MAT1348-23 Final hints

Discrete Mathematics for Computing (University of Ottawa)

Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Discrete Math for Computing MAT 1348 A

Final Exam

April 26, 2023

Prof. Hai Yan Liu (Jack)

You must **sign below** to confirm that you have read, understand, and will follow these **instructions**:

- This is a 180-minute **closed-book** exam; no notes are allowed. **Calculators and notes are not permitted.**
- The exam consists of 19 questions on 15 pages, with a maximum of 40 points. Page 14 contains table of set identity. Page 15 provides additional work space. If you need more additional space, you can use the backs of any of the pages. You may detach page 14 and 15 for your convenience, but **do not detach any other pages**.
- Questions 1–10 are multiple-choice questions worth 1 point each. **Put your answer to these questions in the table on page 2.** There is no penalty for an incorrect answer.
- Questions 11–14 are short-answer questions worth 1 point each. Write your answer in the box provided. Any rough work will not be graded.
- Questions 15–19 are long-answer questions worth points as indicated. You must use the technique that the question asks for and show all relevant steps in order to obtain full marks.
- Please raise your hand and ask a proctor if you need extra paper or to use the restroom. Do not get up from your seat unless instructed to do so.
- **Cellular phones** and other electronic devices **are not permitted** during this exam. Phones and other devices must be turned off completely and stored out of reach. Do not keep them in your possession, such as in your pockets. If you are caught with such a device, the following may occur: academic fraud allegations will be filed which may result in your obtaining a 0 (zero) for the exam.

For marker's use only:

Seat number: ______________________

Family name: ______________________

First name: ______________________

Student number: ______________________

Signature: ______________________

Q	Score	Max
1–10		10
11–14		4
15		5
16		5
17		5
18		6
19		5
Total		40

Multiple-Choice Questions

*For questions 1 to 10, **enter your answer in the table below**. Each correct answer is worth 1 point, and each incorrect answer is worth 0 points. You do not have to justify your answers.*

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10

Q1. Suppose there are 110 students at a university. Suppose that, among these 110 students, 45 students take MAT 1348, that 21 students take MAT 2375, and that 13 students take both courses. How many students do not take either of these two courses?

A. 73

B. 79

C. 53

D. 97

E. 44

F. 57 correct answer

Answer is F

Q2. Let $A = \{1, 2, 3, 4\}$. Which of the following sets is equivalent to the relation

$$\mathcal{R} = \{(a, b) \mid a \text{ divides } b\}?$$

A. $\mathcal{R} = \Big\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\Big\}$ correct answer

B. $\mathcal{R} = \Big\{(1,2),(1,3),(1,4),(2,4)\}\Big\}$

C. $\mathcal{R} = \Big\{(1,2),(2,1),(1,3),(3,1),(1,4),(4,1),(2,4),(4,2)\Big\}$

D. $\mathcal{R} = \Big\{(1,1),(1,2),(1,3),(1,4),(2,4),(3,3),(4,4)\Big\}$

E. $\mathcal{R} = \Big\{(1,2),(1,4),(1,3),(2,4),(2,2),(4,4)\Big\}$

F. $\mathcal{R} = \Big\{(1,2),(1,3),(1,4),(4,1),(2,2),(3,3),(4,4)\Big\}$

answer is A

Q3. Define the following propositions:

R : "It's raining."

W : "I will walk to the store."

B : "I will take the bus to the store."

Choose the compound proposition that best translates to, "(It's raining is a sufficient condition for me to take the bus to the store) only if I don't walk there."

A. $\neg W \to \neg(R \to B)$

B. $\neg W \to (B \to R)$

C. $\neg W \to (R \to B)$

D. $\neg(R \to B) \to \neg W$

E. $(B \to R) \to \neg W$

F. $(R \to B) \to \neg W$ correct answer

answer is F

Q4. Walking on the Island of Knights & Knaves, you encounter inhabitants A and B.

