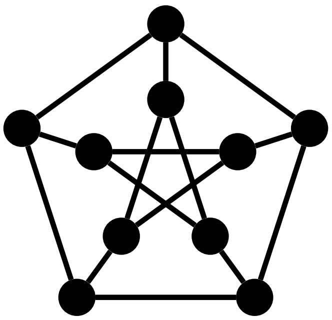
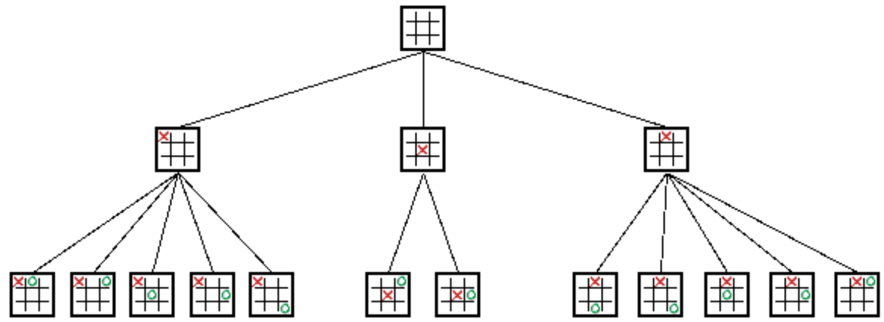
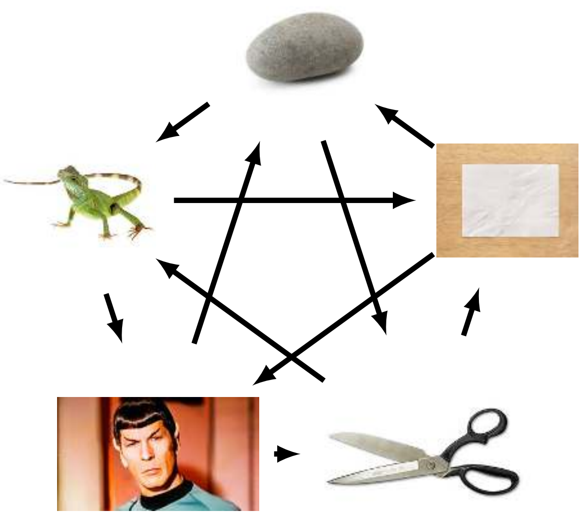
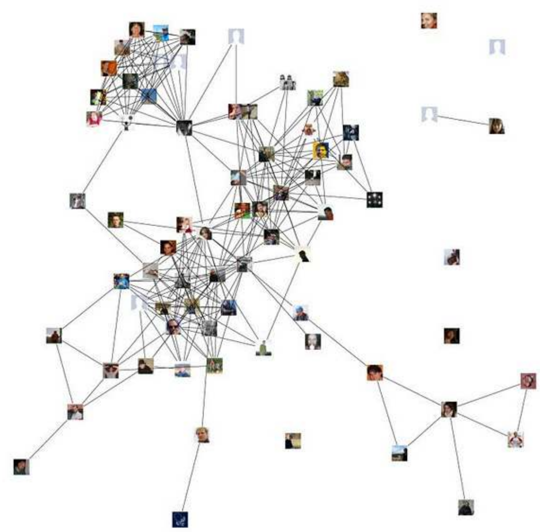
21. Introduction to Graph Theory

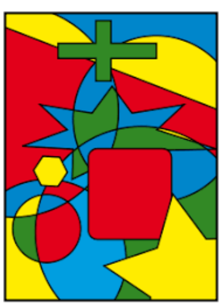
GRAPH THEORY EXAMPLES (BEFORE WE TALK ABOUT THE FORMAL DEFINITIONS)

