MAT 1348 Final Exam

Discrete Mathematics for Computing (University of Ottawa)

MAT1348B Winter 2020 Final Exam Cover Page Instructions

This is a **2-hour open-book** final exam. The exam will be available for 3 hours in total; this includes the time to scan and upload and submit all your answers in Brightspace.

You may use a **basic scientific calculator**. Basic scientific calculators are permitted during this exam.

Although it is an open-book exam, you **must sign a statement** to declare that you will comply with the following **terms of the MAT1348 exam**:

- **All** work done during the exam will be done **entirely by yourself, with no help from others**.
- You will **not** communicate with anybody except the professor during the exam (for exam-related questions).
- You will not consult any people, sources, or writings other than the course textbook, your self-made review sheets, your course notes, the course documents made available on Brightspace, the video lectures, and the exam itself.
- You will not share information about the exam's contents before the exam period is over on April 24.
- You certify that all solutions are entirely your own work, that you did not consult people or unauthorized resources during the examination, and did not share information with others during the exam.

In your uploaded exam files, you must include a copy (photograph or scan) of a page with the following student declaration, written in your own handwriting and signed and dated with your student id card:

Student's Declaration:

I ____*Your full name printed here*____ have read and I agree to honour the terms of the MAT1348 exam.

SIGNATURE:__________ STUDENT #:________ DATE:________

Include image of your student id card on your student declaration page.

Failure to comply with the terms of this exam may result in academic fraud allegations being filed and may result in you obtaining zero for this exam.

Final Exam Submission Checklist:

- ☐ You have included your Student Declaration Page written, signed, with Student Photo ID attached.

- ☐ Your answers to MULTIPLE-CHOICE Questions 1–9 are included, with question numbers and selected answers clearly labelled.

- ☐ Your solutions to LONG-ANSWER Questions 10–14 are included, with question numbers clearly labelled.

- ☐ Your NAME, STUDENT NUMBER and SIGNATURE appears ON EVERY PAGE you submit.

- ☐ Your photos/scanned copies of your work are each WELL LIT and IN FOCUS and ORIENTED CORRECTLY.

- ☐ You have uploaded ALL of your work, each time you submitted files (remember that updating your submissions will over-write all files previously submitted).

General Guidelines:

You do not need to copy any of the questions.

If you have a printer, or tablet, you may fill in your answers on the exam itself; otherwise, regular paper with hand-written answers is fine.

Please ensure your work will be clearly visible when you scan your solutions.

Should you have any issues, email your professor as soon as possible `mlemir2@uottawa.ca`

The exam consists of 9 multiple-choice questions worth a total of 19 points and 5 long-answer questions worth a total of 21 points.

Multiple Choice Questions

Select the correct response(s).

You do not need show your work for the multiple-choice questions.

Write your answers to multiple-choice questions using the following template:

Q1. ☐

Q2. ☐

Q3. ☐

Q4. ☐

Q5. ☐ ☐ ☐

Q6. ☐

Q7. ☐

Q8. ☐

Q9. ☐

Question 1. [2 points]

Which one of the following propositions is **not** logically equivalent to the following proposition P ?

$$P: \quad (a \wedge b) \rightarrow \neg c$$

Select the correct response:

A. $\neg a \vee (\neg b \vee \neg c)$

B. $\neg c \rightarrow (a \wedge b)$

C. $\neg(a \wedge (b \wedge c))$

D. $c \rightarrow \neg(a \wedge b)$

E. $a \rightarrow (b \rightarrow \neg c)$

F. $(a \rightarrow \neg c) \vee (b \rightarrow \neg c)$

Question 2. [2 points]

Consider the following propositional variables and argument:

H: "The wolf will howl."
M: "The moon is full."
S: "The wolf stalks its prey."

Argument:

The wolf stalks its prey if and only if the moon is not full.
A necessary condition for the wolf to howl is that the moon is full.

$\therefore$ The wolf will howl if it stalks its prey.

Fact: The above argument is **not** valid.

Which one of the following truth assignments is a **counterexample** to certify this fact?

