4. The Laws of Logical Equivalences & DNF

Example 4.1. Using a truth table, verify each of the following laws from The Table of Logical Equivalences:

- ◇ $p \wedge \mathbf{T} \equiv p$ (Identity Law)
- ◇ $p \vee \mathbf{F} \equiv p$ (Identity Law)
- ◇ $p \vee \mathbf{T} \equiv \mathbf{T}$ (Domination Law)
- ◇ $p \wedge \mathbf{F} \equiv \mathbf{F}$ (Domination Law)
- ◇ $p \vee \neg p \equiv \mathbf{T}$ (Negation Law)
- ◇ $p \wedge \neg p \equiv \mathbf{F}$ (Negation Law)

Identity Laws:

p	$p \wedge T$	$p \wedge T \leftrightarrow p$
T	T	T
F	F	T

$p \wedge T \equiv p$ because $p \wedge T \leftrightarrow p$ is a tautology.

p	$p \vee F$	$p \vee F \leftrightarrow p$
T	T	T
F	F	T

$p \vee F \equiv p$ because $p \vee F \leftrightarrow p$ is a tautology.

CORRECTIONS

Domination Laws:

p	$p \vee T$	$p \vee T \leftrightarrow T$
T	T	T
F	T	T

$p \vee T \equiv T$ because $p \vee T \leftrightarrow T$ is a tautology.

p	$p \wedge F$	$p \wedge F \leftrightarrow F$
T	F	T
F	F	T

$p \wedge F \equiv F$ because $p \wedge F \leftrightarrow F$ is a tautology.

Negation Laws:

p	$p \vee \neg p$	$p \vee \neg p \leftrightarrow T$
T	T	T
F	T	T

$p \vee \neg p \equiv T$ because $p \vee \neg p \leftrightarrow T$ is a tautology.

p	$p \wedge \neg p$	$p \wedge \neg p \leftrightarrow F$
T	F	T
F	F	T

$p \wedge F \equiv F$ because $p \wedge F \leftrightarrow F$ is a tautology.

* These notes are solely for the personal use of students registered in MAT1348.

The Table of Logical Equivalences

P→Q is T when P is F or Q is T	1.	$P \to Q \equiv \neg P \vee Q$	Implication Law
2 ways to think about $\leftrightarrow$	2. 3.	$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$ $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$	Biconditional Laws
out of $P/\neg P$ exactly one is T and other is F.	4. 5.	$P \vee \neg P \equiv \mathbf{T}$ $P \wedge \neg P \equiv \mathbf{F}$	Negation Laws
	6. 7.	$P \vee \mathbf{F} \equiv P$ $P \wedge \mathbf{T} \equiv P$	Identity Laws
	8. 9.	$P \vee \mathbf{T} \equiv \mathbf{T}$ $P \wedge \mathbf{F} \equiv \mathbf{F}$	Domination Laws
	10. 11.	$P \vee P \equiv P$ $P \wedge P \equiv P$	Idempotent Laws
	12.	$\neg\neg P \equiv P$	Double Negation Law
	13. 14.	$P \vee Q \equiv Q \vee P$ $P \wedge Q \equiv Q \wedge P$	Commutative Laws
	15. 16.	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$ $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	Associative Laws
a bit like factoring	17. 18.	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	Distributive Laws
a way to switch $\wedge/\vee$ with $\neg\neg$	19. 20.	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	De Morgan's Laws
	21. 22.	$P \vee (P \wedge Q) \equiv P$ $P \wedge (P \vee Q) \equiv P$	Absorption Laws

