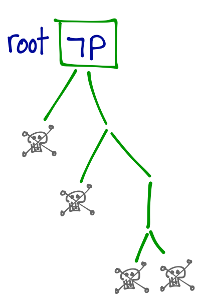
6. Tautology-testing using Truth Trees and Arguments

Things to keep in mind when growing a truth tree:

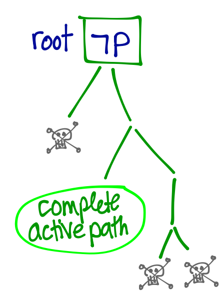
- ♠ From each leaf (at the bottom of the tree so far), there is a unique path up to the root.
- ♠ A path from leaf to root is **alive/active** until there is some atom and its negation on the path. Whenever a path contains an atom and that atom's negation, the path is **dead/inactive**.
- ♠ A path from leaf to root is **complete** (fully grown) after all its propositions are **checked** (their branching rules grow on all paths stemming below the checked proposition, or the proposition is simply a literal), or a path is complete when it is dead/inactive.
- ♠ A properly grown truth tree is always **binary** (at most 2 branches stem from one proposition).
- ♠ You should apply rules to the principal connectives of each proposition (do not use logical equivalences to rewrite a proposition before branching).

Turning a Tautology–Problem into a Contradiction–Problem

- From a tree with $\boxed{P}$ at its root, it is easy to decide whether P is a contradiction
- P is a tautology if and only if $\neg P$ is a contradiction
- To use a tree to decide whether P is a tautology, we will grow a tree with root $\boxed{\neg P}$ and test whether $\neg P$ is a contradiction !



all paths dead tells us root $\neg P$ is a contradiction, which means P itself is a tautology



at least one complete active path tells us root $\neg P$ can be true, which means P itself can be false.

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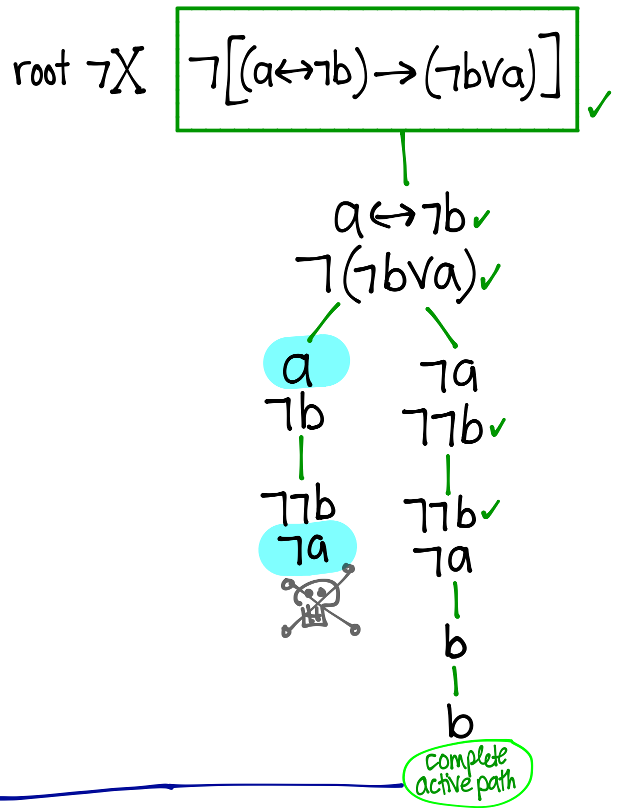
Example 6.1. Using an appropriate truth tree, determine whether each of the following propositions is a tautology? If it is not a tautology, give all counterexamples.

i). $X: \quad (a \leftrightarrow \neg b) \to (\neg b \vee a)$

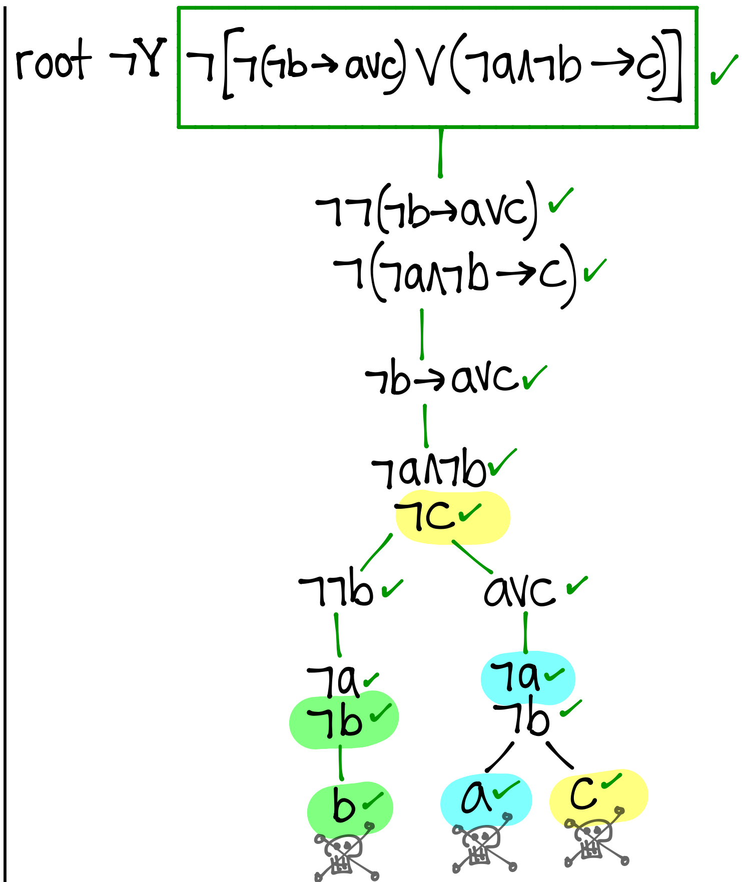
ii). $Y: \quad \big[\neg(\neg b \to (a \vee c))\big] \vee \big[(\neg a \wedge \neg b) \to c\big]$



