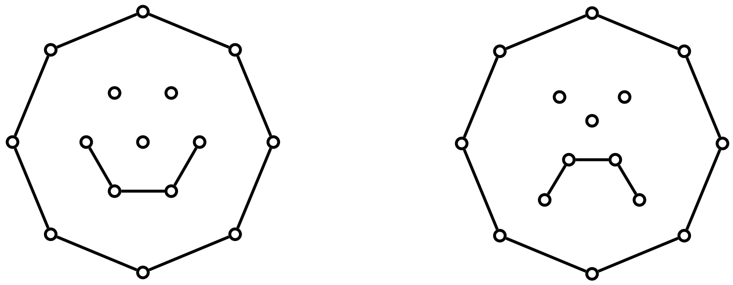
22. Introduction to Graph Theory

Exercise. Give an isomorphism from H to S.



$H \qquad \cong \qquad S$

SUBGRAPHS

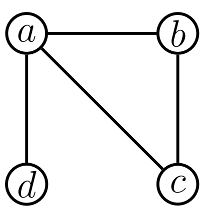
Definition.

Let H and G be graphs.

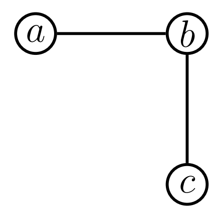
H is called a **subgraph** of G, denoted $H \subseteq G$, if both $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Example 22.1. Verify that H is a subgraph of G.

G



H



$V(G) = \{a, b, c, d\}$

$V(H) = \{a, b, c\}$

$E(G) = \{\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}\}$

$E(H) = \{\{a,b\}, \{b,c\}\}$

Since $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, H is a subgraph of G $(H \subseteq G)$

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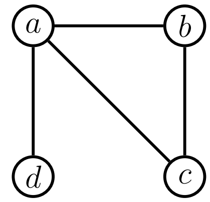
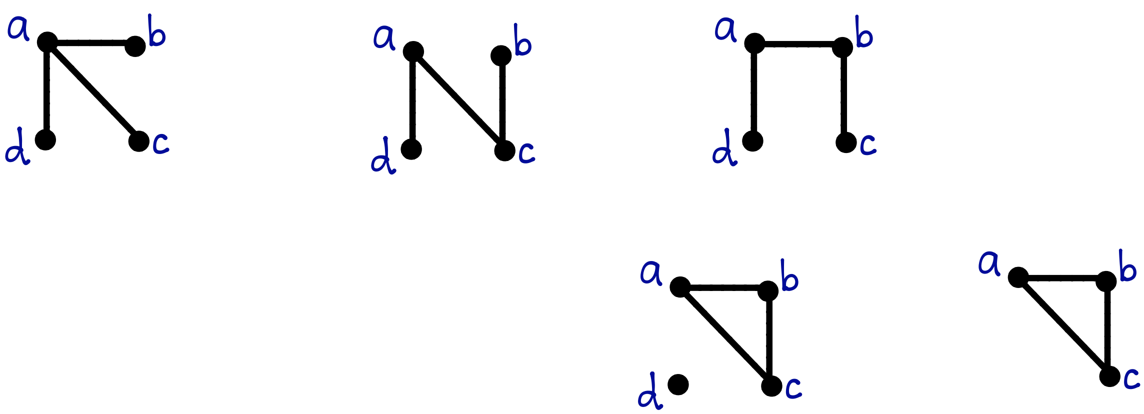
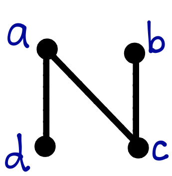
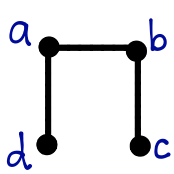
Example 22.2. Draw all the subgraphs of G with exactly 3 edges:

G 



Note these two subgraphs of G are not the same (one has 4 vertices while the other has only 3 vertices).

Which (if any) of these subgraphs of G are isomorphic?

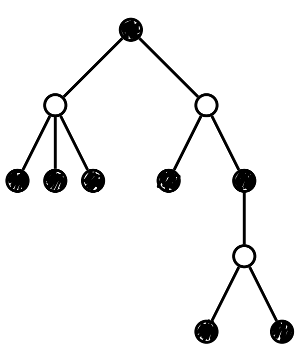
 and  are isomorphic (both are isomorphic to P_3)

BIPARTITE GRAPHS

Definition.

