MAT 1348 – Winter 2024

Exercises 9 – Solutions

Professor: Antoine Poirier

Questions are taken from Discrete Mathematics 8th edition, by Kenneth H. Rosen.

QUESTION 1 **(5.1 # 1)**. There are infinitely many bus stations on a bus line. Suppose the bus stops at the first station, and suppose also that if the bus stops at any station, it will also stop at the next station. Prove that the bus will stop at every station.

Solution: Let $P(n)$ be the statement "the bus stops at the n^{th} station". From the question, we know $P(1)$ is true and that $P(k) \to P(k+1)$ is true for all $k \geq 1$. Therefore, by the induction principle, we have that $P(n)$ is true for all $n \geq 1$, and so the bus stops at every station.

QUESTION 2 **(5.1 # 3)**. For any integer $n \geq 1$, let $P(n)$ be the following proposition: $1^2+2^2+...+n^2 = n(n+1)(2n+1)/6$.

(a) What is $P(1)$?
(b) Show $P(1)$ is true.
(c) Show that if $P(k)$ is true for some $k \geq 1$, then $P(k+1)$ is true.
(d) Explain why we can conclude from the previous steps that $P(n)$ is true for all $n \geq 1$.

Solution:

(a) $P(1) =$ "$1^2 = 1(1+1)(2 \cdot 1+1)/6$"
(b) The two sides of the equality in $P(1)$ are 1, so $P(1)$ is true.
(c) Suppose $P(k)$ is true for some $k \geq 1$, so that $1^2+2^2+...+k^2 = k(k+1)(2k+1)/6$. We show that $P(k+1)$ is true, so we show that $1^2+2^2+...+k^2+(k+1)^2 = (k+1)(k+2)(2(k+1)+1)/6$. Starting from the left side of this equality we get

$$(1^2+2^2+...+k^2)+(k+1)^2 = \frac{k(k+1)(2k+1)}{6}+(k+1)^2 = \frac{k(k+1)(2k+1)+6(k+1)^2}{6} = \frac{(k+1)(k(2k+1)+6(k+1)}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

Therefore, $P(k+1)$ is true.

(d) By the induction principle, since $P(1)$ is true and $P(k) \to P(k+1)$ is also true for all $k \geq 1$, we conclude $P(n)$ is true for all $n \geq 1$.

QUESTION 3 **(5.1 # 5)**. Show that $1^2+3^2+...+(2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ whenever n is a non-negative integer.

Solution: Let $P(n)$ be the statement: $1^2+3^2+...+(2n+1)^2 = (n+1)(2n+1)(2n+3)/3$.

As a base case, we show $P(0)$. $P(0)$ is the statement "$1^2 = (1)(1)(3)/3$", which is true.

For the induction step, suppose $P(k)$ is true for some $k \geq 0$. Therefore, we suppose $1^2+...+(2k+1)^2 = (k+1)(2k+1)(2k+3)/3$. We show $P(k+1)$ is true, so we show that $1^2+...+(2k+1)^2+(2k+3)^2 = (k+2)(2k+3)(2k+5)/3$. Starting from the left side of the previous equality, we get

$$(1^2+\ldots+(2k+1)^2)+(2k+3)^2=\frac{(k+1)(2k+1)(2k+3)}{3}+(2k+3)^2=\frac{(2k+3)((k+1)(2k+1)+3(2k+3))}{3}=$$

$$\frac{(2k+3)(2k^2+3k+1+6k+9)}{3}=\frac{(2k+3)(2k^2+9k+10)}{3}=\frac{(2k+3)(k+2)(2k+5)}{3}$$

This shows $P(k+1)$ is true. By the induction principle, $P(n)$ is true for all $n \geq 0$.

QUESTION 4 **(5.1 # 7)**. Show that $3+3\cdot 5+3\cdot 5^2+\ldots+3\cdot 5^n = 3(5^{n+1}-1)/4$ whenever n is a non-negative integer.

Solution: Let $P(n)$ be the following proposition: "$3+3\cdot 5+3\cdot 5^2+\ldots+3\cdot 5^n = 3(5^{n+1}-1)/4$" for $n \geq 0$.

