11. Definite Integrals & The Fundamental Theorem of Calculus

Lec 10 mini review.

- ◇ **an antiderivative** vs. **the most general antiderivative**
- ◇ **undoing basic rules of differentiation**
- ◇ **some ideas for undoing less basic rules of differentiation**
- ◇ **setup for a Riemann sum with** n **rectangles on** $[a, b]$:

$$\Delta x = \tfrac{b-a}{n} \qquad x_i = a + i\Delta x \qquad \text{sample point} \quad x_i^* \in [x_{i-1}, x_i]$$

- ◇ **using a Riemann sum to approximate net area** A **between** f **and the** x**-axis on** $[a, b]$:

$$A \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

DEFINITE INTEGRALS

- Let f be a function defined for $a \leq x \leq b$.
- Divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$.
- Let $x_0 = a$ and, for $i = 0, \ldots, n$, let $x_i = a + i\Delta x$ be the endpoints of these subintervals.
- Let x_i^* be *any* **sample point** from the ith subinterval $[x_{i-1}, x_i]$.

Then the **definite integral of** f **from** a **to** b is

provided that this limit exists and gives the same value for all possible choices of sample points.

If it does exist and is equal for all sample point choices, then we say that f is **INTEGRABLE** on $[a, b]$.

Theorem 11.1. If f is

continuous on $[a, b]$, or if f has only a finite number of jump discontinuities on $[a, b]$,

then the definite integral $\int_a^b f(x)\,dx$ (which is a limit!) exists, hence f is integrable on $[a, b]$.

Theorem 11.2. If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i)\,\Delta x \right)$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$ (that is, we can use right endpoints in our Riemann sum).

* These notes are solely for the personal use of students registered in MAT1320.

Example 11.3. Evaluate $\displaystyle\int_0^3 \left(x^3 - 6x\right) dx$ using the (limit) definition of a definite integral.

Properties of Definite Integrals

$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

$$\boxed{\int_a^a f(x)\,dx = 0}$$

1. If $c \in \mathbb{R}$, then $\boxed{\int_a^b c\,dx = c(b-a)}$.

2./4. $\boxed{\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx}$

3. If $c \in \mathbb{R}$, then $\boxed{\int_a^b \big(cf(x)\big)\,dx = c\int_a^b f(x)\,dx}$

5. $\boxed{\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx}$

7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\boxed{\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx}$

8. If $L \leq f(x) \leq U$ for $a \leq x \leq b$, then $\boxed{L(b-a) \leq \int_a^b f(x)\,dx \leq U(b-a)}$

Exercise 11.4. If $\int_1^4 f(x)\,dx = 5$ and $\int_1^4 \big[2f(x) + 3g(x)\big]\,dx = 7$, find $\int_4^1 g(x)\,dx$.

FTC 2

The Fundamental Theorem of Calculus, Part 2

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, F is any function such that $F' = f$.

$\Longrightarrow$ We can forget about computing difficult limits of Riemann sums! FTC 2 gives us a quick way to evaluate definite integrals:

1. find an antiderivative of the integrand
2. subtract our antiderivative at the limits of integration

Notation:

Example 11.5. Evaluate the definite integral $\displaystyle\int_0^3 \left(x^3 - 6x\right) dx$ using FTC 2. Compare this procedure with the limit we used in Example 11.4 to compute the same definite integral.

Example 11.6. Evaluate the definite integral $\displaystyle\int_0^2 \Big((x-1)^2+1\Big)\,dx$ using FTC 2. Compare this with the Riemann sum approximations we obtained in Example 10.8.

Example 11.7. $\displaystyle\int_{-1}^2 x^2\,dx$

Example 11.8. $\displaystyle\int_{-1}^2 (x^2-1)\,dx$

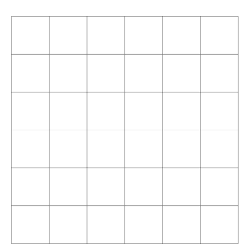
Example 11.9. $\displaystyle\int_0^1 e^x\,dx$

Example 11.10. $\displaystyle\int_1^2 \frac{dx}{x}$

Example 11.11. $\displaystyle\int_0^1 \frac{dx}{x}$

Example 11.12. $\displaystyle\int_{-\pi/2}^{3\pi/4} \sin(t)\,dt$

Example 11.13. $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$ *hint: draw a picture of the net area represented by this definite integral.*



Indefinite vs Definite Integrals

From now on, we will use our integral notation in two ways:

- ◇ We write $\displaystyle\int f(x)\,dx$ to represent **THE MOST GENERAL ANTIDERIVATIVE OF $f(x)$.** That is,

  The integral $\displaystyle\int f(x)\,dx$ is also called **AN INDEFINITE INTEGRAL** .

  In particular, an indefinite integral represents an infinite family of functions, each member of which has derivative equal to $f(x)$.

- ◇ If there are **LIMITS OF INTEGRATION** , $\int_a^b$, then $\displaystyle\int_a^b f(x)\,dx$ is called a **DEFINITE INTEGRAL** .

  By FTC (assuming $f(x)$ is continuous on $[a, b]$), the definite integral $\int_a^b f(x)dx$ equals the difference $F(b) - F(a)$, where F is any antiderivative of f. Thus, a definite integral is a **number**, *not* a family of functions. This number corresponds to the net area between $f(x)$ and the x-axis on the interval $[a, b]$.

Example 11.14. Evaluate each of the following integrals:

$$\int \frac{2}{x^2+1}\,dx$$

$$\int_{-1}^{0} \frac{2}{x^2+1}\,dx$$

Study Guide

- ◇ **definition of the definite integral**(if it exists): $\displaystyle\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$
- ◇ **net area interpretation of definite integral**
- ◇ **evaluating definite integrals using known sums and properties of integrals**

FTC2 If f is continuous on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f.

- ◇ **indefinite integral** vs. **definite integral**