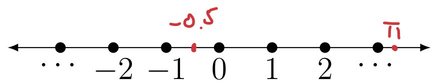
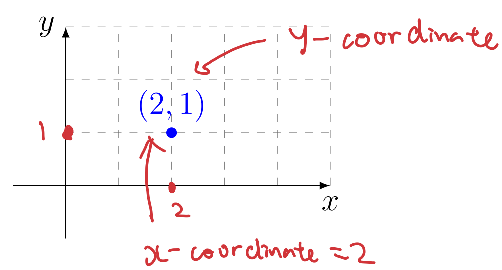
Notes for MAT1341A Fall 2023 Part I

Chapter 2 - Vector Geometry
MAT 1341

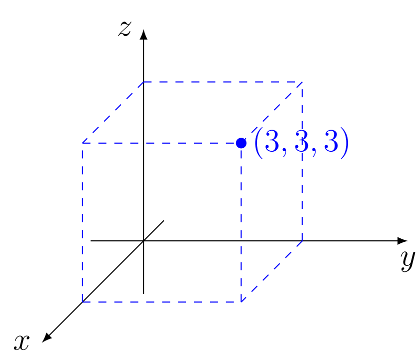
Let $\mathbb{R}$ be the set of real numbers. We can represent it on a line, sometimes called real line.



$\mathbb{R}^2$ represents the 2-dimensional plane. A point in $\mathbb{R}^2$ is represented by 2 coordinates, let's say (a, b), where $a, b \in \mathbb{R}$.



$\mathbb{R}^3$ is the 3-dimensional space. A point in $\mathbb{R}^3$ is represented by (a, b, c) where $a, b, c \in \mathbb{R}$.



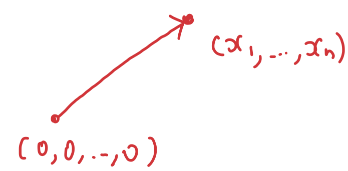
For any integer $n \geq 1$, we have the "n-space", denoted by $\mathbb{R}^n$. An element in $\mathbb{R}^n$ is represented by $(x_1, x_2, ..., x_n)$, where $x_i \in \mathbb{R}$ for $i = 1, 2, ..., n$.

belongs to / is an element of

A **vector** in $\mathbb{R}^n$ represents the displacement between two points.

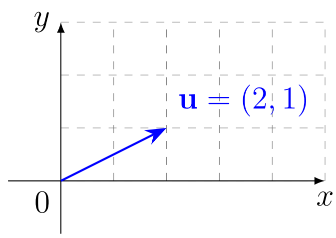
in $\mathbb{R}^n$

We write $\mathbf{u} = (x_1, ..., x_n)$ to represent a vector going from $(0, 0, ..., 0)$ to $(x_1, ..., x_n)$. We can also use the notation



$$\mathbf{u} = (x_1, \ldots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}^\top \qquad (\top = \text{transpose})$$

Sometimes we write $\vec{u}$ (instead of $\mathbf{u}$) to emphasize that the vector encodes a direction.



a vector in $\mathbb{R}^2$

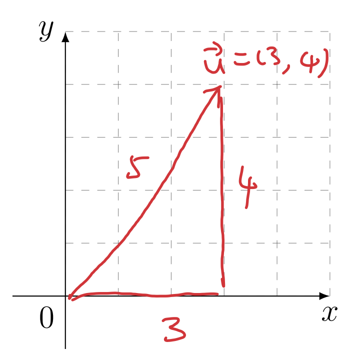
The **magnitude** (or **length**, or **norm**) of a vector is defined by

Alternative notation: $\sqrt{\sum_{i=1}^{n} x_i^2}$

$$||\mathbf{u}|| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \qquad \text{where } \mathbf{u} = (x_1, \ldots, x_n)$$

[*E.g.*] Find the magnitude of $\mathbf{u} = (3, 4)$.

$$||\vec{u}|| = \sqrt{3^2 + 4^2}$$
$$= 5$$



Special elements

We always have the **zero vector** (**null vector**) $\vec{0} = (0, 0, \ldots, 0)$

- In $\mathbb{R}^2$, we have $\hat{i} = (1, 0),\ \hat{j} = (0, 1)$
- In $\mathbb{R}^3$, we have $\hat{i} = (1, 0, 0),\ \hat{j} = (0, 1, 0),\ \hat{k} = (0, 0, 1)$

The notation $\hat{}$ means a unit vector, i.e. it's length is one.

