1. Propositional Logic

WHY MATHEMATICS IS AMAZING!

- Mathematics is a form of *Art*. The masterpieces of math are elegant arguments (proofs) that tell us some fragment of truth.
- Unlike other sciences where results are most often obtained from experimental data and observation, mathematical results are "discovered" when someone is able to prove that something must be true.
- For a proof to be correct, the underlying steps must be a valid logical argument and the language we use must be well-defined and exact.
- It is useful to know when something is true! Many scientific results are based on underlying mathematical models → physics, chemistry, biology, engineering...
- In this course, we will study DISCRETE MATH — the part of mathematics devoted to the study of "discrete" objects (consisting of distinct "whole" parts or unconnected elements).

Ex. DNA sequences = fundamental building blocks of living things

or = finite strings made of 4 symbols from the set {T (U), C, A, G} (RNA)

translation: RNA ex. —AGC $\boxed{\text{AUC}}$ GCAUCGA—

From DNA to RNA to building proteins from amino acids encoded by 3-letter "words" called codons

- Logic itself is especially important in Computer Science; the language of computers is Boolean algebra. Using logic, we are able to translate our intentions so that a machine made of electrical circuits does our bidding.

THE ROUGH PLAN FOR MAT1348

- We will begin by studying Propositional Logic.
- We will work our way towards writing and understanding correct Mathematical Proofs.
- We will study the basics of Set Theory, Functions, and Relations (especially Equivalence Relations).
- We will learn basic methods of Counting, and Proof by Induction — these are fundamental tools for math and computer science.
- We will end with an introduction to Graph Theory, which is an essential tool for modelling many discrete structures.

Warnings!

- This course is very fast-paced.
- You will be given **many** new and precise definitions and techniques to use.
- Having precise (lawyer-like?) language is very important — when we communicate (in the form of written mathematical proofs, computer programs, and otherwise), we want to be certain we understand each other exactly.

* These notes are solely for the personal use of students registered in MAT1348.

- A **proposition** is a declarative sentence that is either **true** or **false** but not both.
- If a proposition is true, then its **truth value** is TRUE, denoted **T**.
- If a proposition is false, then its **truth value** is FALSE, denoted **F**.

Example 1.1. For each of the following sentences, determine whether or not it is a proposition. If it is a proposition, what is its truth value? If it is not a proposition, explain.

(a) Ottawa is the capital of Canada. ← a proposition ✓ It's T.

(b) $1+2=3$. ← a proposition ✓ It's T.

(c) $2+2=3$. ← a proposition ✓ It's F.

(d) $1+x=3$. ← not a proposition unless we know what x is.

(e) There exists an integer x such that $1+x=3$. ← a proposition ✓ It's T.

(f) For all integers x, the equation $1+x=3$ is true. ← a proposition ✓ It's F.

(g) Eat your vegetables. ← not a proposition since it's not a declarative statement

(h) What time is it? ← not a proposition since it's not a declarative statement

(i) All books in the library are yellow.
a proposition ✓ It's F (at uOttawa library)

(j) At least 2 books in the library are yellow.
a proposition ✓ It's T (at uOttawa library)

Propositional variables (a.k.a atoms) and compound propositions.

- For short, we will use variables to represent propositions.
- The variables we use to represent propositions are called **propositional variables** or "**atoms**" for short.

Ex. p : "Tigers are furry animals."

Ex. q : "Elephants are wise."

- A **compound proposition** is formed by combining one or more atoms using **logical connectives.**

Negation. Let p be a proposition.

- The negation of p, denoted $\neg p$, read "not-p", is the proposition "It is not the case that p."
- The truth value of $\neg p$ is the opposite of the truth value of p.

Truth table:

p	$\neg p$
T	F
F	T

Conjunction. Let p and q be propositions.

- The conjunction of p and q, denoted $p \wedge q$, read "p and q", is the proposition

  "p and q"
- $p \wedge q$ is T when both p and q are T; otherwise, $p \wedge q$ is F.

Truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction. Let p and q be propositions.

- The disjunction of p and q, denoted $p \vee q$, read "p or q", is the proposition "p or q"
- $p \vee q$ is F when both p and q are F; otherwise, $p \vee q$ is T.

$\Longrightarrow$ "or" is the inclusive or i.e. "and/or"

Truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or. Let p and q be propositions.

- The exclusive or of p and q, denoted $p \oplus q$, read "p X-OR q", is the proposition

  "p or q but not both"
- $p \oplus q$ is T when exactly one of p and q is T and the other is F; otherwise, $p \oplus q$ is F.

Truth Table:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional Statement / Implication. Let p and q be propositions.

- The conditional statement "if p, then q", is denoted $p \to q$, read "p implies q", is the proposition "If p, then q."

p is the premise $\quad p \to q \quad$ q is the conclusion

- $p \to q$ is F when p is T and q is F; otherwise, $p \to q$ is T.
- $p \to q$ is like a promise or bargain. If p is T, then q is promised to be T. The only way a promise is broken is when p is T but q is F.

