## Lesson 4 – Finding Limits Algebraically

In order to better understand how the slope of the secant gets closer and closer to the slope of the tangent, we need to study a branch of calculus known as limits.
We write that $\lim_{x \to a} f(x) = L$ if when $x \to a$ , $f(x) \to L$ .

We say: *"The limit of f (x) as x approaches $a$ is L, if when x approaches $a$, f (x) approaches L"*

**Note: Very important!** The expression above does not say that *x* ever reaches $a$ , or that *f* (*x*) every reaches $L$. It just says that as *x* gets very close to $a$, *f* (*x*) gets very close to $L$.

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We will simplify the process of solving Limit questions by categorizing them as follows:

1. Substitution
2. $\frac{0}{k}$ where $k \neq 0$
3. $\frac{0}{0}$
4. $\frac{k}{\infty}$
5. $\frac{\infty}{\infty}$
6. $\frac{k}{0^+}$

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**Category 1: Substitution**

To find the value of the limit, simply substitute the value of *a* for *x* in *f* (*x*). For example:

$$\lim_{x\to 3}(5x-2)$$
$$=5(3)-2$$
$$=13$$

This expression says that as $x$ gets closer and closer to 3, $5x-2$ will get closer and closer to 13.

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**Category 2:** $\frac{0}{k}$ where $k \neq 0$

This is essentially the same as Category 1, except that a fraction is involved. The only thing that you need to remember is that $\frac{0}{k}$ is 0 for any value of *k* which is not zero!

$$\lim_{x\to 2}\frac{3x-6}{x+4}$$
$$=\frac{3(2)-6}{2+4}$$
$$=\frac{0}{6}$$
$$=0$$

This expression says that as $x$ gets closer and closer to 2, $\frac{3x-6}{x+4}$ will get closer and closer to 0.

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**Category 3:** $\frac{0}{0}$

This category will require the most work! You need to rewrite the original fraction as an equivalent fraction where something in the numerator and something in the denominator will cancel each other. Hopefully you will then be able to use straight substitution to find your answer. If straight substitution still gives $\frac{0}{0}$, you may have to repeat the process!

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Example 1:

diff of squares

$$\lim_{x \to 2} \frac{x^2-4}{x-2}$$
$$= \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$
$$= \lim_{x \to 2}(x+2)$$
$$= 2+2$$
$$= 4$$

This expression says that as $x$ gets closer and closer to 2, the fraction $\frac{x^2-4}{x-2}$ will get closer and closer to 4.

Here we used difference of squares to get rid of the $\frac{0}{0}$ problem. You may have to use a variety of techniques which include but are not limited to: factoring trinomials, remainder theorem, factor theorem, change of variable polynomial division, and maybe even rationalizing the numerator or denominator!

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Example 2: Change of variable

Sometimes, it may be necessary to introduce a new variable to simplify the expression.

Here, we let $u^3 = \left((x+8)^{1/3}\right)^3$ isolate

It follows that $u^3 = x + 8$

And therefore, $x = u^3 - 8$

$$\lim_{x\to 0}\frac{(x+8)^{1/3}-2}{x} = \frac{u-2}{x}$$

$$\frac{(x+8)^{1/3}-2}{x} = \frac{u-2}{u^3-8} \quad \leftarrow \text{diff of cubes}$$

$$= \frac{\cancel{u-2}}{(\cancel{u-2})(u^2+2u+4)}$$

$$\lim_{x\to 0} = \frac{1}{u^2+2u+4}$$

$x \to 0$

$u \to 2$

$$\lim_{x\to 0}\frac{(x+8)^{1/3}-2}{x} = \boxed{\lim_{u\to 2}\frac{1}{u^2+2u+4}}$$

Since $\boxed{u = (x+8)^{1/3}}$, as $x \to 0$ it follows that $u \to 8^{1/3} = 2$

$$= \frac{1}{2^2+2(2)+4}$$

$$= \frac{1}{12}$$

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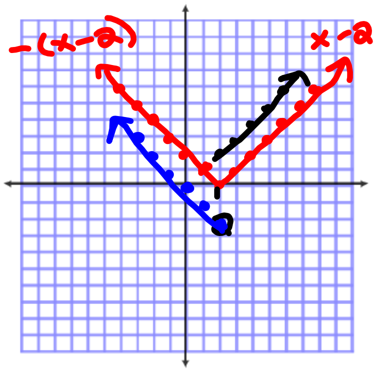
Example 3: Special Case (absolute value function)

Evaluate the limit if it exists. Illustrate the result graphically.

$$\lim_{x\to 2}\frac{x|x-2|}{x-2}$$

$|x-2|$ is either $(x-2)$ or $-(x-2)$




Substitution would yield $\frac{0}{0}$ at $x = 2$.