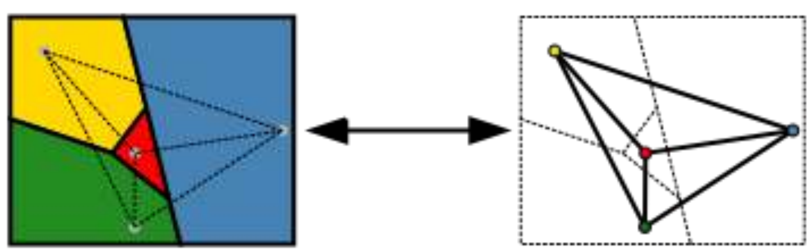


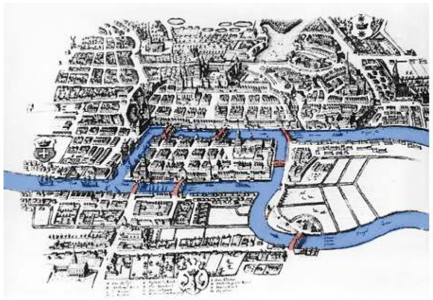


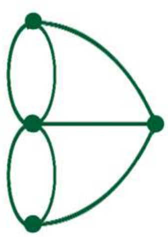


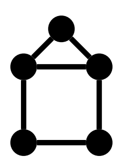


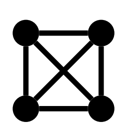


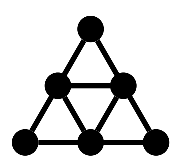












* These notes are solely for the personal use of students registered in MAT1348.

A **graph** G is an ordered pair $G = (V(G), E(G))$, where

- $V(G)$ is a nonempty set whose elements are called **vertices**.
- $V(G)$ is called the **vertex set** of G.

- $E(G)$ is a set whose elements are called **edges**.
- $E(G)$ is called the **edge set** of G.

- $V(G)$ and $E(G)$ are related to each other by a function

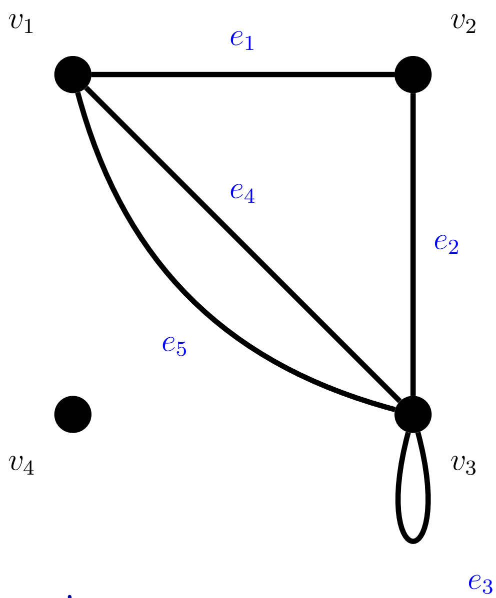
$$_G : E(G) \to \Big\{ \{u, v\} : u, v \in V(G) \Big\}$$

- $_G$ is called the **incidence function** of G
- for each edge $e \in E(G)$, $\quad \psi_G(e) = \{$the endpoint(s) of the edge $e\}$

Remark. Although we are using the word "graph", a graph, as defined above, is not the same as the graph of a function.

Example 21.1. Let G be a graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$ and edge set $E(G) = \{e_1, e_2, e_3, e_4, e_5\}$, whose incidence function ψ_G is defined as follows:

G



e_1, e_2, e_3, and e_4 are links

e_3 is a loop.

$\psi_G(e_1) = \{ V_1, V_2$

$\psi_G(e_2) = \{ V_2, V_3$

$\psi_G(e_3) = \{ \quad V_3$

$\psi_G(e_4) = \{ V_3, V_1$

$\psi_G(e_5) = \{ V_3, V_1$

e_4 and e_5 are parallel edges

G is not a simple graph.

- An edge $e \in E(G)$ is called a **loop** if
$$_G(e) = \{v\}$$
for some vertex $v \in V(G)$ (i.e. the endpoints of e coincide).

