11. Functions

Building new sets from old:

☐ power set of S ☐ Cartesian product of two (or more) sets

$\mathcal{P}(S)$ $S \times T$ $S_1 \times S_2 \times \cdots \times S_t$

Set Operations:

☐ union ☐ intersection ☐ complement ☐ difference ☐ symmetric difference

$S \cup T$ $S \cap T$ $\overline{S}$ $S - T$ $S \oplus T$

Set identities:

☐ verify using membership tables ☐ verify using a rigorous proof

☐ prove other identities using the laws from the Table of Important Set Identities

FUNCTIONS

Let A and B be sets.

A **function** f from A to B is an assignment of **exactly one** element of B **to each** element of A.

We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element $a \in A$.

name of function → $f : A \to B$ ← codomain of f

domain of f

• for $a \in A$, $f(a)$ is the image of a (in particular, $f(a) \in B$)

• $f(A) = \{f(a) : a \in A\}$ is the set of images of all $a \in A$

called the image of the domain ("range" of f)

• for any subset $S \subseteq A$, we may consider the set $f(S) = \{f(a) : a \in S\}$

the image of the set $S \subseteq A$

• for each $b \in B$, define $f^{-1}(b) = \{a \in A : f(a) = b\}$

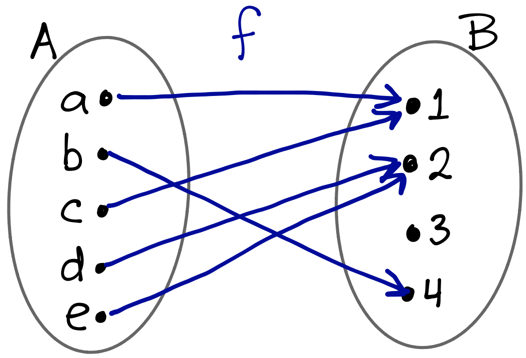
the set $f^{-1}(b)$ is called the preimage of b

* These notes are solely for the personal use of students registered in MAT1348.

Example 11.1. Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$.

Define $f: A \to B$ as follows:

$f(a) = 1$
$f(b) = 4$
$f(c) = 1$
$f(d) = 2$
$f(e) = 2$



$f(A) = \{1, 2, 4\}$

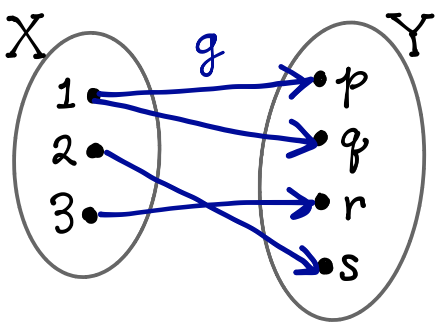
$f(\{c, d, e\}) = \{1, 2\}$

$f^{-1}(2) = \{d, e\}$

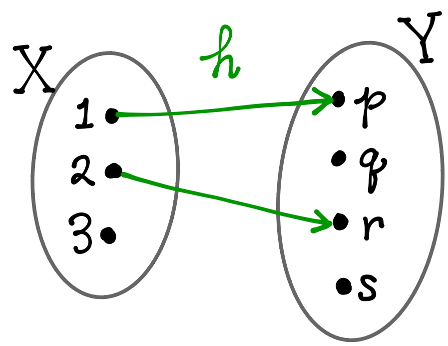
$f^{-1}(3) = \emptyset$

$f^{-1}(4) = \{b\}$

Example 11.2. Let $X = \{1, 2, 3\}$ and $Y = \{p, q, r, s\}$.



g is <u>not</u> a function from X to Y because $1 \in X$ is assigned to more than one element of Y



h is <u>not</u> a function from X to Y because $3 \in X$ is not assigned to any element of Y.

Examples of functions given by rules (instead of arrow diagrams).

<u>Ex</u>. $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$

<u>Ex</u>. $f: \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Q}$
$f(m, n) = \frac{m}{n}$

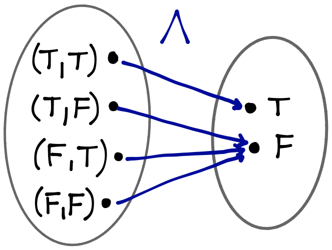
<u>Ex</u>. $D: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$
$D(x, y) = x - y$

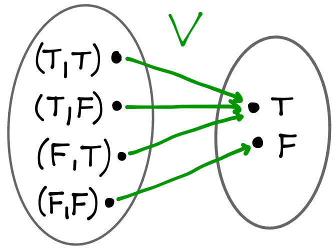
<u>Ex</u>. $S: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
$S(a, b) = a + b$

