12. Functions

Recall:

- A function $f : A \to B$ is called **injective** or "1-1" if

for all $a_1, a_2 \in A$, the implication $(f(a_1) = f(a_2)) \longrightarrow (a_1 = a_2)$ is true.

- A function $f : A \to B$ is called **surjective** or "onto" if

for all $b \in B$, there is at least one $a \in A$ such that $f(a) = b$ ie $f^{-1}(b) \neq \emptyset$.

The properties "injective" and "surjective" are independent properties.
Any combination of these two properties is possible.

Consider the following functions from $\mathbb{Z}$ to $\mathbb{Z}$

$id_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$
$id_{\mathbb{Z}}(k) = k$

$id_{\mathbb{Z}}$ is both injective and surjective

$s : \mathbb{Z} \to \mathbb{Z}$
$s(k) = k + 1$

s is both injective and surjective

* Note the "floor function"
$\lfloor\ \rfloor : \mathbb{R} \to \mathbb{Z}$ is defined by
$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$

Ex $\lfloor 1.8 \rfloor = 1$ Ex $\lfloor -5 \rfloor = -5$
Ex $\lfloor -1.8 \rfloor = -2$ Ex $\lfloor 1.999 \rfloor = 1$

$g : \mathbb{Z} \to \mathbb{Z}$
$g(k) = \lfloor k/2 \rfloor$

g is surjective but not injective

$h : \mathbb{Z} \to \mathbb{Z}$
$h(k) = k^3$

h is injective but not surjective

$f : \mathbb{Z} \to \mathbb{Z}$
$f(k) = 55$

f is neither injective nor surjective

$\boxed{\text{Exercise}}$ give proofs or counterexamples for each of the above functions and properties

* These notes are solely for the personal use of students registered in MAT1348.

A function $f : A \to B$ is called a **bijection** if

f is both injective and surjective.

Example 12.1. Let $g : \mathbb{R}^- \times \mathbb{R}^+ \to \mathbb{R}^- \times \mathbb{R}^-$ be the function defined as follows:

$$g(x, y) = \left(\frac{x}{y}, 3xy\right)$$

Note: since $x \in \mathbb{R}^-$ and $y \in \mathbb{R}^+$ it follows that $x/y \in \mathbb{R}^-$ and $3xy \in \mathbb{R}^-$

Prove that g is a bijection. → we must prove that g is both injective and surjective

Recall: $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$

[Surjective]. Let $(r,s) \in \mathbb{R}^- \times \mathbb{R}^-$ be an arbitrary element of g's codomain.

(goal: prove that there exists some $(x,y) \in \mathbb{R}^- \times \mathbb{R}^+$ such that $g(x,y) = (r,s)$)

We must reverse-engineer what (x,y) need to equal in terms of (r,s)

$\Rightarrow$ we need $(x,y) \in$ g's domain such that $\left(\frac{x}{y}, 3xy\right) = (r,s)$

$\Rightarrow$ we need $\frac{x}{y} = r$ and $\underbrace{3xy = s}_{①}$

$\Rightarrow \underbrace{x = ry}_{②}$ and $\underbrace{3(ry)y = s}_{\text{plug ② into ①}}$ $\Rightarrow y^2 = \underbrace{\frac{s}{3r}}$ $\Rightarrow \underbrace{y = \sqrt{\frac{s}{3r}}}_{③}$ or $\cancel{y = -\sqrt{\frac{s}{3r}}}$ (reject because $y \in \mathbb{R}^+$)

note: $\frac{s}{3r} \in \mathbb{R}^+$ because $r,s \in \mathbb{R}^-$.

$\Rightarrow \underbrace{x = r\sqrt{\frac{s}{3r}}}_{\text{plug ③ into ②}}$

Since $r,s \in \mathbb{R}^-$, it follows that $x = r\sqrt{\frac{s}{3r}} \in \mathbb{R}^-$ and $y = \sqrt{\frac{s}{3r}} \in \mathbb{R}^+$

$\therefore (x,y) \in \mathbb{R}^- \times \mathbb{R}^+$ (g's domain).

