# Midterm 1sols2 - old tests

Calculus III for Engineers (University of Ottawa)

Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

# Calculus III for Engineers
## MAT 2322A - Fall 2017
## Midterm I
## Professor: Victor G. LeBlanc
## Time limit: 80 minutes. Closed books.

Name: Solutions ID Number: ____________

**Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement.**

(Signature): ______________________

### Instructions

- **The only calculators which are allowed are Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.**
- The exam has 8 pages. Read each question carefully before answering.
- Questions 1 to 3 are multiple choice. These questions are worth 2 points each and no partial marks are possible. **Please write your answers in the corresponding boxes in the grid below entitled "Answers to multiple choice Qs".**
- Questions 4 to 6 are long answer questions. Questions 4 and 6 are worth 6 marks each, and question 5 is worth 7 marks, so organize your time accordingly. **A correct answer requires a full, clearly-written and detailed solution.** Answer each question in the space provided, using backs of pages or the extra pages at the end if necessary.
- Do not unstaple the test. Good luck!

Answers to multiple choice Qs

| 1 | 2 | 3 |
|---|---|---|
| D | F | A |

Grid below is used for grading
(do not write in this grid)

| MCQ | 4 | 5 | 6 | Total |
|---|---|---|---|---|
| /6 | /6 | /7 | /6 | /25 |

**1.** What is the value of the double integral $\iint_R f(x,y)dA$, where $f(x,y) = xy + y$ and $R$ is the rectangle in the plane defined by the inequalities $2 \le x \le 3$, $1 \le y \le 3$?

A. 17

B. 16

C. 15

D. 14

E. 13

F. 12

$$\int_{y=1}^{3}\int_{x=2}^{3} (xy+y)\,dx\,dy = \int_1^3 \left(\frac{x^2}{2}y + xy \Big|_2^3\right) dy = \int_1^3 \left[\left(\frac{9y}{2} + 3y\right) - (2y+2y)\right] dy$$

$$= \frac{9}{4}y^2 + \frac{3y^2}{2} - 2y^2 \Big|_1^3 = \left(\frac{81}{4} + \frac{27}{2} - 18\right) - \left(\frac{9}{4} + \frac{3}{2} - 2\right)$$

$$= \frac{81 + 54 - 72 - 9 - 6 + 8}{4} = \frac{56}{4} = 14$$

**2.** What is the value of the directional derivative $D_{\vec{u}}f(\pi, 1)$, if $f(x,y) = \cos x + x^2 y$, and $\vec{u} = \frac{3}{\sqrt{34}}\vec{i} + \frac{5}{\sqrt{34}}\vec{j}$?

A. $\frac{6}{\sqrt{34}}\pi\vec{i}$

B. $\frac{5}{\sqrt{34}}\pi^2$

C. $\frac{6}{\sqrt{34}}\pi$

D. $\frac{5}{\sqrt{34}}\pi^2\vec{j}$

E. $\frac{1}{\sqrt{34}}(6\pi\vec{i} + 5\pi^2\vec{j})$

F. $\frac{1}{\sqrt{34}}(6\pi + 5\pi^2)$

$f_x = -\sin x + 2xy \qquad f_y = x^2 \qquad \vec{u}$ is unit.

$f_x(\pi, 1) = 2\pi \qquad f_y(\pi, 1) = \pi^2$

$$D_{\vec{u}}f(\pi,1) = \vec{\nabla} f(\pi,1) \cdot \vec{u}$$

$$= (2\pi\vec{i} + \pi^2\vec{j}) \cdot \left(\frac{3}{\sqrt{34}}\vec{i} + \frac{5}{\sqrt{34}}\vec{j}\right)$$

$$= \frac{6\pi}{\sqrt{34}} + \frac{5\pi^2}{\sqrt{34}} = \frac{1}{\sqrt{34}}(6\pi + 5\pi^2)$$

**3.** Consider the function $f(x, y) = x^2 + xy + y^2 + 9x - 9y$, defined on the disk

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 2\}.$$

Which of the following is true about the global extrema of $f$ on $D$?

A. $f$ has a global minimum at the point $(x, y) = (-1, 1)$ and a global maximum at the point $(x, y) = (1, -1)$.

B. $f$ has a global minimum at the point $(x, y) = (1, 1)$ and a global maximum at the point $(x, y) = (-1, -1)$.

C. $f$ has a global minimum at the point $(x, y) = (-9, 9)$ and no global maximum.

D. $f$ has a global minimum at the point $(x, y) = (-9, 9)$ and a global maximum at the point $(x, y) = (1, -1)$

E. $f$ has a global minimum at the point $(x, y) = (2, 0)$ and a global maximum at the point $(x, y) = (0, 2)$.

F. $f$ has a global maximum at the point $(x, y) = (-1, 1)$ and a global minimum at the point $(x, y) = (1, -1)$.

$f_x = 2x + y + 9$ $\quad f_y = x + 2y - 9$ $\quad \left.\begin{array}{l} f_x = 0 \\ f_y = 0 \end{array}\right\} \Rightarrow (x, y) = (-9, 9)$, NOT IN D reject.

