MAT 1348 – Winter 2024

Exercises 7 – Solutions

Professor: Antoine Poirier

Questions are taken from Discrete Mathematics 8th edition, by Kenneth H. Rosen.

QUESTION 1 **(2.3 # 1)**. Explain why f is not a function from $\mathbb{R}$ to $\mathbb{R}$ if

(a) $f(x) = 1/x$
(b) $f(x) = \sqrt{x}$
(c) $f(x) = \pm\sqrt{(x^2+1)}$

Solution:

(a) $f(0)$ is undefined.
(b) $f(x)$ is undefined for $x < 0$.
(c) f assigns two values to each element of the domain.

QUESTION 2 **(2.3 # 4, # 7)**. Determine the domain and codomain of the following functions

(a) The function which, for each non-negative integer, assigns its last digit.
(b) The function which assigns to each finite sequence of 0's and 1's the number of 1's in it.
(c) The function which assigns to a pair of positive integer the greatest of the two integers.

Solution:

(a) The domain is $\mathbb{Z}^+$. The codomain is $\{0,1,2,3,4,5,6,7,8,9\}$.
(b) The domain is the set of all finite sequences of 0's and 1's. The codomain is $\mathbb{N} \cup \{0\}$.
(c) The domain is $\mathbb{Z}^+ \times \mathbb{Z}^+$. The codomain is $\mathbb{Z}^+$.

QUESTION 3 **(2.3 # 10, # 11)**. Determine if the following functions from $\{a,b,c,d\}$ to $\{a,b,c,d\}$ are injective and surjective

(a) $f(a) = b,\ f(b) = a,\ f(c) = c,\ f(d) = d$
(b) $f(a) = b,\ f(b) = b,\ f(c) = d,\ f(d) = c$
(c) $f(a) = d,\ f(b) = b,\ f(c) = c,\ f(d) = d$

Solution:

(a) Injective and surjective.
(b) Not injective ($f(a) = f(b)$), not surjective ($f(x) \neq a$ for all x).
(c) Not injective ($f(a) = f(d)$), not surjective ($f(x) \neq a$ for all x).

QUESTION 4 **(2.3 # 12, # 13)**. Determine if the following functions from $\mathbb{Z}$ to $\mathbb{Z}$ are injective and surjective.

(a) $f(n) = n-1$
(b) $f(n) = n^2+1$
(c) $f(n) = n^3$
(d) $f(n) = \lceil n/2 \rceil$. (Here, $\lceil x \rceil$ is the ceiling function, which outputs the smallest integer n satisfying $n \geq x$).

Solution:

(a) f is injective: if $f(x) = f(y)$, then $x-1 = y-1$ and so $x = y$. f is surjective: for all $y \in \mathbb{Z}$ (codomain), we have $f(y+1) = y$.
(b) f is not injective: $f(3) = f(-3)$, but $3 \neq -3$. f is not surjective: there is no $n \in \mathbb{Z}$ for which $f(n) = -1$, since $f(n) = n^2+1 \geq 0+1 = 1$ for all n.
(c) f is injective: if $f(x) = f(y)$, then $x^3 = y^3$ and so $x = y$. f is not surjective: there is no integer n for which $f(n) = 2$. The only n satisfying this equation is $2^{1/3}$, which is not in the domain of the function.
(d) f is not injective: $f(3) = f(4) = 2$, but $3 \neq 4$. f is surjective: for all $y \in \mathbb{Z}$ (codomain), there exists $x \in \mathbb{Z}$ such that $f(x) = y$. Such an x would be $2y$: $f(x) = \lceil 2y/2 \rceil = \lceil y \rceil = y$.

QUESTION 5 **(2.3 # 14)**. Determine if the following functions from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$ are surjective.

(a) $f(m,n) = 2m-n$
(b) $f(m,n) = m^2-n^2$
(c) $f(m,n) = m+n+1$
(d) $f(m,n) = |m|-|n|$
(e) $f(m,n) = m^2-4$

Solution:

(a) f is surjective: for all $y \in \mathbb{Z}$, we have $f(0,-y) = y$.
(b) f is not surjective: $f(m,n) \neq 2$ for all $(m,n) \in \mathbb{Z} \times \mathbb{Z}$. Since $f(m,n) = m^2-n^2 = (m-n)(m+n)$, in order for $(m-n)(m+n) = 2$, which is even, $m-n$ or $m+n$ must be even. However, if of them is even, then so is the other. This implies $(m-n)(m+n)$ is a multiple of 4, and so cannot be equal to 2.
(c) f is surjective: for all $y \in \mathbb{Z}$, we have $f(0,y-1) = y$.
(d) f is surjective: for all $y \in \mathbb{Z}^+$, we have $f(y,0) = |y| = y$. For all $y \in \mathbb{Z}^-$, we have $f(0,y) = -|y| = --y = y$. Finally, we have $f(0,0) = 0$.
(e) f is not surjective, since $f(m,n) = m^2-4 \geq -4$. Therefore, there are no pairs (m,n) for which $f(m,n) = -5$.

QUESTION 6 **(2.3 # 15)**. Determine if the following functions from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$ are surjective.

(a) $f(m,n) = m+n$
(b) $f(m,n) = m^2+n^2$
(c) $f(m,n) = m$
(d) $f(m,n) = |n|$
(e) $f(m,n) = m-n$

Solution:

(a) f is surjective: for all $y \in \mathbb{Z}$, we have $f(y,0) = y$.
(b) f is not surjective since $f(m,n) \geq 0$. Therefore, there is no pair (m,n) for which $f(m,n) = -1$, for instance.
(c) f is surjective: for all $y \in \mathbb{Z}$, we have $f(y,0) = y$.
(d) f is not surjective since $f(m,n) \geq 0$. Therefore, there is no pair (m,n) for which $f(m,n) = -1$, for instance.
(e) f is surjective: for all $y \in \mathbb{Z}$, we have $f(y,0) = y$.