In $\mathbb{R}^n$, we have $e_1 = (1, 0, 0, \ldots, 0)$
$e_2 = (0, 1, 0, \ldots, 0)$
$- \; - - \cdot - \; -$
$e_n = (0, 0, \ldots, 1)$

Manipulation of vector

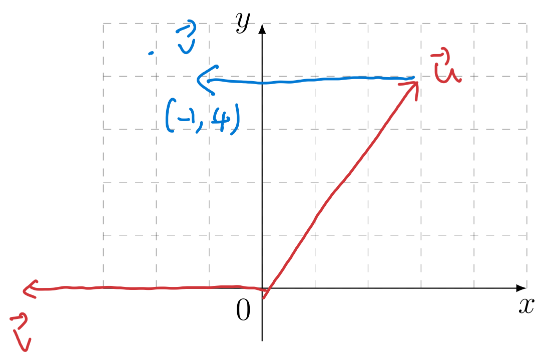
Let $\mathbf{u} = (x_1, \ldots, x_n), \mathbf{v} = (y_1, \ldots, y_n)$.
We can add them

$$\mathbf{u} + \mathbf{v} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

[*E.g.*] Find $\mathbf{u} + \mathbf{v}$, where $\mathbf{u} = (3, 4)$, $\mathbf{v} = (-4, 0)$.

$\vec{u} + \vec{v} = (3 - 4, 4 + 0)$

$= (-1, 4)$



Multiply a vector by a scalar (an element $c \in \mathbb{R}$)

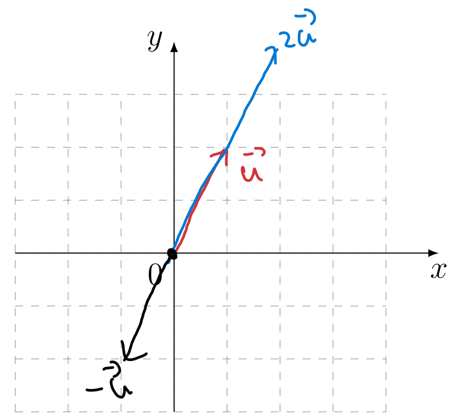
$$c\mathbf{u} = (cx_1, cx_2, \ldots, cx_n)$$

When $c = -1, -\mathbf{u} = (-x_1, \ldots, -x_n)$, this is the opposite of $\mathbf{u}$ (this reversed the direction of $\mathbf{u}$)

[*E.g.*] Find $2\mathbf{u}$ and $-\mathbf{u}$, where $\mathbf{u} = (1, 2)$.

$2\vec{u} = 2(1,2) = (2,4)$

$-\vec{u} = -(1,2) = (-1,-2)$



- If $c = 0$, then $c\vec{u} = \vec{0}$
- $\vec{u} + (-\vec{u}) = \vec{0}$

If $\underbrace{c, d}_{\text{scalars}} \in \mathbb{R}$, we can from a **linear combination** of $\underbrace{\mathbf{u} \text{ and } \mathbf{v}}_{\text{vectors}}$:

$$c\mathbf{u} + d\mathbf{v} = (cx_1 + dy_1, cx_2 + dy_2, \ldots, cx_n + dy_n)$$

If $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are m vectors in $\mathbb{R}^n$, then we can from a linear combination

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_m\mathbf{u}_m \quad \text{where } c_i \in \mathbb{R}$$

