Lesson 1 – The Derivative Function

PART A: The definition of the Derivative

In the previous unit, we studied the limiting value of the difference quotient $\lim_{h\to 0}\frac{\Delta y}{\Delta x} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

as the increment $h \to 0$. We interpreted this limit as both the *slope of the tangent line* to the curve $y = f(x)$ at the point $(a, f(a))$ and as the *rate of change* of *y* with respect to *x*.

The concepts we discussed in Unit 1 have laid the foundation for us to develop a sophisticated operation called *differentiation*, which is one of the **most fundamental and powerful operations of calculus**. The output of this operation is called the **derivative**, which can be used to calculate the slope of the tangent to any point in the functions' domain.

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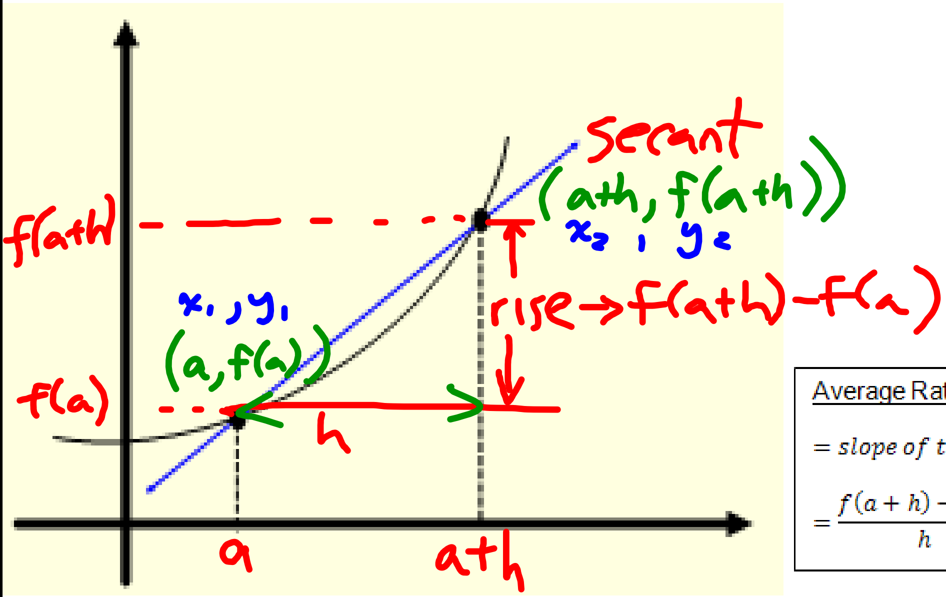

Average Rate of Change

$= slope\ of\ the\ secant\ line$

$= \frac{f(a+h) - f(a)}{h}$

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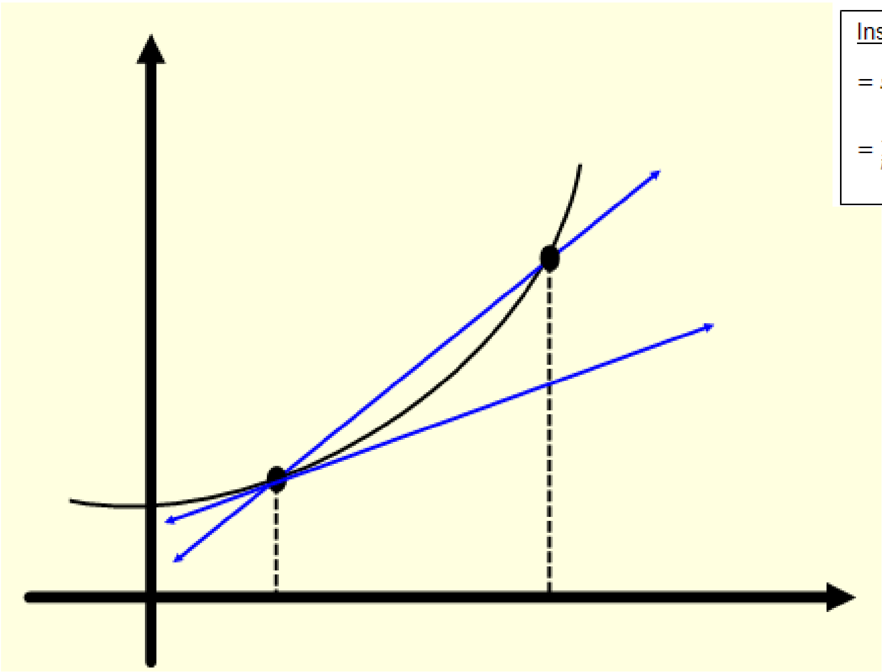

