5. Derivatives: The Definition

Lec 4 mini review.

The Squeeze Theorem:
If $\boxed{f(x) \leq g(x) \leq h(x)}$, and
$\boxed{\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)}$,
then $\boxed{\lim_{x \to a} g(x) = L}$.

Limit Law for Continuous Composite Functions

Limits at Infinity

Continuous at a point a: $\boxed{\lim_{x \to a} f(x) = f(a)}$

Discontinous at a (hole, jump, asymptote)

One-sided Continuity

Continuous on an interval $[a, b]$

Intermediate Value Theorem for a continuous function on $[a, b]$

Horizontal Asymptotes

THE DEFINITION OF THE DERIVATIVE AT A POINT — SLOPE OF A TANGENT

Recall, for $h > 0$, the **slope of the secant** passing through the points $(a, f(a))$ and $(a + h, f(a + h))$ is

Our goal for developing skills with limits was to take the limit as $h \to 0$.

Provided the limit exists, the **derivative of a function** f **at a number** a, denoted $f'(a)$, is

* These notes are solely for the personal use of students registered in MAT1320.

Example 5.1. Find the equation of the tangent line to the graph of $f(x) = \sqrt{x+3}$ at the point $x = 1$.

INSTANTANEOUS RATE OF CHANGE

Recall: If f is **continuous** on the interval $[a, b]$, then **average rate of change of** f **on** $[a, b]$ is

The **instantaneous rate of change of** f **at** $x = a$ is

Example 5.2. The height of a ball, t seconds after it is thrown with an initial upwards velocity of 10 m/s from the upper observation deck of the CN Tower (450 m above the ground), is given by

$$b(t) = -4.9t^2 + 10t + 450$$

What is the ball's instantaneous velocity

- when $t = 3$ s ?
- when the ball hits the ground?
- when $t = 1$ s ?
- at other times?
- If we want to calculate the derivative of a function at several different numbers, the calculations are all quite similar, but with different choices of a.
- Let's delay choosing a specific number a. Once we see the general pattern for $f'(a)$, we can plug in whichever a we want later!

The Derivative as a Function

The derivative of $f(x)$, denoted $f'(x)$, is the function

- The domain of $f'(x)$ is the set of all x in the domain of f, for which the above limit exists.
- The domain of f' can be smaller than the domain of f.
- As a function of x, we can regard the value of $f'(x)$ geometrically as the slope of the tangent line of f at x

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Example 5.3. Using the following graph to draw a rough sketch of the function's derivative below.

graph of $f(x)$

sketch of $f'(x)$

OTHER NOTATION FOR THE DERIVATIVE

For $y = f(x)$, here are several equivalent notations for the derivative $f'(x)$.

DIFFERENTIABILITY

- A function f is **differentiable** at a if (the limit) $f'(a)$ exists.
- f is **differentiable on an open interval** I if $f'(a)$ exists for all $a \in I$.

Example 5.4. Where is the function $f(x) = |x|$ differentiable?

DIFFERENTIABILITY $\Longrightarrow$ CONTINUITY

Theorem 5.5. If f is differentiable at a, then f is continuous at a.

The converse of Theorem 5.5 is false:

HOW FUNCTIONS CAN FAIL TO BE DIFFERENTIABLE

HIGHER-ORDER DERIVATIVES

Let $y = f(x)$ be a function. If, in each case, the preceding derivative exists, we can iterate the process of differentiation to take **higher-order derivatives**.

		Notation		
1st derivative of f:	derivative of f	$f'(x)$	y'	$\frac{dy}{dx}$
2nd derivative of f:	derivative of f'	$f''(x)$	y''	$\frac{d^2y}{dx^2}$
3rd derivative of f:	derivative of f''	$f'''(x)$	y'''	$\frac{d^3y}{dx^3}$
4th derivative of f:	derivative of f'''	$f^{(4)}(x)$	$y^{(4)}$	$\frac{d^4y}{dx^4}$
nth derivative of f:	derivative of $f^{(n-1)}$	$f^{(n)}(x)$	$y^{(n)}$	$\frac{d^ny}{dx^n}$

BASIC RULES

Constants

Powers

$n = 0$

$n = 1$

$n = 2$

$n = 3$

STUDY GUIDE

Important terms and concepts:

- $\diamond$ **Definition of the Derivative at** a: $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

  slope of tangent line   instantaneous rate of change   derivative at a point
- $\diamond$ **Definition of the Derivative as a Function**: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$
- $\diamond$ **Relationship between the graph of** f **and the graph of** f'
- $\diamond$ **Other notation for derivative of** $y = f(x)$: $f'(x)$ y' $\frac{dy}{dx}$ $\frac{df}{dx}$ $\frac{d}{dx}f(x)$ $D_x f(x)$
- $\diamond$ **Differentiability** $\Longrightarrow$ **Continuity**
- $\diamond$ **Failing to be differentiable** (discontinuity, corner, vertical tangent line)
- $\diamond$ **Higher-order derivatives**: f f' f'' f''' $f^{(4)}$ *etc.*