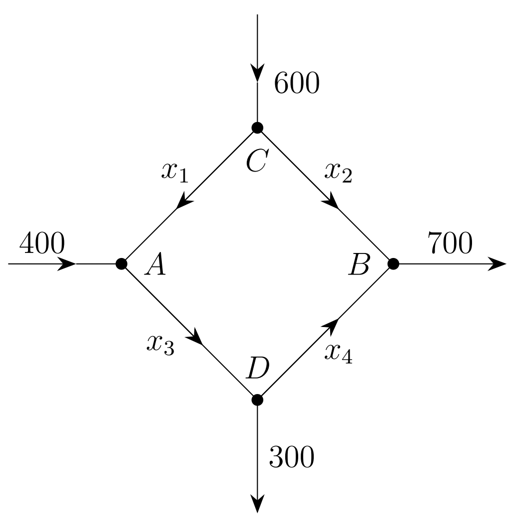
Notes for MAT1341A Fall 2023
Part III

Chapter 13 - Applications of Solving Linear Systems

I. Traffic flow network of one-way street.

The diagram in the figure below represents a network of one-way streets. The numbers on the figure represent the flow of traffic (in cars per hour) along each street, and the intersections are labeled A, B, C and D. The arrows indicate the direction of the flow of traffic. The variables x_1, x_2, x_3, x_4 represent the (unknown) level of traffic on certain streets.



Notice that the variables are traffic flows on internal street, $x_i = \#$ cars per hours.

Goal:

- explain the traffic flow in simple terms (solve).
- answer question / scenarios.

Equations: flow in = flow out

Intersection	**Flow in**	=	**Flow out**
A	$x_1 + 400$	=	x_3
B	$x_2 + x_4$	=	700
C	600	=	$x_1 + x_2$
D	x_3	=	$\cancel{700}$ x_4 + 300

This is the linear system:

$$x_1 - x_3 = -400$$
$$x_2 + x_4 = 700$$
$$x_1 + x_2 = 600$$
$$x_3 - x_4 = 300$$

with augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -400 \\ 0 & 1 & 0 & 1 & 700 \\ 1 & 1 & 0 & 0 & 600 \\ 0 & 0 & 1 & -1 & 300 \end{array}\right] \sim \underset{-R_1+R_3 \to R_3}{\cancel{R_3 \to R_3 - R_1}} \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -400 \\ 0 & 1 & 0 & 1 & 700 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & -1 & 300 \end{array}\right]$$

$$\sim \underset{-R_2+R_3 \to R_3}{\cancel{R_3 \to R_3 - R_2}} \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -400 \\ 0 & 1 & 0 & 1 & 700 \\ 0 & 0 & 1 & -1 & 300 \\ 0 & 0 & 1 & -1 & 300 \end{array}\right] \sim \begin{cases} \cancel{R_1 \to R_1 + R_3} \\ \cancel{R_4 \to R_4 - R_3} \end{cases} -R_3+R_4 \to R_4$$

$$\sim\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -400 \\ 0 & 1 & 0 & 1 & 700 \\ 0 & 0 & 1 & -1 & 300 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{(R}\cancel{\text{R}}\text{EF)} \quad \underset{R_3+R_1 \to R_1}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -100 \\ 0 & 1 & 0 & 1 & 700 \\ 0 & 0 & 1 & -1 & 300 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{(RREF)}$$

our solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -100 + t \\ 700 - t \\ 300 + t \\ t \end{bmatrix}$$

$x_1 - x_4 = -100$

$x_1 = -100 + x_4$

Wait, think about the real-life situation. Try to determine the interval of t.

$$\begin{aligned} x_1 &= -100+t \geq 0 \\ x_2 &= 700-t \geq 0 \\ x_3 &= 300+t \geq 0 \\ x_4 &= t \geq 0 \end{aligned} \implies \begin{aligned} t &\geq 100 \\ t &\leq 700 \end{aligned}$$

, So t has to be between 100 and 700.

Now consider the following questions:

a. What is the minimum flow along AD?

b. What happens if we close AD, will there be a traffic jam?

On AD, we have $x_3 = 300+t \geq 300+100 = 400$

We cannot close AD, since this gives $300+t=0$
$t=-300$
impossible.

We can however close CB, since we get
$t=700$, $x_1 = 600$, $x_3 = 1000$, $x_4 = 700$,
which is possible.

