DGD 8

Q1. EQUIVALENCE RELATION ON $\mathbb{Z} \times \mathbb{Z}^+$

Let $\sim$ be a relation on $\mathbb{Z} \times \mathbb{Z}^+$ defined by the following rule:

$$\text{for all } (a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}^+, \quad (a, b) \sim (c, d) \quad \text{if and only if} \quad \frac{a}{b} = \frac{c}{d}.$$

Prove that $\sim$ is an equivalence relation on $\mathbb{Z} \times \mathbb{Z}^+$.

[ref.] Let $(a,b) \in \mathbb{Z} \times \mathbb{Z}^+$. Then $\frac{a}{b} = \frac{a}{b}$ $\therefore (a,b) \sim (a,b)$

We proved $\Big((a,b) \in \mathbb{Z} \times \mathbb{Z}^+\Big) \longrightarrow \Big((a,b) \sim (a,b)\Big)$ $\therefore \sim$ is reflexive

[Sym.] Let $(a,b), (c,d) \in \mathbb{Z} \times \mathbb{Z}^+$. Assume $(a,b) \sim (c,d)$.

Then $\frac{a}{b} = \frac{c}{d}$

$\Longrightarrow \frac{c}{d} = \frac{a}{b}$ $\therefore (c,d) \sim (a,b)$.

We proved $\Big((a,b) \sim (c,d)\Big) \longrightarrow \Big((c,d) \sim (a,b)\Big)$ $\therefore \sim$ is symmetric.

[trans.] Let $(a,b), (c,d), (e,f) \in \mathbb{Z} \times \mathbb{Z}^+$. Assume $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$.

Then $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$. $\therefore \frac{a}{b} = \frac{e}{f}$ $\therefore (a,b) \sim (e,f)$.

We proved $\Big((a,b) \sim (c,d) \text{ and } (c,d) \sim (e,f)\Big) \longrightarrow \Big((a,b) \sim (e,f)\Big)$

$\therefore \sim$ is transitive

Since $\sim$ is reflexive, symmetric and transitive, it's an equivalence relation on $\mathbb{Z} \times \mathbb{Z}^+$

b) List 4 elements that belong to the equivalence class $[(-1,2)]_\sim$.

$$[(-1,2)]_\sim = \{(a,b) \in \mathbb{Z} \times \mathbb{Z}^+ : (-1,2) \sim (a,b)\} = \{(a,b) \in \mathbb{Z} \times \mathbb{Z}^+ : -\tfrac{1}{2} = \tfrac{a}{b}\}$$

$\therefore (-1,2), (-2,4), (-3,6), (-100,200) \in [(-1,2)]_\sim$ since $-\frac{1}{2} = \frac{-2}{4} = \frac{-3}{6}$ etc...

Q2 . CONGRUENCE $\pmod n$ AN EQUIVALENCE RELATION ON $\mathbb{Z}$

Q2a. Determine the remainder $\pmod 7$ of each element of the set

$$A = \{-15, -12, -7, -6, -4, -1, 0, 1, 2, 7, 8, 9, 10, 11, 12, 13, 14\}$$

remainder: 6 2 0 1 3 6 0 1 2 0 1 2 3 4 5 6 0

* remainder (mod 7) must be an integer r such that $0 \le r < 7$

$x = qm + r$, $0 \le r < 7$

$-15 = (-3)(7) + 6$ $-4 = (-1)(7) + 3$ $2 = (0)(7) + 2$ $10 = (1)(7) + 3$

$-12 = (-2)(7) + 2$ $-1 = (-1)(7) + 6$ $7 = (1)(7) + 0$ $11 = (1)(7) + 4$ $14 = (2)(7) + 0$

$-7 = (-1)(7) + 0$ $0 = (0)(7) + 0$ $8 = (1)(7) + 1$ $12 = (1)(7) + 5$

$-6 = (-1)(7) + 1$ $1 = (0)(7) + 1$ $9 = (1)(7) + 2$ $13 = (1)(7) + 6$

Q2b. Given that $\equiv \pmod 7$ is an equivalence relation on A, which elements of the set A belong to the equivalence class $[-6]_{\equiv \pmod 7}$?

the remainder of $-6 \pmod 7$ is 1 so all elements in A with remainder 1 (mod 7) are in the equivalence class of -6.