Example 4.2. Prove $\big((x \wedge y) \vee (x \wedge \neg y)\big) \vee (\neg x \wedge y) \equiv x \vee y$

$$\big((x\wedge y)\vee(x\wedge\neg y)\big)\vee(\neg x\wedge y) \equiv [x\wedge(y\vee\neg y)]\vee(\neg x\wedge y) \quad \text{(distributive law)}$$

$$\equiv [x\wedge T]\vee(\neg x\wedge y) \quad \text{(negation law)}$$

$$\equiv (x)\vee(\neg x\wedge y) \quad \text{(identity law)}$$

$$\equiv (x\vee\neg x)\wedge(x\vee y) \quad \text{(distributive law)}$$

$$\equiv T\wedge(x\vee y) \quad \text{(negation law)}$$

$$\equiv x\vee y \quad \text{(identity law)}$$

$$\therefore \big((x\wedge y)\vee(x\wedge\neg y)\big)\vee(\neg x\wedge y) \equiv x\vee y$$

Example 4.3. Prove that $(a \wedge \neg b) \wedge (\neg a \vee b)$ is a contradiction <u>Note</u> contradiction $\equiv F$

$$(a\wedge\neg b)\wedge(\neg a\vee b) \equiv a\wedge(\neg b\wedge(\neg a\vee b)) \quad \text{(associative law)}$$

$$\equiv a\wedge[(\neg b\wedge\neg a)\vee(\neg b\wedge b)] \quad \text{(distributive law)}$$

$$\equiv a\wedge[(\neg b\wedge\neg a)\vee F] \quad \text{(negation law)}$$

$$\equiv a\wedge[\neg b\wedge\neg a] \quad \text{(identity law)}$$

$$\equiv a\wedge(\neg a\wedge\neg b) \quad \text{(commutative law)}$$

$$\equiv (a\wedge\neg a)\wedge\neg b \quad \text{(associative law)}$$

$$\equiv F\wedge\neg b \quad \text{(negation law)}$$

$$\equiv F \quad \text{(domination law)}$$

$\therefore (a\wedge\neg b)\wedge(\neg a\vee b) \equiv F$ so $(a\wedge\neg b)\wedge(\neg a\vee b)$ is a contradiction.

Example 4.4. Find a compound proposition that is logically equivalent to $X \wedge Y$ that uses only the logical connectives $\rightarrow$ and $\neg$.

$$
\begin{aligned}
X \wedge Y &\equiv \neg\neg(X \wedge Y) && \text{(double negation law)}\\
&\equiv \neg(\neg X \vee \neg Y) && \text{(De Morgan's law)}\\
&\equiv \neg(X \rightarrow \neg Y) && \text{(implication law)}
\end{aligned}
$$

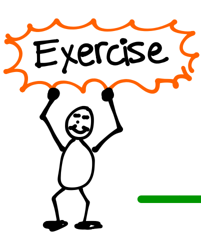
Thus, we found a proposition $\neg(X \rightarrow \neg Y)$ such that $X \wedge Y \equiv \neg(X \rightarrow \neg Y)$ and $\neg(X \rightarrow \neg Y)$ uses only the logical connectives $\rightarrow$ and $\neg$.

Example 4.5. Find a compound proposition that is logically equivalent to $p \rightarrow (q \vee r)$ that uses only the logical connectives $\neg$ and $\wedge$.

$$
\begin{aligned}
p \rightarrow (q \vee r) &\equiv \neg p \vee (q \vee r) && \text{(implication law)}\\
&\equiv \neg\neg[\neg p \vee (q \vee r)] && \text{(double negation)}\\
&\equiv \neg[\neg\neg p \wedge \neg(q \vee r)] && \text{(De Morgan's law)}\\
&\equiv \neg[p \wedge \neg(q \vee r)] && \text{(double negation law)}\\
&\equiv \neg[p \wedge (\neg q \wedge \neg r)] && \text{(De Morgan's law)}
\end{aligned}
$$

Thus, $p \rightarrow (q \vee r) \equiv \neg[p \wedge (\neg q \wedge \neg r)]$ and $\neg[p \wedge (\neg q \wedge \neg r)]$ uses only the logical connectives $\neg$ and $\wedge$.

*A collection of logical connectives is called <u>functionally complete</u> if every compound proposition is logically equivalent to a compound proposition involving only these logical connectives.



Show that $\{\neg, \rightarrow\}$ is functionally complete.

