Lesson 5 – Derivatives of Composite Functions

PARTA: Recall Composition of Functions

$$h(x) = (x^{2} + 3x + 4)^{12}$$

$$h(x) = \sqrt{4x^{2} + 9}$$

$$h(x) = 2^{x^{2} - 4}$$
each can be expressed as $h(x) = f(g(x))$

Definition of a composite function

Given two functions f and g, the **composite function** $(f \circ g)$, is defined by $(f \circ g)(x) = f(g(x)).$

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Example

If
$$f(x) = \sqrt{x}$$
 and $g(x) = x+5$

Find a) g(f(4))

b)
$$f(g(x)) = \sqrt{x+5}$$

PARTB: Taking the Derivative of Composite Functions

We will now look at the Chain Rule which will allow us to take the derivative of composite functions of the form $F = f \circ g$ in terms of the derivatives of f and g.

The Chain Rule

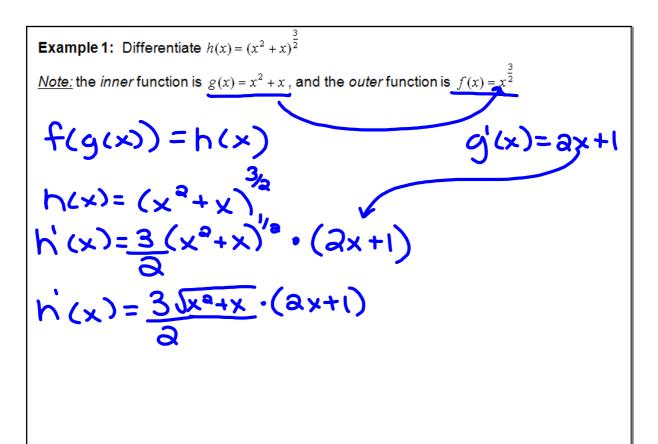
If f and g are functions that have derivatives, then the composite function h(x) = f(g(x))

has a derivative given by:

$$h'(x) = f'(g(x)) \cdot g'(x)$$

$$(x+5)^{3} \cdot (1)$$

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The Chain Rule in Leibniz Notation

If y is a function of u and u is a function of x (so that y is a composite function), then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

provided that $\frac{dy}{du}$ and $\frac{du}{dx}$ exist.

Example 2: If $y = u^3 - 2u + 1$, where $u = 2\sqrt{x}$, find $\frac{dy}{dx}$ at x = 4.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{dy}{dx} \cdot \frac{du}{dx}$$

$$= \frac{dy$$

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PART C: Combining multiple rules to differentiate a function

Let us first reiterate a rule that we stated before, but in more general terms.

The Power of a Function Rule

If u is a function of x, and n is a **real** number, then, $\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$

or

In function notation, if $f(x) = [g(x)]^n$, then $f'(x) = n[g(x)]^{n-1} \cdot g'(x)$

Example 1 Differentiate $h(x) = (x^2 + 3)^4 (4x - 5)^3$. Express answer in <u>simplified factored</u> form. <u>Note:</u> we must use both the product and chain rule.

$$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$$= \left[4(x^{2}+3)^{3}(3x)(4x-5)^{3}\right] + \left[12(x^{2}+3)^{4}\cdot 3(4x-5)^{2}\right]$$

$$= \left[8x(x^{2}+3)^{3}(4x-5)^{3}\right] + \left[12(x^{2}+3)^{4}(4x-5)^{2}\right]$$

$$= \left[4(x^{2}+3)^{3}(4x-5)^{3}\right] \left[2x(4x-5) + 3(x^{2}+3)\right]$$

$$= \left[4(x^{2}+3)^{3}(4x-5)^{3}\right] \left[8x^{2}-10x+3x^{2}+9\right]$$

$$= \left[4(x^{2}+3)^{3}(4x-5)^{3}\right] \left[11x^{2}-10x+9\right]$$

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Example #2 Differentiate
$$g(x) = \left(\frac{1+x^2}{1-x^2}\right)^{10}$$

$$g'(x) = 10\left(\frac{1+x^2}{1-x^2}\right)^q \left(\frac{2x-2x^3+2x+2x^3}{1-x^2}\right)$$

$$= 10\left(\frac{1+x^2}{1-x^2}\right)^q \left(\frac{4x}{1-x^2}\right)^q$$

$$= 10\left(\frac{1+x^2}{1-x^2}\right)^q \left(\frac{4x}{1-x^2}\right)^q$$

$$= 10\left(\frac{1+x^2}{1-x^2}\right)^q \left(\frac{4x}{1-x^2}\right)^q$$

$$= \frac{40x(1+x^2)^q}{(1-x^2)^n}$$