# Notes for MAT1341A Fall 2023 Part II

**Chapter 11 & 12 - Solving systems of linear equations**

A **linear equation** is an equation of the from

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

Here, $a_i$ and $b$ are scalars (real numbers), $x_i$ are **unknowns** or **variables** or **in-determinants**.

$$2x + 3y = 1$$

this is a linear equation with _____ unknowns.

A **linear system** is a collection of **linear equations**.

$$\begin{aligned} x_1 - x_2 + x_3 &= 4 \qquad (1) \\ 2x_2 - x_3 &= 1 \qquad (2) \end{aligned}$$

is a linear system of _____ equations and _____ unknowns.

A **solution** to a linear system is a solution to **all** equations of the system simultaneously.

Notice that, $(4, 0, 0)$ is a solution to equation (1), but not (2), so this is not a solution to the system.

**Definition (11.1.2).** The *general solution* to a linear system is the set of *all* solutions.

**Definition (11.1.6).** A linear system is *consistent* if there exists at last one solution. Otherwise, the system is *inconsistent*.

[*E.g.*] Determine if the following system is consistent.

$$\begin{aligned} x + y &= 1 \\ y + z &= 2 \\ x + 2y + z &= -1 \end{aligned}$$

**Definition (11.1.7).** A linear system is *homogeneous* if all the constant terms are 0. Otherwise, the system is *inhomogeneous*.

[*E.g.*] Determine if the following system is homogeneous.

$$\begin{aligned} x + 2y &= 0 \\ 2y + z &= 0 \end{aligned}$$

***Fact.*** Homogeneous linear systems are always consistent.

**Theorem (11.1.9).** If a linear system is consistent, then it admits either

- a unique solution
- infinitely many solutions

We will see why this is true by learning how to solve linear systems systematically. The method is called **Gaussian elimination** (or **elementary row operations**).

We first put the coefficients of a linear system in **an augmented matrix**, which is a rectangular grid that looks like this:

$$\begin{aligned} 2x + y &= 3 \\ x - y &= 5 \end{aligned} \quad \longrightarrow \quad \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 2 \\ x_1 - 5x_3 &= -4 \end{aligned} \quad \longrightarrow \quad \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix}$$

We can carry out the following operations on an augmented matrix:

1. Add a multiple of one row to another.
2. Exchange two rows.
3. Multiply a row by a nonzero scalar.

[*E.g.*] Perform the elementary row operations to the following augmented matrix and find the solution.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -1 & 1 & 2 \\ 1 & 0 & -5 & -4 \end{array}\right]$$

**Definition (11.3.1).** A matrix is in row echelon form or *REF* if

1. All zero rows (if any) are at the bottom
2. The first nonzero entry in each row is a 1 (called a leading one or a pivot).
3. Each leading 1 is to the right of the leading 1s in the rows above.

A matrix is in reduced row echelon form or *RREF* if it is in *REF* and pivot is the only nonzero entry in its column.

[*E.g.*] Determine if the following matrices are in REF. How about RREF?

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 3 \\ 0 & 1 & 1 & 1 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

**Theorem (12.0.4).** Any matrix can be turned into RREF via elementary row operations. Furthermore, the RREF we get is *unique.*

[*E.g.*] Find the general solution to the linear system with following augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Whenever we have a system given by augmented matrix in RREF and there is a variable that gives a column that contains no leading 1, then this variable will give us a parameter in the general solution. In particular, this gives infinitely many solutions.

$[E.g.]$ Show that the following system is inconsistent.

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -3 & 2 & 0 \end{array}\right]$$

[*E.g.*] Solve the following linear system.

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & -2 & 6 \\ 0 & 0 & 0 & 1 & -4 \\ 1 & 2 & 3 & 0 & -3 \\ 1 & -2 & 3 & 0 & 5 \end{array}\right]$$

**Definition (12.0.1).** We say that two linear systems are *equivalent* if they have the same general solution.

***Fact.*** Every matrix is row equivalent to a matrix in RREF.

**Definition (12.0.3).** Two matrices $A$ is *row equivalent* to $B$, written $A \sim B$ if $B$ can be obtained from $A$ by elementary row operations.

**Definition (12.4.1).** The *rank* of a matrix $A$, denoted $\text{rank}(A)$, is the number of leading 1 (pivots) in any REF of $A$.

***Remark:*** In the Gaussian Algorithm, the passage from the REF to the RREF does not change the number of leading ones.

$$B = \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Then, we have $\text{rank}(B)=$ _____ .

If we have a linear system described by an augmented matrix:

$$B = [\,{\color{blue}A} \mid {\color{red}\mathbf{b}}\,]$$

a matrix

a column vector

What do the ranks of $A$ and $B$ tell us about the linear system?

We always have $\text{rank}(B) \geq \text{rank}(A)$.

If $\text{rank}(B) > \text{rank}(A)$, then the system is inconsistent. For example:

$$B = \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\text{rank}(B) =$
$\text{rank}(A) =$

If $\text{rank}(B) = \text{rank}(A)$, then the system is consistent. There are two sub-cases.

If $\text{rank}(A) = \#$ of unknowns, then we have a unique solution.

$$B = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\text{rank}(B) =$
$\text{rank}(A) =$

If $\text{rank}(A) < \#$ of unknowns, then we have infinitely many solutions.

$$B = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\text{rank}(B) =$
$\text{rank}(A) =$

We always have $\text{rank}(A) \leq \#$ of unknowns