Lesson 3 Vertical and Horizontal Asymptotes

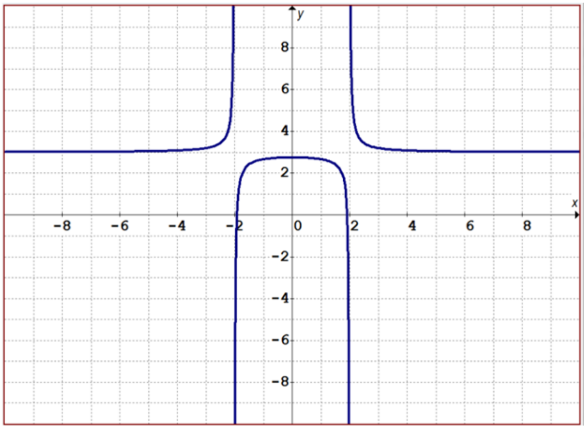
PART A: Vertical Asymptotes

The graph of a function $f(x)$ has a vertical asymptote at $x = a$ if $\lim_{x \to a} f(x) = \pm\infty$.

The easiest way to find vertical asymptotes is to find where the denominator is zero.

Example 1: Consider the function

$$f(x) = \frac{1}{x^2 - 4} + 3.$$



The vertical asymptotes occur when $x^2 - 4 = 0$

$(x-2)(x+2) = 0$

$\left.\begin{array}{l} x = 2 \\ x = -2 \end{array}\right\}$ V.A.

So the graph has vertical asymptotes at *x*=2 and at *x*=-2

PART B: Horizontal Asymptotes

The graph of a function $f(x)$ has a horizontal asymptote at $y = L$ if $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$. The easiest way to find horizontal asymptotes is to take the limit of the function as *x* approaches infinity.

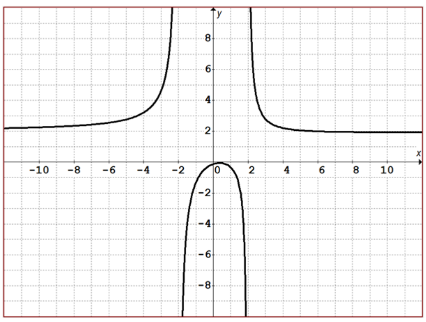
Example 2: Consider the function $f(x) = \dfrac{4x^2 - 3x + 1}{2x^2 - 8}$

The horizontal asymptote occurs where

$$\lim_{x \to \pm\infty} \frac{4x^2 - 3x + 1}{2x^2 - 8}$$

$$\lim_{x \to \infty} \frac{\frac{4\cancel{x^2}}{\cancel{x^2}} - \cancel{\frac{3x}{x^2}}^{0} + \cancel{\frac{1}{x^2}}^{0}}{\frac{2\cancel{x^2}}{\cancel{x^2}} - \cancel{\frac{8}{x^2}}^{0}}$$

$= 2$



So the graph has a horizontal asymptote at *y*=2

$2x^2 - 8 = 0$

$x^2 - 4 = 0$

$(x-2)(x+2) = 0$

PART C: Slant Asymptotes

The graph of a function $f(x)$ has a slant (oblique) asymptote $y = mx + b$ if when you take the limit as x approaches infinity, the result can be written in the form: $y = \lim_{x\to\infty}(mx+b) + \lim_{x\to\infty} g(x)$ where $\lim_{x\to\infty} g(x) = 0$. This can often be achieved by performing polynomial division.

Example 3: Consider the function $\boxed{f(x) = \dfrac{2x^2 + x - 2}{x-1}}$.

