \times K$ by $\varphi(g) = \varphi(hk) = (h, k)$. We know this function is bijective from the assertions above. If $g_1 = h_1k_1$ and $g_2 = h_2k_2$ such that $\varphi(g_1) = \varphi(g_2)$, then $(h_1, k_1) = (h_2, k_2)$ which means $h_1 = h_2$ and $k_1 = k_2$, and we know those expressions are unique so $g_1 = g_2$. Also, since G is closed we know that φ is onto $H \times K$; for any $(h, k) \in H \times K$, we know $h, k \in G$ so the product $hk \in G$ and $\varphi(hk) = (h, k)$. To see the homomorphism property, we let $g_1, g_2 \in G$ where $g_1 = h_1k_1$ and $g_2 = h_2k_2$. Then

$$\varphi(g_1g_2) = \varphi(h_1k_1h_2k_2),$$

and again from property **(b)** (and associativity) we can rearrange the product $k_1h_2 = h_2k_1$ such that

$$\varphi(h_1k_1h_2k_2) = \varphi(h_1h_2k_1k_2) = (h_1h_2, k_1k_2) = (h_1, k_1)(h_2, k_2) = \varphi(g_1)\varphi(g_2).$$

Therefore if G has subgroups H and K satisfying properties **(a)**,**(b)**, and **(c)** then $G \cong H \times K$. $\square$