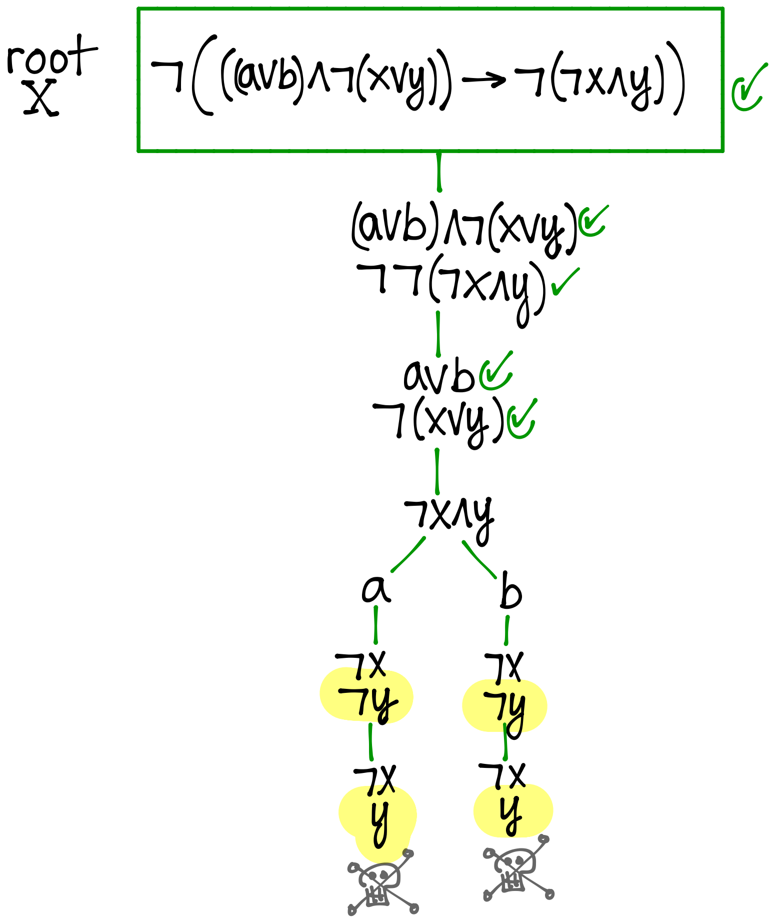
DGD 3

Q1. TRUTH TREES

Use a truth tree to determine whether or not each of the following compound propositions is a contradiction. If it is not a contradiction, give all counterexamples. If it is a contradiction, explain how you know based on the tree.

i. $X: \quad \neg\Big(\big((a \vee b) \wedge \neg(x \vee y)\big) \rightarrow \neg(\neg x \wedge y)\Big)$

