Summer 2017 Final, answers

Calculus II (University of Ottawa)

Solution to Final Examination

MAT1322-3X, Summer 2017

This exam has three versions. The questions are the same. The only difference is the choices in multiple-choice questions.

V1: DFACEDBC
V2: EABDFECD
V3: CCDBAFEB

Part I. Multiple-choice Questions ($3 \times 8 = 24$ marks)

1. Let R be the region between the graph of $y = \sqrt{x}$ and the x-axis, $0 \le x \le 1$. The volume of the solid obtained by revolving R about the line $y = -1$ is

(A) $\frac{15}{8}\pi$; (B) $\frac{16}{3}\pi$; (C) $\frac{5}{4}\pi$; (D) $\frac{11}{6}\pi$; (E) $\frac{11}{3}\pi$; (F) $\frac{7}{6}\pi$.

Answer. (D) $r_{\text{inner}} = 1$, and $r_{\text{outer}} = 1 + \sqrt{x}$. The volume of the solid is

$$V = \pi\int_0^1\left((\sqrt{x}+1)^2 - 1^2\right)dx = \pi\int_0^1(x+2\sqrt{x})dx = \pi\left[\frac{1}{2}x^2 + \frac{4}{3}x^{3/2}\right]_{x=0}^1 = \frac{11}{6}\pi .$$

2. Suppose a vertical cylindrical tank filled with oil of density ρ kg/m^3 is buried underground so that the top is 3 meters under the ground surface. The diameter of the tank is 6 meters, and the height of the tank is 7 meters. Let x be the depth of a layer of oil, i.e., the distance between a layer of oil in the tank and the ground surface, and let g be the acceleration of gravity. Then the work, in Joules, needed to pump out all the oil in the tank to the ground surface is calculated by the integral

(A) $9\pi\rho g\int_0^{10} x^2dx$; (B) $9\pi\rho g\int_3^{10} x^2dx$; (C) $9\pi\rho g\int_0^{10} xdx$;
(D) $9\pi\rho g\int_0^{7} x^2dx$; (E) $9\pi\rho g\int_0^{7} xdx$; (F) $9\pi\rho g\int_3^{10} xdx$.

Solution. (F) A layer of oil at depth x with thickness dx has volume $V(x) = \frac{1}{4}\times 6^2\pi dx = 9\pi dx$. Its weight is $w(x) = \rho g V(x) = 9\pi\rho g dx$. The work needed to pump this layer of oil to the ground is $W(x) = w(x)x = 9\pi\rho g x dx$. The depth of the top layer is $x = 3$ and the depth of the bottom layer is $x = 3 + 7 = 10$. The total work needed is $9\pi\rho g\int_3^{10} xdx$.

3. Consider improper integral $\int_1^\infty \frac{2x-1}{\sqrt{x^4+x^3}}dx$. Which one of the following statement is true?

(A) Because $\dfrac{2x-1}{\sqrt{x^4+x^3}} > \dfrac{x}{\sqrt{2x^4}} = \dfrac{1}{\sqrt{2}x}$ when $x > 1$, and $\int_1^\infty \dfrac{1}{\sqrt{2}x}dx = \dfrac{1}{\sqrt{2}}\int_1^\infty \dfrac{1}{x}dx$ is divergent, this improper integral is divergent.

(B) Because $\dfrac{2x-1}{\sqrt{x^4+x^3}} > \dfrac{x}{\sqrt{2x^4}} = \dfrac{1}{\sqrt{2}x}$ when $x > 1$, and $\int_1^\infty \dfrac{1}{\sqrt{2}x}dx = \dfrac{1}{\sqrt{2}}\int_1^\infty \dfrac{1}{x}dx$ is convergent, this improper integral is convergent.

(C) Because $\dfrac{2x-1}{\sqrt{x^4+x^3}} > \dfrac{x}{\sqrt{2x^4}} = \dfrac{1}{\sqrt{2}x}$ when $x > 1$, and $\int_1^\infty \dfrac{1}{\sqrt{2}x}dx = \dfrac{1}{\sqrt{2}}\int_1^\infty \dfrac{1}{x}dx$ is divergent, this improper integral is convergent.