Truth Table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Note: [F → (anything)] is T.
[(anything) → T] is T.

Biconditional Statement. Let p and q be propositions.

- The biconditional statement "p if and only if q" is denoted $p \leftrightarrow q$, read "p if and only if q".
- $p \leftrightarrow q$ is T when p and q have the same truth value (both T or both F); otherwise, $p \leftrightarrow q$ is F.

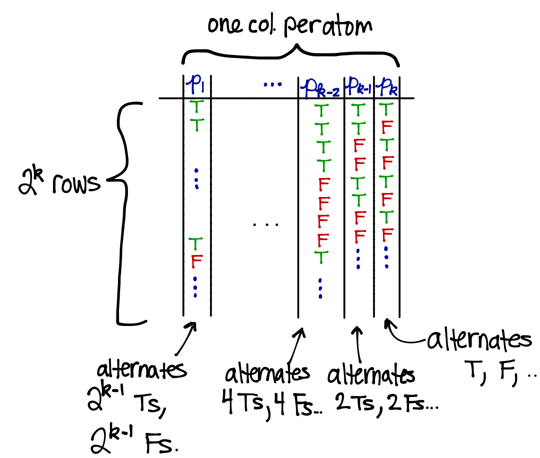
Truth Table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Tables

When making a truth table for a compound proposition consisting of atoms $p_1, p_2, \ldots, p_k$, we will always use the following convention:

- 2^k rows for k atoms $p_1, \ldots, p_k$
- truth values always Ts and Fs (not 0s/1s)
- Each atom has its own column.
- For rightmost atom's column, we alternate T, F, T, F, ... from top to bottom of column
  - Moving one column to the left, the next atom's column alternates twice as many Ts and twice as many Fs as the column on its immediate right



Example 1.2. Write a truth table for each of the following compound propositions:

a. $(p \to q) \leftrightarrow (\neg p \vee q)$

b. $(p \oplus q) \to \neg q$

c. $(p \to q) \wedge (q \to p)$

d. $\neg(\neg p \vee q)$

e. $(p \wedge \neg p) \leftrightarrow (p \vee \neg p)$

f. $q \leftrightarrow (p \leftrightarrow r)$

a)

p	q	$p \to q$	$\neg p \vee q$	$(p \to q) \leftrightarrow (\neg p \vee q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

b)

p	q	$p \oplus q$	$\neg q$	$(p \oplus q) \to \neg q$
T	T	F	F	T
T	F	T	T	T
F	T	T	F	F
F	F	F	T	T

c)

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

d)

p	q	$\neg p$	$\neg p \vee q$	$\neg(\neg p \vee q)$
T	T	F	T	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	F

e)

p	$\neg p$	$p \wedge \neg p$	$p \vee \neg p$	$(p \wedge \neg p) \leftrightarrow (p \vee \neg p)$
T	F	F	T	F
F	T	F	T	F

f)

p	q	r	$p \leftrightarrow r$	$q \leftrightarrow (p \leftrightarrow r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	F
T	F	F	F	T
F	T	T	F	F
F	T	F	T	T
F	F	T	F	T
F	F	F	T	F

TAUTOLOGIES, CONTRADICTIONS, AND CONTINGENCIES

- A compound proposition that is always true (in all rows of its truth table) is called a **tautology**.
- A compound proposition that is always false (in all rows if its truth table) is called a **contradiction**.
- A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

Example 1.3. For each compound proposition given in Example 1.2, determine whether it is a tautology, contradiction, or contingency.

a) $(p \to q) \leftrightarrow (\neg p \vee q)$ is always T $\therefore$ tautology

b) $(p \oplus q) \to \neg q$ is sometimes T, sometimes F $\therefore$ contingency

c) $(p \to q) \wedge (q \to p)$ is sometimes T, sometimes F $\therefore$ contingency

d) $\neg(\neg p \vee q)$ is sometimes T, sometimes F $\therefore$ contingency

e) $(p \wedge \neg p) \leftrightarrow (p \vee \neg p)$ is always F $\therefore$ contradiction

f) $q \leftrightarrow (p \leftrightarrow r)$ is sometimes T, sometimes F $\therefore$ contingency

STUDY GUIDE

- ◇ proposition   propositional variable / atom   compound proposition
- ◇ truth value   truth table (how many rows?)
- ◇ logical connectives: negation $\neg$ disjunction (or) $\vee$ conjunction (and) $\wedge$
  implication $\to$ biconditional (if and only if) $\leftrightarrow$ exclusive or (XOR) $\oplus$
- ◇ tautology   contradiction   contingency

Rosen Exercises	1.1 # 1,3,5,9,11,29,31,33,37 1.3 # 9,15