Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

Calculus I MAT1320

First Midterm Exam

5 October 2022

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Instructions. *You must sign below to confirm that you have read, understand, and will follow them.*

- This is an 80-minute *closed-book* exam; no notes are allowed. Calculators are *not* permitted.
- The exam consists of 7 questions on 8 pages.
- Be sure to read carefully and follow the instructions for the individual problems. To receive full marks, your solution must be correct, complete, and show all relevant details.
- For rough work or additional work space, you may use the back pages. Do not use scrap paper of your own.
- Use proper mathematical notation and terminology.
- You may ask for clarification.
- Unauthorized electronic devices (such as cellular phones) are not permitted during this exam. Such devices must be *turned off completely* and stored out of students' reach. Students found in possession of such a device during the exam will be asked to leave immediately and academic fraud allegations may be filed.

LAST NAME:____________________________

First name:____________________________

Signature:____________________________

Write your student number on the next page.

Circle your DGD (this is where you will pick up your marked exam):

C01	**C02**	**C03**
10:00	11:30	13:00
FTX 361	LMX 219	VNR 1075

Student number:________________

Question	1	2	3	4	5	6	7	Total
Max	2	3	3	3	4	4	3	22
Marks								

1. Determine the domain of the following function. *Briefly explain your reasoning. Give your answer in the form of a union of intervals.*

[2pts]

$$g(x) = \frac{\sqrt{x^2 - 16}}{\ln(x+5)}$$

Solution:

We need

$$\begin{aligned} &1. \quad x^2 - 16 \geq 0, \\ &2. \quad x + 5 > 0, \text{ and} \\ &3. \quad \ln(x+5) \neq 0. \end{aligned}$$

From 1. $x^2 \geq 16$ so $|x| \geq 4$. Thus, $x \in (-\infty, -4] \cup [4, \infty)$.

From 2. $x + 5 > 0$, so $x > -5$. Thus, $x \in (-5, \infty)$.

From 3. $\ln(x+5) \neq 0$, so $x + 5 \neq e^0 = 1$. Thus, $x \neq -4$.

Therefore, the domain of g is $(-5, -4) \cup [4, \infty)$.

2. (a) Find all solutions to the equation $\log_4(x^2 - 6x) = 2$.

Solution:

We have

$$\begin{aligned} \log_4(x^2 - 6x) &= 2 \\ \implies x^2 - 6x &= 4^2 \\ \implies x^2 - 6x - 16 &= 0 \\ \implies (x+2)(x-8) &= 0 \\ \implies x = -2 &\text{ or } x = 8 \end{aligned}$$

(b) Use (a) to find $f^{-1}(2)$ for the function $f(x) = \log_4(x) + \log_4(x-6)$.

[3pts] *Note: It can be verified that f is one-to-one, and hence invertible, but you need not show this.*

Solution:

We have $f^{-1}(2) = x \iff f(x) = 2$, so we need to solve for x in the second equation:

$$\begin{aligned} f(x) &= 2 \\ \implies \log_4(x) + \log_4(x-6) &= 2 \\ \implies \log_4(x(x-6)) &= 2 \\ \implies \log_4(x^2 - 6x) &= 2 \\ \implies x = -2 \text{ or } x &= 8 && \text{from part a)} \\ \implies f^{-1}(2) &= 8 && \text{(since } -2 \text{ is not in the domain of } f\text{)} \end{aligned}$$

3. Find the derivative of

$$f(x) = \sqrt{3x+4}$$

using the definition. *You may not use any of the differentiation rules from class, only the definition involving a limit (i.e. from 1st principles). Show all relevant steps in your solution!*

[3pts]

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3(x+h)+4}-\sqrt{3x+4}}{h} \\
&= \lim_{h\to 0} \left(\frac{\sqrt{3(x+h)+4}-\sqrt{3x+4}}{h}\right)\left(\frac{\sqrt{3(x+h)+4}+\sqrt{3x+4}}{\sqrt{3(x+h)+4}+\sqrt{3x+4}}\right) \\
&= \lim_{h\to 0} \frac{(3(x+h)+4)-(3x+4)}{h(\sqrt{3(x+h)+4}+\sqrt{3x+4})} \\
&= \lim_{h\to 0} \frac{3h}{h(\sqrt{3(x+h)+4}+\sqrt{3x+4})} \\
&= \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)+4}+\sqrt{3x+4}} \\
&= \frac{3}{\sqrt{3(x+0)+4}+\sqrt{3x+4}} \\
&= \frac{3}{\sqrt{3x+4}+\sqrt{3x+4}}
\end{aligned}$$

4. Let A and B be parameters, and define a function

$$g(x) = \begin{cases} \dfrac{A}{x^2-2} & \text{if } x < 2 \\ B & \text{if } x = 2 \\ \dfrac{4x+A}{5x-4} & \text{if } x > 2 \end{cases}.$$

[3pts] *Show all relevant steps when answering the following questions.*

(a) Determine $\lim_{x\to 2^-} g(x)$ and $\lim_{x\to 2^+} g(x)$.

Solution:

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} \frac{A}{x^2-2} = \frac{A}{2^2-2} = \frac{A}{2}$$

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} \frac{4x+A}{5x-4} = \frac{4(2)+A}{5(2)-4} = \frac{8+A}{6}$$

(b) Use your work in (a) to determine all values of A such that $\lim_{x\to 2} g(x)$ exists.

