Table of Points for marking purposes. *Do not write in this table.*

	multiple-choice	Q4	Q5	Q6	Q7	TOTAL
max points possible	13 pts	6 pts	5 pts	6 pts	7 pts	37 points
points obtained						

Multiple-Choice Questions.

Write the question number and the letters of your answer(s). No justification is needed.

Q1. Let $S = \Big\{ a,\ \{a\},\ \{a,\ b\},\ \{a, \{b\}\}\Big\}$ and let $T = \Big\{ a,\ \{a\},\ \{b\} \Big\}$.

a. Compute the following cardinalities [8 points]

$|S \times T| =$. 12

$|\mathcal{P}(T)| =$ 8

$|S \bigoplus T| =$ 3

b. Given each of the following sets in list notation (use set brackets { } where appropriate)

$S \cap T =$. $\{a, \{a\}\}$

$S \oplus T =$. $\{\{a, b\}, \{a, \{b\}\}, \{b\}\}$

c. True or False ? circle your answer, you do not need to justify

$\{\{a\}, \{b\}\} \subseteq \mathcal{P}(T)$ True False

$(\phi, \phi) \in \mathcal{P}(T) \times \mathcal{P}(T)$ True False

$\{a, b\} \in \mathcal{P}(S)$. True False

Q2. Consider the following three functions:

$$f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+ \qquad g : \mathbb{Z}^+ \to \mathbb{Z} \qquad h : \mathbb{R} \times \mathbb{Z}^+ \to \mathbb{Z}^+$$
$$f(m,n) = mn \qquad g(n) = n - 9 \qquad h(x,y) = 7y + 1$$

Which 3 of the following statements are true? *Only 3 statements are true.*

A. f is injective (one-to-one).

B. f is surjective (onto).

C. g is injective (one-to-one).

D. g is surjective (onto).

E. h is injective (one-to-one).

F. h is surjective (onto).

G. g is invertible.

H. The domain of the composition $g \circ f$ is $\mathbb{Z}^+$

I. The codomain of the composition $g \circ f$ is $\mathbb{Z}$

Answers: $\boxed{B}$ $\boxed{C}$ $\boxed{I}$ **[3 points]**

Q3. Let $\mathcal{R}$ be a relation on the set $A = \{1,2,3\}$, defined as follows:

$$\mathcal{R} = \Big\{(1,1),\ (1,2),\ (1,3),\ (2,2),\ (2,3),\ (3,3),\ \Big\}$$

Which one of the following statements is true? *Only 1 statement is true.*

A. $\mathcal{R}$ is reflexive, antisymmetric, and transitive, but $\mathcal{R}$ is not symmetric.

B. $\mathcal{R}$ is reflexive and symmetric, but $\mathcal{R}$ is not antisymmetric nor transitive.

C. $\mathcal{R}$ antisymmetric and transitive, but $\mathcal{R}$ is not reflexive nor symmetric.

D. $\mathcal{R}$ is reflexive, symmetric, and antisymmetric, but $\mathcal{R}$ is not transitive.

E. $\mathcal{R}$ is an equivalence relation on A.

Answer: $\boxed{A}$ **[2 points]**

Long-Answer Questions. Detailed justifications are required.

Please CLEARLY WRITE THE QUESTION NUMBER in your solutions.

You do NOT need to copy the questions.

Q4. Let $A = \{7m+1 : m \in \mathbb{Z}\}$ and $B = \{14n-6 : n \in \mathbb{Z}\}$ be subsets of a universal set $\mathcal{U}$.

a. Justify why $A \not\subseteq B$ by providing a counterexample with a brief explanation. [**1 point**]

$1 \in A$ (choose $m=0$)

but $1 \notin B$ ($14n-6=1 \iff n=\frac{1}{2} \notin \mathbb{Z}$).

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b. For part **b.**, no justification is needed. [**2 points**]

i. List 2 different elements of the set $A \cap B$ (one in each box): $\boxed{8}$ and $\boxed{22}$

ii. List 2 different elements of the set $A \times B$ (one in each box): $\boxed{(8,8)}$ and $\boxed{(22,22)}$

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c. Prove that $B \subseteq A$ using a rigorous proof. [**3 points**]

Important! In each step of your proof make sure it is clear whether what is written is something you are assuming, something you are about to prove, or something that follows from a previous step or definition. If any variables appear in your proof, make sure you clearly write what they represent.

Let $x \in B$. So there exists $n \in \mathbb{Z}$ such that $x = 14n-6$

Then

$x = 14n-7+1 = 7(2n-1)+1$

Take $m = 2n-1 \in \mathbb{Z}$. Then $x = 7m+1$ where $m \in \mathbb{Z}$. Therefore, $x \in A$.

