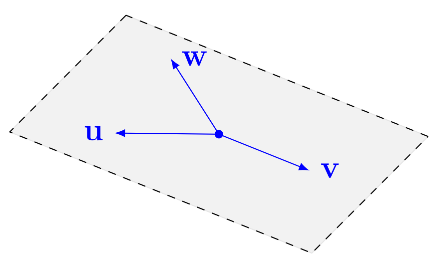
Notes for MAT1341A Fall 2023 Part IV

Chapter 7 - Linear dependence and independence

[*e.g.*] Show that $\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right\}$.

Geometrically, we see that the problem is that all the vectors are collinear (meaning, parallel, or all lying on one line). Similar problems would occur in $\mathbb{R}^3$ if we had three *coplanar* vectors, that is, all lying in a plane.



$\text{span}\{\mathbf{u}, \mathbf{v}\} = \text{span}\{\mathbf{u}, \mathbf{w}\} = \text{span}\{\mathbf{v}, \mathbf{w}\} = \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are coplanar (lying in the same plane).

Definition (7.5.1). Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m \in V$, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m \in V\}$ is *linearly dependent* if and only if there are scalars $a_1, a_2, ..., a_m \in \mathbb{R}$, not all zero such that

$$a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v}_m = \mathbf{0}.$$

[*E.g.*] Show that $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$ is LD.

If the zero vector is in a set of vectors, then this set is always LD since

$$1 \cdot \vec{0} = \vec{0}$$

[*E.g.*] Show that $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is LD.

[*E.g.*] Show that $\{\sin^2 x, 1, \cos^2 x\}$ is LD.

Definition (7.6.1). Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m \in V$, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m \in V\}$ is *linearly independent* if and only if the *only solution* to

$$a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v}_m = \mathbf{0}$$

is the trivial solution $a_1 = 0, \ldots, a_m = 0$.

[*E.g.*] Show that $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ is LI.

[*E.g.*] Show that $\{1 + X, 1 - X\}$ is LI.

Fact. If $\mathbf{v} \in V$, then $\{\mathbf{v}\}$ is LI if and only if $\mathbf{v} \neq \mathbf{0}$.

Fact. If $S = \{\mathbf{v}_1, ..., \mathbf{v}_m\}$ is LD, then any set containing S is also LD.

Proof.

Fact. If $S = \{\mathbf{v}_1, ..., \mathbf{v}_m\}$ is LI, then any subset of S is also LI.

Proof.

Fact. A set with three or more vectors can be LD *even though* no two vectors are multiples of one another.

[*E.g.*] $\{(1,0),(0,1),(1,1)\}$ are coplanar but no two vectors are collinear.

Chapter 8 - Linear independence and spanning sets

Theorem (8.1.1 - Relation between linear dependence and spanning). A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is LD if and only if there is at least one vector $\mathbf{v}_k$ which is in the span of the rest.

What this tells us is that in a LD set, there is an element that is "reductant".

[*E.g.*] Show that the following sets are LD.

a) $\{(1,1,1),(2,1,2),(0,1,0)\}$.

b) $\{x^2, 1+2x, (1+x)^2\} \subset \mathbb{P}_2$.

Theorem (8.2.2 - Reducing spanning sets).
Suppose $W = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$. If $\mathbf{v}_1 \in \text{span}\{\mathbf{v}_2, \ldots, \mathbf{v}_m\}$, then

$$W = \text{span}\{\mathbf{v}_2, \ldots, \mathbf{v}_m\}.$$

[*E.g.*] Show that $\text{span}\{(1,1,1),(2,1,2),(0,1,0)\} = \text{span}\{(1,1,1),(2,1,2)\}$.

Theorem (8.3.1 - Enlarging linearly independent sets).
Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a LI subset of a subspace W. For any $\mathbf{v} \in W$, we have

$$\{\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_m\} \text{ is LI} \quad \Leftrightarrow \quad \mathbf{v} \notin \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}.$$

[*E.g.*] The set $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ is LI. Enlarge it to a LI set with 3 elements.

[*E.g.*] The set $\{x^2, 1+2x\} \subset \mathbb{P}_3$ is LI, show that $\{1, x^2, 1+2x\}$ is also LI.