4. Limits, Limits at Infinity, Continuity, and I.V.T.

Lec 3 mini review.

slope of secant: $\frac{f(b)-f(a)}{b-a}$

average rate of change (AROC): $\frac{f(b)-f(a)}{b-a}$

goal: slope of tangent at a:

goal: instantaneous rate of change (IROC) at a:

$\frac{f(a+h)-f(a)}{h}$ want $h \to 0$

$\frac{f(a+h)-f(a)}{h}$ want $h \to 0$

limits: the intuitive definition

$\lim\limits_{x\to a} f(x) = L$

one-sided limits:

$\lim\limits_{x\to a^-} f(x) \qquad \lim\limits_{x\to a^+} f(x)$

why some limits DNE:
infinite limits (vertical asymptotes)
no unique real number L
different or DNE from left/right

ways to evaluate limits:
numerically graphically
with **Limit Laws** and algebraic tricks
(factoring, rationalizing,...)

SQUEEZING LIMITS

Example 4.1. Recall from last class that $\lim\limits_{x\to 0} \sin\left(\dfrac{\pi}{x}\right)$ DNE.

What about the limit $\lim\limits_{x\to 0} x^2 \sin\left(\dfrac{\pi}{x}\right)$?

[*] These notes are solely for the personal use of students registered in MAT1320.

THE SQUEEZE THEOREM.

Let f, g, and h be functions.

If $\boxed{f(x) \leq g(x) \leq h(x)}$ when x is *near* a, except possibly at a,

and $\boxed{\lim_{x\to a} f(x)} = L = \boxed{\lim_{x\to a} h(x)}$ for some unique real number L,

then

Going back to $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$

CONTINUITY

A function f is **CONTINUOUS AT A NUMBER a** if

In order for $\lim_{x\to a} f(x) = f(a)$, three things must be true (by definition of this limit's existence):

- ◇
- ◇
- ◇

If f is defined *near* a, but f fails to be continuous at a, then f is called **DISCONTINUOUS AT a**, or we say that f **has a discontinuity at** a.

REASONS WHY A FUNCTION COULD BE DISCONTINUOUS

Example 4.2. Consider the graph of f below.

a	Is f continuous at $x = a$? Explain why or why not.
$a = -1.5$	
$a = -1$	
$a = 0$	
$a = 0.5$	
$a = 2$	
$a = \sqrt{7}$	
$a = 3.5$	
$a = 4$	
$a = 5$	

Summary of possible reasons why f could be discontinuous at $x = a$.

- a is not in the domain of f
  (*e.g.* hole, vertical asymptote)
- limit of $f(x)$ as $x \to a$ DNE
  (*e.g.* infinite limit, no unique limit L, different one-sided limits, one-sided limit DNE)
- limit of $f(x)$ as $x \to a$ exists, but isn't equal to $f(a)$
  (*e.g.* jump in the graph of f at $x = a$)

► For discontinuities, look for holes, jumps, and vertical asymptotes.

ONE-SIDED CONTINUITY

► A function $f(x)$ is **CONTINUOUS...**

...FROM THE LEFT AT A NUMBER a IF

...FROM THE RIGHT AT A NUMBER a IF

Example 4.3. Reconsider the function f given in Example 4.2.

a	Is f continuous at $x = a$ from the left, from the right, or neither? Explain.
$a = -1.5$	
$a = -1$	
$a = 0$	
$a = 0.5$	
$a = 2$	
$a = \sqrt{7}$	
$a = 3.5$	
$a = 4$	

Continuous on an Interval

- A function f is **Continuous on an Interval** if f is continuous at every number in the interval.
- If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right, or continuous from the left.
- Informally, f is **Continuous on an Interval** if we can trace the graph of f along the entire interval, without needing to lift our pencil off the paper.

Theorem 4.4. Let k be a constant.

If f and g are continuous at a number a, then the following functions are also continuous at a:

Theorem 4.5. The following types of functions are continuous at every real number <u>in their domains</u>:

A Limit Law for Compositions of Continuous Functions

Theorem 4.6. Let f and g be functions.

If f is continuous at b and $\lim_{x \to a} g(x) = b$, then

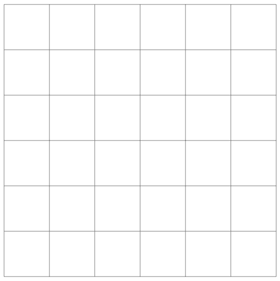
Example 4.7. Evaluate $\lim_{x \to 1} \sin\left(\frac{\pi - \pi\sqrt{x}}{1-x}\right)$

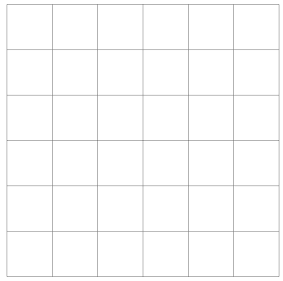
INTERMEDIATE VALUE THEOREM

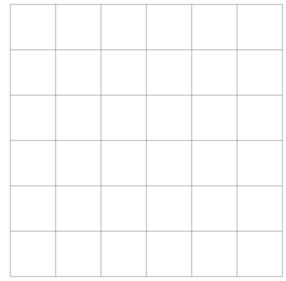
Suppose that f is continuous on the closed interval $[a, b]$ and $f(a) \neq f(b)$. If N is any number between $f(a)$ and $f(b)$, then

The Intermediate Value Theorem may seem obvious, but don't forget that it relies on the fact that f is **continuous** on the interval $[a, b]$.

Exercise 4.8. Think about the ways in which a function can have a discontinuity. Then draw several possibilities in which a function f has the property that $f(3) = -1$, $f(5) = 1$, but there is no point $c \in [3, 5]$ such that $f(c) = 0$.







Example 4.9. Use the **Intermediate Value Theorem** to prove that the equation

$$x^5 - x^4 + x^3 - x - 1 = 0$$

has a root in the interval $[1, 2]$.

LIMITS AT INFINITY & HORIZONTAL ASYMPTOTES

- Let f be a function defined on some interval (a, ∞). Then

  means that the values of $f(x)$ can be made arbitrarily close to a unique real number L so long as x is sufficiently large.

- Let f be a function defined on some interval $(-\infty, a)$. Then

  means that the values of $f(x)$ can be made arbitrarily close to a unique real number L so long as x is a sufficiently large negative number.

- The line $y = L$ is called a **HORIZONTAL ASYMPTOTE** if $\lim\limits_{x \to -\infty} f(x) = L$ or $\lim\limits_{x \to \infty} f(x) = L$.

Useful Fact (Theorem)

- If $r > 0$ is a rational number, then

- If $r > 0$ is a rational number such that x^r is defined for all $x \in \mathbb{R}$, then

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Example 4.10. $\lim\limits_{x\to\infty} \dfrac{8x^3 - x^2}{1 + x - x^3}$

Example 4.11. $\lim\limits_{x\to-\infty} \dfrac{x}{|x|}$

Example 4.12. $\lim\limits_{x\to\infty} \dfrac{\sin(x)}{x}$

Example 4.13. $\lim_{x\to\infty} \sqrt{9x^2+x} - 3x$

Example 4.14. $\lim_{x\to\infty} x^2$

Example 4.15. $\lim_{x\to-\infty} \cos(x)$

STUDY GUIDE

Important terms and concepts:

- ⋄ **The Squeeze Theorem**
- ⋄ **Continuity** continuous at $x = a$ continuous on an interval
- ⋄ **Discontinuity** hole jump vertical asymptote
- ⋄ **Intermediate Value Theorem**
- ⋄ **Limits At Infinity & Horizontal Asymptotes**
