10. Set Operations & Set Identities

Basics of Set Theory:

- ☐ set element when two sets are equal
- ☐ describing a set:
  - set-builder notation
  - list notation (order / multiplicity do not affect an element's membership in a set)
- ☐ when two sets are equal subset proper subset
- ☐ empty set $\emptyset$ universal set $\mathcal{U}$
- ☐ cardinality of a finite set S: $|S|$ power set of a set S: $\mathcal{P}(S)$

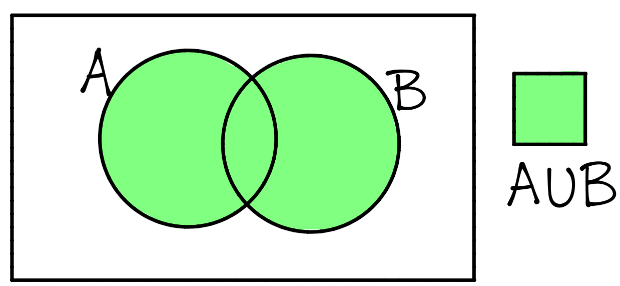
SETS OPERATIONS

Venn diagrams are a graphical method for depicting sets and set operations.

Union.

The union of sets A and B, denoted $A \cup B$, is the set

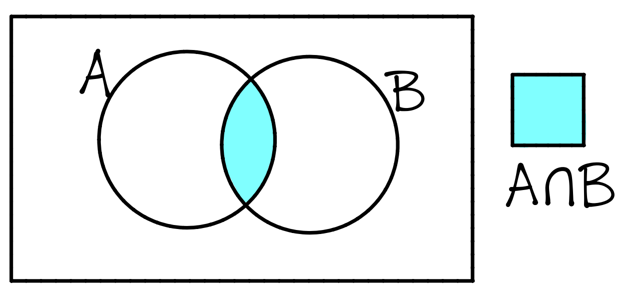
$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$



Intersection.

The intersection of sets A and B, denoted $A \cap B$, is the set

$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$$



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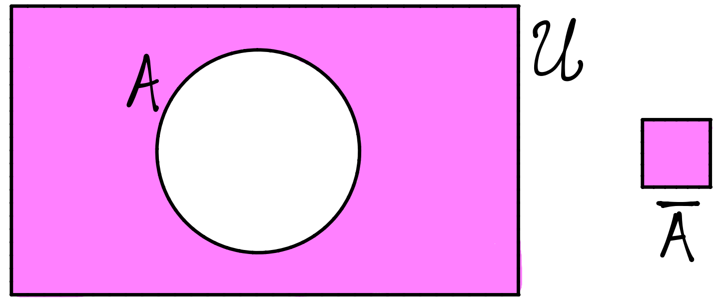
Complement.

The complement of a set A, denoted $\overline{A}$, is the set

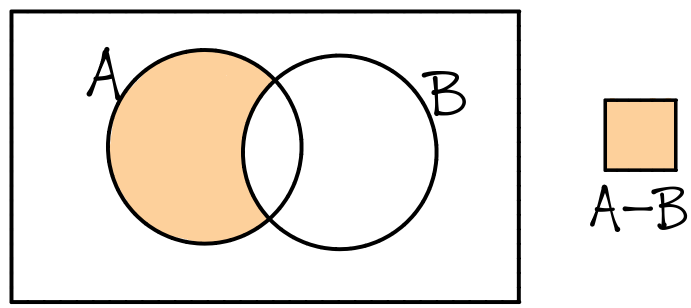
$$\overline{A} = \{x : (x \in \mathcal{U}) \wedge (x \notin A)\}$$



Difference.

