Notes for MAT1341A Fall 2023 Part IV

Chapter 4 - Vector Spaces

Definition (4.2.1). Any set V satisfying the following 10 axioms is called a *vector space*.

Closure

(1) We have an addition on V such that given $\mathbf{x}, \mathbf{y} \in V$, we have $\mathbf{x} + \mathbf{y} \in V$.

(2) There is a multiplication by scale such that given $\mathbf{x} \in V$ and $c \in \mathbb{R}$, we have $c\mathbf{x} \in V$.

Existence

(3) There is a zero vector (or a neutral element, or an additive identity), denoted by $\vec{0}$ or simply $\mathbf{0}$ such that $\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x}, \forall \mathbf{x} \in \mathbb{R}$.

(4) $\forall \mathbf{x} \in V,\ \exists -\mathbf{x} \in V$ s.t. $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = 0$. $-\mathbf{x}$ is called the (additive) inverse of $\mathbf{x}$.

Arithmetic properties

For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and any $c, d \in \mathbb{R}$:

(5) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(6) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

(7) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(8) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(10) $1\mathbf{u} = \mathbf{u}$

[*E.g.*] $\mathbb{R}^n$ is a vector space.

[*E.g.*] Let $\mathbb{P}_n$ be the set of real polynomials of degree at most n.

- x^2 is a poly of degree _____ .
- $x+3$ is a poly of degree _____ .
- x^3-1 is a poly of degree _____ .

$\mathbb{P}_n = \{a_0 + a_1x + \ldots + a_nx^n \mid a_0, a_1, \ldots, a_n \in \mathbb{R}\}$

If

$$f(x) = a_0 + a_1x + \ldots + a_nx^n$$
$$g(x) = b_0 + b_1x + \ldots + b_nx^n$$

then

$$f + g = a_0 + b_0 + (a_1 + b_2)x + \ldots + (a_n + b_n)x^n$$

If $c \in \mathbb{R}$, we have $c \cdot f = ca_0 + ca_1x + \ldots + ca_nx^n$.

If $f, g \in \mathbb{P}_n$, then $f + g \in \mathbb{P}_n$. *Axiom 1*

If $c \in \mathbb{R}, f \in \mathbb{P}_n$, then $c \cdot f \in \mathbb{P}_n$. *Axiom 2*

The zero vector is the zero polynomial $f(x) = 0$. *Axiom 3*

The inverse of $a_0 + a_1x + \ldots + a_nx^n$ is $-a_0 - a_1x - \ldots - a_nx^n$. *Axiom 4*

Try the rest for yourself !

$\vdots$

$\Longrightarrow \mathbb{P}_n$ is a vector space.

[*E.g.*] Let $S = \{x \in \mathbb{R} \mid x \geq 0\}$. Is S a vector space?

[*E.g.*] The line $y = 2x + 1$ in $\mathbb{R}^2$. Is this a vector space?

[*E.g.*] What about the line $L : y = kx$, where k is a scalar?

[*E.g.*] Let $\mathcal{E}$ be the set of *all* linear equations in n variables, equipped with the usual addition and multiplication by scalars. Then $\mathcal{E}$ is a vector space.

[*E.g.*] The set $V = \{\mathbf{0}\} \subset \mathbb{R}^n$, with operations given by the rule $\mathbf{0} + \mathbf{0} = \mathbf{0}$, and $c \cdot \mathbf{0} = \mathbf{0}$, is a vector space. This is called the *zero vector space* or the *trivial vector space*.

Warning: the zero vector in V is not always the same as the zero scalar.

[*E.g.*] $M_{m,n}(\mathbb{R})$, the set of all $m \times n$ matrices is a vector space, for any $m, n \geq 1$.

[*E.g.*] Let $a < b$ be real numbers. The set of continuous functions $f : [a, b] \to \mathbb{R}$ is a vector space.

Chapter 5 - Subspaces

> **Definition (5.1.2).** Suppose V is a vector space. We say that a subset $W \subseteq V$ is a *subspace* if W is also a vector space under the same operations on V.