- An edge $e \in E(G)$ is called a **link** if
$$_G(e) = \{u, v\} \qquad \text{for two distinct vertices } u, v \in V(G),\ u \neq v.$$

- Distinct edges e_1 and e_2 are called **parallel edges** if
$$\psi_G(e_1) = \psi_G(e_2) \qquad \text{(i.e. } e_1 \text{ and } e_2 \text{ share the same endpoints).}$$

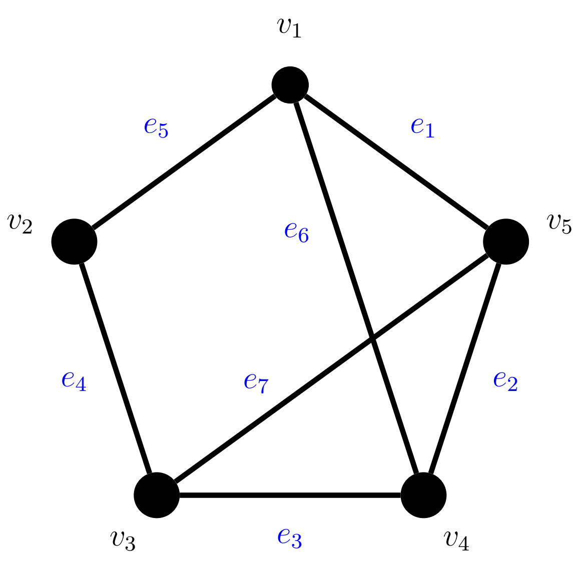
- A graph G is called a **simple graph** if G has no loops and no parallel edges.

Observations:

◇ If G is a graph with no parallel edges, then we can think of an edge e interchangeably with its endpoints (since there is at most one edge joining any set of endpoints).

◇ If G has parallel edges, then we need the incidence function ψ_G to keep track of which edge we are talking about when we consider two endpoints.

Example 21.2.



- v_1 and v_4 are adjacent
- we can also say that v_1 and v_4 are neighbours.
- the neighbours of v_3 are v_2, v_4, and v_5
- e_6 is incident with v_4
- $\psi(e_6) = \{v_1, v_4\}$
- the endpoints of e_6 are v_1 and v_4.

- this graph has no loops ✓
- this graph has no parallel edges ✓

∴ it is a simple graph

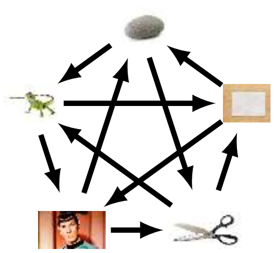
More Graph Terminology and Conventions

Let $G = (V, E)$ be a graph.

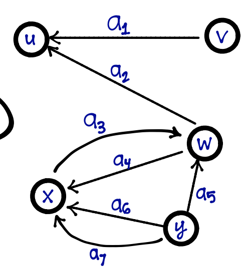
- Vertices $u, v \in V(G)$ are called **adjacent** or **neighbours** if there is some edge $e \in E(G)$ such that $\psi_G(e) = \{u, v\}$.
- **Notation for adjacency**: $u \sim v$ means "u and v are adjacent."
- An edge $e \in E(G)$ is said to be **incident** with its endpoints.
- For **undirected** graphs (which are the types of graphs we will consider) there is no importance placed on the order in which we specify the endpoint(s) of an edge.
- There is also the notion of a **directed graph** where the order of the endpoints of an **arc** (edge) is taken into consideration.

• a_1 is an arc from v to u ← terminal vertex

↖ initial vertex



Ex. D



• $\psi_D(a_1) = (v, u)$

• $\psi_D(a_1) \neq (u, v)$

The Adjacency Matrix of a Graph

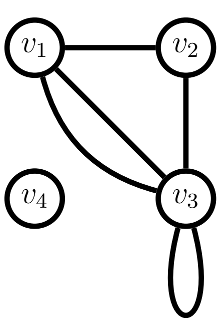
Suppose G is an undirected graph with n vertices labelled $v_1, v_2, \ldots, v_n$.

