Mat1322-practice Test 1-Solutions

Calculus II (University of Ottawa)

MAT1322C Practice Midterm Solutions

Multiple-choice Questions

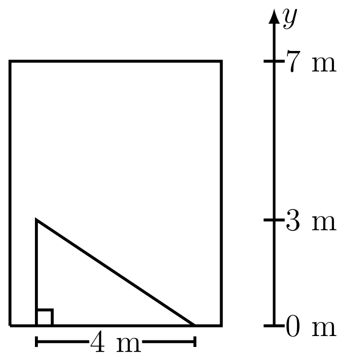
Q1. What is the area of the region bounded by the curves $y = 3x^2$ and $y + 2x = 1$ for $-1 \le x \le \frac{1}{3}$?

A. $\frac{9}{8}$ **B.** $\frac{3}{8}$ **C.** $\frac{7}{9}$ **D.** $\frac{9}{10}$ **E.** $\frac{32}{27}$ **F.** $\frac{1}{4}$

G. None of the above.

Solution: $\boxed{\textbf{Q1. E}}$

Q2. A reservoir has a triangular door located at the bottom of one of its vertical sides, as shown in the diagram below. The reservoir is 7 m high and filled to the top with water. The door is 4 m wide by 3 m high.



Let y represent the height from the bottom of the reservoir. Which of the following integrals represents the hydrostatic force exerted by the water on the door? Note that the density of water is 1000 kg/m^3 and the acceleration due to gravity is 9.8 m/s^2.

A. $9800\int_0^3 \frac{4}{3}(3-y)(y-7)dy$ **B.** $9800\int_0^3 6dy$ **C.** $9800\int_0^3 2(7-y)dy$

D. $9800\int_0^7 \frac{3}{4}(y-7)dy$ **E.** $9800\int_0^3 2(y-7)dy$ **F.** $9800\int_0^7 \frac{4}{3}(3-y)dy$

G. None of the above.

Solution: $\boxed{\textbf{Q2. G}}$

Q3. What is the arc length of the curve $y = 2x^{\frac{3}{2}} - 1$ between $x = 0$ and $x = 4$, rounded to one decimal place?

A. 16.6 **B.** 25.2 **C.** 51.7 **D.** 74.4 **E.** 32.3 **F.** 54.9

G. None of the above.

Solution: $\boxed{\textbf{Q3. A}}$

Q4. Use Euler's method with step size $h = 0.1$ to estimate $y(2.2)$, where $y(x)$ is the solution to the differential equation $y' = 2x^2 - y$ with initial condition $y(2) = 0$.

A. 1.52 **B.** 0.93 **C.** 1.602 **D.** 1.428 **E.** 0.8 **F.** 1.742

G. None of the above. Solution: $\boxed{\textbf{Q4. C}}$

Q5. Let $\mathcal{R}$ be the region bounded by the curve $y = e^{3x+1}$, the x-axis, and the lines $x = 2$ and $x = 5$.

Which of the following integrals represents the volume of the solid obtained by rotating the region $\mathcal{R}$ about the x-axis ?

A. $\displaystyle\int_2^5 2\pi e^{6x+2} dx$ **B.** $\displaystyle\int_2^5 \pi e^{6x+2} dx$ **C.** $\displaystyle\int_2^4 \pi e^{6x} dx$

D. $\displaystyle\int_2^5 \pi e^{(3x+1)^2} dx$ **E.** $\displaystyle\int_0^e \pi e^{3x+1} dx$ **F.** $\displaystyle\int_2^5 \pi e^{3x+1} dy$

G. None of the above. Solution: $\boxed{\textbf{Q5. B}}$

LONG-ANSWER QUESTIONS: Give detailed solutions, clearly showing each of your steps.

Q6. Consider the following differential equation: $\frac{dy}{dx} - y^2 + 4xy^2 = 0$

i. Determine its general solution.

Solution:

$$\begin{aligned}
&\frac{dy}{dx} - y^2 + 4xy^2 = 0 \\
&\implies \frac{dy}{dx} = y^2 - 4xy^2 \\
&\implies \frac{dy}{dx} = (1-4x)y^2 \\
&\implies \frac{1}{y^2} dy = (1-4x)dx \\
&\implies \int \frac{1}{y^2} dy = \int (1-4x)dx \\
&\implies -y^{-1} = x - 2x^2 + C \\
&\implies y = -\frac{1}{x-2x^2+C} \qquad \text{is the general solution}
\end{aligned}$$

ii. Determine the particular solution for which $y(0) = -2$.

Solution: We have

$$-2 = y(0) \implies -2 = -\frac{1}{0 - 2(0^2) + C} \implies C = \frac{1}{2}$$

The particular solution is thus $y = -\dfrac{1}{x - 2x^2 + 1/2}$.

Q7a. Consider the integral $\displaystyle\int_{-\infty}^{5} \frac{x}{(x^2+1)^3}\,dx$.

Explain what makes this an improper integral. Determine whether it converges or diverges. Fully justify your answer using appropriate methods and notation. If it converges, find its exact value.