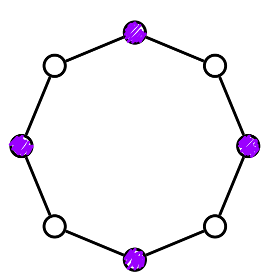
A graph G is called **bipartite** or **2-colourable** if we can colour the vertices of G using 2 colours so that no two neighbours (pair of adjacent vertices) are assigned the same colour.

Example 22.3. Which of the following graphs are bipartite?



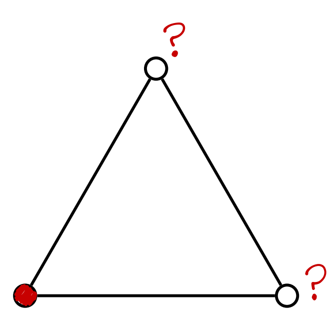
T

this "tree" is bipartite



C_8

C_8 is bipartite



C_3

C_3 is not bipartite

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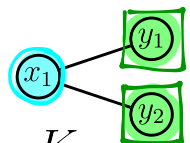
Let m and n be positive integers.

The **complete bipartite graph**, denoted $K_{m,n}$ is defined as follows:

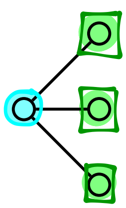
- Let $X = \{x_1, \ldots, x_m\}$ and let $Y = \{y_1, \ldots, y_n\}$ be sets such that
$$|X| = m, \quad |Y| = n, \quad \text{and} \quad X \cap Y = \emptyset.$$

- Let $V(K_{m,n}) = \{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_n\}$
(so $\{X, Y\}$ is a partition of $V(K_{m,n})$ into two classes)

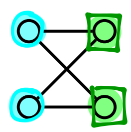
- Then $E(K_{m,n}) = \Big\{ \{x_i, y_j\} : x_i \in X \text{ and } y_j \in Y \Big\}$
(so we have all possible links with one end in X and the other end in Y).



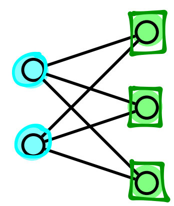
$K_{1,2}$



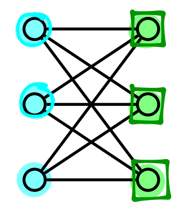
$K_{1,3}$



$K_{2,2}$



$K_{2,3}$



$K_{3,3}$

- $|V(K_{m,n})| = m+n$
- $|E(K_{m,n})| = m \cdot n$
- for each $x_i \in X$, $\deg_{K_{m,n}}(x_i) = n$
- for each $y_i \in Y$, $\deg_{K_{m,n}}(y_i) = m$
- degree sequence of $K_{m,n}$

$$(\underbrace{n, n, \ldots, n}_{\substack{m \text{ times} \\ \text{(degrees of} \\ X \text{ vertices)}}}, \underbrace{m, m, \ldots, m}_{\substack{n \text{ times} \\ \text{(degrees of} \\ Y \text{ vertices)}}})$$

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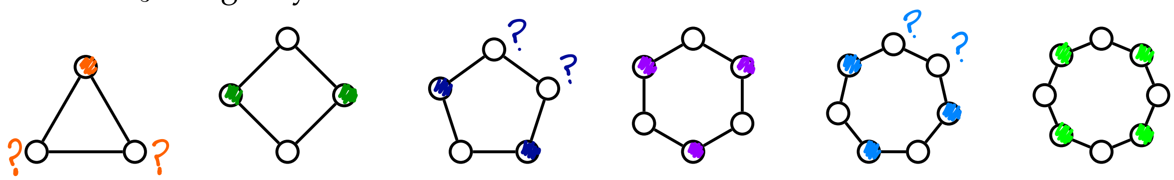
- A graph G is called **bipartite** or **2-colourable** if we can colour the vertices of G using 2 colours so that no two neighbours (pair of adjacent vertices) are assigned the same colour.
- So the cycle C_3 is **not** bipartite. No matter what we try, we cannot properly colour the vertices of C_3 using only 2 colours.



