Lesson 2 - The Derivative of the General Exponential Function $y = b^x$

PART A: Investigation

You will be working on the investigation on page 235-236 from the textbook. Use the tables and space below to record your work.

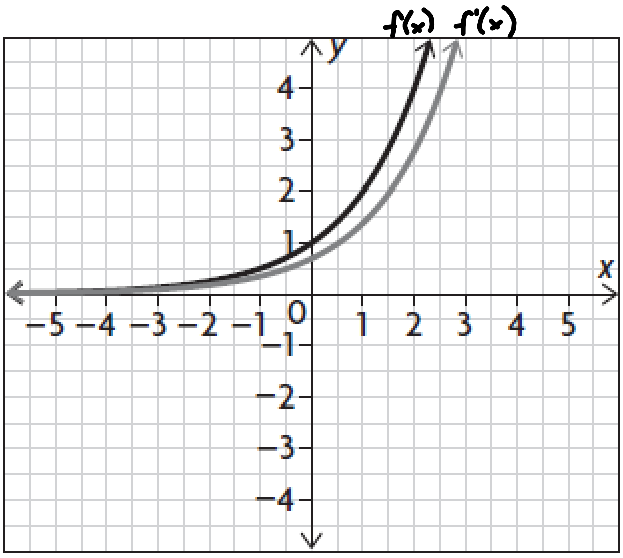
A. Table for $f(x) = 2^x$

x	$f(x)$	$f'(x)$	$\frac{f'(x)}{f(x)}$
-2	$\frac{1}{4}$	0.173	0.693
-1	$\frac{1}{2}$	0.347	0.693
0	1	0.693	0.693
1	2	1.386	0.693
2	4	2.773	0.693
3	8	5.545	0.693

F. ii) the ratio is the same

F. iii) less than 1

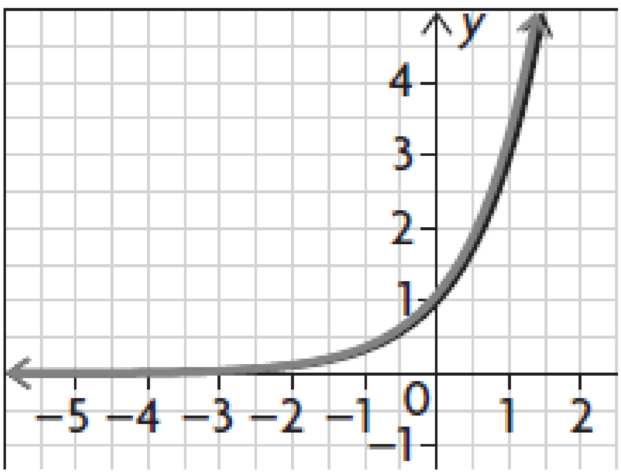
Grid for graph of $f(x) = 2^x$ and $f'(x)$



G. Table for $f(x) = 3^x$

x	$f(x)$	$f'(x)$	$\frac{f'(x)}{f(x)}$
-2	$\frac{1}{9}$	0.122	1.099
-1	$\frac{1}{3}$	0.366	1.099
0	1	1.099	1.099
1	3	3.296	1.099
2	9	9.888	1.099
3	27	29.663	1.099

Grid for graph of $f(x) = 3^x$ and $f'(x)$



F. ii) the ratio is the same

F. iii) greater than 1

J. The derivative of $f(x) = a^x$ is a multiple of a^x.
If $a > e$, the multiple is larger than 1
If $a < e$, the multiple is smaller than 1

PART B: Summary of Investigation

In general, for the exponential function $f(x) = b^x$, we can conclude the following:

- $f(x)$ and $f'(x)$ are both exponential functions
- Slope of the tangent at a point on the curve is proportional to the value of the function at this point
- $f'(x)$ is a vertical stretch or compression of $f(x)$, (depends on value of b)
- The ratio $\dfrac{f'(x)}{f(x)}$ is a constant and is equivalent to the stretch/compression factor

We will now use first principles definition of the derivative to determine the derivative of the exponential function $f(x) = b^x$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{b^{x+h} - b^x}{h}$$

$$= \lim_{h \to 0} \frac{b^x \cdot b^h - b^x}{h}$$

- use exponent laws to separate b^{x+h} into product of two bases

$$= \lim_{h \to 0} \frac{b^x(b^h - 1)}{h}$$

- factor out b^x

$$= b^x \cdot \lim_{h \to 0} \frac{b^h - 1}{h}$$

- b^x is not dependant on h, therefore it can be brought outside of the limit

From today's investigation:

if $f(x) = 2^x$, $f'(x) \doteq 0.69(2^x)$

$f(x) = 3^x$, $f'(x) \doteq 1.10\ (3^x)$

From Lesson 5.1:

if $f(x) = e^x$, $f'(x) = e^x$

If we look at the derivative from first principles:

$$f'(x) = e^x \cdot \underbrace{\lim_{h \to 0} \frac{e^h - 1}{h}}_{=1}$$

$$= e^x$$

Since f(x) and f'(x) are related by a constant of proportionality and exponential functions are the inverse of logarithms, we look to logs for our constant

if $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

- this must be true for the derivative of e^x to be equal to itself

$\log_a a^x = x$

$\log_e e = 1$

$\ln e = 1$

try $\ln 2 \doteq 0.69$

$\ln 3 \doteq 1.10$

Therefore, if $f(x)=b^x$, $f'(x)=(\ln b)b^x$

if $f(x) = e^x$

$f'(x) = \ln e \cdot e^x$

$= 1 \cdot e^x$

$= e^x$

Derivative of $f(x)=b^x$

if $f(x)=b^x$, then $f'(x)=(\ln b)\times b^x$

PART C: Examples

Example 1: Determine the derivative of the following functions:

a) $f(x)=4^x$

$f'(x)=(\ln 4)4^x$

b) $f(x)=125\left(\frac{1}{2}\right)^x$

$f'(x)=125(\ln\frac{1}{2})(\frac{1}{2})^x$

PART D: Derivative of composite exponential functions

Derivative of $f(x)=b^{g(x)}$

if $f(x)=b^{g(x)}$, then $f'(x)=\underline{(\ln b)}\times b^{g(x)}\times g'(x)$

Example 2: Determine the derivative of $f(x)=6x(2)^{2x^2-5x}$.

$f(x)=\underbrace{6x}_{g(x)}\,\underbrace{(2)^{2x^2-5x}}_{h(x)}$

$f'(x)=g'(x)h(x)+g(x)h'(x)$

$=6(2)^{2x^2-5x}+6x(\ln 2\cdot 2^{2x^2-5x}\cdot(4x-5))$

$=6(2)^{2x^2-5x}(1+\ln 2(4x-5)x)$