Since ¬X can be True, it means X can be False. ∴ X is not a tautology.

Counterexample (to show X is not a tautology):

when a=F, b=T, X is F.



All paths are dead with root ¬Y

∴ ¬Y is a contradiction

∴ Y itself is a tautology!

Exercise 6.2. We know already that determining whether P and Q are logically equivalent amounts to determining whether the biconditional statement $P \leftrightarrow Q$ is a tautology.

Suppose we want to determine whether P and Q are logically equivalent using a truth tree. What should we place at the root of the tree and why? Explain.

P is logically equivalent to Q if and only if (P↔Q) is a tautology

if and only if ¬(P↔Q) is a contradiction

Strategy Put $\boxed{\neg(P \leftrightarrow Q)}$ as root and test whether ¬(P↔Q) is a contradiction.

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ARGUMENTS

- An **argument** is a set of propositions in which one (called the **conclusion**) is claimed to follow from the other propositions (called the **premises**). In other words, an argument is a compound proposition of the form

argument: $(P_1 \wedge P_2 \wedge \cdots \wedge P_k) \rightarrow C$

where $P_1, P_2, \ldots, P_k$ are the premises; C is the conclusion

$\Rightarrow$ an argument is a conditional statement whose overall "big" premise is the conjunction of one or more "little" premises $P_1 \wedge P_2 \wedge \cdots \wedge P_k$

- Sometimes, arguments are written vertically like this:

$$\begin{array}{l} P_1 \\ P_2 \\ \vdots \\ \underline{P_k \quad} \\ \therefore C \end{array}$$

VALID ARGUMENTS

- An argument $(P_1 \wedge P_2 \wedge \cdots \wedge P_k) \rightarrow C$ is called **a valid argument** if the conclusion C is true whenever all the premises $P_1, \ldots, P_k$ are true.

- In other words, $(P_1 \wedge P_2 \wedge \cdots \wedge P_k) \rightarrow C$ is a **valid argument** if and only if $(P_1 \wedge P_2 \wedge \cdots \wedge P_k) \rightarrow C$ is a tautology.

Example 6.3. Prove that the following argument is valid:

2 premises: P_1: $A \vee B$

P_2: $\neg A$

Conclusion $\therefore C$: $\therefore B$

(this argument is called *Disjunctive Syllogism*)
(it is one of the *Rules of Inference*)

◇ To show this is a valid argument, we must prove that $[(A \vee B) \wedge (\neg A)] \rightarrow B$ is a tautology.

A	B	P_1: $A \vee B$	P_2: $\neg A$	C: B	$P_1 \wedge P_2 \rightarrow C$: $[(A \vee B) \wedge (\neg A)] \rightarrow B$
T	T	T	F	T	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

Since $[(A \vee B) \wedge (\neg A)] \rightarrow B$ is a tautology, this argument is valid

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Example 6.4.

(2 premises)

P_1 $\quad A \vee B$

P_2 $\quad \neg B$

$\therefore C$ $\quad \therefore \neg A$

Is this argument valid? If not, give all counterexamples.

$\boxed{\text{It's a valid argument if and only if } [(A \vee B) \wedge (\neg B)] \to (\neg A) \text{ is a tautology.}}$

A	B	premise P_1 $A \vee B$	premise P_2 $\neg B$	conclusion C $\neg A$	argument $(P_1 \wedge P_2) \to C$ $[(A \vee B) \wedge (\neg B)] \to (\neg A)$
T	T	T	F	F	T
T	F	T	T	F	F
F	T	T	F	T	T
F	F	F	T	T	T

there is one counterexample: when A=T, B=F, both premises are T but conclusion is F.

the argument is not a tautology so it is invalid.

DETERMINING THE VALIDITY OF ARGUMENTS WITH TRUTH TREES

How would we use a truth tree to determine whether or not the argument $(P_1 \wedge \cdots \wedge P_k) \to C$ is **valid**?