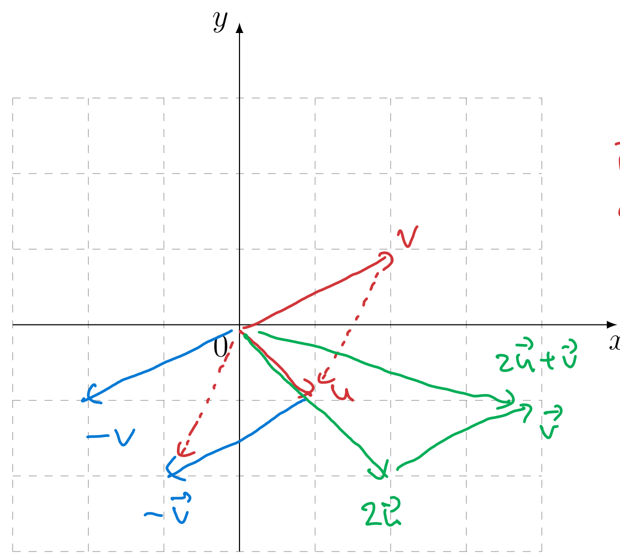
[*E.g.*] $\mathbf{u} = (1, -1)$ and $\mathbf{v} = (2, 1)$. Find $\mathbf{u} - \mathbf{v}$ and $2\mathbf{u} + \mathbf{v}$.

$u - v = (1,-1) - (2,1)$

$= (-1, -2)$

$2u + v = 2(1,-1) + (2,1)$

$= (4, -1)$



$\vec{u} - \vec{v}$ is the difference between the two vectors

Fact. Every element in $\mathbb{R}^2$ is a linear combination of $\hat{\imath}$ and $\hat{\jmath}$.
If $(a, b) \in \mathbb{R}^2$, then

$$\begin{aligned}(a, b) &= (a, 0) + (0, b) \\ &= a(1, 0) + b(0, 1) \\ &= a\hat{\imath} + b\hat{\jmath}\end{aligned}$$

In $\mathbb{R}^3$, (a,b,c)
$= a\hat{i} + b\hat{j} + c\hat{k}$.
Every element of $\mathbb{R}^3$ is a linear combination of $\hat{i}$, $\hat{j}$ and $\hat{k}$.

Similarly, in $\mathbb{R}^3$, every element is a linear combination of $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$.

[*E.g.*] Show that $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is **not** a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Suppose $\vec{u}$ is a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Then $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ for some $a, b \in \mathbb{R}$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ -b \\ a+b \end{bmatrix}$$

$a = 1$, $b = -2$, $a + b = 3$

$1 + (-2) = 3$, which is a contradiction

So $\vec{u}$ is not a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Basic properties

For any $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$

- $\mathbf{u} + \vec{0} = \vec{0} + \mathbf{u} = \mathbf{u}$
- $\mathbf{u} + (-\mathbf{u}) = \vec{0}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ [associativity]
- $(cc')\mathbf{u} = c(c'\mathbf{u})$, where $c, c \in \mathbb{R}$
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$, $(c + c')\mathbf{u} = c\mathbf{u} + c'\mathbf{u}$ [distributivity]
- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ [commutativity]

Dot products (inner products)

We can take $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and form the dot product $\mathbf{u} \cdot \mathbf{v}$ which is given by the following formula:

If $\mathbf{u} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{v} = (y_1, y_2, \ldots, y_n)$

Then $\mathbf{u} \cdot \mathbf{v} = x_1y_1 + x_2y_2 + \ldots + x_ny_n \in \mathbb{R}$

Alternative notation

$\sum_{i=1}^{n} x_i y_i$

[*E.g.*] $\mathbf{u} = (1, -2, 1), \mathbf{v} = (-3, 0, 5)$. Find $\mathbf{u} \cdot \mathbf{v}$

$(1,-2,1) \cdot (-3,0,5)$

$= 1 \times (-3) + (-2) \times 0 + 1 \times 5$

$= -3 + 0 + 5$

$= 2$

If $\mathbf{u} = \mathbf{v}$, then $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$

$u \cdot u = x_1 x_1 + x_2 x_2 + \cdots + x_n x_n$

$= x_1^2 + x_2^2 + \cdots + x_n^2$

$= ||u||^2$

Notice that

- $\mathbf{u} = \vec{0}$ if and only if $\mathbf{u} \cdot \mathbf{u} = 0$.
- $\mathbf{u} \cdot \mathbf{u}$ is always ≥ 0, and it is 0 if and only if $x_i = 0$ for all i.