The difference of sets A and B, denoted $A - B$, is the set

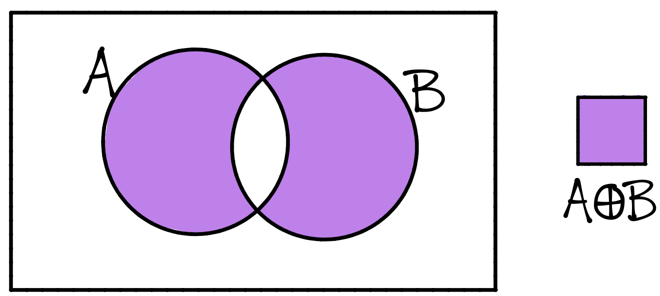
$$A - B = \{x : (x \in A) \wedge (x \notin B)\}$$



Symmetric Difference.

The symmetric difference of sets A and B, denoted $A \oplus B$, is the set

$$A \oplus B = \{x : (x \in A) \oplus (x \in B)\}$$



Example 10.1. Let $A = \{1, 2, 3\}$ $B = \{1, 3, 5, 7\}$ $C = \{5, 7\}$ be subsets of a universal set $\mathcal{U} = \{1,2,3,4,5,6,7,8\}$.

$B \cap C = \{5,7\} = C$

$A - B = \{2\}$

$A \cup B = \{1,2,3,5,7\}$

$B \cup C = \{1,3,5,7\} = B$

$B - C = \{1,3\}$

$A \cap B = \{1,3\}$

$\overline{B} = \{2,4,6,8\}$

$A \oplus B = \{2,5,7\}$

$A \cap C = \{\} = \emptyset$

$\overline{A} = \{4,5,6,7,8\}$

$B \oplus C = \{1,3\}$

Disjoint Sets.

Sets A and B are called **disjoint** if $A \cap B = \emptyset$.

Ex. A and C are disjoint. (from above example)

A **set identity** is an equation involving sets and set operations that is true *no matter what* particular sets we consider.

Example 10.2. For all sets A, B, C, the following equation is true:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

look familiar?
▶ distributive law!

Ways to verify set identities

How can we verify that a set identity will be true no matter what the sets A, B, and C are?

◇ Recall that two sets S and T are equal if

for all $x \in \mathcal{U}$, the biconditional statement $(x \in S) \leftrightarrow (x \in T)$ is true.

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1. We can verify a set identity using a **membership table**.

- Membership tables are similar to truth tables, but more like attendance sheets.
- If there are n sets involved in an identity, then the table will have 2^n rows.
- Each row corresponds to one possible "location" of an element $x \in \mathcal{U}$, *relative to* the sets in the identity.

Example 10.3. Using a membership table, prove $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (De Morgan)

A	B	$A \cup B$	$\overline{A \cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

Since membership is the same for $\overline{A \cup B}$ and $\overline{A} \cap \overline{B}$ it follows that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

for 2 sets A and B, these 4 rows cover <u>all</u> possible cases for the membership of an arbitrary element x of the universal set relative to A and B.

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2. We can verify a set identity using what is called a **rigorous proof** (essentially a *proof of equivalence* of the definition of set equality).

▷▷▷ To prove $S = T$ with a **rigorous proof**, we must prove 2 things:

i) for all $x \in \mathcal{U}$, $(x \in S) \rightarrow (x \in T)$ (ie prove $S \subseteq T$)

and

ii) for all $x \in \mathcal{U}$, $(x \in T) \rightarrow (x \in S)$ (ie prove $T \subseteq S$)

∴ we will have completed a proof of equivalence that $(x \in S) \leftrightarrow (x \in T)$ which is the definition of two sets S and T being equal.

Example 10.4. Use a rigorous proof to prove $\overline{A \cap B} = \overline{A} \cup \overline{B}$ (De Morgan)

To prove $\overline{A \cap B} = \overline{A} \cup \overline{B}$ with a rigorous proof, we must prove 1. $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and 2. $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

i. ($\subseteq$) Let $x \in \mathcal{U}$. Assume $x \in \overline{A \cap B}$.

By def. (of complement), this means that $x \notin A \cap B$.

