8. Implicit Differentiation

Lec 7 mini review.

Two Special Limits:

$$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1} \qquad \boxed{\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0}$$

Trig Rules:

$$\boxed{\frac{d}{dx}[\sin x]=\cos x} \qquad \boxed{\frac{d}{dx}[\cos x]=-\sin x} \qquad \boxed{\frac{d}{dx}[\tan x]=\sec^2 x}$$

$$\boxed{\frac{d}{dx}[\csc x]=-\csc x\cot x} \qquad \boxed{\frac{d}{dx}[\sec x]=\sec x\tan x} \qquad \boxed{\frac{d}{dx}[\cot x]=-\csc^2 x}$$

The Chain Rule:

$$\boxed{\frac{d}{dx}\Big[f\big(g(x)\big)\Big]=f'\big(g(x)\big)g'(x)}$$

Power Chain Rule:

$$\boxed{\frac{d}{dx}\Big[\big(g(x)\big)^n\Big]=n\big(g(x)\big)^{n-1}g'(x)}$$

Exponential Chain Rule:

$$\boxed{\frac{d}{dx}\Big[e^{g(x)}\Big]=e^{g(x)}g'(x)}$$

WARM-UP FOR IMPLICIT DIFFERENTIATION

Differentiate each of the following expressions:

$f(x)=x^3g(x)+[h(x)]^5$

$V(t)=\pi[R(t)]^2H(t)$

$y=\left(\sqrt[3]{v(x)}\right)\Big[u(x)\Big]^{10}$

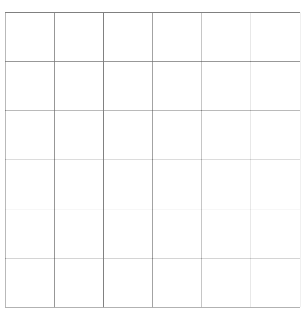
* These notes are solely for the personal use of students registered in MAT1320.

$$p(x) = e^{x^3} + e^{g(x)/h(x)} + \frac{h(x)}{\cos(x)} + \sin\big(f(x) + g(x)\big)$$

GRAPHS

- ◇ Any equation in two variables (let's use x and y) has a graph.
- ◇ The graph consists of all pairs of the form (x, y) that satisfy the equation.
- ◇ It might not be possible to isolate y and write an *explicit* formula $y = f(x)$.
- ◇ Nonetheless, we still think of y as an **implicit** function of x (where "function" is being used loosely; the graph of the equation could fail the vertical line test, hence not technically be a function of x)

Example 8.1. Consider the equation $x^2 + y^2 = 4$.



Implicit Differentiation

1. Start with some equation with x's and y's.

2. Implicitly differentiate both sides of the equation:

- Treat y as a "mystery" function of x (an implicitly defined function)
- When you need to write the derivative of y, just write $\frac{dy}{dx}$
- Differentiate x's as usual.
- When you are done differentiating both sides, you will have a new equation that may contain some x's, some y's, and some $\frac{dy}{dx}$'s. Because you performed the same operation (differentiation) to both sides of the original equation, the new equation is still a valid equation.

3. Isolate $\frac{dy}{dx}$ from your new equation:

- Put all terms that have a $\frac{dy}{dx}$ on one side of the equation, and put all other terms on the other side of the equation.
- Factor out $\frac{dy}{dx}$ from all terms on the $\frac{dy}{dx}$-side of the equation, then divide to isolate $\frac{dy}{dx}$.

Example 8.2. **a.** Find $\frac{dy}{dx}$ for the equation $x^2 + y^2 = 4$.

b. What is the slope of the tangent line to the graph of $x^2 + y^2 = 4$ at the point $(-1, \sqrt{3})$? What is it at $(-1, -\sqrt{3})$?

Example 8.3. For the following equation, find $\frac{dy}{dx}$ at the point $(1, 0)$: $\quad e^{2y+x} + x^2y^3 = e^x$

Example 8.4. Find $\frac{dy}{dx}$ if $\sin(x + y) = y^2 \cos(x)$.

Inverse Trig Derivatives

Derivative of Arcsine

Derivative of Arctangent

Inverse Trig Rules

Derivatives of Logarithms

Log Chain Rules

Example 8.5. Find $f'(x)$ if $f(x) = \ln|x|$.

Example 8.6. Differentiate each of the following:

$f(x) = \arctan(3e^x - 2x^5)$

$y = \ln(x)\sin^{-1}(x)$

LOGARITHMIC DIFFERENTIATION

Example 8.7. Find $f'(x)$ where $f(x) = x^x$.

Exercise 8.8. Use logarithmic differentiation to prove that the Power Rule (which we've been using for all sorts of powers $n \in \mathbb{R}$) is in fact valid.

That is, prove $\frac{d}{dx}[x^n] = nx^{n-1}$ for all $n \in \mathbb{R}$.

Study Guide

- **implicit differentiation strategy**
- **derivative rules for inverse trig functions**:
  $\boxed{\dfrac{d}{dx}[\sin^{-1} x] = \dfrac{1}{\sqrt{1-x^2}}}$ $\boxed{\dfrac{d}{dx}[\tan^{-1} x] = \dfrac{1}{1+x^2}}$ (and others!)
- **derivative rules for logs**: $\boxed{\dfrac{d}{dx}[\ln(x)] = \dfrac{1}{x}}$ $\boxed{\dfrac{d}{dx}[\log_b(x)] = \left(\dfrac{1}{\ln b}\right)\dfrac{1}{x}}$
- **log chain rule**: $\boxed{\dfrac{d}{dx}[\ln\big(g(x)\big)] = \dfrac{g'(x)}{g(x)}}$
- **logarithmic differentiation strategy**