UNIVERSITY OF LONDON

BSc EXAMINATION 2024

For Internal Students of
Royal Holloway

DO NOT TURN OVER UNTIL TOLD TO BEGIN

CS2900: Multi-dimensional Data Processing
CS2900R: Multi-dimensional Data Processing – for FIRSTSIT/RESIT CANDIDATES

Time Allowed: **TWO hours**

Answer ALL questions
Calculators are NOT permitted

1. The following questions require short definitions of technical terms in the context of vectors and matrices. Component-wise definitions are sufficient. You may assume that $\underline{u}$ and $\underline{v}$ are real valued column vectors in N-dimensions (N is finite) and that $\mathbf{M}$ is a real valued $N \times N$ matrix.

   (a) List three properties of the dot product of two vectors that are true in any finite number of dimensions. [9 marks]

   (b) If $\underline{u} = \mathbf{M}\underline{v}$ then what is is the corresponding expression for the ith component of $\underline{u}$, u_i? [3 marks]

   (c) Define the following.

      i. The length of a vector. [3 marks]

      ii. A vector multiplied by a scalar. [3 marks]

      iii. Diagonal matrix. [3 marks]

      iv. Symmetric matrix. [3 marks]

2. (a) What is the null vector in 6 dimensions? [3 marks]

(b) Compute the dot product of the following vectors.

$$\begin{pmatrix} -\sqrt{2} \\ 1 \\ 0 \\ 1 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ \sqrt{5} \\ -1 \\ 1 \end{pmatrix}.$$

[4 marks]

(c) Show that these vectors are orthogonal to each other.

$$\begin{pmatrix} -3 \\ 0 \\ 1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 6 \\ 1 \\ -4 \end{pmatrix}.$$

[4 marks]

(d) Construct a unit vector $\hat{\underline{n}}$ pointing in the same direction as the following 4 dimensional real-valued vector. [4 marks]

$$\underline{n} = \begin{pmatrix} 1 \\ 2 \\ \sqrt{2} \\ \sqrt{2} \end{pmatrix}$$

(e) Compute the projection of the vector $\underline{p}$, defined below, onto $\hat{\underline{n}}$. [6 marks]

$$\underline{p} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

(f) A ship has travelled from the origin to a point which is represented as the following vector

$$\begin{pmatrix} 5 \\ 4 \end{pmatrix},$$

where

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

represents travel of one kilometre in due-north direction and

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

represents travel of one kilometre in due-east direction.

Compute the length it has travelled along the North-East direction. You must compute this using vector notation. [6 marks]

(g) If

$$\mathbf{M_1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \end{pmatrix},$$

$$\mathbf{M_2} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$\mathbf{M_3} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Compute each the following or demonstrate why it is not possible to do so $\mathbf{M_1 M_2}$, $\mathbf{M_2 M_3}$, $\mathbf{M_1 M_3}$. [9 marks]

3. (a) The following is with respect to the projection of vectors onto embedded sub-spaces.

   i. Demonstrate that the unit vectors that lie in the same direction of $(-1, 1, 1, 1)^\intercal$ and $(1, 1, 1, -1)^\intercal$ can form an orthonormal basis. [2 marks]

   ii. Compute the projection of the vector $(1\ -1\ 2\ 1)^\intercal$ onto the 2-dimensional plane that lies in a 4-dimensional space and passes through the origin and the points $(-1, 1, 1, 1)$ and $(1, 1, 1, -1)$. The projected vector is to be represented in the original 4-dimensional space. Show all the necessary steps to compute this. [8 marks]

(b) What is the rank of the following matrices? Demonstrate your reasoning. [6 marks]

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}$$

(c) A graph has the following adjacency matrix.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

   i. Draw the equivalent graph, giving the vertices labels. [3 marks]

   ii. Using the adjacency matrix $\mathbf{A}$ list all the pairs of vertices that do not have a path of length two between them. Show your reasoning. [5 marks]

4. (a) The following questions are with respect to Singular Value Decomposition (SVD).

The matrix $\mathbf{X}$ can be expressed as the following product of matrices (it is not necessary to explicitly compute this).

$$\mathbf{X} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 20 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

i. Outline how you would show that the above expression for $\mathbf{X}$ is an SVD? [6 marks]

ii. What is the condition number of $\mathbf{X}$? [2 marks]

iii. Compute the pseudo-inverse of $\mathbf{X}$. [8 marks]

END