

## 16. Counting: Product Rule, Permutations & Combinations

### THE PRODUCT RULE FOR $k$ TASKS

Suppose a procedure can be broken down into a sequence of  $k$  tasks  $T_1, T_2, \dots, T_k$ , such that

- there are  $n_1$  ways to carry out task  $T_1$
- and, for  $i=2,\dots,k$ , there are  $n_i$  ways to carry out task  $T_i$   
after each of the tasks  $T_1, T_2, \dots, T_{i-1}$  has been done in a certain way.

Then there are  $\underline{(n_1)(n_2) \cdots (n_k)}$  ways to carry out the procedure.

**Example 16.1.** How many (new) Ontario licence plates are there consisting of four uppercase letters followed by 3 digits? (let's ignore the possibility that some words/sequences might be forbidden)



$T_1$ : task of choosing 1st letter (26 ways)

$T_2$ : task of choosing 2nd letter (26 ways)

$T_3$ : task of choosing 3rd letter (26 ways)

$T_4$ : task of choosing 4th letter (26 ways)

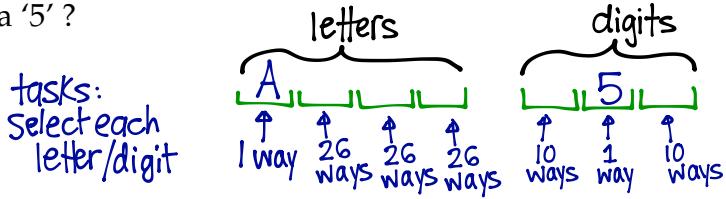
$T_5$ : task of choosing 1st digit (10 ways)

$T_6$ : task of choosing 2nd digit (10 ways)

$T_7$ : task of choosing 3rd digit (10 ways)

∴ there are  $(26)(26)(26)(26)(10)(10)(10) = 26^4 10^3$  ways to create such a licence plate.

**Example 16.2.** How many (new) Ontario licence plates are there such that the first letter is an 'A' and the second digit is a '5'?



∴ there are  $(1)(26)(26)(26)(10)(1)(10) = 26^3 \cdot 10^2$  ways to create such a licence plate.

**Example 16.3.** Let  $A = \{1, 2, 3, 4, 5\}$  and  $B = \{x, y, z\}$ .

How many distinct functions are there from  $A$  to  $B$ ?

Procedure: build a function  $f: A \rightarrow B$  by choosing images  $f(1), f(2), f(3), f(4), f(5)$

For  $1 \leq i \leq 5$ , let  $T_i$  be the task

$T_i$ : choose the image  $f(i) \in B$  in one of 3 ways

∴ there are  $3^5 = 243$  functions from  $A$  to  $B$ .

**Example 16.4.** Let  $A$  and  $B$  be finite sets with  $|A| = m$  and  $|B| = n$ .

How many functions  $f: A \rightarrow B$  are there? Let  $A = \{a_1, \dots, a_m\}$

Procedure: to create a function  $f: A \rightarrow B$

choose  $f(a_1), \dots, f(a_m) \in B$

i.e. choose exactly one image in  $B$  for each element in the domain  $A$ .

task  $T_1$ : choose  $f(a_1) \in B$  (in one of  $|B| = n$  ways)

task  $T_2$ : choose  $f(a_2) \in B$  (in one of  $|B| = n$  ways)

⋮

task  $T_m$ : choose  $f(a_m) \in B$  (in one of  $|B| = n$  ways)

$$\begin{aligned} \# \text{ ways to perform } (T_1, T_2, \dots, T_m) &= |B| \cdot |B| \cdot \dots \cdot |B| \\ &= (n)(n) \dots (n) \\ &= n^m \\ &= |B|^{|A|} \end{aligned}$$

∴ there are  $|B|^{|A|}$  ( $= n^m$ ) functions  $f: A \rightarrow B$

**Example 16.5.** Let  $A$  and  $B$  be finite sets with  $|A| = m$  and  $|B| = n$ .

How many functions  $f : A \rightarrow B$  are **injective**?

Let  $A = \{a_1, \dots, a_m\}$

Procedure: to create an injective function  $f : A \rightarrow B$

$\hookrightarrow$  choose  $f(a_1), \dots, f(a_m) \in B$  all distinct (so that  $f$  is 1-1)

task  $T_1$ : choose  $f(a_1) \in B$  in one of  $|B| = n$  ways

task  $T_2$ : choose  $f(a_2) \in B$  after  $f(a_1)$  has been chosen in one of  $n-1$  ways  
(so that  $f(a_2) \neq f(a_1)$ ).

task  $T_3$ : choose  $f(a_3) \in B$  after  $f(a_1), f(a_2)$  have been chosen in one of  $n-2$  ways  
(so that  $f(a_3) \neq f(a_1)$  and  $f(a_3) \neq f(a_2)$ ).