Solution:

For the $\lim_{x\to 2} g(x)$ to exist, we need the left limit to equal the right limit. Thus, we need:

$$\frac{A}{2} = \frac{8+A}{6}$$
$$6A = 2(8+A) = 16 + 2A$$
$$4A = 16$$
$$A = 4$$

For the limit to exist, we need $A = 4$.

(c) Use your work in (b) to determine all values of B such that g is continuous at 2.

Solution:

To be continuous at 2, we need $\lim_{x\to 2} g(x) = g(2)$.

From (b), we find that $\lim_{x\to 2} g(x) = \frac{A}{2} = \frac{4}{2} = 2$.

By definition of g, we have $g(2) = B$. Thus, we need $B = 2$.

5. Evaluate the following limits using rigorous mathematical methods seen in class. Show all relevant steps in your solution. *You may use any technique we have seen so far in the course. Even if you know L'Hospital's Rule — please do not use it.*

[4pts]

1. $\lim_{x\to\infty} \dfrac{2x^2+\sqrt{x^4+6}}{-7x^2+4}$

Solution:

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x^2+\sqrt{x^4+6}}{-7x^2+4} &= \lim_{x\to\infty} \frac{2x^2+\sqrt{x^4(1+\frac{6}{x^4})}}{-7x^2+4} \\
&= \lim_{x\to\infty} \frac{2x^2+\sqrt{x^4}\sqrt{1+\frac{6}{x^4}}}{-7x^2+4} \\
&= \lim_{x\to\infty} \frac{2x^2+|x^2|\sqrt{1+\frac{6}{x^4}}}{-7x^2+4} \\
&= \lim_{x\to\infty} \frac{2x^2+x^2\sqrt{1+\frac{6}{x^4}}}{-7x^2+4} && \text{since } x^2 \geq 0 \\
&= \lim_{x\to\infty} \frac{x^2(2+\sqrt{1+\frac{6}{x^4}})}{x^2(-7+\frac{4}{x^2})} \\
&= \lim_{x\to\infty} \frac{2+\sqrt{1+\frac{6}{x^4}}}{-7+\frac{4}{x^2}} \\
&= \frac{2+\sqrt{1+0}}{-7+0} \\
&= \frac{2+1}{-7} \\
&= -\frac{3}{7}
\end{aligned}$$

2. $\lim_{x\to 10}\left(\frac{1}{x-10}-\frac{20}{x^2-100}\right)$

Solution:

$$\begin{aligned}
\lim_{x\to 10}\left(\frac{1}{x-10}-\frac{20}{x^2-100}\right) &= \lim_{x\to 10}\left(\frac{1}{x-10}-\frac{20}{(x-10)(x+10)}\right)\\
&= \lim_{x\to 10}\left(\frac{1(x+10)}{(x-10)(x+10)}-\frac{20}{(x-10)(x+10)}\right)\\
&= \lim_{x\to 10}\frac{1(x+10)-20}{(x-10)(x+10)}\\
&= \lim_{x\to 10}\frac{x-10}{(x-10)(x+10)}\\
&= \lim_{x\to 10}\frac{1}{x+10}\\
&= \frac{1}{10+10}\\
&= \frac{1}{20}
\end{aligned}$$

6. Determine each of the following derivatives. *You may use any technique we have seen so far in the course. You do not need to simplify your answers.*

[4pts]

1. $\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{5t\cos t + 2\pi}{3t^4 - 7t}\right]$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{5t\cos t + 2\pi}{3t^4 - 7t}\right] = \frac{\Big(5\cos t + 5t(-\sin t) + 0\Big)\Big(3t^4 - 7t\Big) - \Big(5t\cos t + 2\pi\Big)\Big(3(4t^3) - 7\Big)}{(3t^4 - 7t)^2}$$
$$= \frac{\Big(5\cos t - 5t\sin t\Big)\Big(3t^4 - 7t\Big) - \Big(5t\cos t + 2\pi\Big)\Big(3(4t^3) - 7\Big)}{(3t^4 - 7t)^2}$$

2. $\dfrac{\mathrm{d}}{\mathrm{d}x}\Big[\tan(5\sin(\sqrt{x}))\Big]$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[\tan(5\sin(\sqrt{x}))\Big] = \sec^2(5\sin(\sqrt{x}))(5\cos(\sqrt{x}))\left(\frac{1}{2}x^{-1/2}\right)$$
$$= \frac{\sec^2(5\sin(\sqrt{x}))(5\cos(\sqrt{x}))}{2\sqrt{x}}$$

7. Determine the x-coordinates of all points on the curve $f(x) = (x-4)^2 e^{x+5}$ where the tangent line is horizontal. *Show all relevant steps in your solution.*

[3pts]

Solution:

$$f(x) = (x-4)^2 e^{x+5}$$

$$\implies f'(x) = 2(x-4)^1 e^{x+5} + (x-4)^2 e^{x+5}$$

For horizontal tangents, we solve $f'(x) = 0$

$$\implies 2(x-4)e^{x+5} + (x-4)^2 e^{x+5} = 0$$

$$\implies (x-4)e^{x+5}\Big[2 + (x-4)\Big] = 0$$

$$\implies (x-4)e^{x+5}(x-2) = 0$$

$$\implies x = 4 \text{ or } x = 2 \qquad (e^{x+5} \neq 0 \text{ for all } x \in \mathbb{R})$$

The tangent line to f is horizontal when $x = 4$ or $x = 2$.

End of the exam!