14. Relations, Equivalence Relations & Equivalence Classes

Let A and B be sets.

- ★ A **relation from** A **to** B is a subset $\mathcal{R}$ of $A \times B$.
- ★ A **relation on** A is a relation from A to itself, i.e., a subset of $A \times A$.
- ★ Given a relation $\mathcal{R}$ from A to B, we write $a \mathcal{R} b$ if and only if $(a, b) \in \mathcal{R}$.

  $a \mathcal{R} b$ means "a is related to b by the relation $\mathcal{R}$".

  $a \not\mathcal{R} b$ means "a is **not** related to b by the relation $\mathcal{R}$".

NUMBER OF RELATIONS

Theorem 14.1. Let A and B be finite sets.

Then

- there are $\boxed{2^{|A|\cdot|B|}}$ relations from A to B, and
- there are $\boxed{2^{|A|^2}}$ relations on A.

proof. $\mathcal{R}$ is a relation from A to B $\Leftrightarrow$ $\mathcal{R}$ is a subset of AxB

$$\therefore \begin{pmatrix}\#\text{ relations}\\ \text{from A to B}\end{pmatrix} = \begin{pmatrix}\#\text{ subsets}\\ \text{of } A\times B\end{pmatrix} = |\mathcal{P}(A\times B)| = 2^{|A\times B|} = 2^{|A|\cdot|B|}$$

Example 14.2. Let $A = \{1, 2, 3\}$ and let $B = \{x, y\}$

How many relations from A to B are there?

There are $2^{|A|\cdot|B|} = 2^{(3)(2)} = 2^6 = 64$ relations from A to B

How many relations on A are there?

There are $2^{|A|^2} = 2^{3^2} = 2^9 = 512$ relations on A.

* These notes are solely for the personal use of students registered in MAT1348.

Example 14.3. Let $\mathcal{R}_3$ be a relation on $\mathbb{Z}$ defined by $\mathcal{R}_3 = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : m|n$

Recall "m divides n" $\Leftrightarrow$ $n = km$ for some $k \in \mathbb{Z}$ and $m \neq 0$

correction

In class, I forgot $\boxed{\text{this}}$ detail of the definition of "divides." I wrote that $0|0$ but, that was incorrect. By the definition of "divides", $0 \nmid 0$. By excluding $m=0$ in the definition, it let's us view the relation $m|n$ equivalenty as $\frac{n}{m} \in \mathbb{Z}$ which is clearly undefined when $m=0$.

[transitive] We must prove $x\mathcal{R}_3 y \wedge y\mathcal{R}_3 z \rightarrow x\mathcal{R}_3 z$

ie $[(x|y) \wedge (y|z)] \rightarrow [x|z]$

Let $x,y,z \in \mathbb{Z}$.

Assume $x|y$ and $y|z$ (goal: prove $x|z$)

Then $y = kx$ and $z = \ell y$ for some integers $k, \ell \in \mathbb{Z}$ (def of divides)

Consequently $z = \ell(kx) = (\ell k)x = jx$ where $j = \ell k$ $\therefore j \in \mathbb{Z}$

$\therefore x|z$ (def of divides)

We proved $x|y$ and $y|z \rightarrow x|z$ $\therefore \mathcal{R}_3$ ("divides") is transitive

◇ $\mathcal{R}_3$ is ~~reflexive and~~ transitive, but not symmetric nor antisymmetric, nor reflexive.

correction correction

Example 14.4. Let $\mathcal{R}_4$ be a relation on $\mathbb{Z}^+$ defined by $\mathcal{R}_4 = \{(m,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : m|n$

(same rule as $\mathcal{R}_3$ but different set)

[antisymmetric] We must prove $(x\mathcal{R}_4 y \wedge y\mathcal{R}_4 x) \rightarrow (x=y)$

ie $[(x|y) \wedge (y|x)] \rightarrow [x=y]$

Let $x,y \in \mathbb{Z}^+$

Assume $x|y$ and $y|x$ (goal: prove $x=y$)

Then $y = kx$ and $x = \ell y$ for some integers $k, \ell \in \mathbb{Z}$ (def. of divides)

Note: since $x,y \in \mathbb{Z}^+$, it follows that $k, \ell \in \mathbb{Z}^+$.

