

DGD 4**Q1. PROOF STRATEGIES — DIRECT PROOF**

Definition. Let d and n be integers and let $d \geq 1$. We say that d divides n if there exists an integer k such that $n = kd$. It means the **remainder** of n divided by d is 0.

Note. If d does not divide n , then there exist positive integers k and r such that $n = kd + r$, and $0 < r < d$. We call r the **remainder** of n divided by d . Indeed, if d divides n , then $n = kd + 0$ for some integer k , i.e. the remainder is $r = 0$.

For example, 2 divides -100 because $-100 = (-50)(2)$ and -50 is an integer.

For example, 3 does not divide 22 because $22 = (7)(3) + 1$ (so the remainder of 22 divided by 3 is not 0).

Give a **direct proof** of the following theorem:

Theorem 1. Let n be an integer. If $\underbrace{3 \text{ divides } (n+1)}_{P}$, then $\underbrace{3 \text{ divides } (n^3 + n + 2)}_{Q}$.

Direct proof of Theorem 1. Let n be an integer.

Assume P is true. i.e Assume 3 divides $n+1$. (goal: prove 3 divides n^3+n+2)

Then $n+1=3k$ for some integer k (by def. of divides)

Consequently, $n=3k-1$

$$\text{So } n^3+n+2 = (3k-1)^3 + (3k-1) + 2$$

$$= 27k^3 - 27k^2 + 9k - 1 + 3k - 1 + 2$$

$$= 27k^3 - 27k^2 + 12k$$

$$= 3[9k^3 - 9k^2 + 4k]$$

= $3j$ where $j = 9k^3 - 9k^2 + 4k$ so j is an integer.

$\therefore n^3+n+2$ is an integer multiple of 3 i.e $3 \mid (n^3+n+2)$

so Q is true. \therefore we proved $P \rightarrow Q$ is true



Q2. PROOF STRATEGIES — INDIRECT PROOF

Prove the following theorem with an **indirect proof**:

Theorem 2. Let n be an integer. If $\underbrace{n^3 + 5 \text{ is odd}}_P$, then $\underbrace{n+2 \text{ is even}}_Q$.

To prove $P \rightarrow Q$ with an indirect proof, we prove $\neg Q \rightarrow \neg P$ (the contrapositive $\neg Q \rightarrow \neg P$ is $\equiv P \rightarrow Q$)

$$\neg Q: n+2 \text{ is odd.} \quad \neg P: n^3 + 5 \text{ is even.}$$

Indirect Proof of Theorem 3.

Let n be an integer.

Assume $\neg Q$ is True. i.e. Assume $n+2$ is odd. (goal is to prove $\neg P: n^3 + 5$ is even)

Then $n+2 = 2m+1$ for some integer m .

$$\begin{aligned} \Rightarrow n &= 2m+1-2 \\ &= 2m-1 \end{aligned}$$

$$\text{Consequently, } n^3 + 5 = (2m-1)^3 + 5$$

$$= ((2m)^3 - 3(2m)^2 + 3(2m) - 1) + 5$$

$$= 8m^3 - 12m^2 + 6m + 4$$

$$= 2[4m^3 - 6m^2 + 3m + 2]$$

$$= 2j \text{ for } j = 4m^3 - 6m^2 + 3m + 2. \text{ Thus } j \in \mathbb{Z}$$

Thus, by def. $n^3 + 5$ is even. i.e. $\neg P$ is True.

Therefore, we proved $\neg Q \rightarrow \neg P$ is true.

$$\equiv P \rightarrow Q.$$



Q3. PROOF STRATEGIES — PROOF BY CONTRADICTION

Prove the following theorem using a **proof by contradiction**:

Theorem 3. Let x and y integers. If $\underbrace{x+y=2018}_{P}$, then $x < 1010$ or $y < 1010$.

$$P \rightarrow Q$$

To prove $P \rightarrow Q$ by contradiction, we assume $\neg(P \rightarrow Q)$ is true
(which is $\equiv P \wedge \neg Q$)

$\neg Q$ says: "It is not the case that $(x < 1010 \text{ or } y < 1010)$ "
ie $\neg(\underbrace{Q_1 \vee Q_2}_{\text{which is } \equiv \neg Q_1 \wedge \neg Q_2})$

In other words: $\neg Q$ says " $x \geq 1010$ and $y \geq 1010$ " (De Morgan's Law)

Proof.

Let x and y be integers.

Assume P is true and assume $\neg Q$ is true.

That is, assume $x+y=2018$ and assume $x \geq 1010$ and $y \geq 1010$.

Then $2018 = x+y$ (since P is true)

$\geq 1010 + 1010$ (since $\neg Q$ is true)

$= 2020$

so $2018 \geq 2020$

\therefore The assumption $\neg(P \rightarrow Q)$ is T leads to contradiction.

$\therefore P \rightarrow Q$ must be true.



Q4. PROOF STRATEGIES — PROOF OF EQUIVALENCE

Prove the following theorem:

Theorem 4. Let n be an integer. Then $\underbrace{5 \mid n^2}_{P}$ if and only if $\underbrace{5 \mid n}_{Q}$.