[*E.g.*] We know $V = \mathbb{R}^2$ is a vector space. $W = \{(x, kx) | x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ is also a vector space under the same operations as the ones on $\mathbb{R}^2$. So W is a subspace of $\mathbb{R}^2$.

> **Theorem (5.1.4 - Subspace Test).** If V is a vector space and $W \subseteq V$, then W is a subspace of V if and only if the following 3 conditions hold:
>
> 1. $\mathbf{0} \in W$
> 2. $\forall \mathbf{u}, \mathbf{v} \in W, \mathbf{u} + \mathbf{v} \in W$
> 3. $\forall c \in \mathbb{R}, \forall \mathbf{u} \in W$, we have $c\mathbf{u} \in W$

[*E.g.*] Is $S = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ a subspace of $\mathbb{R}^2$?

[*E.g.*] Let S be a linear system with n unknowns. We suppose that S is homogeneous. Show that set of general solutions to S is a subspace of $\mathbb{R}^n$.

[*E.g.*] Is $L = \{(x, y \in \mathbb{R}^2) | x - 3y = 1\}$ closed under addition?

[*E.g.*] Let $F(\mathbb{R})$ be the set of continuous function on $\mathbb{R}$. Is the set $T = \{f \in F(\mathbb{R}) | f(1) = 2\}$ closed under multiplication by scalar?

[*E.g.*] Let $\mathbb{P}$ be the set of polynomials with coefficients in $\mathbb{R}$. Is $S = \{f(x) \in \mathbb{P} | f(2) = 0\}$ a subspace of $\mathbb{P}$?

Chapter 6 - The Span of Vectors

Recall. If $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ are k elements in a vector space V, a linear combination of these elements is of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$$

where c_i are scalar.

[*E.g.*] $\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1 \cdot \hat{\imath} + 3 \cdot \hat{\jmath}$ is a linear combination of $\hat{\imath}$ and $\hat{\jmath}$.

Note that a linear combination may be written as $A \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$, where $A = [\mathbf{v}_1 \ldots \mathbf{v}_k]$.

[*E.g.*] Let V be the plane defined by $x + 2y - z = 0$. Pick $x = s, y = t$ for any real numbers s, t. Then, we have $z = s + 2t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ t \\ s+2t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

So elements in V are the linear combinations of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Definition (6.3.1). In general, we write $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for the set of *all* linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. We say that this set is spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

In our last example: $x + 2y - z = 0$, we have $V = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.

In particular, V is spanned by $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

[*E.g.*] Show that $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ is a plane and find its cartesian equation.

Definition (6.5.3). Given an $n \times n$ matrix A, define the *trace* of a matrix to be the sum of the entries on the diagonal. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the trace of A is $a + d$, denoted $\text{tr}(A)$.

[*E.g.*] Let $S = \{A \in M_{2,2} \mid \text{tr}(A) = 0\}$. Find a set of matrices so that S is spanned by this set.

[*E.g.*] Let the set $S = \{A \in M_{2,2}, A^\top = -A\}$, show that

$$S = \text{span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

Theorem (6.4.1 aka THE BIG THEOREM). Let V be a vector space. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset V$, define $U = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Then

(1) U is *always* a subspace of V.

(2) If W is any subspace of V which contains all the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$, then in fact $U \subseteq W$.

[*E.g.*] Show that $(0, 1, 1)$ and $(1, 0, 1)$ belong to $\text{span}\{(1, 1, 2), (-1, 1, 0)\}$.

[*E.g.*] Show that $\text{span}\{(0, 1, 1), (1, 0, 1)\} = \text{span}\{(1, 1, 2), (-1, 1, 0)\}$.

Recall. $\mathbb{P}_n = \{\text{polynomials of degree at most } n\}$. An element of $\mathbb{P}_3$ is of the form $a_0 + a_1x + a_2x^2 + a_3x^3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$, so it's a linear combination of $1, x, x^2, x^3$. $\mathbb{P}_3 = \text{span}\{1, x, x^2, x^3\}$.

[*E.g.*] Find a spanning set for $U = \{f(x) \in \mathbb{P}_2 | f(3) = 0\}$.