DGD 5

Q1. SETS: ELEMENTS, SUBSETS, CARDINALITY, POWER SET, CARTESIAN PRODUCT

Let $A = \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ let $B = \{\emptyset, \{\emptyset\}\}$ and let $C = \{a, e, i, o, u\}$

i. Determine the cardinalities of A, B, and C.

$|A| = 2 \qquad |B| = 2 \qquad |C| = 5$

ii. What is $|\mathcal{P}(A)|$? What is $|\mathcal{P}(\mathcal{P}(B))|$? What is $|\mathcal{P}(C)|$?

$|\mathcal{P}(A)| = 2^{|A|} = 2^2 = 4$

$|\mathcal{P}(C)| = 2^{|C|} = 2^5 = 32$

$|\mathcal{P}(\mathcal{P}(B))| = 2^{|\mathcal{P}(B)|} = 2^{(2^{|B|})} = 2^{(2^2)} = 2^4 = 16$

Compute the power set of A and the power set of B. List at least 4 elements of $\mathcal{P}(C)$.

$\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{\{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}\}$

$\mathcal{P}(B) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

$\emptyset \in \mathcal{P}(C)$ because $\emptyset \subseteq C$

$C \in \mathcal{P}(C)$ because $C \subseteq C$

$\{a, o\} \in \mathcal{P}(C)$ because $\{a, o\} \subseteq C$

$\{i\} \in \mathcal{P}(C)$ because $\{i\} \subseteq C$

etc...

$\mathcal{P}(C) = \{\emptyset, \{a\}, \{e\}, \{i\}, \{o\}, \{u\}, \{a,e\}, \{a,i\}, \{a,o\}, \{a,u\}, \{e,i\}, \{e,o\}, \{e,u\}, \{i,o\}, \{i,u\}, \{o,u\}, \{a,e,i\}, \{a,e,o\}, \{a,e,u\}, \{a,i,o\}, \{a,i,u\}, \{a,o,u\}, \{e,i,o\}, \{e,i,u\}, \{e,o,u\}, \{i,o,u\}, \{a,e,i,o\}, \{a,e,i,u\}, \{a,e,o,u\}, \{a,i,o,u\}, \{e,i,o,u\}, \{a,e,i,o,u\}\}$

iii. List all the elements of $B \times B$.

$(\emptyset, \emptyset) \in B \times B \qquad (\emptyset, \{\emptyset\}) \in B \times B$

$(\{\emptyset\}, \emptyset) \in B \times B \qquad (\{\emptyset\}, \{\emptyset\}) \in B \times B$

Thus,

$B \times B = \{(\emptyset, \emptyset), (\emptyset, \{\emptyset\}), (\{\emptyset\}, \emptyset), (\{\emptyset\}, \{\emptyset\})\}$

iv. Determine the cardinalities of $A \times B$ and $B \times \mathcal{P}(A)$ and $A \times B \times C$.

$|A \times B| = |A| \cdot |B| = 2 \cdot 2 = 4$

$|B \times \mathcal{P}(A)| = |B| \cdot |\mathcal{P}(A)| = |B| \cdot 2^{|A|} = 2 \cdot 2^2 = 8$

$A \times B \times C = |A| \cdot |B| \cdot |C| = 2 \cdot 2 \cdot 5 = 20$

v. Which of the following statements are true for the sets A, B, C given below?

$$A = \{\emptyset, \{\emptyset, \{\emptyset\}\}\} \qquad B = \{\emptyset, \{\emptyset\}\} \qquad C = \{a, e, i, o, u\}$$

$\emptyset \subseteq B$ true	$\emptyset \in B$ true	$\{\{\emptyset\}\} \subseteq B$ true
$\emptyset \subseteq C$ true	$\emptyset \in C$ false	$\emptyset \subseteq \mathcal{P}(C)$ true
$\{a, i\} \in C$ false	$\{a, i\} \subset C$ true	$\{a, i\} \subseteq C$ true
$B \subseteq A$ false	$\{\emptyset\} \in \mathcal{P}(A)$ true	$\{\{\{\emptyset\}\}\} \subseteq \mathcal{P}(B)$ true
$B \in A$ true	$B \in \mathcal{P}(A)$ false	$\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \in \mathcal{P}(A)$ false
$\emptyset \in A$ true	$\{\emptyset\} \in A$ false	$\{\emptyset\} \subseteq A$ true
$(\emptyset, \emptyset) \in A \times B$ true	$(\emptyset, \emptyset) \subseteq A \times B$ false	$\emptyset \subseteq A \times B$ true
$\{(\emptyset, \emptyset)\} \subset B \times A$ true	$\{(i, i), (a, e)\} \in C \times C$ false	$(e, a) \in C \times C$ true