Constraint on boundary $g(x,y) = x^2 + y^2 - 2 = 0$

$$\left.\begin{array}{l} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{array}\right\} \quad \begin{array}{l} 2x + y + 9 = 2\lambda x \\ x + 2y - 9 = 2\lambda y \\ x^2 + y^2 = 2 \end{array} \Rightarrow \begin{array}{l} 2x(1-\lambda) + y = -9 \\ x + 2y(1-\lambda) = 9 \end{array}$$

ADD

$(2x + 2y)(1-\lambda) + (x + y) = 0$

$[2(1-\lambda) + 1](x + y) = 0$

$\Rightarrow y = -x$

or $\lambda = 3/2$

$\lambda = 3/2$

$-x + y = -9$

$x - y = 9$

$\Rightarrow y = 9 + x$

$x^2 + (9 + x)^2 - 2 = 0$

$x^2 + 81 + 18x + x^2 - 2 = 0$

$2x^2 + 18x + 79 = 0$

discriminant $18^2 - 8 \cdot 79$

$= -308 < 0$

NO SOLUTIONS

$y = -x$

$x^2 + x^2 = 2$

$2x^2 = 2$

$x = \pm 1$

$\Rightarrow (1, -1)$ and $(-1, 1)$

| $(x,y)$ | $f$ | |
|---|---|---|
| $(1,-1)$ | 19 | MAX |
| $(-1,1)$ | $-17$ | MIN |

**4.** Find and classify all critical points of the function $f(x, y) = 12x^2 - 4x^3 + 6y^2 - 12xy - 2$.

$f_x = 24x - 12x^2 - 12y$

$f_y = 12y - 12x \Rightarrow$ $2x - x^2 - y = 0$

$y - x = 0$

$y = x$

$2x - x^2 - x = 0$

$-x^2 + x = 0$

$-x(x-1) = 0$ $x = 0$ or $x = 1$

Two critical points:

$(0,0)$ and $(1,1)$

$f_{xx} = 24 - 24x$

$f_{xy} = -12$

$f_{yy} = 12$

$\underline{(0,0)}$; $f_{xx}(0,0) f_{yy}(0,0) - (f_{xy}(0,0))^2 =$

$(24)(12) - (-12)^2 = 144 > 0$

$f_{xx}(0,0) = 24 > 0$

$\Rightarrow (0,0)$ local minimum

$\underline{(1,1)}$; $f_{xx}(1,1) f_{yy}(1,1) - (f_{xy}(1,1))^2 =$

$0 \cdot 12 - (-12)^2 = -144 < 0$

$\Rightarrow (1,1)$ saddle point.

**5.** Consider the function $f(x,y) = \dfrac{x+y}{x^2+y^2+1}$.

(a) What is the domain of $f$?

(b) Find the equation of the tangent plane to the graph of $z = f(x,y)$ at the point $(2,-1,1/6)$.

(c) Use local linearization to compute an approximate value for $f$ at the point $(2.1,-0.9)$ to 3 decimal places. What is the exact value of $f$ (to 3 decimal places) at the point $(2.1,-0.9)$?

a) Denominator is always $\geq 1 > 0 \Rightarrow$ Domain $= \mathbb{R}^2$

b) $f_x = \dfrac{1(x^2+y^2+1) - 2x(x+y)}{(x^2+y^2+1)^2} \Rightarrow f_x(2,-1) = \dfrac{6-4}{36} = \dfrac{2}{36} = \dfrac{1}{18}$

$f_y = \dfrac{1(x^2+y^2+1) - 2y(x+y)}{(x^2+y^2+1)^2} \Rightarrow f_y(2,-1) = \dfrac{6+2}{36} = \dfrac{+8}{36} = \dfrac{+2}{9}$

$\Rightarrow$ eqn of tangent plane is $\boxed{z = \frac{1}{18}(x-2) + \frac{2}{9}(y+1) + \frac{1}{6}}$

c) $f(2.1,-0.9) \approx \dfrac{1}{18}(2.1-2) + \dfrac{2}{9}(-0.9+1) + \dfrac{1}{6}$

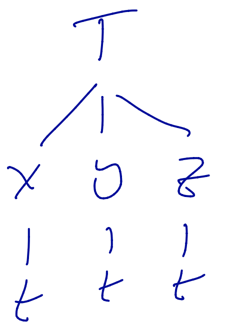
$= \dfrac{0.1}{18} + \dfrac{0.2}{9} + \dfrac{1}{6} = 0.194$

Exact value $f(2.1,-0.9) = \dfrac{2.1-0.9}{(2.1)^2+(-0.9)^2+1} = 0.193$

**6.** The temperature in space is given by the function $T(x, y, z) = 100 - 3x^2 - 2y^2 + 5z$. A satellite is moving through space with position coordinates given by $(x(t), y(t), z(t))$ where $x(t)$, $y(t)$ and $z(t)$ are differentiable functions. At $t = 5$, the satellite is at position $(x(5), y(5), z(5)) = (2, -1, 4)$, and its velocity is given by $(x'(5), y'(5), z'(5)) = (-1, 0, -2)$. What is the time rate of change of the temperature experienced by the satellite at $t = 5$, i.e. what is

$$\frac{d}{dt}T(x(t), y(t), z(t))\Big|_{t=5}?$$

Do not worry about the units.



$$\frac{dT}{dt} = \frac{\partial T}{\partial x}(x(t),y(t),z(t))\frac{dx}{dt}(t) + \frac{\partial T}{\partial y}(x(t),y(t),z(t))\frac{dy}{dt}(t) + \frac{\partial T}{\partial z}(x(t),y(t),z(t))\frac{dz}{dt}(t)$$

$$\frac{\partial T}{\partial x} = -6x \qquad \frac{\partial T}{\partial y} = -4y \qquad \frac{\partial T}{\partial z} = 5$$

$$\frac{\partial T}{\partial x}(2,-1,4) = -12 \qquad \frac{\partial T}{\partial y}(2,-1,4) = 4 \qquad \frac{\partial T}{\partial z}(2,-1,4) = 5$$

$$\Rightarrow \frac{dT}{dt}\Big|_{t=5} = (-12)(-1) + 4(0) + 5(-2)$$

$$= 12 - 10 = 2$$

(Extra page)

(Extra page)