For the base case, we show $P(0)$. $P(0)$ is the statement "$3 = 3(5^1-1)/4$" which is true since both sides of the equality are 3.

For the induction step, suppose $P(k)$ is true for some $k \geq 0$. Therefore, $3+3\cdot 5+3\cdot 5^2+\ldots+3\cdot 5^k = 3(5^{k+1}-1)/4$. We now show that $P(k+1)$ is true, so we show $3+3\cdot 5+3\cdot 5^2+\ldots+3\cdot 5^{k+1} = 3(5^{k+2}-1)/4$. Starting from the left side, we get

$$(3+3\cdot 5+3\cdot 5^2+\ldots+3\cdot 5^k)+3\cdot 5^{k+1}=\frac{3}{4}(5^{k+1}-1)+3\cdot 5^{k+1}=\frac{3}{4}(5^{k+1}-1+4\cdot 5^{k+1})=\frac{3}{4}(5\cdot 5^{k+1}-1)=\frac{3}{4}(5^{k+2}-1)$$

which shows $P(k+1)$ is true. By the principle of induction, we conclude that $P(n)$ is true for all $n \geq 0$.

QUESTION 5 **(5.1 # 9)**.

(a) Find a formula for the sum of the first n even positive numbers.
(b) Show, using induction, that the formula found in a) is valid for all $n \geq 1$.

Solution:

(a) We have

$$\begin{aligned} 2 &= 2 \\ 2+4 &= 6 \\ 2+4+6 &= 12 \\ 2+4+6+8 &= 20 \\ 2+4+6+8+10 &= 30 \end{aligned}$$

We conjecture that: $2+4+\ldots+2n = n^2+n$.

(b) Let $P(n)$ be the proposition "$2+4+\ldots+2n = n^2+n$", for $n \geq 1$.

As a base case, we show $P(1)$. $P(1)$ is the proposition "$2 = 1^2+1$", which is true since both sides are equal to 2.

For the induction step, suppose $P(k)$ is true for some $k \geq 1$, so suppose $2+4+\ldots+2k = k^2+k$. We want to show $P(k+1)$ is true, so we want to show $2+4+\ldots+2k+(2k+2) = (k+1)^2+(k+1)$. Starting from the left side, we have

$$(2+4+\ldots+2k)+(2k+2)=k^2+k+2k+2=(k^2+2k+1)+(k+1)=(k+1)^2+(k+1)$$

which shows $P(k+1)$ is true. By the principle of induction, we conclude $P(n)$ is true for all $n \geq 1$.

QUESTION 6 **(5.1 # 11)**.

(a) Find a formula for

$$\frac{1}{2}+\frac{1}{4}+\dots\frac{1}{2^n}$$

by computing the sum for small values of n.

(b) Show, using induction, that the formula found in a) is valid for all $n \geq 1$.

Solution:

(a) We have

$$\begin{aligned}\frac{1}{2} &= \frac{1}{2}\\ \frac{1}{2}+\frac{1}{4} &= \frac{3}{4}\\ \frac{1}{2}+\frac{1}{4}+\frac{1}{8} &= \frac{7}{8}\\ \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} &= \frac{15}{16}\end{aligned}$$

We conjecture that $\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n}=1-\frac{1}{2^n}$.

(b) Let $P(n)$ be the following proposition: "$\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n}=1-\frac{1}{2^n}$", for $n \geq 1$.

As a base case, we verify $P(1)$. $P(1)$ is the statement "$\frac{1}{2}=1-\frac{1}{2}$", which is true since both sides are $\frac{1}{2}$.

For the induction step, suppose $P(k)$ is true for some $k \geq 1$, so suppose $\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^k}=1-\frac{1}{2^k}$. We show that $P(k+1)$ is true, so we show that $\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^{k+1}}$. Starting from the left side of the previous equality, we get

$$\left(\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}=1-\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^{k+1}}(2-1)=1-\frac{1}{2^{k+1}}$$

This shows $P(k+1)$ is true. By the induction principle, we conclude that $P(n)$ is true for all $n \geq 1$.

QUESTION 7 **(5.1 # 13)**. Show that $1^2-2^2+3^2-\dots+(-1)^{n-1}n^2=(-1)^{n-1}n(n+1)/2$ whenever n is a positive integer.

