MAT 1348 – Winter 2023

Exercises 10 – Solutions

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Questions are taken from Discrete Mathematics 8th edition, by Kenneth H. Rosen.

QUESTION 1 **(6.2 # 3)**. A drawer contains 12 brown socks and 12 black socks. We take socks from this drawer at random.

(a) How many socks must we take from this drawer to ensure we have taken a pair of socks of the same colour?
(b) How many socks must we take from this drawer to ensure we have taken a pair of black socks?

Solution:

(a) 3. If we take only 2 socks, it is possible to have one brown sock and one black sock, in which case we do not have a pair of socks of the same colour. With 3 socks (objects) and 2 colours of socks (boxes), the pigeonhole principle ensures that we will have two socks of the same colour.
(b) 14. If we take only 13 socks, it is possible to take 12 borwn socks and only one black sock. By taking 14 socks, if we still do not have 2 black socks, then we must have at least 13 brown socks, which is impossible.

QUESTION 2 **(6.2 # 4)**. A bowl contains 10 red marbles and 10 blue marbles. We take marbles out of the bowl at random.

(a) How many marbles must we draw to be certain that we have drawn 3 marbles of the same colour?
(b) How many marbles must we draw to be certain that we have drawn 3 blue marbles?

Solution:

(a) 5. If we only draw 4 marbles, it is possible to draw 2 red marbles and 2 blue marbles, in which case we dot not have 3 marbles of the same colour. By drawing $N = 5$ marbles (objexts) that have $k = 2$ different colours (boxes), the generalized pigeonhole principle ensures we will have $\lceil N/k \rceil = \lceil 5/2 \rceil = 3$ marbles of the same colour.
(b) 13. If we only draw 12 marbles, it is possible to have 10 red marbles and 2 blue marbles, in which case we do not have 3 blue marbles. By drawing 13 marbles, if we still do not have 3 blue marbles, then we must have drawn at least 11 red marbles, which is impossible.

QUESTION 3 **(6.2 #5)**. At a university, there are students in 1st year, 2nd year, 3rd year and 4th year. The university has 21 programs. How many students are needed to ensure there are two students in the same year and the same program?

Solution: There are $4 \cdot 21 = 84$ possible combinations of year and program. There must be at least 85 students to ensure there are two students in the same year and program.

QUESTION 4 **(6.2 #7)**. Show that any set of 5 integers must contain two integers whose difference is divisible by 4.

Solution: Let $A_0 = \{..., -8, -4, 0, 4, 8, ...\}$, $A_1 = \{..., -7, -3, 1, 5, 9, ...\}$, $A_2 = \{..., -6, -2, 2, 6, ...\}$ et $A_3 = \{..., -5, -1, 3, 7, ...\}$. We notice that the difference between any two elements from a single set is divisible by 4, and that every integer belongs to one of the four sets. By the pigeonhole principle, if we take 5 integers (objects), at least two of them will belong to the same set A_i (boxes). Therefore, the difference of those two numbers will be divisible by 4.

QUESTION 5 **(6.2 # 15)**. Let $A = \{1,2,3,4,5,6,7,8\}$

(a) Show that any subset of A of size 5 contains two numbers whose sum is 9.
(b) If we instead take subsets of size 4, is the previous statement still true?

Solution:

(a) We decompose A in the following boxes: $A_1 = \{1,8\}$, $A_2 = \{2,7\}$, $A_3 = \{3,6\}$, $A_4 = \{4,5\}$. The sum of the two elements in each box is 9.

Let S be a subset of A of size 5. The elements of S are the objects. Since we have 5 objects but only 4 boxes, S must contain two objects belonging to the same box (pigeonhole principle). The sum of these two numbers is 9.

(b) The statement becomes false. If $S = \{1,2,3,4\}$, then no pair of elements of S adds up to 9.

QUESTION 6 **(6.2 # 18)**. Let $A = \{1,3,5,7,9,11,13,15\}$. What is the smallest value of n for which the following statement is true?

"Every subset of A of size n contains two numbers whose sum is 16"

Solution: We decompose $A = \{1,3,5,7,9,11,13,15\}$ into the following boxes: $A_1 = \{1,15\}$, $A_2 = \{3,13\}$, $A_3 = \{5,11\}$, $A_4 = \{7,9\}$. Two elements from the same box add up to 16.

