Lesson 5 – Differentiation Rules For Sinusoidal Functions

PART A: Spreading the good word!

Recall: $f(x) = \cos x$ $\quad g(x) = \sin x$
$f'(x) = -\sin x$ $\quad g'(x) = \cos x$

Good news! The Constant Multiple Rule and Sum and Difference Rule that we learned in Unit 2 also apply for the derivatives of sinusoidal functions.

The Constant Multiple Rule

If $f(x) = kg(x)$, where k is a constant, then $f'(x) = kg'(x)$

or

In Leibniz notation, $\frac{d}{dx}(ky) = k\frac{dy}{dx}$

The Sum/Difference Rule

If functions $p(x)$ and $q(x)$ are differentiable, and $f(x) = p(x) \pm q(x)$, then

$f'(x) = p'(x) \pm q'(x)$

or

In Leibniz notation, $\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) \pm \frac{d}{dx}(q(x))$

Example 1: Find each derivative with respect to x.

a) $y = 3\sin x$

$$\frac{dy}{dx} = 3\frac{d(\sin x)}{dx}$$
$$= 3\cos x$$

b) $f(x) = -4\cos x$

$$f'(x) = -4(-\sin x)$$
$$= 4\sin x$$

c) $y = \sin x + \cos x$

$$y' = \cos x + (-\sin x)$$
$$= \cos x - \sin x$$

d) $f(x) = 7\sin x - 3\cos x + 20$

$$f'(x) = 7\cos x - 3(-\sin x)$$
$$= 7\cos x + 3\sin x$$

Example 2: Find the equation of the tangent line to the function $y = cosx + 2$ that passes through the point $\left(\frac{2\pi}{3}, \frac{3}{2}\right)$.

① Determine slope of tangent at $x = 2\pi/3$ (ie. $\left.\frac{dy}{dx}\right|_{x=2\pi/3}$)

$\frac{dy}{dx} = -\sin x$

$\left.\frac{dy}{dx}\right|_{x=\frac{2\pi}{3}} = -\sin\frac{2\pi}{3}$

$m = -\frac{\sqrt{3}}{2}$

② Use point and slope to write the equation of tangent line:

$y - y_1 = m(x - x_1)$; $\left(\frac{2\pi}{3}, \frac{3}{2}\right)$ x_1, y_1 $m = -\frac{\sqrt{3}}{2}$

$y - \frac{3}{2} = -\frac{\sqrt{3}}{2}\left(x - \frac{2\pi}{3}\right)$

PART B: Still spreading the good word!

More good news! The Product Rule, Power of a Function Rule and Chain Rule that we learned in Unit 2 all apply to sinusoidal functions.

The Product Rule

If $m(x) = f(x) \cdot g(x)$, then $m'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

or

In Leibniz notation,
$\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + f(x) \cdot \frac{d}{dx}[g(x)]$

The Power of a Function Rule

If u is a function of x, and n is an integer, then, $\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$,

or

In function notation, if $f(x) = [g(x)]^n$, then $f'(x) = n[g(x)]^{n-1} \cdot g'(x)$

The Chain Rule

If *f* and *g* are functions that have derivatives, then the composite function $h(x) = f(g(x))$ has a derivative given by:

$$h'(x) = f'(g(x)) \cdot g'(x)$$

The Chain Rule in Leibniz Notation

If *y* is a function of *u* and *u* is a function of *x* (so that *y* is a composite function), then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

provided that $\frac{dy}{du}$ and $\frac{du}{dx}$ exist.

Example 3: Find the derivate of each function, with respect to x. State the rule(s) used.

a) $y = sin5x$ (chain)

let $u = 5x$

$y = \sin u$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$\frac{dy}{dx} = \cos u \cdot 5$

$= \cos 5x \cdot 5$

$= 5\cos 5x$

$y' = \cos 5x \cdot 5$

$= 5\cos 5x$

b) $y = cos\pi x$ (chain rule)

$y' = -\sin(\pi x) \cdot \pi$

$= -\pi \sin(\pi x)$

c) $y = 6sin3x - sinx$ (chain and difference)

$y' = 6\cos(3x) \cdot 3 - \cos x$

$= 18\cos(3x) - \cos x$

d) $y = sin^3x - 2cosx$ (power and difference and constant)

$y' = 3(\sin x)^2(\cos x) - 2(-\sin x)$

$y' = 3\sin^2 x \cos x + 2\sin x$

$y' = \sin x(3\sin x\cos x + 2)$

e) $y = \frac{1}{\sin x}$ (quotient)

$y' = \frac{\sin x\,(0) - 1\,(\cos x)}{\sin^2 x}$

$y' = -\frac{\cos x}{\sin^2 x}$

$(\sin x)^2$

$\cancel{\sin x^2}$

f) $y = \cos(-\pi x + 3)$ chain

$y' = -\sin(-\pi x + 3) \cdot (-\pi)$

$y' = \pi \sin(-\pi x + 3)$

Example 4: Differentiate with respect to *t*.

a) $y = sin^2(t - 3)$

$y = [\sin(t-3)]^2$

$y' = 2[\sin(t-3)]^1 \cdot \cos(t-3) \cdot (1)$

$y' = 2\sin(t-3) \cdot \cos(t-3)$

b) $f(t) = \sin(3t^2) + cos4t$

$f'(t) = \cos(3t^2) \cdot (6t) - \sin(4t) \cdot (4)$

$f'(t) = 6t\cos(3t^2) - 4\sin(4t)$

c) $f(t) = \sin(\underbrace{\cos^2 t}_{g(t)})$

$f'(t) = \cos(\cos^2 t) \cdot 2\cos^1 t \cdot (-\sin t)(1)$

$f'(t) = -2\cos(\cos^2 t) \cdot \cos t \cdot \sin t$

Example 5: Find the slope of the tangent line to $y = 2\underbrace{sinxcosx}$ at $x = \pi$.

① determine derivative and evaluate at $x = \pi$

$$y' = 2[\cos x \cdot \cos x + \sin x(-\sin x)]$$
$$y' = 2[\cos^2 x - \sin^2 x]$$
$$y' = 2[\cos 2x]$$
$$y' = 2\cos 2x$$

$$y'\big|_{x=\pi} = 2\cos(2\pi)$$
$$= 2(1)$$
$$= 2$$