# Part 1

**Exercise 13.5.** Show that if $\mathcal{A}$ is a basis for a topology on $X$, then the topology generated by $A$ equals the intersection of all topologies on $X$ that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

*Proof.* Let $\mathcal{T}$ be the topology generated by $\mathcal{A}$ and $\{\mathcal{T}_\beta\}_{\beta \in H}$ be the collection of all topologies on $X$ that contain $\mathcal{A}$.

($\subset$) Obviously $\mathcal{A} \subset \mathcal{T}$ so $\mathcal{T} = \mathcal{T}_\beta$ for some $\beta \in H$ and it must be the case that $\cap_{\beta \in H} \mathcal{T}_\beta \subset \mathcal{T}$.

($\supset$) Conversely suppose $U \in \mathcal{T}$. By Lemma 13.1, $U$ is a union of basis elements in $\mathcal{A}$. But for any $\beta \in H$ we know $\mathcal{T}_\beta$ contains $\mathcal{A}$ and is closed under unions. Hence $U \in \cap_{\beta \in H} \mathcal{T}_\beta$ and we conclude $\mathcal{T} \subset \cap_{\beta \in H} \mathcal{T}_\beta$.

Therefore the topology $\mathcal{T}$ generated by $\mathcal{A}$ equals the intersection of all topologies on $X$ that contain $\mathcal{A}$. □

*Proof.* Suppose $\mathcal{A}$ is a subbasis for a topology $\mathcal{T}$ on $X$. Let $\{\mathcal{T}_\beta\}_{\beta \in H}$ be the collection of topologies on $X$ that contain $\mathcal{A}$.

($\subset$) Just as in the preceding proof, note that since $\mathcal{T}$ contains $\mathcal{A}$, $\mathcal{T} = \mathcal{T}_\beta$ for some $\beta \in H$, and therefore $\cap_{\beta \in H} \mathcal{T}_\beta \subset \mathcal{T}$.

($\supset$) Conversely suppose $U \in \mathcal{T}$. Then $U$ is a union of finite intersections of elements of $\mathcal{A}$. But for any $\beta \in H$ we know $\mathcal{T}_\beta$ contains $\mathcal{A}$ and is closed under finite intersections and arbitrary unions. Hence $U \in \cap_{\beta \in H} \mathcal{T}_\beta$ and we conclude $\mathcal{T} \subset \cap_{\beta \in H} \mathcal{T}_\beta$.

Therefore the topology $\mathcal{T}$ generated by the subbasis $\mathcal{A}$ equals the intersection of all topologies on $X$ that contain $\mathcal{A}$. □

**Exercise 13.7.** Consider the following topologies on $\mathbb{R}$:

$$
\begin{aligned}
\mathcal{T}_1 &= \text{ the standard topology} \\
\mathcal{T}_2 &= \text{ the topology of } \mathbb{R}_K \\
\mathcal{T}_3 &= \text{ the finite complement topology} \\
\mathcal{T}_4 &= \text{ the upper limit topology, having all sets } (a, b] \text{ as basis} \\
\mathcal{T}_5 &= \text{ the topology having all sets } (-\infty, a) = \{x : x < a\} \text{ as basis.}
\end{aligned}
$$

Determine, for each of these topologies, which of the others it contains.

**(1)** $\mathcal{T}_1 \supset \mathcal{T}_3, \mathcal{T}_5$

**(2)** $\mathcal{T}_2 \supset \mathcal{T}_1, \mathcal{T}_3, \mathcal{T}_5$

**(3)** $\mathcal{T}_3$ does not contain any of the others

**(4)** $\mathcal{T}_4 \supset \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_5$

**(5)** $\mathcal{T}_5$ does not contain any of the others

Most of these relations are very straightforward to verify. One that I feel needs justification is $\mathcal{T}_2 \subset \mathcal{T}_4$. Consider an arbitrary basis element of $\mathcal{T}_2$ which has the form $(a, b)$ or $(a, b) - K$. If $x \in (a, b)$ then obviously $x \in (a, x] \subset (a, b)$. If $x \in (a, b) - K$, we need to be more careful because we're not guaranteed that $(a, x] \subset (a, b) - K$. However, in this case we can choose the least $n$ such that $\frac{1}{n} < x$ and pick a real number $c$ such that $\frac{1}{n} < c < x$. Then $x \in (c, x] \subset (a, b) - K$. By Lemma 13.3 we conclude that $\mathcal{T}_2 \subset \mathcal{T}_4$.