Solution: Let $P(n)$ be the following proposition: "$1^2-2^2+3^2-\dots+(-1)^{n-1}n^2=(-1)^{n-1}n(n+1)/2$" for $n \geq 1$. We show that $P(n)$ is true for all $n \geq 1$ by induction.

For the base case, we show $P(1)$. $P(1)$ is the proposition "$1^2=(-1)^{1-1}1(1+1)/2$" which is true since both sides of the equality are 1.

For the induction step, assume $P(k)$ is true for some $k \geq 1$. Therefore, suppose $1^2-2^2+3^2-\dots+(-1)^{k-1}k^2=(-1)^{k-1}k(k+1)/2$. We show that $P(k+1)$ is true, so we show that $1^2-2^2+3^2-\dots+(-1)^k(k+1)^2=(-1)^k(k+1)(k+2)/2$. Starting from the left side of the previous equality, we get

$$(1^2-2^2+3^2-\dots+(-1)^{k-1}k^2)+(-1)^k(k+1)^2=(-1)^{k-1}\frac{k(k+1)}{2}+(-1)^k(k+1)^2=(-1)^k(k+1)((-1)^{-1}\frac{k}{2}+(k+1))=(-1)^k(k+1)($$

which shows $P(k+1)$ is true. By the induction principle, we conclude that $P(n)$ is true for all $n \geq 1$.

QUESTION 8 **(6.4 # 3)**. Expand $(x+y)^6$

Solution: $(x+y)^6 = \binom{6}{0}x^6y^0 + \binom{6}{1}x^5y^1 + \binom{6}{2}x^4y^2 + \binom{6}{3}x^3y^3 + \binom{6}{4}x^2y^4 + \binom{6}{5}x^1y^5 + \binom{6}{6}x^0y^6$.

QUESTION 9 **(6.4 # 7)**. What is the coefficient of x^9 in $(2-x)^{19}$?

Solution: We have $(2-x)^{19} = \sum_{i=0}^{19}\binom{19}{i}2^{19-i}(-x)^i$. The term x^9 is obtained by replacing i by 9, which gives $\binom{19}{9}2^{10}(-1)^9x^9$. The coefficien of x^9 is therefore $\binom{19}{9}2^{10}(-1)^9$.

QUESTION 10 **(6.4 # 9)**. What is the coefficient of $x^{101}y^{99}$ in $(2x-3y)^{200}$?

Solution: We have $(2x-3y)^{200} = \sum_{i=0}^{200}\binom{200}{i}(2x)^i(-3y)^{200-i}$. The term $x^{101}y^{99}$ is obtained by replacing i by 101, which gives $\binom{200}{101}2^{200}x^{200}(-3)^{99}y^{99}$. The coefficient of $x^{101}y^{99}$ is therefore $\binom{200}{101}2^{101}(-3)^{99}$.

QUESTION 11 **(6.4 # 15)**. Find a formula for the coefficient of x^k in $(\frac{x^2-1}{x})^{100}$, where k is an integer.

Solution: We have

$$(\frac{x^2-1}{x})^{100} = \frac{1}{x^{100}}(x^2-1)^{100} = \frac{1}{x^{100}}\sum_{i=0}^{100}\binom{100}{i}x^{2i}(-1)^{100-i} = \sum_{i=0}^{100}\binom{100}{i}x^{2i-100}(-1)^{100-i}$$

To obtain the coefficient of x^k, we must choose i such that $2i-100=k$. Therefore, we must choose $i=\frac{k+100}{2}$. Therefore, the coefficient of x^k is $\binom{100}{\frac{k+100}{2}}(-1)^{100-\frac{k+100}{2}}$.

However, this formula is only valid for $0 \leq i \leq 100$, so when $0 \leq \frac{k+100}{2} \leq 100$. This formula is therefore only valid when $-100 \leq k \leq 100$. Furthermore, k must be even, otherwise $\frac{k+100}{2}$ is not an integer.

QUESTION 12 **(6.4 # 23)**. Show Pascal's identity using the formula for $\binom{n}{k}$.