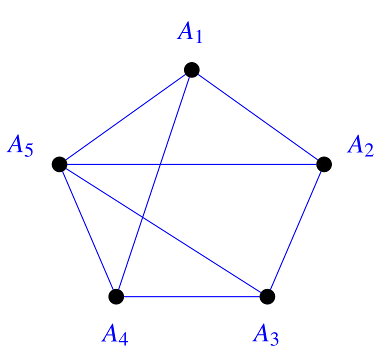
let S be a subset of A of size 5. The elements of S are the objects. Since we have 5 objects but only 4 boxes, S contains two elements belonging to the same box (pigeonhole principle). The sum of these two numbers add up to 16. So $n \leq 5$

If we take $n = 4$, then the statement is false since $S = \{1,3,5,7\}$ does not contain two numbers whose sum is 16. Therefore, the answer is $n = 5$.

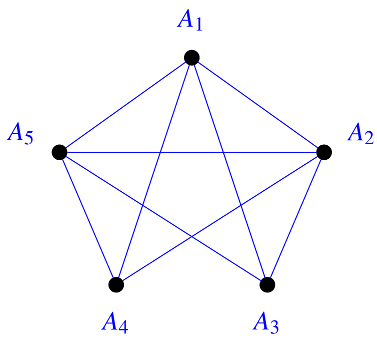
QUESTION 7 **(1.4 # 1)**. The intersection graph of a family of sets $\{A_1, A_2, ..., A_n\}$ is the graph whose vertices are labeled $A_1, A_2, ..., A_n$. Two vertices A_i and A_j are adjacent if and only if $A_i \cap A_j \neq \emptyset$. Draw the intersection graph for the following families.

(a) $A_1 = \{0,2,4,6,8\}, A_2 = \{0,1,2,3,4\}, A_3 = \{1,3,5,7,9\}, A_4 = \{5,6,7,8,9\}, A_5 = \{0,1,8,9\}$.
(b) $A_1 = \{..., -4,-3,-2,-1,0\}, A_2 = \{..., -2,-1,0,1,2,...\}, A_3 = \{..., -6,-4,-2,0,2,4,6,...\}, A_4 = \{..., -5,-3,-1,1,3,5,...\}$, $A_5 = \{...., -9,-6,-3,0,3,6,9,...\}$.
(c) $A_1 =]-\infty,0[, A_2 =]-1,0[, A_3 =]0,1[, A_4 =]-1,\infty[, A_5 = \mathbb{R}$.

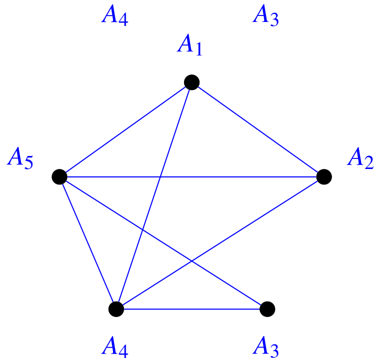
Solution:



(a)



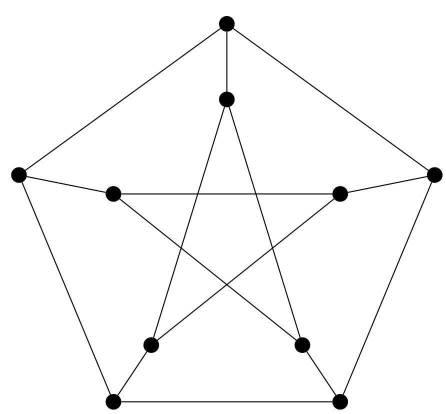
(b)



(c)

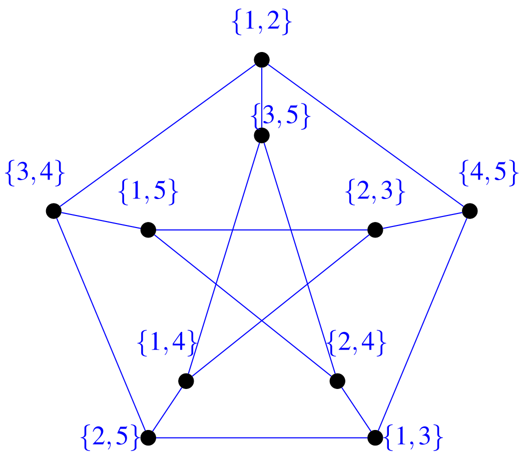
QUESTION 8 **(1.4 # 3)**. The Petersen graph P is defined as follows. Its vertices are the subsets of $\{1,2,3,4,5\}$ of size 2, and two vertices A_1 and A_2 are adjacent in P if and only if $A_1 \cap A_2 = \emptyset$.

