Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Introduction to Linear Algebra MAT 1341A

Final Exam

December 13, 2023

You must **sign below** to confirm that you have read, understand, and will follow these **instructions**:

- This is a 3-hour **closed-book** exam; no notes are allowed. **Calculators are not permitted.**
- The exam consists of 9 questions on 13 pages. Page 13 provides additional work space. If you need additional space, you can use the backs of any of the pages. Indicate clearly where your answer can be found if it is not written down immediately after where the question is given.
- Questions 1–5 are short-answer questions. No justification is required for these questions.
- Questions 6–9 are long-answer questions. You must show all relevant steps and give appropriate justifications in order to obtain full marks.
- On Page 12, there is a bonus question. Only attempt it after you have finished all the other questions.
- Cellular phones, smart watches, unauthorized electronic devices, or course notes are **not** allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

  By signing below, you acknowledge that you have ensured that you are complying with the above statement.

LAST NAME:________________

First name:________________

Student number:________________

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Q. 1	Q. 2	Q. 3	Q. 4	Q. 5	Q. 6	Q. 7	Q. 8	Q. 9	Bonus	Total
$\overline{}$ 16	$\overline{}$ 5	$\overline{}$ 5	$\overline{}$ 5	$\overline{}$ 16	$\overline{}$ 12	$\overline{}$ 15	$\overline{}$ 15	$\overline{}$ 11	$\overline{}$ 5	$\overline{}$ 100

(1) C1. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set in a vector space, then $\mathbf{u} \in$Span$\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}\}$.

C2. Let A be an $n \times n$ matrix. If the columns of A form an orthogonal set, then rank$(A) = n$.

C3. If A is a matrix such that the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a unique solution, then the columns of A are linearly dependent.

C4. The set of functions on $\mathbb{R}$ $\{1, \sin^2 x, \cos^2 x\}$ is linearly dependent.

C5. Let $\mathbf{u}$ be an element of Null(A) for some matrix A, then $\mathbf{u}$ is orthogonal to every row of A.

C6. If A is a 2×2 matrix such that $\det(A) = 3$, then $\det(2A) = 6$.

C7. If A is an invertible square matrix, then A is diagonalizable.

C8. The matrix $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable.

In each box below, circle T if the statement is true or F if the statement if alse. [16]

C1	C2	C3	C4	C5	C6	C7	C8
T/F	T/F	T/F	T/F	T/F	T/F	T/F	T/F

(2) Let $V = \mathbb{P}_2$ be the vector space of polynomials of degree at most 2. Which of the following sets are subspaces of V? There may be more than one correct answer.

A. $\{f \in \mathbb{P}_2 : f(-2)f(2) = 0\}$

B. $\{f \in \mathbb{P}_2 : f(0) = 0\}$

C. $\{f \in \mathbb{P}_2 : f(1) = f(0)\}$

D. $\{f \in \mathbb{P}_2 : f(1) = -1\}$

E. $\text{Span}\{X^3, 1 + X\}$

Answer: B, C [5]

(3) Let V be a vector space, and let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be elements of V. Which of the following statements are equivalent to saying that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent? There may be more than one correct answer.

A. If $a = b = c = 0$, then $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$.

B. If $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$, then $a = b = c = 0$.

C. The subspace $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is of dimension 3.

D. None of the three elements $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is a linear combination of the other two elements in the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

E. None of the three elements $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is a scalar mulitple of another element in the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

Answer: B, C, D [5]

(4) Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$. Which of the following statements are correct? There may be more than one correct answer.

A. A is not invertible.

B. The first row of A^{-1} is $(-1, 0, 1)$.

C. The second row of A^{-1} is $(0, 2, 1)$.

D. The third row of A^{-1} is $(1, 1, -1)$.

E. The second column of A^{-1} is $(0, -2, 1)$.

Answer: $\boxed{\text{B , D, E}}$ [5]

(5) (a) What is the dimension of $U = \{A \in M_{3,3} : A^T = -A\}$? $\dim U = \boxed{3}$ [2]

