1. (2 points) Consider the following function

$$f(x,y) = 27x^3 + x^3y^3 + 27y^3 - 1\,.$$

Which of the following statements is true for f?

$\boxed{\text{A}}$ (0,0) is a local maximum and $(-3,-3)$ is a local minimum.

$\boxed{\text{B}}$ (0,0) is a local minimum and $(-3,-3)$ is a local minimum.

$\boxed{\text{C}}$ (0,0) is a local maximum and $(-3,-3)$ is a local maximum.

$\boxed{\text{D}}$ $(-3,-3)$ is a saddle point and nothing can be said about the point (0,0).

$\boxed{\text{E}}$ (0,0) is a saddle point and nothing can be said about the point $(-3,-3)$.

$\boxed{\text{F}}$ (0, 0) is not a critical point and $(-3,-3)$ is a local minimum.

Solution: (D).
$f_x(x,y) = 81x^2 + 3x^2y^3$, $f_y(x,y) = 3x^3y^2 + 81y^2$.
Setting $f_x = 0$ and $f_y = 0$:

$$81x^2 + 3x^2y^3 = 0, \quad 3x^2(27 + y^3) = 0, \quad x = 0, or, y = -3.$$

$$3x^3y^2 + 81y^2 = 0, \quad 3y^2(27 + x^3) = 0, \quad x = -3, or, y = 0.$$

Thus critical points are:

$$(0,0), \quad (-3,-3).$$

$$f_{xx} = 162x + 6xy^3, \quad f_{xy} = 9x^2y^2, \quad f_{yy} = 6x^3y + 162y.$$

$f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 0$, nothing can be said about the point (0,0).
$f_{xx}(-3,-3)f_{yy}(-3,-3) - f_{xy}(-3,-3)^2 < 0$, $(-3,-3)$ is a saddle point.

2. (2 points) Find the arc length of the curve parametrized by $\vec{r}(t) = (3 + 2t^2, 1 + t^3,\ 2)$, $0 \le t \le 1$.

$\boxed{\text{A}}$ $\dfrac{99}{23}$ $\boxed{\text{B}}$ $\dfrac{89}{27}$ $\boxed{\text{C}}$ $\dfrac{89}{23}$ $\boxed{\text{D}}$ $\dfrac{98}{27}$ $\boxed{\text{E}}$ $\dfrac{77}{27}$ $\boxed{\text{F}}$ $\dfrac{37}{27}$

Solution: Any choice is correct.

$$L = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt$$

$$= \int_0^1 \sqrt{(4t)^2 + (3t^2)^2}dt = \int_0^1 \sqrt{16t^2 + 9t^4}dt = \int_0^1 t\sqrt{16 + 9t^2}dt$$

$$= \int_{16}^{25} \frac{1}{18}\sqrt{u}\ du \qquad u = 16 + 9t^2, du = 18tdt$$

$$= \frac{1}{27}u^{3/2}|_{16}^{25} = \frac{125 - 64}{27} = \frac{61}{27}.$$

3. (2 points) Which of the following integrals gives $\int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} x^2 y \, dy dx$ in polar coordinates ?

$\boxed{\text{A}}$ $\int_0^{\pi} \int_{\sqrt{1-r^2}}^{2} r^3 \cos^2\theta \sin\theta d\theta dr$

$\boxed{\text{B}}$ $\int_0^{\pi/4} \int_0^{2} r^3 \cos^2\theta \sin\theta dr d\theta$

$\boxed{\text{C}}$ $\int_{\pi/4}^{\pi/2} \int_0^{2} r^4 \cos^2\theta \sin\theta dr d\theta$

$\boxed{\text{D}}$ $\int_{\pi/4}^{\pi/2} \int_{-\sqrt{2}}^{\sqrt{2}} r^4 \cos^2\theta \sin\theta dr d\theta$

$\boxed{\text{E}}$ $\int_{\pi/2}^{3\pi/4} \int_{-\sqrt{2}}^{2} r^4 \cos^2\theta \sin\theta dr d\theta$