Thus $y = k(\ell y) \Rightarrow y - k\ell y = 0$

$\Rightarrow y(1-k\ell) = 0$

$\Rightarrow \cancel{y=0}$ or $k\ell = 1$

reject because $y \in \mathbb{Z}^+$

$\therefore k = \ell = 1$ because the only integer factors of 1 are ± 1 but since $k, \ell \in \mathbb{Z}^+$, the only option is $k = \ell = 1$.

$\therefore y = kx \Rightarrow y = 1 \cdot x = x$ $\therefore y = x$

We proved $x|y$ and $y|x \rightarrow x = y$ $\therefore \mathcal{R}_4$ is antisymmetric.

◇ As a relation on $\mathbb{Z}^+$ (instead of on $\mathbb{Z}$), $\mathcal{R}_4$ is reflexive, antisymmetric, and transitive, but not symmetric. (verify this!)

Example 14.5. Let $\mathcal{R}_5$ be a relation on $\mathbb{Q}$ defined by

$$q \, \mathcal{R}_5 \, r \quad \Longleftrightarrow \quad q < r$$

Is $\mathcal{R}_5$ reflexive?

No. Counterexample: $5.98 \in \mathbb{Q}$ but "$5.98 < 5.98$" is false. $\therefore (5.98, 5.98) \notin \mathcal{R}_5$

Thus, it is not the case that $(q,q) \in \mathcal{R}_5$ for all $q \in \mathbb{Q}$. $\therefore \mathcal{R}_5$ is not reflexive.

Is $\mathcal{R}_5$ antisymmetric?

Yes! proof Let $x,y \in \mathbb{Q}$. Then there are 3 cases for x and y:

Case 1 Assume $x=y$. Then $x<y$ is False, thus $[(x<y) \wedge (y<x)] \to [x=y]$ is vacuously true.

Case 2 Assume $x<y$. Then $y<x$ is False, thus $[(x<y) \wedge (y<x)] \to [x=y]$ is vacuously true.

Case 1 Assume $x>y$. Then $x<y$ is False, thus $[(x<y) \wedge (y<x)] \to [x=y]$ is vacuously true.

$\therefore$ for all $x,y \in \mathbb{Q}$, the implication $[(x<y) \wedge (y<x)] \to [x=y]$ is true. $\therefore \mathcal{R}_5$ is antisymmetric.

◇ $\mathcal{R}_5$ is transitive and antisymmetric, but not reflexive, nor symmetric. (verify this!)

Example 14.6. Let $\mathcal{R}_6$ be a relation on $\mathbb{Z}$ defined by

$$(r,s) \in \mathcal{R}_6 \quad \Longleftrightarrow \quad s = 2r + 1$$

Is $\mathcal{R}_6$ symmetric?

No. Counterexample: $0, 1 \in \mathbb{Z}$

$1 = 2(0) + 1 \Rightarrow (0,1) \in \mathcal{R}_6$

$0 \neq 2(1) + 1 \Rightarrow (1,0) \notin \mathcal{R}_6$

Thus, it is not the case that $(r,s) \in \mathcal{R}_6 \to (s,r) \in \mathcal{R}_6$ for all $r,s \in \mathbb{Z}$

$\therefore \mathcal{R}_6$ is not symmetric.

◇ $\mathcal{R}_6$ is antisymmetric, but not reflexive, nor symmetric, nor transitive. (verify this!)

Example 14.7. Let $\mathcal{R}_7$ be a relation on $A = \{1,2,3,4\}$ defined by

$$\mathcal{R}_7 = \{(1,1), (2,2), (2,3), (3,2), (3,3), (4,4)\}$$

Is $\mathcal{R}_7$ reflexive?

Yes! proof: $(1,1), (2,2), (3,3), (4,4) \in \mathcal{R}_7 \quad \therefore (a \in A) \to (a \mathcal{R} a)$ is true

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Is $\mathcal{R}_7$ antisymmetric?

