10. Antiderivatives & Areas

ANTIDERIVATIVES

A function $F(x)$ is called an **antiderivative** of $f(x)$ if

Example 10.1. Guess an antiderivative for each of the following functions. Then check your answer by differentiating.

function $f(x)$	guess an antiderivative $F(x)$ of f	verify that $F'(x) = f(x)$
(a) $f(x) = 2x$		
(b) $f(x) = 3x^2$		
(c) $f(x) = e^x$		
(d) $f(x) = \cos(x)$		
(e) $f(x) = 0$		

Fact. If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, then $F(x) - G(x) = C$, for some constant C.

- Thus, if F and G are two distinct antiderivatives of $f(x)$, then the graphs of F and G are identical, up to a vertical translation.

Theorem 10.2. If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $\quad F(x) + C \quad$ where C is an arbitrary constant.

Integral Notation:

"UNDOING" BASIC RULES OF DIFFERENTIATION

Constants.

Powers.

The special power $n = -1$

Constant multiples

Sums and differences.

Example 10.3. Find $f(x)$ given that $f'(x) = \dfrac{5}{x^6} + 8x^3 - \dfrac{2}{x}$ and $f(1) = 2$.

Trig functions.

Example 10.4. Verify that $F(x) = -\ln|\cos x| + C$ is an antiderivative of $\tan x$.

Exponential functions.

Inverse trig functions.

"UNDOING" LESS BASIC RULES OF DIFFERENTIATION

Example 10.5. Verify that $x \ln x - x$ is an antiderivative of $\ln(x)$.

Example 10.6. Try to determine (guess!) an antiderivative for each of the following functions. Verify (check!) your answer by differentiating.

(a) $2xe^{x^2}$

(b) $3x^2 \sin x + x^3 \cos x$

(c) $\cot(x)$

(d) e^{ax+b} where a and b are constant real numbers.

(e) $\frac{g'(x)}{g(x)}$

(f) $e^x \left[e^x + 8\right]^{10}$

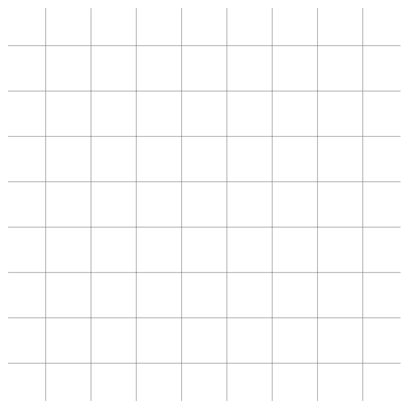
Exercise 10.7. Determine an equation for the height of a ball at time t if the ball is thrown with an initial upwards velocity of v_0 m/s, from an initial height of h_0 m above the ground, on Earth where acceleration due to gravity is -9.8 m/s^2.

Example 10.8. Let $f(x) = (x-1)^2 + 1$.

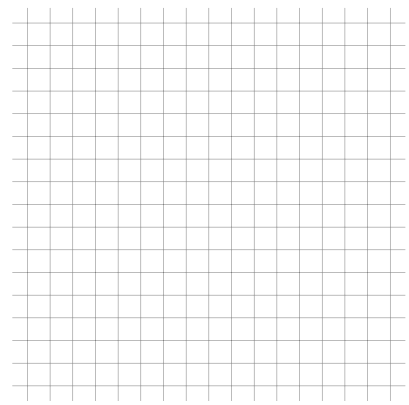
What is the area of the region under the graph of f, above the x-axis between $x = 0$ and $x = 2$?

This is not a standard shape, so we do not have an area formula.