(D) Because $\dfrac{2x-1}{\sqrt{x^4+x^3}} < \dfrac{2x}{\sqrt{x^4}} = \dfrac{2}{x}$ when $x > 1$, and $\int_1^\infty \dfrac{2}{x}dx = 2\int_1^\infty \dfrac{1}{x}dx$ is divergent, this improper integral is divergent.

(E) Because $\dfrac{2x-1}{\sqrt{x^4+x^3}} < \dfrac{2x}{\sqrt{x^4}} = \dfrac{2}{x}$ when $x > 1$, and $\int_1^\infty \dfrac{2}{x}dx = 2\int_1^\infty \dfrac{1}{x}dx$ is convergent, this improper integral is convergent.

(F) Because $\dfrac{2x-1}{\sqrt{x^4+x^3}} < \dfrac{2x}{\sqrt{x^4}} = \dfrac{2}{x}$ when $x > 1$, and $\int_1^\infty \dfrac{2}{x}dx = 2\int_1^\infty \dfrac{1}{x}dx$ is convergent, this improper integral is divergent.

Solution. (A)

4. Suppose Euler's method with step size $h = 0.25$ is used to find an approximation of $y(1.5)$, where $y(t)$ is the solution to the initial-value problem $y' = \sqrt{2y-t}$, $y(1) = 2$. Which one of the following is closest to the answer?

(A) 2.43; (B) 2.65; (C) 2.91; (D) 3.11; (E) 3.43; (F) 3.76.

Solution. (C)

i	t_i	y_i
0	1	2
1	1.25	$2 + 0.25 \times \sqrt{4-1} = 2.433$
2	1.5	$2.433 + 0.25 \times \sqrt{4.866-1.25} \approx 2.908$.

5. If $y = f(t)$ is the solution of the initial-value problem $y' = (1 + y)(5 - y)$, $y(0) = -2$. Solve this initial-value problem to find which one of the following values is closest to the value $y\left(\frac{1}{6}\right)$?

(A) -2.8; (B) -3.1; (C) -3.6; (D) -3.9; (E) -4.8; (F) -5.2.

Solution. (E) Separating variables, $\int \frac{dy}{(1+y)(5-y)} = \int dt$. Hence,

$\int \frac{dy}{(1+y)(5-y)} = \frac{1}{6}\int \left(\frac{1}{1+y} + \frac{1}{5-y}\right)dy = \frac{1}{6}\ln\left|\frac{1+y}{5-y}\right| = t + C$. Then $\frac{1+y}{5-y} = Ke^{6t}$, where $K =$ $\pm e^{6C} \neq 0$. By the initial condition, $K = -\frac{1}{7}$. $\frac{1+y}{5-y} = -\frac{1}{7}e^{6t}$, $y = \frac{5e^{6t}+7}{e^{6t}-7}$. $y\left(\frac{1}{6}\right) = \frac{5e+7}{e-7} \approx -4.8$.

6. $\sum_{n=0}^{\infty} \frac{3^n - (-1)^n}{2^{2n}} =$

(A) $\frac{24}{5}$; (B) $\frac{16}{7}$; (C) $\frac{24}{7}$; (D) $\frac{16}{5}$; (E) $\frac{63}{35}$; (F) $\frac{86}{35}$.

Solution. (D) $\sum_{n=0}^{\infty} \frac{3^n - (-1)^n}{2^{2n}} = \sum_{n=0}^{\infty} \frac{3^n}{2^{2n}} - \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} = \frac{1}{1-3/4} - \frac{1}{1+1/4} = 4 - \frac{4}{5} = \frac{16}{5}$.

7. The power series $\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{2^n n}$ is convergent in interval

(A) $-1 < x \le 3$; (B) $-3 < x \le 1$; (C) $-1 \le x \le 3$; (D) $-3 \le x < 1$;
(E) $-3 \le x \le 1$, (F) $-1 \le x < 3$.

Solution. (B) Use the ratio test.

Let $\lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{2^{n+1}(n+1)}\frac{2^n n}{(x+1)^n}\right| = \lim_{n\to\infty}\left|\frac{x+1}{2}\left(\frac{n}{n+1}\right)\right| = \left|\frac{x+1}{2}\right| < 1$. Then $|x + 1| < 2$, or $-2 < x + 1 < 2$, $-3 < x < 1$. The series in absolutely convergent in $(-3, 1)$ and is divergent when $x < -3$ or $x > 1$.

When $x = -3$, this series becomes $\sum_{n=0}^{\infty} \frac{1}{n}$, which is divergent.

