Lesson 1 – Connecting Secant and Tangent Lines

PART A: Recall

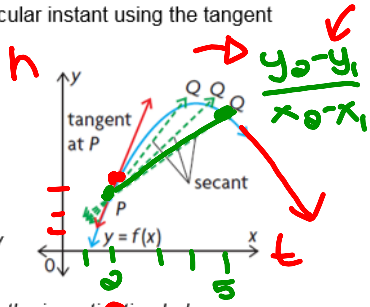
Secant Line: A line which passes through at least two points of a curve. As such, it can be used to determine the average slope between these two points of the curve.

Tangent Line: A line that just touches the curve at a single point and provides a straight line approximation to the curve at that point. The tangent line can be used to calculate the slope of the curve at a given point.

ARC: Average rate of change of a function over an interval between two points is the slope of the secant line connecting the two points.

IRC: The instantaneous rate of change of a function at a particular instant using the tangent of the curve at that point.

We discovered in Advanced Functions that the line joining point P and point Q on a curve is called a **secant**. It is easy to calculate the slope of a secant since we are given two end points to work with. A **tangent** is used to find the slope at a particular point on a curve.



Q. *How can we calculate the slope of a tangent when we only have one point?*

A. *To better understand the answer to this question, complete the investigation below.*

$P(2, 3)$

$Q(2.001, y_2)$

2.01

$m =$ 5 m/s

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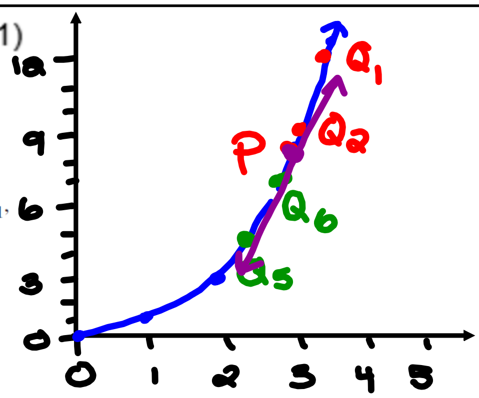
PART B: Investigation #1 from the textbook (page 11)



A. Determine the y-coordinates of the following points that lie on the graph of the parabola $y = x^2$: ← $y = (3.5)^2$

i) $Q_1(3.5, y)$ ii) $Q_2(3.1, y)$ iii) $Q_3(3.01, y)$ iv) $Q_4(3.001, y)$

B. Calculate the slopes of the secants through $P(3, 9)$ and each of the points Q_1, Q_2, Q_3, and Q_4.

C. Determine the y-coordinates of each point on the parabola, and then repeat part B using the following points.

i) $Q_5(2.5, y)$ ii) $Q_6(2.9, y)$ iii) $Q_7(2.99, y)$ iv) $Q_8(2.999, y)$

D. Use your results from parts B and C to estimate the slope of the tangent at point $P(3, 9)$.

$h \quad \frac{y_2-y_1}{x_2-x_1}$

Point	x	y	Secant	Δy	Δx	Slope
Q_1	3.5	12.25	PQ_1	-3.25	-0.5	6.5
Q_2	3.1	9.61	PQ_2	-0.61	-0.1	6.1
Q_3	3.01	9.0601	PQ_3	-0.0601	-0.01	6.01
Q_4	3.001	9.006001	PQ_4	-0.006001	-0.001	6.001
P	3	9				
Q_8	2.999	8.994001	PQ_8	0.005999	0.001	5.999
Q_7	2.99	8.9401	PQ_7	0.0599	0.01	5.99
Q_6	2.9	8.41	PQ_6	0.59	0.1	5.9
Q_5	2.5	6.25	PQ_5	2.75	0.5	5.5

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From the investigation, we can see that the slopes of the secants PQ_n are approaching a better approximation of the slope of the tangent at point P, as we slide the secant closer and closer to the tangent at that point. What this means, is that the slope of the tangent at P approaches ___6___ as h approaches zero (using the notation "$h \to 0$").

PART C: Investigation #3 from the textbook (page 12)

Determine an expression for the slope of the secant PQ through points $P(3, 9)$ and $Q(3 + h, (3 + h)^2)$.

