Final April 2019, questions and answers

Calculus II (University of Ottawa)

University of Ottawa
Department of Mathematics and Statistics

MAT1322 (Calculus II) Final exam

Instructor: Vadim Kaimanovich

April 23, 2019
Duration: 3 hours

Read the following information before starting the exam:

- Verify that your copy of the exam contains 9 pages, including this one. Do not detach any pages.
- Write your name and student number on this page.
- Each problem is worth 2 points (the right answer and a correct solution with possible minor mistakes - 2 points, the right answer with a solution which contains a major mistake but still demonstrates a general understanding of the subject - 1 point, either a wrong answer or the correct answer with a missing or completely unsatisfactory solution - 0 points).
- Show all work, clearly and in order, if you want to get full credit. Points may be taken off if it is not clear how you arrived at your answer (even if your final answer is correct).
- Please keep your written answers brief; be clear and to the point. Points may be taken off for rambling and for incorrect or irrelevant statements.
- Circle or otherwise indicate your final answers.
- Use both sides if necessary, but make sure you clearly indicate where the rest of your answer is.

The Faculty of Science requires that you read and sign the following statement:

Cellular phones, calculators, other electronic devices or course notes are not allowed during this test. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

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Last Name: ______________________________

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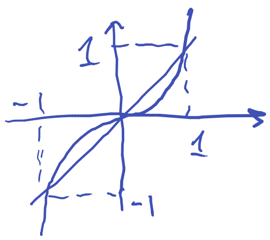
Student Number: ______________________________

Problem	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Total (30)
Points																

1. Find the area of the region enclosed by the curves $y = x, y = x^3$.

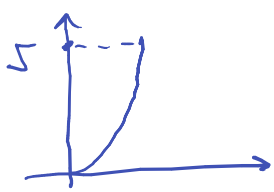
A. $1/4$ B. $-1/4$ C. 0 D. $-1/8$ E. 1 F. $1/2$ G. $-1/2$ H. none of the above



$$A = \int_{-1}^{1} |x^3 - x| \, dx = 2\int_0^1 (x - x^3)\, dx = 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}$$

2. Find the volume of the solid obtained by rotating the region bounded by the curves $x = 2\sqrt{y}, x = 0, y = 5$ about the y-axis.

A. $\pi/2$ B. 50π C. 0 D. 25π E. -50π F. πr^2 G. -25π H. none of the above



$$V = \int_0^5 \pi r^2(y)\, dy = \int_0^5 4\pi y \, dy$$

$$= 2\pi y^2 \Big|_{y=0}^{y=5} = 50\pi$$

3. Find the average value of the function $g(t) = t/\sqrt{3+t^2}$ on the interval $[1, 3]$.

A. $2\sqrt{3}-1$ B. $\sqrt{3}-2$ C. 1 D. $\sqrt{3}-1$ E. $2\sqrt{3}-2$ F. $t^2/2$ G. $1-\sqrt{3}$ H. none of the above

$\overline{g} = \frac{I}{3-1} = \frac{1}{2} I$ with

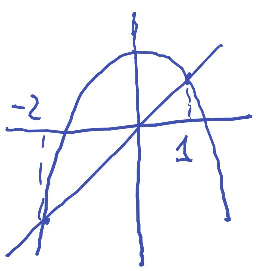
$I = \int_1^3 \frac{t\,dt}{\sqrt{3+t^2}}$ $u = 3+t^2$ $du = 2t\,dt$

$= \frac{1}{2}\int_4^{12} \frac{du}{\sqrt{u}} = u^{1/2}\Big|_{u=4}^{u=12} = \sqrt{12} - \sqrt{4} = 2(\sqrt{3}-1),$

whence $\overline{g} = \sqrt{3} - 1$

4. Find the centroid of the region bounded by the curves $y = 2 - x^2, y = x$.

A. $(-1, 2)$ B. $(1/2, -2/5)$ C. 0 D. $(-1/2, 1/2)$ E. $(1/2, -1/2)$ F. $1/2$ G. $(-1/2, 2/5)$
H. none of the above



The area $A = \int_{-2}^{1}(2 - x^2 - x)\,dx$

$= \left(2x - \frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_{x=-2}^{x=1}$

