MAT 1348 – Winter 2024

Exercises 8 – Solutions

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Questions are taken from Discrete Mathematics 8th edition, by Kenneth H. Rosen.

QUESTION 1 **(6.1 # 1)**. At a university, there are 18 math students and 325 computer science students.

(a) In how many ways can we choose two representatives, namely one math student and one computer science student?
(b) In how many ways can we choose one representative? (either a math student or a computer science student).

Solution:

(a) $18 \cdot 325 = 5850$.
(b) $18 + 325 = 343$

QUESTION 2 **(6.1 # 7)**. How many different three letter initials are there?

Solution: $26^3 = 17576$

QUESTION 3 **(6.1 # 11)**. How many sequences of 0's and 1's of length 8 are there?

Solution: For each position in the sequence, we must choose if it's a 0 or a 1 (2 options). By the product principle, the answer is $2^8 = 256$.

QUESTION 4 **(6.1 # 16)**. How many sequences of 4 letters contain the letter x?

Solution: We count the total number of sequences of 4 letters, and we subtract those that do not contain any x. The total number of sequences is 26^4. The number of sequences with no x is 25^4. Therefore, the answer is $26^4 - 25^4 = 66351$

QUESTION 5 **(6.1 # 19)**. An RNA sequence contains the letters A,C,U and G. How many RNA sequences of length 6 have the following properties?

(a) The sequence does not contain any U's.
(b) The sequence ends with GU.
(c) The sequence starts with C.
(d) The sequence contains only A's and U's.

Solution:

(a) 3^6
(b) 4^4
(c) 4^5
(d) 2^6

QUESTION 6 **(6.1 # 25)**. How many sequences of 3 digits

(a) are not composed of the same digit three times?
(b) start with an odd digit?
(c) contains exactly two 4's?

Solution:

(a) We count all the sequences of 3 digits and subtract those that contain the same digit three times. There are 10^3 sequences of 3 digits, and 10 of them are composed of the same digit three times (000, 111,...,999). The answer is therefore $10^3 - 10$.
(b) Choose the first digit, which must be odd: we have 5 options. Choose the second digits, which has no restriction: we have 10 options. Similarly, we have 10 options for the third digit. By the product principle, the answer is $5 \cdot 10 \cdot 10 = 500$.
(c) Choose the number which is not 4: we have 9 options. Once chosen, place it in the sequence: we have 3 options. Place the two 4's in the remaining two positions: there is only one option here. Therefore, the answer is $9 \cdot 3 \cdot 1 = 27$.

QUESTION 7 **(6.1 # 33)**. How many sequences of 8 letters

(a) contain no vowels, and letters may be repeated?
(b) contain no vowels, but letters cannot be repeated?
(c) start with a vowel, and letters may be repeated?
(d) start with a vowel, but letters cannot be repeated?
(e) contains at least one vowel, and letters may be repeated?
(f) contains exactly one vowel, and letters may be repeated?
(g) start with X, contain at least one vowel, and letters may be repeated?
(h) start and end with X, contain at least one vowel, and letters may be repeated?

Solution:

(a) The vowels are a, e, i, o, u, y, so there are 20 consonants. There are therefore 20^8 sequences of 8 letters without vowels.
(b) This is an 8-permutation of the 20 consonants: the answer is therefore $(20)(19)(18)...(13)$.
(c) Choose the first letter, which must be a vowel: we have 6 options. Choose the remaining 7 letters without restriction: we have 26^7 options. The answer is therefore $6 \cdot 26^7$.
(d) Choose the first letter, which must be a vowel: we have 6 options. For the remaining 7 letters, we have a 7-permutation of the 25 letters (we exclude the vowel chosen at the previous step). The answer is therefore $(6)(25)(24)(23)(22)(21)(20)(19)$.
(e) Count the total number of sequences (26^8), and subract those that do not contain any vowels (20^8). The answer is therefore $26^8 - 20^8$.
(f) Choose a vowel: we have 6 options. Choose its position in the sequence: we have 8 options. For the remaining 7 positions, we place consonants: we have 20^7 options. The answer is therefore $(6)(8)(20^7)$.
(g) Count all the sequences with repetition starting with X, and subtract those that start with X and contain vowels. The number of sequences with repetition starting with X is 26^7, and the number of sequences with repetition starting with X and have no vowels is 20^7. The answer is therefore $26^7 - 20^7$.
(h) Count all the sequences with repetitions starting and ending with X, and subtract the sequences with repetitions starting and ending with X and have no vowels. The number of sequences with repetitions starting and ending with X is 26^6, and the number of sequences with repetitions starting and ending with X and have no vowels is 20^6. The answer is therefore $26^6 - 20^6$.

QUESTION 8 **(6.1 # 34)**. How many functions from A to B are there, if $|A| = 10$ and

(a) $|B| = 2$?
(b) $|B| = 3$?
(c) $|B| = 4$?
(d) $|B| = 5$?

Solution:

(a) 2^{10}
(b) 3^{10}
(c) 4^{10}
(d) 5^{10}

QUESTION 9 **(6.1 # 35)**. How many injective functions from A to B are there, if $|A| = 5$ and

(a) $|B| = 4$?
(b) $|B| = 5$?
(c) $|B| = 6$?
(d) $|B| = 7$?

Solution:

(a) 0
(b) $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
(c) $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$
(d) $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$

QUESTION 10 **(6.1 # 37)**. How many functions from $A = \{1, 2, ..., n\}$ to $B = \{0, 1\}$

(a) are injective?
(b) are such that $f(1) = f(n) = 0$?

