Lesson 5 – Derivatives of Composite Functions

PART A: Recall Composition of Functions

$h(x) = (x^2 + 3x + 4)^{12}$

$h(x) = \sqrt{4x^2 + 9}$

$h(x) = 2^{x^2 - 4}$

each can be expressed as $h(x) = f(g(x))$

Definition of a composite function

Given two functions *f* and *g*, the **composite function** $(f \circ g)$, is defined by $(f \circ g)(x) = f(g(x))$.

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Example

If $f(x) = \sqrt{x}$ and $g(x) = x + 5$

Find a) $g(f(4))$

$g(f(x)) = \sqrt{x} + 5$

$g(f(4)) = \sqrt{4} + 5$

$= 2 + 5$

$= 7$

$f(4) = \sqrt{4} = 2$

$g(2) = 2 + 5$

$= 7$

b) $f(g(x))$ $= \sqrt{x+5}$

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PART B: Taking the Derivative of Composite Functions

We will now look at the Chain Rule which will allow us to take the derivative of composite functions of the form $F = f \circ g$ in terms of the derivatives of f and g.

The Chain Rule

If *f* and *g* are functions that have derivatives, then the composite function $h(x) = f(g(x))$ has a derivative given by:

$$h'(x) = f'(g(x)) \cdot g'(x)$$

$(x+5)^3$

$3(x+5)^2 \cdot (1)$

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Example 1: Differentiate $h(x) = (x^2 + x)^{\frac{3}{2}}$

Note: the *inner* function is $g(x) = x^2 + x$, and the *outer* function is $f(x) = x^{\frac{3}{2}}$

$f(g(x)) = h(x)$

$g'(x) = 2x + 1$

$h(x) = (x^2 + x)^{3/2}$

$h'(x) = \frac{3}{2}(x^2+x)^{1/2} \cdot (2x+1)$

$h'(x) = \frac{3\sqrt{x^2+x}}{2} \cdot (2x+1)$

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The Chain Rule in Leibniz Notation

If y is a function of u and u is a function of x (so that y is a composite function), then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

provided that $\frac{dy}{du}$ and $\frac{du}{dx}$ exist.

Example 2: If $y = u^3 - 2u + 1$, where $u = 2\sqrt{x}$, find $\frac{dy}{dx}$ at $x=4$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= \frac{dy}{du}(u^3 - 2u + 1)\frac{du}{dx}(2\sqrt{x})$$
$$= (3u^2 - 2)\left(\frac{1}{\sqrt{x}}\right)$$
$$= [3(2\sqrt{x})^2 - 2]\left(\frac{1}{\sqrt{x}}\right)$$
$$= [3(2\sqrt{4})^2 - 2]\left(\frac{1}{\sqrt{4}}\right)$$
$$= [3(2(2))^2 - 2]\left(\frac{1}{2}\right)$$
$$= [3(16) - 2]\left(\frac{1}{2}\right)$$
$$= (48 - 2)\left(\frac{1}{2}\right)$$
$$= 46\left(\frac{1}{2}\right)$$
$$= 23$$

$$\frac{du}{dx} = 2\sqrt{x}$$
$$= 2(x)^{1/2 - 1}$$
$$= 2\left(\frac{1}{2}x^{-1/2}\right)$$
$$= \cancel{2}\left(\frac{1}{\cancel{2}\sqrt{x}}\right)$$
$$= \frac{1}{\sqrt{x}}$$

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PART C: Combining multiple rules to differentiate a function

Let us first reiterate a rule that we stated before, but in more general terms.

The Power of a Function Rule

If u is a function of x, and n is a ***real*** number, then, $\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$

or

In function notation, if $f(x) = [g(x)]^n$, then $f'(x) = n[g(x)]^{n-1} \cdot g'(x)$

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Example 1 Differentiate $h(x) = (x^2+3)^4(4x-5)^3$. Express answer in simplified factored form. *Note:* we must use both the product and chain rule.

$$h'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

$$= \left[4(x^2+3)^3(2x)(4x-5)^3\right] + \left[(x^2+3)^4\cdot 3(4x-5)^2(4)\right]$$

$$= \left[8x(x^2+3)^3(4x-5)^3\right] + \left[12(x^2+3)^4(4x-5)^2\right]$$

$$= \left[4(x^2+3)^3(4x-5)^2\right]\left[2x(4x-5)+3(x^2+3)\right]$$

$$= \left[4(x^2+3)^3(4x-5)^2\right]\left[8x^2-10x+3x^2+9\right]$$

$$= \left[4(x^2+3)^3(4x-5)^2\right]\left[11x^2-10x+9\right]$$

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Example #2 Differentiate $g(x) = \left(\frac{1+x^2}{1-x^2}\right)^{10}$

$$g'(x) = 10\left(\frac{1+x^2}{1-x^2}\right)^9\left[\frac{(2x)(1-x^2)-(1+x^2)(-2x)}{(1-x^2)^2}\right]$$

$$g'(x) = 10\left(\frac{1+x^2}{1-x^2}\right)^9\left(\frac{2x-2x^3+2x+2x^3}{(1-x^2)^2}\right)$$

$$= 10\left(\frac{1+x^2}{1-x^2}\right)^9\left(\frac{4x}{(1-x^2)^2}\right)$$

$$= 10\,\frac{(1+x^2)^9}{(1-x^2)^9}\left(\frac{4x}{(1-x^2)^2}\right)$$

$$= \frac{40x(1+x^2)^9}{(1-x^2)^{11}}$$

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