7. Proof Strategies

- A **theorem** is a mathematical statement that is true.
- The statement of a theorem can be an atomic proposition

Ex. V2 is irrational.

 \circ The statement of a theorem is often in the form of a logical argument (a set of premises that imply some conclusion), which, in its simplest form is a conditional statement $P \to Q$.

Ex. If n is an odd integer, then n2 is an odd integer.

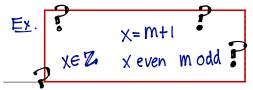
The premises of a mathematical argument can be a conjunction of several axioms
(assumptions accepted without proof), new or known definitions, as well as other
previously proved theorems.

Some Remarks:

- ♦ In propositional logic, the variables only have two truth values (**T** or **F**). We could (with enough time), check all possible truth assignments to solve our problems or prove our claims.
- In mathematical proofs, the variables can represent numbers and other more complicated mathematical objects, each having specific definitions and properties, or infinitely many possibilities.
- We will need to make use of some math (addition, subtraction, multiplication, division, real numbers, integers, rational numbers, etc...) and reasoning in words.
- ♦ To get some practice writing proofs, we will prove fairly simple "facts" of math. In general, the "theorems" we will prove in MAT1348 are not themselves very important... we just need some material to practice with.
- What is important is understanding the †strategy of your proof, that it is correct, and that anyone else who reads your proof will understand your reasoning, and see at each step why your claims are logical and correct!
- ⇒ Writing a proof is a bit like writing a recipe. You've baked a cake and you want to write down instructions so that others can bake the same cake you did.

Make sure to clearly indicate what you are assuming vs. What you are proving.

It is not the marker's job to fill in the gaps of your proof (even if they seem "obvious")



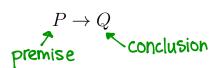
Let XEZ, and let m=X-1 Assume X is even. Then m must be odd... Let meZ, and let x=m+1.
Assume mis odd.
Then x must be even...

† For now, our proof-writing practice will seem a bit like painting-by-numbers. It's a starting point. Later on, you will be free to prove (or read proofs of) more interesting mathematical results, which involve more creativity and original thought, but which will nonetheless have a logical foundation.

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NAME	PROPOSITION	Strategy
OF Strategy	TO Prove	OF Proof
Direct Proof	P→Q	First step: Assume P is True Then prove that Q being true must follow from P.
Indirect Proof	P→Q	•Prove 7Q→7P directly (because 7Q→7P = P→Q)
(Proof by Contraposition)	=7Q→7p	First Step: •Assume 7Q is True (i.e. Assume Q is False)
		Then prove that 7P being true must follow from 7Q
Proof by Contradiction	P	First step: Negate Pentirely and assume 7p is True.
	(Pcould be a compound Proposition such as X→Y)	Then prove that contradiction always follows from 7P % P must be True
Proof by Cases	(P ₁ V…VP _k)→Q	We prove $(P_1 V P_2 V \cdots V P_k) \rightarrow \emptyset$ by proving each of
		$P_1 \rightarrow Q$, $P_2 \rightarrow Q$,, $P_k \rightarrow Q$
Proof of Equivalence	P↔Q	Prove both $P \rightarrow Q$ and its converse $Q \rightarrow P$
		because $P \leftrightarrow Q \equiv (P \rightarrow Q) \land (Q \rightarrow P)$
Vacuous Proof	P→Q	Simply prove P is False
		because it will mean $P \rightarrow Q \equiv F \rightarrow Q \equiv T$
Trivial Proof	P→Q	Simply prove Q is True
		because it will mean
		$P \rightarrow Q \equiv P \rightarrow T \equiv T$

CONDITIONAL STATEMENTS



- if P, then Q
- P only if Q
- Q if P
- *P* is a sufficient condition for *Q*
- Q is a necessary condition for P



In a <u>direct proof</u> of $P \rightarrow Q$, we are proving that this row Cannot happen.