$\therefore$ it is not the case that $(x \in A) \wedge (x \in B)$

$\therefore x \notin A$ or $x \notin B$

We are making use of De Morgan's Law $\neg((x \in A) \wedge (x \in B)) \equiv \neg(x \in A) \vee \neg(x \in B)$

By def. of complement, this means that $x \in \overline{A}$ or $x \in \overline{B}$

By def. of union, this means that $x \in \overline{A} \cup \overline{B}$

$\therefore$ We proved that $(x \in \overline{A \cap B}) \rightarrow (x \in \overline{A} \cup \overline{B})$ is True. Hence, $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

ii. ($\supseteq$) Let $x \in \mathcal{U}$. Assume $x \in \overline{A} \cup \overline{B}$.

Then (by def. of $\cup$) $x \in \overline{A}$ or $x \in \overline{B}$

$\Rightarrow$ (by def. of $\overline{}$) $x \notin A$ or $x \notin B$

$\therefore x \notin A \cap B$

(not being an element of A or not being an element of B is sufficient to guarantee that x is not an element of $A \cap B$)

$\Rightarrow$ (by def. of $\overline{}$) $x \in \overline{A \cap B}$

$\therefore$ We proved that $(x \in \overline{A} \cup \overline{B}) \rightarrow (x \in \overline{A \cap B})$ is True. Hence $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

Since $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$, we have proved that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ ■

Using the Table of Set Identities

Example 10.5. Let A, B, and C be subsets of a universal set $\mathcal{U}$.

Using set identities, prove that $\overline{(B \cup C) - A} = (\overline{C} \cap \overline{B}) \cup A$

$LS = \overline{(B \cup C) - A}$

$= \overline{(B \cup C) \cap \overline{A}}$ (Difference Law)

$= \overline{(B \cup C)} \cup \overline{\overline{A}}$ (De Morgan's Law)

$= (\overline{B} \cap \overline{C}) \cup \overline{\overline{A}}$ (De Morgan's Law)

$= (\overline{B} \cap \overline{C}) \cup A$ ((Double) Complementation Law)

$= (\overline{C} \cap \overline{B}) \cup A$ (Commutative Law)

$= RS$

$\therefore LS = RS$

Example 10.6. Prove the following theorem:

Theorem 10.6. Let A and B be subsets of the universal set.

Then $\underbrace{\overline{A} \subseteq \overline{B}}_{P}$ if and only if $\underbrace{B \subseteq A}_{Q}$.

Note: P says "for all $x \in \mathcal{U}$, $(x \in \overline{A}) \to (x \in \overline{B})$"
Equivalently (contrapos.): $(x \notin \overline{B}) \to (x \notin \overline{A})$ ☆☆☆

Q says "for all $x \in \mathcal{U}$, $(x \in B) \to (x \in A)$"
Equivalently (contrapos.): $(x \notin A) \to (x \notin B)$ ✧✧✧

Proof of Theorem 11.2 (using a proof of equivalence)

($\Rightarrow$) We will prove $P \to Q$ with a direct proof.

Assume P is True. ie Assume $\overline{A} \subseteq \overline{B}$. (goal is to prove Q is True, ie $B \subseteq A$).

Let $x \in \mathcal{U}$.

Assume $x \in B$. Then $x \notin \overline{B}$ since $x \in B$.

$\Rightarrow x \notin \overline{A}$ since we assumed $\overline{A} \subseteq \overline{B}$. ☆☆☆

$\Rightarrow x \in A$

Thus, we proved $(x \in B) \to (x \in A)$ $\therefore$ $B \subseteq A$ (ie Q is True).

Overall, we proved $(\overline{A} \subseteq \overline{B}) \to (B \subseteq A)$

($\Leftarrow$) We will prove $Q \to P$ with a direct proof.

Assume Q is True. ie Assume $B \subseteq A$ (goal is to prove P is True, ie $\overline{A} \subseteq \overline{B}$).

Let $x \in \mathcal{U}$.

