MAT 1348 – Winter 2023

Exercises 2 – Solutions

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Questions are taken from Discrete Mathematics 8th edition, by Kenneth H. Rosen.

QUESTION 1 **(1.2 # 3)**. Consider the following variables:

g = "You graduate"
m = "You owe money to the university"
r = "You passed all the required classes"
b = "You have books that must be returned to the library"

Use the previous variables to translate the following into logic.

"You graduate only if you passsed all the required classes, you don't owe money to the university and you don't have books that must be returned to the library"

Solution: $g \rightarrow (r \wedge \neg m \wedge \neg b)$

QUESTION 2 **(1.2 # 7)**. Consider the following variables:

p = "The message is verified by the antivirus software" q = "The message was sent by an unknown system"

Use the previous variables to translate the following into logic.

(a) "The message is verified by the antivirus software when the message is sent by an unknown system."
(b) "The message was sent by an unknown system, but it was not verified by the antivirus software."
(c) "It is necessary that the message is verified by the antivirus software when it is sent by an unknown system."
(d) "If the message is not sent by an unknown system, then it is not verified by the antivirus software."

Solution:

(a) $q \rightarrow p$
(b) $q \wedge \neg p$
(c) $q \rightarrow p$
(d) $\neg q \rightarrow \neg p$

QUESTION 3 **(1.2 # 23, # 25, # 27)**. On the island of knights and knaves, you meet two inhabitants, A et B. A knight always tells the truth, and a knave always lies. In each of the following situations, determine the identities of A and B.

(a) A says "At least one of us is a knave". B says nothing.
(b) A says "I am a knave or B is a knight". B says nothing.
(c) A says "We are both knaves". B says nothing.

Solution:

(a) A is a knight and B is a knave.
(b) A and B are knights.
(c) A is a knave and B is a knight.

QUESTION 4 **(1.3 # 1)**. Use truth tables to prove the following logical equivalences:

(a) $p \wedge T \equiv p$
(b) $p \vee F \equiv p$
(c) $p \wedge F \equiv F$
(d) $p \vee T \equiv T$
(e) $p \vee p \equiv p$
(f) $p \wedge p \equiv p$

Solution: The following table shows all 6 equivalences:

p	$p \wedge T$	$p \vee F$	$p \wedge F$	$p \vee T$	$p \vee p$	$p \wedge p$
T	T	T	F	T	T	T
F	F	F	F	T	F	F

QUESTION 5 **(1.3 # 3)**. Use truth tables to verify the commutativity rules

(a) $p \vee q \equiv q \vee p$
(b) $p \wedge q \equiv q \wedge p$

Solution:

(a)

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

(b)

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

QUESTION 6 **(1.3 # 5)**. Use truth tables to determine the distributivity laws

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

Solution:

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

QUESTION 7 **(1.3 # 35)**. Show $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent.

Solution: When p, q and r are all false, then $(p \to q) \to r$ is false, but $p \to (q \to r)$ is true.

QUESTION 8 **(1.3 # 37)**. Show $(p \to q) \to (r \to s)$ and $(p \to r) \to (q \to s)$ are not logically equivalent.

Solution: When q is true and p, r and s are false, then $(p \to q) \to (r \to s)$ will be true, but $(p \to r) \to (q \to s)$ will be false.

QUESTION 9 **(1.3 # 7)**. Use De Morgan's law to find the negation of the following statements:

(a) Jan is rich an happy.
(b) Carlos will take his bike or run tomorrow.
(c) Mei takes the bus or walks to get to school.
(d) Ibrahim is smart and hard-working.

Solution:

(a) Jan is not rich or Jan is not happy.
(b) Carlos will not take his bike and will not run tomorrow.
(c) Mei does not take the bus and does not walk to get to school.
(d) Ibrahim is not smart or he is not hard-working.

QUESTION 10 **(1.3 #15)**. Use the laws of logic to show that the following compound propositions are tautologies (Do not use a truth table):

(a) $(p \wedge q) \to p$
(b) $p \to (p \vee q)$
(c) $\neg p \to (p \to q)$
(d) $(p \wedge q) \to (p \to q)$
(e) $\neg(p \to q) \to p$
(f) $\neg(p \to q) \to \neg q$

Solution:

(a) $p \wedge q \to p \equiv \neg(p \wedge q) \vee p \equiv \neg p \vee \neg q \vee p \equiv (p \vee \neg p) \vee \neg q \equiv T \vee \neg q \equiv T$
(b) $p \to (p \vee q) \equiv \neg p \vee (p \vee q) \equiv (p \vee \neg p) \vee q \equiv T \vee q \equiv T$
(c) $\neg p \to (p \to q) \equiv p \vee (p \to q) \equiv p \vee (\neg p \vee q) \equiv (p \vee \neg p) \vee q \equiv T \vee q \equiv T$
(d) $(p \wedge q) \to (p \to q) \equiv \neg(p \wedge q) \vee (\neg p \vee q) \equiv \neg p \vee \neg q \vee \neg p \vee q \equiv (\neg p \vee \neg p) \vee (\neg q \vee q) \equiv \neg p \vee T \equiv T$
(e) $\neg(p \to q) \to p \equiv (p \to q) \vee p \equiv \neg p \vee q \vee p \equiv (\neg p \vee p) \vee q \equiv T \vee q \equiv T$
(f) $\neg(p \to q) \to \neg q \equiv (p \to q) \vee \neg q \equiv \neg p \vee q \vee \neg q \equiv \neg p \vee (q \vee \neg q) \equiv \neg p \vee T \equiv T$

QUESTION 11 (). On the island of knights and knaves, you meet two inhabitants: A and B. A says "I am a knave or B is a knight". What are the identities of A and B?

Solution: Let $a =$ "A is a knight" and $b =$ "B is a knight", then A says $\neg a \vee b$.

a	b	$\neg a \vee b$
T	T	T
T	F	F
F	T	T
F	F	T

The only row where a has the same truth value as $\neg a \vee b$ is the first, where a and b are true. We conclude that A and B are knights.

QUESTION 12 (). On the island of knights and knaves, you meet three inhabitants: A, B and C. A says "We are all knaves". B says "Exactly one of us is a knave". What are the identities of each inhabitant?

Solution: Let $a =$ "A is a knight", $b =$ "B is a knight" and $c =$ "C is a knight". A says $P = \neg a \wedge \neg b \wedge \neg c$. B says $Q = (\neg a \wedge b \wedge c) \vee (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c)$.

a	b	c	P	Q
T	T	T	F	F
T	T	F	F	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	F
F	F	F	T	F

The 5th and 7th row have the property that a has the same truth value as P and b has the same truth value as Q. The two possible outcomes are $a = F, b = c = T$ or $a = b = F, c = T$. We conclude that A is a knave, C is a knight, and that we cannot determine the identity of B.

QUESTION 13 (). On the island of knights and knaves, you meet three inhabitants: A, B and C. A says "B is a knave". B says "A and C are of the same type (two knights or two knaves)". What are the identities of each inhabitant?

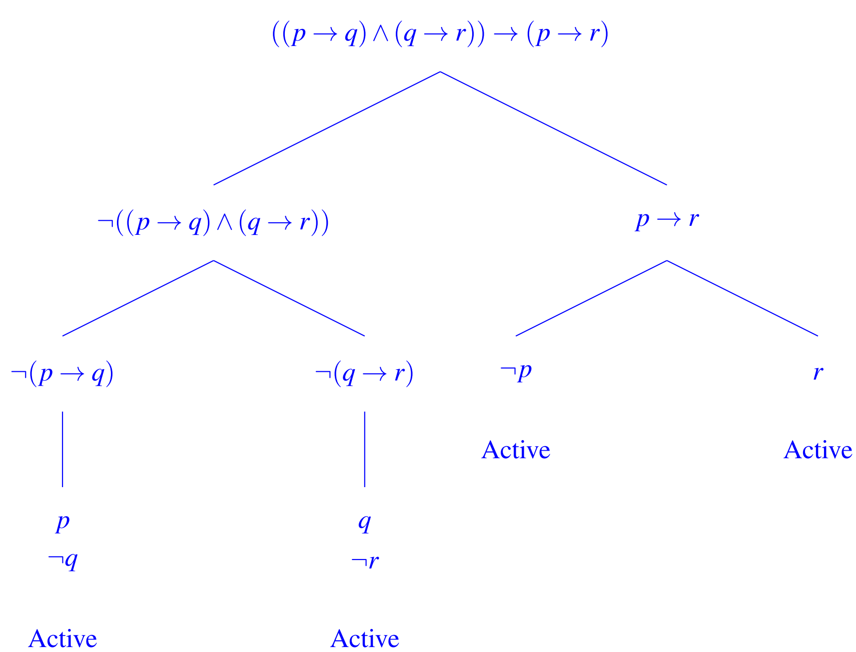
Solution: Let $a =$ "A is a knight", $b =$ "B is a knight" and $c =$ "C is a knight". A says $\neg b$, and B says $a \leftrightarrow c$.

a	b	c	$\neg b$	$a \leftrightarrow c$
T	T	T	F	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	F
F	T	T	F	F
F	T	F	F	T
F	F	T	T	F
F	F	F	T	T

The 4th and 6th rows have the property that a has the same truth value as $\neg b$ and b has the same truth value as $a \leftrightarrow c$. The two possible outcomes are $a = T, b = c = F$ or $a = c = F, b = T$. Therefore, the only conclusion we can draw is that C is a knave.