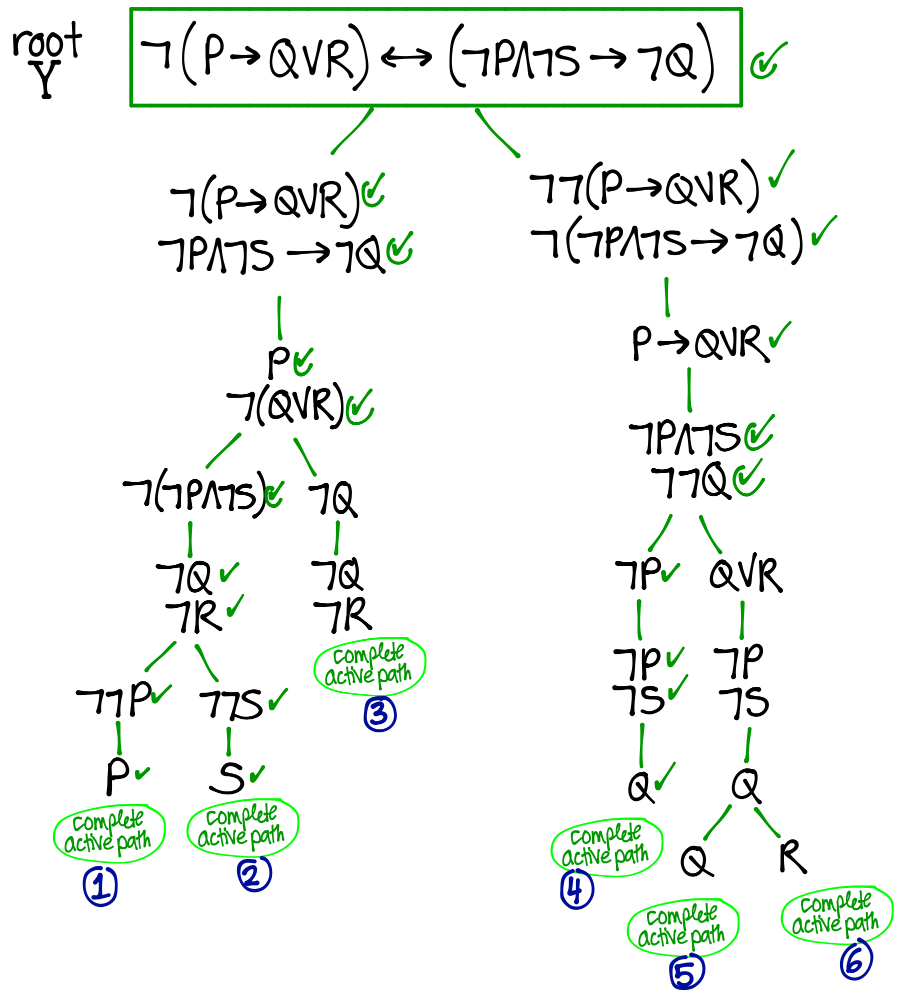


In the fully grown tree with root X, all paths are dead/inactive.

∴ the root X is a contradiction.

Based on your tree, give a DNF for X: F

ii. $Y: \quad \neg(P \to (Q \vee R)) \leftrightarrow ((\neg P \wedge \neg S) \to \neg Q)$



In the fully grown tree with root Y, there exists at least 1 complete path. ∴ the root Y is <u>not</u> a contradiction.

counterexamples:

① P=T, Q=F, R=F	④ P=F, Q=T, S=F
② P=T, Q=F, R=F, S=T	⑤ P=F, Q=T, S=F
③ P=T, Q=F, R=F	⑥ P=F, Q=T, R=T, S=F

Each of these truth assignments make the root Y true (which certifies that Y is <u>not</u> a contradiction)

Based on your tree, give a DNF for Y:

$\underbrace{(P\wedge\neg Q\wedge\neg R)}_{①}\vee\underbrace{(P\wedge\neg Q\wedge\neg R\wedge S)}_{②}\vee\underbrace{(P\wedge\neg Q\wedge\neg R)}_{③}\vee\underbrace{(\neg P\wedge Q\wedge\neg S)}_{④}\vee\underbrace{(\neg P\wedge Q\wedge\neg S)}_{⑤}\vee\underbrace{(\neg P\wedge Q\wedge R\wedge\neg S)}_{⑥}$

<u>Note</u> this DNF for Y has some redundancies, but it's still a DNF...

Q2. Consistent Set of Propositions

- A set $\{P_1, P_2, \ldots, P_n\}$ of compound propositions is called **consistent** if there exists at least one truth assignment that makes all the propositions $P_1, \ldots, P_n$ true at the same time.
- If $\{P_1, P_2, \ldots, P_n\}$ is not consistent, then, for each possible truth assignment, at least one of the propositions P_i is false; in this case, the set $\{P_1, P_2, \ldots, P_n\}$ is called **inconsistent**.

