Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Calculus I MAT1320

Second Midterm Exam

16 November 2022

Prof. Elizabeth Maltais

Instructions. *You must sign below to confirm that you have read, understand, and will follow them.*

- This is an 80-minute *closed-book* exam; no notes are allowed. Calculators are *not* permitted.
- The exam consists of 8 questions on 8 pages.
- Be sure to read carefully and follow the instructions for the individual problems. To receive full marks, your solution must be correct, complete, and show all relevant details.
- For rough work or additional work space, you may use the back pages. Do not use scrap paper of your own. Do not detach any pages.
- Use proper mathematical notation and terminology.
- You may ask for clarification.
- Unauthorized electronic devices (such as cellular phones) are not permitted during this exam. Such devices must be *turned off completely* and stored out of students' reach. Students found in possession of such a device during the exam will be asked to leave immediately and academic fraud allegations may be filed.

LAST NAME:____________________________

First name:____________________________

Signature:____________________________

Write your student number on the next page.

	C01	C02	C03
Circle your DGD section:	10:00am	11:30am	1:00pm
	FTX 361	LMX 219	VNR 1075

Possibly useful formulas

$$\sin^2(x) + \cos^2(x) = 1 \qquad \sin(2x) = 2\sin(x)\cos(x) \qquad \sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\tan^2(x) + 1 = \sec^2(x) \qquad \cos(2x) = \cos^2(x) - \sin^2(x) \qquad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

Student number:________________

Question	1	2	3	4	5	6	7	8	Total
Max	2	3	2	2	2	2	5	4	22
Marks									

[2pts] **1.** Consider the function: $y = \left(4x^4 + 6\right)^{\sin(3x)}$

Use **logarithmic differentiation** to find an expression for y' purely in terms of x.

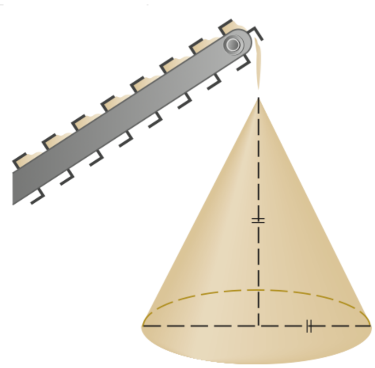
Solution:

$$
\begin{aligned}
y &= \left(4x^4 + 6\right)^{\sin(3x)} \\
\implies \quad \ln(y) &= \ln\left(\left(4x^4 + 6\right)^{\sin(3x)}\right) \\
\implies \quad \ln(y) &= \sin(3x)\ln\left(4x^4 + 6\right) \\
\implies \quad \frac{y'}{y} &= 3\cos(3x)\ln\left(4x^4 + 6\right) + \sin(3x)\left(\frac{4(4x^3)}{4x^4 + 6}\right) \\
\implies \quad y' &= y\left[3\cos(3x)\ln\left(4x^4 + 6\right) + \sin(3x)\left(\frac{4(4x^3)}{4x^4 + 6}\right)\right] \\
\implies \quad y' &= \left(4x^4 + 6\right)^{\sin(3x)}\left[3\cos(3x)\ln\left(4x^4 + 6\right) + \sin(3x)\left(\frac{4(4x^3)}{4x^4 + 6}\right)\right]
\end{aligned}
$$

2. Sand is being dumped from a conveyor belt at a rate of 4 m^3 / minute. The coarseness of the sand is such that it forms a pile in the shape of a cone whose **base diameter** and

[3pts] **height** are always equal (see figure below).

How fast is the height of the sand pile increasing when the pile is 6 m high?



Note: the volume V of a cone with height H and base radius R is $V = \frac{1}{3}\pi R^2 H$

Solution:

Let V denote the volume of the sand pile (in m^3), let R denote its radius, and let H denote its height.

We are given that $\frac{dV}{dt} = 4$ m^3 / minute.

Since the base diameter D is always equal to the pile's height H, we have $D = 2R = H$. Thus, $R = \frac{1}{2}H$. Thus,

$$V = \frac{1}{3}\pi R^2 H = \frac{1}{3}\pi \left(\frac{1}{2}H\right)^2 H$$

$$\implies \quad V = \frac{\pi}{12}H^3$$

$$\implies \quad \frac{dV}{dt} = \frac{3\pi}{12}H^2\frac{dH}{dt}$$

When $H = 6$ m, we have

$$\begin{aligned} \frac{dV}{dt} &= \frac{3\pi}{12}H^2\frac{dH}{dt} \\ \implies 4 &= \frac{\pi}{4}(6)^2\frac{dH}{dt} \\ \implies \frac{dH}{dt} &= \frac{4}{36\pi}(4) \\ &= \frac{16}{36\pi} \text{ m / minute} \end{aligned}$$

Therefore, when the pile is 6 m high, its height is increasing by $\frac{16}{36\pi}$ m / minute.

