MAT 1348 – Winter 2023

Exercises 6 – Solutions

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Questions are taken from Discrete Mathematics 8th edition, by Kenneth H. Rosen.

QUESTION 1 **(9.1 # 1)**. List the pairs in the following relations from $A = \{0,1,2,3,4\}$ to $B = \{0,1,2,3\}$.

(a) $(a,b) \in R$ if and only if $a = b$.
(b) $(a,b) \in R$ if and only if $a+b = 4$.
(c) $(a,b) \in R$ if and only if $a > b$.
(d) $(a,b) \in R$ if and only ig $a|b$.
(e) $(a,b) \in R$ if and only if a and b do not have any common factors
(f) $(a,b) \in R$ if and only if the least common multiple of a and b is 2.

Solution:

(a) $R = \{(0,0),(1,1),(2,2),(3,3)\}$
(b) $R = \{(1,3),(2,2),(3,1),(4,0)\}$.
(c) $R = \{(1,0),(2,0),(2,1),(3,0),(3,1),(3,2),(4,0),(4,1),(4,2),(4,3)\}$
(d) $R = \{(1,0),(1,1),(1,2),(1,3),(2,0),(2,2),(3,0),(3,3),(4,0)\}$
(e) $R = \{(0,1),(1,0),(1,1),(1,2),(1,3),(2,1),(2,3),(3,1),(3,2),(4,1),(4,3)\}$
(f) $R = \{(1,2),(2,1),(2,2)\}$

QUESTION 2 **(9.1 # 3)**. For each of the following relations R on $\{1,2,3,4\}$, determine if it is reflexive, symmetric, antisymmetric and transitive.

(a) $\{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$.
(b) $\{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\}$.
(c) $\{(2,4),(4,2)\}$.
(d) $\{(1,2),(2,3),(3,4)\}$.
(e) $\{(1,1),(2,2),(3,3),(4,4)\}$.
(f) $\{(1,3),(1,4),(2,3),(2,4),(3,1),(3,4)\}$.

Solution:

(a) Transitive
(b) Reflexive, symmetric, transitive
(c) Symmetric
(d) Antisymmetric
(e) Reflexive, symmetric, antisymmetric, transitive
(f) None

QUESTION 3 **(9.1 # 5)**. Determine if the following relations R on the set of all webpages are reflexive, symmetric, antisymmetric and transitives. For two webpages a and b:

(a) $(a,b) \in R$ if and only if whoever visited page a also visited page b.
(b) $(a,b) \in R$ if and only if a and b do not have a link to a common webpage.
(c) $(a,b) \in R$ if and only if a and b have a link to a common webpage.
(d) $(a,b) \in R$ if and only if there exists a webpage with a link to a and a link to b.

Solution:

(a) Reflexive, transitive
(b) Symmetric
(c) Symmetric
(d) Symmetric

QUESTION 4 **(9.1 # 6)**. Determine if the following relations R on the set of real numbers are reflexive, symmetric, antisymmetric and transitive.

(a) $(x,y) \in R$ if and only if $x + y = 0$
(b) $(x,y) \in R$ if and only if $x = \pm y$
(c) $(x,y) \in R$ if and only if $x - y$ is rationnal.
(d) $(x,y) \in R$ if and only if $x = 2y$
(e) $(x,y) \in R$ if and only if $xy \geq 0$
(f) $(x,y) \in R$ if and only if $xy = 0$
(g) $(x,y) \in R$ if and only if $x = 1$
(h) $(x,y) \in R$ if and only if $x = 1$ or $y = 1$

Solution:

(a) Symmetric
(b) Reflexive, symmetric, transitive
(c) Reflexive, symmetric, transitive
(d) Antisymmetric
(e) Reflexive, symmetric
(f) Symmetric
(g) Antisymmetric, transitive
(h) Symmetric

QUESTION 5 **(9.1 # 7)**. Determine if the following relations R on the set of integers are reflexive, symmetric, antisymmetric and transitive.

(a) $(x,y) \in R$ if and only if $x \neq y$
(b) $(x,y) \in R$ if and only if $xy \geq 1$
(c) $(x,y) \in R$ if and only if $x = y+1$ or $x = y-1$
(d) $(x,y) \in R$ if and only if $x - y$ is divisible by 7
(e) $(x,y) \in R$ if and only if x is a multiple of y
(f) $(x,y) \in R$ if and only if x and y are both negative, or both positive.
(g) $(x,y) \in R$ if and only if $x = y^2$
(h) $(x,y) \in R$ if and only if $x \geq y^2$

Solution:

(a) Symmetric

(b) Symmetric, transitive
(c) Symmetric
(d) Reflexive, symmetric, transitive
(e) Reflexive, antisymmetric, transitive
(f) Reflexive, symmetric, transitive
(g) Antisymmetric
(h) Antisymmetric, transitive

QUESTION 6 **(9.1 # 8)**. Show that the relation $R = \emptyset$ on a non-empty set S is symmetric and transitive, but not reflexive.