Moreover,

$$g(x,y) = g\left(r\sqrt{\frac{s}{3r}}, \sqrt{\frac{s}{3r}}\right) = \left(\frac{r\sqrt{\frac{s}{3r}}}{\sqrt{\frac{s}{3r}}}, 3r\sqrt{\frac{s}{3r}} \cdot \sqrt{\frac{s}{3r}}\right) = (r,s).$$

$\therefore$ g is surjective

[injective]. Let $(a,b),(c,d) \in \mathbb{R}^- \times \mathbb{R}^+$ be arbitrary elements of g's domain.

Assume $g(a,b) = g(c,d)$. (goal: prove $(a,b)=(c,d)$)

Then $\left(\frac{a}{b}, 3ab\right) = \left(\frac{c}{d}, 3cd\right)$ (by def of g)

$\Rightarrow \frac{a}{b} = \frac{c}{d}$ and $3ab = 3cd$

$\Rightarrow \underbrace{a = \frac{cb}{d}}_{①} \Rightarrow 3\left(\frac{cb}{d}\right)b = 3cd$

$\Rightarrow b^2 = d^2$

$\Rightarrow b = d$ or $\cancel{b = -d}$ ← since $b, d \in \mathbb{R}^+$, they cannot have opposite signs

$\therefore \underbrace{b=d}_{②} \Rightarrow a = \frac{cb}{b}$ (plug ② into ①) $\Rightarrow a = c$

$\therefore$ we proved that $(g(a,b)=g(c,d)) \rightarrow ((a,b)=(c,d))$ $\therefore$ g is injective.

Since g is both surjective and injective, it's a bijection. ▨

Cardinalities of Infinite Sets

Note. If A and B are finite sets and $f : A \to B$ is a bijection, then $|A| = |B|$.

For infinite sets, the way we compare their cardinality is through bijections. We define the notion of equality of cardinalities of infinite sets as follows:

$$|A| = |B| \quad \text{if and only if} \quad \text{there exists a bijection from } A \text{ to } B.$$

An infinite set S is called **countable** if $|S| = |\mathbb{N}|$.

$$\boxed{\begin{array}{c} (\text{injective}) \wedge (\text{surjective}) \\ \downarrow \qquad\qquad \downarrow \\ (|A| \le |B|) \wedge (|B| \le |A|) \end{array}}$$

Ex. $f : \mathbb{Z} \to \mathbb{N}$ defined by $f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ 2|n|-1 & \text{if } n < 0 \end{cases}$ is 1-1 and onto (verify this!)

$\therefore$ there is a bijection from $\mathbb{Z}$ to $\mathbb{N}$

$\therefore |\mathbb{Z}| = |\mathbb{N}|$

$$f \left(\begin{array}{ccccccccc} \mathbb{Z} & 0 & -1 & 1 & -2 & 2 & -3 & 3 & -4 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbb{N} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \right. \cdots$$

Fact. There is no bijection from $\mathbb{R}$ to $\mathbb{N}$. Therefore, the set of real numbers is called **uncountable**.

The identity function.

Let A be any set.

The **identity function on** A, denoted id_A, is the function $\mathrm{id}_A : A \to A$ defined by

$$\mathrm{id}_A(x) = x \quad \text{for all } x \in A.$$

In particular, id_A is a bijection from the set A to itself.

Compositions of Functions

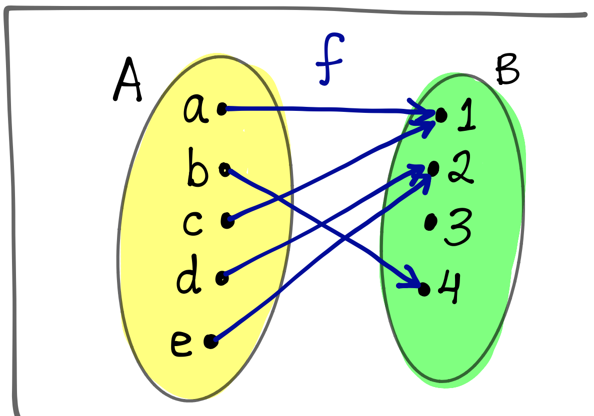
Let $f : A \to B$ and let $g : B \to C$ be functions.