$\therefore [-6]_{\equiv(\text{mod } 7)} = \{-6, 1, 8\}$

Q2c. Determine the partition of A into equivalence classes with respect to the equivalence relation $\equiv \pmod 7$ on A.

$[-6]_{\equiv(\text{mod } 7)} = \{-6, 1, 8\}$ so $[1]_{\equiv(\text{mod } 7)} = \{-6, 1, 8\}$ and $[8]_{\equiv(\text{mod } 7)} = \{-6, 1, 8\}$.

Now, look at an element of A that is not related to $-6, 1, 8$, and continue this process until we have partitioned A into its equivalence classes.

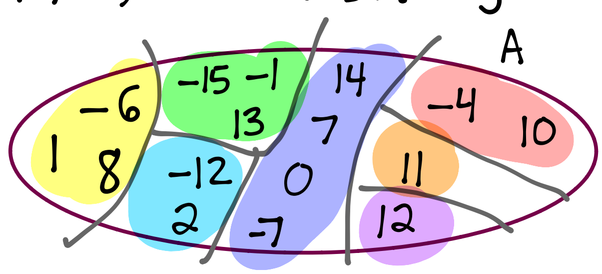
$[-15]_{\equiv(\text{mod } 7)} = \{-15, -1, 13\}$ $[-4]_{\equiv(\text{mod } 7)} = \{-4, 10\}$

$[-12]_{\equiv(\text{mod } 7)} = \{-12, 2\}$ $[11]_{\equiv(\text{mod } 7)} = \{11\}$

$[-7]_{\equiv(\text{mod } 7)} = \{-7, 0, 7, 14\}$ $[12]_{\equiv(\text{mod } 7)} = \{12\}$

$\therefore$ the partion of A into equiv. classes with respect to $\equiv(\text{mod } 7)$ is

$$\mathcal{P} = \{\{-6,1,8\}, \{-15,-1,13\}, \{-12,2\}, \{-7,0,7,14\}, \{-4,10\}, \{11\}, \{12\}\}$$



Q3. Partitions of a Set into Equivalence Classes

Let $A = \{-4, -3, -1, 0, 2, 5, 6, 7, 8, 15, 23\}$.

i. Determine the partition of A into equivalence classes of the equivalence relation $\equiv \pmod 7$.

	-4	-3	-1	0	2	5	6	7	8	15	23
remainder (mod7)	3	4	6	0	2	5	6	0	1	1	2

partition of A into equivalence classes (mod7) $\mathcal{P} = \{\{-4\}, \{-3\}, \{-1, 6\}, \{0, 7\}, \{2, 23\}, \{5\}, \{8, 15\}\}$

ii. Determine the partition of A into equivalence classes of the equivalence relation $\mathcal{S}$ given by the rule for all $x, y \in A$, $\quad x \,\mathcal{S}\, y \quad$ if and only if $\quad x + 3y$ is even.

$[-4]_{\mathcal{S}} = \{-4, 0, 2, 6, 8\}$ $\qquad$ $[-3]_{\mathcal{S}} = \{-3, -1, 5, 7, 15, 23\}$

partition of A into equivalence classes of $\mathcal{S}$ $\mathcal{P} = \{\{-4, 0, 2, 6, 8\}, \{-3, -1, 5, 7, 15, 23\}\}$

iii. Verify that the relation $\mathcal{T}$ given below is an equivalence relation on A.

$$\mathcal{T} = \Big\{(-4,-4), (-3,-3), (-1,-1), (0,0), (0,2), (0,5), (2,0), (2,2), (2,5), (5,0), (5,2), (5,5), (6,6), (6,7), (7,6), (7,7), (8,8), (15,15), (23,23)\Big\}$$

Give the equivalence classes $\quad [-4]_{\mathcal{T}} \quad$ and $\quad [5]_{\mathcal{T}}$.

