Final 1322 F14

Calculus II (University of Ottawa)

Calculus II (MAT1322-D)

Final Exam December 7, 2014

Family name: ____________________

First name: ____________________

Student ID: ____________________

- Length: 3 hours.
- Notes are not permitted.
- Only scientific calculators of model TI-30, TI34, Casio fx-260 and Casio fx-300 are permitted.
- Questions 1–10 are multiple choice, each worth 4 points. Circle the correct response. Numerical responses are rounded to the indicated decimal.
- Questions 11–16 are long answer questions. They require a detailed response with all solutions clearly justified.
- Write your solutions in the space provided. Use the back of the page for scratch work if necessary. Scratch work will not be taken into account for solutions of multiple choice questions, except in case of suspicion of academic fraud.
- The exam is marked out of 100.

1. [**4 points**] What is the value of the improper integral $\int_1^\infty 2xe^{-x^2}\,dx$?

A) 0 B) $4e^{-2}$ C) $3e^{-2}$ D) e^{-1} E) $3e$ F) ∞

The correct answer is D.

$$\int_1^\infty 2xe^{-x^2}\,dx = -\lim_{t\to\infty}\left[e^{-x^2}\right]_1^t = -\lim_{t\to\infty} e^{-t^2} + e^{-1} = e^{-1}.$$

2. [**4 points**] Determine the volume of the solid whose base is the region in the xy-plane bounded by $y = x$ and $y = x^2$ and with cross-sections perpendicular to the x-axis being squares.

A) 1/30 B) 1/15 C) 2/3 D) 3/4 E) 5/6 F) 5/4

The correct answer is A.

The volume of each cross-section is $(x - x^2)^2 dx$. So, we have

$$\int_0^1 (x - x^2)^2 dx = \int_0^1 (x^2 - 2x^3 + x^4) dx = \left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right]_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{1}{30}.$$

3. [**4 points**] Use Euler's Method with step size $h = 0.1$ to estimate $y(0.2)$, where y is the solution of the initial value problem $y' = y^2 + x,\ y(0) = 1$.

A) 1.573 B) 2.589 C) 1.231 D) 2.623 E) 3.634 F) 2.653

The correct answer is C.

The formula for the approximations is

$$y_n = y_{n-1} + hf\left(x_{n-1}, y_{n-1}\right)$$

So, we have

$$y_1 = y_0 + (0.1)(y_0^2 + x_0) = 1 + (0.1)(1^2 + 0) = 1 + 0.1 = 1.1$$

$$y_2 = y_1 + (0.1)(y_1^2 + x_1) = 1.1 + (0.1)((1.1)^2 + (0.1)) = 1.231$$

4. [**4 points**] A population of bacteria starts with 1000 individuals and grows at a rate proportional to its size. After one hour, there are 1500 individuals in the population. After how many hours will the population be 2000?

A) 3.64 B) 1.68 C) 1.72 D) 3.76 E) 1.71 F) 3.84

The correct answer is E.

$$P = P_0 e^{kt} \Rightarrow \quad 1.5 = e^k \Rightarrow k = \ln(1.5).$$

$$2000 = 1000e^{kT} \Rightarrow T = \frac{\ln 2}{\ln(1.5)} \approx 1.71.$$

5. [**4 points**] Determine $y(\pi)$, where $y(t)$ is the solution of the separable differential equation

$$\frac{dy}{dt} = 2ty, \quad y(0) = 1.$$

A) $\sqrt{3}$ B) e C) $\sqrt{8}$ D) $\ln(3)$ E) $\ln(5)$ F) $e+1$

The correct answer is B.

$$\frac{dy}{y} = 2tdt \Rightarrow \ln y = t^2 + c \Rightarrow y = Ce^{t^2} \quad \Rightarrow y(0) = C = 1.$$

So the solution of the IVP is $y = e^{t^2}$ and $y(1) = e$.

6. [**4 points**] Determine the first three nonzero terms of the Maclaurin series of $(1-x)e^x$.