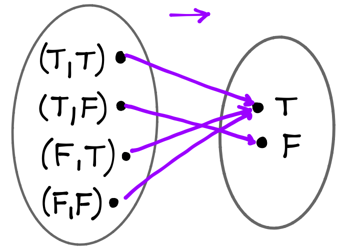
<u>Ex</u>. $P: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
$P(n, m) = nm$

<u>Ex</u>. $f: \mathbb{R} \to \mathbb{Z} \times \mathbb{Q}$
$f(x) = (8, 1.2)$
(a <u>constant</u> function)

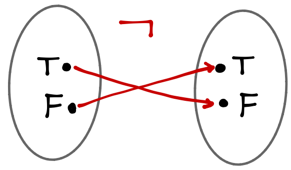
More examples: domain $\{T,F\} \times \{T,F\}$ codomain $\{T,F\}$







Ex. domain $\{T,F\}$
codomain $\{T,F\}$



Example 11.3. Suppose X is a compound proposition consisting of n propositional variables.

We can think of X as a function with domain $\{T,F\}^n$ and codomain $\{T,F\}$

Thus, each element of X's "domain" is a truth assignment to X's n variables

The "image" of a truth assignment is X's truth value for that truth assignment

Ex. Let X be the proposition $\neg(p \rightarrow (q \wedge r))$

Think of X as a function of 3 variables: $X(p,q,r)$ or $X: \{T,F\}^3 \rightarrow \{T,F\}$

Ex $X(T,T,F) = T$ because X is T when $p=T, q=T, r=F$

Injective (One-to-One) Functions

A function $f : A \to B$ is called **injective** or **one-to-one**, if for all $x, y \in A$, the implication

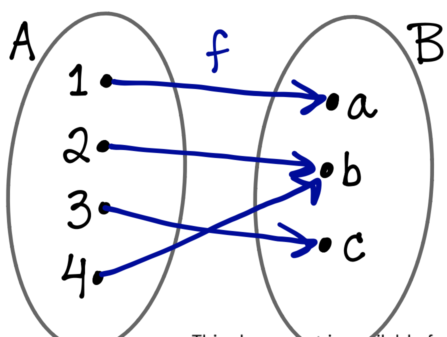
$$\Big(f(x) = f(y)\Big) \rightarrow \Big(x = y\Big) \qquad \text{is True.}$$

Equivalently (contrapositive form)

for all $x, y \in A$, $(x \neq y) \rightarrow (f(x) \neq f(y))$ is True.

Thus, each distinct element of A is assigned its own distinct unique element of B.

Ex.



f is not injective because $f(2) = f(4)$

Example 11.4. Is $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = x^2$ injective?

Let $a, b \in \mathbb{R}^+$ (the domain of f)

Assume $f(a) = f(b)$. Then $a^2 = b^2$

$$\Rightarrow a^2 - b^2 = 0$$

$$\Rightarrow (a-b)(a+b) = 0$$

$a = b$ or $\cancel{a = -b}$

Since, $a, b \in \mathbb{R}^+$, the only possible solution is $a = b$

Note. Since, for this function, the domain is $\mathbb{R}^+$, we know that neither a nor b are negative... so $a = -b$ is not actually possible.

Thus, we proved $(f(a) = f(b)) \to (a = b)$ is True for all $a, b \in \mathbb{R}^+$.

$\therefore$ f is injective (1-1).

Example 11.5. Is $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ injective?

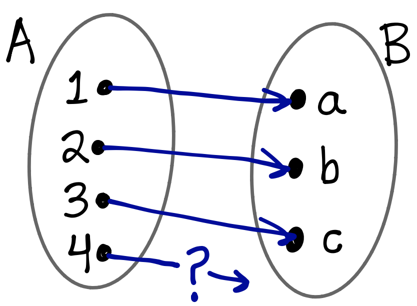
No! Counterexample: 1 and -1 are two distinct elements of the domain $\mathbb{R}$, yet $g(1) = g(-1) = 1$.

$\therefore$ g is not injective (1-1).

Example 11.6. Let A and B be sets such that $|A| = 4$ and $|B| = 3$.

Does there exist an injective function $f : A \to B$?

No!



Informal explanation:

If $|A| > |B|$, then, at some point, B will "run out" of "new" distinct images for the elements of A.

Theorem 11.7. Let A and B be sets.
If there exists an injective function $f : A \to B$, then $|A| \leq |B|$.

