DGD 11

Q1. PIGEONHOLE PRINCIPLE

i. Without knowing birthdays in advance, how many people do we need in order to guarantee that at least two among them were born on the same day of the week and in the same month (but possibly a different year)?

boxes = {day of week} × {month}
$k = 7 \times 12 = 84$

objects = people
N is to be determined

goal: we want at least $\boxed{2}$ objects (people) to be in the same box (have same day of week & month of birth)

Worst-case scenario: • 84 is too few in worst-case scenario when exactly 1 person was born on each of the 84 distinct day of week/month combos.

Using the Pigeonhole Principle: • 85 is enough because GPP guarantees at least one box will contain at least $\lceil \frac{N}{k} \rceil = \lceil \frac{85}{84} \rceil = 2$ objects.

Answer: We need $\boxed{N=85.}$

ii. How many people do we need in order to guarantee that at least three among them were born on the same month?

boxes = months
$k = 12$

objects = people
N is to be determined

goal: we want at least $\boxed{3}$ objects (guests) to be in the same box (have same month of birth)

Worst-case scenario: 24 is too few in the worst-case scenario when exactly two guests were born in each of the 12 different months.

Using the Pigeonhole Principle: 25 is enough since the GPP guarantees at least one box will contain at least $\lceil \frac{N}{k} \rceil = \lceil \frac{25}{12} \rceil = 3$ objects

Answer: We need $\boxed{N=25.}$

iii. Let k and r be positive integers. Suppose we have k boxes and some objects are being placed into the boxes, but we do not have control over which objects will end up in which boxes. What is the minimum number N of objects we need in order to guarantee that, when N objects are placed somehow into the k boxes, at least one box will contain at least r objects?

boxes = boxes
$k = k$

objects = objects
N is to be determined

goal: we want at least $\boxed{r}$ objects to be in the same box

Worst-case scenario:

with only $k(r-1)$ objects, we could end up in the worst-case scenario:

each of the k boxes contains $r-1$ objects

$r-1$ objects	$r-1$ objects	...	$r-1$ objects
box 1	box 2		box k

(each box is just 1 object short of the goal).

$\therefore$ $k(r-1)$ objects is too few.

Using the Pigeonhole Principle:

If we have $N = k(r-1)+1$ objects placed into k boxes, then GPP guarantees that at least one box will contain at least $\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{k(r-1)+1}{k} \right\rceil = \left\lceil \underbrace{r-1}_{\in \mathbb{Z}} + \frac{1}{k} \right\rceil = r$ objects

$\frac{1}{k} > 0$ since $k \in \mathbb{Z}^+$
so the ceiling of
$r - 1 + \frac{1}{k}$ will equal r

$\therefore$ $N = k(r-1)+1$ is enough

$\therefore$ $N = k(r-1)+1$ is exactly the minimum threshold to guarantee that at least one box contains at least r objects.

Q2. Degree Sequences Does there exist

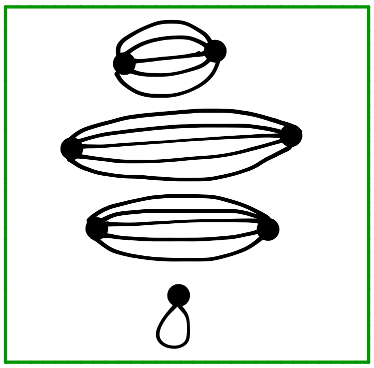
i. a graph

ii. a **simple** graph

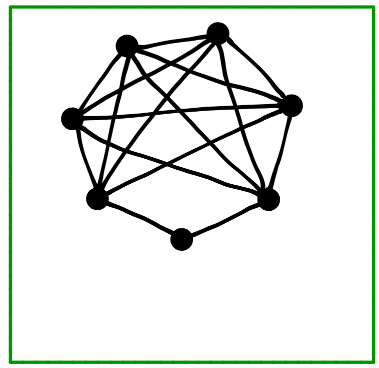
whose degree sequence is the given sequence?