⋮

task  $T_i$ : choose  $f(a_i) \in B$  after  $f(a_1), \dots, f(a_{i-1})$  have been chosen in one of  $n-(i-1)$  ways  
(so that  $f(a_i) \neq f(a_1), \dots, f(a_i) \neq f(a_{i-1})$ ).

⋮

task  $T_m$ : choose  $f(a_m) \in B$  after  $f(a_1), \dots, f(a_{m-1})$  have been chosen in one of  $n-(m-1)$  ways  
(so that  $f(a_m) \neq f(a_1), \dots, f(a_m) \neq f(a_{m-1})$ ).

∴ # ways to carry out  $(T_1, T_2, \dots, T_m) = n(n-1)(n-2) \dots (n-(i-1)) \dots (n-(m-1))$

∴ there are  $n(n-1) \dots (n-m+1)$  injective functions from  $A$  to  $B$

\*\*Note If  $|B| < |A|$  (i.e. if  $n < m$ ), then the answer is 0 (zero).

## PERMUTATIONS

A **permutation** is an ordered arrangement of some (or all) of the elements of a finite set.

Let  $n \geq r \geq 0$  be integers.

An  **$r$ -permutation of an  $n$ -element set  $S$**  is an ordered arrangement of  $r$  elements from  $S$ .

**Example 16.6.** Write all 3-permutations of the 3-element set  $S = \{1, 2, 3\}$

123  
132  
213

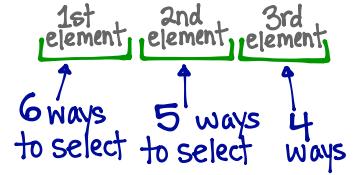
231  
312  
321

In total, there are six  
3-permutations of  $S$ .

**Example 16.7.** How many 3-permutations are there of the 6-element set  $S = \{1, 2, 3, 4, 5, 6\}$ ? Write some of these down.

Procedure: build a 3-permutation of a 6-element set:

In total, there are  $6 \cdot 5 \cdot 4 = 120$  3-permutations of a 6-element set.



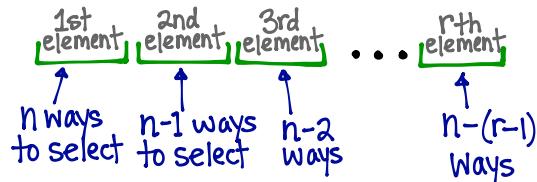
Ex 123 124 125 126 132 134 135 136 142 143 145 146 ... 651 ... etc

Let  $n \geq r \geq 0$  be integers.

$P(n, r) = \#$  of  $r$ -permutations of an  $n$ -element set

$P(n, n) = \#$  of permutations of an  $n$ -element set

$P(n, r) = ?$  Procedure: build an  $r$ -permutation of an  $n$ -element set:



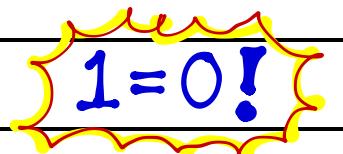
By the Product Rule:

$$P(n, r) = (n)(n-1)(n-2)\dots(n-r+1)$$

In particular, for  $r=n$ ,

$$\begin{aligned} P(n, n) &= n(n-1)(n-2)\dots(n-n+1) \\ &= n(n-1)(n-2)\dots(1) \\ &= n! \end{aligned}$$

FACTORIALS



" $n$  factorial", denoted  $n!$ , is defined for each  $n \in \mathbb{N}$  as follows:

$$0! = 1$$

For  $n \geq 1$ ,  $n!$  is defined recursively:

$$n! = n(n-1)!$$

$$= n(n-1)(n-2)\dots(2)(1)$$

Examples

$$\begin{aligned} 1! &= 1(0!) \\ &= 1(1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} 2! &= 2(1!) \\ &= 2(1) \\ &= 2 \end{aligned}$$

$$3! = 3(2!)$$

$$\begin{aligned} &= 3(2)(1!) \\ &= 3(2)(1) \\ &= 6 \end{aligned}$$

$$4! = 4(3!)$$

$$\begin{aligned} &= 4(3)(2!) \\ &= 4(3)(2)(1) \\ &= 24 \end{aligned}$$

## COMBINATIONS

**Example 16.8.** List all possible 3-element subsets of the set  $S = \{1, 2, 3, 4\}$

$\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$

3-element subsets of  $S$   
"3-combinations from a 4-element set"

Let  $n \geq r \geq 0$  be integers.

