

20. The Pigeonhole Principle

EXAMPLES OF THE PIGEONHOLE PRINCIPLE

Suppose we have a standard deck of 52 playing cards (with jokers removed).

- Each card is of one of 2 colours: **red** or **black**
- Each card is of one of 13 ranks: **A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K**
- Each card is of one of 4 suits: **♥ ♦ ♣ ♠**

A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦
A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣
A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠

Example 20.1. How many cards do I need to draw from the deck in order to guarantee that at least two cards are **black**?

Worst-case scenario
(27 is too few)

We draw 1 black card and all 26 red cards before getting a 2nd black card.

28 cards is enough Since there are only 26 red cards that could thwart us, 28 cards leaves us with at least $28 - 26 = 2$ cards that must both be black.

∴ we need to draw 28 cards in order to guarantee that at least 2 cards are black.

Example 20.2. How many cards do I need to draw from the deck in order to guarantee that at least two cards are the same colour?

Worst-case Scenario : we draw 1 of each colour before getting a 2nd card of one of the 2 colours



3 cards is enough

If we draw 3 cards, but there are only 2 possible colours, then we will end up with at least 2 that are the same colour.

∴ we need to draw 3 cards in order to guarantee that at least 2 are the same colour.

Example 20.3. How many cards do I need to draw from a standard deck of 52 playing cards in order to guarantee that at least three cards are the same suit?

Worst-case scenario: we draw 2 of each suit before getting a 3rd card of one of the 4 suits
(8 is too few)



9 cards is enough:

If we draw 9 cards, but there are only 4 possible suits, then we will end up with at least 3 that are the same suit.

Proof (by contradiction) that 9 cards is enough to guarantee at least 3 of the same suit.

Assume we drew 9 cards and did not get at least 3 of the same suit.

Then, of each suit, we drew at most 2 cards.

$$\begin{aligned} \text{Now, } (\text{total \# cards drawn}) &= (\#\text{ \heartsuit's drawn}) + (\#\text{ \diamond's drawn}) + (\#\text{ \clubsuit's drawn}) + (\#\text{ \spadesuit's drawn}) \\ &\leq 2 + 2 + 2 + 2 \end{aligned}$$

But we drew a total of 9 cards. This means $9 \leq 2+2+2+2 = 8$ ↗

Since $9 \leq 8$ is a contradiction, 9 cards really must have been enough

THE PIGEONHOLE PRINCIPLE

If a flock of $k+1$ pigeons flies into a set of k pigeonholes, then at least one pigeonhole contains at least two pigeons



THE CEILING FUNCTION (a function from \mathbb{R} to \mathbb{Z})

for $x \in \mathbb{R}$, the ceiling of x , denoted $\lceil x \rceil$, is defined as follows:

$$\lceil x \rceil = \min\{n \in \mathbb{Z} : x \leq n\} \quad \text{"rounding up"}$$

Ex. $\lceil 2 \rceil = 2$ Ex. $\lceil 2.3 \rceil = 3$ Ex. $\lceil -2.3 \rceil = -2$

THEOREM: THE GENERALIZED PIGEONHOLE PRINCIPLE (G.P.P.)

Theorem 20.4. (The Generalized Pigeonhole Principle) Let k and N be positive integers.

If N objects are placed into k boxes, then there is at least one box that contains at least $\lceil \frac{N}{k} \rceil$ objects.

(P.P.) Theorem 20.5. (THE PIGEONHOLE PRINCIPLE) Let k be a positive integer.

If $k+1$ objects are placed into k boxes, then there is at least one box that contains at least 2 objects.



The Pigeon Version of Theorem:

If a flock of $k+1$ pigeons flies into a set of k pigeonholes to roost, then at least one pigeonhole will contain more than one pigeon (because there are more pigeons than pigeonholes).

Proof by Contradiction of Theorem 19.6

Let $k \in \mathbb{Z}^+$ (k is an integer and $k \geq 1$). Assume the Pigeonhole Principle is false.

Assume $k+1$ objects are placed into k boxes and assume that it is not the case that at least one box contains at least 2 objects.

Then each box contains at most 1 object.

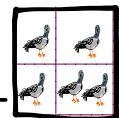
Now, $k+1 = (\text{total # objects})$

$$\begin{aligned} &= (\# \text{ objects in Box 1}) + (\# \text{ objects in Box 2}) + \dots + (\# \text{ objects in Box } k) \\ &\leq 1 + 1 + \dots + 1 \\ &= k \end{aligned}$$

Since our assumption leads to the contradiction $k+1 \leq k$, we were wrong.

∴ the Pigeonhole Principle is true.

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Lemma 20.6. Let $x \in \mathbb{R}$. Then $\lceil x \rceil < x + 1$.

Proof. Let $x \in \mathbb{R}$. We consider 2 cases: either x is an integer or not.

Case 1. Assume x is an integer. Then $\lceil x \rceil = x$ which is $< x+1$.

\therefore in case 1, $\lceil x \rceil < x+1$ is true.

Case 2. Assume x is not an integer. Then we can write $x = n + \delta$ for some integer n and some real number δ such that $0 < \delta < 1$. That is, we can write x in decimal form as an integer plus whatever comes after the decimal point.

Then $\lceil x \rceil = \lceil n + \delta \rceil = n + 1 < n + 1 + \delta = x + 1$

\therefore in case 2, $\lceil x \rceil < x+1$ is true.