The **adjacency matrix of** G is an $n \times n$ matrix $A = [a_{ij}]$ whose (i, j)-entry a_{ij} (the entry in row i and column j) is

$$a_{ij} = \Big|\{e \in E(G) : \ _G(e) = \{v_i, v_j\}\}\Big|$$

That is, $a_{ij} = \#$ edges whose endpoints are v_i and v_j.

Example 21.3.



$$\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Let G be a graph.

The **degree** of a vertex $u \in V(G)$, denoted $\deg_G(u)$, is

$\deg_G(u) =$ # of edges incident with u,
where each loop incident with u is counted twice
(ie each loop incident with u contributes 2 to $\deg_G(u)$).

- if $\deg_G(u) = 0$, then u is called **isolated.**
- if $\deg_G(u) = 1$, then u is called a **leaf** or **pendant vertex**.

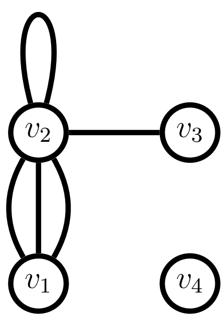
Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$.

The **degree sequence** of G is the sequence:

$$(\deg_G(v_1), \deg_G(v_2), \ldots, \deg_G(v_n))$$

Note. The entries of a degree sequence may be listed in any order, but most often, we list the degrees in a non-decreasing order.

Example 21.4.



G

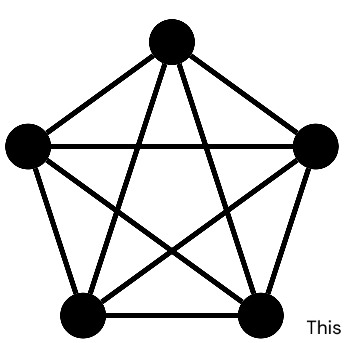
$\deg_G(v_1) = 3$

$\deg_G(v_2) = 6$ ← notice how the loop adds 2 to degree of v_2.

$\deg_G(v_3) = 1$ ← v_3 is a leaf

$\deg_G(v_4) = 0$ ← v_4 is isolated

degree sequence of G_1: $(3,6,1,0)$

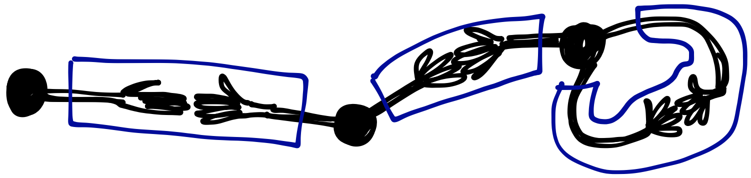


H

degree sequence of H: $(4,4,4,4,4)$

Theorem 21.5. Let G be any graph. Then

$$\sum_{u \in V(G)} \deg_G(u) = 2|E(G)|.$$



Proof. Let G be a graph. Let e be any edge of G. Then either e is a loop, or e is a link.

Case 1. Assume e is a link. Then e has two distinct endpoints, say u and v, and e will contribute 1 to $\deg_G(u)$ and e will contribute 1 to $\deg_G(v)$ in the sum.

Case 2. Assume e is a loop. Then e has only one endpoint, say u, and e will contribute 2 to $\deg_G(u)$ in the sum.

In both possible cases, e will contribute 2 in total to the degree sum.

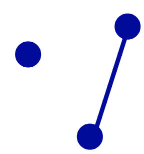
Since this is true of any edge $e \in E(G)$, it follows that the total degree sum must be equal to twice the number of edges. ☐

Exercise. Verify that the equation of The Handshaking Theorem is true for each of the undirected graphs in these notes.