$[-4]_{\mathcal{T}} = \{-4\}$ $\qquad$ $[5]_{\mathcal{T}} = \{5, 0, 2\}$

$[-3]_{\mathcal{T}} = \{-3\}$ $\quad$ $[-1]_{\mathcal{T}} = \{-1\}$ $\quad$ $[6]_{\mathcal{T}} = \{6, 7\}$ $\quad$ $[8]_{\mathcal{T}} = \{8\}$ $\quad$ $[15]_{\mathcal{T}} = \{15\}$ $\quad$ $[23]_{\mathcal{T}} = \{23\}$

Determine the partition of A into equivalence classes of $\mathcal{T}$.

$\mathcal{P} = \{\{-4\}, \{-3\}, \{-1\}, \{0, 2, 5\}, \{6, 7\}, \{8\}, \{15\}, \{23\}\}$

Q4. Properties of Relations

Let $\mathcal{R}$ be a relation on $\mathbb{Z}$ defined by

$$(r, s) \in \mathcal{R} \iff s = 2r + 1$$

Is $\mathcal{R}$ ... reflexive? ... symmetric? ... antisymmetric? ... transitive?
Justify your answers either by giving a proof, or providing a counterexample.

- $\mathcal{R}$ is <u>not</u> reflexive.
  Counterex: $1 \in \mathbb{Z}$ but $1 \not\mathcal{R} 1$ since $1 \neq 2(1)+1$.

- $\mathcal{R}$ is <u>not</u> symmetric.
  Counterexample: $1, 3 \in \mathbb{Z}$ $3 \mathcal{R} 1$ since $3 = 2(1)+1$
  but $1 \not\mathcal{R} 3$ since $1 \neq 2(3)+1$

- $\mathcal{R}$ is antisymmetric.
  proof. Let $x, y \in \mathbb{Z}$. Assume $x \mathcal{R} y$ and $y \mathcal{R} z$ (goal: prove $x=y$)
  Then $\underbrace{x = 2y+1}_{①}$ and $\underbrace{y = 2x+1}_{②}$ (by $\mathcal{R}$'s rule)

  So $x = 2(2x+1)+1$ (by plugging in ② to ①)
  $\Rightarrow x = 4x+3$
  $\Rightarrow -3x = 3$
  $\Rightarrow x = -1$ Consequently, $y = 2(-1)+1 = -1$

  so the only $x, y \in \mathbb{Z}$ such that $x \mathcal{R} y$ and $y \mathcal{R} x$ are $x = -1$ and $y = -1$.
  (In particular, $x = y$)

  $\therefore$ for all $x, y \in \mathbb{Z}$, $[x \mathcal{R} y \wedge y \mathcal{R} z] \rightarrow (x = y (= -1))$ so $\mathcal{R}$ is antisymmetric.

- $\mathcal{R}$ is not transitive
  Counterexample: $1, 3, 7 \in \mathbb{Z}$ $7 \mathcal{R} 3$ since $7 = 2(3)+1$
  and $3 \mathcal{R} 1$ since $3 = 2(1)+1$
  but $7 \not\mathcal{R} 1$ since $7 \neq 2(1)+1$.

Q5. Give an example of a relation on the set $A = \{1, 2, 3\}$

- that is both symmetric and antisymmetric:

$\mathcal{R} = \{(1,1)\}$ is symmetric and antisymmetric (there are other possible answers)

- that is neither symmetric nor antisymmetric:

$\mathcal{S} = \{(1,2),(2,1),(2,3)\}$ is neither symmetric, nor antisymmetric

(there are other possible answers)

Q5. Give an **indirect proof** of the following theorem:

Theorem. Let $\mathcal{R}$ be a relation on a set A.

If $\underbrace{\mathcal{R} \text{ is both symmetric and antisymmetric}}_{P}$, then, $\underbrace{\text{for all } a, b \in A \text{ such that } a \neq b,\ (a, b) \notin \mathcal{R}}_{Q}$.

