MAT2322 B Midterm 1 solutions - year 2018/2019

Calculus III for Engineers (University of Ottawa)

Key

Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 2322 B – Midterm I   Professor: S. Molladavoudi.

October 4, 2018   Time: 80 minutes.

Last name: ______________ First name: ___________ Student #: ___________

INSTRUCTIONS

Please, read the following instructions carefully:

- Questions 1 to 4 are multiple choice. These questions are worth 2 points each and no partial marks are possible.
- Questions 5 to 7 are long answer questions. Questions 5 and 7 are worth 6 marks each, and question 6 is worth 5 marks, so organize your time accordingly. **A correct answer requires a full, clearly-written and detailed solution.** Answer each question in the space provided, using backs of pages or the last page if necessary.
- **The only calculators which are allowed are Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.**
- **This is a closed book exam, and no notes of any kind are allowed. Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement.**

Signature: ___________________

THIS SPACE IS RESERVED FOR THE MARKER

Question	1	2	3	4	5	6	7	Total
Mark								
Out of	2	2	2	2	6	5	6	25

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Question 1. Which of these expressions corresponds to the equation of the tangent plane to the graph of $z = f(x, y) = x^3y^2$ at the point $(-1, 1, -1)$? `(2 points)`

`cross (X) the correct answer:`

- [A] $z = 3x - 2y - 1$
- [B] $z = 3x^2y^2(x+1) + 2x^3y(y-1) - 1$
- [C] $z = 3(x+1)\,\mathbf{i} - 2(y-1)\,\mathbf{j}$
- [D] This function does not admit a tangent plane at the indicated point.
- [E] $z = 0$
- [X] $z = 3(x+1) - 2(y-1) - 1$

$(x_0, y_0, z_0) = (-1, 1, -1)$

Tangent plane: $z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

$f_x(x,y) = 3x^2y^2 \Rightarrow f_x(-1,1) = 3$

$f_y(x,y) = 2x^3y \Rightarrow f_y(-1,1) = -2$

$z = 3(x+1) - 2(y-1) - 1$

Question 2. Which of the following statements is true concerning the critical points of the function $f(x, y) = x^3 + 18xy - 3y^2$? `(2 points)`

`cross (X) the correct answer:`

- [X] f has a saddle point at $(0, 0)$ and a local maximum at $(-18, -54)$.
- [B] f has a local minimum at $(0, 0)$ and a local maximum at $(-18, -54)$.
- [C] f does not have any critical points.
- [D] f has a saddle point at $(0, 0)$ and a local minimum at $(-18, -54)$.
- [E] f has a local maximum at $(0, 0)$.
- [F] f has a local maximum at $(0, 0)$ and a saddle point at $(-18, -54)$.

$f_x = 3x^2 + 18y = 0 \Rightarrow x^2 + 6y = 0 \Rightarrow x^2 + 18x = 0 \Rightarrow \begin{cases} x = 0, -18 \\ y = 0, -54 \end{cases}$

$f_y = 18x - 6y = 0 \Rightarrow 3x = y$

Critical points $(0,0), (-18, -54)$

$f_{xx} = 6x$, $f_{xy} = 18$

$f_{yy} = -6$ $\Rightarrow D(x,y) = f_{xx}f_{yy} - [f_{xy}]^2$

$D(0,0) < 0$ (saddle), $D(-18,-54) > 0$ & $f_{xx}(-18,-54) < 0$ local max.

Question 3. Consider the function $f(x,y) = x^3y + 2xy^2$. If you are at the point $(1,-1)$, in which direction $\mathbf{u}$ should you move in order to get the largest possible value of the directional derivative $D_{\mathbf{u}}f(1,-1)$? (2 points)

cross (X) the correct answer:

- [A] $\mathbf{u} = \frac{3}{\sqrt{10}}\mathbf{i} - \frac{1}{\sqrt{10}}\mathbf{j}$
- [B] $\mathbf{u} = \mathbf{i}$
- [C] $\mathbf{u} = \mathbf{j}$
- [D] This function does not have directional derivatives.
- [X] $\mathbf{u} = -\frac{1}{\sqrt{10}}\mathbf{i} - \frac{3}{\sqrt{10}}\mathbf{j}$
- [F] $\mathbf{u} = \frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$

$D_{\vec{u}} f(x,y) = \vec{\nabla} f(x,y) \cdot \vec{u}$

The max value of $D_{\vec{u}} f$ is $\|\vec{\nabla} f\|$, which occurs when $\vec{u}$ and $\vec{\nabla} f$ are parallel.

