18. Min/Max Values

Lec 17 mini review.

- ◇ **Midpoint Rule**: $\boxed{M_n = \sum_{i=1}^{n} f(\overline{x_i})\Delta x}$ $\quad (\overline{x_i} = \frac{1}{2}(x_{i-1}+x_i))$
- ◇ **Trapezoidal Rule**: $\boxed{T_n = \frac{\Delta x}{2}\Big[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\Big]}$
- ◇ **Simpson's Rule (n even)**:

$$\boxed{S_n = \frac{\Delta x}{3}\Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big]}$$

Error Bounds:

- If $|f''(x)| \le K$ for all $a \le x \le b$, then $\left|T_n - \int_a^b f(x)dx\right| \le \frac{K(b-a)^3}{12n^2}$ and $\left|M_n - \int_a^b f(x)dx\right| \le \frac{K(b-a)^3}{24n^2}$
- If $|f^{(4)}| \le K$ for all $a \le x \le b$, then $\left|S_n - \int_a^b f(x)dx\right| \le \frac{K(b-a)^5}{180n^4}$

MAX/MIN VALUES

A function $y = f(x)$ has...

...an **ABSOLUTE/GLOBAL MAXIMUM AT $x = c$** if

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The value $f(c)$ is called the...

GLOBAL MAXIMUM VALUE of f.

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- ◇ The max/min values of f are called **EXTREME VALUES** .
- ◇ A function f can attain global max/min values at many numbers.

⋄ Some functions do not have max and/or min values.

LOCAL VS. GLOBAL EXTREME VALUES

A function $y = f(x)$ has...

...a **LOCAL MAXIMUM AT** $x = c$ if

...a **LOCAL MINIMUM AT** $x = c$ if

⋄ Some local max/min are also absolute extreme points.

⋄ Not every local max/min is an absolute max/min.

⋄ Not every absolute max/min is a local max/min.

Extreme Value Theorem

Theorem 18.1. (Extreme Value Theorem)
If $y = f(x)$ is continuous on the closed interval $[a, b]$, then, restricted to the interval $[a, b]$, f has an absolute maximum and an absolute minimum on $[a, b]$.

Note. If f is not continuous on $[a, b]$, then the Extreme Value Theorem is not applicable. Even if f is continuous, if the interval is not closed, then the Extreme Value Theorem is not applicable.

Critical Numbers and How to Find Extreme Values

The Extreme Value Theorem tells us of a situation in which absolute extreme values must exist, but it does not tell us how to find them. For that, we need Fermat's Theorem.

Theorem 18.2. (Fermat's Theorem)

A number x is called a **Critical Number** of a function $f(x)$ if either

- x is in the domain of f and $f'(x) = 0$ (type 1)
- or x is in the domain of f but x is **not** in the domain of f'. (type 2)

Graphically, these two types of critical numbers correspond to numbers x such that

- $f(x)$ has a horizontal tangent at x (type 1)
- $f(x)$ has a corner, jump discontinuity, or vertical tangent line at x (type 2)

Example 18.3. Find all the critical numbers of $h(x) = x^{2/3}(x-2)^2$.

The Closed Interval Method

Follow these steps to find the absolute extrema of a <u>continuous</u> function $f(x)$ on a closed interval $[a,b]$

1. Find the critical numbers of f.
2. For each critical number c such that $c \in [a,b]$, compute its value $f(c)$.
3. For the endpoints $x = a$ and $x = b$ of $[a,b]$, compute the values $f(a)$ and $f(b)$.
4. The absolute maximum value of f on $[a,b]$ is largest value computed in steps 2 and 3.
5. The absolute minimum value of f on $[a,b]$ is smallest value computed in steps 2 and 3.

Example 18.4. Find the absolute/global extreme points of $h(x) = x^{2/3}(x-2)^2$ on the closed interval $[-1,1]$ (the function from Ex. 18.3)

Example 18.5. After an antibiotic tablet is taken, the concentration of the antibiotic in the bloodstream is modelled by the function

$$C(t) = 8(e^{-0.4t} - e^{-0.6t})$$

where the time t is measured in hours and C is measured in μg/ mL. What is the maximum concentration of the antibiotic during the first 12 hours?

Example 18.6. Find the absolute maximum and absolute minimum of $f(x) = \sin(x)\cos(x)$ on the interval $[-\pi, \pi]$.

MEAN VALUE THEOREM

Theorem 18.7. (Mean Value Theorem) If $f(x)$ is continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b), then there exists a number $c \in (a, b)$ such that

Example 18.8. Consider $f(x) = x^3$ on the closed interval $[0, 1]$. Calculate the slope of the secant line joining the endpoints of this interval. What does the Mean Value Theorem guarantee? Verify that it holds for this particular function.

Example 18.9. Try to find the critical numbers of $f(x) = \sin(x)\ln(x)$.

Example 18.10. Using the Mean Value Theorem, explain why $f(x) = \sin(x)\ln(x)$ has at least one critical number on the interval $[1, \pi]$.

Theorem 18.11. (Rolle's Theorem)
If $f(x)$ is continuous on the closed interval $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there exists a number $c \in (a, b)$ such that $f'(c) = 0$.

- ◇ Rolle's Theorem can be viewed as a special case of the Mean Value Theorem:

STUDY GUIDE

- ◦ **absolute/global minimum** **absolute/global maximum**
- ◦ **relative/local minimum** **relative/local maximum**
- ◦ **critical numbers** and how to find them (type 1 and type 2).
- ◦ **Extreme Value Theorem** and how to find absolute extrema of a continuous function on a closed interval