When $x = 1$, this series becomes $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n}$, which is convergent by alternating series test.

Hence, this series is convergent in interval $-3 < x \le 1$.

8. Let $z = xe^{2x-y}$. Then, when $x = 1$ and $y = 2$, $\dfrac{\partial^2 z}{\partial x \partial y} =$

(A) -1; (B) 2; (C) -3; (D) 4; (E) 5; (F) -6.

Solution. (C) $z_x = e^{2x-y} + 2xe^{2x-y} = (1 + 2x)e^{2x-y}$. $z_{xy} = -(1 + 2x)e^{2x-y}$. When $x = 1$ and $y = 2$, $z_{xy} = -3$.

Part II. Long Answer Questions (26 marks)

1. (4 marks) Find the centroid of the region between the graph of $y = \dfrac{1}{x^2}$ and the x-axis, $1 \le x \le 2$.

Solution. Assume this region has the unit density $\rho = 1$. The mass is

$$m = \int_1^2 \frac{1}{x^2}\,dx = \frac{1}{2}.$$

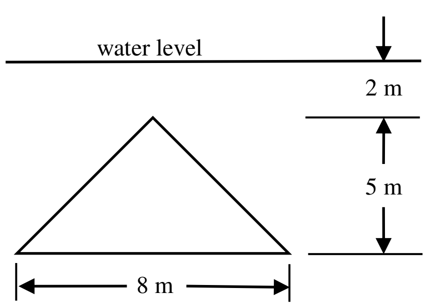
The moments are

$$M_x = \frac{1}{2}\int_1^2 \left(\frac{1}{x^2}\right)^2 dx = \frac{7}{48}.$$

$$M_y = \int_1^2 x\left(\frac{1}{x^2}\right)dx = \ln 2.$$

The centroid is at $(\bar{x}, \bar{y}) = \left(2\ln 2, \dfrac{7}{24}\right)$.

2. (4 marks) Suppose a triangular surface is vertically submerged into water (density $\rho = 1000$ kg / m^3) so that the top is 2 meters under the water surface as shown in the following figure:



Find the total force, in Newtons, acting on this surface. (Use $g = 9.81$ m /sec^2).

Solution. A stripe of the surface h meters under the water level with height dh has area $A(h) = \frac{8}{5}(h-2)\,dh$. The depth of this stripe is h. The pressure on this slice is $P(h) = \rho gh$. The force acting on this slice is $F(h) = P(h)A(h) = \frac{8}{5}\rho gh(h-2)\,dh$. The total force acting on this slice is $F = \frac{8}{5}\rho g\int_2^7 h(h-2)dx \approx 1.046 \times 10^6$ Newton.

Alternative solutions:

If you define x to be the distance between a stripe on the surface and the top of the triangle, the integral will be

$$\frac{8}{5}\rho g\int_0^5 x(x+2)dx\,.$$

If you define x to the distance between a stripe on the surface and the bottom of the triangle, then the integral will be

$$\frac{8}{5}\rho g\int_0^5 (5-x)(7-x)dx\,.$$

Of course, the final result will be the same.

3. (4 marks) Air with 0.05% carbon dioxide flows into a room with volume 200 m^3 at a rate 2 m^3 / min and well mixed air flows out at the same rate. Let $Q(t)$ be the amount of carbon dioxide, in m^3, in the room at time t.

(a) (2 marks) Construct a differential equation that is satisfied by the function $Q(t)$.

(b) (2 marks) Assume, at time $t = 0$, the air in the room contains 0.2% carbon dioxide. Solve this initial-value problem to find the function $Q(t)$.

Solution. (a) $\text{Rate}_{\text{in}} = 0.05\% \times 2 = 0.001$. $\text{Rate}_{\text{out}} = 2 \times Q(t) / 200 = 0.01Q(t)$.

The differential equation is

$Q'(t) = 0.001 - 0.01Q(t)$.

(b) $\displaystyle\int \frac{1}{0.001-0.01Q}dQ = -\frac{1}{0.01}\ln|0.001-0.01Q| = \int dt = t + C\,.$

$|0.001 - 0.01Q| = K_1e^{-0.01t}$, where $K_1 = e^{-0.01C} > 0$.

$0.001 - 0.01Q = Ke^{-0.01t}$, where $K = \pm K_1 \neq 0$.