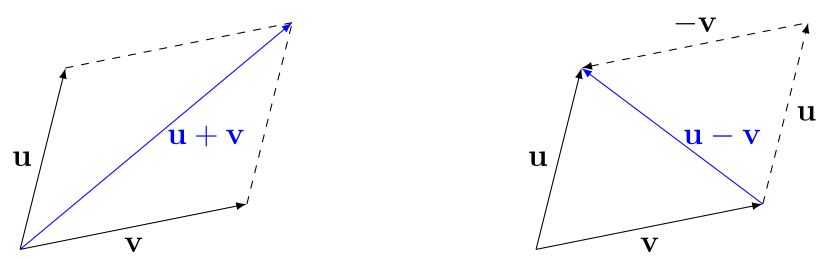
$u \cdot u = x_1^2 + \cdots + x_n^2 \geq 0$

This is 0 if and only if $x_1 = \cdots = x_n = 0$

Basic properties

(i) If $c \in \mathbb{R}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we have $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$

(ii) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (commutativity)



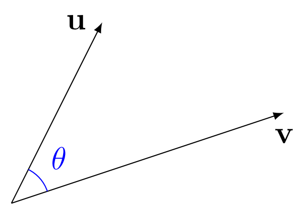
$||\mathbf{u} - \mathbf{v}||$ tells you the distance between the two end points of $\mathbf{u}$ and $\mathbf{v}$.

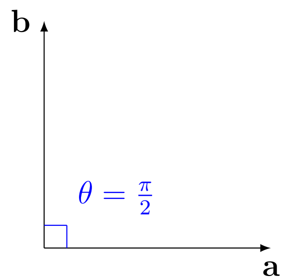
Orthogonality

We say that two vectors in $\mathbb{R}^n$ are **orthogonal** (or perpendicular) if their dot product is 0.

right-angle between two vectors.

More generally, $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos\theta$.
If $\mathbf{u} \cdot \mathbf{v} = 0$ and $||\mathbf{u}||, ||\mathbf{v}|| \neq 0$, then $\cos\theta = 0$, so $\theta = \frac{\pi}{2}$.





[*E.g.*] Find the angle between $\mathbf{u} = (3, 4)$ and $\mathbf{v} = (1, 1)$.

$$\cos\theta = \frac{u \cdot v}{||u|| \, ||v||} = \frac{(3,4)\cdot(1,1)}{||(3,4)|| \, ||(1,1)||}$$

$$= \frac{3+4}{\sqrt{3^2+4^2}\sqrt{1^2+1^2}}$$

$$= \frac{7}{5\sqrt{2}} \qquad \theta = \cos^{-1}\left(\frac{7}{5\sqrt{2}}\right)$$

> **Theorem** (**2.7.1** Cauchy-Schwartz inequality)**.** For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then
>
> Textbook
>
> $$|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}|| \cdot ||\mathbf{v}||$$
>
> absolute of a real number

$\mathbf{u} = (1, 1), \mathbf{v} = (-1, -1)$

$u \cdot v = -1 - 1 = -2$

$||u|| = \sqrt{1^2+1^2} = \sqrt{2}$

$||v|| = \sqrt{1^2+1^2} = \sqrt{2}$

LHS:
$|u \cdot v| = 2$

RHS
$\sqrt{2} \cdot \sqrt{2} = 2$

We have equality in this example.

