17. Approximate Integration

- Recall: If f is integrable on $[a, b]$, then the definite integral $\int_a^b f(x)dx$ is the net area between $f(x)$ and the x-axis:

▶ We started by approximating such areas using Riemann sums:

- If the limit exists and is independent of our choice of sample points $x_i^* \in [x_{i-1}, x_i]$, then

- FTC 2 gave us an "easy" way to evaluate definite integrals without using the limit of the Riemann sum:

  but, FTC 2 requires us to know an antiderivative of the integrand.

- What if we don't know how to find (or it's impossible to find!) an antiderivative for a particular function?!

◀ Then it's back to approximating with rectangles, or more sophisticated approximations such as using trapezoids, or slices whose tops are parabolas.

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To approximate a definite integral $\int_a^b f(x)dx$ using a Riemann sum:

- ◇ Choose n (the # rectangles).
- ◇ Subdivide the interval $[a, b]$ into n subintervals of equal width:
- ◇ Choose a **sample point** $x_i^* \in [x_{i-1}, x_i]$ in the ith subinterval.
- ◇ Typical "good" sample points:

Left endpoint **Right endpoint**

MIDPOINT RULE

Trapezoidal Rule

Error Bounds

- The actual error might be smaller than these bounds.
- If we know a bound on $f''(x)$ for $a \leq x \leq b$, then this knowledge gives us a worst-case-scenario error bound. This allows us to choose n sufficiently large to guarantee that the error is no worse than $K(b-a)^3/12n^2$, or $K(b-a)^3/24n^2$, respectively.
- Notice that the error bound on T_n is twice the error bound on M_n (so typically, we have better guarantees from the Midpoint Rule).

Example 17.1. Use T_5, then M_5 to approximate $\displaystyle\int_1^2 \frac{1}{x}dx$

Example 17.2. How large should n be in order to guarantee that the error using T_n and M_n to estimate $\int_1^2 \frac{1}{x} dx$, respectively, is within 0.0001 ?

SIMPSON'S RULE

SIMPSON'S RULE ERROR BOUND

Suppose that $|f^{(4)}(x)| \leq K_4$ for $a \leq x \leq b$. Then

Example 17.3. Compute S_6 to approximate $\int_1^2 \frac{1}{x} dx$, then determine the smallest n needed in order to guarantee that S_n is within 0.0001 of the exact value of $\int_1^2 \frac{1}{x} dx$.

STUDY GUIDE

- $\diamond$ To estimate $\int_a^b f(x)dx$, choose n, compute $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$, $i = 0, 1, 2, \ldots, n$
- $\diamond$ $\boxed{L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x}$ $\quad (x_i^* = x_{i-1})$ $\qquad$ $\boxed{R_n = \sum_{i=1}^{n} f(x_i)\Delta x}$ $\quad (x_i^* = x_i)$
- $\diamond$ **Midpoint Rule**: $\boxed{M_n = \sum_{i=1}^{n} f(\overline{x_i})\Delta x}$ $\quad (\overline{x_i} = \frac{1}{2}(x_{i-1} + x_i))$
- $\diamond$ **Trapezoidal Rule**: $\boxed{T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]}$
- $\diamond$ **Simpson's Rule (n even)**:

$$\boxed{S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]}$$