$(P_1 \wedge \cdots \wedge P_k) \to C$ is a valid argument

if and only if $(P_1 \wedge \cdots \wedge P_k) \to C$ is a tautology

if and only if $\neg[(P_1 \wedge \cdots \wedge P_k) \to C]$ is a contradiction

<u>Strategy</u> Put $\boxed{\begin{matrix} P_1 \\ P_2 \\ \vdots \\ P_k \\ \neg C \end{matrix}}$ as root and test whether it's a contradiction.

all premises ($P_1, P_2, \ldots, P_k$)

negation of conclusion → $\neg C$

Note: this root (list) is like starting with the branching of $\neg[(P_1 \wedge \cdots \wedge P_k) \to C]$

which is $\equiv P_1 \wedge P_2 \wedge \cdots \wedge P_k \wedge \neg C$

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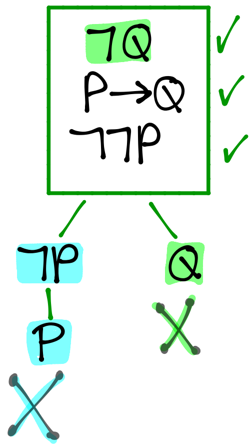
Example 6.5. Use a truth tree to determine whether or not each of the following arguments is valid.

$$\begin{array}{l} \neg Q \\ P \to Q \\ \hline \therefore \neg P \end{array}$$

root ≡ negation of argument

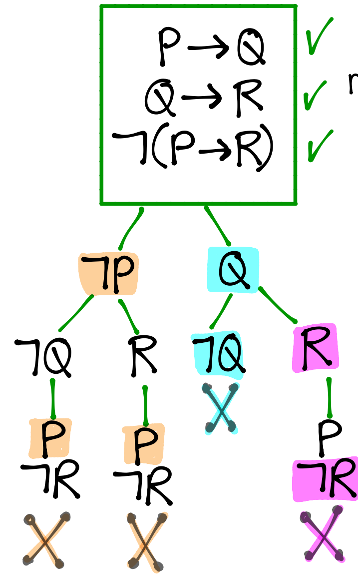


Since all paths are dead, it means the root is a contradiction. Since the root is logically equivalent to the argument's negation, the argument itself must be a tautology

hence, the argument is <u>valid</u>

$$\begin{array}{l} P \to Q \\ Q \to R \\ \hline \therefore P \to R \end{array}$$

root ≡ negation of argument

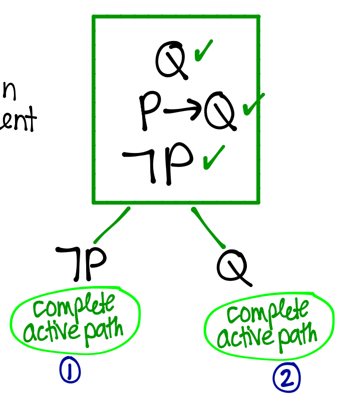


Since all paths are dead, it means the root is a contradiction. Since the root is logically equivalent to the argument's negation, the argument itself must be a tautology

hence, the argument is <u>valid</u>

$$\begin{array}{l} Q \\ P \to Q \\ \hline \therefore P \end{array}$$

root ≡ negation of argument



Since there exists at least one complete active path, it means the root is <u>not</u> a contradiction. The root is equivalent to the argument's negation, so the argument itself is not a tautology ∴ this argument is <u>invalid</u>

counterexamples:

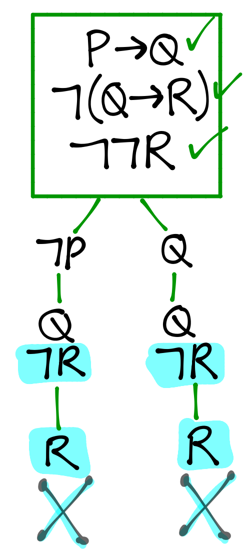
① P=F, Q=T

② Q=F, P=T

for each of these truth assignments, both premises are true but the conclusion is false.

$$\begin{array}{l} P \to Q \\ \neg(Q \to R) \\ \hline \therefore \neg R \end{array}$$

root ≡ negation of argument



The argument's negation is a contradiction since all paths are dead.

∴ the argument itself is a tautology, hence it is a valid argument.

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Go to DGD 4 to see solution!

Exercise 6.6. Translate the following argument into propositional logic. Then use a truth tree to determine whether it's valid or not. If it's valid, explain how you know this based on your tree. If it's invalid, give all counterexamples.

If it is hungry, then the bear eats berries or the bear eats trout. The bear eats berries only if it does not see trout. Whenever the bear sees trout and is hungry, it eats trout. The bear eats berries. Therefore, the bear is not hungry.

Use the propositional variables:

h: The bear is hungry.
b: The bear eats berries.
t: The bear eats trout.
s: The bear sees trout.

Study Guide

Important terms and concepts:

- **argument**: $P_1 \wedge \cdots \wedge P_k \rightarrow C$ **valid** vs. **invalid** argument
- truth trees (semantic tableaux) branching rules active vs. dead/inactive paths
- using a truth tree to determine whether a proposition is a tautology
- using a truth tree to determine whether an argument is valid

Exercises	Sup.Ex. §1 # 5, 6 (with trees and truth tables) Rosen (8th) §1.6 # 35

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