Winter 2020 Mat1322C Practice midterm 1 - with solutions

Calculus II (University of Ottawa)

Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 1322C Practice Midterm Exam #1

February 3rd 2020

Professor: Guy Beaulieu

LAST NAME:_________________________ **First name:**_________________________

Student number:_________________________

Instructions:

- You have 75 minutes to complete the exam.
- It is a test with closed books and without calculator. The use of cellphones, pagers or any other device that can transmit or store information **is not allowed.**
- Read each question carefully before answering it.
- This exam is divided into two parts:
  - Part one: It includes 6 multiple choice questions, each 2 marks. You must enter your answers in the table provided on page 2 of the exam. There will be no partial points for multiple choice questions.
  - Part two: It includes 2 development questions. The correct answer requires legible and logical written justification. You have to convince me that you know why your solution is the right one. **Clearly** frame your final responses.
- Use the space specified to answer each question. If you don't have enough space you may use the back of any other page. In this case indicate clearly where the solution continues and where is the final response.
- Do not detach the exam.
- Good Luck!

It is prohibited to use cell phones, unauthorized electronic devices or course notes (unless it is an open book exam). Phones and devices must be closed and stored in your bag: you cannot leave them in your pockets or on you. Otherwise, you may be asked to leave the room immediately and allegations of academic fraud may be made and the result may be 0 (zero) for the exam.

By signing, you acknowledge that you have complied with the above statement.

Signature: ____________________

"Answers to multiple choice questions"

Q1	Q2	Q3
Q4	Q5	Q6

"Do not write anything in this table"

Question	QCM	7	8	Total
Maximum	12	5	3	20
Mark				

Question 1. [5 points] Determine whether the integral is convergent or divergent. Evaluate the integral if it is convergent.

$$\int_3^{11} \frac{6}{(x-3)^{2/3}}\,dx$$

Solution.

This integral is improper, since the function $\frac{6}{(x-3)^{2/3}}$ is not defined when $x = 3$.
If we let $u = x - 3$, we find:

$$\begin{aligned}
\int \frac{6}{(x-3)^{2/3}}\,dx &= \int \frac{6}{u^{2/3}}\,du \\
&= 6\int u^{-2/3}\,du \\
&= \frac{6}{1-2/3}u^{1-2/3} + C \\
&= 18(x-3)^{1/3} + C.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_3^{11} \frac{6}{(x-3)^{2/3}}\,dx &= \lim_{t\to 3^+}\int_t^{11} \frac{6}{(x-3)^{2/3}}\,dx \\
&= 18\lim_{t\to 3^+}\left((11-3)^{1/3} - (t-3)^{1/3}\right) \\
&= 18(8)^{1/3} \\
&= 36.
\end{aligned}$$

Hence, the integral is convergent and its value is 36.

Question 2. [2 points] We wish to use the Comparison Theorem to determine if

$$\int_1^\infty \frac{5s^2 + 2s\cos(s)}{s^6 + 2}\, ds$$

is convergent, and if it is, an upper bound for its value. Choose the correct argument.

(A) The integral is convergetn since for all $s \geq 1$, we have $5s^2 + 2s\cos(s) \leq (5+2)s^2 = 7s^2$ and $s^6 + 2 \geq s^6$, thus

$$\int_1^\infty \frac{5s^2 + 2s\cos(s)}{s^6 + 2}\, ds \leq \int_1^\infty \frac{7}{s^4}\, ds = \frac{7}{3}.$$

(B) The integral is divergent since for all $s \geq 1$, we have $5s^2 + 2s\cos(s) \geq 3s^2$ and $s^6 + 2 \geq 3$, thus

$$\int_1^\infty \frac{5s^2 + 2s\cos(s)}{s^6 + 2}\, ds \geq \int_1^\infty s^2\, ds = \infty.$$

(C) The integral is divergent since for all $s \geq 1$, we have $5s^2 + 2s\cos(s) \geq 5s^2$ and $s^6 + 2 \geq 3$, thus

$$\int_1^\infty \frac{5s^2 + 2s\cos(s)}{s^6 + 2}\, ds \geq \int_1^\infty \frac{5}{3}s^2\, ds = \infty.$$

(D) The integral is convergent since for all $s \geq 1$, we have $5s^2 + 2s\cos(s) \leq (5+2)s^2 = 7s^2$ and $s^6 + 2 \leq 3s^6$, thus

$$\int_1^\infty \frac{5s^2 + 2s\cos(s)}{s^6 + 2}\, ds \leq \int_1^\infty \frac{7}{3s^4}\, ds = \frac{7}{9}.$$