**Exercise 13.8 (a).** Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) : a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

*Proof.* Let $\mathcal{T}$ denote the standard topology on $\mathbb{R}$. First note that $\mathcal{B}$ is clearly a collection of open sets in $\mathcal{T}$. Next let $U \in \mathcal{T}$ and $x \in U$. Since

$$\mathcal{C} = \{(a, b) : a < b, a \text{ and } b \text{ real}\}$$

is a basis for $\mathcal{T}$ we know that $U$ is a union of open intervals in $\mathbb{R}$. It follows that $x \in (a, b) \subset U$ for some real numbers $a, b$. Since $\mathbb{Q}$ is dense in $\mathcal{R}$ we can find rational numbers $p, q$ such that

$$a < p < x < q < b$$

so $x \in (p, q) \subset (a, b) \subset U$. By Lemma 13.2, $\mathcal{B}$ is a basis for $\mathcal{T}$. $\square$

**Exercise 13.8 (b).** Show that the collection

$$\mathcal{C} = \{[a, b) : a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

*Proof.* First we will show that $\mathcal{C}$ is indeed a basis for a topology on $\mathbb{R}$. First, for any $x \in \mathbb{R}$, we have $x \in \big[\lfloor x \rfloor, \lceil x \rceil + 1\big) \in \mathcal{C}$. Next suppose that $x \in [a_1, b_1) \cap [a_2, b_2)$ for some $a_1, b_1, a_2, b_2 \in \mathbb{Q}$. Let $c = \max\{a_1, a_2\}$ and $d = \min\{b_1, b_2\}$. Then $[c, d) \in \mathcal{C}$ and

$$x \in [c, d) \subset [a_1, b_1) \cap [a_2, b_2).$$

Therefore $\mathcal{C}$ is indeed a basis for a topology on $\mathbb{R}$.

Consider the interval $[\pi, 4)$ which is an element of the basis $\mathcal{B} = \{[a, b) : a < b, a, b \in \mathbb{R}\}$ of the lower limit topology on $\mathbb{R}$. Notice $\pi \in [\pi, 4)$ and there is no $[a, b) \in \mathcal{C}$ such that $\pi \in [a, b) \subset [\pi, 4)$. By Lemma 13.3 we know that the topology generated by $\mathcal{C}$ does not contain the lower limit topology, and thus must be different than the lower limit topology. $\square$

**Exercise 16.4.** A map $f : X \to Y$ is said to be an open map if for every open set $U$ of $X$, the set $f(U)$ is open in $Y$. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

*Proof.* Let $W$ be open in $X \times Y$. Then $W$ is a union of basis elements

$$W = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha$$

where $U_\alpha$ and $V_\alpha$ are open sets in $X$ and $Y$ respectively. Notice that

$$\begin{aligned} x \times y \in \bigcup_{\alpha \in J} U_\alpha \times V_\alpha &\iff x \times y \in U_\alpha \times V_\alpha \text{ for some } \alpha \in J \\ &\iff x \in U_\alpha \text{ and } y \in V_\alpha \text{ for some } \alpha \in J \\ &\iff x \in \bigcup_{\alpha \in J} U_\alpha \text{ and } y \in \bigcup_{\alpha \in J} V_\alpha \\ &\iff x \times y \in \bigcup_{\alpha \in J} U_\alpha \times \bigcup_{\alpha \in J} V_\alpha \end{aligned}$$

Therefore $W = \cup U_\alpha \times \cup V_\alpha$, so $\pi_1(W) = \cup U_\alpha$ and $\pi_2(W) = \cup V_\alpha$. Since each $U_\alpha$ and $V_\alpha$ is open in $X$ and $Y$ respectively, and these topologies are closed under unions, we see that $\pi_1(W)$ and $\pi_2(W)$ are both open. Therefore $\pi_1$ and $\pi_2$ are open maps. $\square$

## Part 2

Let $X = \{p \in \mathbb{Z}_+ : p \text{ is prime}\} = \{2, 3, 5, 7, 11, \ldots\}$. Given $n \in \mathbb{Z}_+$, let $B_n = \{p \in x : p \text{ is not a divisor of } n\}$. As an example, $B_{42} = \{5, 11, 13, 17, 19, 23, 29, \ldots\} = X - \{2, 3, 7\}$. Let $\mathcal{B} = \{B_n : n \in \mathbb{Z}_+\}$.

