DGD 9

Q1. COUNTING: BINARY STRINGS **i.** How many **binary strings of length 9** are there?

Procedure:
Build a binary string of length 9
Task T_i: choose ith entry

2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1

There are $2^9 = 512$ binary strings of length 9.

ii. How many **binary strings of length 9** are there that begin with 4 zeros?

Procedure:
Build a binary string of length 9 that begins with four 0's.

0 0 0 0

1 choice: 0 | 1 choice: 0 | 1 choice: 0 | 1 choice: 0 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1

Task T_i: choose ith entry

There are $1^4 \cdot 2^5 = 32$ such strings.

iii. How many **binary strings of length 9** are there that contain **exactly** 4 zeros?

Task T_1: choose 4 of the 9 entries in the string to place the zeros
(in one of $C(9,4) = \binom{9}{4} = \frac{9!}{4!5!} = \frac{\cancel{9}^{3} \cdot \cancel{8} \cdot 7 \cdot 6}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 21 \cdot 6 = 126$ ways)

Task T_2: fill the other 5 entries with ones (in 1 way)

$\therefore$ there are $\binom{9}{4} \cdot (1) = 126$ such binary strings.

iv. How many **binary strings of length 9** are there that contain **at most** 4 zeros?

We will break this up into 5 separate cases:

Case 0: String contains no zeros (hence 9 ones)
There are $\binom{9}{0} = 1$ binary strings of length 9 with no zeros

Case 1: string contains exactly 1 zero (hence 8 ones)
There are $\binom{9}{1} = 9$ binary strings of length 9 with exactly 1 zero

Case 2: string contains exactly 2 zeros (hence 7 ones)
There are $\binom{9}{2} = 36$ binary strings of length 9 with exactly 2 zeros

Case 3: string contains exactly 3 zeros (hence 6 ones)
There are $\binom{9}{3} = 84$ binary strings of length 9 with exactly 3 zeros

Case 4: string contains exactly 4 zeros (hence 5 ones)
There are $\binom{9}{5} = 126$ binary strings of length 9 with exactly 4 zeros

$\therefore$ there are $1 + 9 + 36 + 84 + 126 = 256$ binary strings with at most 4 zeros.

Q2. COUNTING

i. How many strings consisting of 7 distinct letters (of the English alphabet) contain the substring 'MATH' (i.e. the letters MATH appear consecutively in this order somewhere in the string) ?

Procedure: Build such a 7-letter string.

T_1: choose a location in the string for the 4 consecutive letters MATH in one of 4 ways

1. M A T H _ _ _
2. _ M A T H _ _
3. _ _ M A T H _
4. _ _ _ M A T H

T_2: select and arrange 3 letters from the other $26-4=22$ letters of the alphabet in one of $P(22,3)=22\cdot 21\cdot 20$ ways (the other 3 letters go in the remaining 3 positions after T_1 has been carried out).

$\Rightarrow$ there are $4\cdot P(22,3)=4\cdot 22\cdot 21\cdot 20=36\,960$ ways to create such a string.

ii. How many strings of 7 distinct letters (of the English alphabet) do not contain the substring 'MATH' ?

$$\begin{pmatrix}\text{\# strings of 7 distinct}\\ \text{letters that do not contain}\\ \text{the substring 'MATH'}\end{pmatrix} = \begin{pmatrix}\text{total \# of strings}\\ \text{of 7 distinct letters}\end{pmatrix} - \begin{pmatrix}\text{\# strings of 7 distinct}\\ \text{letters that \underline{do} contain}\\ \text{the substring 'MATH'}\end{pmatrix}$$

$$= P(26,7) \qquad - \text{(answer from i.)}$$

$$= P(26,7) - 4\cdot P(22,3)$$

$$= 26\cdot 25\cdot 24\cdot 23\cdot 22\cdot 21\cdot 20 \ - 4\cdot 22\cdot 21\cdot 20$$ $\leftarrow$ without a calculator, you can write your answer like this

$$= 3315312000 - 36960$$

$$= 3315275040$$

iii. How many strings of 7 distinct letters (of the English alphabet) contain the letters 'M', 'A', 'T' and 'H', not necesarrily consecutively, not in any particular order ?