We must prove $P \rightarrow Q$ and $Q \rightarrow P$.

Indirect proof of $P \rightarrow Q$.

Let n be an integer.

Assume $\neg Q$ is true. i.e. Assume $5 \nmid n$. (goal: prove $\neg P$ is true i.e. prove $5 \nmid n^2$)

Then there are 4 possibilities for a nonzero remainder when n is divided by 5:

remainder 1, 2, 3, or 4.

Case 1 (remainder 1). Assume $n=5k+1$ for some integer k . (goal: prove $\neg P$ i.e. prove $5 \nmid n^2$)

$$\begin{aligned} \text{Then } n^2 &= (5k+1)^2 \\ &= 25k^2 + 10k + 1 \\ &= 5[5k^2 + 2k] + 1 \\ &= 5j + 1 \quad \text{where } j = 5k^2 + 2k \text{ so } j \in \mathbb{Z} \end{aligned}$$

Thus 5 does not divide n^2 in case 1 (i.e. $\neg P$ is true).

Case 2 (remainder 2). Assume $n=5k+2$ for some integer k . (goal: prove $\neg P$ i.e. prove $5 \nmid n^2$)

$$\begin{aligned} \text{Then } n^2 &= (5k+2)^2 \\ &= 25k^2 + 20k + 4 \\ &= 5[5k^2 + 4k] + 4 \\ &= 5j + 4 \quad \text{where } j = 5k^2 + 4k \text{ so } j \in \mathbb{Z} \end{aligned}$$

Thus 5 does not divide n^2 in case 2 (i.e. $\neg P$ is true).

Case 3 (remainder 3). Assume $n=5k+3$ for some integer k . (goal: prove $\neg P$)
Then $n^2 = (5k+3)^2$ Le prove $5 \nmid n^2$)

$$\begin{aligned} &= 25k^2 + 30k + 9 \\ &= 5[5k^2 + 6k + 1] + 4 \\ &= 5j + 4 \quad \text{where } j = 5k^2 + 6k + 1 \quad \text{so } j \in \mathbb{Z} \end{aligned}$$

Thus 5 does not divide n^2 in case 3 (ie $\neg P$ is true).

Case 4 (remainder 4). Assume $n=5k+4$ for some integer k . (goal: prove $\neg P$)
Then $n^2 = (5k+4)^2$ Le prove $5 \nmid n^2$)

$$\begin{aligned} &= 25k^2 + 40k + 16 \\ &= 5[5k^2 + 8k + 3] + 1 \\ &= 5j + 1 \quad \text{where } j = 5k^2 + 8k + 3 \quad \text{so } j \in \mathbb{Z} \end{aligned}$$

Thus 5 does not divide n^2 in case 4 (ie $\neg P$ is true).

Since $\neg Q \equiv (\text{case 1}) \vee (\text{case 2}) \vee (\text{case 3}) \vee (\text{case 4})$ and we proved Case 1 $\rightarrow \neg P$
Case 2 $\rightarrow \neg P$
Case 3 $\rightarrow \neg P$
Case 4 $\rightarrow \neg P$
it follows that we proved $\neg Q \rightarrow \neg P$.

$\therefore P \rightarrow Q$ is true.

Direct proof of $Q \rightarrow P$

Let n be an integer. Assume Q is true. ie Assume $5 \mid n$. (goal: prove P is T)
ie prove $5 \mid n^2$)

Then $n = 5k$ for some integer k .

Thus $n^2 = (5k)^2 = 25k^2 = 5(5k^2) = 5j$ where $j = 5k^2$ so $j \in \mathbb{Z}$.

$\therefore 5 \mid n^2$ (by def of divides) so P is true.

$\therefore Q \rightarrow P$ is true

Overall, we proved $P \rightarrow Q$ and $Q \rightarrow P$ \therefore we proved $P \leftrightarrow Q$ is true

ie we proved Theorem 5.1.

Q5. PROOF STRATEGIES

Prove the following theorem:

Theorem 5. The equation $x^3 + 3x + 5 = 0$ has no rational roots.

P

proof by contradiction

Assume $\neg P$ is true (goal: show this leads to contradiction).

i.e. Assume it is not the case that the eqⁿ: $x^3 + 3x + 5 = 0$ has no rational roots.

Then there must exist at least one root of this equation that is rational.

Assume r is a rational root of the equation.

Then, there exist integers n and d such that $r = \frac{n}{d}$ and $d \neq 0$ (def of rational)

We may assume without loss of generality, that n and d have no common factors other than ± 1 , i.e. the fraction $\frac{n}{d}$ is in lowest terms.

* In particular, n and d are not both even.

Since $r = \frac{n}{d}$ is a root, it follows that

$$\left(\frac{n}{d}\right)^3 + 3\left(\frac{n}{d}\right) + 5 = 0$$

consequently, $\frac{n^3}{d^3} + \frac{3n}{d} + 5 = 0 \Rightarrow n^3 + 3nd^2 + 5d^3 = 0$ (multiply both sides of equation by d^3)

We will consider all possible cases for the parity of n and d :

Case 1 (n even d odd)

Assume n is even and assumed d is odd.

