15. Equivalence Relations, Classes, & Partitions

- An **equivalence relation** on a set A is a relation that is reflexive, symmetric, and transitive.
- Suppose $\mathcal{R}$ is an equivalence relation on A. For each element $a \in A$, the **equivalence class of** a is the set $[a]_{\mathcal{R}} = \{x \in A : a \,\mathcal{R}\, x\}$

General Observations on Equivalence Classes of an Equivalence Relation.

Let $\mathcal{R}$ be an equivalence relation on a set A. Then:

i. $a \in [a]_{\mathcal{R}}$ for all $a \in A$.

ii. $[a]_{\mathcal{R}} = [b]_{\mathcal{R}}$ if and only if $(a, b) \in \mathcal{R}$.

iii. $[a]_{\mathcal{R}} \cap [b]_{\mathcal{R}} = \emptyset$ if and only if $(a, b) \notin \mathcal{R}$.

*In fact, these properties turn out to give us what is called a **partition** of A.

PARTITIONS

A **partition** of a set A is a collection $\mathcal{P} = \{S_1, S_2, \dots\}$ of subsets $S_i \subseteq A$ such that the following three properties hold:

i. $S_i \neq \emptyset$ for all i (S_i are non-empty subsets of A)

ii. $A = S_1 \cup S_2 \cup \cdots$ (union of all S_i is all of A)

iii. $S_i \cap S_j = \emptyset$ for all $i \neq j$ (pairwise disjoint)

Example 15.1. Let $A = \{1, 2, 3, 4, 5\}$ $\mathcal{P}_1 = \{\{3,4,1\},\{2\},\{5\}\}$ is a partition of A.

$\mathcal{P}_2 = \{\{3,4\},\{2\},\{5\}\}$ is not a partition of A (fails property ii)

$\mathcal{P}_3 = \{\{3,4,1\},\{2\},\emptyset,\{5\}\}$ is not a partition of A (fails property i)

$\mathcal{P}_4 = \{\{3,4,1\},\{1,2\},\{5\}\}$ is not a partition of A (fails property iii)

Example 15.2. Here are two partitions of $\mathbb{Z}$:

$\mathcal{P}_1 = \{\mathbb{Z}^-, \{0\}, \mathbb{Z}^+\}$ is a partition of $\mathbb{Z}$

$\mathcal{P}_2 = \{\{0\},\{1,-1\},\{2,-2\},\{3,-3\},\dots\}$ is a partition of $\mathbb{Z}$

* These notes are solely for the personal use of students registered in MAT1348.

Correspondence between Equivalence Relations and Partitions

Theorem 15.3. Let A be a set.

(I) If $\mathcal{R}$ is an equivalence relation on A, then the collection of equivalence classes of $\mathcal{R}$ forms a partition of A.

(II) If $\mathcal{P} = \{S_1, S_2, \dots\}$ is a partition of A, then the relation $\mathcal{S}$ on A defined by the rule

$$\text{For all } a, b \in A \quad (a,b) \in \mathcal{S} \iff \{a,b\} \subseteq S_i \quad \text{for some } S_i \in \mathcal{P}$$

is an equivalence relation on A.

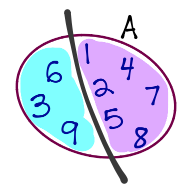
Example 15.4. Let $A = \{1,2,3,4,5,6,7,8,9\}$ and let $\mathcal{R}$ be the relation on A defined by the rule

$$\text{for all } x, y \in A \quad x \,\mathcal{R}\, y \quad \text{if and only if} \quad 3 \mid (x^2 + 2y^2).$$

Exercise: Prove that $\mathcal{R}$ is an equivalence relation on A.

Determine the corresponding partition of A into equivalence classes.

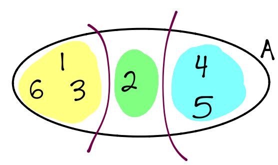
$[1]_{\mathcal{R}} = \{1,2,4,5,7,8\}$ $\qquad$ $[3]_{\mathcal{R}} = \{3,6,9\}$



$\therefore$ the partion of A into equivalence classes is $\mathcal{P} = \big\{\{3,6,9\}, \{1,2,4,5,7,8\}\big\}$

Example 15.5. Let $A = \{1,2,3,4,5,6\}$.

Here is a partition of A: $\mathcal{P} = \{\overset{S_1}{\{1,3,6\}}, \overset{S_2}{\{2\}}, \overset{S_3}{\{4,5\}}\}$



Here is the corresponding equivalence relation $\mathcal{S}$ on A:

rule for $\mathcal{S}$: $(a,b) \in \mathcal{S} \iff \{a,b\} \subseteq S_i$ for some $S_i \in \mathcal{P}$

$\mathcal{S}$ as a finite list: $\mathcal{S} = \{\underbrace{(1,1),(1,3),(1,6),(3,1),(3,3),(3,6),(6,1),(6,3),(6,6)}_{\text{Corresponds to } \{a,b\} \subseteq S_1 = \{1,3,6\}}, \underbrace{(2,2)}_{\text{Corresponds to } \{a,b\} \subseteq S_2 = \{2\}}, \underbrace{(4,4),(4,5),(5,4),(5,5)}_{\text{Corresponds to } \{a,b\} \subseteq S_3 = \{4,5\}}\}$

Exercise 15.6. Consider the **equivalence relation** $\mathcal{R}$ on the set $A = \{-6,-5,-2,0,1,3,5,7\}$ defined as follows:

$$a \,\mathcal{R}\, b \quad \iff \quad a^2 \equiv b^2 \pmod 5$$

i. Prove that $\mathcal{R}$ is an equivalence relation on A.
ii. Determine the partition of A into equivalence classes with respect to $\mathcal{R}$

Study Guide

Important terms and concepts:	**equivalence relations:** **equivalence classes:**	reflexive, symmetric, & transitive $[a]_{\mathcal{R}} = \{x \in A : x \,\mathcal{R}\, a\}$
partition $\mathcal{P} = \{S_1, S_2, \dots\}$ **of a set** A 1. $S_i \neq \emptyset$ for all i	2. $A = S_1 \cup S_2 \cup \cdots$	3. $S_i \cap S_j = \emptyset$ for all $i \neq j$
Exercises		Sup.Ex. §7 # 1b, 2, 3, 4, 6, 8, 9, 10, 11 Rosen §9.5 # 1, 3, 7, 11, 15, 17, 25, 26, 29, 41, 47