MAT 1348 – Winter 2023

Exercises 5 – Solutions

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Questions are taken from Discrete Mathematics 8th edition, by Kenneth H. Rosen.

QUESTION 1 **(2.1 # 1)**. List all the elements from the following sets.

(a) $\{x \mid x \text{ is a real number such that } x^2 = 1\}$
(b) $\{x \mid x \text{ is a positive integer less than } 12\}$
(c) $\{x \mid x \text{ is the square of an integer and } x < 100\}$
(d) $\{x \mid x \text{ is an integer such that } x^2 = 2\}$

Solution:

(a) $\{-1, 1\}$
(b) $\{1,2,3,4,5,6,7,8,9,10,11\}$
(c) $\{0,1,4,9,16,25,36,49,64,81\}$
(d) $\emptyset$

QUESTION 2 **(2.1 # 2)**. Use set-builder notation to describe the following sets:

(a) $\{0,3,6,9,12\}$
(b) $\{-3,-2,-1,0,1,2,3\}$
(c) $\{m,n,o,p\}$

Solution:

(a) $\{3n \mid n \in \{0,1,2,3,4\}\}$
(b) $\{x \in \mathbb{Z} \mid -3 \leq x \leq 3\}$
(c) $\{x \mid \text{x is a letter between m and p}\}$.

QUESTION 3 **(2.1 # 7)**. Determine if the following pairs of sets are equal.

(a) $\{1,3,3,3,5,5,5,5,5\}$ and $\{5,3,1\}$.
(b) $\{\{1\}\}$ and $\{1,\{1\}\}$.
(c) $\emptyset$ and $\{\emptyset\}$.

Solution:

(a) Yes
(b) No
(c) No

QUESTION 4 (**2.1 # 8**). Let $A = \{2,4,6\}, B = \{2,6\}, C = \{4,6\}$ and $D = \{4,6,8\}$. Determine which of the previous sets is a subset of another.

Solution:

$B \subseteq A, C \subseteq A, C \subseteq D$.

QUESTION 5 (**2.1 # 9, # 10**). Determine if 2 and $\{2\}$ belong to the following sets.

(a) $\{x \in \mathbb{R} \mid x \text{ is an integer greater than } 1\}$
(b) $\{x \in \mathbb{R} \mid x \text{ is the square of an integer }\}$
(c) $\{2, \{2\}\}$
(d) $\{\{2\}, \{\{2\}\}\}$
(e) $\{\{2\}, \{2, \{2\}\}\}$
(f) $\{\{\{2\}\}\}$

Solution:

(a) Yes, No
(b) No, No
(c) Yes, Yes
(d) No, Yes
(e) No, Yes
(f) No, No

QUESTION 6 (**2.1, # 11**). Determine if the following statements are true or false.

(a) $0 \in \emptyset$
(b) $\emptyset \in \{0\}$
(c) $\{0\} \subset \emptyset$
(d) $\emptyset \subset \{0\}$
(e) $\{0\} \in \{0\}$
(f) $\{0\} \subset \{0\}$
(g) $\{\emptyset\} \subseteq \{\emptyset\}$

Solution:

(a) False
(b) False
(c) False
(d) True
(e) False
(f) False
(g) True

QUESTION 7 **(2.1 # 12)**. Determine if the following are true or false.

(a) $\emptyset \in \{\emptyset\}$
(b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$
(c) $\{\emptyset\} \in \{\emptyset\}$
(d) $\{\emptyset\} \in \{\{\emptyset\}\}$
(e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$
(f) $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$
(g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

Solution:

(a) True
(b) True
(c) False
(d) True
(e) True
(f) True
(g) False

QUESTION 8 **(2.1 # 13)**. Determine if the following are true or false.

(a) $x \in \{x\}$
(b) $\{x\} \subseteq \{x\}$
(c) $\{x\} \in \{x\}$
(d) $\{x\} \in \{\{x\}\}$
(e) $\emptyset \subseteq \{x\}$
(f) $\emptyset \in \{x\}$

Solution:

(a) True
(b) True
(c) False
(d) True
(e) True
(f) False

QUESTION 9 **(2.1 # 21)**. What is the cardinality of the following sets?

(a) $\{a\}$
(b) $\{\{a\}\}$
(c) $\{a, \{a\}\}$
(d) $\{a, \{a\}, \{a, \{a\}\}\}$

Solution:

(a) 1
(b) 1
(c) 2
(d) 3

QUESTION 10 **(2.1 # 22)**. What is the cardinality of the following sets?