T_1: choose the letters M, A, T, and H to go in the string in one way

T_2: choose 3 letters from among the $26-4=22$ other letters in one of $C(22,3)$ ways

T_3: choose an arrangement of the 7 letters chosen in tasks T_1 and T_2 in one of $P(7,7)$ ways

$\Rightarrow$ there are $(1)\cdot C(22,3)\cdot P(7,7) = \dfrac{22!}{3!\,19!}\cdot 7!$

$$= \frac{22\cdot 21\cdot 20\cdot \cancel{19!}\cdot 7\cdot 6\cdot 5\cdot 4\cdot \cancel{3!}}{\cancel{3!}\,\cancel{19!}}$$

$= 22\cdot 21\cdot 20\cdot 7\cdot 6\cdot 5\cdot 4$ (without calculator)

$= 7761600$ such strings

Q3. COUNTING A 9-member fellowship is to be formed by selecting its members from among a group of 15 people consisting of 1 wizard, 5 hobbits, 5 elves, 2 men, and 2 dwarves.

i. How many different 9-member fellowships could be formed from these people?

• choose 9 out of 15 people (in one of $\binom{15}{9} = \frac{15!}{9!6!} = 5005$ ways)

∴ there are 5005 different 9-member fellowships that could be formed

ii. Suppose one member will be distinguished from the rest as a "ring-bearer". How many different 9-member fellowships with one ring-bearer can be formed from these people?

T_1: choose the ring-bearer (in one of $\binom{15}{1} = 15$ ways)

T_2: choose 8 other members from among the remaining 14 people (in one of $\binom{14}{8} = 3003$ ways)

∴ there are $15 \cdot 3003 = 45045$ different 9-member fellowships that could be formed containing one ring-bearer.

iii. How many fellowships are there that include 1 wizard, 1 hobbit as the ring-bearer and 3 other hobbits, 1 elf, 2 men, and 1 dwarf?

T_1: choose 1 wizard (in $\binom{1}{1} = 1$ way)

T_2: choose one ring-bearer from among the 5 hobbits (in one of $\binom{5}{1} = 5$ ways)

T_3: choose 3 other hobbits from among the 4 remaining hobbits (in one of $\binom{4}{3} = 4$ ways)

T_4: choose an elf (in one of $\binom{5}{1} = 5$ ways)

T_5: choose 2 men (in $\binom{2}{2} = 1$ way)

T_6: choose 1 dwarf (in one of $\binom{2}{1} = 2$ ways).

∴ there are $1 \cdot 5 \cdot 4 \cdot 5 \cdot 1 \cdot 2 = 200$ such fellowships that could be formed.

iv. After the fellowship has been selected in some way, its nine members will line up for a group photo before they depart on a perilous quest. How many ways can these 9 people be arranged for the photo?

There are $P(9,9) = 9! = 362\,880$ different arrangements (permutations) of the 9 chosen members

v. How many fellowships are there that include 1 hobbit as the ring-bearer and an equal number of dwarves and elves?

We will break this up into cases where # dwarves = # elves.

Case 1 0 elves & 0 dwarves means we need 9 members from the 1 wizard, 5 hobbits, and 2 men (not possible)

Case 2: 1 dwarf + 1 elf means we need 7 members from the 1 wizard, 5 hobbits, and 2 men

T_0: choose one hobbit ring-bearer (in one of $\binom{5}{1} = 5$ ways)

T_1: choose 1 dwarf (in one of $\binom{2}{1} = 2$ ways)

T_2: choose 1 elf (in one of $\binom{5}{1} = 5$ ways)

T_3: choose 6 other members from among the 7 others (1 wizard, 4 hobbits, 2 men) (in one of $\binom{7}{6} = 7$ ways)

Case 3: 2 dwarves + 2 elves means we need 5 members from the 1 wizard, 5 hobbits, and 2 men

T_0: choose one hobbit ring-bearer (in one of $\binom{5}{1} = 5$ ways)

T_1: choose 2 dwarves (in $\binom{2}{2} = 1$ way)

T_2: choose 2 elves (in one of $\binom{5}{2} = 10$ ways)

T_3: choose 4 other members from among the 7 others (1 wizard, 4 hobbits, 2 men) (in one of $\binom{7}{4} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$ ways)

$\therefore$ there are $5 \cdot 2 \cdot 5 \cdot 7 + 5 \cdot 1 \cdot 10 \cdot 35 = 350 + 1750 = 2100$ ways to form such a fellowship.