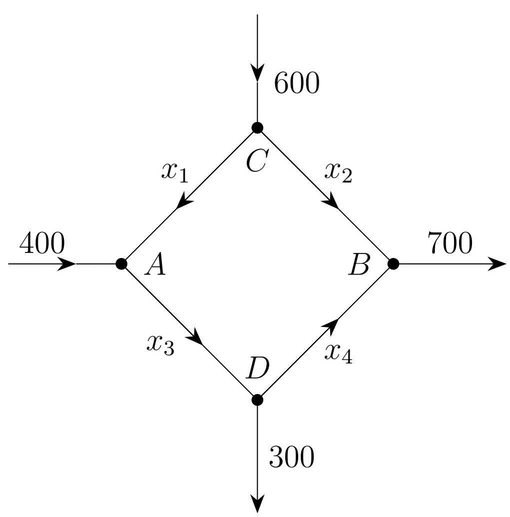
Notes for MAT1341A Fall 2023 Part III

Chapter 13 - Applications of Solving Linear Systems

I. Traffic flow network of one-way street.

The diagram in the figure below represents a network of one-way streets. The numbers on the figure represent the flow of traffic (in cars per hour) along each street, and the intersections are labeled A, B, C and D. The arrows indicate the direction of the flow of traffic. The variables x_1, x_2, x_3, x_4 represent the (unknown) level of traffic on certain streets.



Notice that the variables are traffic flows on internal street, $x_i = \#$ cars per hours.

Goal:

- explain the traffic flow in simple terms (solve).
- answer question / scenarios.

Equations: flow in = flow out

Intersection	**Flow in**	=	**Flow out**
A	$x_1 + 400$	=	x_3
B	$x_2 + x_4$	=	700
C	600	=	$x_1 + x_2$
D	x_3	=	$\cancel{700} + 300$ x_4

This is the linear system:

$$
\begin{aligned}
x_1 - x_3 &= -400 \\
x_2 + x_4 &= 700 \\
x_1 + x_2 &= 600 \\
x_3 - x_4 &= 300
\end{aligned}
$$

with augmented matrix:

$$
\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -400 \\ 0 & 1 & 0 & 1 & 700 \\ 1 & 1 & 0 & 0 & 600 \\ 0 & 0 & 1 & -1 & 300 \end{array}\right] \sim \underset{-R_1+R_3 \to R_3}{\cancel{R_3 \to R_3 - R_1}} \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -400 \\ 0 & 1 & 0 & 1 & 700 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & -1 & 300 \end{array}\right]
$$

$$
\sim \underset{-R_2+R_3 \to R_3}{\cancel{R_3 \to R_3 - R_2}} \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -400 \\ 0 & 1 & 0 & 1 & 700 \\ 0 & \cancel{1}\ 0 & 1 & -1 & 300 \\ 0 & 0 & 1 & -1 & 300 \end{array}\right] \sim \begin{cases} \cancel{R_1 \to R_1 + R_3} & -R_3 + R_4 \to R_4 \\ \cancel{R_4 \to R_4 - R_3} \end{cases}
$$

$$
\sim\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -400 \\ 0 & 1 & 0 & 1 & 700 \\ 0 & 0 & 1 & -1 & 300 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{(R}\cancel{\text{R}}\text{EF)} \quad \underset{R_3 + R_1 \to R_1}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -100 \\ 0 & 1 & 0 & 1 & 700 \\ 0 & 0 & 1 & -1 & 300 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

(RREF)

our solution is

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -100 + t \\ 700 - t \\ 300 + t \\ t \end{bmatrix}
$$

$x_1 - x_4 = -100$

$x_1 = -100 + x_4$

Wait, think about the real-life situation. Try to determine the interval of t.

$$x_1 = -100 + t \geq 0 \qquad\qquad t \geq 100$$
$$x_2 = 700 - t \geq 0 \quad \Rightarrow \quad t \leq 700 \quad \text{, So } t \text{ has to be}$$
$$x_3 = 300 + t \geq 0 \qquad\qquad \text{between } 100 \text{ and } 700.$$
$$x_4 = t \geq 0$$

Now consider the following questions:

a. What is the minimum flow along AD?

b. What happens if we close AD, will there be a traffic jam?