A) $1 - \frac{1}{2}x^2 - \frac{1}{3}x^3$ B) $1 - \frac{1}{2}x^2 - \frac{4}{3}x^3$ C) $1 - \frac{1}{2}x^2 + \frac{2}{3}x^3$

D) $1 - \frac{1}{2}x - \frac{1}{3}x^2$ E) $1 - \frac{1}{2}x - \frac{4}{3}x^2$ F) $1 - \frac{1}{2}x + \frac{2}{3}x^2$

The correct answer is A.

$$(1-x)e^x = (1-x)\sum_{n=0}^{\infty} = (1-x)(1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\ldots) = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\ldots-x-x^2-\frac{x^3}{2}+\frac{x^4}{6}+\ldots$$

$$= 1 - \frac{x^2}{2} - \frac{x^3}{3} - \ldots$$

7. [**4 points**] Find the power series representation for $f(x) = \dfrac{x}{x^2+16}$ and determine the radius of convergence R.

A) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{16^{n+1}}, \quad R=4$ B) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{16^{n+1}}, \quad R=2$ C) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{4^{n+1}}, \quad R=4$

D) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{n}}{16^{n}}, \quad R=1$ E) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{16^{n}}, \quad R=4$ F) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{4^{n+1}}, \quad R=1$

The correct answer is A.

$$f(x) = \frac{x}{16}\frac{1}{1-\left(-\frac{x^2}{16}\right)} = \frac{x}{16}\sum_{n=0}^{\infty}\left(-\frac{x^2}{16}\right)^n = \frac{x}{16}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{16^n} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{16^{n+1}}.$$

$$\left|-\frac{x^2}{16}\right| < 1 \Rightarrow x^2 < 16 \Rightarrow R = 4.$$

8. [**4 points**] The radius r and height h of a cylinder vary as functions of time t. Set $S = 2\pi rh + 2\pi r^2$, the function giving its surface area. Given that at time $t=2$ we have

$$r = 5 \text{ cm}, \quad h = 20 \text{ cm}, \quad \frac{dr}{dt} = 1 \text{ cm/s} \quad \text{and} \quad \frac{dh}{dt} = 2 \text{ cm/s},$$

determine the rate of change $\dfrac{dS}{dt}$ of the surface area at that instant.

A) 180π B) 100π C) 80π D) 40π E) 60π F) 120π

The correct answer is C.

$$\frac{dS}{dt} = 2\pi\frac{dr}{dt}h + 2\pi r\frac{dh}{dt} + 4\pi r\frac{dr}{dt} = 2\pi(1)(20) + 2\pi(5)(2) + 4\pi(5)(1) = 80\pi$$

9. [**4 points**] Match each function with the corresponding series expansion.

$\dfrac{1}{1+x^3}$ •	• $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{3n}$
$\cos(4x)$ •	• $\displaystyle\sum_{n=1}^{\infty} n^3 x^{n-1}$
$\displaystyle\int_0^x \frac{1}{1-t}dt$ •	• $\displaystyle\sum_{n=0}^{\infty}\frac{(-16)^n x^{2n}}{(2n)!}$
$\dfrac{d}{dx}\left(\displaystyle\sum_{n=0}^{\infty} n^2 x^n\right)$ •	• $\displaystyle\sum_{n=1}^{\infty}\frac{x^{n+1}}{n+1}$

10. [**4 points**] Find the directional derivative of the function $f(x,y) = 2xy - 3y^2$ at the point $(5,5)$ in the direction of the vector $\vec{v} = \langle 4,3\rangle = 4\hat{\mathbf{\imath}} + 3\hat{\mathbf{\jmath}}$.

A) 2 B) -6 C) 10 D) 3 E) 9 F) -4

The correct answer is F.

$f_x = 2y$, $f_y = 2x - 6y$, and $|v| = 5$, so we have

$$D_{\vec{v}} f(5,5) = f_x(5,5)\frac{4}{5} + f_y(5,5)\frac{3}{5} = (10)\frac{4}{5} - (20)\frac{3}{5} = -4.$$

11. [10 points] Let $\mathcal{R}$ denote the region of the plane which is bounded by the curve $y = x^3$ and the horizontal lines $y = 8$ and $y = 0$ (the x-axis). Set $\mathcal{S}$ to be the solid of revolution obtained by rotating the region $\mathcal{R}$ about the vertical line $x = -3$.