(Recall that knights always tell the truth, and knaves always lie. You should also recall that when two inhabitants are of the same type, that means they are both knights or both knaves; when they are of different types, that means one is a knight and the other is a knave.)

A says to you: "B and I are of the same type."
B then says: "A and I are of different types."

What, if anything, can we conclude about A and B?

A. A is a knight and B is a knight.

B. A is a knight and B is a knave.

C. A is a knave and B is a knight. correct answer

D. A is a knave and B is a knave.

E. A could be either and B could be either.

F. None of the above statements is accurate.

Answer is C

Q5. Let x and y be propositional variables. Which of the following propositions is **not** a tautology?

A. $x \to x$

B. $(x \to y) \vee (y \to x)$

C. $\neg(x \wedge \neg x)$

D. $y \to (x \vee \neg x)$

E. $x \to \neg x$ correct

F. $(y \wedge \neg y) \to x$

Answer is E

Q6. How many numbers, at a minimum, should we choose at random from $S = \{-4, -3, -2, -1, 1, 2, 3, 4\}$ in order to guarantee that two of the numbers we have chosen have a sum equal to 0?

A. 2

B. 3

C. 4

D. 5 correct answer

E. 6

F. None of the above.

answer is D

Q7. Suppose $A = \{2, 3, 4, 5\}$, $B = \{2, 4, 6, 8\}$, $C = \{1, 3, 5\}$.

Which of the following statements is **false**?

A. $B \cap C = \emptyset$

B. $|A \cap C| = |A \cap B|$

C. $|P(A)| = 16$

D. $A - B \subseteq C$

E. $B \subseteq A \cup C$ correct answer

F. $|A \times C| = 12$

Answer is E.

Q8. Let x and y be propositional variables, and let

$$\begin{aligned} P_1 &: x \vee y \\ P_2 &: x \wedge \neg y \\ P_3 &: x \to y \\ C &: y \end{aligned}$$

Answer the following three questions, in order:

- Is the argument $(P_1 \wedge P_2 \wedge P_3) \to C$ valid?
- Is the set $\{P_1, P_2, P_3\}$ consistent?
- Are P_2 and $\neg P_3$ logically equivalent?

A. Yes, yes, yes.

B. Yes, no, yes. correct answer

C. Yes, no, no.

D. No, yes, yes.

E. No, yes, no.

F. No, no, yes.

Answer is B.

Q9. Recall that a *binary string* is a sequence of 0's and 1's. How many binary strings of length 8 have exactly three 1's?

A. $8!$

B. 2^8

C. $\frac{8!}{5!}$

D. $\frac{8!}{3!5!}$ correct answer

E. 8^2

F. $8 \cdot 7 \cdot 6$

Answer is D.

Q10. Let $f : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $f(x) = (x, x+2)$. Answer the following three questions, in order

- f is injective?
- f is surjective?
- f is bijective?

A. Yes, no, no correct answer

B. Yes, no, yes

C. Yes, yes, yes

D. No, yes, yes

E. No, yes, no

F. No, no, no

Answer is A.

Short-Answer Questions

Questions 11 to 14 are short-answer questions. Write your final answer in the box provided. You do not have to justify your answers.

Q11. (1 point) What is the coefficient of x^5 in $(2x + \frac{3}{x})^8$?

0

Q12. (1 point) A graph G has only vertices of degree 3 and degree 5. If G has 22 edges and 10 vertices, how many vertices of degree 3 does G have?
$x + y = 10$ and $3x + 5y = 44$, $x = 3$

Q13. (1 point) Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ which is injective, but not surjective. $f(x) = 2x$

Q14. (1 point) Give a disjunctive normal form of the proposition $(p \leftrightarrow q) \wedge \neg(p \vee q)$.

$((p \wedge q) \vee (\neg p \wedge \neg q)) \wedge (\neg p \wedge \neg q) = \neg p \wedge \neg q$

Long-Answer Questions

Questions 15 to 19 are long-answer questions. You must justify your answers by showing all your steps clearly.