$\vec{\nabla} f(x,y) = \langle 3x^2y + 2y^2, x^3 + 4xy \rangle \Big|_{(1,-1)} = \langle -1, -3 \rangle$

$\vec{u} = \frac{1}{\|\vec{u}\|}(-\hat{i} - 3\hat{j})$, $\|\vec{u}\| = \sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$

$\Rightarrow \vec{u} = \frac{1}{\sqrt{10}}(-\hat{i} - 3\hat{j})$

Question 4. Let $z = f(x,y)$ be a function such that $f_x(2,6) = 3$ and $f_y(2,6) = 4$. Suppose that $x = x(u,v) = uv$ and $y = y(u,v) = u^2 + 2v^3$, and consider the composite function $z(u,v) = f(x(u,v), y(u,v))$. What is the value of $z_v(2,1)$? (2 points)

cross (X) the correct answer:

- [A] 18
- [B] −18
- [C] 19
- [D] −19
- [X] 30
- [F] −30

$z_v(u,v) = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$

$(u,v) = (2,1) \Rightarrow x(2,1) = 2,\ y(2,1) = 6$

$\frac{\partial z}{\partial x}\Big|_{(2,6)} = f_x(2,6) = 3,\quad \frac{\partial z}{\partial y}\Big|_{(2,6)} = f_y(2,6) = 4$

$\frac{\partial x}{\partial v}\Big|_{(2,1)} = u\Big|_{(2,1)} = 2,\quad \frac{\partial y}{\partial v}\Big|_{(2,1)} = 6v^2\Big|_{(2,1)} = 6$

$\Rightarrow z_v(2,1) = 3 \times 2 + 4 \times 6 = 30$

Question 5. Consider the function $f(x, y) = x^2 + y^2 + 3$ defined over the disk

$$D = \{(x, y) \in \mathbb{R}^2 \mid (x+1)^2 + (y-1)^2 \leq 8\}.$$

Find the absolute maximum and the absolute minimum of f on D. For part of your solution, you must use the method of Lagrange multipliers. **(6 points)**

(Clearly identify all steps of the optimization algorithm.)

$$\left.\begin{array}{l} f_x = 2x = 0 \\ f_y = 2y = 0 \end{array}\right\} \text{c.p. inside } D: (0,0) \Rightarrow f(0,0) = 3.$$

1

To find the extreme values of f on the boundary of D, we use the method of Lagrange multipliers:

$\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow f_x = \lambda g_x$ and $f_y = \lambda g_y$ 1

$g(x,y) = 0 \Rightarrow (x+1)^2 + (y-1)^2 = 8$

$$\left.\begin{array}{l} (1)\ 2x = 2\lambda(x+1) \Rightarrow x - \lambda x = \lambda \Rightarrow x = \lambda/(1-\lambda) \\ (2)\ 2y = 2\lambda(y-1) \Rightarrow y - \lambda y = -\lambda \Rightarrow y = \lambda/(\lambda - 1) \end{array}\right\} \text{plug into (3)}$$

$(3)\ (x+1)^2 + (y-1)^2 = 8$ 1

$$\left(\frac{\lambda}{1-\lambda} + 1\right)^2 + \left(\frac{\lambda}{\lambda-1} - 1\right)^2 = 8 \Rightarrow \frac{2}{(1-\lambda)^2} = 8 \Rightarrow \underline{\lambda = 1/2,\ 3/2}$$

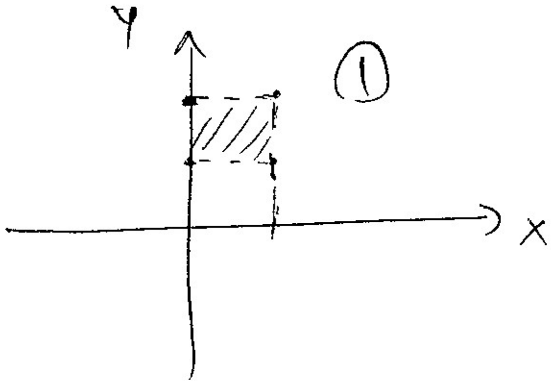
If $\lambda = 1/2 \Rightarrow x = 1,\ y = -1$ 1 and if $\lambda = 3/2 \Rightarrow x = -3,\ y = 3$ 1