Calculate the volume of $\mathcal{S}$ by the disc/washer method or the method of cylindrical shells (your choice). **Sketch the region $\mathcal{R}$, the section of the solid $\mathcal{S}$ which intersects the xy-plane, and an element of volume (disc/washer or cylinder) with its dimensions.**

Solution:

1. Using washer method:

   - Using the washer method with respect to the y variable :

     $$V = \int_c^d \pi(R^2(y) - r^2(y))\,dy$$

   - limits:
     - $c : y = (0)^3 = 0$
     - $d : y = 8$
   - radii:
     - outer radius: $y = x^3 \Rightarrow x = \sqrt[3]{y}$

       $$R(y) = \sqrt[3]{y} + 3$$

     - inner radius: $r(y) = 3$
   - volume:

     $$\begin{aligned} V = \int_0^8 \pi\left((\sqrt[3]{y}+3)^2 - (3)^2\right)\,dy &= \pi \int_0^8 (y^{2/3} + 6y^{1/3} + 9) - 9\,dy \\ &= \pi \int_0^8 y^{2/3} + 6y^{1/3}\,dy \\ &= \pi \left[\frac{3}{5}y^{5/3} + 6\left(\frac{3}{4}y^{4/3}\right)\right]_0^8 \\ &= \pi\left(\frac{3(32)}{5} + \frac{6(3)(16)}{4}\right) - 0 = \frac{456\pi}{5} \end{aligned}$$

2. Using cylindrical shells method:

- Using the cylindrical shells method with respect to the x variable :

$$V = \int_a^b 2\pi(\text{shell radius})(\text{shell height})\,dx$$

- limits:
  - $a : x = 0$
  - $b : x^3 = 8 \Rightarrow x = 2$
- shell info:
  - shell radius $= x + 3$
  - shell height $= 8 - x^3$
- volume:

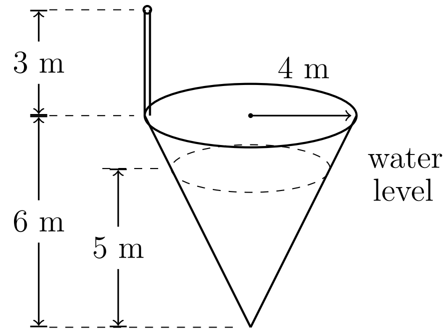
$$\begin{aligned} V = \int_0^2 2\pi(x+3)(8-x^3)\,dx &= 2\pi \int_0^2 -x^4 - 3x^3 + 8x + 24\,dx \\ &= 2\pi\left[-\frac{x^5}{5} - \frac{3x^4}{4} + 4x^2 + 24x\right]_0^2 \\ &= 2\pi\left(-\frac{32}{5} - \frac{3(16)}{4} + 16 + 48\right) - 0 = \frac{456\pi}{5} \end{aligned}$$

12. [**10 points**] A reservoir is built in the form of a right circular cone pointing down, as in the figure to the right.

Its height is 6 m and the radius at the base (that is, the top of the reservoir) is 4 m. It is full of water to a depth of 5 m.

We wish to pump all of this water to a height of 3 m above the reservoir.

Denote by x the height in meters measured **from the bottom of the reservoir**.



(a) What is, at first approximation, the volume ΔV of the layer of water between the heights x and $x + \Delta x$?

Response: $\Delta V \cong$

$$\begin{aligned} V &\approx \pi \cdot r_x^2 \cdot \triangle x \\ &= \pi \left(\frac{2}{3}x\right)^2 \triangle x \\ &= \frac{4}{9}\pi x^2 \triangle x \end{aligned}$$

$$\begin{aligned} \frac{r_x}{x} &= \frac{4}{6} \\ r_x &= \frac{2}{3}x \end{aligned}$$

(b) What is, at first approximation, the work ΔW required to pump that layer of water to a height of 3 m above the reservoir? Recall that the density of water is 1000 kg/m^3, and that $g \cong 9.8$ m/s^2.

