Mat1322 Midterm 2

Calculus II (University of Ottawa)

Wednesday, March 15, 2023 **Instructor:** Fabrizio Donzelli

MAT1322[E] Midterm 2

Instructions

- This is an **80-minute closed book test**.
- The only calculators permitted are one of the following four Faculty-approved models: **Texas Instruments TI-30, Texas Instruments TI-34, Casio fx-260, Casio fx-300**.
- If you are caught with any other make or model of calculator, then it will be confiscated, allegations of academic fraud may be filed, and/or you may receive zero for this test.
- The exam consists of **10 pages** including this cover page and page 10 which is for additional workspace or scrap work.
- Maximum points possible = **25 points**. Each question is worth the indicated number of points.
- Read each question carefully.
- Questions 1–4 are **multiple choice**. Record your answer in the box. No justification is needed.
- Questions 5–7 are **long answer**. You must justify your answers with a clear and complete solution.
- You may use the backs of pages and page 10 for rough work. **Do not use any of your own scrap paper.**
- **Cellular phones, unauthorized electronic devices, or course notes are not allowed during this test. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession such as in your pockets. If caught with such a device or document, academic fraud allegations may be filed which may result in you obtaining zero for this test.**

† **By signing below, you acknowledge that you have read, understand, and will comply with the above instructions.**

Family Name:	Student Number:
First Name:	†Signature:

Good luck!

MULTIPLE-CHOICE QUESTIONS Questions 1–4 are **multiple-choice questions** worth 2 points each. Your answers to multiple-choice questions do not need to be justified. When you reach your answer, clearly indicate the question number and write the letter of your response beside the question number:

For example: (*write out your scrap work, but it will not be graded*)

(*clearly indicate your final choice*) **Q1.** [letter of your choice]

Q1. An object with a temperature of $50°C$ is put in a $200°C$ oven and heats up according to the differential equation

$$\frac{dH}{dt} = K(H - 200)$$

where $H(t)$ represents the object's temperature t minutes after being placed in the oven. After 48 minutes the temperature of the object is $130°C$. What is the value of K to 3 decimal places?

A. 0.056 **B.** 0.027 **C.** -0.016 **D.** 0.0761 **E.** -0.285 **F.** 0.047 **G.** -0.192

Solution: $\boxed{\text{C}}$

We have

$$\frac{dH}{dt} = K(H - 200)$$
$$\frac{dH}{(H-200)} = Kdt$$
$$\int \frac{dH}{(h-200)} = \int Kdt$$
$$\ln|H - 200| = Kt + C$$
$$|H - 200| = e^{Kt}e^{C}$$

Let $A = \pm e^{C}$. Then

$$H - 200 = Ae^{Kt}$$
$$H = Ae^{Kt} + 200$$

According to the initial condition $H(0) = 50°C$. Then

$$50 = Ae^{0} + 200$$
$$50 = A + 200$$
$$A = -150$$

Also, we know that $H(48) = 130°C$. Then

$$130 = -150e^{48K} + 200$$
$$150e^{48K} = 70$$
$$48K = \ln(70/150)$$
$$K = \frac{\ln(70/150)}{48} \approx -0.016$$

Q2. For all $n \geq 1$, let $a_n = \dfrac{e^2 n^5 - en + 18}{e^4 n^5 + 8n^4 - 19n^2 - 3}$. Which one of the following statements is true?

A. The sequence $\{a_n\}_{n=1}^{\infty}$ converges, and the series $\sum_{n=1}^{\infty} a_n$ also converges.

B. The sequence $\{a_n\}_{n=1}^{\infty}$ converges, but the series $\sum_{n=1}^{\infty} a_n$ diverges.

C. The sequence $\{a_n\}_{n=1}^{\infty}$ diverges, and the series $\sum_{n=1}^{\infty} a_n$ also diverges.

D. The series $\sum_{n=1}^{\infty} a_n$ converges, but the sequence $\{a_n\}_{n=1}^{\infty}$ diverges.

Solution: $\boxed{\textbf{B}}$

The sequence converges, and, in particular, $\lim_{n\to\infty} a_n \neq 0$, hence the series diverges by the Test for Divergence.

Q3. Fact: the series $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{7/3}}$ converges by virtue of the Alternating Series Test.

According to the Alternating Series Estimation Theorem, what is the smallest value of N for which the error $|R_N| = |S - S_N|$ is at most 0.001 ?