- In fact, no cycle of odd length is bipartite, whereas every cycle of even length is bipartite. (why? think about this!)
- Given that it is impossible to properly 2-colour the vertices of any cycle of odd length, it follows that all graphs which contain an odd cycle as a subgraph cannot be properly 2-coloured.
- In other words, in order for a graph G to be 2-colourable, it is **necessary that** G **contains no odd cycle as a subgraph**.
- What may be surprising is that the above necessary condition **is also sufficient.**

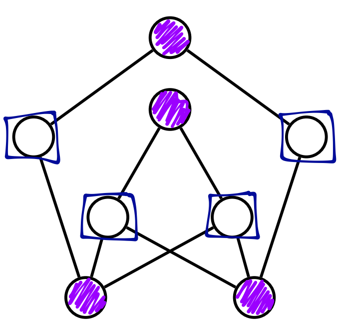
Bipartite Graph Theorem.

Let G be a graph. Then

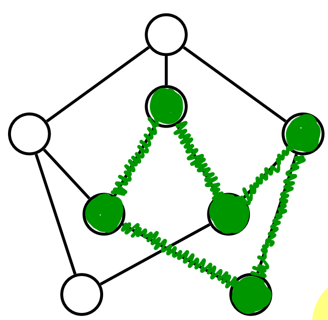
G is bipartite/2-colourable **if and only if** G has no odd-length cycle as a subgraph.

Thus, you can either properly 2-colour the vertices of a graph G, or else you are guaranteed to find at least one subgraph of G that is (isomorphic to) a cycle of odd length.

Example 22.4. Which of the following graphs are bipartite? Either give a proper 2-colouring or find an odd cycle to justify your answer.

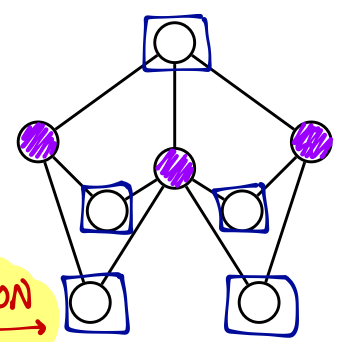


G_1



G_2

CORRECTION



G_3

G_1 is bipartite.
(see the proper 2-colouring of the vertices of G_1)

G_2 is not bipartite
(see the odd cycle of length 5 in G_2)

G_3 is bipartite.
(see the proper 2-colouring of the vertices of G_3)

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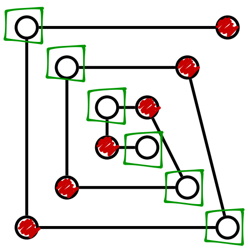
Exercise.

Determine whether each of the following graphs is bipartite or not. Justify your answer. For each graph below, can you determine the **minimum number** of colours we would need in order to properly colour the vertices of the graph so that no pair of adjacent vertices are assigned the same colour?

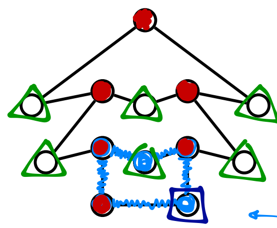
For your interest: for a given graph G, this minimum number of colours is called the **chromatic number of** G, and is denoted $\chi(G)$. The 4-colour Theorem is a theorem about the chromatic number of a special class of graphs called **planar graphs**. It says, if G is a planar graph, then $\chi(G) \leq 4$.

P_{11}



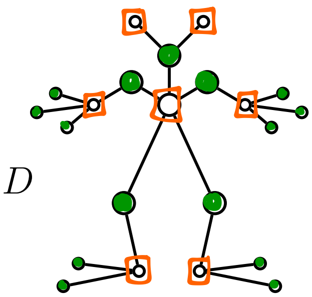
The path P_{11} is bipartite
$\chi(P_{11}) = 2$

G



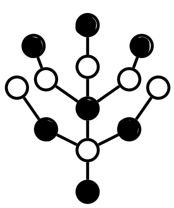
G contains an odd cycle C_5
$\Rightarrow$ not bipartite
$\chi(G) = 3$

D



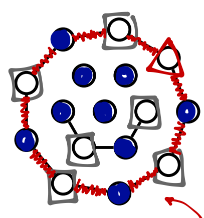
D is bipartite
$\chi(G) = 2$

F



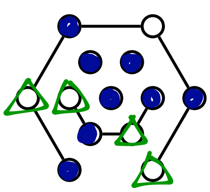
F is bipartite
$\chi(F) = 2$

H



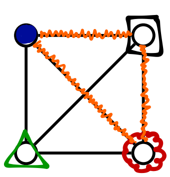
H contains an odd cycle C_9
$\Rightarrow$ not bipartite
$\chi(H) = 3$