Solution: $R = \emptyset$ is symmetric, since the hypothesis in the implication $(x,y) \in R \rightarrow (y,x) \in R$ is always false, hence the implication is true.

$R = \emptyset$ is transitive, since the hypothesis in $[(x,y) \in R \wedge (y,z) \in R] \rightarrow (x,z) \in R$ is always false, hence the implication is true.

Since S is not empty, let $s \in S$. We have that $(s,s) \notin R$, hence R is not reflexive.

QUESTION 7 **(9.1 # 9)**. Show that the relation $R = \emptyset$ on the empty set $S = \emptyset$ is reflexive, symmetric and transitive.

Solution: $R = \emptyset$ is reflexive, since it is true that all elements of $S = \emptyset$ are related to themselves. In other words, there is no element of S which is not in relation with itself.

$R = \emptyset$ is symmetric, since the hypothesis in the implication $(x,y) \in R \rightarrow (y,x) \in R$ is always false, hence the implication is true.

$R = \emptyset$ is transitive, since the hypothesis in $[(x,y) \in R \wedge (y,z) \in R] \rightarrow (x,z) \in R$ is always false, hence the implication is true.

QUESTION 8 **(9.1 # 10)**. Give an example of a relation which is

(a) Symmetric and antisymmetric.
(b) Neither symmetric nor antisymmetric.

Solution:

(a) $\{(0,0),(1,1)\}$ on the set $\{0,1\}$.
(b) $\{(0,1),(1,0),(2,3)\}$ on the set $\{0,1,2,3\}$.

QUESTION 9 **(9.1 # 44,45,46)**.

(a) List all 16 relations on the set $\{0,1\}$.
(b) How many relations on $\{0,1\}$ contain the pair $(0,1)$?
(c) Which of the relations found in (a) are reflexive?
(d) Which of the relations found in (a) are symmetric?
(e) Which of the relations found in (a) are antisymmetric?
(f) Which of the relations found in (a) are transitive??

Solution:

(a)
1. $\emptyset$
2. $\{(0,0)\}$
3. $\{(0,1)\}$
4. $\{(1,0)\}$
5. $\{(1,1)\}$
6. $\{(0,0),(0,1)\}$
7. $\{(0,0),(1,0)\}$
8. $\{(0,0),(1,1)\}$
9. $\{(0,1),(1,0)\}$
10. $\{(0,1),(1,1)\}$
11. $\{(1,0),(1,1)\}$
12. $\{(0,0),(0,1),(1,0)\}$
13. $\{(0,0),(0,1),(1,1)\}$
14. $\{(0,0),(1,0),(1,1)\}$
15. $\{(0,1),(1,0),(1,1)\}$
16. $\{(0,0),(0,1),(1,0),(1,1)\}$

(b) 8
(c) 8,13,14 and 16
(d) 1,2,5,8,9,12,15 and 16
(e) 1,2,3,4,5,6,7,8,10,11,13 and 14
(f) 1,2,3,4,5,6,7,8,10,11,13,14 and 16

QUESTION 10 **(9.5 # 1, #26)**. Which of the following relations on $\{0,1,2,3\}$ are equivalence relations? For the ones that are equivalence relations, list the equivalence classes.

(a) $\{(0,0),(1,1),(2,2),(3,3)\}$
(b) $\{(0,0),(0,2),(2,0),(2,2),(2,3),(3,2),(3,3)\}$
(c) $\{(0,0),(1,1),(1,2),(2,1),(2,2),(3,3)\}$
(d) $\{(0,0),(1,1),(1,3),(2,2),(2,3),(3,1),(3,2),(3,3)\}$
(e) $\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,2),(3,3)\}$

Solution:

(a) It is an equivalence relation and $\{0\},\{1\},\{2\}$ and $\{3\}$ are the equivalence classes
(b) This is not an equivalence relation since it is not reflexive and not transitive.
(c) It is an equivalence relation and $\{0\},\{1,2\},\{3\}$ are the equivalence classes.
(d) This is not an equivalence relation since it is not transitive.
(e) This is not an equivalence relation since it is not symmetric.

QUESTION 11 **(9.5 # 3)**. Which of the following relations on the set of functions from $\mathbb{Z}$ to $\mathbb{Z}$ are equivalence relations?