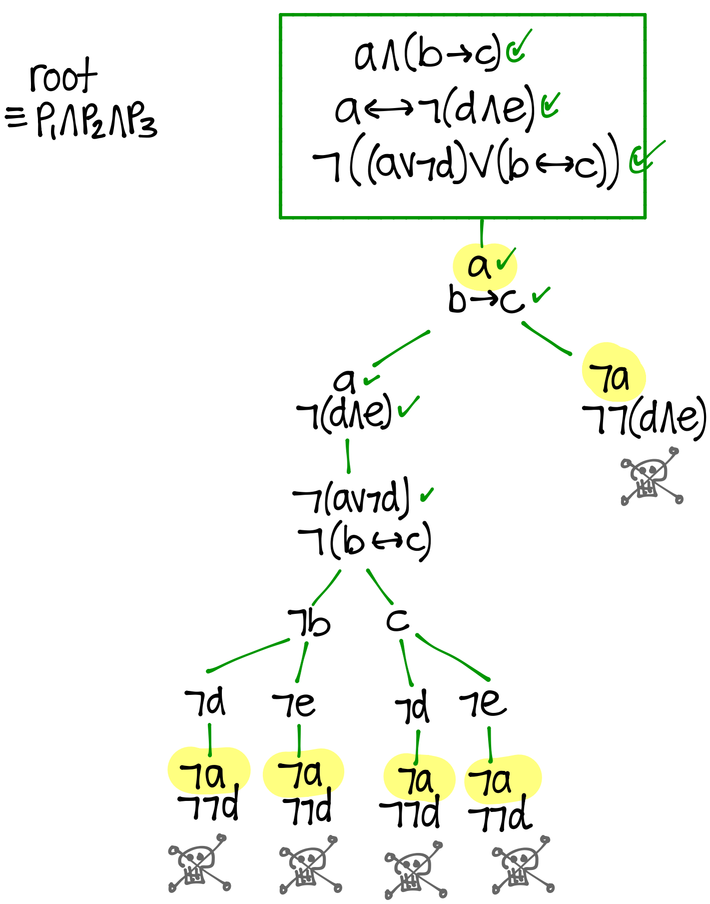
  Equivalently, a set $\{P_1, P_2, \ldots, P_n\}$ is...

  ▷ **consistent** if the conjunction $P_1 \wedge \cdots \wedge P_n$ is *not* a contradiction.

  ▷ **inconsistent** if the conjunction $P_1 \wedge \cdots \wedge P_n$ *is* a contradiction.

Using an appropriate truth tree, determine whether each of the following sets of propositions is consistent. If you claim the set is consistent, give all truth assignments that certify the set is consistent. If you claim the set is inconsistent, explain based on your tree.