Q2. Let $A = \{n \in \mathbb{Z} : 3 \mid n\}$, $B = \{n \in \mathbb{Z} : 12 \mid n\}$, and $C = \{n \in \mathbb{Z} : 2 \mid n\}$.

i. Is $A \subseteq B$? If so, prove it rigorously. If not, give a concrete counterexample and briefly explain.

No. Counterexample (there are many other possible counterexamples)

$15 \in A$ since $3 \mid 15$ is true $(15 = 5 \cdot 3)$

but $15 \notin B$ since $12 \nmid 15$.

$\therefore$ the implication $(x \in A) \to (x \in B)$ is false for $x = 15$.

(hence is not true for all x)

$\therefore A \not\subseteq B$.

ii. Is $B \subseteq C$? If so, prove it rigorously. If not, give a concrete counterexample and briefly explain.

Yes. Proof. Assume $x \in B$. (goal: prove $x \in C$)

Then $12 \mid x$ by def. of B.

$\Longrightarrow x = 12k$ for some integer $k \in \mathbb{Z}$ (def. of $12 \mid x$)

$\Longrightarrow x = 2(6k)$

$\Longrightarrow x = 2m$ where $m = 6k$. Since $k \in \mathbb{Z}$, we know $m = 6k \in \mathbb{Z}$.

$\therefore 2 \mid x$ $\therefore x \in C$ (by def. of C).
(goal!)

We proved that $(x \in B) \to (x \in C)$ is true for arbitrary x, thus $B \subseteq C$. ∎

iii. Use set-builder notation to describe the sets $A \cap B$, $A \cap C$, $A \cup B$, $A - C$, and $A \oplus C$.

$A \cap B = \{x \in \mathbb{Z} : 3 \mid x \text{ and } 12 \mid x\} = \{x \in \mathbb{Z} : 12 \mid x\} = B$. (because $B \subseteq A$)

$A \cap C = \{x \in \mathbb{Z} : 3 \mid x \text{ and } 2 \mid x\} = \{x \in \mathbb{Z} : 6 \mid x\}$ (because 6 is the lowest common multiple of 3 and 2)

$A \cup B = \{x \in \mathbb{Z} : 3 \mid x \text{ or } 12 \mid x\} = \{x \in \mathbb{Z} : 3 \mid x\} = A$ (because $B \subseteq A$)

$A - C = \{x \in \mathbb{Z} : 3 \mid x \text{ and } 2 \nmid x\} = \{x \in \mathbb{Z} : x = 3m \text{ for some } m \in \mathbb{Z} \text{ and } x \text{ is odd}\}$
$= \{x \in \mathbb{Z} : x = 6k + 3 \text{ for some } k \in \mathbb{Z}\}$

$A \oplus C = \{x \in \mathbb{Z} : 3 \mid x \text{ or } 2 \mid x \text{ but not both}\} = \{x \in \mathbb{Z} : 3 \mid x \text{ or } 2 \mid x \text{ but } 6 \nmid x\}$.

Q3. PROOF INVOLVING SETS

Prove the following theorem using an appropriate proof strategy:

Theorem 6.2. Let X and Y be sets. Then $\underbrace{\mathcal{P}(X) \subseteq \mathcal{P}(Y)}_{P}$ if and only if $\underbrace{X \subseteq Y}_{Q}$.

$P \longleftrightarrow Q$

To prove $P \leftrightarrow Q$ we must prove $P \to Q$ and the converse $Q \to P$. (proof of equivalence)

<u>Proof</u>. ($\Longrightarrow$) To prove $P \to Q$ we will use an indirect proof. Assume $\neg Q$ is T and prove $\neg P$ must follow.

$\neg Q$: $X \not\subseteq Y$ $\quad\quad$ $\neg P$: $\mathcal{P}(X) \not\subseteq \mathcal{P}(Y)$

Assume $\neg Q$ is T. <u>ie</u> Assume $X \not\subseteq Y$.