$= \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - 2\right) = \frac{7}{6} + 6 - \frac{16}{6} = \frac{9}{2}$

The coordinates of the centroid $(\overline{x}, \overline{y})$ are

$\overline{x} = \frac{I_x}{A}$, $I_x = \int_{-2}^{1} x(2 - x^2 - x)\,dx = \int_{-2}^{1}(2x - x^3 - x^2)\,dx$

$= \left(x^2 - \frac{x^4}{4} - \frac{x^3}{3}\right)\Big|_{x=-2}^{x=1} = \left(1 - \frac{1}{4} - \frac{1}{3}\right) - \left(4 - 4 + \frac{8}{3}\right) = \frac{5}{12} - \frac{8}{3} = -\frac{9}{4} \Rightarrow \overline{x} = -\frac{1}{2}$

$\overline{y} = \frac{I_y}{A}$, $I_y = \frac{1}{2}\int_{-2}^{1}\left[(2-x^2)^2 - x^2\right]dx = \frac{1}{2}\int_{-2}^{1}(x^4 - 5x^2 + 4)\,dx = \frac{1}{2}\left(\frac{x^5}{5} - \frac{5}{3}x^3 + 4x\right)\Big|_{x=-2}^{x=1}$

$= \frac{1}{2}\left(\frac{1}{5} - \frac{5}{3} + 4 + \frac{32}{5} - \frac{40}{3} + 8\right) = \frac{1}{2}\left(\frac{33}{5} - \frac{45}{3} + 12\right) = \frac{1}{2}\left(\frac{33}{5} - 3\right) = \frac{9}{5} \Rightarrow \overline{y} = \frac{2}{5}$

5. Which of the following improper integrals are convergent? (I) $\int_0^\infty \frac{dx}{\sqrt{1+x}}$ (II) $\int_0^1 \frac{dx}{x}$ (III) $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

A. (I) only B. (II) only C. (III) only D. none E. (I) and (II) only F. all G. (II) and (III) only H. none of the above

(I) $\int_0^\infty \frac{dx}{\sqrt{1+x}} = 2(1+x)^{1/2}\Big|_{x=0}^{\infty}$ diverges

(II) $\int_0^1 \frac{dx}{x} = \ln x\Big|_{x=0}^{x=1}$ diverges

(III) $\int_{-\infty}^{\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_0^\infty xe^{-x^2}dx$

$= \frac{1}{2}\int_{-\infty}^{0} e^{-x^2}d(x^2) + \frac{1}{2}\int_0^\infty e^{-x^2}d(x^2)$

$= \underbrace{-\frac{1}{2}\int_0^\infty e^{-t}dt}_{=-\frac{1}{2}} + \underbrace{\frac{1}{2}\int_0^\infty e^{-t}dt}_{=\frac{1}{2}}$ converges

6. Determine whether the sequence $a_n = n^2 \cos n/(1+n^2)$ converges or diverges. If it converges, find its limit L.

A. diverges, $L=1$ B. converges, $L=1$ C. converges. $L=\cos n$ D. diverges, $L=\cos n$ E. converges, $L=\pi$ F. converges, $L=0$ G. diverges H. none of the above

$a_n = \frac{n^2 \cos n}{n^2+1} = \underbrace{\frac{n^2}{n^2+1}}_{\to 1} \cdot \cos n$, so that

the convergence of a_n is equivalent to the convergence of $\cos n$

7. Which of the following series converge? (I) $\sum_{k=2}^{\infty} \frac{k^2 \cos k}{k^4-1}$, (II) $\sum_{n=0}^{\infty} (-1)^{n-1} \frac{n^2}{n^3+1}$ (III) $\sum_{n=1}^{\infty} \frac{n^2-n}{3n^3+2}$

A. (I) only B. (II) only C. (III) only D. none E. (I) and (II) only F. all G. (II) and (III) only H. none of the above

(I) $\left| \frac{k^2 \cos k}{k^4-1} \right| \le \frac{k^2}{k^4-1}$ which converges by the comparison test

(II) $\left(\frac{x^2}{x^3+1}\right)' = \frac{2x(x^3+1)-3x^4}{(x^3+1)^2} = \frac{2x-x^4}{(x^3+1)^4} < 0$ for $x \geq 2$, so that $\frac{n^2}{n^3+1}$ is decreasing for $n \geq 2$, and $\frac{n^2}{n^3+1} \to 0$, whence the series converges

(III) diverges by the comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$ and using the limit comparison test

8. How many terms n of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ should one take so that the error R_n do not exceed 0.001?