II. Solving systems with parameters.

For what values of a does the system with the following augmented matrix have a unique solution?

$$A = \begin{bmatrix} a & 2 & 2 & | & -2 \\ 1 & 1 & 3 & | & a \\ 2 & a & a & | & 2 \end{bmatrix}$$

III. Solving vector equations.

What are all the vectors in $\mathbb{R}^3$ that are a linear combination of

$$\{(1,2,1),(3,4,4),(2,6,1)\}\ ?$$

Chapter 14 - Matrices
MAT 1341

A matrix can be thought of as:

- a table of numbers
- the augmented matrix of a linear system
- a collection of column vectors
- a collection of row vectors
- a mathematical object in its own right

Definition. A matrix with m rows and n columns is called an m by n matrix, it has size $m \times n$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ — } 2 \times 3 \text{ matrix}$$

$(2, 3)$ entry row 2, column 3

You can add matrices componentwise if they have the same size.

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{2\times 2} + \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{2\times 2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \text{but } \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{2\times 2} + \underbrace{\begin{bmatrix} 5 \\ 6 \end{bmatrix}}_{2\times 1} \text{ not allowed}$$

You can multiply a matrix by a scale ($k \in \mathbb{R}$)

$$2\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{2\times 3} = \underbrace{\begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}}_{2\times 3}$$

You have a zero matrix in every size

$$\mathbf{0}_{2\times 3} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{2\times 3} \qquad \mathbf{0}_{1\times 2} = \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}}_{1\times 2}$$

Definition (Matrix transpose)**.** If A is $m \times n$ then the "A-transpose" $A^\top$ is $n \times m$, and the rows of A are the columns of $A^\top$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^\top = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Note that the transpose operation on matrices satisfies

- $(A + B)^\top = A^\top + B^\top$
- $(kA)^\top = kA^\top, k \in \mathbb{R}$
- $(A^\top)^\top = A$

Definition (**14.1.2**)**.** If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then their *product* AB is the $m \times p$ matrix whose (i, j) entry is the dot product of the ith row of A with the jth column of B.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 6 & 5 \\ -1 & 15 & 11 \end{bmatrix}$$

$[E.g.]$ Find the matrix product of the following matrices

a) $A = \begin{bmatrix} 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 2 & 0 & -4 \end{bmatrix}$

• We can express a linear system as a matrix equation:

$$\begin{aligned} x + 2y + z &= 1 \\ 4x + 5y + 6z &= 2 \\ 7x + 8y + 9z &= 3 \end{aligned}$$

is equivalent to $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

The augmented matrix of the system is $[A|\vec{b}]$.

• We can express a linear combination as a matrix multiplication:

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n = [\vec{u}_1 \ \vec{u}_2 \ \cdots \ \vec{u}_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

[*E.g.*] One can check

$$a \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix} + c \begin{bmatrix} 4 \\ -2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 1 & -2 \\ 2 & -2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

There are some ways in which matrix multiplication is **different** from number multiplication:

1. Is $AB = BA$?

2. If $AB = 0$, must A or B be the zero-matrix?

3. If $AB = AC$, can we cancel A to get $B = C$?

Set $I_k = k \times k$ matrix with 1s on diagonal and 0s elsewhere.

$$I_1 = 1 \qquad\qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad\qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

called the identity matrix of size k.

$[E.g.]$ Find the matrix product AI_3, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Theorem (14.3.1 - Properties of the matrix product). Let A, B and C be matrices and let k be a scalar. Then, whenever defined, we have

1. $(AB)C = A(BC)$ (Associativity)
2. $A(B + C) = AB + AC$ (Distributivity on the right)
3. $(B + C)A = BA + CA$ (Distributivity on the left)
4. $k(AB) = (kA)B = A(kB)$
5. $(AB)^\top = B^\top A^\top$ (NOTE the reversal of order!)
6. $AI = A$ and $IB = B$
7. If A is $m \times n$, then $A0_{n\times p} = 0_{m\times p}$ and $0_{q\times m}A = 0_{q\times n}$.

Now we can do basic algebra:

i. $(A + B)(C + D) =$

ii. $(A + B)(A - B) =$

Definition. If a matrix has size $m \times m$, we say that it is a *square matrix*.

Given a square matrix and a positive integer n, we define

$$A^n = \underbrace{A \cdots A}_{n \text{ times}}.$$

[*E.g.*] Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Calculate A^{2023} and B^{2023}.

$[E.g.]$ Calculate $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{2023}$.