$\boxed{\text{F}}$ $\int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} r^3 \cos^2\theta \sin\theta dr d\theta$

Solution: (C).
In polar coordinates, $x = r\cos\theta, y = r\sin\theta,\ dA = r dr d\theta$.
$y \geq x, r\sin\theta \geq r\cos\theta, \tan\theta \geq 1, \theta \geq \frac{\pi}{4}$.
$y \leq \sqrt{4-x^2}, y^2 \leq 4 - x^2, x^2 + y^2 \leq 4, \quad r^2 \leq 4,\ 0 \leq r \leq 2$.
$\{0 \leq x \leq \sqrt{2},\ x \leq y \leq \sqrt{4-x^2}\} = \{0 \leq r \leq 2, \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}\}$.

4. (7 points) Using the Lagrange multiplier method, determine the absolute maximum and the absolute minimum of the function $f(x,y) = x^2 - y^3$ on the circle $x^2 + y^2 = 9$.

Solution: Let $g(x,y) = x^2 + y^2 - 9$. Then

$$\nabla f = (2x, -3y^2), \quad \nabla g = (2x, 2y).$$

By

$$\nabla f = \lambda \nabla g, \quad g(x,y) = 0,$$

we imply that

$$\begin{cases} 2x = \lambda 2x \quad (1) \\ -3y^2 = \lambda 2y \; (2) \\ x^2 + y^2 = 9, (3) \end{cases}$$

It is easy to see that $\lambda \neq 0$.

If $x = 0$, by (3), $y = \pm 3$.

If $y = 0$, by (3), $x = \pm 3$.

If $x \neq 0$, $y \neq 0$, then by (1), $\lambda = 1$, from (2), $y = -\frac{2}{3}$. Put into (3), $x = \pm\frac{\sqrt{77}}{3}$.

Thus critical points are

$$(x,y) = (0,3),\ (0,-3),\ (3,0),\ (-3,0), (\pm\frac{\sqrt{77}}{3}, -\frac{2}{3}).$$

$f(0,3) = -27$,
$f(0,-3) = 27$,
$f(\pm 3, 0) = 9$,
$f(\pm\frac{\sqrt{77}}{3}, -\frac{2}{3}) = \frac{239}{27}$.

The maximum is 27, the minimum is -27.

5. (6 points) Evaluate the following integral.

$$\int_0^2 \int_y^2 e^{-x^2}\ dx\ dy$$

Solution:

$$\{(x,y) : 0 \le y \le 2, y \le x \le 2\} \to \{(x,y) : 0 \le x \le 2, 0 \le y \le x\}$$

$$\int_0^2 \int_y^2 e^{-x^2}\ dx\ dy = \int_0^2 \int_0^x e^{-x^2}\ dy\ dx$$

$$= \int_0^2 \left(ye^{-x^2}\right)\Big|_{y=0}^{x}\ dx = \int_0^2 xe^{-x^2}\ dx$$

$$= -\frac{1}{2}e^{-x^2}\Big|_{x=0}^{2} = \frac{1}{2}(1-e^{-4}).$$

6. (6 points) Calculate the volume of the solid bounded by the parabola $z = 16 - x^2 - y^2$ and the plane $z = 12$.

Solution: The intersection between $z = 16 - x^2 - y^2$ and $z = 12$ is $12 = 16 - x^2 - y^2$, i.e., $x^2 + y^2 = 4$. Thus

$$R = \{(x, y) : x^2 + y^2 \leq 4\}.$$

$$V = \iint\limits_R (z - 12)dA = \iint\limits_R (16 - x^2 - y^2 - 12)dA.$$

In polar coordinates, $x = r\cos\theta, y = r\sin\theta$, $dA = rdrd\theta$. The domain R becomes

$$0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi.$$

$$V = \int_0^{2\pi} \int_0^2 (4 - r^2)rdrd\theta = 2\pi(2r^2 - \frac{1}{4}r^4)|_0^2 = 8\pi.$$