No. Counterexample:

$2,3 \in A$ and $2 \neq 3$ but both $(2,3) \in \mathcal{R}_7$ and $(3,2) \in \mathcal{R}_7$

Thus, it is not the case that $(a \neq b) \to ((a,b) \notin \mathcal{R}_7$ or $(b,a) \notin \mathcal{R}_7)$ for all $a,b \in A$. $\therefore \mathcal{R}_7$ is not antisymmetric.

◇ $\mathcal{R}_7$ is reflexive, symmetric, and transitive, but not antisymmetric. (verify this!)

In particular, $\mathcal{R}_7$ is an equivalence relation on $\{1,2,3,4\}$

Note the set $\quad$ $\mathcal{R}_8$ relates pairs of ordered pairs to each other by this rule

Example 14.8. Let $\mathcal{R}_8$ be a relation on $\mathbb{N} \times \mathbb{N}$ defined by

$$\big((n_1, n_2), (m_1, m_2)\big) \in \mathcal{R}_8 \quad \Longleftrightarrow \quad n_2 = m_1$$

Is $\mathcal{R}_8$ transitive?

No. Counterexample: $(1,5), (5,0), (0,9) \in \mathbb{N}\times\mathbb{N}$

$(1,5) \, \mathcal{R}_8 \, (5,0)$ and $(5,0) \, \mathcal{R}_8 \, (0,9)$

but $(1,5) \, \cancel{\mathcal{R}_8} \, (0,9)$ $\qquad$ $\therefore$ $\mathcal{R}_8$ is not transitive.

◇ $\mathcal{R}_8$ is not reflexive, nor symmetric, nor antisymmetric, nor transitive. (verify this!)

Exercise 14.9. Give an example of a relation on the set $A = \{1, 2, 3\}$

- that is both symmetric and antisymmetric:

  $\mathcal{R} = \{(1,1)\}$ is symmetric and antisymmetric (there are other possible answers)

- that is neither symmetric nor antisymmetric:

  $\mathcal{S} = \{(1,2), (2,1), (2,3)\}$ is neither symmetric, nor antisymmetric

  (there are other possible answers)

Exercise 14.10. There are 16 relations on the set $A = \{1, 2\}$. Here they all are:

$\mathcal{R}_1 = \emptyset$	$\mathcal{R}_9 = \{(1,2), (2,1)\}$
$\mathcal{R}_2 = \{(1,1)\}$	$\mathcal{R}_{10} = \{(1,2), (2,2)\}$
$\mathcal{R}_3 = \{(1,2)\}$	$\mathcal{R}_{11} = \{(2,1), (2,2)\}$
$\mathcal{R}_4 = \{(2,1)\}$	$\mathcal{R}_{12} = \{(1,1), (1,2), (2,1)\}$
$\mathcal{R}_5 = \{(2,2)\}$	$\mathcal{R}_{13} = \{(1,1), (1,2), (2,2)\}$
$\mathcal{R}_6 = \{(1,1), (1,2)\}$	$\mathcal{R}_{14} = \{(1,1), (2,1), (2,2)\}$
$\mathcal{R}_7 = \{(1,1), (2,1)\}$	$\mathcal{R}_{15} = \{(1,2), (2,1), (2,2)\}$
$\mathcal{R}_8 = \{(1,1), (2,2)\}$	$\mathcal{R}_{16} = \{(1,1), (1,2), (2,1), (2,2)\}$

◇ Which of these relations is reflexive? $\mathcal{R}_i$ for all $i \in \{8, 13, 14, 16\}$

◇ Which of these relations is symmetric? $\mathcal{R}_i$ for all $i \in \{1, 2, 5, 8, 9, 12, 15, 16\}$

◇ Which of these relations is antisymmetric? $\mathcal{R}_i$ for all $i \in \{1,2,3,4,5,6,7,8,10,11,13,14\}$

◇ Which of these relations is transitive? $\mathcal{R}_i$ for all $i \in \{1,2,3,4,5,6,7,8,10,11,13,14,16\}$

◇ Which of these relations is an equivalence relation on A? $\mathcal{R}_i$ for all $i \in \{8, 16\}$

Logical Equivalence:

Let $\mathscr{A}$ be the set of all compound propositions. Logical equivalence $\equiv$ is a relation on $\mathscr{A}$ given by the rule:

$$\text{for all } P, Q \in \mathscr{A}, \quad P \equiv Q \quad \text{if and only if} \quad P \leftrightarrow Q \text{ is a tautology.}$$

Exercise 14.11. Prove that $\equiv$ is an equivalence relation on $\mathscr{A}$.