[2pts] **3.** Let $f(x) = \sqrt{3+x}$.

(a) Give the **linearization** of $f(x)$ at 1.

Solution:

We have $f(1) = \sqrt{3+1} = \sqrt{4} = 2$.

Also, $f'(x) = \frac{1}{2}(3+x)^{-1/2} = \frac{1}{2\sqrt{3+x}}$, so $f'(1) = \frac{1}{2\sqrt{3+1}} = \frac{1}{4}$.

Thus, $L(x) = f(1) + f'(1)(x-1) = 2 + \frac{1}{4}(x-1)$.

(b) Use this linearization to estimate $\sqrt{4.02}$. *You do not need to simplify your answer.*

Solution:

We have $\sqrt{4.02} = f(1.02) \approx L(1.02) = 2 + \frac{1}{4}(1.02-1) = 2 + \frac{0.02}{4}$.

[2pts] **4.** Suppose you know that

$$\int_{-2}^{3} f(x)\,dx = 5, \qquad \int_{-2}^{3} g(x)\,dx = 10, \qquad \text{and} \qquad \int_{3}^{5} \Big(3f(x) + 2g(x)\Big)\,dx = -1.$$

Find $\displaystyle\int_{-2}^{5} \Big(3f(x) + 2g(x)\Big)\,dx$.

Solution:

$$\begin{aligned}
\int_{-2}^{5} \Big(3f(x) + 2g(x)\Big)\,dx &= \int_{-2}^{3} \Big(3f(x) + 2g(x)\Big)\,dx + \int_{3}^{5} \Big(3f(x) + 2g(x)\Big)\,dx \\
&= 3\left(\int_{-2}^{3} f(x)\,dx\right) + 2\left(\int_{-2}^{3} g(x)\,dx\right) + (-1) \\
&= 3(5) + 2(10) + (-1) \\
&= 34
\end{aligned}$$

[2pts] **5.** Using the Fundamental Theorem of Calculus Part 1, evaluate the derivative:

$$\frac{d}{dx}\int_{2x}^{x^4} e^{1-t^2}\,dt$$

Solution:

$$\begin{aligned}
\frac{d}{dx}\int_{2x}^{x^4} e^{1-t^2}\,dt &= \frac{d}{dx}\left(\int_{2x}^{0} e^{1-t^2}\,dt + \int_{0}^{x^4} e^{1-t^2}\,dt\right) \\
&= \frac{d}{dx}\left(-\int_{0}^{2x} e^{1-t^2}\,dt\right) + \frac{d}{dx}\left(\int_{0}^{x^4} e^{1-t^2}\,dt\right) && u = 2x \\
&= -\frac{d}{dx}\left(\int_{0}^{2x} e^{1-t^2}\,dt\right) + \frac{d}{dx}\left(\int_{0}^{x^4} e^{1-t^2}\,dt\right) && v = x^4 \\
&= -\frac{d}{du}\left(\int_{0}^{u} e^{1-t^2}\,dt\right)\frac{du}{dx} + \frac{d}{dv}\left(\int_{0}^{v} e^{1-t^2}\,dt\right)\frac{dv}{dx} \\
&= -\left(e^{1-u^2}\right)(2) + \left(e^{1-v^2}\right)(4x^3) \\
&= -2e^{1-(2x)^2} + 4x^3 e^{1-(x^4)^2} \\
&= -2e^{1-4x^2} + 4x^3 e^{1-x^8}
\end{aligned}$$

[2pts] **6.** For each function $g(x)$ below, give its most general antiderivative $G(x)$.

For this question only, no justification is required.