Corollary (Triangle inequality). $||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}||+||\mathbf{v}||$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

Proof.

$$\begin{aligned}
||u+v||^2 &= (u+v)\cdot(u+v) \\
&= u\cdot u + u\cdot v + v\cdot u + v\cdot v \\
&= ||u||^2 + u\cdot v + u\cdot v + ||v||^2 \\
&= ||u||^2 + 2u\cdot v + ||v||^2 \\
&\leq ||u||^2 + 2\,|u\cdot v| + ||v||^2 \qquad (\text{if } x \in \mathbb{R},\ |x| \geqslant x) \\
&\leq ||u||^2 + 2||u||\,||v|| + ||v||^2 \qquad (\text{by Cauchy-Schwarz inequality}) \\
&= (||u|| + ||v||)^2
\end{aligned}$$

Take square roots: $||u+v|| \leq ||u|| + ||v||$.

Orthogonal projection

Let $\mathbf{u}$ be a non-zero vector in $\mathbb{R}^n$. If $\mathbf{v}$ is a vector in $\mathbb{R}^n$, we define the orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$, denoted

$$\text{proj}_{\mathbf{u}}(\mathbf{v})$$

is the unique vector which satisfies

- $\text{proj}_{\mathbf{u}}(\mathbf{v})$ is parallel to $\mathbf{u}$
- $\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v}) \perp \mathbf{u}$ (is orthogonal to $\mathbf{u}$)

We decompose $\mathbf{v}$ as a sum

$$\mathbf{v} = (\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})) + \text{proj}_{\mathbf{u}}(\mathbf{v})$$

Using either trigonometry, or just solving directly from the above two conditions, we get:

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{v}\cdot\mathbf{u}}{||\mathbf{u}||^2}\mathbf{u}$$

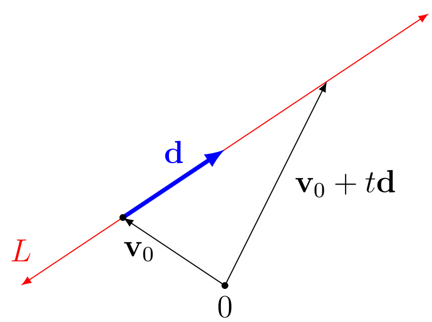
Proof.

[*E.g.*] If $\mathbf{u} = (1, 2), \mathbf{v} = (3, -1)$. Find $\text{proj}_{\mathbf{u}}(\mathbf{v})$ and $\text{proj}_{\mathbf{v}}(\mathbf{u})$.

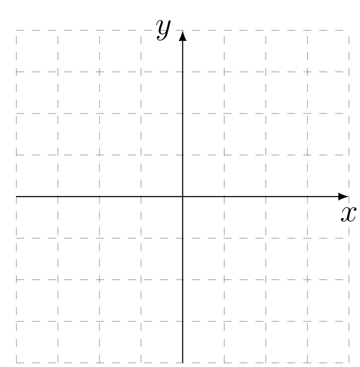
Chapter 3 - Lines and Planes

In $\mathbb{R}^n$, we can describe a line using parametric equations.
A line L going through the tip of $\mathbf{v}_0$ and such that $\mathbf{d}$ is a vector parallel to the direction of L can be described as the set

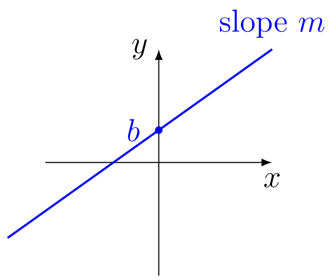
Any point on this line L can be written as $\mathbf{v}_0 + t\mathbf{d}$, where $t \in \mathbb{R}$



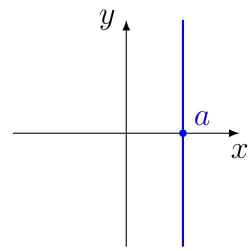
[*E.g.*] Find the parametric equation of the line in $\mathbb{R}^2$ passing through $P = (1, 2)$ and $Q = (3, -2)$



A line in $\mathbb{R}^2$ can be described by a Cartesian equation as well.