L



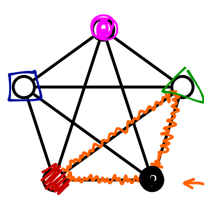
L is bipartite
$\chi(L) = 2$

K_4



K_4 contains an odd cycle: C_3
$\Rightarrow$ not bipartite
$\chi(K_4) = 4$

K_5

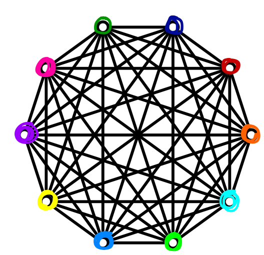


K_5 contains an odd cycle: C_3
$\Rightarrow$ not bipartite
$\chi(K_5) = 5$

K_1

K_1 is bipartite
$\chi(K_1) = 1$

K_{10}



not bipartite
$\chi(K_{10}) = 10$

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Let $G = (V, E)$ be a graph with incidence function ψ_G.

Let x and y be vertices of G and let k be an integer such that $k \geq 0$.

Definition. An (x, y)-**walk of length** k **in** G is an alternating sequence of vertices and edges of G

$$v_0\, e_1\, v_1\, e_2\, v_2\, \ldots\, v_{k-1} e_k v_k$$

such that:

- $v_0 = x$ and $v_k = y$ (so the sequence starts with x and ends with y)
- $v_0, v_1, \ldots, v_k \in V(G)$
- $e_1, \ldots, e_k \in E(G)$
- for each $i = 1, 2, \ldots, k$, $\quad \psi_G(e_i) = \{v_{i-1}, v_i\}$ (that is, v_{i-1} and v_i are the endpoints of the edge e_i between them in this walk)

In particular, an (x, y)-walk of length k, say $W = v_0\, e_1\, v_1\, e_2\, v_2\, \ldots\, v_{k-1} e_k v_k$, is called

- **closed** if $v_0 = v_k$

  (i.e. the walk starts and ends on the same vertex)

- **open** if $v_0 \neq v_k$

  (i.e. the walk starts and ends on two distinct vertices)

- a **trail** if its edges are pairwise distinct

  (i.e. we never "walk along" the same edge more than once)

- a **path** if its vertices are pairwise distinct

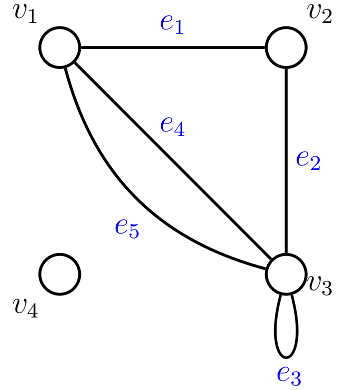
  (i.e. we never "walk along" the same vertex more than once)

- a **cycle** if $v_0 = v_k$ but all of its "internal vertices" $v_1, \ldots, v_{k-1}$ are pairwise distinct

  (i.e. we never "walk along" the same vertex more than once, *except* we end at the same vertex on which we started)

Example 22.5.

G



$V_3\, e_3\, V_3\, e_2\, V_2\, e_1\, V_1$ ← open (V_3, V_1)-trail of length 3

$V_3\, e_5\, V_1\, e_4\, V_3\, e_4\, V_1\, e_1\, V_2\, e_2\, V_3$

closed walk, not a trail, not a cycle

$V_3\, e_2\, V_2\, e_1\, V_1$ (V_3, V_1)-path of length 2

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COUNTING WALKS WITH THE ADJACENCY MATRIX – **an exercise (for those interested in a fun challenge!)** Let G be a graph and let A be its **adjacency matrix**. Assume $V(G) = \{v_1, \ldots, v_n\}$ and write A so that its (i,j)-entry corresponds to the number of edges joining v_i and v_j.

Prove that the (i,j)-entry of the matrix A^k (A matrix-multiplied by itself k times) is equal to the number of (v_i, v_j)-walks of length k in G.

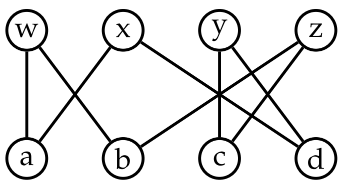
CONNECTION

Definition. A graph G is called **connected** if

for any vertices $u, v \in V(G)$, there exists a (u,v)-walk in G.

