9. Basics of Set Theory

Important Proof Strategies:
☐ Direct Proof ☐ Indirect Proof ☐ Proof by Contradiction
☐ Proof by Cases ☐ Proof of Equivalence

Sets and Set Concepts

A **set** is a well-defined unordered collection of objects called **elements**.

Ex. $A = \{1, 2, a, \text{Ottawa}, ♡\}$
name of set
list of elements contained in set

Ex. {people in this room}

<u>Notation for set membership:</u> $1 \in A$ means "1 is an element of A"
(greek letter epsilon)
$3 \notin A$ means "3 is not an element of A"

Two sets are **equal** if they contain the same elements (regardless of the order and multiplicity).

Ex. $S = \{a, b, c\}$ $T = \{b, c, a\}$ $U = \{a, a, b, c\}$ (U contains 3 distinct elements, namely a, b, and c)

<u>Note.</u> $S = T = U$.

A set either contains some element or it does not.
↳ order elements are listed does not affect "is an element of"
↳ repeating an element more than once does not affect "is an element of"

Different ways to Describe a Set:

List elements: <u>ex.</u> $A = \{a, e, i, o, u\}$

Set-builder notation: $A = \{\ell : \ell \text{ is a vowel of the English alphabet}\}$

<u>ex.</u> $B = \{3, 6, 9, \ldots, 36\}$ $B = \{3n : n \in \mathbb{Z}, 1 \leq n \leq 12\}$

<u>ex.</u> $C = \{3, 4, 5, 6, \ldots\}$ $C = \{n : n \in \mathbb{Z}, n \geq 3\}$

<u>ex.</u> $\mathcal{D} = \{x \in \mathbb{R} : x \neq 5\}$ $\mathcal{D} = (-\infty, 5) \cup (5, \infty)$ (special notation for intervals of real #s)

* These notes are solely for the personal use of students registered in MAT1348.

Important Sets of Numbers

$\mathbb{N}=\{0,1,2,3,\dots\}$ (The set of natural numbers)

$\mathbb{Z}=\{\dots,-2,-1,0,1,2,3,\dots\}$ (the set of integers)

$\mathbb{Q}=\{\frac{p}{q} : p\in\mathbb{Z}, q\in\mathbb{N}, q\neq 0\}$ (the set of rational numbers)

$\mathbb{R}=\{r : r \text{ is a real number}\}$ (the set of real numbers)

$\mathbb{Z}^+=\{n\in\mathbb{Z} : n>0\}$ (the set of positive integers)

$\mathbb{Z}^-=\{n\in\mathbb{Z} : n<0\}$ (the set of negative integers)

Similarly, $\mathbb{Q}^+, \mathbb{Q}^-, \mathbb{R}^+, \mathbb{R}^- \dots$

The Empty Set. $\emptyset$

The empty set, denoted $\emptyset$, is the set with no elements. i.e. $\emptyset=\{\}$

Note. $\{\emptyset\}$ is not the empty set because it does contain an element

(its one element happens to be a set, the empty set in fact. Regardless of what the element is, $\{\emptyset\}$ does contain an element)

The Universal Set. $\mathcal{U}$

The universal set, denoted $\mathcal{U}$, is the set of all objects under consideration.

Subsets.

Let A and B be sets.
Then A is said to be a **subset** of B (written $A \subseteq B$) if every element of A is also an element of B.
i.e. for all $x\in\mathcal{U}$, the implication $(x\in A)\to(x\in B)$ is true. exercise

Write the negation of this definition.
ie What is the definition for
"A is not a subset of B" ?

Ex. $\mathbb{N}\subseteq\mathbb{Z}\subseteq\mathbb{Q}\subseteq\mathbb{R}$

Ex. $A=\{a,b,c\}$ $B=\{a,c\}$ $C=\{a,\{b\},c\}$

"is not a subset of"

$B\subseteq A$ $\qquad$ $A\nsubseteq B$ because $b\in A$ but $b\notin B$

$B\subseteq C$ $\qquad$ $A\nsubseteq C$ because $b\in A$ but $b\notin C$

$C\nsubseteq A$ because $\{b\}\in C$ but $\{b\}\notin A$

$A\neq C$ because A and C do not contain the same elements

just a letter of the English alphabet → b vs. $\{b\}$ ← a set containing one element, namely b.