Select the correct response:

A. $H = \mathbf{T}$, $M = \mathbf{T}$, and $S = \mathbf{T}$

B. $H = \mathbf{T}$, $M = \mathbf{T}$, and $S = \mathbf{F}$

C. $H = \mathbf{T}$, $M = \mathbf{F}$, and $S = \mathbf{T}$

D. $H = \mathbf{T}$, $M = \mathbf{F}$, and $S = \mathbf{F}$

E. $H = \mathbf{F}$, $M = \mathbf{T}$, and $S = \mathbf{T}$

F. $H = \mathbf{F}$, $M = \mathbf{T}$, and $S = \mathbf{F}$

G. $H = \mathbf{F}$, $M = \mathbf{F}$, and $S = \mathbf{T}$

H. $H = \mathbf{F}$, $M = \mathbf{F}$, and $S = \mathbf{F}$

Question 3. [2 points]

Let x, y, z be propositional variables, and let P and Q be the following compound propositions:

$P: \quad \neg(x \wedge z) \oplus (y \leftrightarrow \neg x)$ $\qquad$ $Q: \quad ((x \wedge y) \rightarrow z) \rightarrow z$

What types of propositions are P and Q?

Select the correct response:

A. P and Q are both contingencies (neither a tautology or a contradiction).

B. P is a contingency (neither a tautology or a contradiction) and Q is a contradiction.

C. P is a contingency (neither a tautology or a contradiction) and Q is a tautology.

D. P is a contradiction and Q is a contingency (neither a tautology or a contradiction).

E. P and Q are both contradictions.

F. P is a tautology and Q is a contingency (neither a tautology or a contradiction).

G. P is a tautology and Q is a contradiction.

H. P and Q are both tautologies.

Question 4. [2 points]

Let A, B, and C be finite sets.

Which one of the following statements is always true?

Select the correct response:

A. $|\mathcal{P}(A) \times \mathcal{P}(B)| = |\mathcal{P}(A \times B)|$

B. $A \cap B \subseteq A \cup C$

C. $A \cup B \subseteq B \cup C$

D. $A \cap (B \cup C) = (A \cap B) \cup C$

E. If $A - B = \emptyset$, then $A = B$.

F. If $B - A = C - A$, then $B = C$.

G. $A \subseteq \mathcal{P}(A)$

Question 5. [3 points]

Consider the following functions:

$$f: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \qquad\qquad g: \mathbb{R} \to \mathbb{Z}$$
$$f(s,t) = s + 2t \qquad\qquad g(x) = 2\lceil x \rceil + 1$$

Recall: $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ denotes the ceiling function, namely $\lceil x \rceil = \min\{m \in \mathbb{Z} : m \geq x\}$.

Which three of the following statements are true?

Select the **three** correct responses:

A. f is injective (one-to-one).

B. f is surjective (onto).

C. f is a bijection.

D. g is injective (one-to-one).

E. g is surjective (onto).

F. g is invertible.

G. The domain of the composition $g \circ f$ is $\mathbb{R} \times \mathbb{R}$

H. The codomain of the composition $g \circ f$ is $\mathbb{Z}$

Question 6. [2 points]

Consider the following relation $\mathcal{R}$ on the set $A = \{1, 2, 3, 4, 5\}$.

$$\mathcal{R} = \{(1,1),\ (2,2),\ (2,3),\ (3,2),\ (3,3),\ (4,4),\ (4,5),\ (5,4),\ (5,5)\}$$

Given that $\mathcal{R}$ is an equivalence relation on A, which of the following is the partition of A into equivalence classes?

Select the correct response.

A. $\mathcal{P} = \Big\{\{1\}, \{1,2\}, \{3\}, \{3,4\}, \{4\}, \{5\}\Big\}$

B. $\mathcal{P} = \Big\{\{1,2,3,4,5\}\Big\}$

C. $\mathcal{P} = \Big\{\{1,2\}, \{3,4\}, \{5\}\Big\}$

D. $\mathcal{P} = \Big\{\{1\}, \{2,3\}, \{4,5\}\Big\}$

E. $\mathcal{P} = \Big\{\{1,2,3\}, \{4,5\}\Big\}$

F. $\mathcal{P} = \Big\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\Big\}$

G. $\mathcal{P} = \Big\{\{1,2\}, \{3\}, \{4,5\}\Big\}$

Question 7. [2 points]

How many 6-symbol strings are there consisting of symbols from the set

$$\{1, 2, 3, 4, a, b, c, d\}$$

that start with two letters **and** end with '3d' ? Symbols may be repeated.

Select the correct response:

A. 972

B. 1024

C. 49

D. 448

E. 441

F. 324

G. 64

H. 40

I. 36

Question 8. [2 points]

Suppose a 5-member committee is being formed by selecting its members from among a group of 12 people consisting of 3 wizards, 4 elves, and 5 hobbits.