Equivalence Relations and Equivalence Classes

Given an equivalence relation $\mathcal{R}$ on A, for each element $a \in A$, we define **the equivalence class of** a **with respect to** $\mathcal{R}$ as follows:

$$\boxed{[a]_{\mathcal{R}} = \{x \in A : a\mathcal{R}x\}}$$

= set of all elements of A which are related to a by $\mathcal{R}$

Example 14.12. Let x and y be propositional variables, and let $A = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8\}$, where the elements $P_i \in A$ are the following compound propositions:

$P_1: \; x \to y$	$P_3: \; \neg(\neg x \vee y)$	$P_5: \; \neg(x \to y)$	$P_7: \; x \wedge \neg y$
$P_2: \; x \vee y$	$P_4: \; \neg x \vee y$	$P_6: \; x \oplus y$	$P_8: \; \neg(x \leftrightarrow y)$

Let $\mathcal{R}$ be a relation on the set A defined by $(P_i, P_j) \in \mathcal{R} \iff P_i \equiv P_j$

Note: Because $\equiv$ is an equivalence relation on the set of *all* compound propositions, it follows that $\mathcal{R}$ is an equivalence relation on A.

Compute the equivalence class for each element of A.

$[P_1]_{\mathcal{R}} = \{P_1, P_4\}$

$[P_2]_{\mathcal{R}} = \{P_2\}$

$[P_3]_{\mathcal{R}} = \{P_3, P_5, P_7\}$

$[P_4]_{\mathcal{R}} = \{P_1, P_4\}$

$[P_5]_{\mathcal{R}} = \{P_3, P_5, P_7\}$

$[P_6]_{\mathcal{R}} = \{P_6, P_8\}$

$[P_7]_{\mathcal{R}} = \{P_7, P_3, P_5\}$

$[P_8]_{\mathcal{R}} = \{P_8, P_6\}$

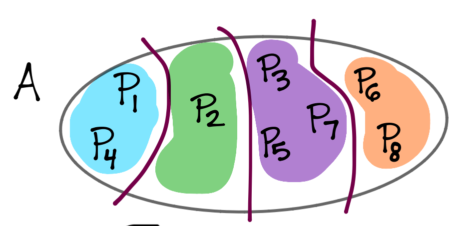
Some observations:

$[P_1]_{\mathcal{R}} = [P_4]_{\mathcal{R}}$ $\quad [P_3]_{\mathcal{R}} = [P_5]_{\mathcal{R}} = [P_7]_{\mathcal{R}}$ $\quad [P_6]_{\mathcal{R}} = [P_8]_{\mathcal{R}}$

$[P_1]_{\mathcal{R}} \cap [P_2]_{\mathcal{R}} = \emptyset$ $\quad [P_2]_{\mathcal{R}} \cap [P_3]_{\mathcal{R}} = \emptyset$

$[P_1]_{\mathcal{R}} \cap [P_3]_{\mathcal{R}} = \emptyset$ $\quad [P_2]_{\mathcal{R}} \cap [P_6]_{\mathcal{R}} = \emptyset$

$[P_1]_{\mathcal{R}} \cap [P_6]_{\mathcal{R}} = \emptyset$ $\quad [P_3]_{\mathcal{R}} \cap [P_6]_{\mathcal{R}} = \emptyset$



Let m be a positive integer, and let $x \in \mathbb{Z}$ be any integer.

- The **remainder of** $x \pmod m$ is the unique integer r such that $0 \le r < m$ and

$$\boxed{x = km + r} \qquad\qquad (k \in \mathbb{Z})$$

- We call m the **modulus**.
- Two integers a and b are called **congruent modulo** m if the remainder of $a \pmod m$ equals the remainder of $b \pmod m$.