(E) The integral is convergentsince for all $s \geq 1$, we have $5s^2 + 2s\cos(s) \leq 5s^2$ and $s^6 + 2 \geq s^6$, thus

$$\int_1^\infty \frac{5s^2 + 2s\cos(s)}{s^6 + 2}\, ds \leq \int_1^\infty \frac{5}{s^4}\, ds = \frac{5}{3}.$$

(F) The integral is convergent since for all $s \geq 1$, we have $5s^2 + 2s\cos(s) \leq 5s^2$ and $s^6 + 2 \leq 3s^6$, thus

$$\int_1^\infty \frac{5s^2 + 2s\cos(s)}{s^6 + 2}\, ds \leq \int_1^\infty \frac{5}{3s^4}\, ds = \frac{5}{9}.$$

Solution. (A) The integral is convergent since for all $s \geq 1$, we have $5s^2 + 2s\cos(s) \leq (5+2)s^2 = 7s^2$ and $s^6 + 2 \geq s^6$, thus

$$\int_1^\infty \frac{5s^2 + 2s\cos(s)}{s^6 + 2}\, ds \leq \int_1^\infty \frac{7}{s^4}\, ds = \frac{7}{3}.$$

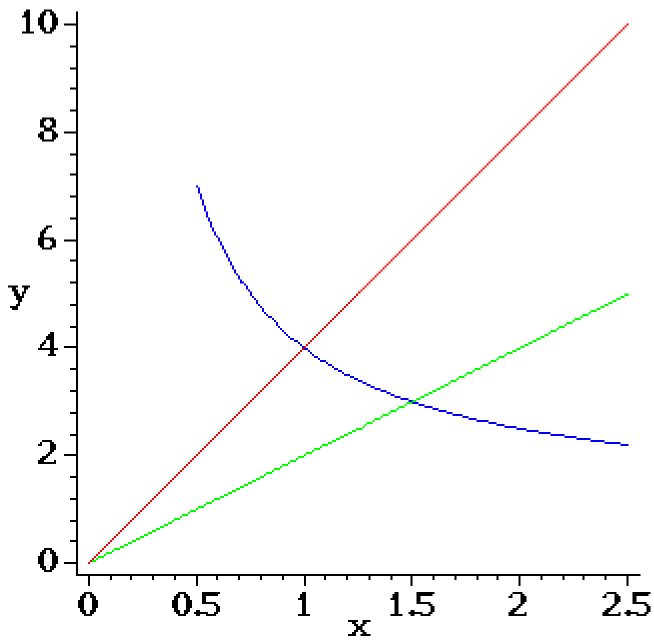
Question 3. [2 points] Consider the region in the first quadrant bounded by the curves $y = 2x, y = 4x$ and $y = \frac{3}{x} + 1$. Construct the integral which would compute the area of this region. **You are not required to compute the actual value of the area of the region.**

Solution. The line $y = 2x$ and the hyperbola $y = \dfrac{3}{x} + 1$ intersect at the points (x, y) where

$$2x = \frac{3}{x} + 1 \Rightarrow 2x^2 - x - 3 = 0.$$

The only positive root of this equation is $x = 3/2$. Therefore their intersection point in the first quadrant is $(x, y) = (3/2, 3)$. Similarly, we find that the line $y = 4x$ crosses the hyperbola at the point $(x, y) = (1, 4)$ in the first quadrant.

The following diagram illustates the region of interest with: in blue, the hypervola $y = \dfrac{3}{x} + 1$, in green, the line $y = 2x$, and in red, the line $y = 4x$.



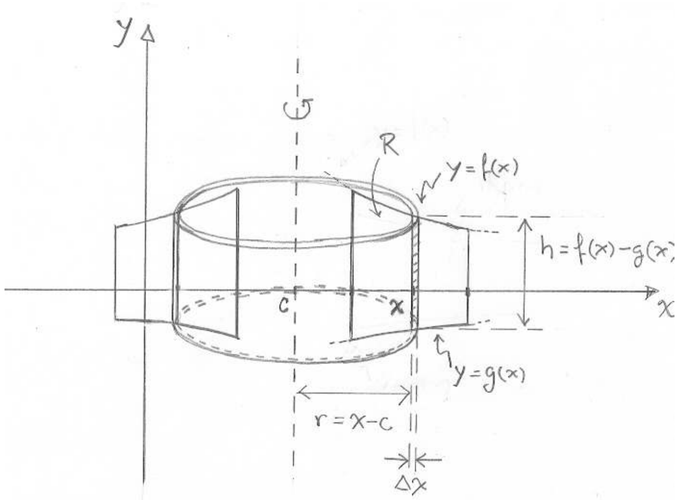
This region can be divided in two parts: the left half describes by $0 \leq x \leq 1$ and $2x \leq y \leq 4x$, and the right half described by $1 \leq x \leq 3/2$ and $2x \leq y \leq \dfrac{3}{x} + 1$.