Q15. (5 points) Let $a_1, a_2, ...$ be the sequence defined recursively by

$$a_1 = 2, \qquad a_n = 3^{n-1} + a_{n-1} \quad \text{if } n \geq 2$$

Use induction to show that $a_n = \dfrac{3^n + 1}{2}$ for all $n \geq 1$. $P(n) : a_n = \dfrac{3^n + 1}{2}$ for all $n \geq 1$

Let begin check $n = 1$ for $P(1)$
$a_1 = \frac{3^1+1}{2} = 2$, we know $a_1 = 1$, so $P(1)$ is true.

Suppose $P(k) : a_k = \frac{3^k+1}{2}$ is true.

Now we need to prove :
$P(k+1) : a_{k+1} = \frac{3^{k+1}+1}{2}$ is true
By condition

$a_{k+1} = 3^{k+1-1} + a_{k+1-1} = 3^k + a_k = 3^k + \frac{3^k+1}{2} = \frac{2\times 3^k + 3^k + 1}{2} = \frac{3^{k+1}+1}{2}$

which means $P(k+1)$ is true.

Q16. (5 points) *(Note: in this counting question, you can leave your final answer as a sum or product of factorials and powers of integers.)*

Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$. How many functions $f : A \to B$...

(a) ... are injective?

$P(8, 5) = 8 \times 7 \times 6 \times 5 \times 4$

(b) ... are not injective?

$8^5 - P(8, 5)$

(c) ... are such that $f(a) = f(b) = f(c)$?

Let first choose $f(a)$, we have 8 choices. Since $f(a) = f(b) = f(c)$, therefore for a, b, c we have 8 choices . for d, e we don't have any limitation, therefore we have 8^2 , so total we have 8^3.

(d) ... are such that exactly three elements of A have 8 as an image?

First we choose 3 elements from A , we have $C(5, 3)$, then rest two elements can not choose 8, so everyone has 7 choices. total $C(5, 3)7^2$

(e) ... are surjective?

Impossible it is zero since $|A| < |B|$.

Q17. (5 points) Recall that the *symmetric difference* of two sets X and Y, denoted $X \oplus Y$, is the set of all elements that are in X or Y, but not both.

Give a **rigorous proof** that $A \oplus (A - B) = A \cap B$ for all sets A and B.

Important! State all of your assumptions clearly. It must be evident how each of your steps comes from a previous step, assumption, or definition. You will be graded based on the correctness and readability of your proof. You can use any method to prove.

From left to right

$x \in A \oplus (A - B)$ separates by two cases:

Case 1 $x \in A$, but $x \notin (A - B)$

Case 2 $x \in A - B$, but $x \notin A$

Case 2 is impossible . Let us to think of case 1:

$x \in A$, but $x \notin (A - B)$, which means $x \in B$, therefore $x \in A \cap B$.

From right to left prove:

If $x \in A \cap B$, which means $x \in A$ and $x \in B$. So, $x \notin A - B$, finally $x \in A \oplus (A - B)$. We finish our prove.

Q18. (6 points) Recall that, in graph, a walk of length $n \geq 0$ from vertex a to b is an alternating sequence of vertices and edges $v_0 e_1 v_1 e_2 v_2 e_3 \ldots v_{n-1} e_n v_n$, where $v_0 = a$ and $v_n = b$, and v_{i-1} and v_i are ends of edge e_i for all $i = 1, 2, 3, \ldots, n$.

Let G be a graph with vertex set V, define a binary relation R on V as follows:

For all $u, v \in V$: $\quad u\mathcal{R}v \quad \Longleftrightarrow \quad$ there exists a walk from u to v.

Prove that **this relation $\mathcal{R}$ is an equivalence relation.**

Reflexive, for any point v we v=can find length 0 walking from v to v, so it is reflexive.

Symmetric, if we have walk $ue_1v_1e_2v_2e_3 \ldots v_{n-1}e_nv$,, then we have a walk

$ve_nv_{n-1}e_{n-1} \ldots e_3v_2e_2v_1e_1u$. So it is symmetric.

