Lesson 3 – Optimization Problems Involving Exponential Functions

PART A: Optimization Strategy

The strategy employed for optimization problems involving exponential functions is the same used for polynomial and rational functions.

- Understand the problem, identify variable quantities
- Determine a function (in one variable) that represents the quantity to be optimized
- Determine domain of function
- Use the algorithm for finding extreme values to find absolute max/min on the domain
- Use above result to answer the question

Example 1: Exponential business model

A mathematical consultant determines that the proportion of people who will have responded to the advertisement of a new product after it has been marketed for t days is given by $f(t) = 0.7(1 - e^{-0.2t})$. The area covered by the advertisement contains 10 million potential customers, and each response to the advertisement results in revenue to the company of \$0.70 (on average), excluding the cost of advertising. The advertising costs \$30 000 to produce and a further \$5000 per day to run.

a) Determine $\lim_{t \to \infty} f(t)$ and interpret the result.

$\lim_{t \to \infty} 0.7(1 - e^{-0.2t})$

$= 0.7$ (b/c $e^{-0.2t} \to 0$)

If the ad was to continue indefinitely, we could expect to reach 70% of the target market.

b) What percent of potential customers have responded after seven days of advertising?

$t = 7$ so: $f(7) = 0.7(1 - e^{-0.2(7)})$

$= 0.52738$

$\doteq 0.53$

About 53% of the potential customers respond after 7 days.

c) Write the function $P(t)$ that represents the profit after t days of advertising. What is the profit after seven days?

$$P(t) = R(t) - C(t)$$

$$= \underbrace{0.7\left[0.7\left(1-e^{-0.2t}\right)\right] \times 10\,000\,000}_{R(t)} - \underbrace{(30000 + 5000t)}_{C(t)}$$

$$= 4900\,000\left(1-e^{-0.2(7)}\right) - 30000 - 5000(7)$$

$$\doteq \$3\,626\,674.88$$

d) For how many full days should the advertising campaign be run in order to maximize the profit? Assume an advertising budget of $200 000.

Restriction

Budget cannot exceed $200 000

$$C(t) \leq 200\,000$$

$$30\,000 + 5000t \leq 200\,000$$

$$\frac{5000t}{5000} \leq \frac{170\,000}{5000}$$

$$\boxed{t \leq 34 \text{ days}}$$

To maximize profit, take derivative of profit equation and set to zero (then solve)

$$P(t) = 4.9\times10^{6}\left(1-e^{-0.2t}\right) - 5000t - 30000$$

$$P'(t) = 4.9\times10^{6}\left(0.2e^{-0.2t}\right) - 5000$$

set $P'(t)=0$ and solve for t

$$0 = 4.9\times10^{6}\left(0.2e^{-0.2t}\right) - 5000$$

$$\frac{5000}{0.2\times4.9\times10^{6}} = \frac{\cancel{4.9\times10^{6}}\left(\cancel{0.2}e^{-0.2t}\right)}{\cancel{4.9\times10^{6}}\times\cancel{0.2}}$$

$$\frac{1}{196} = e^{-0.2t}$$

$$\ln\left(\frac{1}{196}\right) = \ln e^{-0.2t}$$

$$\frac{-5.278}{-0.2} = \frac{-0.2t}{-0.2}\cancel{\ln e}^{1} \qquad \therefore 26 \text{ days}$$

$$\therefore t = 26.4 \text{ days}$$

Algorithm for extreme values

$P(0) = \$-30\,000$ (losing money)

$P(26) = \$4\,712\,968.83$

$P(34) = \$4\,694\,542.50$

Therefore, maximum profit occurs at t = 26 days