Part 1

Exercise 28.3. Let X be limit point compact.

(a) If $f : X \to Y$ is continuous, does it follow that $f(X)$ is limit point compact?

Solution. No. As seen in Example 28.1, the space $X = \mathbb{Z}_+ \times \{0, 1\}$ with $\{0, 1\}$ indiscrete is limit point compact. However the projection $\pi_1 : X \to \mathbb{Z}_+$ is continuous, while the image $f(X) = \mathbb{Z}_+$ is not limit point compact because it is discrete. □

(b) If A is a closed subset of X, does it follow that A is limit point compact?

Solution. Yes. Suppose $B \subset A$ is infinite. Then B has a limit point $x \in X$. Since A is closed, $x \in \overline{B} \subset \overline{A} = A$, so B has a limit point in A. Therefore A is limit point compact. □

(c) If X is a subspace of the Hausdorff space Z, does it follow that X is closed in Z?

Solution. No. As seen in Example 28.2, S_Ω is limit point compact, and it is a subspace of the Hausdorff space $\overline{S}_\Omega$, but of course S_Ω is not closed in $\overline{S}_\Omega$, particularly because it does not contain the limit point Ω. □

Exercise 28.4. A space X is said to be *countably compact* if every countable open covering of X has a finite subcollection that covers X. Show that for a T_1 space X, countable compactness is equivalent to limit point compactness.

Proof. Suppose X is T1.

($\Longrightarrow$) Suppose X is not limit point compact and let A be an infinite subset of X with no limit points. Let B be a countably infinite subset of A; notice B still has no limit points, otherwise they would be limit points of A. Thus B is closed and $X - B$ is open. Since B has no limit points, for each $x_i \in B$ we can find a neighborhood U_i of x_i that does not intersect $B - \{x_i\}$. Then

$$\mathcal{A} = (X - B) \cup \{U_i\}_{x_i \in B}$$

is a countable open covering of X with no finite subcover. This is because a finite subcollection of $\mathcal{A}$ would only have finitely many U_i's and $B \cap U_i = \{x_i\}$, hence the subcollection would only contain finitely many elements from B. Therefore X is not countably compact.

($\Longleftarrow$) Suppose X is not countably compact, so that there exists a countable open covering $\mathcal{C} = \{U_n\}_{n \in \mathbb{Z}_+}$ with no finite subcover. Then for any n, $\cup_{i=1}^n U_i \neq X$. Furthermore, if the difference were finite, so that $X - \cup_{i=1}^n U_i = \{x_1, \ldots, x_m\}$, then we could find $U_{x_i} \in \mathcal{C}$

so $x_i \in U_{x_i}$ and $X = (\cup_{i=1}^n U_i) \cup (\cup_{i=1}^m U_{x_i})$, which contradicts $\mathcal{C}$ lacking a finite subcover. Hence $X - \cup_{i=1}^n U_i$ is infinite for any $n \in \mathbb{Z}_+$, so we can construct a sequence

$$\begin{aligned} &a_1 \in X - U_1 \\ &a_2 \in X - (U_1 \cup U_2) - \{a_1\} \\ &\quad\vdots \\ &a_n \in X - (U_1 \cup \cdots \cup U_n) - \{a_1, \ldots, a_{n-1}\} \end{aligned}$$

of distinct terms. Then the set $A = \{a_n\}_{n \in \mathbb{Z}_+}$ is infinite. Let $x \in X$. Let n be the least integer such that $x \in U_n$. By construction, $a_m \notin U_n$ for any $m \geq n$, so A intersects U_n in at most a finite number of places. By Theorem 17.9, x is not a limit point of A. Since x was arbitrary, we conclude A has no limit points. Therefore X is not limit point compact. □

Exercise 29.3. Let X be a locally compact space. If $f : X \to Y$ is continuous, does it follow that $f(X)$ is locally compact? What if f is both continuous and open?

Solution. No, continuity alone is not enough to preserve local compactness. First note that every infinite discrete space X is locally compact but not compact; it is locally compact because for any $x \in X$, $x \in \{x\}$ where $\{x\}$ is both open and compact (finite spaces are always compact), but it is not compact because $\{\{x\} : x \in X\}$ is a covering with no finite subcover. To construct a counter example we look for a space X is that is locally compact, but not compact, and a function f which is continuous but not open. Let's choose $f : \mathbb{R}^\omega \to \mathbb{R}^\omega$ as the identity function which maps from the discrete topology to the product topology. Since f is the identity, we have $f(\mathbb{R}^\omega) = \mathbb{R}^\omega$, and since the domain is discrete, we have f is continuous, and by the argument above, the domain is locally compact. However, as seen in Example 29.2, the image $\mathbb{R}^\omega$ is not locally compact in the product topology.

If f is both continuous and open, then $f(X)$ is locally compact. For any $f(x) \in f(X)$, there exists U open and C compact in X so that $x \in U \subset C$, so $f(x) \in f(U) \subset f(C)$. Since f is open, $f(U)$ is open, and since f is continuous, $f(C)$ is compact. Hence $f(X)$ is locally compact. □

Exercise 29.4. Show that $[0,1]^\omega$ is not locally compact in the uniform topology.