Therefore there exists at least one $x \in \mathcal{U}$ such that $x \in X$ but $x \notin Y$.

Consequently, $\{x\} \subseteq X$ but $\{x\} \not\subseteq Y$.

Therefore, $\{x\} \in \mathcal{P}(X)$ but $\{x\} \notin \mathcal{P}(Y)$

Therefore $\mathcal{P}(X)$ contains an element that is not also an element of $\mathcal{P}(Y)$

<u>ie</u> $\mathcal{P}(X) \not\subseteq \mathcal{P}(Y)$ Thus $\neg P$ is true.

So we proved $\neg Q \to \neg P$ which is $\equiv$ $P \to Q$.

($\Longleftarrow$) To prove the converse $Q \to P$ we will use a direct proof. Assume Q is T and prove P must follow

Assume Q is T. <u>ie</u> Assume $X \subseteq Y$.

Then $(x \in X) \to (x \in Y)$ is true for all $x \in \mathcal{U}$.

Let S be any element of $\mathcal{P}(X)$. Then, S must be a subset of X by def. of $\mathcal{P}(X)$

Since $S \subseteq X$, we know that

$(x \in S) \to (x \in X)$ is true for all $x \in \mathcal{U}$.

Since $X \subseteq Y$, we also know that $(x \in X) \to (x \in Y)$ for all $x \in \mathcal{U}$.

for all $x \in \mathcal{U}$ we have

$$\begin{array}{l} (x \in S) \to (x \in X) \\ (x \in X) \to (x \in Y) \\ \hline \therefore (x \in S) \to (x \in Y) \end{array}$$

Consequently, $S \subseteq Y$, hence $S \in \mathcal{P}(Y)$.

Thus, we proved that $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$ <u>ie</u> P is True $\therefore Q \to P$ is T.

Since both $P \to Q$ and $Q \to P$ are true, it follows that $P \longleftrightarrow Q$ is True. $\square$

Q4. USING THE LAWS FROM THE TABLE OF IMPORTANT SET IDENTITIES

Prove the following generalization of De Morgan's Law using set identities:

$$\overline{(A \cup B) \cup C} = \left(\overline{A} \cap \overline{B}\right) \cap \overline{C}$$

$$\overline{(A\cup B)\cup C} = \overline{(A\cup B)} \cap \overline{C} \quad \text{(De Morgan's Law)}$$

$$= (\overline{A} \cap \overline{B}) \cap \overline{C} \quad \text{(De Morgan's Law)}$$

Note: Because of the associative law, it is common to omit parentheses when writing several unions (or intersections, respectively) in a row.

ie. we might write this identity as $\overline{A\cup B\cup C} = \overline{A}\cap\overline{B}\cap\overline{C}$

since $(A\cup B)\cup C = A\cup(B\cup C)$ and $(\overline{A}\cap\overline{B})\cap\overline{C} = \overline{A}\cap(\overline{B}\cap\overline{C})$

Q5. VERIFYING A SET IDENTITY

Prove that $C - (\overline{A} \cap B) = (C \cap A) \cup (C - B)$

i. using the laws from the Table of Important Set Identities

$$C-(\overline{A}\cap B) = C\cap\overline{(\overline{A}\cap B)} \quad \text{(Difference Law)}$$

$$= C\cap(\overline{\overline{A}}\cup\overline{B}) \quad \text{(De Morgan's Law)}$$

$$= C\cap(A\cup\overline{B}) \quad \text{(Double Complementation Law)}$$

$$= (C\cap A)\cup(C\cap\overline{B}) \quad \text{(Distributive Law)}$$

$$= (C\cap A)\cup(C-B) \quad \text{(Difference Law)}$$

ii. using a membership table

A	B	C	$\overline{A}\cap B$	$C-(\overline{A}\cap B)$	$C\cap A$	$C-B$	$(C\cap A)\cup(C-B)$
1	1	1	0	1	1	0	1
1	1	0	0	0	0	0	0
1	0	1	0	1	1	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	0	1	0	1	1
0	0	0	0	0	0	0	0

Since membership is the same for $C-(\overline{A}\cap B)$ and $(C\cap A)\cup(C-B)$, it follows that $(C-(\overline{A}\cap B)) = (C\cap A)\cup(C-B)$