In both possible cases, we showed that $\lceil x \rceil < x+1$ $\therefore \lceil x \rceil < x+1$ is true for all $x \in \mathbb{R}$ 

Proof of Theorem 20.4 (The Generalized Pigeonhole Principle).

We will prove GPP by contradiction.

GPP Says: $\underbrace{\left(\text{We place } N \text{ objects into } k \text{ boxes} \right)}_{P} \rightarrow \underbrace{\left(\begin{array}{l} \text{at least one box contains} \\ \text{at least } \lceil \frac{N}{k} \rceil \text{ objects} \end{array} \right)}_{Q.}$

Assume the negation of GPP is true. That is, assume $\neg(P \rightarrow Q) \equiv P \wedge \neg Q$.

Assume we place N objects into k boxes and it is not the case that at least one box contains at least $\lceil \frac{N}{k} \rceil$ objects.

Then each box contains at most $\lceil \frac{N}{k} \rceil - 1$ objects.

$$\begin{aligned} \Rightarrow (\text{total # of objects}) &= (\text{# objects in box 1}) + (\text{# objects in box 2}) + \dots + (\text{# objects in box } k) \\ &\leq \lceil \frac{N}{k} \rceil - 1 + \lceil \frac{N}{k} \rceil - 1 + \dots + \lceil \frac{N}{k} \rceil - 1 \\ &= k(\lceil \frac{N}{k} \rceil - 1) \\ &< k\left(\frac{N}{k} + 1 - 1\right) \quad \text{by Lemma 20.6} \\ &= N \cancel{\downarrow} \quad \text{this is a contradiction since total # objects} = N \text{ and is} < N. \end{aligned}$$

\therefore the negation of GPP must be false since it leads to contradiction

\therefore the GPP must be true 

CONSEQUENCE OF THE GENERALIZED PIGEONHOLE PRINCIPLE

Let k and r be positive integers. Suppose some objects are to be placed into k boxes.

QUESTION:

What is the minimum number N of objects that we need to place into k boxes in order to guarantee that at least one box contains at least r objects?

ANSWER:

The minimum number N that is needed to guarantee that at least one box contains at least r objects is

$$N = \underline{k(r-1)} + 1$$

EXPLANATION:

By G.P.P., at least one box is guaranteed to contain at least $\left\lceil \frac{N}{k} \right\rceil$ objects.

With $N = k(r-1) + 1$, we get

$$\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{k(r-1)+1}{k} \right\rceil = \left\lceil (r-1) + \frac{1}{k} \right\rceil = r \text{ objects}$$

$\therefore N = k(r-1) + 1$ is enough.

On the other hand, with only $k(r-1)$ objects, we could end up in the worst-case scenario:

each of the k boxes contains $r-1$ objects



(each box is just 1 object short of the goal).

$\therefore k(r-1)$ objects is too few.

$\therefore N = \underline{k(r-1)} + 1$ is exactly the minimum threshold to guarantee that at least one box contains at least r objects.

Example 20.7. How many people do we need to invite to a party in order to guarantee that at least three of our guests were born on the same month?

$$\text{boxes} = \text{months}$$

$$k = 12$$

objects = guests
N is to be determined

goal: we want at least $r=3$ objects to be in the same box
(guests) (have same month of birth)

$$\begin{aligned}\text{so we need } N &= k(r-1)+1 \\ &= (12)(3-1)+1 \\ &= \underline{\underline{25 \text{ guests}}}\end{aligned}$$

Why 25?

- ↳ 24 is too few in the worst-case scenario when exactly two guests were born in each of the 12 different months.
- ↳ $N=25$ is enough since the GPP guarantees at least one box will contain at least $\lceil \frac{N}{k} \rceil = \lceil \frac{25}{12} \rceil = 3$ objects

Example 20.8. How many people do we need in order to guarantee that at least 2 have the same two initials (first letter of first name and first letter of family name)?

$$\text{boxes} = \text{all possible 2-letter initials}$$

$$k = \underline{\underline{26 \cdot 26}} = 26^2 = 676$$

objects = people
N is to be determined

goal: we want at least $r=2$ objects to be in the same box
(people) (have same 1st-name/last-name initials).

$$\begin{aligned}\text{so we need } N &= k(r-1)+1 \\ &= (676)(2-1)+1 \\ &= \underline{\underline{676+1}} \\ &= \underline{\underline{677 \text{ people}}}\end{aligned}$$

Why 677?

- 676 is too few in the worst-case scenario when we have exactly one person with each of the possible 676 2-letter initials.
- By the P.P., placing $k+1=677$ people into $k=676$ boxes guarantees that at least one box will contain at least 2 people.

STUDY GUIDE

◊ The Pigeonhole Principle

$k+1$ objects k boxes

worst-case scenario analysis

◊ the ceiling function $\lceil x \rceil$

◊ The Generalized Pigeonhole Principle

N objects k boxes

at least one box contains at least $\lceil N/k \rceil$ objects

examples with the pigeonhole principle: how many do we need in order to guarantee that at least have the same.... ? worst-case analysis

Exercises

Sup.Ex. §11 # 1, 2, 3, 4, 5, 7, 8, 9, 10, 14

Rosen §2.4 # 1, 3, 9

Sup.Ex. §9 # 3, 4, 5, 6, 8, 12

Rosen §5.1 # 3, 4, 5, 6, 7, 9, 11, 13, 15, 19, 21, 31, 33, 35