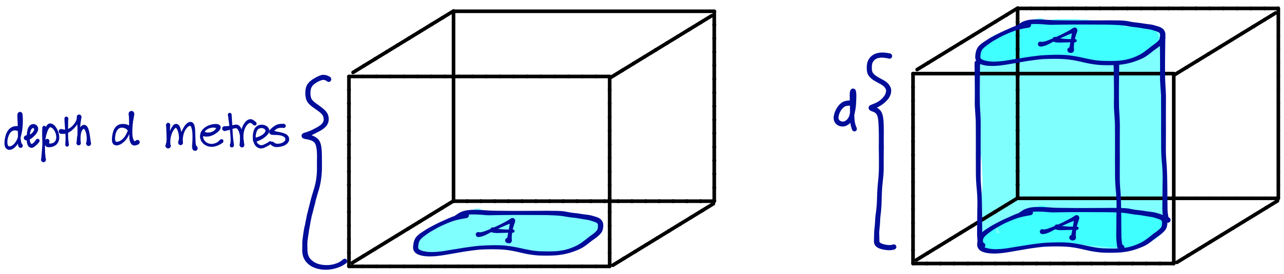
6. Applications: Hydrostatic Force, Centre of Mass

HYDROSTATIC PRESSURE AND FORCE

- Suppose a horizontal plate of area A is submerged in a fluid (at rest) of *mass density* ρ kg/m^3 at a depth d m below the surface of the fluid.



- What is the force exerted on the plate?

- the fluid directly above the plate has volume $V = A \cdot d$
  hence mass $m = \rho V = \rho A d$ (ρ = mass density)

$\therefore$ the force acting on the plate due to mass of water above it and gravity is $\boxed{F = mg = \rho g A d}$

- The **pressure** P on the plate is defined to be the force per unit area:

$P = \dfrac{F}{A} \Rightarrow$ hydrostatic pressure is $\boxed{P = \dfrac{\rho g A d}{A} = \rho g d}$

Important things to know about pressure in any fluid:

- at any point in a liquid, the pressure is the same in all directions
- pressure increases with depth
- since pressure is force/area, we have $P = \frac{F}{A} = \rho d g$.

Beware of units and mass vs. weight density!

units $\underset{\text{(pascals)}}{\text{Pa}} = \dfrac{N}{m^2}$ compare to units in $\rho g d$: $\underbrace{\left(\frac{kg}{m^3}\right)}_{\text{mass density}} \underbrace{\left(\frac{m}{s^2}\right)}_{\text{gravity}} \underbrace{(m)}_{\text{depth}}$

Alternatively,
$P = \delta d$: $\underbrace{\left(\frac{lb}{ft^3}\right)}_{\text{weight density}} \underbrace{(ft)}_{\text{depth}}$

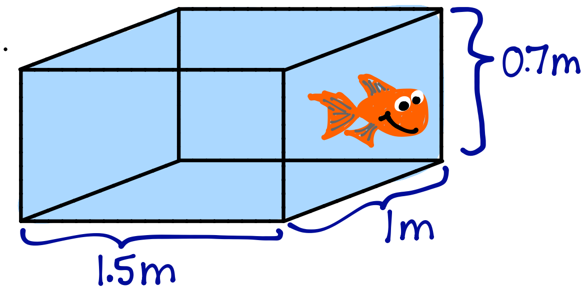
Note the weight density δ of a fluid already incorporates gravity.

Ex weight density of water is 62.5 lb/ft^3
mass density of water is 1000 kg/m^3

* Recall lbs are a unit of force!

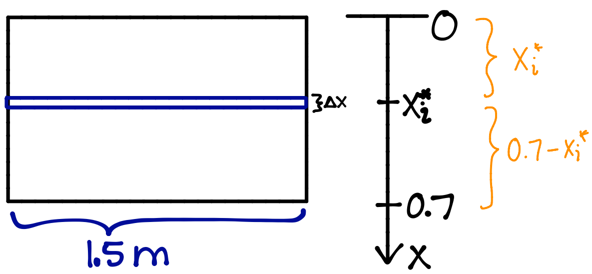
* These notes are solely for the personal use of students registered in MAT1322.

Example 6.1. A fish tank is 1.5 m long by 1 m wide by 0.7 m deep.

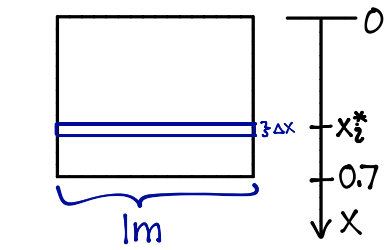


(a) Find the hydrostatic force on the bottom of the tank.