$f(x) = (x-1)^2 + 1$



4 rectangles



8 rectangles

We can try to estimate the area using several rectangles:

- If we use $n = 4$ rectangles, then each rectangle has width $\Delta x = 0.5$ (this is because we are chopping the interval $[0, 2]$ into four equal pieces).
- If we use $n = 8$ rectangles instead, then each rectangle has width 0.25 (this is because we are chopping the interval $[0, 2]$ into eight equal pieces).
- For the height of each rectangle, we will use the y-coordinate of f at the appropriate value of x (for now, we'll use the x-coordinate at the bottom left corner of each rectangle).

$$
\begin{aligned}
L_4 &= \begin{bmatrix} \text{Estimate of area using} \\ \text{4 rectangles with} \\ \text{left endpoint heights} \end{bmatrix} \\
&\approx \begin{pmatrix} \text{area of 1st} \\ \text{rectangle} \end{pmatrix} + \begin{pmatrix} \text{area of 2nd} \\ \text{rectangle} \end{pmatrix} + \begin{pmatrix} \text{area of 3rd} \\ \text{rectangle} \end{pmatrix} + \begin{pmatrix} \text{area of 4th} \\ \text{rectangle} \end{pmatrix} \\
&= \left(\underbrace{0.5}_{\substack{\text{width} \\ \text{1st rect.}}} \times \underbrace{f(0)}_{\substack{\text{height} \\ \text{1st rect.}}} \right) + \left(\underbrace{0.5}_{\substack{\text{width} \\ \text{2nd rect.}}} \times \underbrace{f(0.5)}_{\substack{\text{height} \\ \text{2nd rect.}}} \right) + \left(\underbrace{0.5}_{\substack{\text{width} \\ \text{3rd rect.}}} \times \underbrace{f(1)}_{\substack{\text{height} \\ \text{3rd rect.}}} \right) + \left(\underbrace{0.5}_{\substack{\text{width} \\ \text{4th rect.}}} \times \underbrace{f(1.5)}_{\substack{\text{height} \\ \text{4th rect.}}} \right) \\
&= (0.5 \times 2) + (0.5 \times 1.25) + (0.5 \times 1) + (0.5 \times 1.25) \\
&= [2 + 1.25 + 1 + 1.25](0.5) = 2.75
\end{aligned}
$$

$$L_8 = \begin{bmatrix} \text{Estimate of area using} \\ \text{8 rectangles with} \\ \text{left endpoint heights} \end{bmatrix}$$

$$\begin{aligned}
&\approx \begin{pmatrix} \text{area of 1st} \\ \text{rectangle} \end{pmatrix} + \begin{pmatrix} \text{area of 2nd} \\ \text{rectangle} \end{pmatrix} + \begin{pmatrix} \text{area of 3rd} \\ \text{rectangle} \end{pmatrix} + \begin{pmatrix} \text{area of 4th} \\ \text{rectangle} \end{pmatrix} \\
&\qquad + \begin{pmatrix} \text{area of 5th} \\ \text{rectangle} \end{pmatrix} + \begin{pmatrix} \text{area of 6th} \\ \text{rectangle} \end{pmatrix} + \begin{pmatrix} \text{area of 7th} \\ \text{rectangle} \end{pmatrix} + \begin{pmatrix} \text{area of 8th} \\ \text{rectangle} \end{pmatrix} \\
&= \big(0.25 \times f(0)\big) + \big(0.25 \times f(0.25)\big) + \big(0.25 \times f(0.5)\big) + \big(0.25 \times f(0.75)\big) \\
&\qquad + \big(0.25 \times f(1)\big) + \big(0.25 \times f(1.25)\big) + \big(0.25 \times f(1.5)\big) + \big(0.25 \times f(1.75)\big) \\
&= \big[1 + 1.5625 + 1.25 + 1.0625 + 1 + 1.0625 + 1.25 + 1.5625\big]0.25 = 2.6875
\end{aligned}$$

General Observations:

- If we chopped the interval into even more (thinner) rectangles, we would get better and better estimates of the actual area.
- Using n rectangles, each of width Δx, where Δx is the interval length divided by n, we can estimate the net area between any continuous function and the x-axis from $x = a$ to $x = b$.

- The bigger n gets, the smaller Δx gets (more rectangles, but they are thinner).
- For the height of the ith rectangle, we could just as well have used the x-coordinate at the bottom right corner of each rectangle, or the x-coordinate at the midpoint of the rectangles two sides, or some other sample point x_i^*.

- The widths of the rectangles did not all have to be equal (but having them all of equal width made the calculation more straightforward).

STUDY GUIDE

- **an antiderivative** vs. **the most general antiderivative**
- **undoing basic rules of differentiation**
- **some ideas for undoing less basic rules of differentiation**
- **approximating area (or distance) using a Riemann sum**: $A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$