DGD 10

Q1. INDUCTION Use **Mathematical Induction** to prove that 8 divides $m^2 - 1$ for all **odd** integers $m \geq 1$. (*hint: think of odd numbers in the form $2n+1$, starting from $n = 0$, and define your proposition $P(n)$ corresponding to the n-th odd number*)

★ For each integer $n \geq 0$, define the proposition $P(n)$ as follows:

$P(n)$: " 8 divides $(2n+1)^2 - 1$."

ie $P(n)$ says "8 divides the nth positive odd integer."

★★ B.I. $n_0 = 0$ (ie $2(0)+1 = 1$ is the 1st positive odd number)

$P(0)$ says " 8 divides $(2(0)+1)^2 - 1$"

ie $P(0)$: " 8 divides $1^2 - 1$ "

ie $P(0)$: " 8 divides 0"

Since $0 = (0)(8)$ and $0 \in \mathbb{Z}$, it is indeed true that $8 \mid 0$.

$\therefore P(0)$ is true.

★★★ I.S Let $k \geq n_0 = 0$ and prove $P(k) \rightarrow P(k+1)$.

★★★★ I.H. Assume $P(k)$ is true for some $k \geq n_0 = 0$.

ie assume 8 divides $(2k+1)^2 - 1$

(goal: prove $P(k+1)$ follows from $P(k)$)

ie. prove 8 divides $(2(k+1)+1)^2 - 1$.

Thus, $4k^2 + 4k + 1 - 1 = 8\ell$ for some $\ell \in \mathbb{Z}$ (by def of divides)

$\Rightarrow$ $4k^2 + 4k = 8\ell$ for some $\ell \in \mathbb{Z}$] I.H.

Let's prove $P(k+1)$ is true using our I.H.

First, what does $P(k+1)$ say? $P(k+1)$: " $(2(k+1)+1)^2 - 1$ is divisible by 8"

$$
\begin{aligned}
\text{Well, } (2(k+1)+1)^2 - 1 &= (2k+3)^2 - 1 \\
&= 4k^2 + 12k + 9 - 1 \\
&= 4k^2 + 12k + 8 \\
&= \underbrace{4k^2 + 4k} + 8k + 8 \\
&= 8\ell + 8k + 8 \quad \text{using the I.H. } 4k^2 + 4k = 8\ell \\
&= 8(\ell + k + 1) \\
&= 8j \quad \text{for } j = \ell + k + 1. \text{ Since } \ell, k, 1 \in \mathbb{Z}, \text{ so is } j \in \mathbb{Z}.
\end{aligned}
$$

$\therefore$ 8 divides $(2(k+1)+1)^2 - 1$ $\therefore$ $P(k+1)$ is true and so we proved that $P(k) \rightarrow P(k+1)$.

★★★★★ Conclusion Since we proved $P(0)$ is true and we proved $P(k) \rightarrow P(k+1)$ for any $k \geq 0$, it follows from Mathematical Induction that $P(n)$ is true for all $n \geq 0$.

Q2. INDUCTION Using a **Proof by Induction**, prove that the following inequality is true for all integers $n \geq 5$. Clearly define the proposition $P(n)$ to be proved and include all relevant steps and details such as the **Basis of Induction**, **Induction Hypothesis**, and the **Induction Step.**

hint: you may find it useful to know that for any $n \in \mathbb{N}$, we have $\frac{2n+1}{n+1} < \frac{2n+2}{n+1} = 2$.

Prove that $\binom{2n}{n} < 4^{n-1}$ for all integers $n \geq 5$.

1. For each integer $n \geq 5$, define the proposition $P(n)$ as follows:

$$P(n): \text{"} \binom{2n}{n} < 4^{n-1} \text{"}$$

2. BI $n_0 = 5$. $P(5)$ says "$\binom{2(5)}{5} < 4^{5-1}$"

$LS = \binom{10}{5} = \frac{\cancel{10}^{2} \cdot 9 \cdot \cancel{8}^{2} \cdot 7 \cdot \cancel{6}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3 \cdot 2} \cdot 1} = 252$ $\quad RS = 4^4 = 256$ $\quad LS < RS \therefore P(5)$ is true.

3. IS. Let k be an integer such that $k \geq 5$. We must prove $P(k) \to P(k+1)$.

4. IH. Assume $P(k)$ is true. goal: prove $P(k+1)$ follows from $P(k)$

ie assume $\binom{2k}{k} < 4^{k-1}$ ie prove $\binom{2(k+1)}{k+1} < 4^{k+1-1}$

$$\text{LS of } P(k+1) = \binom{2(k+1)}{k+1} = \binom{2k+2}{k+1} = \frac{(2k+2)!}{(k+1)!(2k+2-(k+1))!}$$

$$= \frac{(2k+2)(2k+1)(2k)!}{(k+1)!\,(k+1)!} = \frac{(2k+2)(2k+1)\,(2k)!}{(k+1)k!(k+1)\,k!} = \frac{(2k+2)(2k+1)}{(k+1)(k+1)} \cdot \frac{(2k)!}{k!k!}$$