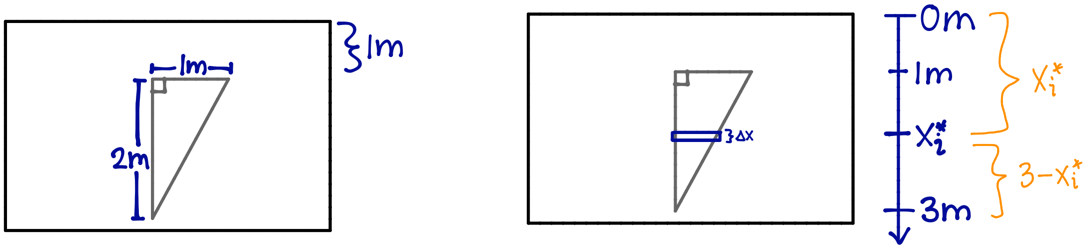
(b) Find the hydrostatic force on each of the four sides of the tank.

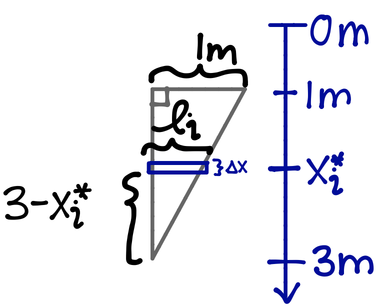


Similarly, for each end of the tank we compute the hydrostatic force.



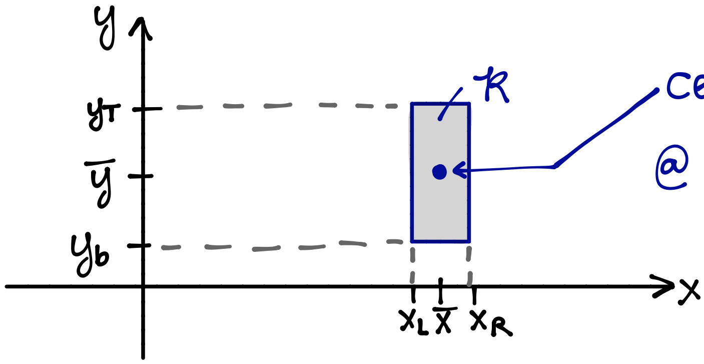
Exercise 6.2. Suppose a vertical plate is submerged 1 m under water, where the plate has dimensions as given in the following diagram:





What is the hydrostatic force acting on the plate?

Suppose a flat rectangular plate of uniform density ρ is situated in the xy-plane as follows:



centre of mass of R is located @ $(\bar{x}, \bar{y}) = \left(\frac{1}{2}(x_L + x_R), \frac{1}{2}(y_b + y_T)\right)$
(x-midpoint, y-midpoint).

The mass of R is the product of its area and its (uniform) density.

The moment of R about the x-axis is defined as the product

$$M_x = (\text{mass of } R) \cdot (\text{(signed) distance from } R\text{'s centre of mass to the x-axis})$$

$$\Rightarrow M_x = (\text{density}) \cdot (\text{area of } R) \cdot (\bar{y})$$

M_x measures the tendency of R to tip over or rotate around the x-axis.

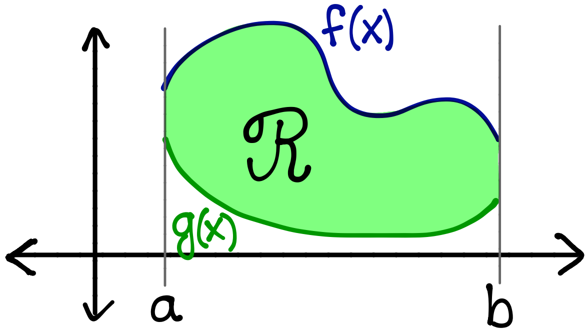
Similarly, the moment of R about the y-axis is defined as the product

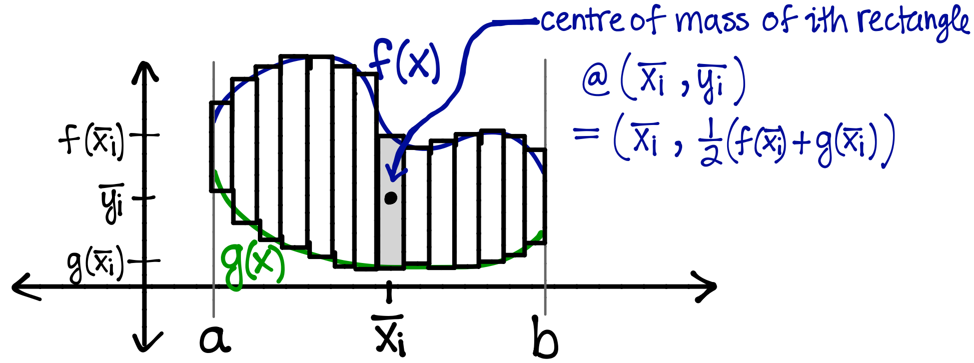
$$M_y = (\text{mass of } R) \cdot (\text{(signed) distance from } R\text{'s centre of mass to the y-axis})$$