In how many ways can this be done if the 5-member committee must contain

exactly 2 of the hobbits **and** at most 2 of the wizards?

Select the correct response:

A. 1152

B. 270

C. 1320

D. 340

E. 310

F. 300

G. 276

H. 1348

Question 9. [2 points]

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$.

How many **injective** (one-to-one) functions $f : A \to B$ have $f(1) = 1$ **or** $f(2) = 5$?

Select the correct response:

A. 21

B. 30

C. 36

D. 40

E. 45

F. 48

G. 55

H. 60

I. 65

Long-Answer Questions

Important: In all proofs, for each step, you must clearly indicate whether you are assuming something, or whether what you wrote is something that follows from a definition or a previous step of your proof. If any variables appear in your proof, make sure you clearly indicate what they represent.

Question 10. [4 points]

Let S and T be the following subsets of the universal set $\mathcal{U} = \mathbb{Z}$:

$S = \{n \in \mathbb{Z} : 4 \text{ does not divide } n\}$ and $T = \{m \in \mathbb{Z} : 8 \text{ does not divide } m\}$.

(a) List two distinct elements of the set $S \cap T$.

(b) For this part, you will prove that $S \subseteq T$. Use an **Indirect Proof** to prove that the implication $(x \in S) \rightarrow (x \in T)$ holds true for any $x \in \mathbb{Z}$.

Question 11. [4 points]

Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers defined by the **recurrence relation** given below:

$$a_1 := 1$$

$$\text{for each integer } n \geq 2, \quad a_n = \tfrac{1}{2}a_{n-1} + 4$$

Using a **Proof by Induction** prove that, for all integers $n \geq 1$, the nth term of the above sequence equals $-14 \cdot 2^{-n} + 8$.

That is, prove the following statement $P(n)$ holds true for all integers $n \geq 1$:

$$P(n): \quad a_n = -14 \cdot 2^{-n} + 8$$

Important: Include all relevant details in your proof. If variables appear in your proof, clearly indicate what they represent. Clearly indicate your Induction Hypothesis and exactly when it is used in the proof of your Induction Step.

Question 12. [4 points]

(a) Let $\mathcal{R}$ be a relation on the set $\mathbb{Z} \times \mathbb{Z}$ defined by the following rule:

$\mathcal{R} = \{((a,b),(c,d)) \in \mathbb{Z} \times \mathbb{Z} \mid a + c = b + d\}$.

Is $\mathcal{R}$ **reflexive** ? Answer with 'YES' or 'NO'.

If your answer is 'YES', give a detailed proof. If your answer is 'NO', provide an appropriate counterexample and explain.

(b) Let $\mathcal{S}$ be a relation on the set $\mathbb{Z}$, defined by the following rule:

$\mathcal{R} = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid 5a + b \text{ is even } \}$.

Is $\mathcal{S}$ **transitive**? If so, give a detailed proof. If not, provide an appropriate counterexample and explain.

Question 13. [4 points]

Let Q, R, S be propositional variables, and let P_1 and P_2 be the following compound propositions:

$$P_1 : (Q \vee R) \to \neg S$$
$$P_2 : \neg Q \leftrightarrow (R \wedge S)$$

Is the set $\{P_1, P_2\}$ a **consistent** set of propositions?

Show your work. For this question, you may use either a **truth table** or a **truth tree** (using the official branching rules).

If your answer is 'YES', then give **all** truth assignments of Q, R, S that support your answer, **and** briefly explain.

If your answer is 'NO', then **explain** how you arrived at this answer, based on your truth table or truth tree, and the definition of 'consistent'.

Question 14. [5 points]

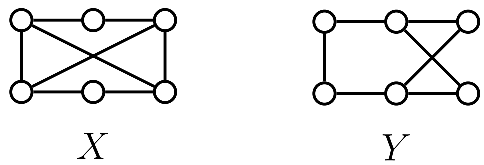
(a) Draw a picture of a simple graph G whose **adjacency matrix** is $A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$.

(1 point)

(b) Does there exist a graph with **degree sequence** $(1, 1, 2, 3, 3)$? (1 point)

If so, draw a picture of one such graph. If not, briefly explain.

(c) Consider the following 2 graphs



Is X a bipartite graph? Is Y a bipartite graph? For each graph, justify why it is or why it is not. (2 points)

(d) Consider a simple graph with 7 vertices and 10 edges. If each vertex is of degree 2 or 3, how many vertices of each degree are there? Justify. (1 point)