$x_1 \quad y_1$

$x_0 \quad y_0$

$$m = \frac{y_0 - y_1}{x_0 - x_1}$$

$$m = \frac{(3+h)^2 - 9}{3+h-3}$$

$$m = \frac{9+6h+h^2-9}{3+h-3}$$

$$m = \frac{6h+h^2}{h}$$

$$m = \frac{\cancel{h}(6+h)}{\cancel{h}}$$

$$m = 6 + h$$

$$m = 6 + 0$$

$$m = 6$$

$(3+h)(3+h)$

$$\frac{9+3h+3h+h^2}{3}$$

As $h \to 0$

$m \to 6$

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The slope of the tangent at point P is the limiting slope of the secant line PQ as Q gets closer and closer to P, or as 'h' gets closer to zero. We can write this as follows:

$$\lim_{h \to 0} (slope\ of\ the\ secant\ PQ)$$

We read this as "*the limiting value of the slope of PQ as h approaches zero*".

In general to find the slope of the tangent for any function $y = f(x)$ we can find the limit of the **difference quotient**. Let P (a, $f(a)$) be a fixed point on the graph of $y = f(x)$ and point Q ($a + h$, $f(a + h)$) be another point on the graph, then the slope of the tangent at P (a, $f(a)$) can be written as:

$y_0 \quad y_1$

$$\boxed{m_{tangent} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

$y_0 - y_1$

$3 \quad x_0 - x_1$

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PART D

Example 1: Determine the slope of the tangent for the functions given at the point shown.

a) $f(x) = -x^2 + 4x + 1$ at $x = 3$

$$f(3) = -(3)^2 + 4(3) + 1$$
$$= -9 + 12 + 1$$
$$= 4$$

$$m = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

$$f(3+h) = -(3+h)^2 + 4(3+h) + 1$$
$$= -(9 + 6h + h^2) + 12 + 4h + 1$$
$$= -9 - 6h - h^2 + 13 + 4h$$
$$= -h^2 - 2h + 4$$

$$m = \lim_{h \to 0} \frac{-h^2 - 2h + 4 - 4}{h}$$

$$m = \lim_{h \to 0} \frac{-h^2 - 2h}{h}$$

$$m = \lim_{h \to 0} \frac{-\cancel{h}(h+2)}{\cancel{h}}$$

$$m = \lim_{h \to 0} -(\cancel{h}^{0} + 2)$$

$$m = -2$$

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b) $f(x) = \sqrt{x}$ at $x = 9$

$$f(9) = \sqrt{9} = 3$$
$$f(9+h) = \sqrt{9+h}$$

$$m = \lim_{h \to 0} \frac{f(9+h) - f(9)}{h}$$

* Rationalize the numerator

$$m = \lim_{h \to 0} \frac{(\sqrt{9+h} - 3)}{h} \times \frac{(\sqrt{9+h} + 3)}{(\sqrt{9+h} + 3)}$$

$$m = \lim_{h \to 0} \frac{9 + h + 3\sqrt{9+h} - 3\sqrt{9+h} - 9}{h(\sqrt{9+h} + 3)}$$

$$m = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{9 + \cancel{h}^{0}} + 3)}$$

$$m = \frac{1}{\sqrt{9+0} + 3}$$

$$m = \frac{1}{\sqrt{9} + 3}$$

$$m = \frac{1}{6}$$

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c) $f(x)=\frac{3x+6}{x}$ at (2, 6)

$f(2)=\frac{3(2)+6}{2}=\frac{12}{2}=6$

$f(2+h)=\frac{3(2+h)+6}{2+h}$

$=\frac{6+3h+6}{2+h}$

$=\frac{12+3h}{2+h}$

$m=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$

$m=\lim_{h\to 0}\frac{12+3h}{2+h}-\frac{6}{1}\frac{(2+h)}{(2+h)}\div h$

$m=\lim_{h\to 0}\frac{12+3h}{2+h}-\frac{6(2+h)}{2+h}\div h$

$m=\lim_{h\to 0}\frac{12+3h-12-6h}{2+h}\div\frac{h}{1}$

$m=\lim_{h\to 0}\frac{-3h}{2+h}\times\frac{1}{h}$

$m=\lim_{h\to 0}\frac{-3\cancel{h}}{\cancel{h}(2+h)}$

$m=\lim_{h\to 0}-\frac{3}{2+\cancel{h}^{\,0}}$

$m=-\frac{3}{2}$

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