(a) $\emptyset$
(b) $\{\emptyset\}$
(c) $\{\emptyset, \{\emptyset\}\}$
(d) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

Solution:

(a) 0
(b) 1
(c) 2
(d) 3

QUESTION 11 **(2.1 # 23)**. Find the power set of each of the following. (Here, assume a and b are distinct)

(a) $\{a\}$
(b) $\{a, b\}$
(c) $\{\emptyset, \{\emptyset\}\}$

Solution:

(a) $\{\emptyset, \{a\}\}$
(b) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
(c) $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

QUESTION 12 **(2.1 # 25)**. How many elements do the following sets contain? Here, assume a and b are distinct.

(a) $\mathscr{P}(\{a, b, \{a, b\}\})$
(b) $\mathscr{P}(\{\emptyset, a, \{a\}, \{\{a\}\}\})$
(c) $\mathscr{P}(\mathscr{P}(\emptyset))$

Solution:

(a) 8
(b) 16
(c) 2

QUESTION 13 **(2.1 # 27)**. Show that $\mathscr{P}(A) \subseteq \mathscr{P}(B)$ if and only if $A \subseteq B$.

Solution: Suppose $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. Let $a \in A$. Therefore $\{a\} \in \mathscr{P}(A)$ and since $\mathscr{P}(A) \subseteq \mathscr{P}(B)$, we get $\{a\} \in \mathscr{P}(B)$.This implies $a \in B$. Therefore, for all $a \in A$, we have $a \in B$: we conclude $A \subseteq B$.

Suppose that $A \subseteq B$. Let $C \in \mathscr{P}(A)$. In that case, $C \subseteq A$. Every element of C is an element of A, and every element of A is an element of B: we conclude that every element of C is an element of B, and so $C \subseteq B$. In that case, $C \in \mathscr{P}(B)$. So, for all $C \in \mathscr{P}(A)$, we have $C \in \mathscr{P}(B)$: we conclude that $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

QUESTION 14 **(2.1 # 29)**. Let $A = \{a,b,c,d\}$ and $B = \{y,z\}$. Find

(a) $A \times B$
(b) $B \times A$

Solution:

(a) $\{(a,y),(b,y),(c,y),(d,y),(a,z),(b,z),(c,z),(d,z)\}$
(b) $\{(y,a),(y,b),(y,c),(y,d),(z,a),(z,b),(z,c),(z,d)\}$

QUESTION 15 **(2.1 # 33)**. Let A be a set. Show that $\emptyset \times A = A \times \emptyset = \emptyset$

Solution: $\emptyset \times A = \{(x,y) \mid x \in \emptyset \wedge y \in A\}$. Since no element x satisfies the condition $x \in \emptyset$, we conclude that $\emptyset \times A = \emptyset$. Similairly, $A \times \emptyset = \{(x,y) \mid x \in A \wedge y \in \emptyset\}$. Since no element y satisfies $y \in \emptyset$, we conclude $A \times \emptyset = \emptyset$.

QUESTION 16 **(2.1 # 34)**. Let $A = \{a,b,c\}$, $B = \{x,y\}$ and $C = \{0,1\}$. List the elements of

(a) $A \times B \times C$
(b) $C \times B \times A$
(c) $C \times A \times B$
(d) $B \times B \times B$

Solution:

(a) $\{(a,x,0),(a,x,1),(a,y,0),(a,y,1),(b,x,0),(b,x,1),(b,y,0),(b,y,1),(c,x,0),(c,x,1),(c,y,0),(c,y,1)\}$
(b) $\{(0,x,a),(0,x,b),(0,x,c),(0,y,a),(0,y,b),(0,y,c),(1,x,a),(1,x,b),(1,x,c),(1,y,a),(1,y,b),(1,y,c)\}$
(c) $\{(0,a,x),(0,a,y),(0,b,x),(0,b,y),(0,c,x),(0,c,y),(1,a,x),(1,a,y),(1,b,x),(1,b,y),(1,c,x),(1,c,y)\}$
(d) $\{(x,x,x),(x,x,y),(x,y,x),(x,y,y),(y,x,x),(y,x,y),(y,y,x),(y,y,y)\}$

QUESTION 17 **(2.1 # 35)**. Find A^2 if $A = \{0,1,3\}$

Solution: $\{(0,0),(0,1),(0,3),(1,0),(1,1),(1,3),(3,0),(3,1),(3,3)\}$

QUESTION 18 **(2.1 # 36)**. Find A^3 if $A = \{a\}$

Solution: $\{(a,a,a)\}$

QUESTION 19 **(2.1 # 41)**. Explain why $A \times B \times C \neq (A \times B) \times C$.