**(a)** Prove that $\mathcal{B}$ is a basis for a topology on $X$.

*Proof.* Let $p \in X$. Then $p$ does not divide 1, so $p \in B_1$. We will show that $B_{nm} = B_n \cap B_m$ for any $n, m \in \mathbb{Z}_+$. So let $p \in B_{nm}$, which means that $p \nmid nm$. Then of course $p \nmid n$ and $p \nmid m$, so that $p \in B_n \cap B_m$. Conversely if $p \in B_n \cap B_m$, which means $p \nmid n$ and $p \nmid m$, since $p$ is prime we know that $p \nmid nm$ so that $p \in B_{nm}$. Therefore $B_{nm} = B_n \cap B_m$. Of course, this means that for any $p \in B_n \cap B_m$ we have $p \in B_{nm} \subset B_n \cap B_m$. Therefore $\mathcal{B}$ is a basis. $\square$

**(b)** The topology generated by $\mathcal{B}$ is a familiar topology. Which one is it? Explain.

*Solution.* This is the finite complement topology on $X$. To see why, we can use Lemma 13.2. For any $B_n \in \mathcal{B}$, $n$ has only finitely many prime divisors $p_1, \ldots, p_k$, so that $X - B_n = \{p_1, \ldots, p_k\}$ is finite. Thus $\mathcal{B}$ is indeed a collection of open sets in the finite complement topology. Next let $U$ be an open set in the finite complement topology. Then $X - U$ is either equal to $X$ or finite. If $X - U = X$ then $U$ is empty and it is vacuously true that for any $x \in U$ we have $x \in B \subset U$ for some $B \in \mathcal{B}$. In the more interesting case, suppose $X - U$ is finite. Then $X - U = \{p_1, \ldots, p_k\}$ for some finite number of primes $p_1, \ldots, p_k \in X$. The prime divisors of the integer $n = p_1 \cdots p_k$ are precisely the primes $p_1, \ldots, p_k$. Thus we have

$$B_n = X - \{p_1, \ldots, p_k\} = X - (X - U) = U,$$

which of course implies that for any $x \in U$ we have $x \in B_n \subset U$. Therefore $\mathcal{B}$ is a basis for the finite complement topology. $\square$

## Part 3

Let $X = \{1, 2, 3, \ldots, n\}$. Let $\mathcal{T}$ be a topology on $X$.

**(a)** For each $k \in X$, let $B_k$ be the intersection of all elements of $\mathcal{T}$ which contain $k$. Show that $\mathcal{B} = \{B_k : k \in X\}$ is a basis for $\mathcal{T}$.

*Proof.* We will use Lemma 13.2 yet again. First note that $\mathcal{T} \subset \mathcal{P}(X)$ is finite, so the collection of open sets containing $k$ is finite, and thus each $B_k$ is a finite intersection of open sets. Therefore $\mathcal{B}$ is a collection of open sets. Next let $U \in \mathcal{T}$ and $x \in U$. Then $x \in B_x$ and since $B_x$ is the intersection of all open sets containing $x$, it must be the case that $x \in B_x \subset U$. Therefore $\mathcal{B}$ is a basis for $\mathcal{T}$. $\square$

**(b)** Show that any topology on an $n$-point set must have some basis containing $n$ or fewer open sets. What is the largest possible number of open sets in $\mathcal{T}$? (for a fixed $n$)

*Proof.* Any $n$-point set $Y$ is isomorphic to $X$, so let $f : X \to Y$ be a bijection. Then it follows from part (a) that the collection $\mathcal{C} = \{B_{f(k)} : k \in X\}$ is a basis for the topology on $Y$. Obviously this collection has size $n$ or less (less is possible when $B_j = B_k$ for some $j \neq k$). The largest possible number of open sets is the size of the discrete topology $|\mathcal{P}(X)| = 2^n$. $\square$

**(c)** Find an infinite space $X$ (and choice of topology) for which the process described in part (a) will produce a basis for the topology.

*Solution.* Let $X = \mathbb{Z}_+$ and $\mathcal{T}$ be the order topology. Then we know that $\mathcal{T}$ is the discrete topology which contains all singleton sets. Thus each $B_k = \{k\}$ and $\mathcal{B}$ is a basis for $\mathcal{T}$. $\square$

**(d)** Find an infinite space $X$ (and choice of topology) for which the process described in part (a) will not produce a basis for the topology.

*Solution.* Consider the finite complement from Part 2. When constructing $B_p$, notice that for any prime $q \neq p$ we have $p \in X - \{q\} \in \mathcal{T}$. Therefore

$$B_p \subset \bigcap_{q \neq p} X - \{q\} = X - \bigcup_{q \neq p} \{q\} = \{p\},$$

and it follows that $B_p = \{p\}$. But $B_p$ is not even open in $\mathcal{T}$, since $X - B_p = X - \{p\}$ is neither finite nor equal to $X$. Thus $\mathcal{B}$ does not generate $\mathcal{T}$. $\square$