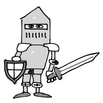
Q1. THE ISLAND OF KNIGHTS & KNAVES

Strolling along on The Island of Knights & Knaves, we meet A, B and C, each of whom is a knight or a knave. Note: knaves always lie and knights always speak the truth. Two people are said to be of the same type if they are both knights or both knaves. A and B make the following statements:



A says: "B is a knave." B says: "A and C are of the same type."

What (if anything) can you conclude?

Solution As an inhabitant of The Island of Knights and Knaves, either A is a Knight or a Knave.

Case 1. A is a Knight ⇒ "B is a Knave" must be true.
⇒ B's statement "A and C are of the same type" must be false.
⇒ A and C must be of different types
⇒ C is a Knave.
∴ Case 1 is possible (A Knight, B Knave, C Knave)

Case 2. A is a Knave. ⇒ "B is a Knave" must be false. ⇒ B must be a Knight.
⇒ B's statement "A and C are of the same type" must be true.
⇒ A and C must be of the same type.
⇒ C is a Knave.
∴ Case 2 is also possible (A Knave, B Knight, C Knave)

- Although we cannot determine the types of A and B, we do know that C must be a Knave in both possible cases.
- We also know that A and B must be opposite types of each other.

Truth Table Method

(1.) Define I-am-Knight atoms for each inhabitant:
a: "A is a Knight."
b: "B is a Knight."
c: "C is a Knight."

(2.) Translate all statements:
A says: "$\neg b$"
B says: "$a \leftrightarrow c$"
there are other logically equivalent ways to translate B's statement

(3.) Truth Table

↯ means "contradiction"

a	b	c	A says $\neg b$	B says $a \leftrightarrow c$	
T	T	T	F	T	← in this row, A is a Knight, but A's statement is F ↯
T	T	F	F	F	← in this row, A is a Knight, but A's statement is F ↯
T	F	T	T	T	← in this row, B is a Knave, but B's statement is T ↯
T	F	F	T	F	← this row is possible on The Island.
F	T	T	F	F	← in this row, B is a Knight, but B's statement is F ↯
F	T	F	F	T	← this row is possible on The Island.
F	F	T	T	F	← in this row, A is a Knave, but A's statement is T ↯
F	F	F	T	T	← in this row, A is a Knave, but A's statement is T ↯

compare these columns

compare these columns

Conclusion: In both possible cases, C is a Knave.
We cannot determine the types of A and B but we know A and B are of opposite type.

Q2. COUNTEREXAMPLES

i. Suppose two compound propositions, say X and Y, are **not** logically equivalent. Describe what a counterexample would be.

A counterexample would be a truth assignment that certifies that $X \leftrightarrow Y$ is not a tautology.

Ex. Is $P \vee Q \to R \equiv R \to P \wedge Q$? If not, provide *all* counterexamples.

P	Q	R	$P\vee Q$	X $P\vee Q\to R$	$P\wedge Q$	Y $R\to P\wedge Q$	$X\leftrightarrow Y$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	F
T	F	T	T	T	F	F	F
T	F	F	T	F	F	T	F
F	T	T	T	T	F	F	F
F	T	F	T	F	F	T	F
F	F	T	F	T	F	F	F
F	F	F	F	T	F	T	T

X is not logically equivalent to Y

counterexamples:

① $P=T, Q=T, R=F$
② $P=T, Q=F, R=T$
③ $P=T, Q=F, R=F$
④ $P=F, Q=T, R=T$
⑤ $P=F, Q=T, R=F$
⑥ $P=F, Q=F, R=T$

For each of these truth assignments, $X \leftrightarrow Y$ is F $\therefore$ $X \not\equiv Y$

ii. Suppose a compound proposition X is **not** a tautology. Describe what a counterexample would be.

A counterexample would be a truth assignment for which the truth value of X is F.

Ex. Is $\neg(\neg P \wedge \neg Q) \leftrightarrow P \wedge Q$ a tautology ? If not, provide *all* counterexamples.

P	Q	$\neg P\wedge\neg Q$	$\neg(\neg P\wedge\neg Q)$	$P\wedge Q$	X $\neg(\neg P\wedge Q)\leftrightarrow P\wedge Q$
T	T	F	T	T	T
T	F	F	T	F	F
F	T	F	T	F	F
F	F	T	F	F	T

X is not a tautology.
counterexamples:

① $P=T, Q=F$

② $P=F, Q=T$

For each of these truth assignments, the truth value of X is F
$\therefore$ X is not a tautology.

iii. Suppose a compound proposition X is **not** a contradiction. Describe what a counterexample would be.

A counterexample would be a truth assignment for which the truth value of X is T.

Ex. Is $P \to (Q \vee \neg P)$ a contradiction ? If not, provide *all* counterexamples.

P	Q	$Q\vee\neg P$	X $P\to(Q\vee\neg P)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

X is not a contradiction
counterexamples:

① $P=T, Q=T$

② $P=F, Q=T$

③ $P=F, Q=F$

For each of these truth assignments, the truth value of X is T
$\therefore$ X is not a contradiction.

