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Math 335
Questions on S15
1/5/11

Theorem 15.18: M is a maximal normal subgroup of G if and only if G/M is simple.

Proof. ($\Rightarrow$) His forward direction is pretty straightforward. Let M be a maximal normal subgroup of G and X be a nontrivial proper normal subgroup of G/M. We know first that $\gamma^{-1}[X]$ must contain M, because if $m \in M$, then $\gamma(m) = mM = M$, which is the identity of G/M and therefore is in the subgroup X. Since X is nontrivial, it also contains an element $aM = \gamma(a)$ where $a \notin M$ and therefore $M \subset \gamma^{-1}[X]$. We also know $\gamma^{-1}[X] \subset G$ because since X is a proper subgroup of G/M, there exists an element $gM = \gamma(g) \notin X$, such that $g \notin \gamma^{-1}[X]$ where $g \in G$. Thus $M \subset \gamma^{-1}[X] \subset G$, and since $\gamma^{-1}[X]$ is normal by Theorem 15.16, this contradicts M being maximal. Therefore there is no nontrivial proper normal subgroups of G/M, which must then be simple.

($\Leftarrow$) On the other hand, I have trouble understanding his proof of the converse. If N is a normal subgroup of G properly containing M, then $\gamma[N]$ is normal in G/M by Theorem 15.16. He then claims that if $N \neq G$, then

$$\gamma[N] \neq G/M \qquad \text{and} \qquad \gamma[N] \neq \{M\}.$$

The second statement is clear from the reasoning in the the forward direction: since N properly contains M, there exists $n \in N \backslash M$ such that $\gamma(n) = nM \neq M$, so $\{M\} \subset \gamma[N]$. However I do not see how the first statement follows; just because there exists $g \in G \setminus N$ doesn't mean that $\gamma(g) \notin \gamma[N]$, because we can easily have $gM = nM$ for some $n \in N$ can't we? In other words, how do we know that a proper subgroup N of G can't generate the cosets of M such that

$$\{nM : n \in N\} = G/M.$$

It only takes $(G : M)$ elements of N to do it, so... am I missing something? $\square$

Theorem 15.20: Let G be a group. The set of all commutators $aba^{-1}b^{-1}$ for $a, b \in G$ generates a subgroup C (the commutator subgroup) of G. This subgroup C is a normal subgroup of G. Furthermore, if N is a normal subgroup of G, then G/N is abelian if and only if $C \leq N$.

Proof. "The commutators certainly generate a subgroup C." Well, I'd argue it's not so certain. I have trouble showing that C is closed. If we let two elements of C be denoted by $aba^{-1}b^{-1}$ and $cdc^{-1}d^{-1}$ for $a, b, c, d \in G$, their product

$$aba^{-1}b^{-1}cdc^{-1}d^{-1}$$

is not easily reduced into the form $wxw^{-1}x^{-1}$. Or maybe it is and I don't see it. But certainly "certainly" is a strong word here.

Luckily, as Fraleigh says "the rest of the theorem is obvious if we have acquired the proper feeling for factor groups," I felt that way too. If $C \leq N$ then we know that *at least* the commutators are sent to the identity in the factor group, which allows for at the very least commutativity between all of the elements. $\square$

I'm going to go ahead and start the homework for this section; while these uncertainties bother me, I'm sure it won't interfere too much with the problems.