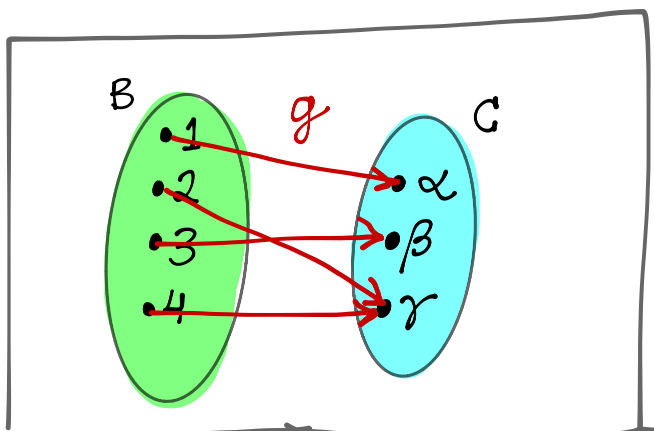
The **composition** g of f, denoted $g \circ f$, is the function $g \circ f : A \to C$ defined by

$$(g \circ f)(a) = g(f(a)) \quad \text{for all } a \in A.$$

Example 12.2. Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4\}$, and $C = \{\alpha, \beta, \gamma\}$.

Let $f : A \to B$ and $g : B \to C$ be functions defined as follows:





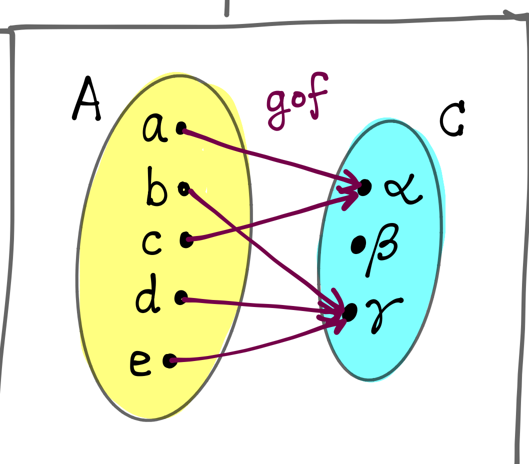
$g \circ f(a) = g(f(a)) = g(1) = \alpha$

$g \circ f(b) = g(f(b)) = g(4) = \gamma$

$g \circ f(c) = g(f(c)) = g(1) = \alpha$

$g \circ f(d) = g(f(d)) = g(2) = \gamma$

$g \circ f(e) = g(f(e)) = g(2) = \gamma$



$f : A \to B \qquad g : B \to C$

$g \circ f : A \to C$

Question. In Example 12.2, does $f \circ g$ make sense? If so, what is $f \circ g$?

$f \circ g(x) = f(g(x))$ so x needs to be in g's domain B.

however, $g(x) \in C$ (g's codomain) and $C \not\subseteq A$ (f's domain)

$\therefore$ $f \circ g$ is not defined.

Note. In order for the composition $f \circ g$ to be defined, we need the image of the domain of g (a subset of the codomain of g) to be a subset of the domain of f.

Informally, for $f \circ g$ to make sense, we need g to "give" f elements that are in the domain of f.

Example 12.3. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 5x - 7$ and $g(x) = x^2$.

