Lesson 5 – Rates of Change Using Limits

PART A: Investigation – Average and Instantaneous Velocity

Let us examine the problem of determining the average and instantaneous velocity of a falling object

Problem: Suppose that Laura drops a marble through a hole in the Sky Deck on Willis Tower (formerly Sears Tower, renamed by insurance broker Willis Group Holdings Ltd who obtained naming rights in 2009).

Laura is very curious to know how fast the ball is travelling after 5 seconds. She wants to determine:

a) the average velocity *after* five seconds of free fall
b) the instantaneous velocity of the marble *at* five seconds.

Solution: A little bit of physics background is required here. The average velocity of a free falling object calculated using the formula:

$$Average\ velocity = \frac{distance\ travelled}{time\ ellapsed} \quad \text{or} \quad V_{avg} = \frac{\Delta d}{\Delta t}$$

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The distance travelled by a falling object (if we ignore air resistance) can be found using the formula:

$d = \frac{1}{2}gt^2$, where g represents the acceleration due to gravity and $g = 9.8\,m/s^2$

$d = \frac{1}{2}(9.8)t^2$

Distance travelled by a falling object = $d(t) = 4.9t^2$

Therefore, the average velocity would now have the following form:

$$V_{avg} = \frac{d(t_2) - d(t_1)}{t_2 - t_1}$$

a) Now, if we assume the object started from rest, and we are measuring time from the moment Laura let go of the marble, $t_1 = 0\ and\ t_2 = 5$.

$$V_{avg} = \frac{d(5) - d(0)}{5 - 0}$$

$$= \frac{(4.9(5)^2) - (4.9(0)^2)}{5}$$

$$= \frac{122.5 - 0}{5}$$

$$= 24.5\ m/s$$

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Although we now know the average velocity of the marble after it fell for five seconds, a more meaningful measure would be the velocity of the object at exactly 5 seconds.

b) In order to determine the instantaneous velocity of the marble at five seconds, lets use the average velocity formula to see what happens as we decrease the interval of time Δt. $d(t)=4.9t^2$

Fill in the table below and answer the question that follows:

Time interval t_1	t_2	$d(t_1)$	$d(t_2)$	Δd	Δt	V_{avg}
5	6	122.5	176.4	53.9	1	53.9
5	5.1	122.5	127.449	4.949	0.1	49.49
5	5.05	122.5	124.96225	2.46225	0.05	49.245
5	5.01	122.5	122.99049	0.49049	0.01	49.049
5	5.001	122.5	122.549049	0.049	0.001	49.0049

$h \to 0$

What conclusions can you draw from observing the results in the table above?

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Let us now look at the average velocity over a general time interval, where the time increment is represented by h $d=4.9t^2$

$$V_{avg}=\frac{d(t_2)-d(t_1)}{t_2-t_1}$$

$$V_{avg}=\frac{d(5+h)-d(5)}{h}$$

$$V_{avg}=\frac{4.9(25+10h+h^2)-122.5}{h}$$

$$V_{avg}=\frac{49h+4.9h^2}{h}$$

$$V_{avg}=\frac{\cancel{h}(49+4.9h)}{\cancel{h}}$$

Factor out h and cancel

$$V_{avg}=49+4.9\cancel{h}^{\,0}$$

As can be seen from the formula, if the time interval is very short, then h is very small and $4.9h \to 0$, and the average velocity is close to 49 m/s.

The instantaneous velocity when $t = 5$ seconds is defined as the limiting value of these average velocities as $h \to 0$.

Therefore,

$$V=\lim_{h\to 0}(49+4.9h)$$

$$V = 49\ m/s$$

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From this example, we can define the instantaneous velocity as:

$$V(a) = \lim_{\Delta t \to 0} \frac{\Delta d}{\Delta t} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

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PART B: Example

The displacement, in metres, of a particle moving in a straight line is given by $d = t^2 + 2t$, where t is measured in seconds. Find the velocity of the particle at 3 seconds.

Solution: $d(t) = t^2 + 2t$

$$V(3) = \lim_{h \to 0} \frac{[(3+h)^2 + 2(3+h)] - [3^2 + 2(3)]}{h}$$

$$V(3) = \lim_{h \to 0} \frac{(9 + 6h + h^2 + 6 + 2h) - (9+6)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 8h + 15 - 15}{h}$$

$$= \lim_{h \to 0} \frac{h^{\cancel{2}} + 8\cancel{h}}{\cancel{h}}$$

$$= \lim_{h \to 0} \overset{0}{\cancel{h}} + 8$$

$$= 0 + 8$$

$$= 8 \text{ m/s}$$

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PART C: Rates of Change in the real world

Whenever you solve a problem involving tangent lines, you are not just solving a problem in geometry, but rather you are implicitly solving a great variety of problems involving rates of change in the natural and social sciences as well as in engineering.

- Engineers may be interested in the rate of change of displacement with respect to time (called velocity).
- Chemists studying chemical reactions are interested in the rate of change in the concentration of a reactant with respect to time (called rate of reaction).
- A textile manufacturer is interested in the rate of change of the cost of producing x square metres of fabric per day with respect to x.

All these rates of change can be interpreted as slopes of tangents.

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