Hence, f has possible extreme values at $(1,-1)$ and $(-3,3)$. Evaluating f at these points and at $(0,0)$, we find that:

min: $\underline{f(0,0) = 3}$, max: $\underline{f(-3,3) = 21}$ 1

Question 6. Consider the function $f(x, y) = x^2 + y^3 - x$. Compute $\iint_D f(x, y)\, dA$, where D is the rectangular region defined by the inequalities (5 points)

$$0 \le x \le 1, \quad 1 \le y \le 2.$$



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$$I = \int_0^1 \int_1^2 (x^2 + y^3 - x)\, dy\, dx$$

1

$$I = \int_0^1 \underbrace{\left(x^2 y + \frac{y^4}{4} - xy\right)\Big|_1^2}\, dx$$

1 $\quad 2x^2 + 4 - 2x - x^2 - \frac{1}{4} + x = x^2 - x + \frac{15}{4}$

$$I = \int_0^1 x^2 - x + \frac{15}{4}\, dx = \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{15}{4}x\right)\Big|_0^1 \quad 1$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{15}{4} = -\frac{1}{6} + \frac{15}{4}$$

$$= \frac{43}{12} \quad 1$$

Question 7. Find the area inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$ (as shown below). **(6 points)**

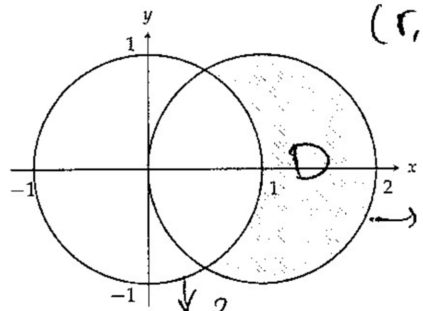
Recall:

$r^2 = x^2 + y^2$

$x = r\cos\theta$

$y = r\sin\theta$

(r,θ): Polar Coordinates



$\rightarrow r^2 = 2r\cos\theta \Rightarrow r = 2\cos\theta$

$r^2 = 1 \Rightarrow r = 1$

(Hint: according to the double angle formula $\cos 2\theta = 2\cos^2\theta - 1$*).*

1

parametrize D in polar coordinates: $x^2+y^2=1 \Rightarrow r^2=1 \Rightarrow r=1$ (1)

$(x-1)^2+y^2=1 \Rightarrow (r\cos\theta-1)^2+r^2\sin^2\theta=1$ 1

$\Rightarrow r^2(\underbrace{\cos^2\theta+\sin^2\theta}_{1}) - 2r\cos\theta + 1 = 1 \Rightarrow r^2 = 2r\cos\theta$ (2)

To find the bounds on (r,θ) we need to find the intersection of these two curves. From (1) we know that $r=1$ and if we replace it into (2), we have: $1 = 2\cos\theta \Rightarrow \cos\theta = 1/2 \Rightarrow \boxed{\theta = \pm\pi/3}$ 1

Also, for points inside the circle $r=2\cos\theta$, we have $r \leq 2\cos\theta$ and for points outside the circle $r=1$, we have $r \geq 1$. Hence, 1

$1 \leq r \leq 2\cos\theta$, and

1

$$A = \int_{-\pi/3}^{\pi/3}\int_{1}^{2\cos\theta} dA = \int_{-\pi/3}^{\pi/3}\int_{1}^{2\cos\theta} r\,dr\,d\theta = \int_{-\pi/3}^{\pi/3} \frac{1}{2}r^2\Big|_{1}^{2\cos\theta} d\theta$$

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3}(4\cos^2\theta - 1)\,d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3}(2(\cos 2\theta + 1) - 1)\,d\theta$$

$$A = \left(\frac{1}{2}\sin 2\theta + \frac{1}{2}\theta\right)\Big|_{-\pi/3}^{\pi/3} = \sqrt{3}/2 + \pi/3 = A$$ 1