A. 1999 B. 2001 C. 500 D. 501 E. 1000 F. 100 G. 2000 H. none of the above

Since $f(x) = \frac{1}{x^2}$ is monotone decreasing,

$$R_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k \le \int_n^{\infty} \frac{dx}{x^2} = \left(-\frac{1}{x}\right)\Big|_{x=n}^{x=\infty} = \frac{1}{n},$$

whence $R_n \le 0.001$ if $\frac{1}{n} \le 0.001$

9. Find the convergence radius R and the convergence interval I of the power series $\sum_{n=0}^{\infty} \frac{n}{2^n(n^2+1)} x^n$.

A. $R = 1, I = [-1, 1]$ B. $R = -1, I = [-2, 2)$ C. $R = 2, I = [-2, 2)$ D. $R = 2, I = (-2, 2]$
E. $R = -2, I = (-2.2)$ F. $R = 1, I = [-1, 1)$ G. $R = 1, I = (-1, 1]$ H. none of the above

For $a_n = \frac{n x^n}{2^n(n^2+1)}$

$\left|\frac{a_{n+1}}{a_n}\right| = \underbrace{\frac{(n+1)(n^2+1)}{((n+1)^2+1)\,n}}_{\to 1} \cdot \left|\frac{x}{2}\right| \to \left|\frac{x}{2}\right|$, so that the series converges for $|x|<2$, diverges for $|x|>2$, and $R=2$

At the endpoints:

for $x=-2$ $a_n = (-1)^n \frac{n}{n^2+1}$, and the series converges by the alternating test

for $x=2$ $a_n = \frac{n}{n^2+1}$, and the series diverges by the comparison with the harmonic series

10. What is the third order entry a_3 of the Taylor series of the function $f(x) = \sqrt[3]{1-3x}$ at the point 0? By using the Taylor series find the limit $L = \lim_{x\to 0}(\sqrt[3]{1-3x} - 1 + x)/x^2$.

A. $a_3 = -\frac{5}{3}x^3, L = -1$ B. $a_3 = x^3, L = 1$ C. $a_3 = x^3/2, L = -1/2$ D. $a_3 = \frac{5}{3}x^3, L = 1$
E. $a_3 = x^3, L = 0$ F. $a_3 = x^3, L = -1$ G. $a_3 = \frac{1}{2}x^3, L = 1$ H. none of the above

$f(x) = (1-3x)^{1/3}$ $f(0) = 1$

$f'(x) = -(1-3x)^{-2/3}$ $f'(0) = -1$

$f''(x) = -2(1-3x)^{-5/3}$ $f''(0) = -2$

$f'''(x) = -10(1-3x)^{-8/3}$ $f'''(0) = -10$

The Taylor series is $1 - x - x^2 - \frac{5}{3}x^3 - \ldots \implies a_3 = -\frac{5}{3}x^3$

$\frac{f(x) - 1 + x}{x^2} = -1 - \frac{5}{3}x - \ldots \implies L = -1$

11. Find the solution of the differential equation $y' = xe^y$ that satisfies the initial condition $y(0) = 0$.

A. $y = \ln(1 - x^2/2)$ B. $y = 2\ln x$ C. $y = e^{-x}$ D. $y = 1/x$ E. $y = -2\ln x$ F. $y = e^{x^2-1}$
G. $y = -\ln(1 - x^2/2)$ H. none of the above

$\frac{dy}{dx} = xe^y \Rightarrow e^{-y}dy = x\,dx \Rightarrow \int e^{-y}dy = \int x\,dx + C$

$\Rightarrow -e^{-y} = \frac{x^2}{2} + C$; for $x=0, y=0$ $-1 = C$

$\Rightarrow e^{-y} = 1 - \frac{x^2}{2} \Rightarrow y = -\ln\left(1 - \frac{x^2}{2}\right)$

12. Which of the following functions $f = f(x, y)$ have the property that $f_{xx} + f_{yy} = 0$?
(I) $e^{x^2-y^2}$, (II) $x^3 + 3xy^2$, (III) $\ln(x^2 + y^2)$, (IV) $e^x \sin y - e^{-y}\cos x$?