- An **atom** is a proposition containing no logical connectives (just a propositional variable).
- A **literal** is an atom or the negation of an atom.
- A **conjunctive clause** is a compound proposition that contains only literals and (possibly) the connective $\wedge$, and no atom appears more than once.
- A compound proposition is said to be in **disjunctive normal form (DNF)** if it is a disjunction of conjunctive clauses.

Facts about DNF.

- Every compound proposition is logically equivalent to a proposition in DNF.
- DNF is not unique, but all DNF of a given compound proposition X are logically equivalent to X, hence logically equivalent to each other.

Ex. $a \to b \equiv (a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b)$ ← one possible DNF of $a \to b$

$a \to b \equiv (\neg a) \vee (b)$ ← another possible DNF of $a \to b$

Using a Truth Table to Obtain a DNF for a given Proposition.

- Let X be a compound proposition consisting of the propositional variables $p_1, \ldots, p_k$.
- Construct a complete truth table for X.
- For each row in which X is **T**, write a conjunctive clause corresponding to the truth value of each of the atoms $p_1, \ldots, p_k$.
- The disjunction of these conjunctive clauses is a DNF for X.

Example 4.6. Use a truth table to determine a DNF for the compound proposition X, defined as follows:

$$X : \quad \neg(p \leftrightarrow q) \vee \neg q$$

p	q	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$\neg q$	X: $\neg(p \leftrightarrow q) \vee \neg q$	conjunctive clauses for each row where X is T:
T	T	T	F	F	F	—
T	F	F	T	T	T	$p \wedge \neg q$
F	T	F	T	F	T	$\neg p \wedge q$
F	F	T	F	T	T	$\neg p \wedge \neg q$

∴ a DNF for X is $(p \wedge \neg q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$

Thus, $\neg(p \leftrightarrow q) \vee \neg q \equiv (p \wedge \neg q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$

Example 4.7. The truth table of a *mystery compound proposition* X consisting of propositional variables p, q, r, s is given below.

p	q	r	s	X
T	**T**	**T**	**T**	**F**
T	**T**	**T**	**F**	**F**
T	**T**	**F**	**T**	**T**
T	**T**	**F**	**F**	**F**
T	**F**	**T**	**T**	**T**
T	**F**	**T**	**F**	**F**
T	**F**	**F**	**T**	**F**
T	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**F**
F	**T**	**T**	**F**	**F**
F	**T**	**F**	**T**	**F**
F	**T**	**F**	**F**	**F**
F	**F**	**T**	**T**	**F**
F	**F**	**T**	**F**	**F**
F	**F**	**F**	**T**	**T**
F	**F**	**F**	**F**	**F**

conjunctive clauses for each row where X is T:

$p \wedge q \wedge \neg r \wedge s$

$p \wedge \neg q \wedge r \wedge s$

$\neg p \wedge \neg q \wedge \neg r \wedge s$

i. Determine a DNF for X.

∴ a DNF for X is

$$(p \wedge q \wedge \neg r \wedge s) \vee (p \wedge \neg q \wedge r \wedge s) \vee (\neg p \wedge \neg q \wedge \neg r \wedge s)$$

ii. Is X a tautology, contradiction, or contingency? Explain.

X is a contingency because it is not always true nor always false.

exercise: Try to find a compound proposition that is logically equivalent to X that uses only the connectives $\rightarrow$ and $\neg$ (and parentheses wherever appropriate).

Study Guide

Important terms and concepts:

- ⋄ The Laws from the Table of Logical Equivalences
- ⋄ DNF (atoms, literals, conjunctive clauses)   how to find DNF from a truth table

Exercises	Sup.Ex. §1 # 1, 2, 3, 7ac (using the Laws), 8 Rosen §1.3 # 7, 8, 15, and using the Laws: # 20–34 **For each of the following, find a compound proposition that uses only the connectives $\neg$ and $\rightarrow$ i. $p \vee q$ ii. $p \wedge q$ iii. $p \oplus q$ iv. $p \leftrightarrow q$ Rosen §1.3 optional: # 47–56