5. Derivatives: The Definition

Lec 4 mini review.

The Squeeze Theorem:

If $f(x) \le g(x) \le h(x)$, and

 $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$

then $\lim_{x\to a} g(x) = L$.

Limit Law for Continuous Composite Functions

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Continuous at a point *a*:

 $\lim_{x \to a} f(x) = f(a)$

Discontinous at *a* (hole, jump, asymptote)

One-sided Continuity

Continuous on an interval [a, b]

Intermediate Value Theorem

for a continuous function on [a, b]

Limits at Infinity Horizontal Asymptotes

THE DEFINITION OF THE DERIVATIVE AT A POINT — SLOPE OF A TANGENT

Recall, for h > 0, the **slope of the secant** passing through the points (a, f(a)) and (a + h, f(a + h)) is

Our goal for developing skills with limits was to take the limit as $h \to 0$.

Provided the limit exists, the **derivative of a function** f **at a number** a, denoted f'(a), is

^{*} These notes are solely for the personal use of students registered in MAT1320.

Example 5.1. Find the equation of the tangent line to the graph of $f(x) = \sqrt{x+3}$ at the point x=1.

INSTANTANEOUS RATE OF CHANGE

Recall: If f is **continuous** on the interval [a, b], then **average rate of change of** f **on** [a, b] is

The instantaneous rate of change of f at x = a is

Example 5.2. The height of a ball, *t* seconds after it is thrown with an initial upwards velocity of 10 m/s from the upper observation deck of the CN Tower (450 m above the ground), is given by

$$b(t) = -4.9t^2 + 10t + 450$$

What is the ball's instantaneous velocity

- when t = 3 s? when the ball hits the ground?
- when t = 1 s? at other times?
- If we want to calculate the derivative of a function at several different numbers, the calculations are all quite similar, but with different choices of *a*.
- Let's delay choosing a specific number a. Once we see the general pattern for f'(a), we can plug in whichever a we want later!

THE DERIVATIVE AS A FUNCTION

The derivative of f(x), denoted f'(x), is the function

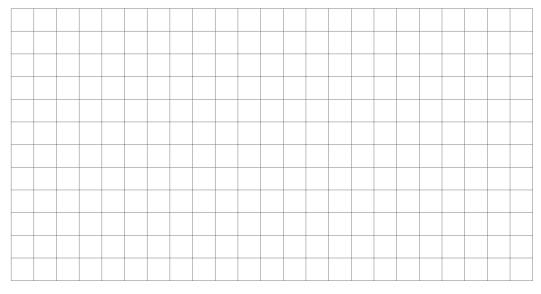
- The domain of f'(x) is the set of all x in the domain of f, for which the above limit exists.
- The domain of f' can be smaller than the domain of f.
- As a function of x, we can regard the value of f'(x) geometrically as the slope of the tangent line of f at x

Example 5.3. Using the following graph to draw a rough sketch of the function's derivative below.

graph of f(x)



sketch of f'(x)



OTHER NOTATION FOR THE DERIVATIVE

For y = f(x), here are several equivalent notations for the derivative f'(x).

DIFFERENTIABILITY

- ▶ A function f is **differentiable** at a if (the limit) f'(a) exists.
- ▶ f is differentiable on an open interval I if f'(a) exists for all $a \in I$.

Example 5.4. Where is the function f(x) = |x| differentiable?

Differentiability \Longrightarrow Continuity				
Theorem 5.5. If f is differentiable at a , then f is continuous at a .				
The converse of Theorem 5.5 is false:				
How Functions Can Fail to Be Differentiable				

HIGHER-ORDER DERIVATIVES

Let y=f(x) be a function. If, in each case, the preceding derivative exists, we can iterate the process of differentiation to take **higher-order derivatives**.

Notation

1st derivative of <i>f</i> :	derivative of f	f'(x)	y'	$\frac{dy}{dx}$
2nd derivative of f :	derivative of f'	f''(x)	y''	$\frac{d^2y}{dx^2}$
3rd derivative of <i>f</i> :	derivative of f''	f'''(x)	y'''	$\frac{d^3y}{dx^3}$
4th derivative of <i>f</i> :	derivative of f'''	$f^{(4)}(x)$	$y^{(4)}$	$\frac{d^4y}{dx^4}$
nth derivative of f :	derivative of $f^{(n-1)}$	$f^{(n)}(x)$	$y^{(n)}$	$\frac{d^ny}{dx^n}$

BASIC RULES

Constants

Powers

n = 0

n = 1

n=2

n = 3

STUDY GUIDE

Important terms and concepts:

- ♦ **Definition of the Derivative at** a: $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$
 - slope of tangent line instantaneous rate of change derivative at a point
- \diamond Definition of the Derivative as a Function: $f'(x) = \lim_{h \to 0} \frac{f(x+h) f(x)}{h}$
- \diamond Relationship between the graph of f and the graph of f'
- \diamond Other notation for derivative of y=f(x): f'(x) y' $\frac{dy}{dx}$ $\frac{df}{dx}$ $\frac{d}{dx}f(x)$ $D_xf(x)$
- $\diamond \ Differentiability \Longrightarrow Continuity$
- ♦ Failing to be differentiable (discontinuity, corner, vertical tangent line)
- \diamond Higher-order derivatives: f f' f'' f''' $f^{(4)}$ etc.