(a) $\{(f,g) \mid f(1) = g(1)\}$
(b) $\{(f,g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$
(c) $\{(f,g) \mid f(x) - g(x) = 1 \text{ for all } x \in \mathbb{Z}\}$
(d) $\{(f,g) \mid \text{There exists } C \in \mathbb{Z}, \text{for all } x \in \mathbb{Z},\ f(x) - g(x) = C\}$
(e) $\{(f,g) \mid f(0) = g(1) \text{ and } f(1) = g(0)\}$

Solution:

(a) Equivalence relation.
(b) Not transitive.
(c) Not reflexive, not symmetric and not transitive.
(d) Equivalence relation.
(e) Not reflexive, not transitive.

QUESTION 12 **(9.5 # 7)**. Show that the logical equivalence relation is a an equivalence relation on the set of all propositions. What are the equivalence classes of T and F?

Solution: The proposition P is logically equivalent to the proposition Q if their truth tables are identical. This relation is reflexive: the truth table of P is identical to the truth table of P. This relation is symmetric: if P has the same truth table as Q, then Q has the same truth table as P. This relation is transitive: if P and Q have the same truth table, and Q and R as well, then P and R have the same truth table.

The equivalence class of T is the set of all tautologies. The equivalence class of F is the set of all contradictions.

QUESTION 13 **(9.5 # 11)**. Let X be the set of all sequences of 0's and 1's of length at least 3. Show that the following relation R is an equivalence relation on X. We have xRy if and only if the first three digits of x and y are the same.

Solution: Since x has the same first three digits as itself, we have xRx. Therefore, R is reflexive. Suppose now that xRy. So, x has the same first three digits as y, and so y has the same first three digits as x. We conclude yRx, and so the relation is symmetric. Suppose now that xRy and yRz. Then, x has the same first three digits as y and y has the same first three digits as z. We conclude that x has the same first three digits as z, hence xRz. We conclude that this relation is transitive, and so it is an equivalence relation.

QUESTION 14 **(9.5 # 15)**. Let R be the following relation on pairs of positive integers: $(a,b)R(c,d)$ if and only if $a+d = b+c$. Show that R is an equivalence relation.

Solution: To show it is reflexive, we notice that $(a,b)R(a,b)$ since $a+b = b+a$. To show it is symmetric, assume $(a,b)R(c,d)$, Then $a+d = b+c$ and so $c+b = d+a$, which implies $(c,d)R(a,b)$. For transitivity, assume $(a,b)R(c,d)$ and $(c,d)R(e,f)$. So, $a+d = b+c$ and $c+f = d+e$. By adding the two equations, we get $a+d+c+f = b+c+d+e$ which simplifies as $a+f = b+e$, hence $(a,b)R(e,f)$.

QUESTION 15 **(9.5 # 25, # 29)**. Let X be the set of all sequences of 0's and 1's. Show that the following relation on X is an equivalence relation: xRy if and only if x and y contain the same number of 1's. What are the equivalence classes?

Solution: This relation is reflexive: x has the same number of 1's as itself. This relation is symmetric: if x has the same number of 1's as y, then y has the same number of 1's as x. This relation is transitive: if x and y have the same number of 1's, and y and z also have the same number of 1's, then x,y and z all have the same number of 1's.

The equivalence classes are $X_i = \{x \in X \mid x \text{ has } i \text{ 1's}\}$ for any i.

QUESTION 16 **(9.5 # 55)**. Find the smallest equivalence relatoin on $\{a,b,c,d,e\}$ which contains $\{(a,b),(a,c),(d,e)\}$.

Solution: $\{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c),(d,d),(d,e),(e,d),(e,e)\}$

QUESTION 17 **(9.5 # 41)**. Which of the following sets are partitions of $\{1,2,3,4,5,6\}$?

(a) $\{\{1,2\},\{2,3,4\},\{4,5,6\}\}$
(b) $\{\{1\},\{2,3,6\},\{4\},\{5\}\}$
(c) $\{\{2,4,6\},\{1,3,5\}\}$
(d) $\{\{1,4,5\},\{2,6\}\}$

Solution: b and c

QUESTION 18 **(9.5 # 47)**. List the pairs in the equivalence relations on $\{0,1,2,3,4,5\}$ obtained from the following partitions.

(a) $\{\{0\},\{1,2\},\{3,4,5\}\}$
(b) $\{\{0,1\},\{2,3\},\{4,5\}\}$
(c) $\{\{0,1,2\},\{3,4,5\}\}$
(d) $\{\{0\},\{1\},\{2\},\{3\},\{4\},\{5\}\}$

Solution:

(a) $\{(0,0),(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(3,5),(4,3),(4,4),(4,5),(5,3),(5,4),(5,5)\}$
(b) $\{(0,0),(0,1),(1,0),(1,1),(2,2),(2,3),(3,2),(3,3),(4,4),(4,5),(5,4),(5,5)\}$
(c) $\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2),(3,3),(3,4),(3,5),(4,3),(4,4),(4,5),(5,3),(5,4),(5,5)\}$
(d) $\{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)\}$