On AD, we have $x_3 = 300 + t \geq 300 + 100 = 400$

We cannot close AD, since this gives $300 + t = 0$
$t = -300$
impossible.

We can however close CB, since we get
$t = 700$, $x_1 = 600$, $x_3 = 1000$, $x_4 = 700$,
which is possible.

II. Solving systems with parameters.

a parameter

For what values of a does the system with the following augmented matrix have a unique solution?

$$A = \left[\begin{array}{ccc|c} a & 2 & 2 & -2 \\ 1 & 1 & 3 & a \\ 2 & a & a & 2 \end{array}\right]$$

$$\underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & a \\ a & 2 & 2 & -2 \\ 2 & a & a & 2 \end{array}\right]$$

$$\begin{array}{c} -aR_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \\ \sim \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 0 & 2-a & 2-3a & -2-a^2 \\ 0 & a-2 & a-6 & 2-2a \end{array}\right]$$

$$\underset{\sim}{R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 0 & 2-a & 2-3a & -2-a^2 \\ 0 & 0 & -4-2a & -a^2-2a \end{array}\right]$$

The system has a unique solution if and only if the matrix on the LHS has rank 3.

This is the same as saying $2-a \neq 0$ and $-4-2a \neq 0$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$a \neq 2 \qquad\qquad a \neq -2$$

Therefore, when $a \neq 2, -2$, the system has a unique solution.

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if $a=2$, we have $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 0 & -4 & -6 \\ 0 & 0 & -8 & -8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$

inconsistent

if $a=-2$, $\left[\begin{array}{ccc|c} 1 & 1 & 3 & -2 \\ 0 & 4 & 8 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

this is consistent, with infinitely many solutions.

Another eg: $\left[\begin{array}{ccc|c} 1 & 1 & 5 & a \\ 0 & 2 & 2-a & a^2 \\ 0 & 0 & a & 1-a \end{array}\right]$ When is it inconsistent?

Answer: $a=0$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$ is consistent

but

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$ is inconsistent.

eg: $\left[\begin{array}{ccc|c} 1 & t & s & s \\ 0 & -1 & 2 & 2+t \\ 0 & 0 & t-s & t^2 \end{array}\right]$ When is it inconsistent?

Answer: $t=s$ and $t\neq 0$

III. Solving vector equations.

What are all the vectors in $\mathbb{R}^3$ that are a linear combination of

$$\{(1,2,1),(3,4,4),(2,6,1)\}\ ?$$

In other words, for which $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ can we find

$a,b,c \in \mathbb{R}$ such that $a\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} + c\begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\Leftrightarrow \begin{cases} a+3b+2c = x \\ 2a+4b+6c = y \\ a+4b+c = z \end{cases}$$

(a,b,c are unknowns; x,y,z are parameters)

We want to know when is this system consistent.

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & x \\ 2 & 4 & 6 & y \\ 1 & 4 & 1 & z \end{array}\right]$$

$$\begin{array}{c} -2R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3 \\ \sim \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 2 & x \\ 0 & -2 & 2 & y-2x \\ 0 & 1 & -1 & z-x \end{array}\right]$$

$$\begin{array}{c} R_2 \leftrightarrow R_3 \\ \sim \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 2 & x \\ 0 & 1 & -1 & z-x \\ 0 & -2 & 2 & y-2x \end{array}\right]$$

$$\begin{array}{c} 2R_2+R_3 \to R_3 \\ \sim \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 2 & x \\ 0 & 1 & -1 & z-x \\ 0 & 0 & 0 & -4x+y+2z \end{array}\right]$$

$y-2x+2(z-x)$

For this to be consistent, we have to have

$-4x+y+2z=0$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a linear combination of the given vectors if and only if it lies in the plane $-4x+y+2z=0$.