$$\Rightarrow M_y = (\text{density}) \cdot (\text{area of } R) \cdot (\bar{x})$$

M_y measures the tendency of R to tip over or rotate around the y-axis.

Now, suppose a flat plate of uniform density ρ is shaped like the region $\mathcal{R}$ which is bounded by the curves $f(x)$ and $g(x)$ for $a \le x \le b$, as follows:





We can slice $\mathcal{R}$ up into n rectangles and find the centre of mass of each rectangle as well as each rectangle's moments about the x- and y-axis.

FACT: The moments of the union of two or more pairwise-non-overlapping regions is the sum of each region's moments.

We can approximate the moment of $\mathcal{R}$ about the x-axis as follows:

$$M_x \approx \sum_{i=1}^{n} (\text{moment } M_x \text{ of } i\text{th rectangle})$$

$$= \sum_{i=1}^{n} (\text{density}) \cdot (\text{area of } i\text{th rectangle}) \cdot (\text{(signed) distance of } i\text{th rectangle's centre of mass from } x\text{-axis})$$

$$= \sum_{i=1}^{n} \rho \cdot \left(f(\bar{x}_i) - g(x_i)\right) \cdot \Delta x \cdot \frac{1}{2}\left(f(\bar{x}_i) + g(\bar{x}_i)\right)$$

$\therefore$ exact moment of $\mathcal{R}$ about x-axis is

$$M_x =$$

$$\Longrightarrow M_x =$$

Similarly, we can approximate the moment of $\mathcal{R}$ about the y-axis as follows:

$$M_y \approx \sum_{i=1}^{n} (\text{moment } M_y \text{ of ith rectangle})$$

$$= \sum_{i=1}^{n} (\text{density}) \cdot (\text{area of ith rectangle}) \cdot (\text{(signed) distance of ith rectangle's centre of mass from } y\text{-axis})$$

$$= \sum_{i=1}^{n} \rho \cdot \left(f(\overline{x}_i) - g(x_i)\right) \cdot \Delta x \cdot \overline{x}_i$$

$\therefore$ exact moment of $\mathcal{R}$ about y-axis is

$$M_y =$$

$$\Longrightarrow \quad M_y =$$

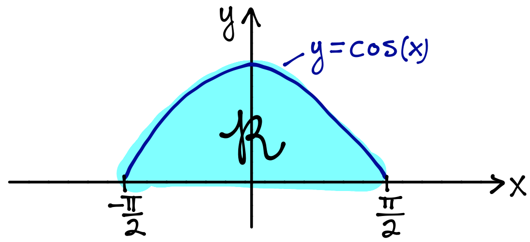
The total mass of $\mathcal{R}$ is $\quad m = (\text{density}) \cdot (\text{area of } \mathcal{R}).$ $\quad$ Therefore

A point mass of mass m that is located at the centre of mass of $\mathcal{R}$ should produce the same moments as $\mathcal{R}$.

Suppose $(\overline{x}, \overline{y})$ are the coordinates of $\mathcal{R}$'s centre of mass (or "centroid"). Then

Thus, $\mathcal{R}$'s centre of mass (or "centroid") is at the point $(\overline{x}, \overline{y})$ where

Example 6.3. Find the moments M_x and M_y and the centre of mass (centroid) of a flat plate of uniform density $\rho = 1$, shaped like the region $\mathcal{R}$ that is bounded by the curve $y = \cos(x)$ and the x-axis for $-\pi/2 \leq x \leq \pi/2$.



STUDY GUIDE

☐ **Hydrostatic Pressure** $\boxed{P = \dfrac{F}{A} = \rho g d = \delta d}$ ☐ **Hydrostatic Force** $\boxed{F = mg = \rho g A d = \delta A d}$

☐ **Centroid** $\boxed{\overline{x} = \dfrac{1}{A} \displaystyle\int_a^b x[f(x) - g(x)]dx}$ $\boxed{\overline{y} = \dfrac{1}{A} \displaystyle\int_a^b \dfrac{1}{2}\{[f(x)]^2 - [g(x)]^2 \ \ dx}$