Using polynomial division, we can write:

$$
\begin{array}{r}
2x+3 \\
x-1\overline{)\,2x^2+x-2} \\
\underline{2x^2-2x} \\
3x-2 \\
\underline{3x-3} \\
1
\end{array}
$$

You can also use synthetic division as long as your divisor is *linear*

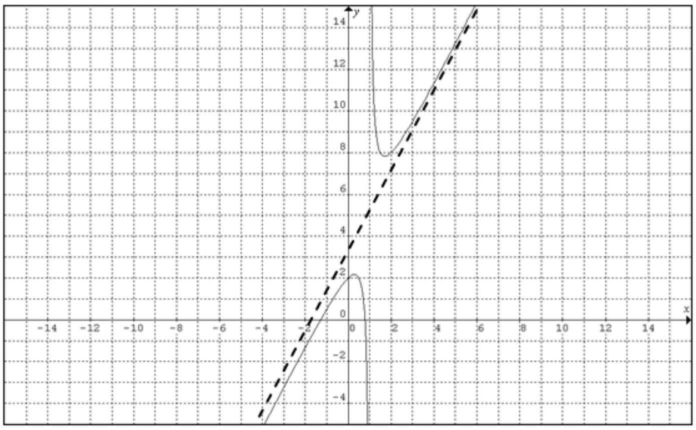
The division statement can be written as:

$$\frac{2x^2+x-2}{(x-1)} = \frac{(x-1)(2x+3)}{(x-1)} + \frac{1}{(x-1)}$$

OR

$$\boxed{\frac{2x^2+x-2}{x-1}} = \boxed{(2x+3) + \frac{1}{x-1}}$$

$$\lim_{x\to\infty} f(x)$$
$$= \lim_{x\to\infty} \frac{2x^2+x-2}{x-1}$$
$$= \lim_{x\to\infty}\left[(2x+3) + \frac{1}{x-1}\right]$$
$$= \lim_{x\to\infty}(2x+3) + \lim_{x\to\infty}\frac{1}{x-1}$$
$$= \lim_{x\to\infty}(2x+3) + \quad 0$$
$$= \lim_{x\to\infty}(2x+3)$$



So the line $y = 2x+3$ is a slant asymptote for this graph. Note the graph is approaching the dotted line having the equation $y = 2x+3$

** We need to verify that the quotient we get from above is the oblique asymptote by using limits (*it usually is*)

Note

For slant asymptotes, the degree of the numerator is higher than that of the denominator.

PART D: Analyzing behaviour of a rational function near its asymptotes

Example 4: For the function $f(x) = \frac{3x}{x^2-x-6}$, determine the equations of all asymptotes and illustrate the behaviour of the graph as it approaches the asymptotes.

For VA

$x^2-x-6=0$

$(x-3)(x+2)=0 \rightarrow$ VA $\rightarrow x=3, x=-2$

values of x	$3x$	$x-3$	$x+2$	$f(x)$	$f(x)\rightarrow$
-2^-	$-$	$-$	$-$	$-$	$-\infty$
-2^+	$-$	$-$	$+$	$+$	∞
3^-	$+$	$-$	$+$	$-$	$-\infty$
3^+	$+$	$+$	$+$	$+$	∞

For HA

- can't use a/c
- use limits

$\lim_{x\to\infty} \frac{3x}{x^2-x-6}$

$= \lim_{x\to\infty} \frac{\frac{3x}{x^2}\to 0}{\frac{x^2}{x^2}-\frac{x}{x^2}\to 0-\frac{6}{x^2}\to 0}$

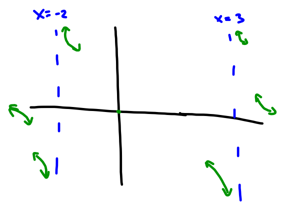
$= \frac{0}{1-0-0}$

$= 0$

Similarly,

$\lim_{x\to-\infty} \frac{3x}{x^2-x-6} = 0$

$\therefore$ HA: $y=0$



For HA

$\lim_{x\to+\infty} \frac{3x}{x^2-x-6}$

$= \lim_{x\to\infty} \frac{\frac{3x}{x^2}}{\frac{x^2}{x^2}-\frac{x}{x^2}-\frac{6}{x^2}}$

$= \lim_{x\to+\infty} \frac{\cancel{\frac{3}{x}}\to 0}{1-\cancel{\frac{1}{x}}\to 0-\cancel{\frac{6}{x^2}}\to 0}$