(b) Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $B = \begin{bmatrix} 4g & a & d-2a \\ 4h & b & e-2b \\ 4i & c & f-2c \end{bmatrix}$. If $\det(A) = 3$, what is the value of $\det(B)$? $\det(B) = \boxed{12}$ [2]

(c) Let A be a 25×23 matrix. If the set of solutions to $A\mathbf{x} = \mathbf{0}$ has 5 free variables, what is the rank of A? rank$(A) = \boxed{18}$ [2]

(d) Let $\mathbf{u}_1 = (0, 3, 4)$, $\mathbf{u}_2 = (1, 0, 0)$ and $\mathbf{u}_3 = (0, 4, -3)$. Find $a, b, c \in \mathbb{R}$ such that $(0, -1, -1) = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$.

$a = \boxed{\frac{-7}{25}}$ $b = \boxed{0}$ $c = \boxed{\frac{-1}{25}}$ [6]

(e) If $A = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 3 & -1 \\ 3 & 0 & x \end{bmatrix}$ is not invertible, what is the value of x? $x = \boxed{3}$ [2]

(f) If U is a subspace of $\mathbb{R}^{2023}$ such that $\dim(U) = 100$, what is the dimension of $U^\perp$?

$\dim(U^\perp) = \boxed{1923}$ [2]

(6) Consider the linear system

$$\begin{cases} x + y + kz = t \\ x + y + 2kz = 2t \\ x - y + 3kz = 3t \end{cases}$$

where x, y, z are unknowns and k, t are parameters.

(a) For what value(s) of k and t does the system have a unique solution? [4]

$$\left[\begin{array}{ccc|c} 1 & 1 & k & t \\ 1 & 1 & 2k & 2t \\ 1 & -1 & 3k & 3t \end{array}\right] \xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & k & t \\ 0 & 0 & k & t \\ 0 & -2 & 2k & 2t \end{array}\right]$$

$$\underset{\sim}{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & k & t \\ 0 & -2 & 2k & 2t \\ 0 & 0 & k & t \end{array}\right] \underset{\sim}{-\tfrac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & k & t \\ 0 & 1 & -k & -t \\ 0 & 0 & k & t \end{array}\right] \quad 2$$

unique sol $\iff$ matrix on the left is of rank 3

So $k \neq 0$ 1, $t \in \mathbb{R}$. 1

(b) For what value(s) of k and t does the system have infinitely solutions? [4]

Infinitely many solns $\iff$ both matrix on the left and the augmented matrix have rank 2. 2

$k = t = 0$
1 1

(c) For what value(s) of k and t does the system have no solution? [4]

No soln $\iff$ matrix on the left has rank 2 and the augmented matrix has rank 3. 2

$k = 0, t \neq 0$
1 1

(7) Let $\mathbf{u}_1 = (1,0,0,-1)$, $\mathbf{u}_2 = (1,-1,0,0)$, $\mathbf{u}_3 = (1,0,1,0)$ and $U =$Span$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

(a) Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis of U. [4]

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (2)$$

The matrix has rank 3, so LI. (2)
(a basis)

Alternatively, solve $c_1u_1 + c_2u_2 + c_3u_3 = 0$

(b) Find an orthogonal basis of U. [7]

$w_1 = (1,0,0,-1)$ (1)

$w_2 = u_2 - \text{proj}_{w_1} u_2$ (1)

$= (1,-1,0,0) - \frac{1}{2}(1,0,0,-1)$

$= (\frac{1}{2}, -1, 0, \frac{1}{2})$ (2)

Set $w_2' = (1,-2,0,1)$

$w_3 = u_3 - \text{proj}_{w_1} u_3 - \text{proj}_{w_2'} u_3$ (1)

$= (1,0,1,0) - \frac{1}{2}(1,0,0,-1) - \frac{1}{6}(1,-2,0,1)$

$= (\frac{1}{3}, \frac{1}{3}, 1, \frac{1}{3})$ (2)

Set $w_3' = (1,1,3,1)$

So $\{w_1, w_2', w_3'\}$ is an orthogonal basis.