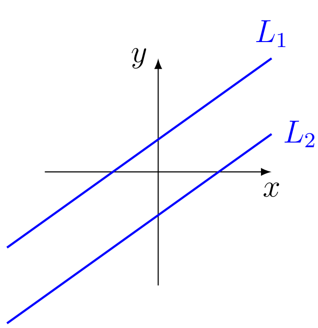
(a) $y = mx + b$

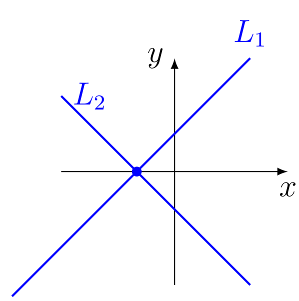


(b) $x = a$

[*E.g.*] Find the intersection of $L_1 = \{(1, 2) + t(-1, 1)) | t \in \mathbb{R}\}$ and $L_2 : y = 2x - 1$

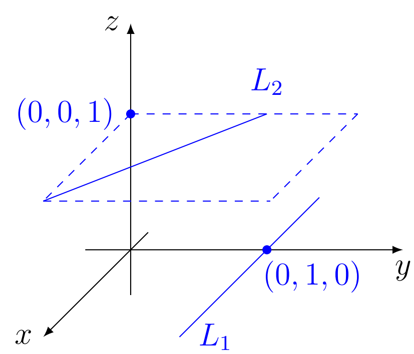
In $\mathbb{R}^2$, two distinct line, they are either parallel or have an intersection.





In $\mathbb{R}^3$, two distinct lines can be parallel to each other, or they can have an intersection, or they are skewed (they are not parallel and lie in two planes).

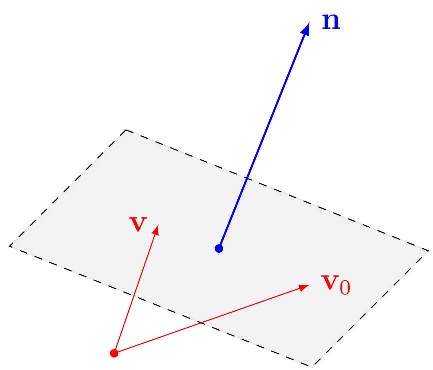
[*E.g.*]



A plane in $\mathbb{R}^3$ has a **Cartesian equation** of the form

$$ax + by + cz = d$$

where $a, b, c, d \in \mathbb{R}$



There is a vector in $\mathbb{R}^3$ that is $\perp$ to the plane. We call this a **normal vector** of the plane. If the plane passes through $\mathbf{v}_0$.

Given another point $\mathbf{v} = (x, y, z)$, then $\mathbf{v} - \mathbf{v}_0 \perp \mathbf{n}$. We have

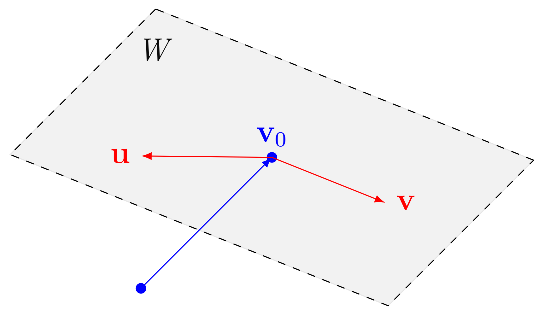
$$\begin{aligned}(\mathbf{v} - \mathbf{v}_0) \cdot \mathbf{n} &= 0 \\ \mathbf{v} \cdot \mathbf{n} - \mathbf{v}_0 \cdot \mathbf{n} &= 0 \\ \mathbf{v} \cdot \mathbf{n} &= \mathbf{v}_0 \cdot \mathbf{n}\end{aligned}$$

So if $\mathbf{n} = (a, b, c)$, then

$$\begin{aligned}(x, y, z) \cdot (a, b, c) &= \mathbf{v}_0 \cdot \mathbf{n} \\ ax + by + cz &= \mathbf{v}_0 \cdot \mathbf{n}\end{aligned}$$

[*E.g.*] Find the Cartesian of the plane passes $\mathbf{v}_0 = (0, 1, 0)$ with normal vector $\mathbf{n} = (-1, 2, 2)$.

