Lesson 3 – Continuity

PART A: Recall

Through our graphical investigation of limits of various functions, we came across one of the main limit properties which states that a **limit exists** if, and only if the **LHL** (Left Hand Limit) and **RHL** (Right Hand Limit) are the **same**. That is to say that the limit exists if the dependant value that we approach from the left side is the same as the dependant value that we approach from the right hand side. It can be summarized as follows:

In order for $\lim_{x \to a} f(x)$ to exist, one of the conditions is that $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$

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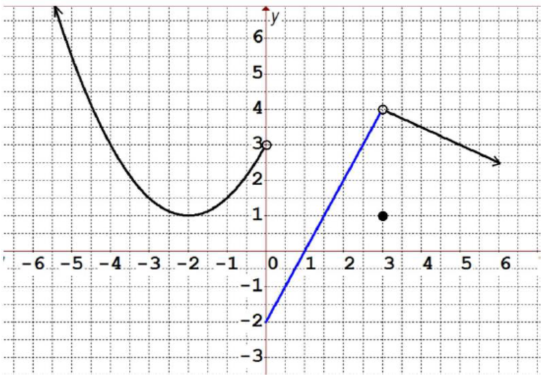
PART B: Investigation of Piecewise Functions and Discontinuities in Graphs

Consider the following piecewise function which is defined by the functions and domains:



$$f(x) = \begin{cases} \frac{1}{2}(x+2)^2 + 1, & (-\infty, 0) \\ 2x - 2, & [0, 3) \\ -\frac{1}{2}x + \frac{11}{2}, & (3, \infty) \\ 1 & if\ x = 3 \end{cases}$$

Answer the following questions about the graph above:

a) $\lim_{x \to 0^-} f(x) =$ 3

b) $\lim_{x \to 0^+} f(x) =$ -2

c) $\lim_{x \to 0} f(x) =$ dne

d) $f(0) =$ -2

e) $\lim_{x \to 3^-} f(x) =$ 4

f) $\lim_{x \to 3^+} f(x) =$ 4

g) $\lim_{x \to 3} f(x) =$ 4

h) $f(3) =$ 1

Note: It is not important what happens ***at*** x for **limits** to exist

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The piecewise function considered in the previous example illustrates a few of the elements that cause discontinuities in functions. The function had holes and a jump discontinuity. We also know that a discontinuity can arise if a vertical asymptote is present as well.

Let us define continuity. Let $f(x)$ be a function, we say $f(x)$ is continuous at $x = a$ if

$$\boxed{\begin{array}{l} 1).\ f(a) \text{ is defined} \\ 2).\ \lim_{x\to a} f(x) \text{ exists} \\ 3).\ \lim_{x\to a} f(x) = f(a) \end{array}}$$

Further to the definition of continuity above, if a function is continuous at every point in its domain, we call it a continuous function.

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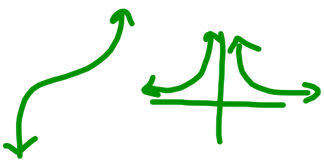
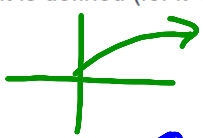
PART C: Examples of Continuous Functions

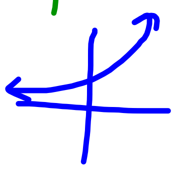


All **Power** functions are continuous

- $f(x) = x^3$
- $g(x) = x^{-2} = \frac{1}{x^2}$, is continuous everywhere it is defined (for $x \neq 0$)
- $h(x) = x^{1/2} = \sqrt{x}$, is continuous for $x \geq 0$

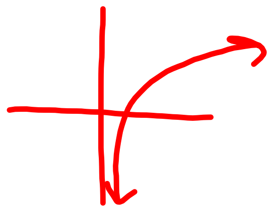


All **Exponential** functions are continuous



- $f(x) = 3^x$
- $g(x) = 10^x$
- $h(x) = \left(\frac{1}{2}\right)^x$

$\left.\right\}$ Continuous on $(-\infty, +\infty)$



All **Logarithmic** functions are continuous

- $f(x) = log x$
- $g(x) = \log_{1/3} x$
- $h(x) = \log_2 x$

$\left.\right\}$ Continuous for $x > 0$

What about Polynomial functions? Are they continuous? The simple answer is YES! Let's see why.

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Problem: Prove that $p(x) = 3x^2 - x + 10$ is continuous.

Answer: $x^2, x, 10$ are continuous because they are power functions.
$3x^2 \;\; and \;\; -x$ are continuous because of a property of limits which can be summarized as follows:

$3x^2 \quad 3 \quad x^2$

$$\boxed{\lim_{x \to a}[cf(x)] = c\left[\lim_{x \to a} f(x)\right], \text{ for any constant c}}$$

$p(x) = 3x^2 - x + 10$ is continuous because of a property of limits which can be summarized as follows:

$$\boxed{\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)}$$

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All **Polynomial** functions are continuous

- $f(x) = 2x^2 - x + 10$
- $g(x) = -6500x^3 + 150x^2 - 20$
- $h(x) = 5x$

$\Big\}$ Continuous on $(-\infty, +\infty)$

All **Rational** functions are continuous wherever they are ***defined***

- $r(x) = \frac{24x}{x^2-9}$, is continuous for $x \neq \pm 3$

$(x+3)(x-3)$

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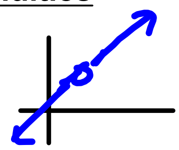
PART D: Summary

- A function $f(x)$ is continuous at $x = a$ if:

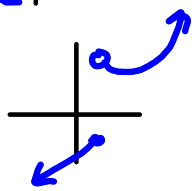
1). $f(a) is\ defined$
2). $\lim_{x \to a} f(x)\ exists$
3). $\lim_{x \to a} f(x) = f(a)$

Basic Discontinuities

Point Discontinuity

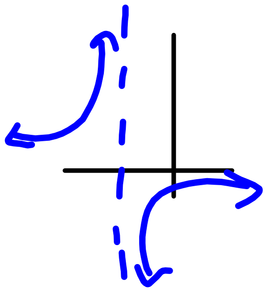


Jump Discontinuity



- Discontinuities arise in graphs when there is some type of break (ie. Hole, jump, VA)
- Polynomial functions are continuous for all real numbers
- Rational functions $h(x) = \frac{f(x)}{g(x)}$ are continuous for all values except those that make the denominator zero $(g(a) \neq 0)$

Infinite Discontinuity



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