Solution: The interval of integration is unbounded, which makes this an improper integra.

We have

$$\begin{aligned}
\int_{-\infty}^{5} \frac{x}{(x^2+1)^3}\,dx &= \lim_{t\to-\infty}\int_{t}^{5} \frac{x}{(x^2+1)^3}\,dx && u = x^2+1 \implies du = 2x\,dx\\
&= \lim_{t\to-\infty}\int_{t^2+1}^{26} \frac{x}{u^3}\left(\frac{dx}{2x}\right) && x=t \implies u=t^2+1,\ x=5 \implies u=26\\
&= \lim_{t\to-\infty}\int_{t^2+1}^{26} \frac{1}{2}u^{-3}\,du\\
&= \lim_{t\to-\infty} \tfrac{1}{2}\left[-\tfrac{1}{2}u^{-2}\right]_{t^2+1}^{26}\\
&= \lim_{t\to-\infty} -\tfrac{1}{4}\left[\tfrac{1}{(26)^2}-\tfrac{1}{(t^2+1)^2}\right]\\
&= -\frac{1}{4}\left[\frac{1}{(26)^2}-0\right]\\
&= -\frac{1}{2704}
\end{aligned}$$

Since the limit exists, this improper integral converges to $-\frac{1}{2704}$

Q7b. Using the comparison test, determine whether the following improper integral converges or not:

$$\int_{1}^{\infty} \frac{\sin^2(x)+x}{x^3+e^{5x}}\,dx$$

Solution: For all $x\in[1,\infty]$, we have

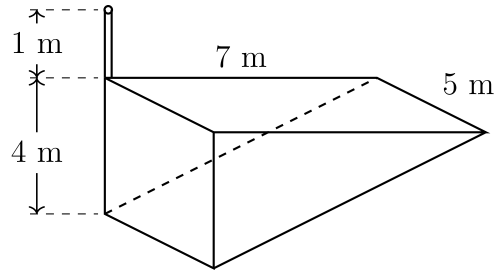
$$0 \le \frac{\sin^2(x)+x}{x^3+e^{5x}} \le \frac{1+x}{x^3} \le \frac{x+x}{x^3} = \frac{2x}{x^3} = 2\frac{1}{x^2}.$$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges (as it is of the form $\int_1^\infty \frac{1}{x^p}\,dx$ with $p>1$), we conclude that $2\int_1^\infty \frac{1}{x^2}\,dx$ converges.

Since $0 \le \int_1^\infty \frac{\sin^2(x)+x}{x^3+e^{5x}}\,dx \le 2\int_1^\infty \frac{1}{x^2}\,dx$, we conclude that $\int_1^\infty \frac{\sin^2(x)+x}{x^3+e^{5x}}\,dx$ must also converge.

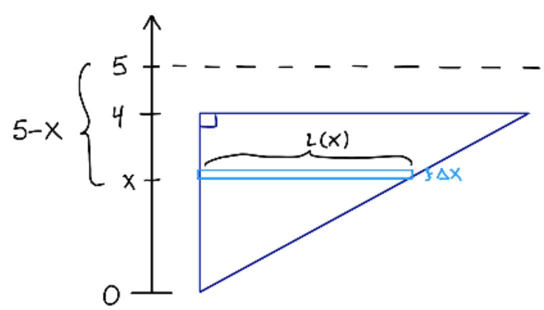
Q8. A pool as shown in the picture is filled with water. What work is done by pumping the water 1 m above the top of the pool? Clearly define all variables that enter into your solution and provide a diagram which shows their meaning.

Note that the density of water is 1000 kg/m^3 and the acceleration due to gravity is 9.8 m/s^2.



Solution:

Let x denote the height (in m), measured from the bottom of the pool.



For a thin layer of water (between heights x and $x + \Delta x$), we use similar triangles to express the approximate length $L(x)$ of this layer, as follows:

$$\frac{L(x)}{x} = \frac{7}{4} \implies L(x) = \frac{7}{4}x$$

Thus, the approximate volume of this layer is $V(x) \approx 5L(x)\Delta x = 5\left(\frac{7}{4}\right)x\Delta x$.

The approximate distance this layer must be lifted is $D(x) \approx 4 + 1 - x = 5 - x$.

The approximate work to lift this layer 1 m above the tank is thus $W(x) \approx \rho g V(x)D(x) \approx 9800(5\left(\frac{7}{4}\right)x\Delta x)(5 - x)$.

Summing over all such layers of water, where $x \in [0, 4]$, we find the total work to pump this water 1 m above the pool, as follows:

$$\begin{aligned} W &= \int_0^4 9800(5\left(\tfrac{7}{4}\right)x)(5 - x)\,dx \\ &= 85750 \int_0^4 (5x - x^2)\,dx \\ &= 85750\left[\tfrac{5}{2}x^2 - \tfrac{1}{3}x^3\right]_0^4 \\ &= 85750\left[\tfrac{5}{2}(4^2) - \tfrac{1}{3}(4^3) - (0 - 0)\right] \\ &\approx 1\,600\,666.67 \text{ J} \end{aligned}$$