Transitive, we have two walks:

$ue_1v_1e_2v_2e_3 \ldots v_{n-1}e_nv$

$ve_1^{'}v_1^{'}e_2^{'}v_2^{'}e_3^{'} \ldots v_{m-1}^{'}e_m^{'}w$, then we have walk:

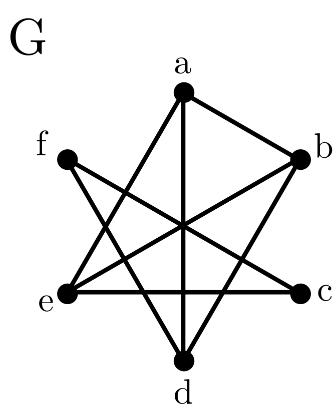
$ue_1v_1e_2v_2e_3 \ldots v_{n-1}e_nve_1^{'}v_1^{'}e_2^{'}v_2^{'}e_3^{'} \ldots v_{m-1}^{'}e_m^{'}w$.

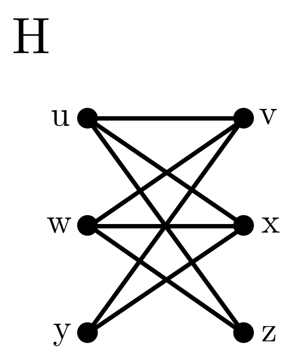
So it is transitive

Q19. (a) (1 point) Give the definition of an *isomorphism* from a graph G to a graph H.

(b) (2 points) Consider the graphs G and H below. Are G and H isomorphic?

- If yes, give an isomorphism from G to H. You don't need to prove that it is an isomorphism.
  No, at G , f, c degree are 2, and they are connected directly, however at H, y, z degree are 2, but they are not connected directly.
- If no, explain why. If you claim that a graph does not have a certain feature, you must demonstrate that concretely.





(c) (2 points) Consider the degree sequence $(0, 1, 2, 3, 4)$. For each of the following, if the answer is yes, draw an example. If the answer is no, explain why.

(i) Does there exist a graph with this degree sequence?
Since $0 + 1 + 2 + 3 + 4 = 10$ is even number therefore there exist a graph with this degree sequence .

(ii) Does there exist a *connected* graph with this degree sequence?

No, since one vertices degree is zero, it is isolated , so there is no *connected* graph with this degree sequence?

MAT 1348 TABLE OF SET IDENTITY

You may detach this page for your convenience.

1.	$A \cup \emptyset = A$	Identity Laws
2.	$A \cap \mathcal{U} = A$	
3.	$A \cup \mathcal{U} = \mathcal{U}$	Domination Laws
4.	$A \cap \emptyset = \emptyset$	
5.	$A \cup A = A$	Idempotent Laws
6.	$A \cap A = A$	
7.	$\overline{(\overline{A})} = A$	(Double) Complementation Law
8.	$A \cup B = B \cup A$	Commutative Laws
9.	$A \cap B = B \cap A$	
10.	$A \cup (B \cup C) = (A \cup B) \cup C$	Associative Laws
11.	$A \cap (B \cap C) = (A \cap B) \cap C$	
12.	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive Laws
13.	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	
14.	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's Laws
15.	$\overline{A \cap B} = \overline{A} \cup \overline{B}$	
16.	$A \cup (A \cap B) = A$	Absorption Laws
17.	$A \cap (A \cup B) = A$	
18.	$A \cup \overline{A} = \mathcal{U}$	Complement Laws
19.	$A \cap \overline{A} = \emptyset$	
20.	$A - B = A \cap \overline{B}$	Difference
21.	$A \oplus B = (A - B) \cup (B - A)$	Symmetric difference law
22.	$A \oplus B = (A \cup B) - (A \cap B)$	

This page is left blank for your rough work. You may detach it and do not need to submit it.