Lesson 2 – The Derivatives of Polynomial Functions

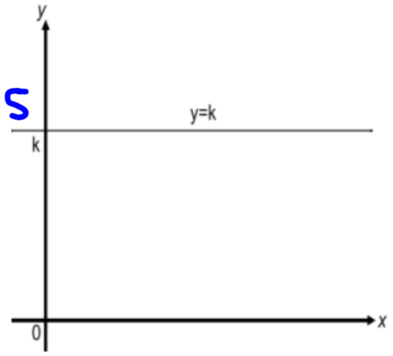
PART A: Stop the Insanity!!!!

If you are a fan of the sitcom Seinfeld (I know, it was before your time), you are familiar with the phrase “Serenity now!!!” If you are, you must be saying it to yourself as you brace yourself for more calculations of the derivative involving first principles and thus limits.

Well, you can breathe a sigh of relief because the sheer monotony and tedium of limit calculations to determine the derivative encouraged the gods of math to develop a more direct approach. The process for calculating derivatives based on these rules that were developed is called *differentiation*.

PART B: The Constant Function Rule

The graph of a constant function $f(x) = k$ is the horizontal line $y = k$. As can be seen from the graph below, a tangent of a point on this line is the line itself and since a horizontal line has a slope of zero, the slope of the tangent line is zero everywhere on the line.



The Constant Function Rule y=5

If $f(x) = k$, where k is a constant, then $f'(x) = 0$

or

In Leibniz notation, $\frac{d}{dx}(5) = 0$

Feb 18-8:09 AM

Proof using first principles:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

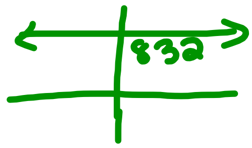
$$= \lim_{h\to 0}\frac{k-k}{h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

Example:

If $f(x) = 832$, determine $f'(x)$.



$f'(x)=0$

or

$\frac{d}{dx}(832)=0$

Feb 18-8:10 AM

The Power Rule

If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$

$y = x^5$ $= 5x^{5-1}$

or $= 5x^4$

In Leibniz notation, $\frac{d}{dx}(x^n) = nx^{n-1}$ x^5

Proof:

The proof uses first principles and the factorization of a difference of nth powers:

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b \ldots + ab^{n-2} + b^{n-1})$$

However, it will not be required of you to prove the power rule in general. You will only be required to prove the power rule using first principles for specific examples of power functions.

Example 1: Prove using first principles that the derivative of $f(x) = x^3$ is $f'(x) = 3x^2$.

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)(x+h)(x+h) - x^3}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)(x^2 + 2xh + h^2) - x^3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3 - \cancel{x^3}}{h}$$

$$= \lim_{h \to 0} \frac{3x^2\cancel{h} + 3xh^{\cancel{2}} + h^{\cancel{3}2}}{\cancel{h}}$$

$$= \lim_{h \to 0} 3x^2 + 3x\cancel{h}^0 + \cancel{h}^{2\,0}$$

$$= 3x^2 \qquad \therefore f'(x) = 3x^2$$

Feb 18-8:10 AM

Example 2: Determine the derivative of $f(x) = x^4$ using the power rule.

$$f'(x) = 4x^{4-1}$$
$$= 4x^3$$

$$\frac{d}{dx}(x^4) = 4x^{4-1}$$
$$= 4x^3$$

Example 3: Determine the derivative of $f(x) = \frac{1}{x^5}$ using the power rule.

$$f(x) = x^{-5}$$
$$= -5x^{-5-1}$$
$$= -5x^{-6}$$
$$= \frac{-5}{x^6}$$

$$\frac{d}{dx}\left(\frac{1}{x^5}\right) = x^{-5}$$
$$= -5x^{-5-1}$$
$$= -5x^{-6}$$

Feb 18-8:11 AM

The Constant Multiple Rule

If $f(x) = kg(x)$, where k is a constant, then $f'(x) = kg'(x)$

$=2x^4$ $=2(4x^3)$

or $=8x^3$

In Leibniz notation, $\frac{d}{dx}(ky) = k\frac{dy}{dx}$

Proof using first principles:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{kg(x+h) - kg(x)}{h}$$

$$= \lim_{h \to 0} k\left[\frac{g(x+h) - g(x)}{h}\right]$$

$$= kg'(x)$$

Example:

a) If $f(x) = 7x^2$, determine $f'(x)$

$f'(x) = 7(2x^{2-1})$

$= 7(2x)$

$= 14x$

$\frac{d}{dx}(7x^2) = 7\frac{d}{dx}(x^2)$

$= 7(2x)$

$= 14x$

Feb 18-8:11 AM

The Sum Rule

$x^3 + 3x^2 + 5$

If functions $p(x)$ and $q(x)$ are differentiable, and $f(x) = p(x) + q(x)$, then $f'(x) = p'(x) + q'(x)$

$= 3x^2 + 6x$

or

In Leibniz notation, $\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) + \frac{d}{dx}(q(x))$

The Difference Rule

$x^3 - 3x^2$

If functions $p(x)$ and $q(x)$ are differentiable, and $f(x) = p(x) - q(x)$, then $f'(x) = p'(x) - q'(x)$

$3x^2 - 6x$

or

In Leibniz notation, $\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) - \frac{d}{dx}(q(x))$

Feb 18-8:12 AM