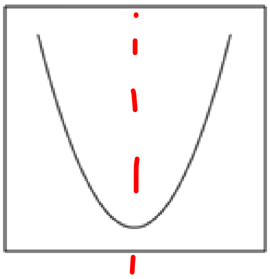
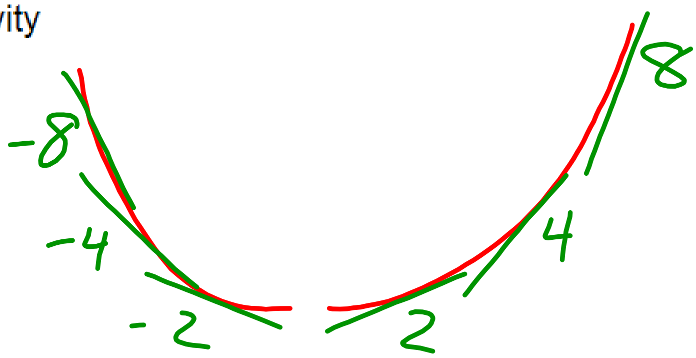
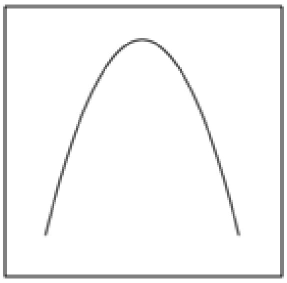
Lesson 4 - Concavity and Points of Inflection

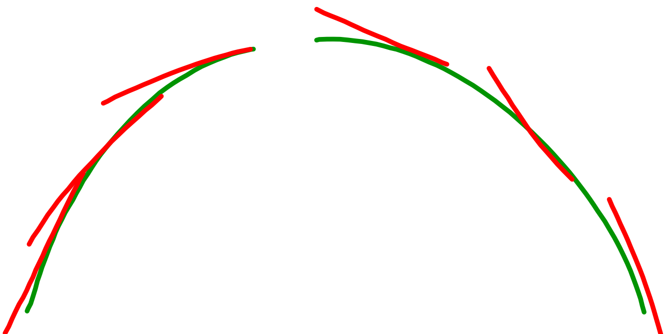
PART A: Concavity



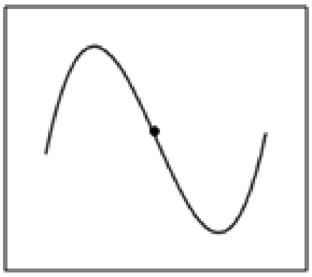


Concave up: the graph of a function $f(x)$ is concave up on the interval $a \le x \le b$ if all the tangents are below the curve. The graph curves upward. Slope of tangent is increasing ($f'(x)$ is ↑).





Concave down: the graph of a function $f(x)$ is concave down on the interval $a \le x \le b$ if all the tangents on the interval are above the curve. The graph curves downward. Slope of tangent is decreasing ($f'(x)$ is ↓).



Point of Inflection: a point at which the graph changes from being concave up to concave down, or vice versa.

Example 1: Identify the intervals over which the graph is concave up and the intervals over which it is concave down.

Concave Up

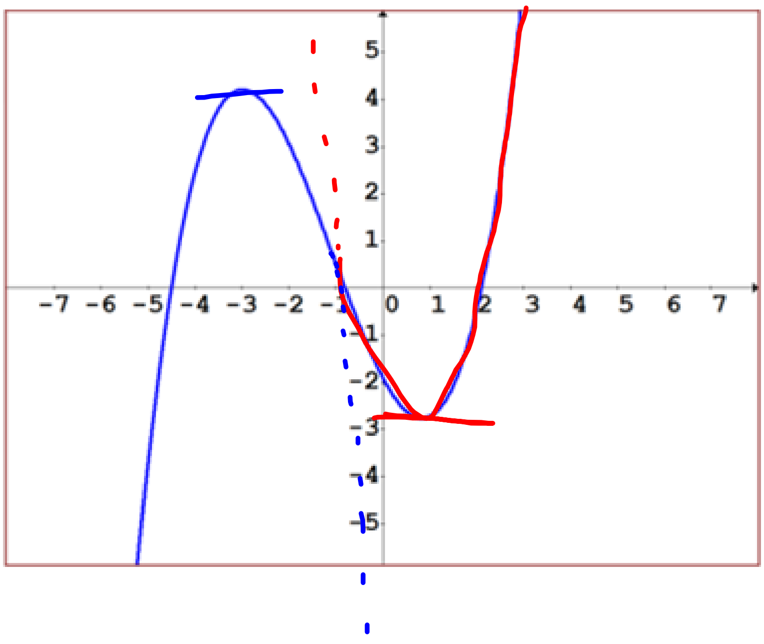
$(-1, 0.8)$

$(0.8, \infty)$

Concave Down

$(-\infty, -3)$

$(-3, -1)$



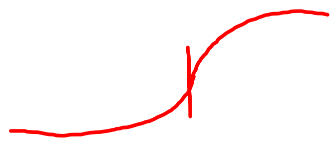
PART B: The Second Derivative Test

Intervals of concavity can be found using the second derivative test or by examining the graph of $f''(x)$.

- A graph is **concave up** on an interval if the second derivative is positive on that interval. If $f'(c) = 0$ and $\boldsymbol{f''(c) > 0}$, there is a local minimum at $[c, f(c)]$
- A graph is **concave down** on an interval if the second derivative is negative on that interval. If $f'(c) = 0$ and $\boldsymbol{f''(c) < 0}$, there is a local maximum at $[c, f(c)]$
- If $f''(c) = 0$ and $f''(c)$ changes sign at $x = c$ there is a **point of inflection** at $[c, f(c)]$

or if

$f''(x) \to DNE$ a change of signs occurs.



Example 2: Find the intervals of concavity and the coordinates of any points of inflection for the function $g(x) = x^4 - 4x^3$.

$g'(x) = 4x^3 - 12x^2$

$g''(x) = 12x^2 - 24x$

$0 = 12x(x-2)$

$x = 0$ or $x = 2$

	$x<0$	$x=0$	$0<x<2$	$x=2$	$x>2$
Test Value	-1	0	1	2	3
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$ Concavity	C.U. $\cup$	P.O.I. $(0,0)$	C.D. $\cap$	P.O.I. $(2,-16)$	C.U. $\cup$

Example 3: Sketch the graph of the function $f(x) = x^{1/3}$.

$f'(x) = \frac{1}{3}x^{-2/3}$

$f''(x) = -\frac{2}{3} \cdot \frac{1}{3}x^{-5/3}$

$0 = \frac{-2}{9x^{5/3}}$

$x = \text{DNE}$

	$x<0$	$x=0$	$x>0$
T.V.	-1	0	1
$f''(x)$	$+$	DNE	$-$
$f(x)$	C.U. $\cup$	P.O.I. $(0,0)$	C.D. $\cap$

y-int (set $x=0$)
$f(0) = 0$
$(0,0)$

x-int (set $y=0$)
$0 = x^{1/3}$
$(0,0)$