Solution: $\boxed{\mathbf{A}}$

By ASET, $|R_N| \leq b_{N+1} = \frac{1}{(N+1)^{7/3}}$, so it suffices to find N such that $\frac{1}{(N+1)^{7/3}} \leq 0.001 \implies (1000)^{3/7} \leq N + 1$ Thus, we need $19.307... \leq N + 1$. Since N must be an integer, we need $N \geq 19$.

A. $N = 19$ **B.** $N = 11$ **C.** $N = 31$ **D.** $N = 42$ **E.** $N = 7$ **F.** $N = 27$

Q4. If it is convergent, find the sum of the geometric series $\displaystyle\sum_{m=1}^{\infty} 9\frac{2^{2m-2}}{5^{m-1}}$.

A. 40 **B.** 60 **C.** -30 **D.** 15 **E.** 5 **F.** 45 **G.** It diverges.

Solution: $\boxed{\textbf{F}}$

This geometric series has first term $a = 9$ and common ratio $r = \frac{4}{5}$ with $|r| < 1$. Therefore, it converges to $\frac{a}{1-r} = \frac{9}{1-(4/5)} = 45$.

LONG-ANSWER QUESTIONS Questions 5–7 are **long-answer questions** worth a total of 17 points. For long-answer questions, **all of your work must be justified and your steps must be written in a clear and logical order**. Clearly indicate Question numbers.

For example: $\boxed{\textbf{Q5(a).}}$ [`write a fully justified solution`].

Q5. [**5 points**] Consider the power series: $$\sum_{n=1}^{\infty} \frac{(x+1)^n}{2^{2n}(n^2+3)}$$

(a) Find its radius of convergence. Show your work.

Solution: We use the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x+1)^{n+1}}{2^{2(n+1)}((n+1)^2+3)}}{\frac{(x+1)^n}{2^{2n}(n^2+3)}}\right| = \lim_{n\to\infty}\left|\frac{(x+1)(n^2+3)}{4((n+1)^2+3)}\right| = \frac{|x+1|}{4} \text{ since } \lim_{n\to\infty}\frac{(n^2+3)}{((n+1)^2+3)} = 1.$$

By the Ratio Test, this power series converges when $\frac{|x+1|}{4} < 1$, hence for $|x+1| < 4$. We conclude that the radius of convergence is $R = 4$.

(b) Find its interval of convergence. Show your work.

Solution: From (a), we know this power series converges for $-4 < x+1 < 4$. We need to check convergence at its endpoints.

When $x = -5$, we have
$$\sum_{n=1}^{\infty} \frac{(-5+1)^n}{2^{2n}(n^2+3)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2+3)}$$

This is an alternating series with $b_n = \dfrac{1}{(n^2+3)} > 0$.

Since $((n+1)^2+3) > (n^2+3)$, we have $b_{n+1} < b_n$. Also $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{1}{(n^2+3)} = 0$.

By the Alternating Series Test, this series converges.

When $x = 3$, we have
$$\sum_{n=1}^{\infty} \frac{(3+1)^n}{2^{2n}(n^2+3)} = \sum_{n=1}^{\infty} \frac{1}{(n^2+3)}$$

This series has positive terms and we can apply the Comparison Test or the Limit Comparison Test.

For instance, we have $0 < \dfrac{1}{(n^2+3)} < \dfrac{1}{n^2}$, since $n^2+3 > n^2$ for all $n \geq 1$. Since $\sum \frac{1}{n^2}$ is a p-series with $p = 2 > 1$, the larger series converges. We conclude that the smaller series must also converge.

Therefore, the interval of convergence of the above power series is $-5 \leq x \leq 3$ or $[-5, 3]$.

Q6. [**6 points**] For each integer $k \geq 1$, let $a_k = \ln\left(\frac{\sqrt{3k+2}}{\sqrt{3k+5}}\right)$.

a) Consider the **sequence** $\{a_k\}_{k=1}^{\infty} = \left\{\ln\left(\frac{\sqrt{3k+2}}{\sqrt{3k+5}}\right)\right\}_{k=1}^{\infty}$

Is this sequence convergent or divergent? If it converges, find its exact value. Show all your work to justify your answer. Be sure to use appropriate mathematical notation.