Example 21.6. For each of the following sequences, determine whether there exists a graph with that sequence as its degree sequence. If so, does there exist a **simple** graph with that sequence as its degree sequence? In each case, either draw such a graph or justify why it cannot exist.

i. $(0, 1, 1)$ (# vertices) = 3 (# edges) = $\frac{1}{2}(0+1+1) = 1$ edge

Here is an example of a simple graph with degree sequence (0,1,1)



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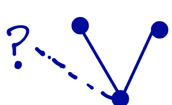
ii. $(1, 2, 3)$ (# vertices) = 3 (# edges) = $\frac{1}{2}(1+2+3) = 3$



needs degree 3...

There is no simple graph on 3 vertices with this degree sequence because, in a simple graph on 3 vertices, any vertex has at most 2 distinct neighbours

On the other hand, there does exist a non-simple graph with this degree sequence...

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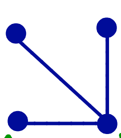
iii. $(1, 2, 2, 3)$ (# vertices) = 4 (# edges) = $\frac{1}{2}(1+2+2+3) = 4$

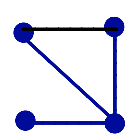
W.L.O.G.



a vertex of degree 1

a vertex of degree 3

Now we need one more edge and the other 2 vertices to have degree 2



✓

here is a simple graph with degree sequence (1,2,2,3)

Can you find a non-simple graph with this degree sequence ?

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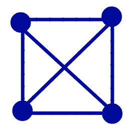
iv. $(3,3,3,3)$ $(\#\text{vertices}) = 4$ $(\#\text{edges}) = \frac{1}{2}(3+3+3+3) = 6$

all vertices have degree 3 and the graph has 4 vertices ∴ each vertex must be adjacent to each of the other 3 vertices



here is a simple graph with degree sequence $(3,3,3,3)$

Can you find a non-simple graph with this degree sequence?

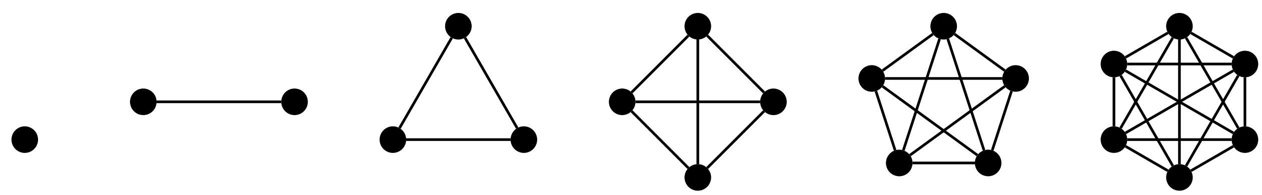
SOME IMPORTANT FAMILIES OF GRAPHS

Complete Graphs:

Let n be a positive integer.

- The **complete graph on n vertices** is denoted K_n.

- $|V(K_n)| = n$ $\qquad E(K_n) = \Big\{ \{u,v\} : u, v, \in V(K_n),\ u \neq v \Big\}$



K_1 $\quad K_2$ $\quad K_3$ $\quad K_4$ $\quad K_5$ $\quad K_6$

Question: How many edges does K_n have?

- In K_n, every subset of 2 distinct vertices forms the endpoints of one edge. ∴ there are $\binom{n}{2}$ edges in K_n.

- We also note that each vertex in K_n is adjacent to all $n-1$ other vertices, so the degree of each vertex in K_n is $n-1$

$\Rightarrow$ degree sequence of K_n is $(\underbrace{n-1, \dots, n-1}_{n \text{ times}})$

By the Handshaking Theorem

$$\left(\begin{matrix}\#\text{edges}\\ \text{in } K_n\end{matrix}\right) = \frac{1}{2} \sum_{u \in V(K_n)} \deg_{K_n}(u) = \frac{1}{2} n(n-1)$$

