3. Secants, Tangents, and Limits

Lec 2 mini review.

composition $(f \circ g)(x) = f(g(x))$

logarithmic functions: laws of logs
base $a > 0$ $y = \log_a x$
natural logarithm $y = \ln x$
inverse relationship between $\log_a x$ and a^x

trig ratios: $\sin\theta$, $\cos\theta$, $\tan\theta$, $\csc\theta$, $\sec\theta$, $\cot\theta$

trig functions: domain, range

inverse $f^{-1}(y) = x \Longleftrightarrow f(x) = y$
one-to-one function horizontal line test

exponential functions: laws of exponents
base $a > 0$ $y = a^x$
natural base $e = 2.718\ldots$ $y = e^x$

identities: $\cos^2(x) + \sin^2(x) = 1$... and others

inverse trig functions: $\arcsin x$, $\arccos x$, $\arctan x$

Example 3.1. Recall the height function $h(t) = -4.9t^2 + v_0 t + h_0$ of an object thrown with initial upwards velocity v_0, from an initial height h_0, t seconds after being thrown.

A ball is thrown up with an initial velocity of 10 m/s from the upper observation deck of the CN Tower, 450 m above the ground. What is the average velocity of the ball over the time interval $[3, 4]$? Estimate the instantaneous velocity of the ball 3 seconds after being thrown. How could you improve your estimate?

SLOPE OF A SECANT — AVERAGE RATE OF CHANGE OVER AN INTERVAL

Let f be a function that is "continuous" (to be defined precisely later) on an interval $[a, b]$.

Then

SLOPE OF TANGENT — INSTANTANEOUS RATE OF CHANGE AT A POINT

Goal: We want the **instantaneous rate of change** of $f(x)$ at a point $x = a$.

In this case, the "interval" we are interested in is $[a, a]$. That is, we only care what happens when x is exactly equal to a.

Obstacle: The formula for the average rate of change of f on the interval $[a, a]$ does not work — we get the **indeterminate form** $\frac{0}{0}$.

Observation: If $h > 0$, then we can calculate the average rate of change over the interval $[a, a + h]$, even when h is extremely tiny.

So, h can **approach** 0, written $h \to 0$, without ever actually equalling zero. At the same time, if the average rate of change of f over the interval $[a, a + h]$ **approaches** a particular number, then that number is called the **instantaneous rate of change of** f **at** $x = a$.

We need to formalize the idea of $h \to 0$. In fact, we will develop a framework for evaluating limits in general, not just those for instantaneous rates of change.

LIMITS: THE INTUITIVE DEFINITION

Suppose $f(x)$ is defined when x is "near" a number a (this means that f is defined on some open interval that contains the number a, except possibly at a itself; a might not be in the domain of f, but at all other points in the neighbourhood of this open interval, f is defined).

- If we can make the values $f(x)$ arbitrarily close to a unique real number L by restricting x (on either side of a) to be sufficiently close to a but not equal to a, then

  [read: "**the limit of** $f(x)$**, as** x **approaches** a**, exists and equals** L"]

  Informally, we can guarantee that $f(x)$ gets arbitrarily close to a unique real number L as long as we make sure that x is close enough to a (without actually letting x equal a).

- If there is no such unique real number L, then the limit of $f(x)$ as x approaches a **does not exist (DNE)**.

Example 3.2. Consider the rational function $f(x) = \dfrac{2x^2 - 2x}{x - 1}$ and the limit $\lim\limits_{x \to 1} f(x)$.

- ◇ What happens if we just plug in $x = 1$ to $f(x)$?

- ◇ Test how $f(x)$ behaves for values of x *near* $x = 1$ by filling in the chart:

(from the left $x \to 1$) (1 $\leftarrow x$ from the right)

x	0.5	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25	1.5
$f(x)$						$\frac{0}{0}$ eek!					

- ◇ As x approaches 1, does $f(x)$ seem to be approaching a specific number?

- ◇ If x is any number other than 1, what does the graph of f look like?

- ◇ Use the graph of f to evaluate the limit: $\lim\limits_{x \to 1} f(x) =$

Reasons Why Some Limits Do Not Exist

Example 3.3. $\lim_{x\to 0} \frac{1}{x^2}$

Infinite Limits (Vertical Asymptotes)

- Let f be a function defined on both sides of a, except possibly at a itself. Then

  means that the values of $f(x)$ grow arbitrarily large as x approaches a.
- **Graphically**: f has a **Vertical Asymptote** as x approaches a.
- Same idea for $\lim_{x\to a} f(x) = -\infty$
- **Note.** Since ∞ is **not** a real number $L \in \mathbb{R}$, infinite limits **DNE**.