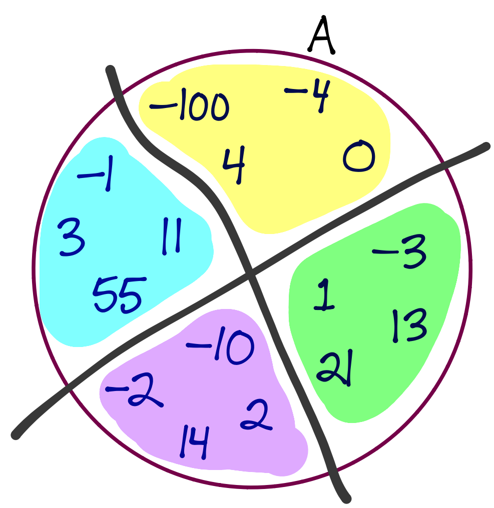
Notation: For short, we write $a \equiv b \pmod m$ whenever a and b are congruent moduluo m.

Example 14.13. Let $m = 4$ be our modulus. For each element of the following set A, compute its remainder $\pmod 4$. Determine which integers in A are congruent to each other modulo 7.

$$A = \{-100,\ -10,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 11,\ 13,\ 14,\ 21,\ 55\}$$

remainder (mod 4) 0 2 0 1 2 3 0 1 2 3 0 3 1 2 1 3

x = $km + r$	x = $km + r$
$-100 = (-25)(4) + 0$	$2 = (0)(4) + 2$
$-10 = (-3)(4) + 2$	$3 = (0)(4) + 3$
$-4 = (-1)(4) + 0$	$4 = (1)(4) + 0$
$-3 = (-1)(4) + 1$	$11 = (2)(4) + 3$
$-2 = (-1)(4) + 2$	$13 = (3)(4) + 1$
$-1 = (-1)(4) + 3$	$14 = (3)(4) + 2$
$0 = (0)(4) + 0$	$21 = (5)(4) + 1$
$1 = (0)(4) + 1$	$55 = (13)(4) + 3$



Fact: $\equiv \pmod 4$ is an equivalence relation on A.

Ex The equivalence class of -100 is $[-100]_{\equiv (\text{mod} 4)} = \{-100, -4, 0, 4\}$

Ex The equivalence class of 2 is $[2]_{\equiv (\text{mod} 4)} = \{-10, -2, 2, 14\}$

Theorem 14.14. Let a and b be integers, and let m be a positive integer.

Then $\quad a \equiv b \pmod m \quad$ if and only if $\quad m$ divides $a - b$.

Exercise 14.15. Prove Theorem 14.14.

Example 14.16. Let m be a positive integer.

Prove that $\equiv \pmod m$ is an equivalence relation on $\mathbb{Z}$.

We will use the fact that $a \equiv b \pmod m \iff m \mid (a-b)$ (Theorem 14.14)

[reflexive] Let $x \in \mathbb{Z}$.

Then $x - x = 0 = (0)(m) + 0$ $\therefore m \mid (x-x)$ $\therefore x \equiv x \pmod m$

we proved $[x \in \mathbb{Z}] \to [x \equiv x \pmod m]$ $\therefore \equiv \pmod m$ is reflexive.

[Symmetric] Let $x, y \in \mathbb{Z}$.

Assume $x \equiv y \pmod m$ (goal: prove $y \equiv x \pmod m$).

Then $m \mid (x-y)$ (by Theorem 14.14)

$\Rightarrow$ $x - y = km$ for some integer k (def of divides)

$\Rightarrow$ $y - x = (-k)m$. Since $k \in \mathbb{Z}$, so too is $-k \in \mathbb{Z}$

$\therefore m \mid (y-x)$ $\therefore y \equiv x \pmod m$ (by Theorem 14.14)

we proved $[x \equiv y \pmod m] \to [y \equiv x \pmod m]$ $\therefore \equiv \pmod m$ is symmetric.

[transitive] Let $x, y, z \in \mathbb{Z}$

Assume $x \equiv y \pmod m$ and $y \equiv z \pmod m$. (goal: prove $x \equiv z \pmod m$).