An  $r$ -combination from an  $n$ -element set  $S$  is an  $r$ -element subset of an  $n$ -element set  $S$ .

$$C(n, r) = \# \text{ of } r\text{-combinations from an } n\text{-element set}$$
$$= \# \text{ of subsets } T \text{ of } S \text{ such that } |T|=r \text{ and } |S|=n$$

**Theorem 16.9.** Let  $n, r \in \mathbb{N}$  with  $n \geq r \geq 0$ .

Then

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Proof First, note that  $\frac{n!}{r!(n-r)!} = \frac{P(n, r)}{P(r, r)}$  (verify this!)

We will now prove that  $P(n, r) = C(n, r) \cdot P(r, r)$

which is equivalent to  
proving  $C(n, r) = \frac{P(n, r)}{P(r, r)}$

Let's count  $P(n, r)$  by the following 2-task procedure for building  $r$ -permutations of an  $n$ -element set.

task  $T_1$ : choose an  $r$ -element subset  $T$  of  $S$  (in one of  $C(n, r)$  possible ways)

task  $T_2$ : arrange the  $r$  elements of  $T$  (in one of  $P(r, r)$  possible ways)

By the Product Rule,

the # of ways to carry out  $(T_1, T_2)$  =  $C(n, r) \cdot P(r, r)$

i.e.  $P(n, r) = C(n, r) \cdot P(r, r)$

$$\therefore C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!}{r!(n-r)!}$$

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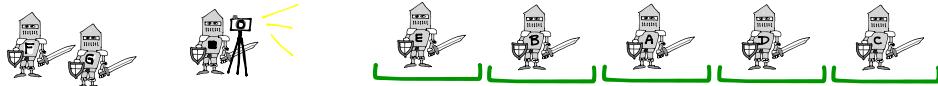
**Example 16.10.** A photographer is taking group photos of 7 inhabitants of The Island of K&K, namely A, B, C, D, E, F, and G.

- In how many ways can the photographer select 5 out of 7 of these inhabitants to be in the photo?

The photographer wants to select a 5-element subset of  $\{A, B, C, D, E, F, G\}$  (ie to select 5 inhabitants to be "in the photo".)

There are  $C(7,5) = \frac{7!}{5!2!} = \frac{7 \cdot 6 \cdot 5!}{5! \cdot 2 \cdot 1} = \frac{7 \cdot 6^3}{2} = 21$  ways to do this.

- How many distinct ways can the photographer arrange 5 out of 7 of these inhabitants in a line?



The photographer wants to select a 5-permutation of  $\{A, B, C, D, E, F, G\}$  (ie to select 5 inhabitants to be "in the photo" and arrange them in some order).

There are  $P(7,5) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$  ways to do this

- In how many distinct ways can the photographer line up 5 out of 7 of these inhabitants so that A and B are in the photo and A stands next to B?

$T_1$ : choose 2 side-by-side positions in the lineup for A and B to stand (4 ways)

$T_2$ : choose an ordering of A and B ( $P(2,2) = 2! = 2$  ways, namely AB and BA)

$T_3$ : choose a 3-permutation of the 5 remaining inhabitants to fill the remaining 3 places in the lineup ( $P(5,3) = 5(4)(3) = 60$  ways)

so there are  $(4)(P(2,2))(P(5,3)) = (4)(2)(60) = 480$  possible photo lineups in which A stands next to B.

## STUDY GUIDE

product rule for  $k$  tasks: there are  $n_1 n_2 \cdots n_k$  ways to carry out the sequence of tasks  $(T_1, T_2, \dots, T_k)$  given that there are  $n_i$  ways to carry out task  $T_i$  after each of the previous tasks has been carried out in some way.

permutations

$$P(n, k) = n(n - 1) \cdots (n - (k - 1))$$

factorials

$$n! = n(n - 1) \cdots (2)(1)$$

combinations

$$C(n, k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Exercises

Sup.Ex. §8 # 2, 3, 4abcd, 12

Rosen §6.1 # 1, 7, 11, 16, 19, 25, 33, 34, 35, 37ab, 49

Sup.Ex. §10 # 6a, 8a, 9abd, 10bc, 11

§6.3 # 1, 2, 3, 4, 5, 6, 7, 8, 11, 19, 21