Q5. Let A, B, and C be subsets of a universal set $\mathcal{U}$. For this question, you will prove the following set identity in two ways:

$$C - (\overline{A} \cap B) = \big(C \cap A\big) \cup \big(C - B\big)$$

i. [**3 points**] Use the laws from the Table of Set Identities provided on page 8. Use at most **one law per step**, and write the name of the law you are using at each step. *Points may be deducted for combining more than one law in one step or for failing to write the name of the law used in each step. Make sure you apply the laws* <u>*precisely*</u> *as they are written in the provided table.*

$$\begin{aligned} C - (\overline{A} \cap B) &= C \cap \overline{(\overline{A} \cap B)} && (20) \\ &= C \cap (\overline{\overline{A}} \cup \overline{B}) && (15) \\ &= C \cap (A \cup \overline{B}) && (7) \\ &= (C \cap A) \cup (C \cap \overline{B}) && (12) \\ &= (C \cap A) \cup (C - B) && (20) \end{aligned}$$

ii. [**2 points**] Complete this **membership table**. No explanation is needed.

A	B	C	$\overline{A}$	$\overline{A} \cap B$	$C - (\overline{A} \cap B)$	$C \cap A$	$C - B$	$(C \cap A) \cup (C - B)$
1	1	1	0	0	1	1	0	1
1	1	0	0	0	0	0	0	0
1	0	1	0	0	1	1	1	1
1	0	0	0	0	0	0	0	0
0	1	1	1	1	0	0	0	0
0	1	0	1	1	0	0	0	0
0	0	1	1	0	1	0	1	1
0	0	0	1	0	0	0	0	0

Q6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function defined by $f(t,s) = \big(1+3t, 2s-t\big)$. **[6 points]**

a. Carefully prove that f is **injective** (one-to-one).

Important! In each step of your proof make sure it is clear whether what is written is something you are assuming, something you are about to prove, or something that follows from a previous step. If any variables appear in your proof, make sure you clearly write what they represent.

Let $(t,s) \in \mathbb{R}^2$ and $(u,v) \in \mathbb{R}^2$ such that $f(t,s) = f(u,v)$. Therefore

$(1+3t, 2s-t) = (1+3u, 2v-u)$

$$\Rightarrow \begin{cases} 1+3t = 1+3u \Rightarrow 3t = 3u \Rightarrow t = u. \\ 2s-t = 2v-u \Rightarrow 2s-u = 2v-u \\ \qquad \Rightarrow 2s = 2v \Rightarrow s = v \end{cases}$$

So $t=u$ and $s=v$ $\Rightarrow (t,s) = (u,v)$.

b. Carefully prove that f is **surjective** (onto). Justify your answer!

Let $(a,b) \in \mathbb{R}^2$. We find $(t,s) \in \mathbb{R}^2$ such that $f(t,s) = (a,b)$.

$$(1+3t, 2s-t) = (a,b) \Rightarrow \begin{cases} 1+3t = a \\ 2s-t = b \end{cases}$$

$$\Rightarrow \begin{cases} t = \frac{a-1}{3} \\ s = \frac{b+t}{2} = \frac{b+\frac{a-1}{3}}{2} \end{cases}$$

For all $(a,b) \in \mathbb{R}^2$, $f\left(\frac{a-1}{3}, \frac{b+\frac{a-1}{3}}{2}\right) = (a,b)$.

Q7. Let $\mathcal{R}$ be a relation on the set $\mathbb{Z}$ defined by the following rule: **[7 points]**

for all $a, b \in \mathbb{Z}$, $\quad a \,\mathcal{R}\, b \quad$ if and only if $\quad 5$ divides $a - b$.

Prove that $\mathcal{R}$ is an **equivalence relation** on $\mathbb{Z}$.

Important! In each step of your proof make sure it is clear whether what is written is something you are assuming, something you are about to prove, or something that follows from a previous step or definition. If any variables appear in your proof, make sure you clearly write what they represent.

R is reflexive: aRa since 5 divides $a-a=0$.
(There exists $k \in \mathbb{Z}$ such that $5k=0$) $(k=0)$

R is symmetric: Suppose aRb. so 5 divides $a-b$
There exists $k \in \mathbb{Z}$ such that $5k = a-b$
Then $b-a = (-k)5$. since $-k \in \mathbb{Z}$, 5 divides $b-a$.
hence bRa.

R is transitive: Suppose aRb and bRc.
There exist $k, k' \in \mathbb{Z}$ such that $a-b = 5k$ and $b-c = 5k'$.

$\Rightarrow a-c = a-b+b-c = 5k+5k' = 5(k+k')$
Since $k+k' \in \mathbb{Z}$, we conclude 5 divides $a-c$,
hence aRc

Since R is reflexive, symmetric and transitive, it is an equivalence relation

1. 2.	$A \cup \emptyset = A$ $A \cap \mathcal{U} = A$	Identity Laws
3. 4.	$A \cup \mathcal{U} = \mathcal{U}$ $A \cap \emptyset = \emptyset$	Domination Laws
5. 6.	$A \cup A = A$ $A \cap A = A$	Idempotent Laws
7.	$\overline{(\overline{A})} = A$	(Double) Complementation Law
8. 9.	$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
10. 11.	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
12. 13.	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Laws
14. 15.	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's Laws
16. 17.	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Laws
18. 19.	$A \cup \overline{A} = \mathcal{U}$ $A \cap \overline{A} = \emptyset$	Complement Laws
20.	$A - B = A \cap \overline{B}$	Difference Law
21. 22.	$A \oplus B = (A - B) \cup (B - A)$ $A \oplus B = (A \cup B) - (A \cap B)$	Symmetric Difference Laws