Solution: We have

$$\begin{aligned}
\lim_{k\to\infty} a_k &= \lim_{k\to\infty} \ln\left(\frac{\sqrt{3k+2}}{\sqrt{3k+5}}\right) \\
&= \lim_{k\to\infty} \ln\left(\sqrt{\frac{3k+2}{3k+5}}\right) \\
&= \ln(1) \qquad \text{since } \lim_{k\to\infty} \frac{3k+2}{3k+5} = 1 \text{ and functions } \ln(x), \sqrt{x} \text{ are continuous at } x = 1. \\
&= 0
\end{aligned}$$

Thus, the sequence $\{a_k\}_{k=1}^{\infty}$ converges to 0.

b) Consider the **telescopic** series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\left(\ln\left(\frac{\sqrt{3k+2}}{\sqrt{3k+5}}\right)\right)$.

Find a simplified expression for the nth partial sum S_n, by cancelling terms telescopically. Show all your work!

Solution:

$$\begin{aligned}
S_n = \sum_{k=1}^{n} a_k &= \sum_{k=1}^{n}\Big(\ln(\sqrt{3k+2}) - \ln(\sqrt{3k+5})\Big) \\
&= \sum_{k=1}^{n} \ln(\sqrt{3k+2}) - \sum_{k=1}^{n} \ln(\sqrt{3k+5}) \\
&= \Big(\ln(\sqrt{5}) + \ln(\sqrt{8}) + \cdots + \ln(\sqrt{3n+2})\Big) \\
&\quad - \Big(\ln(\sqrt{8}) + \cdots + \ln(\sqrt{3n+2}) + \ln(\sqrt{3n+5})\Big) \\
&= \ln(\sqrt{5}) - \ln(\sqrt{3n+5})
\end{aligned}$$

c) Determine whether the series $\sum_{k=1}^{\infty} a_k$ converges or diverges. If it converges, find its exact sum; otherwise, briefly explain why it diverges.

Solution: Using the expression from b), we find

$$\sum_{k=1}^{\infty}\{a_k\} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\Big(\ln(\sqrt{5}) - \ln(\sqrt{3n+5})\Big) = -\infty$$

Thus, this series diverges because the limit, as $n \to \infty$ of its nth partial sum, does not exist.

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Q7. [**6 points**] Use appropriate series tests to determine whether each of the following series is convergent or divergent.

If you apply a certain test, name it, and state and show your work to verify the conditions needed in order for the test to apply.

Show all your work. Clearly explain your conclusion.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2\cos^2(n)+6}{9^n}$ (b) $\displaystyle\sum_{m=1}^{\infty} \frac{3m}{e^{m^2}}$

Solution: (a) This series is convergent. We have $0 \leq \frac{2\cos^2(n)+6}{9^n} \leq \frac{8}{9^n}$ for all $n \geq 1$. Since $\frac{1}{9^n}$ is a convergent geometric series ($r = \frac{1}{9}$), we conclude that the series $\displaystyle\sum_{n=1}^{\infty} \frac{2\cos^2(n)+6}{9^n}$ converges.

Thus, $\sum_{n=1}^{\infty} \frac{2\cos^2(n)+6}{9^n}$ is convergent.

Solution: (b)

First note:

$$\frac{3m}{e^{m^2}} = 3me^{-m^2}$$

Let $f(x) = 3xe^{-x^2}$. Then f is continuous and positive for all $x \geq 1$.

We see that $f'(x) = 3(e^{-x^2} + x(-2xe^{-x^2})) = 3e^{-x^2}(1-2x^2) < 0 \iff 1-2x^2 < 0$. Thus, $f'(x) < 0$ for all x such that $x^2 > \frac{1}{2}$. Therefore, $f(x)$ is decreasing for all $x \geq 1$.

We can apply the Integral Test:

We have

$$\begin{aligned}
\int_1^{\infty} xe^{-x^2}\,dx &= \lim_{T\to\infty} \int_1^T xe^{-x^2}\,dx & u = -x^2 \implies du = -2xdx \\
&= \lim_{T\to\infty} \left[-\tfrac{1}{2}e^{-x^2}\right]_1^T & \textstyle\int xe^{-x^2}\,dx = \int -\tfrac{1}{2}e^u\,du = -\tfrac{1}{2}e^u + C = -\tfrac{1}{2}e^{-x^2} + C \\
&= \lim_{T\to\infty} \left[-\tfrac{1}{2}e^{-T^2} - \left(-\tfrac{1}{2}e^{-1^2}\right)\right] \\
&= 0 + \tfrac{1}{2}e^{-1}
\end{aligned}$$

Thus, the improper integral $3\int_1^{\infty} xe^{-x^2}\,dx$ converges. By the Integral Test, we conclude that the series $\sum_{m=1}^{\infty} me^{-m^2}$ must also converge.

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