15. Integrals of Rational Functions & Partial Fractions

Lec 14 mini review.

useful trig identities:

$\cos^2(x) + \sin^2(x) = 1$

$1 + \tan^2(x) = \sec^2(x)$

$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

$\sin(2x) = 2\sin(x)\cos(x)$
$\cos(2x) = \cos^2(x) - \sin^2(x)$

expression:	**identity**:	**substitution**:
$\sqrt{1-x^2}$	$1 - \sin^2\theta = \cos^2\theta$	$x = \sin\theta$ $(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})$
$\sqrt{1+x^2}$	$1 + \tan^2\theta = \sec^2\theta$	$x = \tan\theta$ $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$
$\sqrt{x^2-1}$	$\sec^2\theta - 1 = \tan^2\theta$	$x = \sec\theta$ $(0 \leq \theta < \frac{\pi}{2}, \quad \pi \leq \theta < \frac{3\pi}{2})$

Strategies for Integrating Rational Functions

Recall: a **Rational Function** is of the form $f(x) = \frac{N(x)}{D(x)}$ where the numerator $N(x)$ and the denominator $D(x)$ are both polynomials.

We already know how to integrate some rational functions:

$$\int \frac{1}{x}\,dx$$

$$\int \frac{a}{bx+c}\,dx$$

$$\int \frac{1}{x^2+1}\,dx$$

$$\int \frac{g'(x)}{g(x)}\,dx$$ (where $g(x)$ is a polynomial)

Observation: the above forms of rational functions all have the property that the degree of the numerator is less than the degree of the denominator.

* These notes are solely for the personal use of students registered in MAT1320.

PARTIAL FRACTIONS

- Now, we consider a new way of expressing a rational function $\frac{N(x)}{D(x)}$ as a sum of simpler fractions.
- Before we can use this idea, we must, if necessary, reduce the integrand into a **PROPER** rational function, meaning one whose numerator $N(x)$ and denominator $D(x)$ satisfy

$$\deg(N) < \deg(D)$$

- If $\deg(N) \geq \deg(D)$, then $\frac{N(x)}{D(x)}$ is called an **IMPROPER RATIONAL FUNCTION** .
- We can use long division to turn any improper rational function into one that is proper.

Example 15.1. $\displaystyle\int \frac{1}{x^2-1}\,dx$

Example 15.2. $\displaystyle\int \frac{2x+3}{x^2+5x+6}\,dx$

Example 15.3. $\displaystyle\int \frac{2x^3 - 4x^2 + 10x + 1}{x^2 - 2x + 5}\, dx$

Partial Fractions with Repeated Factors

Once you have used long division to obtain a proper rational function, you need to factor its denominator $D(x)$.

Every polynomial can be factored into a product of **Linear Factors** (of the form $ax + b$) and **Irreducible Quadratic Factors** (of the form $ax^2 + bx + c$ where $b^2 - 4ac < 0$)

- ⋄ For each distinct **Linear Factor** of the denominator $D(x)$ – which may be a repeated factor (say, to the power r)

  $$(ax + b)^r$$

  the partial fractions decomposition will have r terms corresponding to the factor $(ax + b)^r$:

- ⋄ For each distinct **Irreducible Quadratic Factor** of the denominator $D(x)$ – which may be a repeated factor (say, to the power r) –

  $$(ax^2 + bx + c)^r \quad \text{(where } b^2 - 4ac < 0\text{)}$$

  the partial fractions decomposition will have r terms corresponding to the factor $(ax^2 + bx + c)^r$:

Example 15.4. $\displaystyle\int_2^3 \frac{2x+1}{x(x-1)^2}\,dx$

Example 15.5. Give the partial fraction setup for $\displaystyle\int \frac{2x^2+3x+1}{(x+2)(x-5)^3(x^2+1)^2(x^2-6x+13)}\,dx$

Exercise 15.6. Without solving the system of 10 equations in 10 unknowns, integrate each term of the partial fractions decomposition from Example 15.5.

Example 15.7. $\displaystyle\int \frac{x^2+2x+1}{(x^2+1)^2}\,dx$

STUDY GUIDE

- ◇ **integrating rational functions:** $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|$ $\qquad \int \frac{1}{x^2+1}\,dx = \tan^{-1}(x)$
- ◇ use long division to obtain a proper fraction
- ◇ factor denominator into product of linear and irreducible quadratic factors
- ◇ decompose integrand into its partial fractions