$= \frac{0}{1}$

$= 0$

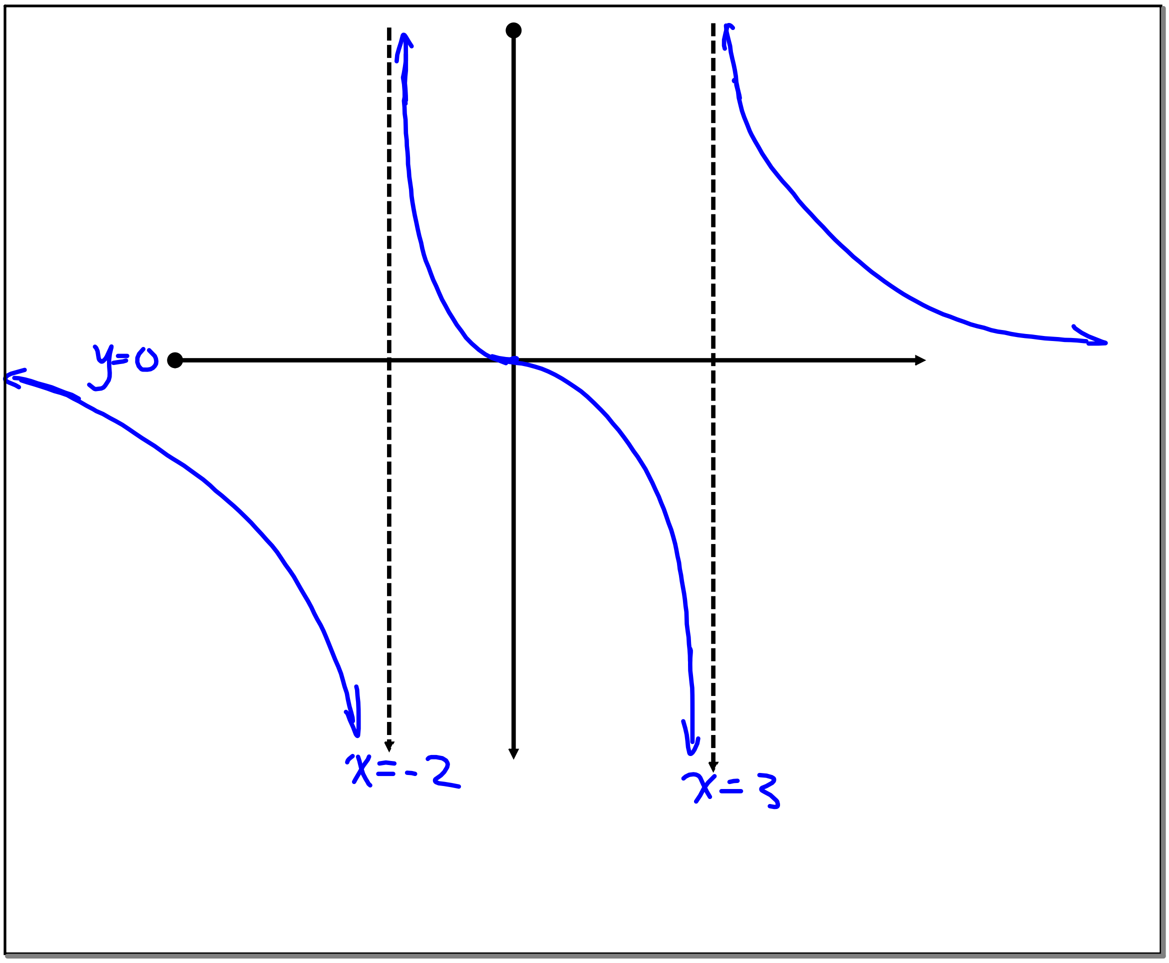
Similarly,

$\lim_{x\to-\infty} f(x) = 0$

$\therefore$ H.A. $\rightarrow y=0$

Behaviour around H.A.

	$x\to-\infty$	$x\to+\infty$
$3x$	$-$	$+$
x^2-x-6	$+$	$+$
$f(x)$	$-$	$+$
	below H.A.	above H.A.



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\# 1, 2, 3ab,

4adf, 5ab,

7ab