(Continued on next page)

(c) Let $\mathbf{w} = (0, 1, 1, 1)$. Calculate the best approximation of $\mathbf{w}$ by a vector of U. [4]

$$\operatorname{proj}_U w = \operatorname{proj}_{w_1} w + \operatorname{proj}_{w_2'} w + \operatorname{proj}_{w_3'} w \quad (1)$$

$$= \frac{-1}{2}(1,0,0,-1) + \frac{-1}{6}(1,-2,0,1) + \frac{5}{12}(1,1,3,1) \quad (1)$$

$$= \left(\frac{-1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{3}{4}\right) \quad (2)$$

(8) Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 0 & 4 \end{bmatrix}$.

(a) Find the characteristic polynomial of A. [5]

$\text{char}A = \det(A - \lambda I)$ ①

$= \begin{bmatrix} 1-\lambda & 0 & -2 \\ 1 & 3-\lambda & 1 \\ 1 & 0 & 4-\lambda \end{bmatrix}$ ①

$= (3-\lambda) \det \begin{bmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{bmatrix}$ ①

$= (3-\lambda)\left[(\lambda-1)(\lambda-4)+2\right]$

$= -(\lambda-3)(\lambda^2-5\lambda+6)$

$= -(\lambda-3)^2(\lambda-2)$ ②

(b) Using the characteristic polynomial found in (a), explain why the eigenvalues of A are 2 and 3. [2]

Since the roots of char A are $2, 3$, they are the e-vals of A.

(Continued on next page)

(c) Find a basis for the eigenspace E_2. [3]

$E_2 = \mathrm{Null}(A - 2I)$ (1)

$= \mathrm{Null}\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ (1)

$x = -2z$, $y = z$, z free

$\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis (1)

(d) Find a basis for the eigenspace E_3. [3]

$E_3 = \mathrm{Null}(A - 3I) = \mathrm{Null}\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ (1)

$x = -z$, y and z free

So $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis.

(1) (1)

(e) Write down an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ (no justification is required). [2]

Answer: $P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(1) (1)

(9) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T(x, y) = (x + y, y - x, 2x - y)$$

(a) Find the standard matrix of T. [2]

$T(x,y) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ (or calculate $T(e_1), T(e_2)$)

1

so the standard matrix is $\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & -1 \end{bmatrix}$.

(b) Find a basis of $\mathrm{Ker}(T)$. [5]

$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ 2

So $\ker(T) = \{0\}$ 2

Basis $= \phi$ 1

(c) Find a basis of $\mathrm{Im}(T)$. [4]

The standard matrix has rank 2, and $\mathrm{Im}(T) =$ column space of standard matrix 2

So $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis 2

Bonus: Let A be an $n \times n$ matrix. Let $\lambda_1, \lambda_2, \lambda_3$ be three distinct eigenvalues of A. Let $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ be elements of the eigenspaces E_{λ_1}, E_{λ_2} and E_{λ_3}, respectively. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent. [5]

Let $c_1, c_2, c_3 \in \mathbb{R}$ s.t. $c_1u_1 + c_2u_2 + c_3u_3 = 0$ (1)

Multiply by A:

$c_1\lambda_1u_1 + c_2\lambda_2u_2 + c_3\lambda_3u_3 = 0$ (1)

Substitute by $c_1u_1 = -c_2u_2 - c_3u_3$:

$\lambda_1(-c_2u_2 - c_3u_3) + c_2\lambda_2u_2 + c_3\lambda_3u_3 = 0$ (1)

$(\lambda_2 - \lambda_1)c_2u_2 + (\lambda_3 - \lambda_1)c_3u_3 = 0$

Multiply by A:

$(\lambda_2 - \lambda_1)c_2\lambda_2u_2 + (\lambda_3 - \lambda_1)c_3\lambda_3u_3 = 0$

Substitute by $(\lambda_2 - \lambda_1)c_2u_2 = -(\lambda_3 - \lambda_1)c_3u_3$ (1)

$-\lambda_2(\lambda_3 - \lambda_1)c_3u_3 + (\lambda_3 - \lambda_1)c_3\lambda_3u_3 = 0$

$(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)c_3u_3 = 0$

$\lambda_1 \neq \lambda_3$, $\lambda_2 \neq \lambda_3$, $u_3 \neq 0$, so $c_3 = 0$ (1)

Similarly for c_1, c_2.