Then $m \mid (x-y)$ and $m \mid (y-z)$ (by Theorem 14.14)

$\Rightarrow$ $x - y = km$ and $y - z = \ell m$ for some integers $k, \ell \in \mathbb{Z}$ (def of divides)

$\therefore x - z = km + y - (\ell m - y) = (k - \ell)m$. Since $k, \ell \in \mathbb{Z}$, so too is $k - \ell \in \mathbb{Z}$

$\therefore m \mid (x-z)$ $\therefore x \equiv z \pmod m$ (by Theorem 14.14)

we proved $[x \equiv y \pmod m \wedge y \equiv z \pmod m] \to [x \equiv z \pmod m]$

$\therefore \equiv \pmod m$ is transitive

Since it's reflexive, symmetric, and transitive,

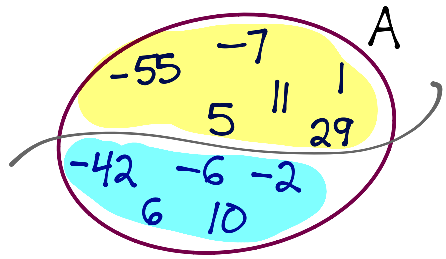
$\equiv \pmod m$ is indeed an equivalence relation on $\mathbb{Z}$.

Example 14.17. Let $A = \{-55, -42, -7, -6, -2, 1, 5, 6, 10, 11, 29\}$.

i. Determine the equivalence class of -55 with respect to the *equivalence* relation $\mathcal{R}_1$ on A defined by the rule:

for all $x, y \in A$, $\quad x\,\mathcal{R}_1$ if and only if $x + y$ is even.

$[-55]_{\mathcal{R}_1} = \{-55, -7, 1, 5, 11, 29\}$



$[-42]_{\mathcal{R}_1} = \{-42, -6, -2, 6, 10\}$

ii. Determine the equivalence class of -55 with respect to the *equivalence* relation $\mathcal{R}_2$ on A defined by the rule:

for all $x, y \in A$, $\quad x\,\mathcal{R}_2$ if and only if $x \equiv y \pmod 9$

$[-55]_{\mathcal{R}_2} = \{-55\}$

	-55	-42	-7	-6	-2	1	5	6	10	11	29
remainder(mod 9)	8	3	2	3	7	1	5	6	1	2	2

iii. Determine the equivalence class of -55 with respect to the *equivalence* relation $\mathcal{R}_3$ on A defined by the rule:

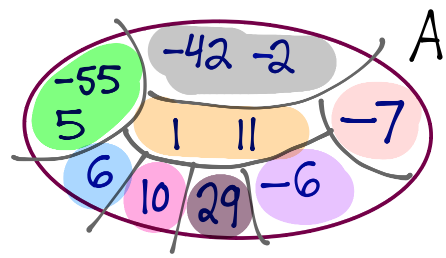
for all $x, y \in A$, $\quad x\,\mathcal{R}_3$ if and only if $x \equiv y \pmod 1$

$[-55]_{\mathcal{R}_3} = \{-55, -42, -7, -6, -2, 1, 5, 6, 10, 11, 29\}$

	-55	-42	-7	-6	-2	1	5	6	10	11	29
remainder(mod 1)	0	0	0	0	0	0	0	0	0	0	0

iv. Determine the equivalence class of each of the elements of A with respect to the *equivalence* relation $\mathcal{R}_4$ on A defined by the rule:

for all $x, y \in A$, $\quad x\,\mathcal{R}_4$ if and only if $x \equiv y \pmod{10}$



$[-55]_{\mathcal{R}_4} = \{-55, 5\}$

$[6]_{\mathcal{R}_4} = \{6\}$

	-55	-42	-7	-6	-2	1	5	6	10	11	29
remainder(mod 10)	5	8	3	4	8	1	5	6	0	1	9

Study Guide

relation on a set A $\mathcal{R} \subseteq A \times A$	**properties of a relation on a set:**	reflexive symmetric antisymmetric transitive
number of relations from A to B $= \lvert\mathcal{P}(A \times B)\rvert = 2^{\lvert A\rvert\lvert B\rvert}$	**equivalence relations:** **equivalence classes:**	reflexive, symmetric, & transitive $[a]_{\mathcal{R}} = \{x \in A : x\,\mathcal{R}\,a\}$
Exercises		Sup.Ex. §7 # 1a, 2, 3, 4a, 6a, 8a, 9, 10a, 11 Rosen §9.5 # 1, 3, 7, 11, 15, 25, 26, 29, 55