

17. Counting: PIE & The Sum Rule

factorial: $0! = 1$ and for $n \geq 1$, $n! = n(n-1)! = n(n-1) \cdots (2)(1)$

r -permutations of an n -set: $P(n, r) = n(n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$ $P(n, n) = n!$

r -combinations of an n -set: $C(n, r) = \binom{n}{r} = \frac{P(n, r)}{P(r, r)} = \frac{n!}{(n-r)!r!}$

THE PRINCIPLE OF INCLUSION–EXCLUSION (PIE)



Theorem 17.1. Let A and B be sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

and

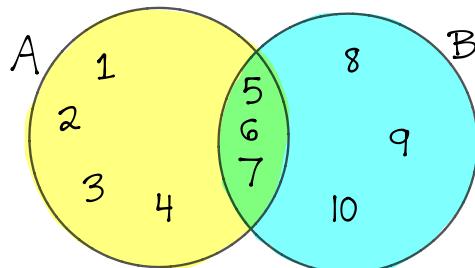
$$\begin{aligned} |\overline{A \cup B}| &= |\mathcal{U}| - |A \cup B| \\ &= |\mathcal{U}| - |A| - |B| + |A \cap B| \end{aligned}$$

Example 17.2. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and let $B = \{5, 6, 7, 8, 9, 10\}$.

$$|\overline{A \cup B}| = 10$$

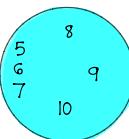
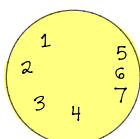
Exercise

Find a PIE formula for $|\overline{A \cup B \cup C}|$. How does PIE generalize for finding the cardinality of the union of n sets?



By PIE,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 7 + 6 - 3 = 10 \end{aligned}$$



Example 17.3. How many 5-letter "words" (strings of length 5 consisting of lowercase letters of the English alphabet) start with the letter **a** or end with the letters **zz**?

Procedure : create 5-letter words with given properties (a sequence of 5 tasks)

Let $A = \{5\text{-letter words starting with 'a'}\}$

$T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5$

$\begin{array}{c} a \\ \boxed{1} \\ \text{way} \end{array} \quad \begin{array}{c} \square \\ 26 \\ \text{ways} \end{array}$

Let $B = \{5\text{-letter words ending with 'zz'}\}$

$\begin{array}{c} \square \\ 26 \\ \text{ways} \end{array} \quad \begin{array}{c} \square \\ 26 \\ \text{ways} \end{array} \quad \begin{array}{c} \square \\ 26 \\ \text{ways} \end{array} \quad \begin{array}{c} z \\ 1 \\ \text{way} \end{array} \quad \begin{array}{c} z \\ 1 \\ \text{way} \end{array}$

Then $A \cap B = \{5\text{-letter words starting with 'a' and ending with 'zz'}\}$

$\begin{array}{c} a \\ \boxed{1} \\ \text{way} \end{array} \quad \begin{array}{c} \square \\ 26 \\ \text{ways} \end{array} \quad \begin{array}{c} \square \\ 26 \\ \text{ways} \end{array} \quad \begin{array}{c} z \\ 1 \\ \text{way} \end{array} \quad \begin{array}{c} z \\ 1 \\ \text{way} \end{array}$

and $A \cup B = \{5\text{-letter words that start with 'a' or end with 'zz'}\}$

Thus, we want $|A \cup B|$. By PIE,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 26^4 + 26^3 - 26^2 \end{aligned}$$

Example 17.4. How many 5-letter "words" do not start with the letter **a** nor end with the letters **zz**?

Observe that $\overline{A \cup B} = \{5\text{-letter words that do not start with 'a' and do not end with 'zz'}\}$

and, in this scenario, the universal set is $\mathcal{U} = \{5\text{-letter words}\}$

Thus, we want $|\overline{A \cup B}|$

$\begin{array}{c} \square \\ 26 \\ \text{ways} \end{array} \quad \begin{array}{c} \square \\ 26 \\ \text{ways} \end{array}$

$$\text{By PIE, } |\overline{A \cup B}| = |\mathcal{U}| - |A \cup B|$$

$$\begin{aligned} &= 26^5 - [26^4 + 26^3 - 26^2] \\ &= 11407500 \end{aligned}$$

Example 17.5. A photographer is taking group photos of 6 inhabitants of The Island of K&K, namely A, B, C, D, E, and F.

In how many different ways can the photographer arrange 5 out of 6 of these inhabitants in a line so that

A and B are both in the photo or B, C, and E are all in the photo ?