Solution: The elements of $A \times B \times C$ are triples of the form (a,b,c) where $a \in A$, $b \in B$ and $c \in C$. The elements of $(A \times B) \times C$ are pairs of the form $((a,b),c)$, where the first coordinate is a pair itself.

QUESTION 20 **(2.1 # 43)**. Prove or disprove the following statement: If A and B are sets, then $\mathscr{P}(A \times B) = \mathscr{P}(A) \times \mathscr{P}(B)$.

Solution: This is false. If we take $A = B = \emptyset$, we have $A \times B = \emptyset$, therefore $\mathscr{P}(A \times B) = \{\emptyset\}$. However, $\mathscr{P}(A) = \mathscr{P}(B) = \{\emptyset\}$ and so $\mathscr{P}(A) \times \mathscr{P}(B) = \{(\emptyset,\emptyset)\}$. We notice that $\mathscr{P}(A \times B) \neq \mathscr{P}(A) \times \mathscr{P}(B)$.

QUESTION 21 **(2.2 # 1)**. Let A be the set of students who live less than 1km away from the university, and let B be the set of students that walk to the university. Describe the following sets

(a) $A \cap B$
(b) $A \cup B$
(c) $A - B$
(d) $B - A$

Solution:

(a) The set of students who live less than 1km away from university and that walk to go there.
(b) The set of students who live less than 1km away from university or that walk to go there (or both).
(c) The set of students who live less than 1km away from university, but do not walk there.
(d) The set of students who walk to go to the university but do not live less than 1km away from there.

QUESTION 22 **(2.2 # 3)**. Let $A = \{1,2,3,4,5\}$ and $B = \{0,3,6\}$. Determine

(a) $A \cap B$
(b) $A \cup B$
(c) $A - B$
(d) $B - A$

Solution:

(a) $\{3\}$
(b) $\{0,1,2,3,4,5,6\}$
(c) $\{1,2,4,5\}$
(d) $\{0,6\}$

QUESTION 23 **(2.2 # 5-10)**. Suppose A is a subset of a universal set $\mathscr{U}$. Show the following equalities

(a) $\overline{\overline{A}} = A$
(b) $A \cup \emptyset = A$
(c) $A \cap \mathscr{U} = A$
(d) $A \cup \mathscr{U} = \mathscr{U}$
(e) $A \cap \emptyset = \emptyset$
(f) $A \cup A = A$
(g) $A \cap A = A$
(h) $A \cup \overline{A} = \mathscr{U}$
(i) $A \cap \overline{A} = \emptyset$
(j) $A - \emptyset = A$
(k) $\emptyset - A = \emptyset$.

Solution:

(a) $\overline{\overline{A}} = \{x \mid \neg(x \in \overline{A})\} = \{x \mid \neg\neg(x \in A)\} = \{x \mid x \in A\} = A.$
(b) $A \cup \emptyset = \{x \mid x \in A \vee x \in \emptyset\} = \{x \mid x \in A \vee F\} = \{x \mid x \in A\} = A.$
(c) $A \cap \mathscr{U} = \{x \mid x \in A \wedge x \in \mathscr{U}\} = \{x \mid x \in A \wedge T\} = \{x \mid x \in A\} = A.$
(d) $A \cup \mathscr{U} = \{x \mid x \in A \vee x \in \mathscr{U}\} = \{x \mid x \in A \vee T\} = \{x \mid T\} = \mathscr{U}.$
(e) $A \cap \emptyset = \{x \mid x \in A \wedge x \in \emptyset\} = \{x \mid x \in A \wedge F\} = \{x \mid F\} = \emptyset.$
(f) $A \cup A = \{x \mid x \in A \vee x \in A\} = \{x \mid x \in A\} = A.$
(g) $A \cap A = \{x \mid x \in A \wedge x \in A\} = \{x \mid x \in A\} = A.$
(h) $A \cup \overline{A} = \{x \mid x \in A \vee \neg(x \in A)\} = \{x \mid T\} = \mathscr{U}.$

(i) $A \cap \overline{A} = \{x \mid x \in A \wedge \neg(x \in A)\} = \{x \mid F\} = \emptyset$.
(j) $A - \emptyset = \{x \mid x \in A \wedge \neg(x \in \emptyset)\} = \{x \mid x \in A \wedge T\} = \{x \mid x \in A\} = A$.
(k) $\emptyset - A = \{x \mid x \in \emptyset \wedge \neg(x \in A)\} = \{x \mid F \wedge \neg(x \in A)\} = \{x \mid F\} = \emptyset$.