Response: $\Delta W \cong$

$$\begin{aligned} \triangle W &\approx g \cdot (1000) \cdot \triangle V \cdot d \\ &= (9.8)(1000)\left(\frac{4}{9}\pi x^2 \triangle x\right)(9 - x) \\ &= \frac{39200\pi}{9}(9x^2 - x^3)\triangle x \end{aligned}$$

(c) What is, in Joules, the work required to pump all the water in the reservoir to a height of 3 meters above it?

$$\begin{aligned} W &= \int_0^5 \frac{39200\pi}{9}(9x^2 - x^3)\,dx \\ &= \frac{39200\pi}{9}\left[3x^3 - \frac{x^4}{4}\right]_0^5 \\ &= \frac{39200\pi}{9}\left(3(125) - \frac{(625)}{4}\right) \\ &= \frac{8575000\pi}{9} \text{ J} \approx 2993239.67 \text{ J} \end{aligned}$$

13. [**10 points**] In a certain cistern, there are 400 L of brine containing 25 kg of dissolved salt. Another brine solution which contains 0.3 kg/L of salt is introduced into the cistern at a rate of 4 L/min. The solution is kept well-mixed and, at the same time, is emptied at such a rate that the volume is unchanged. Let $Q(t)$ denote the quantity (in kg) of salt within the cistern at time t (in minutes).

(i) Find an initial value problem for $Q(t)$.

$$\begin{aligned} \frac{dQ}{dt} &= (4)(0.3) - \left(\frac{Q}{400}\right)(4) \\ \frac{dQ}{dt} &= 1.2 - \frac{Q}{100} \\ \frac{dQ}{dt} &= \frac{120 - Q}{100} \end{aligned}$$

such that $Q(0) = 25$.

(ii) Solve it. Be sure to explain your process.

$$\begin{aligned} \frac{dQ}{dt} &= \frac{120 - Q}{100} \\ \frac{dQ}{120 - Q} &= \frac{dt}{100} \\ -\ln|120 - Q| &= \frac{t}{100} + C \\ |120 - Q| &= Ae^{-t/100} \\ Q &= 120 + Ae^{-t/100} \end{aligned}$$

Since $Q(0) = 25$,

$$\begin{aligned} 120 + A &= 25 \\ A &= -95 \end{aligned}$$

Therefore,

$$Q(t) = 120 - 95e^{-t/100}$$

(ii) How much salt is in the cistern after one hour?

$$\begin{aligned} Q(60) &= 120 - 95e^{-60/100} \\ &\approx 67.86 \text{ kg} \end{aligned}$$

14. (a) [7 points] Determine whether the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n}$ is absolutely convergent, semi-convergent or divergent.

Solution:

1. Is it absolutely convergent?

   - Consider $\sum_{n=1}^{\infty}\frac{\ln n}{n}$.
   - Since $f(x)=\frac{\ln x}{x}$ is
     - positive (since $\ln x \geq 0$ when $x \geq 1$)
     - continuous (since $x \geq 1$)
     - eventually decreasing (since $f'(x)=\frac{1-\ln x}{x^2}$ is negative when $x \geq e$)

     we use the integral test.

$$\begin{aligned}\int_1^{\infty}\frac{\ln x}{x}\,dx &= \lim_{b\to\infty}\int_1^b\frac{\ln x}{x}\,dx \\ &= \lim_{b\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^b \\ &= \lim_{b\to\infty}\left(\frac{(\ln b)^2}{2}\right)=\infty\end{aligned}$$

   - Since the improper integral diverges the series diverges by the Integral Test.

2. Is it semi-convergent?

   - Consider $b_n=\frac{\ln n}{n}$
   - Since
     - b_n are positive
     - $b_n \to 0$
     - b_n is eventually decreasing

     The series converges by the Alternating Series Test.
   - The series is semi-convergent.

(b) [3 points] We wish to calculate $s=\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1+n}{n^2}$ with error no worse than 0.1. What is the number, k, of terms which we must add in order to be certain that the error in approximating s by $s_k=\displaystyle\sum_{n=1}^{k}(-1)^{n+1}\frac{1+n}{n^2}$ satisfies $|\text{error}| \leq 0.1$?