$\underbrace{\text{If there exists an injective function } f : A \to B}_{P} \longrightarrow \underbrace{|A| \leq |B|}_{Q}$

Theorem 12.7 in contrapositive form: Let A and B be sets.

If $\underbrace{|A|>|B|}_{\neg Q}$, then $\underbrace{\text{there does not exist an injective function from } A \text{ to } B}_{\neg P}$.

$\neg Q \longrightarrow \neg P$

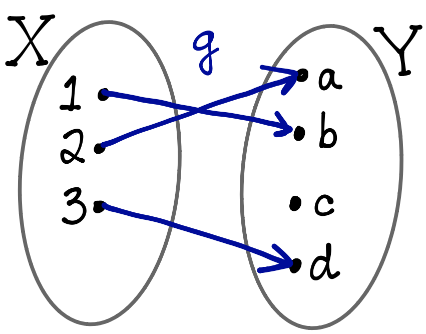
SURJECTIVE (ONTO) FUNCTIONS

A function $f : A \to B$ is called **surjective** or **onto** if, for every element $b \in B$, there exists at least one element $a \in A$ such that $f(a) = b$.

Equivalently, for all $b \in B$, $f^{-1}(B) \neq \emptyset$.

Thus, each element of the codomain B is the image of at least one element of A.

Ex.



g is not surjective because $c \in Y$ (the codomain) and yet $f^{-1}(c) = \{\} = \emptyset$.

Example 11.8. Is $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by $f(m, n) = (2m, n)$ surjective?

Let $(a,b) \in \mathbb{Z} \times \mathbb{Z}$ be an arbitrary element of the codomain of f.

Are we always guaranteed to be able to find some $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ (the domain of f) for which $f(m,n) = (a,b)$?

$(2m, n)$

what if a is odd?

No! f is <u>not</u> surjective (onto).

Counterexample:

No element of the domain $\mathbb{Z} \times \mathbb{Z}$ has $(1,1)$ as its image because every element of the domain gets mapped to an element of the codomain whose first coordinate is <u>even</u>.

Example 11.9. Is $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (2x, y)$ surjective?

Let $(a,b) \in \mathbb{R}^2$ (the codomain).

In order for $f(x,y) = (a,b)$ we need $(2x, y) = (a,b)$

$$\Rightarrow 2x = a \text{ and } y = b$$
$$\Rightarrow x = \tfrac{a}{2} \ (\in \mathbb{R}) \text{ and } y = b \ (\in \mathbb{R})$$
$$\Rightarrow \left(\tfrac{a}{2}, b\right) \in \mathbb{R} \times \mathbb{R} \text{ (the domain)}$$

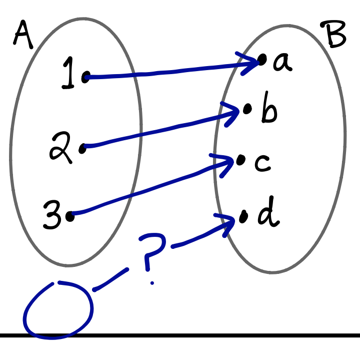
So for any $(a,b) \in \mathbb{R} \times \mathbb{R}$ (the codomain), we can find at least one element of the domain, namely $\left(\frac{a}{2}, b\right) \in \mathbb{R}^2$, such that $f\left(\frac{a}{2}, b\right) = (a,b)$

this is a "constructive" proof

$\therefore$ f is surjective (onto).

Example 11.10. Let A and B be sets such that $|A| = 3$ and $|B| = 4$. Does there exist a surjective function $f : A \to B$?

No!



Informal explanation:

If $|B| > |A|$, then A will "run out" of elements before all elements of B have been assigned as images.

Theorem 11.11. Let A and B be sets.
If $\underbrace{\text{there exists a surjective function } f : A \to B}_{P}$, then $\underbrace{|A| \geq |B|}_{Q}$.

Theorem 12.11 in contrapositive form: Let A and B be sets.

If $\underbrace{|B| > |A|}_{\neg Q}$, then $\underbrace{\text{there does not exist a surjective function from } A \text{ to } B}_{\neg P}$.

STUDY GUIDE

Important terms and concepts:
☐ function ☐ domain ☐ codomain ☐ image ☐ preimage
☐ injective (one-to-one) function ☐ surjective (onto) function

Exercises	Sup.Ex. §5 # 1abd, 3, 4, 5, 7, 9, 10 Rosen §2.3 # 1, 4ac, 7a, 8, 9, 10, 11, 12, 13, 15