At $t = 0$, $Q(0) = 200 \times 0.002 = 0.4$. $K = 0.001 - 0.004 = -0.003$.

$0.01Q = 0.001 + 0.003e^{-0.01t}$, $Q(t) = 0.1 + 0.3e^{-0.01t}$.

4. (6 marks) Use an appropriate test method to determine whether each of the following series is convergent or divergent.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$; (b) $\displaystyle\sum_{n=2}^{\infty} \frac{n-1}{n^2+n}$.

Solution. (a) Since the function $y(x) = \dfrac{1}{x(\ln x)^3}$ is positive, continuous, and decreasing when $x > 2$, we can use the integral test.

Since the improper integral $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty}\int_2^{b} \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty}\int_{\ln 2}^{\ln b} \frac{1}{u^3}\,du = \lim_{b\to\infty}\left[-\frac{1}{2u^2}\right]_{u=\ln 2}^{\ln b}$

$\displaystyle = \frac{1}{2}\lim_{b\to\infty}\left(\frac{1}{(\ln 2)^2} - \frac{1}{(\ln b)^2}\right) = \frac{1}{2(\ln 2)^2} < \infty$ converges, this series converges.

(b) Since this is a positive series, we can use the limit comparison test. Let $a_n = \dfrac{n-1}{n^2+n}$, and let $b_n = \dfrac{1}{n}$. $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n(n-1)}{n^2+n} = 1$. Since $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n}$ diverges, this series diverges.

5. (4 marks) Use the binomial series

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)...(k-n+1)}{n!}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + ...$$

to find the Maclaurin series of the function $F(x) = \displaystyle\int_0^x \sqrt[3]{1+t^2}\,dt$. (Give the first three non-zero terms).

Solution. When $k = \dfrac{1}{3}$, using the binomial series and substituting t^2 for x, we have

$$\sqrt[3]{1+t^2} = 1 + \frac{1}{3}t^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}t^4 + ... = 1 + \frac{1}{3}t^2 - \frac{1}{9}t^4 + ... \,.$$

Integrate this series term by term:

$$\int_0^x \sqrt[3]{1+t^2}\,dt = x + \frac{1}{9}x^3 - \frac{1}{45}x^5 + \dots .$$

6. (4 marks) Suppose a function $z = f(x, y)$ is defined implicitly by the equation $F(x, y, z) = 3$, where $F(x, y, z) = x^3 - y^2z + xyz^3$.

(a) (1 mark) Find the partial derivative z_x and z_y at the point $(1, 2, -1)$.

(b) (1 mark) Find the equation of the tangent plane of the graph of this equation at the point $(1, 2, -1)$.

(c) (1 mark) Find the directional derivative of this function at point $(1, 2, -1)$ in the direction of the vector $\mathbf{u} = \left(\frac{3}{5}, -\frac{4}{5}\right)$.

(d) (1 mark) What is the maximum value of the directional derivative at $(1, 2, -1)$ among all possible directions?

Solution. (a) $F_x = 3x^2 + yz^3$, $F_y = xz^3 - 2yz$, $F_z = 3xyz^2 - y^2$.

$z_x = -\dfrac{F_x}{F_z} = -\dfrac{yz^3 + 3x^2}{3xyz^2 - y^2}$, $z_y = -\dfrac{xz^3 - 2yz}{3xyz^2 - y^2}$. At the point $(1, 2, -1)$, $z_x = -\dfrac{1}{2}$, $z_y = -\dfrac{3}{2}$.

(b) The equation of the tangent plane of the graph of this equation at the point $(1, 2, -1)$ is

$z = -\dfrac{1}{2}(x-1) - \dfrac{3}{2}(y-2) - 1$, or $x + 3y + 2z = 5$.

(c) The directional derivative $z_{\mathbf{u}}(1, 2, -1) = \left(\dfrac{3}{5}, -\dfrac{4}{5}\right)\cdot\left(-\dfrac{1}{2}, -\dfrac{3}{2}\right) = \dfrac{9}{10}$.

(d) The maximum derivative is reached in the direction of the gradient vector, and the maximum derivative is the length of the gradient vector

$$\|\operatorname{grad} f(x, y)\|_{x=1,y=2,z=-1} = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{3}{2}\right)^2} = \frac{1}{2}\sqrt{10}.$$