Solution: We want to show that for any integers k,n such that $0 \leq k$ and $n \geq k+1$, we have

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

By working on the left side, we get

$$\binom{n}{k} + \binom{n}{k+1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} = \frac{n!}{k!(n-k)(n-k-1)!} + \frac{n!}{(k+1)k!(n-k-1)!}$$

$$= \frac{n!(k+1)}{(k+1)k!(n-k)(n-k-1)!} + \frac{n!(n-k)}{(k+1)k!(n-k)(n-k-1)!} = \frac{n!((k+1)+(n-k))}{(k+1)k!(n-k)(n-k-1)!} = \frac{n!(n+1)}{(k+1)!(n-k)!}$$

$$= \frac{(n+1)!}{(k+1)!(n-k)!} = \binom{n+1}{k+1}$$

QUESTION 13 **(6.4 # 25)**. Let n and k be two integers such that $1 \le k \le n$. We want to show that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

(a) Use a combinatorial proof to show this equality. (Show that the two sides of the equality counts the number of ways to choose a subset of size k from a set of size n, and then to choose an element of that subset.)
(b) Use the definition of $\binom{n}{k}$ to show this equality.

Solution:

(a) On the left side, we first choose a subset of size k: there are $\binom{n}{k}$ ways to accomplish this task. Then, choose a special element from that subset. There are $\binom{k}{1} = k$ ways to accomplish this task. By the product principle, we get $k\binom{n}{k}$.

For the right side, we choose the special element first: there are $\binom{n}{1} = n$ ways to accomplish this task. We then create the subset of size k. This subset must contain the special element and $k-1$ other elements from the remaining $n-1$ elements of the set. There are $\binom{n-1}{k-1}$ ways to create this subset. By the product principle, we get $n\binom{n-1}{k-1}$

Since we are counting the same thing in both cases, we conclude that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

(b) We get

$$k\binom{n}{k} = \frac{n!k}{k!(n-k)!} = \frac{n(n-1)!k}{k(k-1)!(n-k)!} = n\frac{(n-1)!}{(k-1)!(n-k)!} = n\frac{(n-1)!}{(k-1)!(n-1-(k-1))!} = n\binom{n-1}{k-1}$$

QUESTION 14 **(6.4 # 26)**. Show that $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, where n, r and k are positive integers such that $k \le r \le n$.

(a) Use a combinatorial proof.
(b) Use the definition of $\binom{n}{r}$.

Solution:

(a) We count N, the number of ways to choose a subset of size r from a set of size n, and then choosing a subset of size k from that subset of size r.

We first choose the subset of size r: there are $\binom{n}{r}$ ways to do that. We then choose the subset of size k: there are $\binom{r}{k}$ ways to do that. By the product principle, $N = \binom{n}{r}\binom{r}{k}$.

Alternatively, we can choose the subset of size k first: there are $\binom{n}{k}$ ways to do that. We then choose the subset of size r. This subset must contain all the elements in the subset of size k: we must therefore only choose the remaining $r-k$ elements, from the remaining $n-k$ elements in the set of size n. There are $\binom{n-k}{r-k}$ ways to do that. By the product principle, $N = \binom{n}{k}\binom{n-k}{r-k}$.

Therefore, $N = \binom{n}{r}\binom{r}{k}$ and $N = \binom{n}{k}\binom{n-k}{r-k}$, so $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$.

(b) We get

$$\binom{n}{r}\binom{r}{k} = \frac{n!}{r!(n-r)!}\frac{r!}{k!(r-k)!} = \frac{n!}{k!(n-r)!(r-k)!} = \frac{n!}{k!(n-k)!}\frac{(n-k)!}{(r-k)!(n-r)!} = \binom{n}{k}\frac{(n-k)!}{(r-k)!((n-k)-(r-k))!} = \binom{n}{k}\binom{n-k}{r-k}$$

QUESTION 15 **(6.4 # 27)**. Show that if n and k are positive integers, then

$$\binom{n+1}{k} = \frac{n+1}{k}\binom{n}{k-1}$$

Solution: We get

$$\binom{n+1}{k} = \frac{(n+1)!}{k!(n+1-k)!} = \frac{(n+1)n!}{k(k-1)!(n-(k-1))!} = \frac{n+1}{k}\frac{n!}{(k-1)!(n-(k-1))!} = \frac{n+1}{k}\binom{n}{k-1}$$