The region's area is thus

$$\int_0^1 (4x - 2x)\, dx + \int_1^{3/2} \left(\frac{3}{x} + 1 - 2x\right)\, dx$$

Question 4. [2 points] Consider S the solid of revolution obtained by rotating around the line $x = 1$ the region R bounded by the curves $x = 3$, $x = 4$, $y = \frac{4}{x}$ and $y = \frac{-2}{x}$. Construct an integral which computes the volume of S.**You are not required to compute the actual value of the volume.**

Solution. We will use the cylindrical shells method. The figure below shows the region R and a typical cross-section:



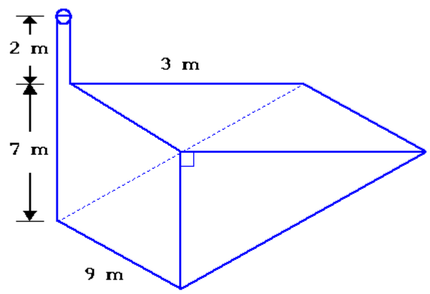
On the figure, we read $c = 1$, $f(x) = 4/x$ and $g(x) = -2/x$. Thus we have $r = x - 3$ and $h = f(x) - g(x) = \frac{6}{x}$.

For the function $A(x)$, it represents the area of the cylinder, therefore $A(x) = 2\pi r h = 2\pi(x-1)\frac{6}{x}$

Since the region R is found between $x = 3$ and $x = 4$, the volume in question is

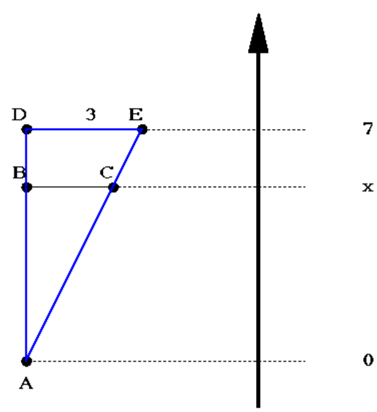
$$\int_3^4 A(x)\,dx \quad = \quad 2\pi \int_3^4 (x-1)\frac{6}{x}\,dx$$

Question 5. [4 points] A reservoir is in the shape of a prism whose extremities are right-hand triangles, as shown in the figure below. It is 7 m high, 9 m wide and 3 m long, and it is full of water. Let x be the height in metres measured from the bottom of the reservoir. Construct an integral which computes the work in Joules required to pump out all the water out an outlet that is 2m metres above the reservoir. (Note that 1 m^3 of water weighs 9800 N) **You are not required to compute the value of the work.**



Solution.

The following figure shows a cross-section of the reservoir parallel to the triangles at its extremities.



The horizontal section of this reservoir at height x is a rectangle of width 9 m and length $|BC|$. By hypothesis, at height $x = |AD| = 7$, we have $|DE| = 3$. Since the triangles $\triangle ABC$ and $\triangle ADE$ are similar, we can deduce that

$$|BC| = \frac{|AB|}{|AD|}|DE| = (\frac{x}{7})3 = (3/7)x.$$

Therefore, for Δx small, the slab of water between the heights x and $x + \Delta x$ looks almost like a rectangular box of width 9 and length (3/7)x. Its volume would be

$$(9)(3/7)x\Delta x = (27/7)x\Delta x \text{ m}^3.$$

Since 1 m^3 of water weighs about 9800 N, the weight of this slab is about

$$9800(27/7)x\Delta x = 37800x\Delta x = P(x)\Delta x \text{ N}$$

with $P(x) = 37800x$.

To pump this slab to 2 m over the reservoir, we must move it from x metres to 9 metres, thus we must raise it $9 - x$ metres. This represents work of

$$(9 - x)P(x)\Delta x = 37800x(9 - x)\Delta x \text{ Joules,}$$

therefore $w(x) = 37800x(9 - x)$.

The work required to pump out all the water is:

$$W = \int_0^7 37800x(9 - x)\,dx$$