# 16. Integration Strategy

## Lec 15 mini review.

**useful trig identities**:

$\cos^2(x) + \sin^2(x) = 1$

$1 + \tan^2(x) = \sec^2(x)$

$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

$\sin(2x) = 2\sin(x)\cos(x)$

| **expression**: | **identity**: | **substitution**: |
|---|---|---|
| $\sqrt{1-x^2}$ | $1 - \sin^2\theta = \cos^2\theta$ | $x = \sin\theta$ $(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})$ |
| $\sqrt{1+x^2}$ | $1 + \tan^2\theta = \sec^2\theta$ | $x = \tan\theta$ $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$ |
| $\sqrt{x^2-1}$ | $\sec^2\theta - 1 = \tan^2\theta$ | $x = \sec\theta$ $(0 \leq \theta < \frac{\pi}{2}, \quad \pi \leq \theta < \frac{3\pi}{2})$ |

**partial fractions:** denominator factors into product of

- distinct linear factors: *partial fractions expression*

$$\frac{N(x)}{D(x)} = \frac{N(x)}{(a_1x+b_1)\dots(a_kx+b_k)} = \frac{A_1}{a_1x+b_1} + \cdots + \frac{A_k}{a_kx+b_k}$$

- distinct irreducible quadratic factors: *partial fractions expression*

$$\frac{N(x)}{D(x)} = \frac{N(x)}{(a_1x^2+b_1+c_1)\dots(a_kx^2+b_kx+c_k)} = \frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \cdots + \frac{A_kx+B_k}{a_kx^2+b_kx+c_k}$$

- if a linear factor is repeated, say $(ax+b)^r$ appears in factorization of denominator, we use

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r} \quad \text{intead of the single term} \quad \frac{A}{ax+b}$$

- if an irreducible quadratic factor is repeated, say $(ax^2+bx+c)^r$ appears in factorization of denominator, we use

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r} \quad \text{intead of the single term} \quad \frac{Ax+B}{ax^2+bx+c}$$

## INTEGRATION STRATEGY

- simple antiderivative? minor simplification first?
- possible $u$-substitution? looks like aftermath of a chain rule?
- $u$-substitution not working? are you sure you can't cancel the old variable by restating $x$ in terms of $u$ in another way?
- maybe integration by parts is needed? what will the new integral look like? possible tie situation in IBP?
- possible substitution in combination with IBP?
- a trig integral? could trig identities be of help?
- in need of a trig substitution?
- a rational function? maybe long division would help? what about partial fractions?

**Example 16.7.** $\displaystyle\int \frac{x^3 - x + \pi}{x^{2/3}} dx$

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**Example 16.8.** $\displaystyle\int 5^x dx$

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**Example 16.9.** $\displaystyle\int \sqrt{x}(1 - x^3 + 2\sqrt{x}) dx$

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**Example 16.10.** $\displaystyle\int \frac{\tan(x)}{\sec(x)} dx$

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**Example 16.11.** $\displaystyle\int \frac{dx}{x^2 - 6x + 14}$

**Example 16.12.** $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}dx$

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**Example 16.13.** $\displaystyle\int \frac{e^{1/x}}{x^3}dx$

**Example 16.14.** $\int \frac{\sin^3(x)}{\cos(x)} dx$

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**Example 16.15.** $\int \frac{x^2}{\sqrt{1-3x^2}} dx$

**Example 16.16.** $\int \frac{2x-3}{x^3+3x} dx$

## MUST-KNOW CONCEPTS/FORMULAS FOR INTEGRATION

- ☐ all basic antiderivatives
- ☐ approximation of definite integral using Riemann sums (with $n$ rectangles)
- ☐ definition of definite integral as infinite limit of Riemann sum
- ☐ how to use FTC 1 to differentiate
- ☐ how to use FTC 2 to evaluate definite integrals
- ☐ interpretation of a definite integral as net area (or net change)
- ☐ difference between definite/indefinite integrals
- ☐ when to write $+C$ versus subtract $F(b) - F(a)$
- ☐ recipe for $u$-substitution
- ☐ formula for integration by parts
- ☐ the useful trig identities
- ☐ the three main strategies for trig substitution
- ☐ general strategies for integrating (*e.g.* using simplifications such as trig identities, completing the square, long division, using IBP, using IBP multiple times, using $u$-substitution, using $u$-substitution in combination with IBP, using trig substitution, other simplifications...)
- ☐ method of partial fractions, including scenarios with repeated factors in the denominator