Let $S = \{ \text{lineups that include both A and B} \}$

T_1 : choose A and B to be in the photo (in $\binom{2}{2} = 1$ way)

T_2 : choose 3 more inhabitants from among C, D, E, F to be in the photo
(in $\binom{4}{3} = 4$ ways)

T_3 : choose an ordering for the lineup of 5 inhabitants chosen in T_1, T_2 .
(in one of $P(5,5) = 5! = 120$ ways)

$$\therefore |S| = (1)(4)(120) = 480$$

Let $T = \{ \text{lineups that include B, C, and E} \}$

T_1 : choose B, C, and E to be in the photo (in $\binom{3}{3} = 1$ way)

T_2 : choose 2 more inhabitants from among A, D, and F to be in the photo
(in $\binom{3}{2} = 3$ ways)

T_3 : choose an ordering for the lineup of 5 inhabitants chosen in T_1, T_2 .
(in one of $P(5,5) = 5! = 120$ ways)

$$\therefore |T| = (1)(3)(120) = 360$$

Thus, $S \cap T = \{ \text{lineups that include A, B, C, and E} \}$

T_1 : choose A, B, C, and E to be in photo ($\binom{4}{4} = 1$ way)

T_2 : choose 1 more from among D and F ($\binom{2}{1} = 2$ ways)

T_3 : choose an ordering of those 5 ($P(5,5) = 120$ ways)

$$\therefore |S \cap T| = (1)(2)(120) = 240$$

$S \cup T = \{ \text{lineups that include A \& B or include B, C, \& E} \}$

By PIE,

$$|S \cup T| = |S| + |T| - |S \cap T| = 480 + 360 - 240$$

∴ there are 600 possible such photo lineups

THE SUM RULE FOR k TASKS

Suppose a procedure can be broken down into k tasks T_1, T_2, \dots, T_k such that

- for $i=1, 2, \dots, k$, the entire procedure can be done by task T_i which can be carried out in one of n_i ways,
- and the tasks T_1, T_2, \dots, T_k are mutually exclusive
ie none of the tasks share a common way to accomplish the procedure

Then there are $n_1 + n_2 + \dots + n_k$ ways to carry out the procedure.

Example 17.6. How many binary strings of length 10 start with '000' and contain at least five 1's ?

0 0 0 _____

Given that 3 entries are 0's and of the 7 remaining entries, at least 5 must be 1's,

We must consider the options for (# 1's):

- ① exactly 5 ones (hence '000' + 2 more 0's)
 - ② exactly 6 ones (hence '000' + 1 more 0)
 - ③ exactly 7 ones (hence '000' + no other 0's)
- } these are mutually exclusive.

- ① choose 5 entries among the 7 remaining after '000' to place 5 1's into, then fill the rest with 0's.

$$\binom{7}{5} = \frac{7!}{5!2!} = \frac{7(6)(5!)}{5!(2)(1)} = 21 \text{ ways}$$

- ② choose 6 entries among the 7 remaining after '000' to place 6 1's into, then fill the last with a 0.

$$\binom{7}{6} = \frac{7!}{6!1!} = 7 \text{ ways}$$

- ③ fill all 7 entries after '000' with 7 1's ($\binom{7}{7} = 1$ way)

By the Sum Rule there are $21+7+1 = 29$ such binary strings in total.

Exercise 17.7. How many binary strings of length 10 start with '000' or contain at least five 1's ?

Exercise 17.8. A 9-member fellowship is to be formed by selecting its members from among a group of 15 people consisting of 1 wizard, 5 hobbits, 5 elves, 2 men, and 2 dwarves.

In how many ways can the fellowship be formed if it must contain at least 2 hobbits or at least 2 men?

Let $A = \{ \text{fellowships with at least 2 hobbits} \}$


PIE

$$\therefore |A| = \underbrace{\binom{5}{2} \cdot \binom{10}{7}}_{\substack{2 \text{ hobbits and } 7 \text{ non-hobbits} \\ \text{from the wizard, 5 elves, 2 men, and 2 dwarves}}} + \underbrace{\binom{5}{3} \cdot \binom{10}{6}}_{\substack{3 \text{ hobbits and } 6 \text{ non-hobbits}}} + \underbrace{\binom{5}{4} \cdot \binom{10}{5}}_{\substack{4 \text{ hobbits and } 5 \text{ non-hobbits}}} + \underbrace{\binom{5}{5} \cdot \binom{10}{4}}_{\substack{5 \text{ hobbits and } 4 \text{ non-hobbits}}} = 4770$$

Let $B = \{ \text{fellowships with at least 2 men} \}$

$$\therefore |B| = \underbrace{\binom{2}{2} \cdot \binom{13}{7}}_{\substack{2 \text{ men and } 7 \text{ others} \\ \text{from the wizard, 5 hobbits, 5 elves, and 2 dwarves}}} = 1716$$

Thus, $A \cap B = \{ \text{fellowships with at least 2 hobbits and at least 2 men} \}$

$$|A \cap B| = \underbrace{\binom{5}{2} \cdot \binom{2}{2} \left(\binom{1+5+2}{5} \right)}_{\substack{2 \text{ hobbits, 2 men, and } 5 \text{ others} \\ \text{from the wizard, 5 elves, and 2 dwarves}}} + \underbrace{\binom{5}{3} \cdot \binom{2}{2} \left(\binom{1+5+2}{4} \right)}_{\substack{4 \text{ hobbits, 2 men, and } 3 \text{ others} \\ \text{from the wizard, 5 elves, and 2 dwarves}}} + \underbrace{\binom{5}{4} \cdot \binom{2}{2} \left(\binom{1+5+2}{3} \right)}_{\substack{5 \text{ hobbits, 2 men, and } 2 \text{ others} \\ \text{from the wizard, 5 elves, and 2 dwarves}}} + \underbrace{\binom{5}{5} \cdot \binom{2}{2} \left(\binom{1+5+2}{2} \right)}_{\substack{3 \text{ hobbits, 2 men, and } 4 \text{ others} \\ \text{from the wizard, 5 elves, and 2 dwarves}}} = 1568$$