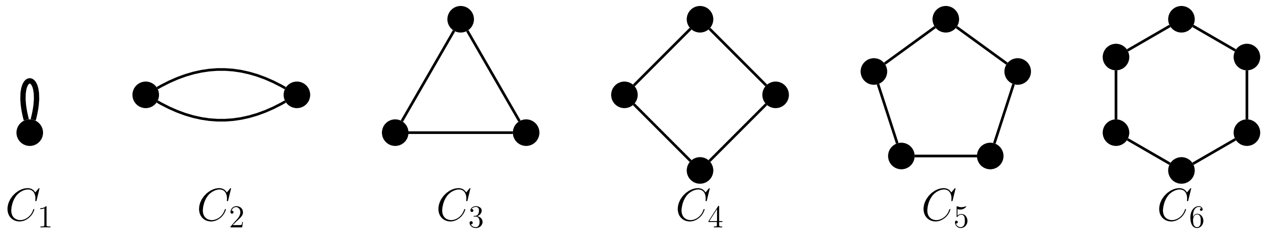
Another question: What does the adjacency matrix of K_n look like?

Cycles:

Let n be a positive integer.

- The **cycle of length** n is denoted C_n.

- $|V(C_n)| = n$.
- Suppose $V(C_n) = \{u_1, u_2, \dots, u_n\}$.
  Then $E(C_n) = \Big\{\{u_1, u_2\}, \{u_2, u_3\}, \dots, \{u_{n-1}, u_n\}, \{u_n, u_1\}\Big\}$



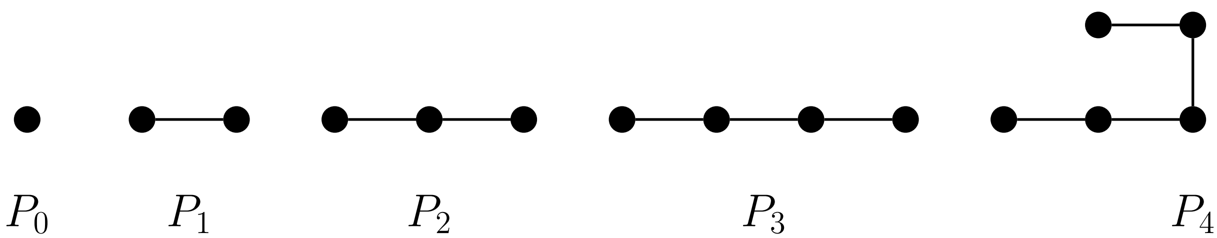
Degree sequence of C_n: $\underbrace{(2, 2, \dots, 2)}_{n \text{ times}}$

Paths:

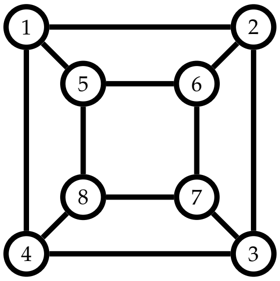
Let n be a positive integer.

- The **path of length** n is denoted P_n.

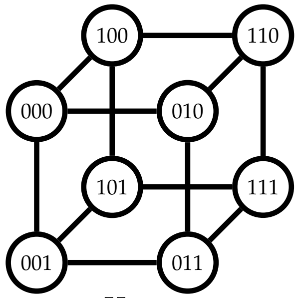
- $|V(P_n)| = n + 1$.
- Suppose $V(P_n) = \{u_0, u_1, u_2, \dots, u_n\}$.
  Then $E(\mathrm{P}_n) = \Big\{\{u_0, u_1\}, \{u_1, u_2\}, \{u_2, u_3\}, \dots, \{u_{n-1}, u_n\}\Big\}$



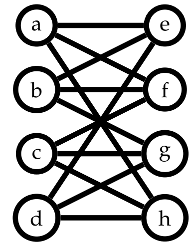
Example 21.7. Click here for an animated example



G



H



I

Let G and H be simple graphs.

- An **isomorphism** from G to H is a bijection $f : V(G) \to V(H)$ such that for all $u, v \in V(G)$

$$\Big(\{u, v\} \in E(G)\Big) \iff \Big(\{f(u), f(v)\} \in E(H)\Big).$$

- Graphs G and H are called **isomorphic** if there exists an isomorphism from G to H.

- **Notation:** $G \cong H$ means "G and H are isomorphic."

