Lesson 3 – The Product Rule

PART A: How NOT to take the derivative of products

Example: Let $m(x) = f(x) \cdot g(x)$, $f(x) = x^4$ and $g(x) = x^6$.
Prove that $m'(x) \neq f'(x) \cdot g'(x)$

$m(x) = (x^4)(x^6)$
$m(x) = x^{10}$
$m'(x) = 10x^9$

$f'(x) = 4x^3$ $g'(x) = 6x^5$
$= (4x^3)(6x^5)$
$= 24x^8$

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PART B: How TO take the derivative of products

So, we have seen how NOT to take the derivative of products, so now let us look at how it is actually done.

The Product Rule

If $m(x) = f(x) \cdot g(x)$, then $m'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

or

In Leibniz notation, $\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + f(x) \cdot \frac{d}{dx}[g(x)]$

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Proof using first principles.

$$m'(x) = \lim_{h \to 0} \frac{m(x+h) - m(x)}{h}$$
$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Note: The purpose of the product rule is to express $m'(x)$ in terms of $f'(x)$ and $g'(x)$. Therefore, we would like to express the right side of the equation above in terms of:

$$\frac{f(x+h) - f(x)}{h} \quad and \quad \frac{g(x+h) - g(x)}{h}$$

This can be done by subtracting and adding $f(x) \cdot g(x+h)$

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Proof Cont'd...

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$
$$= \lim_{h \to 0} \frac{g(x+h)[f(x+h) - f(x)] + f(x)[g(x+h) - g(x)]}{h}$$
$$= \lim_{h \to 0} \frac{g(x+h)[f(x+h) - f(x)]}{h} + \lim_{h \to 0} \frac{f(x)[g(x+h) - g(x)]}{h}$$
$$= \left[\lim_{h \to 0} g(x+h)\right] \cdot \left[\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\right] + \left[\lim_{h \to 0} f(x)\right] \cdot \left[\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}\right]$$
$$= g(x+0)f'(x) + f(x)g'(x)$$
$$= f'(x)g(x) + f(x)g'(x)$$

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$f(x)$ $g(x)$

Example: Differentiate $y=(x^3+5x)(x^2-3x+2)$ using the product rule.

$y'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

$= (3x^2+5)(x^2-3x+2)+(x^3+5x)(2x-3)$

Question: Do you think it is necessary to expand the final answer of the derivative function?

No!! There is nothing to gain by expanding and simplifying this expression.

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PART C: Extension of the Product rule

Let's say you have to determine the derivative of a more complex function of the form:

$$m(x) = f(x)\cdot g(x)\cdot h(x)$$

You could use the extended product rule to differentiate this function:

The Extended Product Rule

If $m(x) = f(x)\cdot g(x)\cdot h(x)$, then
$m'(x) = f'(x)\cdot g(x)\cdot h(x) + f(x)\cdot g'(x)\cdot h(x) + f(x)\cdot g(x)\cdot h'(x)$

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However, this process can be simplified for power functions that take the following form:

$$y = (x^2-3)(x^2-3)(x^2-3)(x^2-3) = \underbrace{(x^2-3)^4}_{(g(x))^n} = 4(x^2-3)^3 \cdot (2x)$$

By using the Power of a Function Rule for Integers

The Power of a Function Rule for Integers

If u is a function of x, and n is an integer, then, $\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$,

or

In function notation, if $f(x) = [g(x)]^n$, then $f'(x) = n[g(x)]^{n-1} \cdot g'(x)$

This rule is a special case of the chain rule (to be explored in lesson 2.5).

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Example 1: Differentiate $f(x) = \underbrace{(2-3x^2)^9}_{(g(x))^n}$ using power of a function rule.

$g(x) = 2-3x^2$

$g'(x) = -6x$

$f'(x) = 9(2-3x^2)^{9-1} \cdot (-6x)$

$= 9(2-3x^2)^8 \cdot (-6x)$

$= -54x(2-3x^2)^8$

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Example 2: Determine the derivative of $f(x) = \frac{2x+5}{(3x-1)}$.

① turn into a product of two functions

$f(x) = \underbrace{(2x+5)}_{g(x)}\underbrace{(3x-1)^{-1}}_{h(x)}$ $g'(x) = 2$ $h'(x) = -1(3x-1)^{-2} \cdot (3)$

② use the product rule combined with power of a function rule to take derivative

$f'(x) = (2)(3x-1)^{-1} + (2x+5)\left[-1(3x-1)^{-2} \cdot (3)\right]$

③ Simplify

$= \frac{2}{(3x-1)} + (2x+5)(-3(3x-1)^{-2})$

$= \frac{2}{3x-1}\frac{(3x-1)}{(3x-1)} + \frac{(2x+5)(-3)}{(3x-1)^2}$

$= \frac{2(3x-1) - 3(2x+5)}{(3x-1)^2}$

$= \frac{6x-2-6x-15}{(3x-1)^2}$

$= \frac{-17}{(3x-1)^2}$

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Example 3: Student council is organizing its annual trip to a football game in Buffalo. For the past four years, the cost of the trip has been \$150 per person. At this price, all 200 of the seats on the buses were filled. This year, student council plans to increase the price of the trip. Based on a student survey, council estimates that for every \$10 increase in price, five fewer students will attend the football game.

a) Write an equation to represent the revenue, R, in dollars, as a function of the number of \$10 increases. Let n represent the number of \$10 increases

R = Price x quantity

$R = (150+10n)(200-5n)$

b) Determine an expression in simplified form, for $\frac{dR}{dn}$ and interpret it for this situation.

$\frac{dR}{dn} = \frac{d}{dn}(150+10n)(200-5n) + (150+10n)\frac{d}{dn}(200-5n)$

$= (10)(200-5n) + (150+10n)(-5)$

$= 2000 - 50n - 750 - 50n$

$= 1250 - 100n$

It represents the revenue per price increase.

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c) What is the change in revenue when the price of the trip is $200? How many will attend the football game at this price?

- How many times was the price increased?

$\frac{200-150}{10} = 5$ times $\boxed{n=5}$

$R'(n) = 1250 - 100n$

$= 1250 - 100(5)$

$= 1250 - 500$

$= \$750$ inc in revenue / price increase

$200 - 5(5)$

$= 200 - 25$

$= 175$ people

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