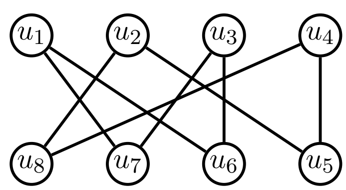
Equivalently, G is **connected** if, for any vertices $u, v \in V(G)$, there exists a path from u to v in G.

Example 22.6. Which of these graphs is connected?



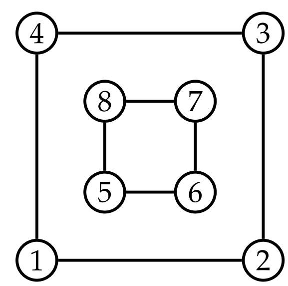
G_1

connected



G_2

disconnected because there is no path from u_1 to u_2 in G_2



G_3

disconnected because there is no path from 1 to 5 in G_3

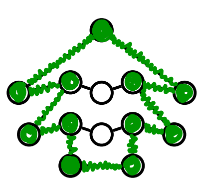
FORESTS & TREES

A graph G is called a **forest** if G has **no cycles** (of any length – even or odd).

A forest which is also a connected graph is called a **tree.**

Example 22.7. Determine whether each of the following graphs is a forest or tree or neither.

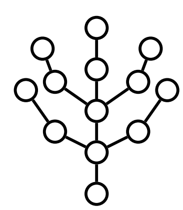
G_1



G_1 is neither a forest nor a tree because G_1 contains cycles (such as the cycle of length 11 which is coloured green).

(* indeed, G_1 is not even bipartite)

G_2



G_2 is a forest (since it has no cycles)

G_2 is a tree (since it is a connected forest)

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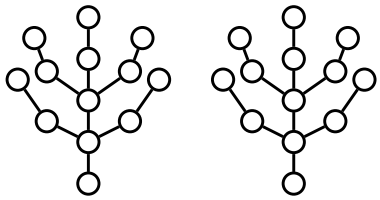
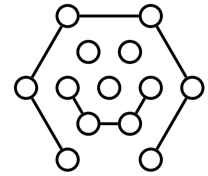
G_3



G_3 is a forest (since it has no cycles) but

G_3 is not a tree (since it is not a connected forest)

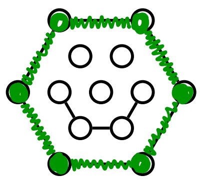
G_4



G_4 is a forest (since it has no cycles)

G_4 is not a tree (since it is not a connected forest)

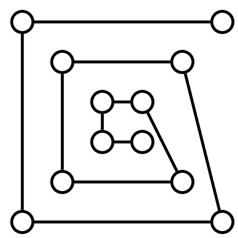
G_5



G_5 is neither a forest nor a tree because G_5 contains a cycle (see the cycle of length 6 which is coloured green).

(* despite not being a forest, G_5 is nevertheless bipartite)

G_6



G_6 is a forest (since it has no cycles)

G_6 is a tree (since it is a connected forest)

G_7

G_7 is a forest (since it has no cycles)

G_7 is a tree (since it is a connected forest)

Facts about Forests & Trees:

- Every tree is a forest.
- All forests are bipartite/2-colourable.
- All trees are bipartite/2-colourable.
- Not every forest is a tree.
- Not every bipartite graph is a forest.
- Not every bipartite graph is a tree.

Theorem. Every tree with at least 2 vertices has at least 2 leaves.

Theorem. Every tree with n vertices has exactly $n - 1$ edges.

Theorem. Let G be a graph. Then

G is a tree **if and only if** for any $u, v \in V(G)$, there is a unique path from u to v in G.

STUDY GUIDE

Important graph theory terms and concepts:

◇ graph (vertex set, edge set) (loops, parallel edges, simple graph)
◇ adjacency matrix degree sequence Handshaking Theorem
◇ special families of graphs: K_n C_n P_n $K_{m,n}$
◇ subgraph graph isomorphism
◇ bipartite/2-colourable graphs
◇ connected graph
◇ trees and forests

Supp. Exercise List (on Brightspace)	§12 # 1, 2, 3, 4, 5, 10, 11, 12	
Graph Theory Notes (on Brightspace)	§2.5 # 1, 2, 3, 4, 7, 8abd, 9, 10, 11abcd, 13 §4.3 # 3 §4.3 # 3a	§3.3 # 7, 8ab §6.4 # 1, 2, 4

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