Let $\mathcal{A}$ denote the set of all finite simple graphs. Let us define a **relation** on $\mathcal{A}$ that relates graphs according to the following rule:

$$\text{for all } G, H \in \mathcal{A}, \quad G \text{ is related to } H \quad \iff \quad G \cong H.$$

Observations:

- For all $G \in \mathcal{A}$, $G \cong G$.

  why?! – The identity function $\mathrm{id}_{V(G)} : V(G) \to V(G)$ turns out to be an isomorphism from G to itself (verify that this is true!)

$\therefore \cong$ is a **reflexive** relation.

- For all $G, H \in \mathcal{A}$, if $G \cong H$, then $H \cong G$.

  why?! – Suppose f is an isomorphism from G to H. Then f is a bijection, hence f is invertible. You can verify the details, but it turns out that f^{-1} will define an isomorphism from H to G.

$\therefore \cong$ is a **symmetric** relation.

- For all $G, H, L \in \mathcal{A}$, if $G \cong H$ and $H \cong L$, then $G \cong L$.

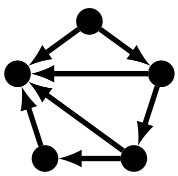
  why?! – Suppose f is an isomorphism from G to H and suppose g is an isomorphism from H to L. Then (verify the details!) it follows that the composition $g \circ f : V(G) \to V(L)$ will define an isomorphism from G to L.

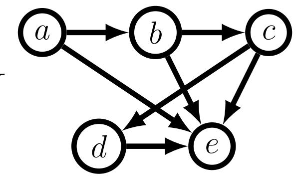
$\therefore \cong$ is a **transitive** relation.

So graph isomorphism defines an **equivalence relation** !

Isomorphism is defined similarly for directed graphs, but we need to make sure that the direction of the "arcs" is also preserved by the isomorphism.

Example 21.8. Click here for a movable directed graph so we can see why $G \cong H$





- To prove that $G \cong H$, we construct an isomorphism from G to H. That is, we give a **bijection** $f : V(G) \to V(H)$ and we verify that

$$\Big(\{u, v\} \in E(G)\Big) \Longleftrightarrow \Big(\{f(u), f(v)\} \in E(H)\Big) \qquad \text{for all } u, v \in V(G).$$

- To prove that $G \not\cong H$ can be tricky. We need to prove that there is something fundamentally different about G and H. Usually, we try to find a property or a "graph invariant" in which G and H differ, such as

  number of vertices, number of edges, degree sequence

  subgraphs contained in one graph that are not contained in the other (e.g. cycles, paths, complete subgraphs,...)

  chromatic number, clique number, independence number, girth, edge chromatic number, connectedness,... and there are many other graph invariants that one could compute and compare.

As of yet, it is not Known whether the Graph Isomorphism problem is polynomial-time solvable or NP-hard.

Study Guide

- ◇ graph G digraph D vertex set $V(G)$ edge set $E(G)$ incidence function ψ_G
- ◇ endpoints of an edge edge loop link parallel edges
- ◇ simple graph
- ◇ adjacent vertices neighbours $u \sim_G v$
- ◇ Adjacency matrix of a graph
- ◇ degree $\deg_G(u)$ degree sequence $\big(\deg_G(v_1), \ldots, \deg_G(v_n)\big)$
- ◇ **Handshaking Theorem**: $\sum_{v \in V(G)} \deg_G(v) = 2|E(G)|$
- ◇ Complete graphs K_n Cycles of length n C_n Paths of length n P_n
- ◇ Graph isomorphism $G \cong H$

Supp. Exercise List (on Brightspace)	§12 # 1, 2, 3, 4, 5, 6, 7(1), 9, 10
Graph Theory Notes (on Brightspace)	§1.4 # 1a, 2ab, 3, 5, 6 §2.5 # 1, 2, 3, 4, 6, 9, 10
Graph Theory Notes (on Brightspace)	§3.3 # (adjacency matrix only) 1, 2a, 3, 4, 7abc