Q3. THE LAWS OF LOGICAL EQUIVALENCES

Using the Laws of Logical Equivalences, naming **one – and only one! –** law per step, prove that

$$\Big(((A \to B) \land \neg B) \to \neg A\Big) \equiv P \lor \neg P$$

$LS = \Big((A \to B) \land \neg B\Big) \to \neg A$

$\equiv \big((\neg A \lor B) \land \neg B\big) \to \neg A$ (Implication Law)

$\equiv \big(\neg B \land (\neg A \lor B)\big) \to \neg A$ (Commutative Law)

$\equiv \big((\neg B \land \neg A) \lor (\neg B \land B)\big) \to \neg A$ (Distributive Law)

$\equiv \big((\neg B \land \neg A) \lor (B \land \neg B)\big) \to \neg A$ (Commutative Law)

$\equiv \big((\neg B \land \neg A) \lor (F)\big) \to \neg A$ (Negation Law)

$\equiv (\neg B \land \neg A) \to \neg A$ (Identity Law)

$\equiv \neg(\neg B \land \neg A) \lor \neg A$ (Implication Law)

$\equiv (\neg\neg B \lor \neg\neg A) \lor \neg A$ (De Morgan's Law)

$\equiv (B \lor A) \lor \neg A$ (Double negation Law (twice))

$\equiv B \lor (A \lor \neg A)$ (Associative Law)

$\equiv B \lor (T)$ (Negation Law)

$\equiv T$ (Domination Law)

$\equiv P \lor \neg P$ (Negation Law)

$\therefore LS \equiv RS.$

Q4. DNF

Using a truth table, find a DNF for each of the compound propositions X and Y, defined as follows:

i. $X:\quad (a \lor \neg b) \oplus (b \land c)$

a	b	c	$a \lor \neg b$	$b \land c$	$(a \lor \neg b) \oplus (b \land c)$ X	conjunctive clauses
T	T	T	T	T	F	
T	T	F	T	F	T	$a \land b \land \neg c$
T	F	T	T	F	T	$a \land \neg b \land c$
T	F	F	T	F	T	$a \land \neg b \land \neg c$
F	T	T	F	T	T	$\neg a \land b \land c$
F	T	F	F	F	F	
F	F	T	T	F	T	$\neg a \land \neg b \land c$
F	F	F	T	F	T	$\neg a \land \neg b \land \neg c$

DNF for X:

$$(a \land b \land \neg c) \lor (a \land \neg b \land c) \lor (a \land \neg b \land \neg c) \lor (\neg a \land b \land c) \lor (\neg a \land \neg b \land c) \lor (\neg a \land \neg b \land \neg c)$$

ii. $Y:\quad \neg(A \to B) \leftrightarrow (\neg A \lor B)$

A	B	$\neg(A \to B)$	$\neg A \lor B$	$\neg(A \to B) \leftrightarrow (\neg A \lor B)$ Y
T	T	F	T	F
T	F	T	F	F
F	T	F	T	F
F	F	F	T	F

← there is no truth assignment for which Y is true (ie Y is a contradiction).

Well, technically 'F' is an atom, technically an atom by itself is a conjuctive clause, and the disjunction of one conjuctive clause is in DNF.

∴ A DNF for Y is F.

Q5 EQUIVALENCES

i. Using the Laws of Logical Equivalences, naming **one – and only one! –** law per step, find a DNF for the compound proposition R, defined as follows:

$$R: \quad (a \vee \neg b) \to \neg(b \wedge c)$$

$R = (a \vee \neg b) \to \neg(b \wedge c)$

$\equiv \neg(a \vee \neg b) \vee \neg(b \wedge c)$ (implication law)

$\equiv (\neg a \wedge \neg\neg b) \vee \neg(b \wedge c)$ (De Morgan's law)

$\equiv (\neg a \wedge b) \vee \neg(b \wedge c)$ (double negation law)

$\equiv \underbrace{(\neg a \wedge b) \vee (\neg b \vee \neg c)}$ (De Morgan's law)

this is a DNF for R

Note: $(\neg a \wedge b) \vee (\neg b \vee \neg c) \equiv (\neg a \wedge b) \vee (\neg b) \vee (\neg c)$ by associativity of $\vee$.

ii. Using the Laws of Logical Equivalences, naming **one – and only one! –** law per step, find a formula for the compound proposition S that uses only the logical connectives $\neg$ and $\to$ and parentheses where appropriate.

$$S: \quad \neg(x \wedge y) \vee \neg(\neg y \vee \neg x)$$

$S = \neg(x \wedge y) \vee \neg(\neg y \vee \neg x)$

$\equiv (\neg x \vee \neg y) \vee \neg(\neg y \vee \neg x)$ (De Morgan's law)

$\equiv (x \to \neg y) \vee \neg(\neg y \vee \neg x)$ (implication law)

$\equiv (x \to \neg y) \vee \neg(y \to \neg x)$ (implication law)

$\equiv \neg(y \to \neg x) \vee (x \to \neg y)$ (commutative law)

$\equiv (y \to \neg x) \to (x \to \neg y)$ (implication law)