QUESTION 7 **(2.3 # 23)**. Determine if the following functions from $\mathbb{R}$ to $\mathbb{R}$ are bijective.

(a) $f(x) = 2x+1$
(b) $f(x) = x^2+1$
(c) $f(x) = x^3$
(d) $f(x) = (x^2+1)/(x^2+2)$

Solution:

(a) f is injective: if $f(x) = f(y)$, then $2x+1 = 2y+1$, which simplifies to $x = y$. f is surjective: for all $y \in \mathbb{R}$ (codomain), we have $f((y-1)/2) = y$. f is therefore bijective.
(b) f is not injective: $f(2) = f(-2) = 5$, but $2 \neq -2$. f is not surjective: we have $f(x) \geq 0+1 \geq 1$, so there is no x for which $f(x) = 0$, for instance. f is not bijective.
(c) f is injective: if $f(x) = f(y)$, then $x^3 = y^3$, which simplifies to $x = y$. f is surjective: for all $y \in \mathbb{R}$ (codomain), there exists $x \in \mathbb{R}$ such that $f(x) = y$. This x is $y^{1/3}$. f is therefore bijective.
(d) f is not injective: $f(2) = f(-2) = 5/6$, but $2 \neq -2$. f is not surjective since $f(x) \geq 0$. There is therefore no $x \in \mathbb{R}$ for which $f(x) = -1$, for instance. f is not bijective.

QUESTION 8 **(2.3 # 33)**. Let $g : A \to B$ and $f : B \to C$.

(a) Show that if f and g are injective, then $f \circ g$ is also injective.
(b) Show that if f and g are surjective, then $f \circ g$ is also surjective.

Solution:

(a) Suppose f and g are injective. We show that $f \circ g$ is also injective. Suppose then $f(g(x)) = f(g(y))$. Since f is injective, it implies $g(x) = g(y)$. Since g is injective, it implies $x = y$. Therefore, $f \circ g$ is injective.
(b) Suppose f and g are surjective. We show that $f \circ g$ is also surjective. Let $z \in C$. Then, since f is surjective, there exists $y \in B$ such that $f(y) = z$. Since g is surjective, there exists $x \in A$ such that $g(x) = y$. When combining the two previous equations, we conclude there is an $x \in A$ such that $f(g(x)) = f(y) = z$. Therefore $f \circ g$ is surjective.

QUESTION 9 **(2.3 #34)**. Let $g : A \to B$ and $f : B \to C$. Show that

(a) If $f \circ g$ is surjective, then f is surjective.
(b) If $f \circ g$ is injective, then g is injective.

Solution:

(a) (Indirect proof) Suppose f is not surjective. Then, there exists $c \in C$ for which $f(y) \neq c$ for all $y \in B$. Therefore, for every $x \in A$, we have $g(x)/inB$, hence $f(g(x)) \neq c$ for all $x \in A$. This indicates $f \circ g$ is not surjective.
(b) (Indirect proof) Suppose g is not injective. Then, there exists $x, y \in A$ such that $g(x) = g(y)$, but $x \neq y$. We apply f to $g(x)$ and $g(y)$ to obtain $f(g(x)) = f(g(y))$, but $x \neq y$. This shows $f \circ g$ is not injective.

QUESTION 10 **(2.3 # 35)**. Find functions f and g such that $f \circ g$ is bijective, but g is not surjective, and f is not injective.

Solution: Let $g : A \to B$ and $f : B \to C$, with $A = \{1,2\}, B = \{p,q,r\}, C = \{a,b\}$. Define $g(1) = p$ and $g(2) = q$, $f(p) = f(r) = a$, $f(q) = b$. Then, g is no surjective since $g(x) \neq r$ for all $x \in A$. f is not injective, since $f(p) = f(r)$. However, $f(g(1)) = a$ and $f(g(2)) = b$, so $f \circ g$ is injective and surjective.

QUESTION 11 **(2.3 # 36)**. If f and $f \circ g$ are injective, is g necessarily also injective?

Solution: Yes. (Indirect proof) Suppose g is not injective. Then, there exists $x, y \in A$ such that $g(x) = g(y)$, but $x \neq y$. Apply f to $g(x)$ and $g(y)$ to obtain $f(g(x)) = f(g(y))$, but $x \neq y$. This shows that $f \circ g$ is not injective, hence "f and $f \circ g$ are injective" is false.

QUESTION 12 **(2.3 # 37)**. If f and $f \circ g$ are surjective, is g necessarily also surjective?

Solution: No. Let $g : A \to B$ and $f : B \to C$, with $A = \{1,2\}, B = \{p,q,r\}, C = \{a,b\}$. Define $g(1) = p$ and $g(2) = q$, $f(p) = f(r) = a$, $f(q) = b$. Therefore, g is not surjective since $g(x) \neq r$ for all $x \in A$. f is surjective. However, $f(g(1)) = a$ and $f(g(2)) = b$, so $f \circ g$ is surjective.

QUESTION 13 **(2.3 # 38)**. Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$. Here, f and g are functions from $\mathbb{R}$ to $\mathbb{R}$.

Solution: $(f \circ g)(x) = f(g(x)) = f(x+2) = (x+2)^2 + 1$, and $(g \circ f)(x) = g(f(x)) = g(x^2+1) = x^2 + 1 + 2 = x^2 + 3$.

QUESTION 14 **(2.3 # 71)**. Find the inverse of $f(x) = x^3 + 1$

Solution: The inverse is $f^{-1}(x) = (x-1)^{1/3}$.