QUESTION 24 **(2.2 # 11)**. Let A and B be two sets. Show that

(a) $A \cup B = B \cup A$
(b) $A \cap B = B \cap A$

Solution:

(a) $A \cup B = \{x \mid x \in A \vee x \in B\} = \{x \mid x \in B \vee x \in A\} = B \cup A$
(b) $A \cap B = \{x \mid x \in A \wedge x \in B\} = \{x \mid x \in B \wedge x \in A\} = B \cap A$

QUESTION 25 **(2.2 # 12)**. Show that $A \cup (A \cap B) = A$.

Solution: Let $x \in A \cup (A \cap B)$. Then $x \in A$ or $x \in A \cap B$. In both cases, $x \in A$. So, $A \cup (A \cap B) \subseteq A$. Conversely, if $x \in A$, then $x \in A \cup (A \cap B)$, so $A \subseteq A \cup (A \cap B)$. We conclude that $A \cup (A \cap B) = A$.

QUESTION 26 **(2.2 # 13)**. Show that $A \cap (A \cup B) = A$.

Solution: Let $x \in A \cap (A \cup B)$. Then $x \in A$ and $x \in A \cup B$. Therefore, $x \in A$. This shows $A \cap (A \cup B) \subseteq A$. Conversely, if $x \in A$, then $x \in A \cup B$. Therefore, $x \in A \cap (A \cup B)$. This shows $A \subseteq A \cap (A \cup B)$. We conclude that $A \cap (A \cup B) = A$.

QUESTION 27 **(2.2 # 14)**. Find sets A and B such that $A - B = \{1,5,7,8\}$, $B - A = \{2,10\}$ and $A \cap B = \{3,6,9\}$.

Solution: $A = \{1,3,5,6,7,8,9\}$, $B = \{2,3,6,9,10\}$

QUESTION 28 **(2.2 # 15)**. Show that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Solution: $\overline{A \cup B} = \{x \mid \neg(x \in A \cup B)\} = \{x \mid \neg(x \in A \vee x \in B)\} = \{x \mid \neg(x \in A) \wedge \neg(x \in B)\} = \{x \mid x \in \overline{A} \wedge x \in \overline{B}\} = \overline{A} \cap \overline{B}$.

QUESTION 29 **(2.2 # 17)**. Show that if A and B are subsets of a universal set $\mathscr{U}$, then $A \subseteq B$ if and only if $\overline{A} \cup B = \mathscr{U}$.

Solution: Suppose $A \subseteq B$. We show that all elements $x \in \mathscr{U}$ are elements of $\overline{A} \cup B$. We have either $x \in \overline{A}$, or $x \in A$. In the first case, $x \in \overline{A} \cup B$. In the second case, $x \in B$, hence $x \in \overline{A} \cup B$.

Suppose now that $\overline{A} \cup B = \mathscr{U}$. Let $x \in A$. Since $x \notin \overline{A}$, we conclude that $x \in B$. This shows $A \subseteq B$.

QUESTION 30 **(2.2 # 19)**. Show that $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$.

Solution: $\overline{A \cap B \cap C} = \{x \mid \neg(x \in A \wedge x \in B \wedge x \in C)\} = \{x \mid \neg(x \in A) \vee \neg(x \in B) \vee \neg(x \in C)\} = \overline{A} \cup \overline{B} \cup \overline{C}$

QUESTION 31 **(2.2 # 21)**. Let A and B be sets. Show that

(a) $A - B = A \cap \overline{B}$
(b) $(A \cap B) \cup (A \cap \overline{B}) = A$

Solution:

(a) $A - B = \{x \mid x \in A \wedge x \notin B\} = \{x \mid x \in A \wedge x \in \overline{B}\} = A \cap \overline{B}$.
(b) $A = A \cap \mathscr{U} = A \cap (B \cup \overline{B}) = (A \cap B) \cup (A \cap \overline{B})$.

QUESTION 32 **(2.2 # 23)**. Show that $(A \cup B) \cup C = A \cup (B \cup C)$.

Solution: $(A \cup B) \cup C = \{x \mid (x \in A \cup B) \vee x \in C\} = \{x \mid (x \in A \vee x \in B) \vee x \in C\} = \{x \mid x \in A \vee (x \in B \vee x \in C)\} = \{x \mid x \in A \vee (x \in B \cup C)\} = A \cup (B \cup C)$.

QUESTION 33 **(2.2 # 31)**. What can you conclude about the sets A and B if

(a) $A \cup B = A$?
(b) $A \cap B = A$?
(c) $A - B = A$?
(d) $A \cap B = B \cap A$?
(e) $A - B = B - A$?

Solution:

(a) $B \subseteq A$
(b) $A \subseteq B$
(c) $A \cap B = \emptyset$
(d) Nothing, this equality is always true.
(e) $A = B$