QUESTION 14 (). Build the truth tree for $X = ((p \to q) \wedge (q \to r)) \to (p \to r)$ and for $\neg X$. What can you conclude about X and $\neg X$?

Solution: For X:



For $\neg X$

$$\neg(((p \to q) \wedge (q \to r)) \to (p \to r))$$

$$(p \to q) \wedge (q \to r)$$
$$\neg(p \to r)$$

$$p \to q$$
$$q \to r$$

$$p$$
$$\neg r$$

$\neg p$ q

Inactive

$\neg q$ r

Inactive   Inactive

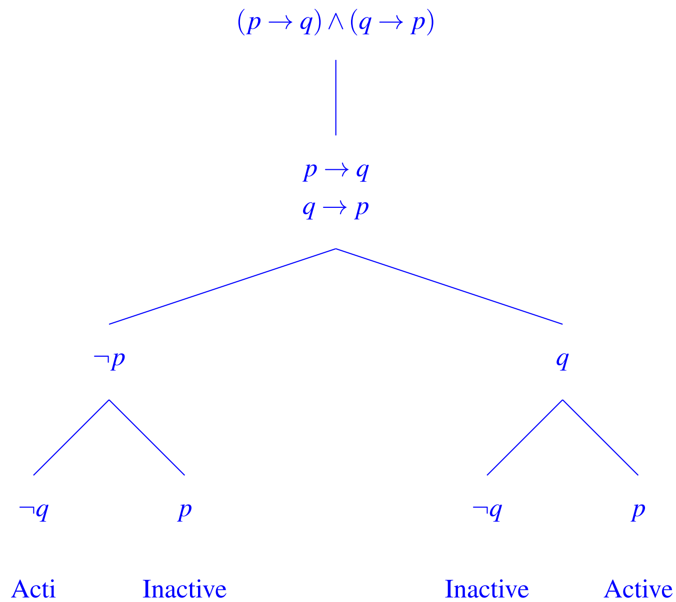
We find that all branches of $\neg X$ are inactive: we conclude that $\neg X$ is a contradiction, and so X must be a tautology.

QUESTION 15 (). Build the truth tree for each of the following propositions. Use the tree to find a disjunctive normal form.

(a) $(p \to q) \wedge (q \to p)$
(b) $(p \to r) \vee (q \to r)$
(c) $(p \to r) \to (q \to s)$

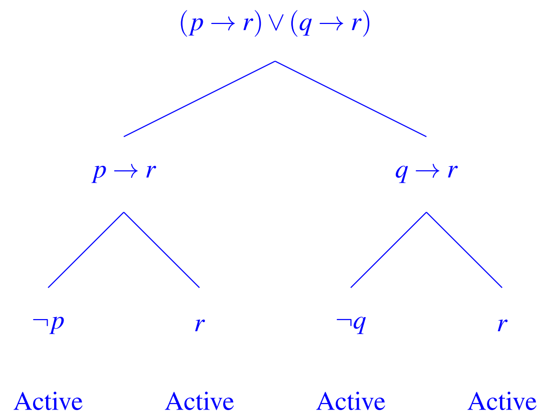
Solution:

(a) The truth tree is



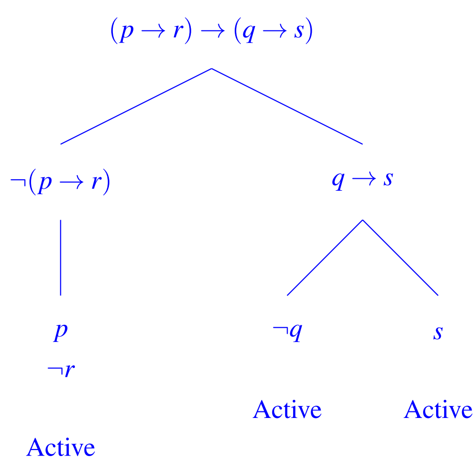
The first active branch gives the conjunction $\neg q\wedge\neg p$, the second active branch gives $p\wedge q$. We therefore have the following disjunctive normal form: $(\neg q\wedge \not p)\vee(p\wedge q)$.

(b) The truth tree is



The active branches give the following conjunctive clauses: $\neg p$, r, $\neg q$ and r. Since we have the same clause twice, r, we do not need to include it twice in the disjunctive normal form. The disjunctive normal form is therefore $\neg p\vee r\vee\neg q$.

(c) The truth tree is



The active branches give the following conjunctive clauses: $\neg r \wedge p$, $\neg q$ and s. The disjunctive normal form is therefore: $(\neg r \wedge p) \vee \neg q \vee s$.

QUESTION 16 (). Find a normal disjunctive form and a normal conjunctive form for the following proposition X, whose truth table is

p	q	r	X
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	T

Solution: A normal disjunctive form is:

$$X \equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

A normal conjunctive form is:

$$X \equiv (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (p \vee \neg q \vee r)$$