Solution : We need

$$|b_{n+1}| \leq 0,1$$

which requires

$$\frac{n+2}{(n+1)^2} \leq 0,1$$

The first value of n for which this inequality is satisfied (by trial and error) is $n = 10$. Hence we must add the first 10 terms of the series in order to achieve the desired precision.

15. [10 points] Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n}$.

Determine its radius and interval of convergence. Take care to justify the convergence or divergence of the series at the endpoints of the interval.

Solution :

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{3^{n+1}(x-2)^{n+1}}{(n+1)} \cdot \frac{n}{3^n(x-2)^n}\right| \\ &= \left|\frac{3n(x-2)}{n+1}\right| \\ &= \frac{3n}{n+1}|x-2| \\ &\to 3|x-2| \end{aligned}$$

- We conclude that $3|x-2| < 1$ whenever $|x-2| < \frac{1}{3}$. Thus the radius of convergence $R = \frac{1}{3}$.
- Thus the series converges within the interval $\frac{5}{3} < x < \frac{7}{3}$ and we now check the endpoints.
- If $x = 5/3$, $\sum_{n=1}^{\infty} \frac{3^n(-\frac{1}{3})^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is the alternating harmonic series which converges. Thus we include the endpoint $x = 5/3$.
- If $x = 7/3$, $\sum_{n=1}^{\infty} \frac{3^n(\frac{1}{3})^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which is the harmonic series which diverges. Thus we do not include the endpoint $x = 7/3$.
- Thus, the interval of convergence is $\left[\frac{5}{3}, \frac{7}{3}\right[$.

16. [10 points]

Consider the function $f(x,y) = \dfrac{1}{x^2+4y^2-1}$.

(a) Determine the domain of the function and skecth it in the xy-plane.

(b) Find the equation of the tangent plane to the surface at the point where $(x,y) = (2,1)$.

(c) Sketch the level curves $f(x,y) = C$ for $C = 1/3, -1, 1/15$.

(d) In which direction does f change the fastest at the point where $(x,y) = (2,1)$? What is the maximal rate of change at that point?

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Solution:

(a) We need that $x^2+4y^2-1 \neq 0$ which implies that $x^2+4y^2 \neq 1$. Which implies that we must not consider the points on the specified ellipse as shown below

(b) The tangent plane is given by the equation

$$z - z_0 = f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0)$$

We know that $z_0 = f(x_0,y_0) = f(2,1) = \frac{1}{4+4-1} = \frac{1}{7}$, as well as

$$f'_x(x,y) = \frac{-2x}{(x^2+4y^2-1)^2} \qquad \text{and } f'_y(x,y) = \frac{-8y}{(x^2+4y^2-1)^2}$$

$$f'_x(2,1) = \frac{-4}{49} \qquad \text{and } f'_y(x,y) = \frac{-8}{49}$$

Thus

$$z - \frac{1}{7} = -\frac{4}{49}(x-2) - \frac{8}{49}(y-1)$$

(c) Since

$$f(x,y) = C \Rightarrow \frac{1}{x^2+4y^2-1} = C \Rightarrow 1 + \frac{1}{C} = x^2 + 4y^2$$

The level curves are ellipses.

- If $C = 1/3$: $\frac{x^2}{2^2} + y^2 = 1$
- If $C = -1$: $x^2 + \frac{y^2}{\left(\frac{1}{2}\right)^2} = 0$
- If $C = 1/15$: $\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$

(d) We compute the gradient :

$$\triangle f = \left(f'_x(x,y), f'_y(x,y)\right) = \left(\frac{-2x}{(x^2+4y^2-1)^2}, \frac{-8y}{(x^2+4y^2-1)^2}\right)$$

$$\triangle f(2,1) = \left(-\frac{4}{49}, -\frac{8}{49}\right)$$

The maximal change occurs in the direction $\left(-\frac{4}{49}, -\frac{8}{49}\right)$ at its rate is

$$\left|\left|\left(-\frac{4}{49}, -\frac{8}{49}\right)\right|\right| = \sqrt{\frac{16}{49^2} + \frac{64}{49^2}} = \frac{\sqrt{80}}{49} \approx 0.1825$$