18. Proof by Induction

Review of Important Concepts and Methods of Counting:

factorial: $\boxed{0! = 1}$ and for $n \geq 1$, $\boxed{n! = n(n-1)! = n(n-1)\cdots(2)(1)}$

 # r-permutations of an n-set: $\boxed{P(n,r) = n(n-1)\cdots(n-r+1) = \dfrac{n!}{(n-r)!}}$ $\boxed{P(n,n) = n!}$

 # r-combinations of an n-set: $\boxed{C(n,r) = \dbinom{n}{r} = \dfrac{P(n,r)}{P(r,r)} = \dfrac{n!}{(n-r)!r!}}$

The Product Rule for k Tasks

The Principle of Inclusion-Exclusion (PIE)

The Sum Rule for k Cases

PROOF BY INDUCTION

The (★★★★★)-Recipe for Mathematical Induction:

★ **Define the Proposition $P(n)$ (that depends on n).**

For $n \in \mathbb{N}$, define a proposition $P(n)$ (which says something involving the number n).

★★ **Basis of Induction (B.I.)**

For an initial **base value** $n_0 \in \mathbb{N}$, prove that $P(n_o)$ is true.

★★★ **Induction Step (I.S.)**

Let $k \geq n_0$ Prove that $P(k) \rightarrow P(k+1)$.

★★★★ **The Induction Hypothesis (I.H.)** Assume $P(k)$ is true.
(I.H. is the 1st step in a direct proof of the I.S.)
Goal: prove that $P(k+1)$ follows from $P(k)$.

★★★★★ **Conclusion** Since $P(n_0)$ is true, and since we proved that $P(k) \rightarrow P(k+1)$ for any $k \geq n_0$, it follows by **Mathematical Induction** that $P(n)$ is true for all $n \geq n_0$.

Example 18.1. Use **Mathematical Induction** to prove that the following formula holds for all integers $n \geq 1$:

$$\boxed{1 + 2 + \cdots + n = \frac{n(n+1)}{2}}$$

★ $\underline{P(n)}$ Define the proposition $\boxed{P(n):\ 1+2+\dots+n = \dfrac{n(n+1)}{2}}$

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* These notes are solely for the personal use of students registered in MAT1348.

★★ B.I. base value: $\boxed{n_0 = 1}$ P(1) says "$\underbrace{1}_{LS} = \underbrace{\frac{1(1+1)}{2}}_{RS}$"

For P(1), LS $= 1$ and RS $= \frac{1(1+1)}{2} = 1$ ∴ P(1) is true.

★★★ I.S. Let $k \geq n_0 = 1$. Now we prove $P(k) \longrightarrow P(k+1)$.

★★★★ I.H. Assume P(k) is true. (goal: prove P(k+1) follows from P(k))

P(k) says "$1 + \dots + k = \frac{k(k+1)}{2}$"

Thus, our induction hypothesis is to assume

I.H. $\boxed{1 + \dots + k = \frac{k(k+1)}{2}}$ for some integer $k \geq n_0 = 1$.

For the I.S. our goal is to show P(k+1) follows from our I.H.

First, observe what P(k+1) says: "$\underbrace{1+2+\dots+k+k+1}_{LS} = \underbrace{\frac{(k+1)(k+1+1)}{2}}_{RS}$"
(so we know what our goal is)

In P(k+1), we have

$$\begin{aligned}
LS &= \underbrace{1+2+\dots+k}+(k+1) \\
&= \frac{k(k+1)}{2} + (k+1) && \text{(using I.H.)} \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} && \text{(common denom. so we can add fractions)} \\
&= \frac{k^2+k+2k+2}{2} \\
&= \frac{k^2+3k+2}{2} \\
&= \frac{(k+1)(k+2)}{2} = RS
\end{aligned}$$

∴ P(k+1) is true! ∴ we proved $P(k) \rightarrow P(k+1)$!

★★★★★ conclusion Since P(1) is true and since we proved $P(k) \rightarrow P(k+1)$ is true for any $k \geq 1$, it follows from Mathematical Induction that P(n) is true for all integers $n \geq 1$.

Example 18.2. Let $a_0, a_1, a_2, a_3, \ldots$ be a sequence of numbers defined according to the following **recurrence relation:**

$$a_0 := 1$$
$$\text{for each integer } n \geq 1, \quad a_n = 5(a_{n-1})^2$$

Using the **recurrence relation**, compute the values for a_1, a_2, a_3, and a_4.

$a_0 := 1$

$a_1 = 5(a_0)^2 = 5(1)^2 = 5$

$a_2 = 5(a_1)^2 = 5(5)^2 = 5^3$

$a_3 = 5(a_2)^2 = 5(5^3)^2 = 5^7$

$a_4 = 5(a_3)^2 = 5(5^7)^2 = 5^{15}$

$a_5 = 5(a_4)^2 = 5(5^{15})^2 = 5^{31}$

What is the **general solution** to this recurrence relation? That is, what does a_n equal as a function of n? Prove this solution using a **Proof by Induction**.

In general, it looks like $\boxed{a_n = 5^{2^n - 1}}$

Let's prove this!

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1. For each integer $n \geq 0$, let $P(n)$ denote the following proposition:

$$P(n): \text{"} a_n = 5^{2^n - 1} \text{"}$$

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2. B.I. $\underline{n_0 = 0}$

$P(0)$ says "$a_0 = 5^{2^0 - 1}$"

According to the recurrence relation,

$a_0 = 1$ and the RS of $P(0)$ is equal to $5^{2^0 - 1} = 5^{1-1} = 5^0 = 1$.

Thus, $P(0)$ is true.

3. I.S. Let $k \geq 0$. We must prove $P(k) \rightarrow P(k+1)$.

4. I.H. Assume $P(k)$ is true. That is, assume $a_k = 5^{2^k - 1}$

(goal: prove $P(k+1)$ follows, that is, prove $a_{k+1} = 5^{2^{k+1} - 1}$)

LS of $P(k+1) = a_{k+1}$

$= 5(a_k)^2$ (by the recurrence relation)

$= 5\left[5^{2^k - 1}\right]^2$ (by the I.H. !)

$= 5 \cdot 5^{(2^k - 1)(2)}$ (by laws of exponents)

$= 5 \cdot 5^{2 \cdot 2^k - 2}$ "

$= 5 \cdot 5^{2^{k+1} - 2}$ "

$= 5^1 \cdot 5^{2^{k+1} - 2}$ "

$= 5^{2^{k+1} - 2 + 1}$ "

$= 5^{2^{k+1} - 1}$

$=$ RS of $P(k+1)$ $\therefore$ $P(k+1)$ does follow from $P(k)$!

5. Conclusion:

Since $P(1)$ is true and since we proved $P(k) \rightarrow P(k+1)$ for any $k \geq 1$, it follows from the principle of Mathematical Induction that $P(n)$ is true for all integers $n \geq 1$.

STUDY GUIDE

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proof by induction:	1. Define $P(n)$ 2. **B.I.** Prove $P(n_0)$ 3. **I.S.** Let $k \geq n_o$. Prove $P(k) \rightarrow P(k+1)$. 4. **I.H.** Assume $P(k)$ is true (goal: prove $P(k+1)$ follows from I.H.) 5. **conclusion**

Exercises Sup.Ex. §11 # 1, 2, 3, 4, 5, 7, 8, 9, 10, 14 Rosen §5.1 # 3, 5, 7, 9, 11, 13, 15