Proof. Notice $\mathbf{0} \in [0,1]^\omega$ but for any neighborhood U of $\mathbf{0}$, there is a basis element $\mathbf{0} \in B(\mathbf{0}, \epsilon) \subset U$ and

$$\overline{B} = \overline{B(\mathbf{0}, \epsilon)} = \overline{\Pi[0, \epsilon)} = \Pi\overline{[0, \epsilon)} = \Pi[0, \epsilon].$$

But if there were a compact set C containing U, then $\overline{B}$ would have to be compact. To see why it is not, consider the covering $\{B(\mathbf{x}, \frac{\epsilon}{2}) : \mathbf{x} \in \overline{B}\}$. Notice $\overline{B}$ contains an infinite number of sequences consisting of 0's and ϵ's, each of which is a distance of ϵ from the others. Thus the covering above which consists of balls of radius $\frac{\epsilon}{2}$ must contain different balls for each of the sequences, and thus cannot have a finite subcover. □

Exercise 30.5(b). Show that every metrizable Lindelöf space has a countable basis.

Proof. Let $\mathcal{A}_n = \{B(x, \frac{1}{n}) : x \in X\}$. For every n the collection $\mathcal{A}_n$ covers X and thus has a countable subcover $\mathcal{A}'_n$. Now let $\mathcal{B} = \cup \mathcal{A}'_n$, which is also countable. To show $\mathcal{B}$ is a basis for X, first note that it consists of open sets, and let U be open with $x \in U$. Pick n big enough that $B(x, \frac{1}{n}) \subset U$. Since $\mathcal{A}'_{2n}$ covers X, there exists $B(y, \frac{1}{2n}) \in \mathcal{B}$ containing x. Also for any $z \in B(y, \frac{1}{2n})$,

$$d(z,x) \leq d(z,y) + d(y,x) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n},$$

hence $x \in B(y, \frac{1}{2n}) \subset B(x, \frac{1}{n}) \subset U$. Therefore $\mathcal{B}$ is a basis. □

Exercise 30.8. Which of the four countability axioms does $\mathbb{R}^\omega$ in the uniform topology satisfy?

Solution. Since the uniform topology is metrizable, we know immediately that it is first-countable. Also the other three axioms are equivalent in a metrizable space, so we can just consider one of them; we will show that the uniform topology is not second-countable.

Let $\mathcal{B}$ be a basis for $\mathbb{R}^\omega$, and let $A \subset \mathbb{R}^\omega$ be the set of all sequences of zeroes and ones. Then A is uncountable and every element in A is a distance of 1 from every other element in A. Thus for each $a \in A$ and open set $B(a, \frac{1}{2})$, there must be a basis element $B_a \in \mathcal{B}$ such that $x \in B_a \subset B(a, \frac{1}{2})$. Given the distance constraints, notice that when $a \neq b$, $B_a \neq B_b$. Thus $\mathcal{B}$ must contain an an uncountable number of basis elements, and we conclude the uniform topology is not second-countable. □

Exercise 30.9. Let A be a closed subset of X. Show that if X is Lindelöf, then A is Lindelöf. Show by example that if X has a countable dense subset, A need not have a countable dense subset.

Proof. Let $\mathcal{A}$ be an open covering of A. Then $X - A$ is open, so $\mathcal{A} \cup \{X - A\}$ is an open covering of X. Since X is Lindelöf there exists a countable subcover $\mathcal{A}' \subset \mathcal{A} \cup \{X - A\}$. Then of course $\mathcal{A} \cap \mathcal{A}'$ is a countable subcover of A. Therefore A is Lindelöf.

The set $I^2 = [0,1] \times [0,1]$ in dictionary order has a countable dense subset $\mathbb{Q}^2 \cap I^2$, but the subspace $I \times \{\frac{1}{2}\}$ is discrete and uncountable, so it cannot have a countable dense subset. □

Part 2

We say that a function $f : X \to Y$ is *sequentially continuous* if for every convergent sequence (x_n) in X, $\lim f(x_n) = f(\lim x_n)$. Every continuous function is automatically sequentially continuous and, if X is first-countable, all sequentially continuous functions are continuous.

(a) S_Ω has a smallest element; call it 0. Define a function $f : \overline{S}_\Omega \to S_\Omega$ by $f(x) = x$ for $x \in S_\Omega$ and $f(\Omega) = 0$. Show that f is sequentially continuous, but not continuous.

Proof. Suppose (x_n) is convergent in $\overline{S}_\Omega$. As explained in Example 28.3, $\lim x_n \neq \Omega$ because $\{x_n\}$ is countable and must have an upper bound in S_Ω. Thus we can choose a subsequence (y_n) of (x_n) that simply removes any elements equal to Ω. Then $(f(y_n))$ is a subsequence of $(f(x_n))$ and

$$\lim f(x_n) = \lim f(y_n) = \lim y_n = \lim x_n = f(\lim x_n).$$

Of course f is not continuous because $\{0\}$ is open in S_Ω but $f^{-1}(\{0\}) = \{\Omega\}$ is not open in $\overline{S}_\Omega$. □

(b) Show that if X is sequentially compact and $f : X \to Y$ is sequentially continuous, then $f(X)$ is sequentially compact.

Proof. Let (y_n) be a sequence in $f(X)$. For each y_n pick $x_n \in X$ so that $f(x_n) = y_n$. Now (x_n) has a convergent subsequence (x_{n_i}) and $\lim f(x_{n_i}) = f(\lim x_{n_i})$, so $(f(x_{n_i}))$ is convergent as well, where $(f(x_{n_i}))$ is a subsequence of (y_n). Therefore $f(X)$ is sequentially compact. □

(c) If X is compact and $f : X \to Y$ is sequentially continuous, then is $f(X)$ necessarily compact?

Proof. Evidently not; as seen in part (a), $\overline{S}_\Omega$ is compact, $f : \overline{S}_\Omega \to S_\Omega$ is sequentially continuous, but $f(X) = S_\Omega$ is not compact. □