Chapter 14 - Matrices
MAT 1341

A matrix can be thought of as:

- a table of numbers
- the augmented matrix of a linear system
- a collection of column vectors
- a collection of row vectors
- a mathematical object in its own right

Definition. A matrix with m rows and n columns is called an m by n matrix, it has size $m \times n$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ — } 2 \times 3 \text{ matrix}$$

2 rows

3 columns

$(2,3)$ entry row 2, column 3

You can add matrices componentwise if they have the same size.

$$\underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} + \underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \qquad \text{but } \underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} + \underset{2\times 1}{\begin{bmatrix} 5 \\ 6 \end{bmatrix}} \text{ not allowed}$$

You can multiply a matrix by a scale $(k \in \mathbb{R})$

$$2 \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} = \underset{2\times 3}{\begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}}$$

(or null matrix)

You have a zero matrix in every size

$$\mathbf{0}_{2\times 3} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{2\times 3} \qquad \mathbf{0}_{1\times 2} = \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}}_{1\times 2}$$

> **Definition** (Matrix transpose)**.** If A is $m \times n$ then the "A-transpose" $A^\top$ is $n \times m$, and the rows of A are the columns of $A^\top$.

$$A = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \qquad A^\top = \underset{3\times 2}{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}}$$

Note that the transpose operation on matrices satisfies

- $(A + B)^\top = A^\top + B^\top$
- $(kA)^\top = kA^\top, k \in \mathbb{R}$
- $(A^\top)^\top = A$

If u is a column vector in $\mathbb{R}^n$
it's an $n \times 1$ matrix.
So u^T is a row vector,
or $1 \times n$ matrix.

> **Definition** (14.1.2)**.** If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then their *product* AB is the $m \times p$ matrix whose (i, j) entry is the dot product of the ith row of A with the jth column of B.

$2\times 4 \quad 4 \times 3$

$\downarrow$

2×3

$$A = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \qquad B = \underset{3\times 3}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}}$$

$$AB = \underset{2\times 3}{\begin{bmatrix} -1 & 6 & 5 \\ -1 & 15 & 11 \end{bmatrix}}$$

$(1\ 2\ 3)\cdot(0\ 1\ -1)$
$= 0 + 2 - 3 = -1$

$(1\ 2\ 3)\cdot(1\ 1\ 1)$
$= 1 + 2 + 3 = 6$

$(1\ 2\ 3)\cdot(0\ 1\ 1)$
$= 2 + 3 = 5$

$(4\ 5\ 6)\cdot(0\ 1\ -1)$
$= 5 - 6 = -1$

$(4\ 5\ 6)\cdot(1\ 1\ 1)$
$= 4 + 5 + 6 = 15$

$(4\ 5\ 6)\cdot(0\ 1\ 1)$
$= 5 + 6 = 11$

BA does not exist

$3\times 3 \quad 2\times 3$

[*E.g.*] Find the matrix product of the following matrices

a) $A = \begin{bmatrix} 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 2 & 0 & -4 \end{bmatrix}$

a) $\underset{1\times 2}{A}\ \underset{1\times 2}{B}$ does not exist

b) $\underset{2\times 1}{A}\underset{1\times 2}{B} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$

c) $AB = \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 2 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 3 & -17 \\ 7 & 14 & 6 & -22 \\ 9 & 18 & 9 & -27 \end{bmatrix}$

- We can express a linear system as a matrix equation:

$$\begin{aligned} x + 2y + z &= 1 \\ 4x + 5y + 6z &= 2 \\ 7x + 8y + 9z &= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

is equivalent to $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

The augmented matrix of the system is $[A|\vec{b}]$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 3 \end{array}\right]$$

- We can express a linear combination as a matrix multiplication:

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n = [\vec{u}_1 \ \vec{u}_2 \ \cdots \ \vec{u}_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

[*E.g.*] One can check

$$\underbrace{a \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix} + c \begin{bmatrix} 4 \\ -2 \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 1 & -2 \\ 2 & -2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

$$\begin{bmatrix} a + 4c \\ -a + b - 2c \\ 2a - 2b \\ 3a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 1 & -2 \\ 2 & -2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

There are some ways in which matrix multiplication is **different** from number multiplication:

1. Is $AB = BA$?

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$AB \neq BA$. So multiplication of matrices is not commutative in general.