Instantaneous Rate of Change

$= slope\ of\ the\ tangent\ line$

$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

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The concept of instantaneous rate of change is so important to calculus it is defined as the derivative:

The definition of the **Derivative:**

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that this limit exists

Note:

- $f'(a)$ – is read as "f prime of a"
- This definition of the derivative is also referred to as the *First Principles Definition of the Derivative*

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Example 1: Determine the value of the derivative of $f(x) = x^2 + 1$ at $x = 3$.

$$f'(3) = \lim_{h \to 0} \frac{[(3+h)^2+1] - [3^2+1]}{h}$$

$$f'(3) = \lim_{h \to 0} \frac{9+6h+h^2+1-10}{h}$$

$$f'(3) = \lim_{h \to 0} \frac{6\cancel{h}+h^{\cancel{2}}}{\cancel{h}}$$

$$f'(3) = \lim_{h \to 0} 6+\cancel{h}^{0}$$

$$f'(3) = 6$$

Note: You can read this answer as
- f prime at three is equal to six
- the derivative of the function at x=3 is 6
- the instantaneous rate of change of the function at x=3 is 6
- the slope of the tangent at x=3 is 6

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PART B: The Definition of the Derivative Function

The definition of the derivative can be used to find the derivative at any point "a" on the domain of that function. However, a more efficient way of doing this would be very helpful, especially when we need to find the derivative at multiple points on one curve. To do this, we need to find the derivative function.

A function f has a derivative $f'(x)$ at every value of x provided the limit exists at that value and therefore:

The Definition of the Derivative Function

The derivative of $f(x)$ with respect to x is the function $f'(x)$, where:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ provided the limit exists.

Disclaimer: You might be having a case of déjà vu at this point, but rest assured, this is a different definition. This function allows us to determine the value of the derivative at any point on the function f (if the limit exists).

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Example 2: Determine a formula for the derivative function $f'(x)$ for $f(x) = x^2 + 1$.

$$f'(x) = \lim_{h\to 0} \frac{[(x+h)^2+1]-[x^2+1]}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2\cancel{+1}\cancel{-x^2}\cancel{-1}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2x\cancel{h}+h^{\cancel{2}}}{\cancel{h}}$$

$$f'(x) = \lim_{h\to 0} 2x+\cancel{h}^{\,0}$$

$$f'(x) = 2x$$

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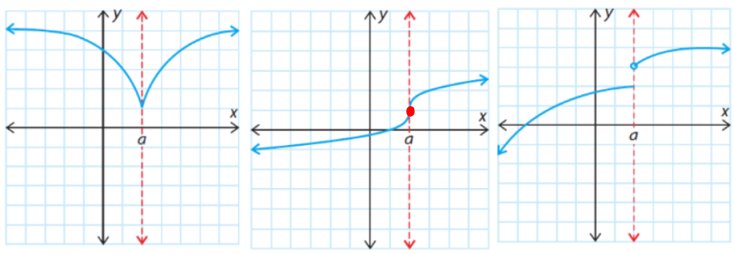
PART C: The Existence of Derivatives

A function is differentiable at "*a*" if $f'(a)$ exists. Three common examples of points where a function would not be differentiable are:



Cusp Vertical Tangent Discontinuity

Question:

Why do you think that the functions above would not be differentiable at the point "*a*"?

All 3 functions can't be differentiated at point "a"
- cusp/vertical tangent have an undefined slope at "a"
- discontinuity has an abrupt change in the rate of change at "a", so you can't determine derivative

Note: A function can be continuous at a point but not differentiable at that point, however, if a function is differentiable at a point you know that it is also continuous at that point.

continuity $\neq$ differentiability

differentiability = continuity

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PART D: Alternate notation

There is an alternative notation for the derivative function of f to the prime notation (f') that we have used in this lesson. The alternative notation was introduced by Leibniz and is thus fittingly called "Leibniz Notation":

> if $y = f(x)$, we write $\dfrac{dy}{dx}$ in place of $f'(x)$. This is read as "dee *y* by dee *x*".

This notation reminds us of the process by which the derivative was obtained – namely, as the limit of the difference quotient $\left(\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\right)$.

Note: the notation $\dfrac{dy}{dx}$ is not a fraction – it is a single symbol for the derivative of $y = f(x)$

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