$A \cup B = \{ \text{fellowships with at least 2 hobbits or at least 2 men} \}$

$$\begin{aligned} \text{By PIE, } |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 4770 + 1716 - 1568 \\ &= 4918 \end{aligned}$$

\therefore there are 4918 different 9-member fellowships with at least 2 hobbits or at least 2 men.



STUDY GUIDE

sum rule for k tasks:

there are $n_1 + n_2 + \dots + n_k$ ways to carry out a procedure given that there are n_i ways to carry out the entire procedure with task T_i , and the tasks T_1, \dots, T_k are mutually exclusive tasks that each carry out the entire procedure.

Exercises

Sup.Ex. §8 # 1, 5, 7, 9, 10, 11

Rosen §6.1 # 9, 15, 29, 31, 49 (treat as inclusive)

Sup.Ex. §10 # 1, 2, 4, 6, 7, 8, 9, 10, 11

§6.3 # 19, 21, 27, 31, 33, 35, 37

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More details on how we computed $|A|$, $|B|$ and $|A \cap B|$

Let $A = \{\text{fellowships with at least 2 hobbits}\}$

We will break A up into cases based on the number of hobbits

Case 1: exactly 2 hobbits

T_1 : select 2 hobbits (in one of $\binom{5}{2} = 10$ ways)

T_2 : select 7 other members from among the wizard, 5 elves, 2 men, 2 dwarves
(in one of $\binom{1+5+2+2}{7} = \binom{10}{7} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 120$ ways)

Case 2: exactly 3 hobbits

T_1 : select 3 hobbits (in one of $\binom{5}{3} = 10$ ways)

T_2 : select 6 other members from among the wizard, 5 elves, 2 men, 2 dwarves
(in one of $\binom{1+5+2+2}{6} = \binom{10}{6} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$ ways)

Case 3: exactly 4 hobbits

T_1 : select 4 hobbits (in one of $\binom{5}{4} = 5$ ways)

T_2 : select 5 other members from among the wizard, 5 elves, 2 men, 2 dwarves
(in one of $\binom{1+5+2+2}{5} = \binom{10}{5} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 252$ ways)

Case 4: exactly 5 hobbits

T_1 : select 5 hobbits (in one of $\binom{5}{5} = 1$ way)

T_2 : select 4 other members from among the wizard, 5 elves, 2 men, 2 dwarves
(in one of $\binom{1+5+2+2}{4} = \binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$ ways)

Let $B = \{\text{fellowships with at least 2 men}\}$

T_1 : select both of the 2 men (in one of $\binom{2}{2} = 1$ way)

T_2 : select 7 other members from among the wizard, 5 hobbits, 5 elves, 2 dwarves
(in one of $\binom{1+5+2+2}{7} = \binom{13}{7} = 1716$ ways)

Thus, $A \cap B = \{\text{fellowships with at least 2 hobbits and at least 2 men}\}$

We will break $A \cup B$ up into cases based on the number of hobbits

Case 1: exactly 2 hobbits

T_1 : select 2 hobbits (in one of $\binom{5}{2} = 10$ ways)

T_2 : select 2 men (in $\binom{2}{2} = 1$ way)

T_3 : select 5 other members from among the wizard, 5 elves, 2 dwarves
(in one of $\binom{1+5+2}{5} = \binom{8}{5} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$ ways)

Case 2: exactly 3 hobbits

T_1 : select 3 hobbits (in one of $\binom{5}{3} = 10$ ways)

T_2 : select 2 men (in $\binom{2}{2} = 1$ way)

T_3 : select 4 other members from among the wizard, 5 elves, 2 dwarves
(in one of $\binom{1+5+2}{4} = \binom{8}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70$ ways)

Case 3: exactly 4 hobbits

T_1 : select 4 hobbits (in one of $\binom{5}{4} = 5$ ways)

T_2 : select 2 men (in $\binom{2}{2} = 1$ way)

T_3 : select 3 other members from among the wizard, 5 elves, 2 dwarves
(in one of $\binom{1+5+2}{3} = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$ ways)

Case 4: exactly 5 hobbits

T_1 : select 5 hobbits (in one of $\binom{5}{5} = 1$ way)

T_2 : select 2 men (in $\binom{2}{2} = 1$ way)

T_3 : select 2 other members from among the wizard, 5 elves, 2 dwarves
(in one of $\binom{1+5+2}{2} = \binom{8}{2} = \frac{8 \cdot 7}{2 \cdot 1} = 28$ ways)