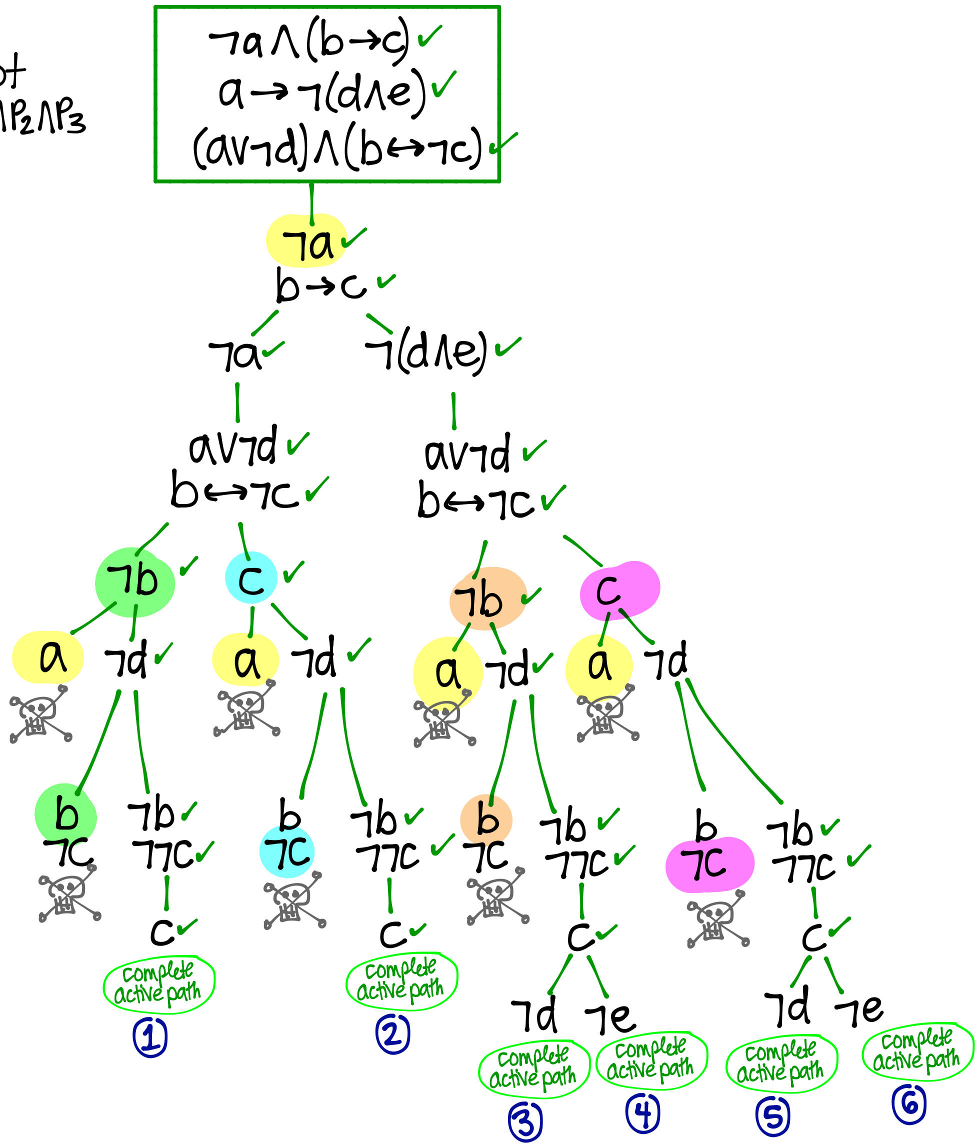
i. $$\Big\{\underbrace{a \wedge (b \to c)}_{P_1},\ \underbrace{a \leftrightarrow \neg(d \wedge e)}_{P_2},\ \underbrace{\neg\big((a \vee \neg d) \vee (b \leftrightarrow c)\big)}_{P_3}\Big\}$$



All paths are dead/inactive ∴ the root (which corresponds to $P_1 \wedge P_2 \wedge P_3$) is a contradiction. ∴ the set $\{P_1, P_2, P_3\}$ is <u>inconsistent</u> (no truth assignment makes all 3 propositions P_1, P_2, P_3 simultaneously true).

ii. $\left\{\underbrace{\neg a \wedge (b \to c)}_{P_1},\ \underbrace{a \to \neg(d \wedge e)}_{P_2},\ \underbrace{(a \vee \neg d) \wedge (b \leftrightarrow \neg c)}_{P_3}\right\}$

root
$\equiv P_1 \wedge P_2 \wedge P_3$



There exists at least one complete active path.

∴ the root (which corresponds to $P_1 \wedge P_2 \wedge P_3$) is <u>not</u> a contradiction.

∴ the set $\{P_1, P_2, P_3\}$ is <u>consistent</u>

Each of the following truth assignments makes all 3 propositions P_1, P_2, P_3 simultaneously true:

① a=F, b=F, c=T, d=F

② a=F, b=F, c=T, d=F

③ a=F, b=F, c=T, d=F

④ a=F, b=F, c=T, d=F, e=F

⑤ a=F, b=F, c=T, d=F

⑥ a=F, b=F, c=T, d=F, e=F

In fact several complete active paths happen to give the same truth assignment (for other sets, your tree might give more than one distinct truth assignment).