- if Pis F, then P→Q is vacuously true.
- if QisT, then P→Qistrivially true.
- Thus, the only chance for $P \rightarrow Q$ to be F is when P is T.
- So, the <u>direct proof</u> strategy is to <u>assume P is T</u>, and then show (using math, definitions, other known theorems, etc) that Q must be true.

In an <u>indirect proof</u> of $P \rightarrow Q$, we are proving that this row Cannot happen.

- if 7Q is F, then 7Q-7P is vacuously true.
- if¬PisT, then 7Q→7P istrivially true.
- Thus, the only chance for 7Q→7P to be F is when 7Q is T.
- So, the <u>indirect proof</u> strategy is to <u>assume 7Q is T</u>, and then show (using math, definitions, other known theorems, etc) that 7P must be true.
- Finally, since $7Q \rightarrow 7P \equiv P \rightarrow Q$, the indirect proof is a way to prove $P \rightarrow Q$.



DIRECT PROOF

To prove a conditional statement, such as $P \to Q$, using a DIRECT PROOF, we will

- Assume *P* is true.
- Then, step-by-step, show that *Q* must follow from *P*.

Definition. An integer n is called **even** if n = 2k for some integer k, and n is called **odd** if n = 2m + 1 for some integer m.

Example 7.1. Give a **direct proof** of the following theorem:

Theorem. Let n be an integer. If n is odd, then n^2 is odd.

<u>proof</u>. Let n be an integer.

Assume Pist. ie Assume n is odd (goal: show Q must bet).

Then, by def. of odd, n=2k+1 for some integer k.

Consequently,
$$n^2 = (2k+1)^2$$

$$=4k^2+4k+1$$

$$= 2[2k^2 + 2k] + 1$$

$$=2\times m+1$$
 for $m=2k^2+2k$

(using a different variable than k)

Since REZ, it follows that m is also an integer.

Thus, by def. of odd, n2 is odd.

: P→Q is true.



Next Indirect proof strategy/Proof by Contraposition

Ex. Contrapositive of Theorem from Ex. 7.1: If n2 is even, then n is even.

We proved the theorem from Ex. 8.1 but it is logically equivalent to its contrapositive.

Think about trying to prove this restatement of the same theorem with a direct proof strategy:

Assume n2 is even. Then n2 = 2k for some integer & (defofeven) and k must be > 0 because $n^2 > 0$.

Thus n= trak -while this is true, it's not

helping us to see that n must be even ...

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INDIRECT PROOF (PROOF BY CONTRAPOSITION)

To prove a conditional statement, such as $P \to Q$, using an INDIRECT PROOF, also called a PROOF BY CONTRAPOSITION, we will

- \circ Assume $\neg Q$ is true (equivalently, assume Q is false).
- \circ Then, step-by-step, show that $\neg P$ must follow from $\neg Q$.

Remark 7.2. Recall that $P \to Q \equiv \neg Q \to \neg P$.

Thus, an indirect proof of $P \to Q$ is simply a direct proof of the **contrapositive** conditional statement $\neg Q \rightarrow \neg P$

Example 7.3. Give an **indirect proof (proof by contraposition)** of the following theorem:

Theorem. Let n be an integer. If 5n + 4 is odd, then n is odd.

$$P \rightarrow Q$$

7P: 5n+4 is even.

70: niseven

For the indirect proof strategy, we want to prove $7Q \rightarrow 7P$ proof. Let n be an integer.

Assume 7Q is T.

ie Assume n is even. (goal: show 7P must be true)

Then, by def. of even, we have n=2k for some integer $k \in \mathbb{Z}$.

Thus,
$$5n+4=5(2k)+4$$

= $10k+4$
= $2(5k+2)$

= 2m where m=5k+2, hence m is an integer.

Thus, 5n+4 is also even (by def. of even). 67 Pis True.

so we proved 7Q→7P which is = P→Q



STUDY GUIDE

 \diamond Proof Strategies: direct proof of $P \to Q$ indirect proof of $P \rightarrow Q$ versus