Suppose we are given a point $\mathbf{v}_0$ in the plane and two vectors $\mathbf{u}$ and $\mathbf{v}$ that are parallel to the plane. Then, we may describe the plane parametrically:



$[E.g.]$ Find a parametric equation for the plane $2x + y - z = 5$.

Cross products

Given $\mathbf{u} = (a, b, c)$ and $\mathbf{v} = (x, y, z)$ in $\mathbb{R}^3$, the **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is denoted by

$$\begin{aligned}\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & c \\ x & y & z \end{vmatrix} &= \hat{\imath}(bz - cy) - \hat{\jmath}(az - cx) + \hat{k}(ay - bx) \\ &= (bz - cy, cx - az, ay - bx)\end{aligned}$$

Recall $\hat{\imath} = (1, 0, 0), \hat{\jmath} = (0, 1, 0), \hat{k} = (0, 0, 1)$. We have:

$$\hat{\imath} \times \hat{\jmath} = \hat{k}, \hat{\jmath} \times \hat{k} = \hat{\imath}, \hat{k} \times \hat{\imath} = \hat{\jmath}.$$

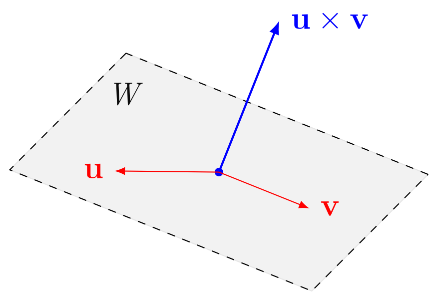
Important properties of the cross product:

- $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.
- $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} =$
- $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$.

What about associativity?

[*E.g.*] Find $(1, 0, 2) \times (1, 2, -1)$.

$\mathbf{u} \times \mathbf{v}$ will give us a vector $\perp$ to $\mathbf{u}$ and $\mathbf{v}$. So $\mathbf{u} \times \mathbf{v}$ gives a normal vector of the plane parallel to $\mathbf{u}$ and $\mathbf{v}$.



[*E.g.*] If a plane W passes through $P = (1, 2, 3), Q = (-3, 2, 1)$ and $R = (2, 4, 5)$. Find a normal vector of W.

Another useful fact: $$\|\|\mathbf{u} \times \mathbf{v}\|\| = \|\|\mathbf{u}\|\| \; \|\|\mathbf{v}\|\| \sin\theta$$ where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

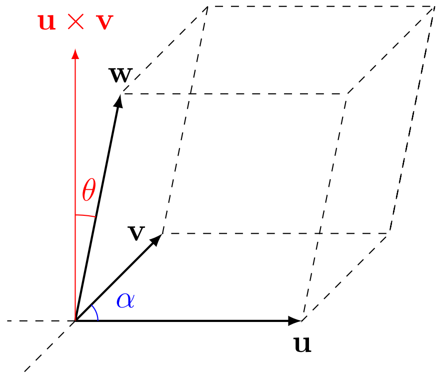
In particular, we can find the area of the triangle formed by $\mathbf{u}$ and $\mathbf{v}$ via

$$S = \frac{1}{2}||\mathbf{u} \times \mathbf{v}||$$

[*E.g.*] Find the area of the triangle formed by $A = (1, 1, 0), B = (2, 1, 2)$, and $C = (2, -1, 1)$.

Theorem (3.7.1). The volume of the parallelepiped with sides $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$ is given by

$$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$$



[*E.g.*] Find the volume of the parallelepiped formed by $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (1, 3, 2)$, and $\mathbf{w} = (1, 2, 2)$.

[*E.g.*] Find the distance form $P = (1, 2, 2)$ to the plane $-x + 2y + 2z = 4$.