Then $n = 2k$ for some $k \in \mathbb{Z}$ and $d = 2j+1$ for some $j \in \mathbb{Z}$ (def of even/odd)

Thus $0 = n^3 + 3nd^2 + 5d^3$

$$\begin{aligned} &= (2k)^3 + 3(2k)(2j+1)^2 + 5(2j+1)^3 \\ &= 8k^3 + 6k(4j^2 + 4j + 1) + 5(8j^3 + 12j^2 + 6j + 1) \\ &= 8k^3 + 6k(4j^2 + 4j + 1) + 40j^3 + 60j^2 + 30j + 5 \\ &= 2 \underbrace{[4k^3 + 3k(4j^2 + 4j + 1) + 20j^3 + 30j^2 + 15j + 2]}_{\text{is integer}} + 1 \end{aligned}$$

$= 2m + 1$ where $m = \text{is integer}$

$\therefore 0 = 2m + 1$ for some $m \in \mathbb{Z}$

$\therefore 0$ is odd $\cancel{\leftarrow}$ ($0 = 2(0)$ so 0 is definitely even)

Case 2 (n odd, d odd)

Assume n is odd and assumed d is odd.

Then $n = 2k+1$ for some $k \in \mathbb{Z}$ and $d = 2j+1$ for some $j \in \mathbb{Z}$ (def of odd)

Thus $O = n^3 + 3nd^2 + 5d^3$

$$\begin{aligned}
&= (2k+1)^3 + 3(2k+1)(2j+1)^2 + 5(2j+1)^3 \\
&= 8k^3 + 12k^2 + 6k + 1 + (6k+3)(4j^2 + 4j + 1) + 5(8j^3 + 12j^2 + 6j + 1) \\
&= 8k^3 + 12k^2 + 6k + 1 + 6k(4j^2 + 4j + 1) + 12j^2 + 12j + 3 + 40j^3 + 60j^2 + 30j + 5 \\
&= 8k^3 + 12k^2 + 6k + 24jk + 24jk + 6k + 12j^2 + 12j + 40j^3 + 60j^2 + 30j + 9 \\
&= 2 \underbrace{[4k^3 + 6k^2 + 3k + 12j^2k + 12jk + 3k + 6j^2 + 6j + 20j^3 + 30j^2 + 15j + 4]}_{\text{is integer}} + 1
\end{aligned}$$

\uparrow

$\therefore O = 2m + 1 \text{ for some } m \in \mathbb{Z}$

$\therefore O \text{ is odd} \cancel{\text{ }} \text{ (} O = 2(m) \text{ so } O \text{ is definitely even)}$

Case 3 (n odd, d even)

Assume n is odd and assumed d is even.

Then $n = 2k+1$ for some $k \in \mathbb{Z}$ and $d = 2j$ for some $j \in \mathbb{Z}$ (def of odd/even)

Thus $O = n^3 + 3nd^2 + 5d^3$

$$\begin{aligned}
&= (2k+1)^3 + 3(2k+1)(2j)^2 + 5(2j)^3 \\
&= 8k^3 + 12k^2 + 6k + 1 + (6k+3)(4j^2) + 5(8j^3) \\
&= 2 \underbrace{[4k^3 + 6k^2 + 3k + (6k+3)(2j^2) + 5(4j^3)]}_{\text{is integer}} + 1
\end{aligned}$$

\uparrow

$\therefore O = 2m + 1 \text{ for some } m \in \mathbb{Z}$

$\therefore O \text{ is odd} \cancel{\text{ }} \text{ (} O = 2(m) \text{ so } O \text{ is definitely even)}$

In all possible cases for the parity of n and d (given that $\frac{n}{d}$ is in lowest terms so they are not both even) we arrive at a contradiction.

\therefore it was wrong to assume P is true

$\therefore P$ itself must actually be true, that is, the eqⁿ has no rational roots.



Q6. PROOF STRATEGIES — PROOF BY CASES

Strolling along on The Island of Knights and Knaves, we meet an inhabitant called A (who is either a knight or a knave).

A says: "There is gold on The Island of Knights and Knaves if and only if I am a knight."

Is there gold on The Island of Knights and Knaves?

hint: break the problem down into two cases based on A's type.

(proof by cases)

Either A is a Knight or A is a Knave

Case 1. Assume A is a Knight.

Then A's statement must be T since Knights never lie.

A says: $(\text{There is gold}) \leftrightarrow (\text{A is a Knight})$

? T in this case

→ to make A's statement T, (There is gold) must be T. (because $(T \leftrightarrow T)$ is T).

∴ If A is a Knight, then there is gold on The Island.

Case 2. Assume A is a Knave.

Then (in this case), A's statement must be F since Knaves always lie

A says: $(\text{There is gold}) \leftrightarrow (\text{A is a Knight})$

? F in this case

→ to make A's statement F, (There is gold) must be T. (because $(T \leftrightarrow F)$ is F)

∴ If A is a Knave, then there is gold on The Island.

In both possible cases, it follows that there is gold on The Island of K&K.

∴ there is gold on The Island.