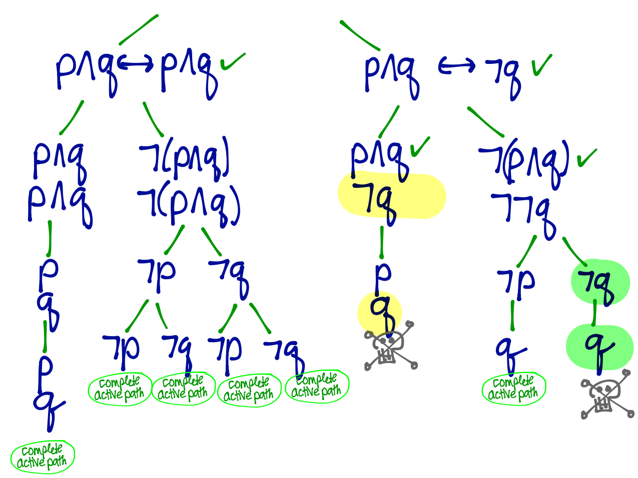
Q3. Possible Misconceptions about Truth Trees

Give examples of compound propositions X and Y, consisting of the propositional variables p and q, such that

i. X is a tautology and the fully grown truth tree with root X contains **at least one** dead/inactive path.

There is more than one correct answer, but here is one example:

$X: ((p\wedge q)\leftrightarrow(p\wedge q))\vee(p\wedge q\leftrightarrow\neg q)$



X is a tautology

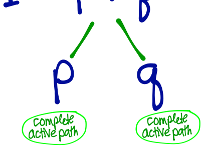
*verify this with a truth table!

Nevertheless, the tree with root X does have at least one dead path

ii. Y is a contingency and the fully grown truth tree with root X contains **no** dead/inactive paths.

There is more than one correct answer, but here is one example:

$Y: p\vee q$



Y is a contingency

*verify this with a truth table!

Nevertheless, the tree with root Y has <u>no</u> dead paths.

iii. Suppose all paths in a fully grown truth tree with root R are complete **active** paths. Does this mean that the root is a tautology? Explain.

No. In Q3ii) we saw that a contingency can have a tree with all paths being complete active paths.

Each complete active path gives a truth assignment that makes the root of the tree true, but does not guarantee that the root is true <u>for all</u> truth assignments.

iv. Suppose all paths in a fully grown truth tree with root R are dead/inactive paths. Does this guarantee that the root is a contradiction? Explain.

Yes.

When <u>all</u> paths are dead, it means there is no truth assignment that makes the root true, hence, for all truth assignments, the root's truth value must be F. ∴ in this situation, we know the root must be a contradiction.

v. Suppose a fully grown truth tree with root R contains at least one complete active path *and* at least one dead/inactive path. Does this guarantee that the root is a contingency? Explain.

No.

At least one complete active path guarantees the root can be true (hence is not a contradiction); however, even a tautology can have a tree with dead paths (such as the proposition X given in Q3i). ∴ the presence of at least one dead path does not imply the root can be false.

Q4. Valid Argument

Q4a. Translate the following argument into propositional logic. Then use an appropriate **truth tree** to determine whether it's valid or not. If it's valid, explain how you know this based on your tree. If it's invalid, give all counterexamples.