Find $g \circ f$.

f's domain, g's codomain

$g \circ f : \mathbb{R} \to \mathbb{R}$

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(5x-7) \\ &= (5x-7)^2 \\ &= 25x^2 - 70x + 49 \end{aligned}$$

Find $f \circ g$.

g's domain, f's codomain

$f \circ g : \mathbb{R} \to \mathbb{R}$

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(x^2) \\ &= 5(x^2) + 7 \\ &= 5x^2 + 7 \end{aligned}$$

Note. In general, $f \circ g \neq g \circ f$, even if both compositions are defined.

Inverse Functions

Let $f: A \to B$ be a function.

The inverse of f (if it exists) is the function $f^{-1}: B \to A$ such that

$$\boxed{f^{-1} \circ f = id_A \quad \text{and} \quad f \circ f^{-1} = id_B}$$

Equivalently,

the inverse of f (if it exists) is the function $f^{-1}: B \to A$ such that

for all $a \in A$, $b \in B$, $\boxed{f^{-1}(b) = a \text{ if and only if } f(a) = b.}$

Example 12.4. For the function $g : \mathbb{R}^- \times \mathbb{R}^+ \to \mathbb{R}^- \times \mathbb{R}^-$, defined by $g(x,y) = \left(\frac{x}{y}, 3xy\right)$, verify that g's inverse $g^{-1} : \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R}^- \times \mathbb{R}^+$ is given by the rule

$$g^{-1}(r,s) = \left(r\sqrt{\frac{s}{3r}}, \sqrt{\frac{s}{3r}}\right)$$

Let $(r,s) \in \mathbb{R}^- \times \mathbb{R}^-$

$$\begin{aligned}(g \circ g^{-1})(r,s) &= g(g^{-1}(r,s)) \\ &= g\left(r\sqrt{\tfrac{s}{3r}}, \sqrt{\tfrac{s}{3r}}\right) \\ &= \left(\frac{r\sqrt{\frac{s}{3r}}}{\sqrt{\frac{s}{3r}}}, 3 \cdot r\sqrt{\tfrac{s}{3r}} \cdot \sqrt{\tfrac{s}{3r}}\right) \\ &= (r,s) \checkmark\end{aligned}$$

$\therefore g \circ g^{-1} = id_{\mathbb{R}^- \times \mathbb{R}^-}$

Let $(x,y) \in \mathbb{R}^- \times \mathbb{R}^+$

$$\begin{aligned}(g^{-1} \circ g)(x,y) &= g^{-1}(g(x,y)) \\ &= g^{-1}\left(\frac{x}{y}, 3xy\right) \\ &= \left(\frac{x}{y} \cdot \sqrt{\frac{3xy}{3\left(\frac{x}{y}\right)}}, \sqrt{\frac{3xy}{3\left(\frac{x}{y}\right)}}\right) \\ &= \left(\frac{x}{y}\sqrt{y^2}, \sqrt{y^2}\right) \\ &= (x,y) \checkmark\end{aligned}$$

$*$ because $y \in \mathbb{R}^+$ we know $y = \sqrt{y^2}$

$\therefore g^{-1} \circ g = id_{\mathbb{R}^- \times \mathbb{R}^+}$

Some facts about inverse functions

- Not every function has an inverse.
- If a function $f : A \to B$ has an inverse, then we call f invertible.
- If f is invertible, then its inverse is unique meaning there is one and only one function from B to A whose compositions with f give the respective identity functions.
- Theorem Let $f : A \to B$ be a function.
  Then f is invertible if and only if f is a bijection.

STUDY GUIDE

Important terms and concepts:

- ☐ bijection
  injective & surjective
- ☐ identity function
  for all $x \in A$, $\mathrm{id}_A(x) = x$
- ☐ composition
  $(f \circ g)(x) = f(g(x))$
- ☐ inverse of $g : A \to B$
  $g^{-1} \circ g = \mathrm{id}_A$
  $g \circ g^{-1} = \mathrm{id}_B$

Exercises

Sup.Ex. §5 # 1, 2, 3, 4, 5, 8, 10, 11
Rosen §2.3 # 1, 9, 10, 11, 12, 13, 14, 15, 33, 34, 35, 36, 37, 38, 71