Theorem 9.1. Let S be any set. Then

1. $S \subseteq S$
2. $\emptyset \subseteq S$

Theorem 9.2. Let A and B be sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Note. To prove $A = B$, we must prove two things: 1. $A \subseteq B$ and 2. $B \subseteq A$.

Proper Subsets.

Let A and B be sets.
Then A is said to be a **proper subset** of B (written $A \subset B$) if $A \subseteq B$ and $A \neq B$.

Ex. $\mathbb{N} \subset \mathbb{Z}$ because $\mathbb{N} \subseteq \mathbb{Z}$, but $\mathbb{N} \neq \mathbb{Z}$.

"is not a proper subset of"

Ex. Let S be a set. Then $S \not\subset S$. Although $S \subseteq S$ is True, $S = S$ is also true.
$\therefore S$ is not a proper subset of itself.

Cardinality.

If a set A has exactly n distinct elements (for some $n \in \mathbb{N}$), then A is called **finite** and the **cardinality** of A is n (its size). The **cardinality** of a set A is denoted $|A|$.

(what about cardinality of infinite sets ? we'll talk about it later...)

Ex. $A = \{a, b, c\}$ $|A| = 3$ "the cardinality of A is 3" because A contains 3 distinct elements.

Ex. $B = \{a, a, b\}$ $|B| = 2$ "the cardinality of B is 2" because B contains 2 distinct elements.

Ex. $G = \{3n : n \in \mathbb{Z}, 1 \leq n \leq 12\}$ $|G| = 12$

Ex. $\emptyset$ $|\emptyset| = 0$

Ex. $\{\emptyset\}$ $|\{\emptyset\}| = 1$

Ex. $D = \{a, \{a\}, \{a, \{a\}\}\}$ $|D| = 3$

these are the 3 elements of D
(one element of D is just a letter, each of the other 2 elements of D happen to be sets)

All Subsets of a Finite Set.

Ex. List all subsets of $A = \{a, b, c\}$ in increasing order of cardinality.

0-element subsets: $\emptyset$

1-element subsets: $\{a\}$, $\{b\}$, and $\{c\}$

2-element subsets: $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$

A has 7 proper subsets

3-element subsets: $\{a, b, c\}$ (ie $A \subseteq A$)

A has 8 subsets

The Power Set. $\mathcal{P}(A)$

Let A be a set.
The **power set of** A, denoted $\mathcal{P}(A)$, is the set of all subsets of A. i.e. $\mathcal{P}(A) = \{S : S \subseteq A\}$

Ex. $A = \{a, b, c\}$ $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, A\}$

In particular, $\emptyset \subseteq A$. Thus $\emptyset \in \mathcal{P}(A)$. Similarly, $A \subseteq A$. Thus $A \in \mathcal{P}(A)$.

Observe also that $A \notin \mathcal{P}(A)$ because, for example, $a \in A$, but $a \notin \mathcal{P}(A)$.

Note. $|A| = 3$ and $|\mathcal{P}(A)| = 2^3 = 8$.

Ex $\mathcal{P}(\emptyset) = \{\emptyset\}$ Note. $|\emptyset| = 0$ and $|\mathcal{P}(\emptyset)| = 2^0 = 1$

Theorem 9.3. Let A be a set. If $|A| = n$, then $|\mathcal{P}(A)| = 2^n$. *(to be proved later)*

Cartesian Product

Let A and B be sets.
The **Cartesian product "A cross B"**, denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

i.e. $A \times B = \{(a,b) : a \in A, b \in B\}$

Ex. The Cartesian plane: $\mathbb{R} \times \mathbb{R} = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}\}$

Ex. $A = \{a, b, c\}$ $B = \{1, 2\}$

$A \times B = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$

Ex. $R = \{A, 2, \ldots, 9, 10, J, Q, K\}$ $S = \{\heartsuit, \diamondsuit, \clubsuit, \spadesuit\}$

$R \times S = \{(A,\heartsuit), (A,\diamondsuit), (A,\clubsuit), (A,\spadesuit), (2,\heartsuit), \ldots, (K,\diamondsuit), (K,\clubsuit), (K,\spadesuit)\}$

$|R \times S| = 13 \times 4 = 52$ Ex $(8,\diamondsuit) \in R \times S$

$S \times R = \{(\heartsuit,A), (\heartsuit,2), (\heartsuit,3), \ldots, (\spadesuit,Q), (\spadesuit,K)\}$ $|S \times R| = 4 \times 13 = 52$.