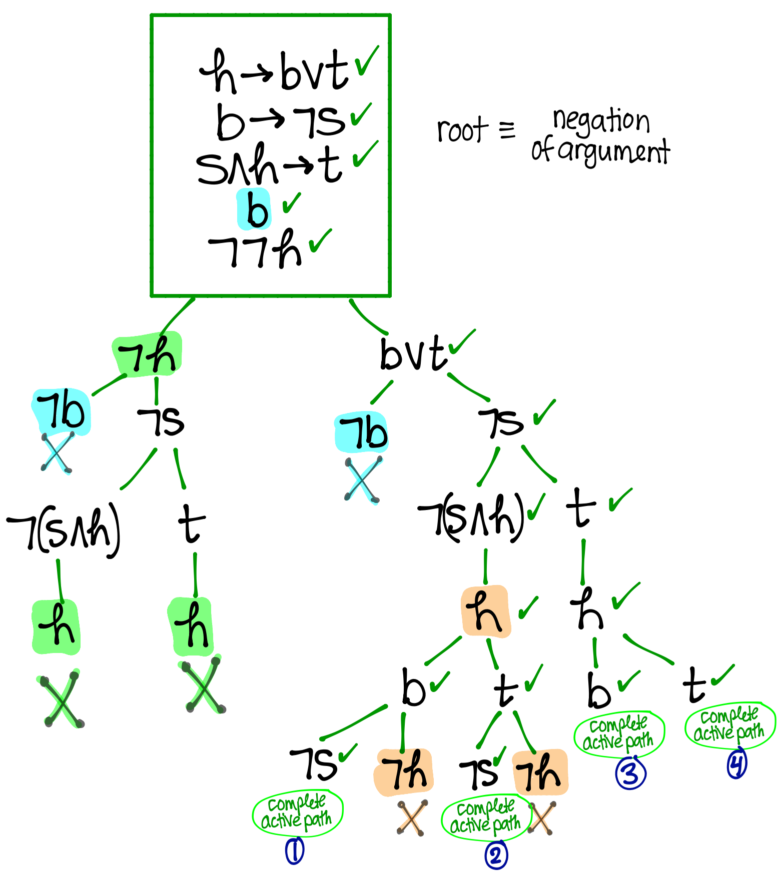
If it is hungry, then the bear eats berries or the bear eats trout. The bear eats berries only if it does not see trout. Whenever the bear sees trout and is hungry, it eats trout. The bear eats berries. Therefore, the bear is not hungry.

Use the propositional variables:

h: The bear is hungry.
b: The bear eats berries.
t: The bear eats trout.
s: The bear sees trout.

translation of argument:

$P_1: h \rightarrow b \vee t$
$P_2: b \rightarrow \neg s$
$P_3: s \wedge h \rightarrow t$
$P_4: b$
$\therefore C: \neg h$



The argument's negation is not a contradiction.

$\therefore$ the argument itself is not a tautology, hence it's an invalid argument.

counterexamples:

① b=T, h=T, s=F
② b=T, h=T, s=F, t=T
③ b=T, h=T, s=F, t=T
④ b=T, h=T, s=F, t=T

For each of the above truth assignments, all premises are true but the conclusion is False.

Q4b. Now, use a truth table to verify your answer from (a). Explain your conclusion based on the table. If the argument is invalid, give all counterexamples. Consider the connections/differences between your solution with the truth-tree approach compared with the truth-table approach.

	h	b	t	s	P_1: $h \to (b \vee t)$	P_2: $b \to \neg s$	P_3: $(s \wedge h) \to t$	P_4: b	C: $\neg h$	
	T	T	T	T	T	F	T	T	F	
(i)	T	T	T	F	T	T	T	T	F	← counterexample (all premises are T but conclusion is F)
	T	T	F	T	T	F	F	T	F	
(ii)	T	T	F	F	T	T	T	T	F	← counterexample (all premises are T but conclusion is F)
	T	F	T	T	T	T	T	F	F	
	T	F	T	F	T	T	T	F	F	
	T	F	F	T	F	T	F	F	F	
	T	F	F	F	F	T	T	F	F	
	F	T	T	T	T	F	T	T	T	
	F	T	T	F	T	T	T	T	T	
	F	T	F	T	T	F	T	T	T	
	F	T	F	F	T	T	T	T	T	
	F	F	T	T	T	T	T	F	T	
	F	F	T	F	T	T	T	F	T	
	F	F	F	T	T	T	T	F	T	
	F	F	F	F	T	T	T	F	T	

There are two distinct counterexamples:

(i) when h=T, b=T, t=T, s=F, the argument is false

(ii) when h=T, b=T, t=F, s=F, the argument is false

Note that the complete active path ① in the truth tree already told us both of the above counterexample since ① implied that the argument is False when ① b=T, h=T, s=F ← because this complete open path did not include a literal for the variable t, it means t can be either T or F while b=T, h=T, s=F, and the argument will be false regardless of the truth value of t. The other 3 complete active path counterexamples did not give us any "new" information.

Observations:

- In a truth table, each distinct row corresponds to a distinct truth assignment
- In a truth tree, distinct complete active paths may actually correspond to the same truth assignment