MAT 1348 – Winter 2023

Exercises 4 – Solutions

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Questions are taken from Discrete Mathematics 8th edition, by Kenneth H. Rosen.

QUESTION 1 **(1.7 # 1)**. Use a direct proof to show that the sum of two odd numbers is even.

Solution: Let n and m two odd numbers. Then, there exists an integer k such that $n = 2k+1$ and there exists an integer l such that $m = 2l+1$. Therefore, $m+n = 2(k+l+1)$. Since $k+l+1$ is an integer, we conclude that $m+n$ is even.

QUESTION 2 **(1.7 # 3)**. Use a direct proof to show that the square of an even number is also even.

Solution: Let n be an even number. Then, there exists an integer k such that $n = 2k$. Therefore, $n^2 = (2k)^2 = 2(2k^2)$. Since $2k^2$ is an integer, we conclude n^2 is even.

QUESTION 3 **(1.7 # 5)**. Show that if $m+n$ and $n+p$ are even, and if m,n and p are integers, then $m+p$ is also even. What type of proof did you use?

Solution: We use a direct proof. Suppose $m+n$ and $n+p$ are even. There exists an integer k such that $m+n = 2k$ and an integer l such that $n+p = l$. If we add the two numbers together, we get $m+2n+p = 2(k+l)$, which becomes $m+p = 2(k+l-n)$. Since $k+l-n$ is an integer, we conclude that $m+p$ is even.

QUESTION 4 **(1.7 # 7)**. Use a direct proof to show that every odd number is the difference of two square numbers. (Hint: compute the difference between the square of $k+1$ and the square of k, where k is any positive integer.)

Solution: Let n be an odd number. Therefore, the exists an integer k such that $n = 2k+1$. Therefore, $(k+1)^2 - k^2 = 2k+1 = n$, so n is the difference between two squares, namely $(k+1)^2$ and k^2.

QUESTION 5 **(1.7 # 13)**. Show that if x is irrationnal, then $\frac{1}{x}$ is irrationnal.

Solution: We use an indirect proof. Suppose $\frac{1}{x}$ is rationnal. There exists two integers p and $q \neq 0$ such that $\frac{1}{x} = \frac{p}{q}$. Since $\frac{1}{x}$ cannot be equal to 0, we get that $p \neq 0$. In this case, $x = \frac{q}{p}$. Therefore, x is rationnal.

QUESTION 6 **(1.7 # 17)**. Use an indirect proof to show that if $x+y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.

Solution: Suppose "$x \geq 1$ or $y \geq 1$" is false, so suppose $x < 1$ and $y < 1$. In this case, $x+y < 1+1 = 2$, which is the negation of $x+y \geq 2$

QUESTION 7 **(1.7 # 18)**. Show that if m and n are integers and mn is even, then m is even or n is even.

Solution: We use an indirect proof. Suppose "m is even or n is even" is false, so m is odd and n is odd. There exists an integer k such that $m = 2k+1$ and an integer l such that $n = 2l+1$. In this case, $mn = (2k+1)(2l+1) = 2(2kl+k+l)+1$. Since $2kl+k+l$ is an integer, we conclude mn is odd, which is the negation of "mn is even".

QUESTION 8 **(1.7 # 19)**. Show that if n is an integer and n^3+5 is odd, then n is even.

Solution: We use an indirect proof. Suppose n is odd, so there exists an integer k such that $n = 2k+1$. In this case, $n^3+5 = (2k+1)^3+5 = 2(4k^3+6k^2+3k+3)$. Since $4k^3+6k^2+3k+3$ is an integer, we conclude n^3+5 is even, which is the negation of n^3+5 is odd".

QUESTION 9 **(1.7 # 29)**. Let n be a positive integer. Show that n is odd if and only if $5n+6$ is odd.

Solution: Let P = "n is odd" and Q = "$5n+6$ is odd". We must show $P \leftrightarrow Q$, which is equivalent to $(P \to Q) \wedge (Q \to P)$. We have two implications to prove.

To show $P \to Q$, we use a direct proof. Suppose n is odd, so there exists an integer k such that $n = 2k+1$. In that case, $5n+6 = 2(5k+5)+1$. Since $5k+5$ is an integer, we conclude that $5n+6$ is odd.

To show $Q \to P$, we use an indirect proof (we show $\neg P \to \neg Q$). Suppose n is even, so there exists an integer m such that $n = 2m$. In that case, $5n+6 = 2(5m+3)$. Since $5m+3$ is an integer, we conclude that $5n+6$ is even, which is the negation of Q.

QUESTION 10 **(1.7 # 41)**. Let $a_1, a_2, ..., a_n$ be real numbers. Show that at least one of these numbers is greater or equal to their average.

Solution: Let A be the average of $a_1, a_2, ..., a_n$. So, $A = \frac{1}{n}(a_1 + a_2 + ... + a_n)$. We use a proof by contradiction. Suppose $a_1, a_2, ..., a_n$ are all less than A. In other words, suppose $a_1 < A$, $a_2 < A$,..., $a_n < A$. In this case, $a_1 + a_2 + ... + a_n < nA$, and so $A = \frac{1}{n}(a_1 + a_2 + ... + a_n) < A$. We get a contradiction ($A < A$). We conclude that the statement must be true.

QUESTION 11 **(1.8 # 1)**. Show that $n^2+1 \geq 2^n$ whenever n is a positive integer such that $1 \leq n \leq 4$.

Solution: We do a proof by cases. If $n = 1$, the inequality $n^2+1 \geq 2^n$ becomes $2 \geq 2$, which is true. If $n = 2$, the inequality becomes $5 \geq 4$, which is true. If $n = 3$, the inequality becomes $10 \geq 8$, which is true. If $n = 4$, the inequality becomes $17 \geq 16$, which is true. We conclude the statement is true.

QUESTION 12 **(1.8 # 3)**. Show that there are no positive integers n such that $n^3 = 100$.

Solution: We show that if n is a positive integer, then $n^3 \neq 100$. We split the proof into two cases: $n \leq 4$ and $n \geq 5$. Suppose first that $n \leq 4$. In this case, $n^3 \leq 4^3 = 64 \leq 100$, and so $n^3 \neq 100$. Suppose now $n \geq 5$. In this case, $n^3 \geq 5^3 = 125 \geq 100$, and so $n^3 \neq 100$.