2. If $AB = 0$, must A or B be the zero-matrix?

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$AB = 0$ but $A, B \neq 0$.

3. If $AB = AC$, can we cancel A to get $B = C$?

If A is the zero matrix, then B and C can be anything!

What if $A \neq 0$?

$$AB = AC \iff A(B - C) = 0$$

Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Then $A(B - C) = 0$, but $A \neq 0$ and $B \neq C$.

So we see that the cancelation law doesn't work for matrices.

$\swarrow$ a square matrix

Set $I_k = k \times k$ matrix with 1s on diagonal and 0s elsewhere.

$$I_1 = [1] \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

called the identity matrix of size k.

[*E.g.*] Find the matrix product AI_3, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

$$\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \qquad AI_3 = A$$

$$\underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \qquad I_2 A = A$$

Theorem (**14.3.1 - Properties of the matrix product**)**.** Let A, B and C be matrices and let k be a scalar. Then, whenever defined, we have

1. $(AB)C = A(BC)$ (Associativity)
2. $A(B + C) = AB + AC$ (Distributivity on the right)
3. $(B + C)A = BA + CA$ (Distributivity on the left)
4. $k(AB) = (kA)B = A(kB)$
5. $(AB)^\top = B^\top A^\top$ (NOTE the reversal of order!)
6. $\underset{m\times n}{A} I_n = A$ and $I_n \underset{n\times p}{B} = B$
7. If A is $m \times n$, then $A0_{n\times p} = 0_{m\times p}$ and $0_{q\times m}A = 0_{q\times n}$.

Suppose A is $m \times n$ and B is $n \times p$

AB is $m \times p$

Then $(AB)^T$ is $p \times m$.

B^T is $p \times n$

A^T is $n \times m$

$\Rightarrow B^T A^T$ is defined and it's of size $p \times m$.

Now we can do basic algebra:

i. $(A + B)(C + D) = AC + AD + BC + BD$

ii. $(A + B)(A - B) = A^2 - \underbrace{AB + BA}_{\text{do not cancel!}} - B^2$

$\underset{m\times \underline{n}}{A} \cdot \underset{\underline{m}\times n}{A}$ makes sense

if and only if $m=n$

a square matrix

Definition. If a matrix has size $m \times m$, we say that it is a *square matrix*.

Given a square matrix and a positive integer n, we define

$$A^n = \underbrace{A \cdots A}_{n \text{ times}}.$$

[*E.g.*] Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Calculate A^{2023} and B^{2023}.

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Each time we multiply by A, the $(1,2)$-entry increases by 1 whereas the other entries stay the same.

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

$$A^{2023} = \begin{bmatrix} 1 & 2023 \\ 0 & 1 \end{bmatrix}.$$

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$B^{2023} = B^{1+2\times 1011}$$

$$= B \cdot (B^2)^{1011}$$

$$= B (I_2)^{1011}$$

$$= B \cdot I_2 = B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$[E.g.]$ Calculate $\left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]^{2023}$.

$$\left[\begin{array}{c|c} A & O_{2\times 2} \\ \hline O_{2\times 2} & B \end{array}\right]^2 = \left[\begin{array}{c|c} A & O_{2\times 2} \\ \hline O_{2\times 2} & B \end{array}\right]\left[\begin{array}{c|c} A & O_{2\times 2} \\ \hline O_{2\times 2} & B \end{array}\right]$$

$$= \left[\begin{array}{c|c} A^2 & O_{2\times 2} \\ \hline O_{2\times 2} & B^2 \end{array}\right]$$

$$\left[\begin{array}{c|c} A & O_{2\times 2} \\ \hline O_{2\times 2} & B \end{array}\right]^{2023} = \left[\begin{array}{c|c} A^{2023} & O_{2\times 2} \\ \hline O_{2\times 2} & B^{2023} \end{array}\right]$$

$$= \begin{bmatrix} 1 & 2023 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$