A. (II) and (IV) B. (III) and (IV) C. (I), (II) and (IV) D. (II), (III) and (IV) E. (I) and (III) F. none
G. all H. none of the above

(I) $f_x = 2xe^{x^2-y^2}$ $f_{xx} = 2e^{x^2-y^2} + 4x^2e^{x^2-y^2}$
$f_y = -2ye^{x^2-y^2}$, $f_{yy} = -2e^{x^2-y^2} + 4y^2e^{x^2-y^2}$

(II) $f_{xx} = 6x$, $f_{yy} = 6x$

(III) $f_x = \frac{2x}{x^2+y^2}$, $f_{xx} = \frac{2(x^2+y^2) - 4x^2}{(x^2+y^2)^2} = \frac{-2x^2+2y^2}{(x^2+y^2)^2}$

Switching x and y, $f_{yy} = \frac{-2y^2+2x^2}{(x^2+y^2)^2}$

(IV) $f_{xx} = e^x \sin y + e^{-y}\cos x$
$f_{yy} = -e^x \sin y - e^{-y}\cos x$

13. For $z = x^2 - xy^2$ with $x = 2s - t + u$ and $y = st^2u^2$ find the partial derivatives $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial u}$ at the point $(s, t, u) = (2, 1, -1)$.

A. 0 B. $(0, -8)$ C. $(1, 4, -4)$ D. $2x - 2xy$ E. $(2, -1, 1)$ F. $(-8, -32, 32)$
G. $(2s - t + u)^2 - (2s - t + u)(st^2u^2)^2$ H. none of the above

$$\frac{\partial x}{\partial s} = 2, \quad \frac{\partial x}{\partial t} = -1, \quad \frac{\partial x}{\partial u} = 1$$

$$\frac{\partial y}{\partial s} = t^2u^2 = 1, \quad \frac{\partial y}{\partial t} = 2stu^2 = 4, \quad \frac{\partial y}{\partial u} = 2st^2u = -4$$

$$x(2,1,-1) = 2, \quad y(2,1,-1) = 2$$

$$\frac{\partial z}{\partial x} = 2x - y^2 = 0, \quad \frac{\partial z}{\partial y} = -2xy = -8$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = -8$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = -32$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = 32$$

14. Find the directional derivative of the function $f(x, y, z) = x^2y - xyz$ at the point $(1, -1, 2)$ in the direction of the vector $\mathbf{v} = (-1, 2, -2)$.

A. 4 B. $4/3$ C. $2/3$ D. 0 E. $-4/3$ F. $-2/3$ G. -2 H. none of the above

$$\nabla f = (2xy - yz, \; x^2 - xz, \; -xy)$$

$$\Rightarrow \nabla f(1,-1,2) = (0, -1, 1)$$

The length of vector $\bar{v}$ is $\sqrt{(-1)^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$, whence its normalization is $\bar{u} = \bar{v}/3 = \left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$,

so that

$$D_{\bar{v}} f(1,-1,2) = D_{\bar{u}} f(1,-1,2) = (0,-1,1) \cdot \left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right) = -\frac{4}{3}$$

15. Find an equation of the tangent plane to the surface $z = e^{x-y}$ at the point $(2, 2, 1)$.

A. $y = x + 1$ B. $z = x + y$ C. $z = x - y + 1$ D. $x + y + z = 1$ E. $x = y + z$ F. $z = e^x - e^y$
G. $z^2 = x - y$ H. $z = x - y$ none of the above

$z = f(x,y)$ with $f(x,y) = e^{x-y}$

$f_x(2,2) = e^{x-y}\Big|_{\substack{x=2\\y=2}} = 1$

$f_y(2,2) = -e^{x-y}\Big|_{\substack{x=2\\y=2}} = -1$, whence the

tangent plane equation is

$z = z_0 + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

$= 1 + 1\cdot(x-2) + (-1)(y-2) = x - y + 1$