Ex $(8,\diamondsuit) \notin S \times R$ although $(\diamondsuit, 8) \in S \times R$.

Note Cartesian product is not commutative in general ($A \times B \neq B \times A$)

Cardinality of $A \times B$.

Theorem 9.4. Let A and B be sets. Then $|A \times B| = |A||B|$.

proof. By definition, $A \times B = \{(a, b) : a \in A,\ b \in B\}$

There are $|A|$ choices for the 1st coordinate

There are $|B|$ choices for the 2nd coordinate.

$\therefore$ total # elements in $A \times B$ is $|A| \cdot |B|$ $\blacksquare$

Cartesian Product is not Associative. meaning $(A \times B) \times C \neq A \times (B \times C)$

because $(A \times B) \times C = \{((a,b), c) : (a,b) \in A \times B,\ c \in C\}$

while $A \times (B \times C) = \{(a, (b,c)) : a \in A,\ (b,c) \in B \times C\}$

So, omitting brackets would not make sense for Cartesian product the way it does make sense for other operations like multiplication; however, we do have a generalization of Cartesian product...

Generalization of Cartesian Product to More Than Two Sets.

Let $A_1, A_2, \ldots, A_n$ be sets.
The **Cartesian product** *"A_1 **cross** A_2 **cross ... cross** A_n"*, denoted $A_1 \times A_2 \times \cdots \times A_n$, is the set...

$$A_1 \times A_2 \times \cdots \times A_n = \{\underbrace{(a_1, a_2, \ldots, a_n)}_{\text{elements are ordered n-tuples}} : a_i \in A_i \text{ for } 1 \leq i \leq n\}$$

elements are ordered n-tuples (n-dimensional vectors)

Notation for Cartesian Product of A with itself n times:

$$\underbrace{A \times A \times \cdots \times A}_{n \text{ times}} = A^n$$

Ex. $\mathbb{R}^3 = \{(x,y,z) : x, y, z \in \mathbb{R}\}$

Example 9.5. Let $A = \{0, 1\}$ and let $B = \{55\}$. List all elements of the following sets:

$A \times B = \{(0,55), (1,55)\}$

$B \times A = \{(55,0), (55,1)\}$

$\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$

$\mathcal{P}(B) = \{\emptyset, \{55\}\}$

$\mathcal{P}(A) \times \mathcal{P}(B) = \{(\emptyset,\emptyset), (\emptyset,\{55\}), (\{0\},\emptyset), (\{0\},\{55\}), (\{1\},\emptyset), (\{1\},\{55\}), (\{0,1\},\emptyset), (\{0,1\},\{55\})\}$

$\mathcal{P}(A \times B) = \{\emptyset, \{(0,55)\}, \{(1,55)\}, \{(0,55), (1,55)\}\}$

$C = \{x : x = ab, a \in A, b \in B\} = \{0, 55\}$

Well-defined Sets and Russell's Paradox.

Note. In the definition of set, we stated "well-defined"...
here is an example of something that seems like a set but is not actually well-defined:

Let S be the set of all sets that do not contain themselves as elements.

i.e. $S = \{x : x \notin x\}$

S might seem okay but
S is <u>not</u> well-defined!

- Suppose $S \in S$. Then $S \notin S$ by definition of S ↯
- Suppose $S \notin S$. Then $S \in S$ by definition of S ↯

Russell's Paradox.
There is a village in which a barber shaves all those villagers and only those who do not shave themselves. Who shaves the barber?

Study Guide

Important terms and concepts:

- ⋄ set element list notation set-builder notation
- ⋄ when two sets are equal
- ⋄ empty set $\emptyset$ universal set $\mathcal{U}$
- ⋄ subset proper subset cardinality power set of S $\mathcal{P}(S)$
- ⋄ Cartesian product of two (or more) sets: $A \times B$ $A_1 \times A_2 \times \cdots \times A_k$

Exercises	Sup.Ex. §4 # 2, 3 Rosen §2.1 # 1, 2,7, 8, 9, 10, 11, 12, 13, 21, 22, 23, 25, 27, 29, 33, 34, 35a, 36a, 41, 43