  Nevertheless, we write $\boxed{\lim_{x\to a} f(x) = \infty}$ or $\boxed{\lim_{x\to a} f(x) = -\infty}$

  because it tells us for short which way the Vertical Asymptote goes.

Example 3.4. $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$

Observations. As $x \to 0$, it happens **infinitely often** that

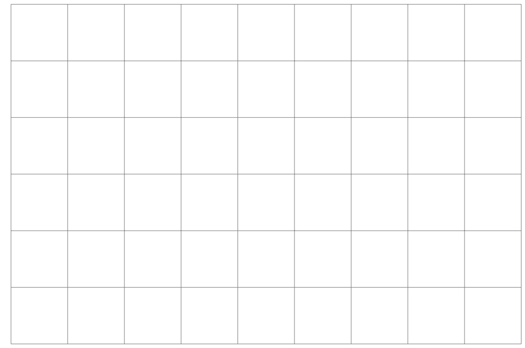
- $\sin\left(\frac{\pi}{x}\right) = 0$

- $\sin\left(\frac{\pi}{x}\right) = 1$

- Since $\sin\left(\frac{\pi}{x}\right)$ does not approach a **unique** real number, as $x \to 0$, this limit **DNE**.

Example 3.5. For all $x \in \mathbb{R}$, **the ceiling function** $\lceil x \rceil$ is defined as $\lceil x \rceil = \min\{n \in \mathbb{Z} : x \leq n\}$.

$\lim_{x \to 2} \lceil x \rceil$



► Since $\lceil x \rceil$ approaches **different real numbers as** $x \to 2$ **from either side**, this limit **DNE**.

ONE-SIDED LIMITS

- ◇ As in Example 3.5, as $x \to a$, the values of $f(x)$ may behave differently from one side than the other.
- ◇ For some functions, a limit as $x \to a$ only makes sense if x approaches a from one side:

To distinguish from which side x approaches a, we use the following notation for **one-sided limits**:

By definition, we can say

$$\boxed{\lim_{x \to a} f(x) = L \quad \text{if and only if both} \quad \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L}$$

EVALUATING LIMITS

- **numerically:** guessing by plugging in nearby values of x
- **graphically:** eyeballing the limit by looking at the graph
- **using the Limit Laws:**

  Let k be a constant real number, and suppose that the limits $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ both exist. Then

$$\lim_{x\to a}\Big[f(x) \pm g(x)\Big] = \Big[\lim_{x\to a} f(x)\Big] \pm \Big[\lim_{x\to a} g(x)\Big]$$

$$\lim_{x\to a}\Big[kf(x)\Big] = k\Big[\lim_{x\to a} f(x)\Big]$$

$$\lim_{x\to a}\Big[f(x)g(x)\Big] = \Big[\lim_{x\to a} f(x)\Big] \cdot \Big[\lim_{x\to a} g(x)\Big]$$

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\Big[\lim\limits_{x\to a} f(x)\Big]}{\Big[\lim\limits_{x\to a} g(x)\Big]} \qquad \textbf{if } \lim_{x\to a} g(x) \neq 0.$$

- **using direct substitution**:

  If f is a polynomial or a rational function and a is in the domain of f, then

$$\boxed{\lim_{x\to a} f(x) = f(a)}.$$

- **using algebraic tricks**:

  If $f(x) = g(x)$ everywhere except when $x = a$, then

$$\boxed{\lim_{x\to a} f(x) = \lim_{x\to a} g(x)} \qquad \text{provided the limit exists.}$$

The above observation allows us to use **algebraic tricks** (such as the following) to evaluate limits:

♣ factoring and cancelling common factors

♡ rationalizing the numerator or denominator

♠ dividing all terms by a common expression

♢ adding/subtracting fractional expressions on a common denominator

Example 3.6. $\lim_{t \to 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$

Example 3.7. $\lim_{x \to 1} \left(\dfrac{1}{x-1} - \dfrac{1}{x^2-x} \right)$

Study Guide

Important terms and concepts:

- ◇ **slope of a secant** **average rate of change of a function**
- ◇ goal: find slope of a tangent instantaneous rate of change of a function
- ◇ **limits and one-sided limits:** $\lim_{x\to a} f(x)$ $\lim_{x\to a^+} f(x)$ $\lim_{x\to a^-} f(x)$
- ◇ **why some limits DNE**: infinite, no unique L, different from left/right
- ◇ **evaluating limits**: numerically, graphically, with Limit Laws and algebraic tricks