$$= \frac{(2k+2)(2k+1)}{(k+1)(k+1)} \cdot \binom{2k}{k} = \frac{2\cancel{(k+1)}}{\cancel{(k+1)}} \cdot \frac{(2k+1)}{(k+1)} \cdot \binom{2k}{k} = 2 \cdot \left(\frac{2k+1}{k+1}\right) \cdot \binom{2k}{k}$$

$$< 2 \cdot 2 \cdot \binom{2k}{k} \text{ by hint.}$$

$$= 4 \cdot \binom{2k}{k} < 4 \cdot 4^{k-1} \text{ by IH}$$

$$= 4^k \qquad \therefore \binom{2(k+1)}{k+1} < 4^{k+1-1} \checkmark$$

5. Conclusion: Since $P(5)$ is true and since we proved $P(k) \to P(k+1)$ is true for any integer $k \geq 5$, it follows from the Principle of Mathematical Induction that $P(n)$ is true for all integers $n \geq 5$.

Q3. INDUCTION Let φ denote the **golden ratio**, that is, $\varphi = \dfrac{1+\sqrt{5}}{2}$.

Use **Mathematical Induction** to prove that the following formula is true for all integers $n \geq 1$:

$$\varphi^{n+1} = \varphi^n + \varphi^{n-1}$$

★ Define the proposition $P(n)$ for each integer $n \geq 1$ as follows:

$$P(n): \text{"} \varphi^{n+1} = \varphi^n + \varphi^{n-1} \text{"}$$

★★ B.I. $n_0 = 1$ $P(1)$ says "$\varphi^2 = \varphi^1 + \varphi^0$" ie $\varphi^2 = \varphi + 1$.

Let's check that $P(1)$ is true:

$LS = \varphi^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2}$ $\qquad RS = \varphi + 1 = \frac{1+\sqrt{5}}{2} + 1 = \frac{1+\sqrt{5}}{2} + \frac{2}{2} = \frac{3+\sqrt{5}}{2}$

Since $LS = RS$, we proved that $P(1)$ is indeed true.

★★★ I.S. Let k be an integer such that $k \geq n_0 = 1$.

We must prove $P(k) \longrightarrow P(k+1)$.

★★★★ I.H. Assume $P(k)$ is true for some $k \geq n_0 = 1$.

ie Assume $\left.\varphi^{k+1} = \varphi^k + \varphi^{k-1}\right]$ I.H.

(goal: prove $P(k+1)$ follows from $P(k)$, ie prove $\varphi^{k+1+1} = \varphi^{k+1} + \varphi^{k+1-1}$

ie prove $\varphi^{k+2} = \varphi^{k+1} + \varphi^k$

Well, LS of $P(k+1) = \varphi^{k+2}$

$= \varphi(\varphi^{k+1})$

$= \varphi[\varphi^k + \varphi^{k-1}]$ using the I.H.: $\varphi^{k+1} = \varphi^k + \varphi^{k-1}$

$= \varphi^{k+1} + \varphi^k$ (by multiplying)

$= RS$ of statement for $P(k+1)$ $\therefore$ $P(k+1)$ is true and so we've proved that $P(k) \rightarrow P(k+1)$.

★★★★★ Conclusion Since we proved $P(1)$ is true and we proved $P(k) \rightarrow P(k+1)$ for any $k \geq 1$, it follows from Mathematical Induction that $P(n)$ is true for all $n \geq 1$.

Q4. BINOMIAL THEOREM What are the coefficients of

i. x^4

ii. x^6

iii. x^{-1}

in the expansion of $\left(\frac{3}{x^2} - x^3\right)^8$?

Binomial Theorem: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} \cdot b^i$

$$\Longrightarrow \quad \left(\frac{3}{x^2} - x^3\right)^8 = \sum_{i=0}^{8} \binom{8}{i} \left(\frac{3}{x^2}\right)^{8-i} (-x^3)^i$$

$$= \sum_{i=0}^{8} \binom{8}{i} \cdot 3^{8-i} (x^{-2})^{8-i} (-1)^i (x^3)^i$$

$$= \sum_{i=0}^{8} \binom{8}{i} 3^{8-i} x^{2i-16} \cdot (-1)^i \cdot x^{3i}$$

$$= \sum_{i=0}^{8} \binom{8}{i} 3^{8-i} \cdot (-1)^i \, x^{5i-16}$$

i. for coeff. of x^4 we need the index i such that $4 = 5i - 16 \Longleftrightarrow 20 = 5i$
$\Longleftrightarrow i = 4$

$\therefore$ coeff. of x^4 is $\binom{8}{4} \cdot 3^4 (-1)^4 = \binom{8}{4} 3^4$ and $i \in \{0, 1, \ldots, 8\}$ ✓

---

ii. for coeff. of x^2 we need the index i such that $2 = 5i - 16$
$\Longleftrightarrow 18 = 5i$
$\Longleftrightarrow i = \frac{18}{5} = 3.6$ ← no integer solution for i in the range $0 \leq i \leq 8$

$\therefore$ coeff. of x^2 is 0 (zero).

---

iii. for coeff. of x^{-1} we need the index i such that $-1 = 5i - 16 \Longleftrightarrow 15 = 5i$
$\Longleftrightarrow i = 3$

$\therefore$ coeff. of x^{-1} is $\binom{8}{3} \cdot 3^5 (-1)^3 = -\binom{8}{3} 3^5$