$P \longrightarrow Q$

proof. Assume it is not the case that [for all $a,b \in A$ such that $a \neq b$, $(a,b) \notin \mathcal{R}$] (ie assume $\neg Q$).

(goal: prove it is not the case that $\mathcal{R}$ is both symmetric and antisymmetric). (ie prove $\neg P$).

Then, there must exist $a, b \in A$ such that $a \neq b$ and $(a,b) \in \mathcal{R}$. (that is what $\neg Q$ being true tells us).

We will consider the possibilities for (b,a): either $(b,a) \in \mathcal{R}$ or $(b,a) \notin \mathcal{R}$.

Case 1: Assume $(b,a) \in \mathcal{R}$. Then $\mathcal{R}$ is not antisymmetric as we have a counterexample: $(a,b) \in \mathcal{R}$ and $(b,a) \in \mathcal{R}$ but $a \neq b$.

Case 2: Assume $(b,a) \notin \mathcal{R}$. Then $\mathcal{R}$ is not symmetric as we have a counterexample: $(a,b) \in \mathcal{R}$ but $(b,a) \notin \mathcal{R}$

In both cases, $\mathcal{R}$ is not (symmetric and antisymmetric) (ie $\neg P$ is true).

We proved $\neg Q \rightarrow \neg P$ $\therefore$ $P \rightarrow Q$ is true ∎

Q6 - bonus. Use the Cantor Diagonalization Argument to prove $\mathbb{R}$ is **uncountable.**

We will prove that there is no bijection from $\mathbb{Z}^+$ to $(0,1)$ which will prove $(0,1)$ is uncountable.

this denotes the open interval of real numbers x such that $0<x<1$

proof (by contradiction)

Assume there exists a bijection $f: \mathbb{Z}^+ \to (0,1)$. Then we can count the numbers in the interval $(0,1)$, written in decimal form, as follows:

$f(1)$: $0.d_{11}d_{12}d_{13}d_{14}\cdots$

$f(2)$: $0.d_{21}d_{22}d_{23}d_{24}\cdots$

$f(3)$: $0.d_{31}d_{32}d_{33}d_{44}\cdots$

$\vdots$

Note: d_{ij} represents the jth digit of $f(i)$. In particular, $d_{ij} \in \{0,1,\dots,9\}$ for all $i,j \in \mathbb{Z}^+$.

This arrangement of real numbers in $(0,1)$ should include each $x \in (0,1)$ once as the image $f(n)$, for some $n \in \mathbb{Z}^+$, since f is a bijection.

Construct a number $x_* \in (0,1)$ as follows: Let $x_* = 0.x_1x_2x_3x_4\cdots$

where x_i denotes the ith digit of x_* and where we have chosen x_i to be any digit in $\{0,1,\dots,8\}$ other than d_{ii} (Note: d_{ii} is the ith digit of $f(i)$)

Consequently, there is no $i \in \mathbb{Z}^+$ such that $f(i) = x_*$ since the ith digit of x_* is different from the ith digit of $f(i)$ for all $i \in \mathbb{Z}^+$ ⚡ this contradicts the assumption that f was a bijection

$\therefore$ there is no bijection from $\mathbb{Z}^+$ to $(0,1)$ so $(0,1)$ is uncountable. ∎

we avoid 9's so we do not construct x_* to end in infinitely many 9's. If it did, then we could construct a number x_* that is already on our list. For example: $0.1999\ldots = 0.200\ldots$
We could construct $x_* = 0.1999\ldots$ but if $0.2000\ldots$ is already on the list, then there would be no contradiction... Avoiding 9's avoids this issue.

Since $|\mathbb{R}| \geq |(0,1)|$, we can conclude that $\mathbb{R}$ must also be uncountable.

Alternatively, we can show that $|\mathbb{R}| = |(0,1)|$ since the function $g: (0,1) \to \mathbb{R}$ defined by $g(x) = \tan(\pi(x - \frac{\pi}{2}))$ is a bijection from $(0,1)$ to $\mathbb{R}$.