Assume $x \in \overline{A}$. Then $x \notin A$ since $x \in \overline{A}$

$\Rightarrow x \notin B$ since we assumed $B \subseteq A$. ✧✧✧

$\Rightarrow x \in \overline{B}$

Thus, we proved $(x \in \overline{A}) \to (x \in \overline{B})$ $\therefore$ $\overline{A} \subseteq \overline{B}$ (ie P is True).

Overall, we proved $(B \subseteq A) \to (\overline{A} \subseteq \overline{B})$

We proved $P \to Q$ and $Q \to P$. $\therefore$ we proved $P \leftrightarrow Q$ is true. ■

Exercise 10.7. Give concrete examples of two sets A and B such that $\overline{A} \not\subseteq \overline{B}$.

Let $\mathcal{U} = \{1,2,3,4,5\}$ and let A and B be the following two sets:

$$A = \{1,2,3\} \qquad B = \{3,4,5\}$$

Then $\overline{A} = \{4,5\}$ $\qquad \overline{B} = \{1,2\}$

and so $\overline{A} \not\subseteq \overline{B}$.

Why doesn't this example contradict Theorem 10.6?

Theorem 11.2 does not say that $\overline{A} \subseteq \overline{B}$ for all sets A, B.

Theorem 11.2 says $\overline{A} \subseteq \overline{B}$ if and only if $B \subseteq A$.

It means: (1.) If $\overline{A} \subseteq \overline{B}$, then $B \subseteq A$ and (2.) If $B \subseteq A$, then $\overline{A} \subseteq \overline{B}$

In our example, B was not a subset of A so the premise of (2) was not fulfilled.

STUDY GUIDE

Basic terms and concepts of Set Theory:

☐ set S ☐ element $x \in S$ ☐ subset $T \subseteq S$ ☐ proper subset $T \subset S$ ☐ equality $S = T$ ☐ cardinality $|S|$

Some important sets:

☐ empty set $\emptyset$ ☐ universal set $\mathcal{U}$ ☐ naturals $\mathbb{N}$ ☐ integers $\mathbb{Z}$ $\mathbb{Z}^-$ $\mathbb{Z}^+$ ☐ rationals $\mathbb{Q}$ $\mathbb{Q}^-$ $\mathbb{Q}^+$ ☐ reals $\mathbb{R}$ $\mathbb{R}^-$ $\mathbb{R}^+$

Building new sets from old:

☐ power set of S $\mathcal{P}(S)$ ☐ Cartesian product of two (or more) sets $S \times T$ $\qquad S_1 \times S_2 \times \cdots \times S_t$

Set Operations:

☐ union $S \cup T$ ☐ intersection $S \cap T$ ☐ complement $\overline{S}$ ☐ difference $S - T$ ☐ symmetric difference $S \oplus T$

Set identities:

☐ verify using membership tables ☐ verify using a rigorous proof

☐ prove other identities using the laws from the Table of Important Set Identities

Exercises Sup.Ex.	§4 # 1, 2, 3, 4, 5, 6(use a rigorous proof), 9, 11
Rosen (8th ed.)	§2.2 # 1, 3, 4, 5–13(using membership tables or rigorous proofs) 14, 15, 17, 19, 21, 23, 31, 41

Table of Important Set Identities

1. 2.	$A \cup \emptyset = A$ $A \cap \mathcal{U} = A$	Identity Laws
3. 4.	$A \cup \mathcal{U} = \mathcal{U}$ $A \cap \emptyset = \emptyset$	Domination Laws
5. 6.	$A \cup A = A$ $A \cap A = A$	Idempotent Laws
7.	$\overline{(\overline{A})} = A$	(Double) Complementation Law
8. 9.	$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
10. 11.	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
12. 13.	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Laws
14. 15.	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's Laws
16. 17.	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Laws
18. 19.	$A \cup \overline{A} = \mathcal{U}$ $A \cap \overline{A} = \emptyset$	Complement Laws
20.	$A - B = A \cap \overline{B}$	Difference Law
21. 22.	$A \oplus B = (A - B) \cup (B - A)$ $A \oplus B = (A \cup B) - (A \cap B)$	Symmetric Difference Laws