Solution:

(a) 0 if $n \geq 3$, $2 \cdot 1$ if $n = 2$ and 2 if $n = 1$.
(b) We must choose $f(2), ..., f(n-1)$ (2 options for each), which gives 2^{n-2} if $n \geq 2$. If $n = 1$, there is only one such function.

QUESTION 11 **(6.3 # 1)**. List all the permutations of $\{a, b, c\}$.

Solution: $abc, acb, bac, bca, cab, cba$

QUESTION 12 **(6.3 # 2)**. How many permutations of $\{a, b, c, d, e, f, g\}$ are there?

Solution: $7!$

QUESTION 13 **(6.3 # 3)**. How many permutations of $\{a,b,c,d,e,f,g\}$ end with a?

Solution: We create such a permutation by first creating a permutation of $\{b,c,d,e,f,g\}$ (6! options) and then adding a at its end (1 option). This gives 6!.

QUESTION 14 **(6.3 # 4)**. Let $S = \{1,2,3,4,5\}$

(a) List all 3-permutations of S.
(b) List all 3-combinations of S.

Solution:

(a) $123, 124, 125, 132, 134, 135, 142, 143, 145, 152, 153, 154, 213, 214, 215, 231, 234, 235, 241, 243, 245, 251, 253, 254, 312, 314, 315, 321, 324, 325, 341, 342, 345, 351, 352, 354, 412, 413, 415, 421, 423, 425, 431, 432, 435, 451, 452, 453, 512, 513, 514, 521, 523, 524, 531, 532, 534, 541, 542, 543$
(b) $\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}$

QUESTION 15 **(6.3 # 5)**. Evaluate the following:

(a) $P(6,3)$
(b) $P(6,5)$
(c) $P(8,1)$
(d) $P(8,5)$
(e) $P(8,8)$
(f) $P(10,9)$

Solution:

(a) 120
(b) 720
(c) 8
(d) 6720
(e) 40320
(f) 3628800

QUESTION 16 **(6.3 # 6)**. Evaluate the following:

(a) $C(5,1)$
(b) $C(5,3)$
(c) $C(8,4)$
(d) $C(8,8)$
(e) $C(8,0)$
(f) $C(12,6)$

Solution:

(a) 5
(b) 10
(c) 70
(d) 1
(e) 1
(f) 924

QUESTION 17 **(6.3 # 7)**. Find the number of 5-permutations of a set of 9 elements.

Solution: $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15120$

QUESTION 18 **(6.3 # 8)**. In how many different orders can 5 runners finish a race?

Solution: This is the number of permutations of 5 runners, so the answer is $5! = 120$.

QUESTION 19 **(6.3 # 11)**. How many sequences of 0's and 1's of length 10 contain

(a) exactly four 1's?
(b) at most four 1's?
(c) at least four 1's?
(d) an equal number of 0's and 1's?

Solution:

(a) Choose the positions of the four 1's: there are $C(10,4)$ options. Then, place the 1's in these positions and the 0's in the remaining positions: there are 1 options. The answer is therefore $C(10,4)$.
(b) There are $C(10,0)$ sequences with zero 1's, $C(10,1)$ sequences with one 1, $C(10,2)$ sequences with two 1's, $C(10,3)$ sequences with three 1's and $C(10,4)$ sequences with four 1's. The answer is the sum $C(10,0) + C(10,1) + C(10,2) + C(10,3) + C(10,4)$.
(c) By a similar reasoning, we conclude the answer is $C(10,4)+C(10,5)+C(10,6)+C(10,7)+C(10,8)+C(10,9)+C(10,10)$.
(d) There are five 1's and five 0's. The answer is therefore $C(10,5)$.

QUESTION 20 **(6.3 # 19)**. We flip a coin ten times and note the results (heads or tails). How many results

(a) are there in total?
(b) have exactly two heads?
(c) have three or more tails?
(d) have the same number of heads as tails?

Solution:

(a) Such a result is a sequence of H (heads) and T (tails) of length 10. There are 2^{10} such sequences.
(b) We must choose the position of the two heads in the sequence: $C(10,2)$ options. Place the two H in these positions and T in the others: we have 1 option. The answer is therefore $C(10,2)$.
(c) $C(10,0) + C(10,1) + C(10,2) + C(10,3)$. The term $C(10,n)$ counts the number of sequences of H and T with exactly n T's.
(d) We need five H and five T: the answer is $C(10,5)$.

QUESTION 21 **(6.3 # 21)**. How many permutations of the letters $ABCDEFG$ contain

(a) the word BCD?
(b) the word $CFGA$?
(c) the words BA and GF?
(d) the words ABC and DE?
(e) the words ABC and DEF?
(f) the words CBA and BED?

Solution:

(a) Consider BCD as one symbol. We must count the permutations of the following five symbols: A, BCD, E, F, G. The answer is $5!$.
(b) Consider $CFGA$ as one symbol. We must count the permutations of the following four symbols: $CFGA, B, D, E$. The answer is $4!$.
(c) We must count the permutations of the following five symbols: BA, C, D, E, GF. The answer is therefore $5!$.
(d) We must count the permutations of the following four symbols: ABC, DE, F, G. The answer is therefore $4!$.
(e) We must count the permutations of the following three symbols: ABC, DEF, G. The answer is therefore $3!$.
(f) Such a permutation does not exist: B must be directly followed by A to have the word CBA, but it must also be directly followed by E to have the word BED. The answer is therefore 0.