Notes for MAT1341A Fall 2023 Part VII

Chapter 15 - Vector spaces associated to Matrices

Definition (15.1.1 & 15.1.2). Let A be an $m \times n$ matrix. The *column space* of A is

$$\text{Col}(A) = \text{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$$

where $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ are the columns of A, viewed as vectors in $\mathbb{R}^m$. The *row space* of A is

$$\text{Row}(A) = \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n\}$$

where $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n\}$ are the rows of A, viewed as vectors in $\mathbb{R}^n$.

[*E.g.*] Consider a matrix is of dimension 2×3

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & 0 \end{bmatrix}$$

Definition (15.1.3). We define the *null space* of A to be the set of vectors in $\mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{0}$. We write $\text{Null}(A)$.

Lemma (15.1.4). $\text{Null}(A)$ is a subspace of $\mathbb{R}^n$.

To find $\text{Null}(A)$, we use Gaussian elimination to solve $A\mathbf{x} = \mathbf{0}$. The set of solutions is given by

$$\{t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \ldots + t_r\mathbf{v}_r \mid t_1, t_2, \ldots, t_r \in \mathbb{R}\}$$

Here, t_i's are parameters and they correspond to the columns in the RREF without a pivot. This gives the dimension of $\text{Null}(A)$.

[*E.g.*]

$$A = \begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(RREF)}$$

Corollary (15.2.3 - Rank-Nullity Theorem). The dimension of the null space of A is equal to the number of non-leading variables of A. That is,

$$\dim \mathrm{Null}(A) + \dim \mathrm{Col}(A) = n$$

$$\dim \mathrm{Null}(A) + \mathrm{rank}(A) = n$$

where n is the number of columns of A.

What can we say about inhomogeneous system?

Theorem (15.3.2). Suppose $A\mathbf{x} = \mathbf{b}$ is a consistent linear system. Suppose $\mathbf{x} = \mathbf{v}_0$ is a solution to the system. Then the general solution to $A\mathbf{x} = \mathbf{b}$ is given by

$$\{\mathbf{v}_0 + \mathbf{v} \mid \mathbf{v} \in \mathrm{Null}(A)\}.$$

[*E.g.*] Give the general solution to the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

Theorem (15.4.1). Let A be an $m \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$. The following statements are equivalent for a system with matrix equation $A\mathbf{x} = \mathbf{b}$:

(1) $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$;

(2) $\operatorname{rank}(A) = m$;

(3) There are no zero rows in the RREF of A;

(4) Every $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of A;

(5) $\operatorname{Col}(A) = \mathbb{R}^m$;

(6) $\dim(\operatorname{Col}(A)) = m$.

[*E.g.*] Consider

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Theorem (15.5.1). Let A be an $m \times n$ matrix, and $\mathbf{b} \in \mathbb{R}^m$. The following statements are equivalent for a consistent system with matrix equation $A\mathbf{x} = \mathbf{b}$:

(1) $A\mathbf{x} = \mathbf{b}$ has a unique solution;

(2) Every variable corresponds to a pivot;

(3) The associated homogeneous system $A\mathbf{x} = \mathbf{0}$ has a unique solution;

(4) The columns of A are linearly independent;

(5) $\mathrm{Null}(A) = \{\mathbf{0}\}$;

(6) $\dim(\mathrm{Col}(A)) = n$;

(7) $\mathrm{rank}(A) = n$.

Given span$\{\mathbf{v}_1, \ldots, V_n\}$. Can we find A such that

$$\mathrm{Null}(A) = \mathrm{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}?$$

[*E.g.*] Let $W = \mathrm{span}\{(1,1,2)\}$. Find a matrix A such that $W = \mathrm{Null}(A)$.

[*E.g.*] Let $W = \mathrm{span}\{(1,0,0,1),(1,1,1,0),(2,1,-1,1)\} \subseteq \mathbb{R}^4$. Find a matrix A such that $W = \mathrm{Null}(A)$.

Chapter 16 - The row and column space algorithms

Given a matrix A, we want to find bases for $\text{Row}(A)$ and $\text{Col}(A)$.

Proposition (16.1.1). If A and B are row equivalent (one can obtain A from B via elementary row operations), then $\text{Row}(A) = \text{Row}(B)$.

The rows in an REF are LI. So, to find a basis of $\text{Row}(A)$, we can apply Gaussian elimination to get a row equivalent matrix which is in REF. The rows will then be a basis of $\text{Row}(A)$.

[*E.g.*] Find a basis of $\text{Row}(A)$, where $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 2 & 2 & 0 \\ 0 & 2 & 2 & 1 \end{bmatrix}$.

As soon as the number of rows is the rank of the matrix, we get a basis.
Not necessary to go all way to RREF.

What about $\mathrm{Col}(A)$? - we can use the fact that $\mathrm{Col}(A) = \mathrm{Row}(A^\top)$.

$[E.g.]$ Find a basis of $\mathrm{Col}(A)$, where $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$.

Sometimes, we would like to find a basis of $\mathrm{Col}(A)$ consisting of the columns in the original matrix.

[*E.g.*] Find a basis of $\mathrm{Col}(A)$ whose elements are columns of A, where

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 2 & 2 & 0 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

[*E.g.*] Let $W = \text{span}\{(1,0,1,1),(0,1,0,0)(1,1,1,1),(0,0,0,1)\}$. Find a basis of W consisting of a subset of the given spanning set.

[*E.g.*] Find a basis for $\text{Row}(A)$ and $\text{Col}(A)$.

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & -1 & 1 & 3 \\ 2 & -1 & 2 & 7 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Corollary (16.2.3). For any matrix A with transpose matrix $A^\top$:

$$\dim\mathrm{Row}(A) = \dim\mathrm{Col}(A) = \dim\mathrm{Col}(A^\top) = \mathrm{rank}(A) = \dim\mathrm{Row}(A^\top)$$

and all are equal to $\mathrm{rank}(A)$.

Chapter 17 - Bases for finite dimensional vector spaces

We would like to find bases of more general finite dimensional vector spaces, not just subspaces of $\mathbb{R}^n$.

[*E.g.*] Find a basis for the subspace W of $\mathbb{P}_3$ spanned by $\{3+x+4x^2+2x^3, 2+4x+6x^2+8x^3, 1+3x+4x^2+6x^3, -1+2x+x^2+4x^3\}$.

[*E.g.*] Find a basis of the subspace of $M_{2,2}$ spanned by the following set:

$$\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$$

[*E.g.*] Consider $\{(1,2,3,1),(1,2,3,2)\}$

(a) Prove that it's a LI set (this spans a 2-dimensional subspace in $\mathbb{R}^4$).

(b) Extend it to a basis of $\mathbb{R}^4$.

We conclude with the following theorem which combines everything we need to know about vector spaces associated to a square matrix.

Theorem (17.3.1). Let A be an $n \times n$ matrix. Then the following are equivalent:

1. $\operatorname{rank}(A) = n$.
2. $\operatorname{rank}(A^\top) = n$.
3. Every linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution.
4. The RREF of A is I_n.
5. $\operatorname{Null}(A) = \{\mathbf{0}\}$.
6. $\operatorname{Col}(A) = \mathbb{R}^n$.
7. $\operatorname{Row}(A) = \mathbb{R}^n$.
8. The columns of A are LI.
9. The rows of A are LI.
10. The columns of A form a basis of $\mathbb{R}^n$.
11. The rows of A form a basis of $\mathbb{R}^n$.