(a) $g(x) = \dfrac{10}{(x+6)^3}$

(b) $g(x) = \dfrac{10}{\sqrt{1-x^2}}$

Solution:

(a) $G(x) = \dfrac{10}{-2}(x+6)^{-2} + C$ (b) $G(x) = 10\arcsin(x) + C$

[5pts] **7.** Evaluate each of the following integrals. *Show all your work!*

(a) $\displaystyle\int_1^4 2x\sqrt{2x^2-1}\,dx$

(b) $\displaystyle\int (4t^2+1)\cos(t)\,dt$

Solution:

(a)

$$\int_1^4 2x\sqrt{2x^2-1}\,dx = \int_{u=1}^{u=44} 2x\sqrt{u}\,\frac{du}{4x} \qquad u = 2x^2-1 \quad du = 4x\,dx \quad dx = \frac{du}{4x}$$

$$x = 1 \implies u = 1$$

$$x = 4 \implies u = 2(4)^2 - 1 = 44$$

$$= \frac{2}{4}\int_1^{44} \sqrt{u}\,du$$

$$= \frac{2}{4}\left[\frac{2}{3}u^{3/2}\right]_1^{44}$$

$$= \frac{2}{6}\left[u^{3/2}\right]_1^{44}$$

$$= \frac{2}{6}\left[(44)^{3/2} - (1)^{3/2}\right]$$

Solution:

(b)

parts:

$$u = 4t^2+1 \quad v' = \cos(t)$$

$$u' = 8t \quad v = \sin(t)$$

$$\int (4t^2+1)\cos(t)\,dt = (4t^2+1)\sin(t) - \int (8t)\sin(t)\,dt$$

parts again:

$$u = 8t \quad v' = \sin(t)$$

$$u' = 8 \quad v = -\cos(t)$$

$$= (4t^2+1)\sin(t) - \left(8t(-\cos(t)) - \int 8(-\cos(t))\,dt\right)$$

$$= (4t^2+1)\sin(t) + 8t\cos(t) - 8\int \cos(t)\,dt$$

$$= (4t^2+1)\sin(t) + 8t\cos(t) - 8\sin(t) + C$$

[4pts] **8.** We wish to evaluate $\displaystyle\int \frac{7}{(16x^2+1)^{5/2}}\,dx$ using trigonometric substitution.

(a) By choosing an appropriate trigonometric substitution and simplifying the result, show that

$$\int \frac{7}{(16x^2+1)^{5/2}}\,dx = \frac{7}{4}\int \cos^3(\theta)\,d\theta.$$

Solution:

$$
\begin{aligned}
\int \frac{7}{(16x^2+1)^{5/2}}\,dx &= 7\int \frac{1}{((4x)^2+1)^{5/2}}\,dx && \text{use trig sub: } 4x=\tan\theta \\
&= 7\int \frac{\frac{1}{4}\sec^2\theta\,d\theta}{(\tan^2\theta+1)^{5/2}} && x=\tfrac{1}{4}\tan\theta \Rightarrow dx=\tfrac{1}{4}\sec^2\theta\,d\theta \\
&= \frac{7}{4}\int \frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^{5/2}} && \text{since } \tan^2\theta+1=\sec^2\theta \\
&= \frac{7}{4}\int \frac{1}{\sec^3\theta}\,d\theta \\
&= \frac{7}{4}\int \cos^3\theta\,d\theta
\end{aligned}
$$

(b) Evaluate the integral $\dfrac{7}{4}\displaystyle\int \cos^3(\theta)\,d\theta$ in terms of θ.

Solution:

$$\frac{7}{4}\int \cos^3\theta\,d\theta = \frac{7}{4}\int \cos\theta\cos^2\theta\,d\theta$$

$$= \frac{7}{4}\int \cos\theta(1-\sin^2\theta)\,d\theta \qquad \text{use } u=\sin\theta \Rightarrow du=\cos\theta\,d\theta$$

$$= \frac{7}{4}\int (1-u^2)\,du$$

$$= \frac{7}{4}\left(u-\frac{1}{3}u^3\right)+C$$

$$= \frac{7}{4}\left(\sin\theta-\frac{1}{3}\sin^3\theta\right)+C$$

(c) According to the trig substitution you applied in part (a), express your answer from (b) in terms of x.

Your final answer should not include any trigonometric or inverse trigonometric functions.

Solution:
Since $4x=\tan\theta$, we have $\sin\theta = \dfrac{4x}{\sqrt{16x^2+1}}$

Thus,

$$\frac{7}{4}\left(\sin\theta-\frac{1}{3}\sin^3\theta\right)+C = \frac{7}{4}\left(\frac{4x}{\sqrt{16x^2+1}}-\frac{1}{3}\left(\frac{4x}{\sqrt{16x^2+1}}\right)^3\right)+C$$

End of the exam!