We must consider the two cases that result from writing:

$$|x-2| = \begin{cases} x-2, & if\ x > 2 \\ -(x-2), & if\ x < 2 \end{cases}$$

And, we must therefore use one-sided limits.

First, if $x > 2$, then  y=x  $x>2$

$$\frac{x|x-2|}{x-2} = \frac{x(\cancel{x-2})}{\cancel{x-2}} = x$$

Therefore, the right hand limit (RHL) would be:

$$\lim_{x\to 2^+}\frac{x|x-2|}{x-2} = \lim_{x\to 2^+} x = 2$$

Second, if $x < 2$, then  $x<2$  y=-x

$$\frac{x|x-2|}{x-2} = \frac{-x(\cancel{x-2})}{\cancel{x-2}} = -x$$

Therefore, the left hand limit (LHL) would be:

$$\lim_{x\to 2^-}\frac{x|x-2|}{x-2} = \lim_{x\to 2^-} -x = -2$$

Since LHL $\neq RHL$, $\lim_{x\to 2}\frac{x|x-2|}{x-2} = DNE$

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**Category 4:** $\frac{k}{\infty}$

In this category, the numerator is staying relatively constant, but the denominator is getting infinitely large. The fraction will get closer and closer to 0. There is no work to show for this question.

$$\lim_{x\to\infty}\frac{5}{x+3}$$
$$=0$$

This expression says that as $x$ gets larger and larger, the fraction $\frac{5}{x+3}$ will get closer and closer to 0.

$$\frac{5}{1+3} = 1.25 \qquad \frac{5}{20+3} = 0.22 \qquad \frac{5}{1000+3} = 0.005$$

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**Category 5:** $\frac{\infty}{\infty}$

In this category, both the numerator and denominator are getting infinitely large. In order to determine this limit, you must divide the numerator and denominator by the largest power of $x$.

Cat 4

$$\lim_{x\to\infty}\frac{5x+10x^3-2x^2+5}{3x+5x^3-2}$$

$$=\lim_{x\to\infty}\frac{\frac{5x^1}{x^3}+\frac{10\cancel{x^3}}{\cancel{x^3}}-\frac{2x^2}{x^3}+\frac{5}{x^3}}{\frac{3x}{x^3}+\frac{5x^3}{x^3}-\frac{2}{x^3}}$$

$$=\lim_{x\to\infty}\frac{\frac{5}{x^2}+10-\frac{2}{x}+\frac{5}{x^3}}{\frac{3}{x^2}+5-\frac{2}{x^3}}$$

$$=\frac{0+10-0+0}{0+5-0}$$

$$=2$$

This expression says that as $x$ gets infinitely large, the fraction will get closer and closer to 2.

Note that each of the fractions which still involves a variable in the denominator will get closer and closer to 0.

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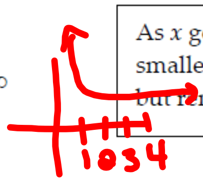
**Category 6:** $\frac{k}{0^+}$ or $\frac{k}{0^-}$

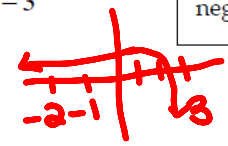
In this category, the numerator stays constant, while the denominator gets closer and closer to 0, either from the right (above zero) or from the left (below zero). Since the denominator is getting closer and closer to 0, the entire fraction will be getting larger and larger. The final answer will be either $+\infty$ or $-\infty$ depending on whether $k$ is positive or negative. There is very little work to show for this category.

$$\lim_{x\to 0^+} \frac{5}{x} = +\infty$$



As $x$ gets closer and closer to zero from the right, we will be dividing 5 by a smaller and smaller positive number. The overall fraction will get infinitely large, but remain positive.

$$\lim_{x\to 3^-} \frac{7}{x-3} = -\infty$$




As $x$ gets closer and closer to 3 from the left, we will be dividing by a smaller and smaller negative number. The overall fraction will get infinitely large, but remain negative.

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**Important:**

When trying to evaluate limits, always try substitution (in your head) first. If there is no problem, just substitute. If substitution does not work, determine which category the limit belongs to, and use the appropriate method.

For good form, once you substitute, you should remove the limit notation.

****** **Be sure to read over the Properties of Limits on page 40 in your textbook** ******

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