If so, draw a picture of such a graph (or simple graph). If not, explain why.

a. $(2,5,5,5,5,5,5)$ (# vertices) = 7 (# edges) = $\frac{1}{2}(2+5+5+5+5+5+5) = 16$



← not a simple graph



← a simple graph

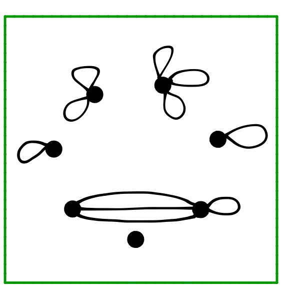
b. $(3,3,3,3,3,5,5)$ (# vertices) = 7

(# edges) = $\frac{1}{2}(3+3+3+3+3+5+5) = 12.5$ ⚡

Since this degree sequence contradicts the Handshaking Theorem, there is no graph (simple or non-simple) with this degree sequence.

c. $(0,2,2,3,4,5,6)$ (# vertices) = 7 (# edges) = $\frac{1}{2}(0+2+2+3+4+5+6) = 11$

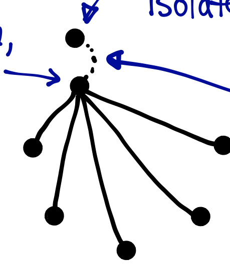
there is no simple graph with this degree sequence:



← not a simple graph

① one vertex must be isolated (degree 0)

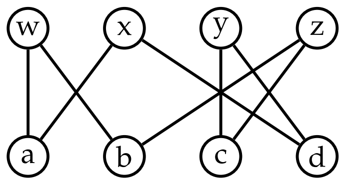
② at the same time, another vertex must have 6 distinct neighbours (degree 6)



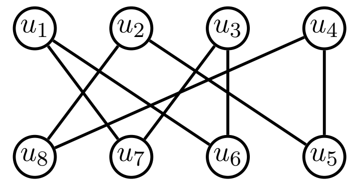
In a simple graph on 7 vertices, ① and ② can't both be true

Q3. GRAPH CONCEPTS

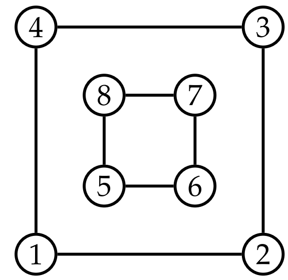
Consider the following 3 graphs:



G_1



G_2



G_3

a. Write the adjacency matrix of each of the above graphs.

G_1:

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} a & b & c & d & w & x & y & z \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \\ w \\ x \\ y \\ z \end{array} &
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

G_2:

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \end{array} \\
\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{array} &
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

G_3:

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} &
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$

b. For each graph, determine its degree sequence and the number of edges.

- All three of these graphs have the same number of vertices: 8 vertices
- All three have the same degree sequence: (2,2,2,2,2,2,2,2)
- All three of these graphs have the same number of edges: 8 edges.

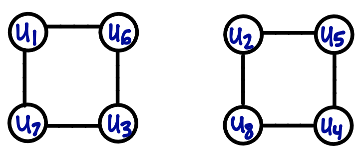
c. Exactly two out of three of these graphs are isomorphic to each other. Which two? Give an isomorphism to prove your claim. Can you explain why the other graph is not isomorphic to the other two?

$\boxed{G_2 \cong G_3}$

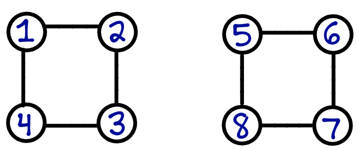
here is an isomorphism $f: V(G_2) \to V(G_3)$

$u_i \in V(G_2)$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
$f(u_i) \in V(G_3)$	1	5	3	7	6	2	4	8

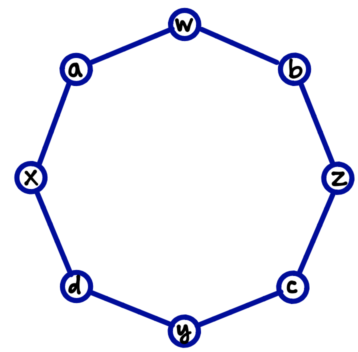
G_2 redrawn:



G_3 redrawn:



G_1 redrawn:



$\boxed{\begin{array}{c} G_1 \cong G_2 \\ G_1 \ncong G_3 \end{array}}$

A fundamental difference between G_1 and the other 2	G_1 is in one "connected" cycle (C_8) whereas G_2 and G_3 each consist of 2 "disconnected" cycles ($C_4 \times 2$)