(a) How many vertices does the Petersen graph have?
(b) Show that the following graph is the Petersen graph by labeling each vertex with a subset of $\{1,2,3,4,5\}$ of size 2.



Solution:

(a) The number of vertices is equal to the number of subsets of $\{1,2,3,4,5\}$ of size 2, which is $\binom{5}{2} = 10$.

(b) 

QUESTION 9 **(2.5 # 1)**. How many edges do the following graphs have?

(a) A graph whose degree sequence is $(0,0,1,1,2,3,4,5,5,7)$
(b) A graph with 6 vertices of degree 3 and 6 vertices of degree 4 (and no other vertices)

Solution:

(a) According to the handshake lemma, the sum of the degrees is equal to twice the number of edges. The sum of the degrees is $0+0+1+1+2+3+4+5+5+7=28$, which means the graph has $28/2=14$ edges.
(b) The sum of the degrees is $6\cdot 3+6\cdot 4=42$, which means there are $42/2=21$ edges.

QUESTION 10 **(2.5 # 2)**. Suppose we have a graph with 11 vertices and 20 edges. If all the vertices have a degree of either 3 or 4, how many vertices of degree 3 does the graph have?

Solution: Let n be the number of vertices of degree 3 and m be the number of vertice of degree 4. Since the graph has 11 vertices, we get $n+m=11$. According to the handshake lemma, $3n+4m=2\cdot 20$. We use the two equations to solve for n and for m. We get $n=4$ and $m=7$. So the graph has $n=4$ vertices of degree 3.

QUESTION 11 **(2.5 # 3)**. Let G be a simple graph with 6 vertices and 10 edges. The degree of each vertex is odd, and there is one more vertex of degree 3 than vertices of degree 5. What is the degree sequence of the graph?

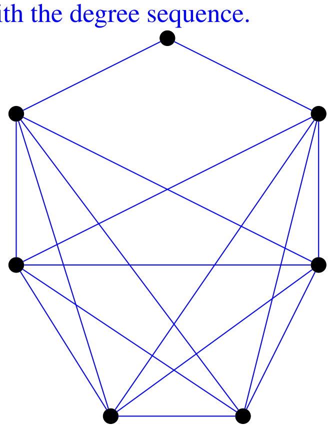
Solution: Let n_i be the number of vertices of degree i. Since the graph is simple and only has 6 vertices, the maximum degree is 5 (such a vertex would be adjacent to the 5 other vertices of the graph). Therefore, since every vertex has an odd degree, $n_1+n_3+n_5=6$. Also $n_3=n_5+1$. By the handshake lemma, $1n_1+3n_3+5n_5=2\cdot 10$. We solve this system of three equations and three unknowns to get $n_1=1$, $n_3=3$ and $n_5=2$. Therefore, the degree sequence is $(1,3,3,3,5,5)$.

QUESTION 12 **(2.5 # 9)**. For each of the following sequences, determine if there exists a graph with that degree sequence, and if there exists a simple graph with that degree sequence.

(a) $(2,5,5,5,5,5,5)$
(b) $(3,3,3,3,3,5,5)$
(c) $(2,2,2,3,4,4,7)$
(d) $(0,2,2,3,4,5,6)$

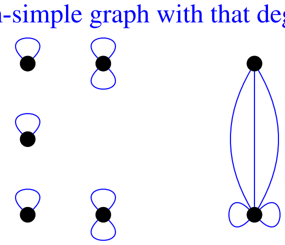
Solution:

(a) Here is a simple graph with the degree sequence $(2,5,5,5,5,5,5)$. So, there exists a graph and a simple graph with the degree sequence.

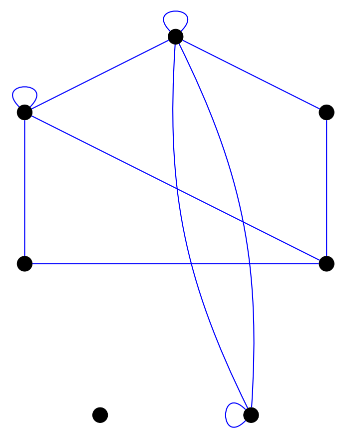


(b) There are no graphs (simple or not) with this degree sequence. According to the handshake lemma, the sum of the degrees must be even. Here, the sum is 25, which is odd.

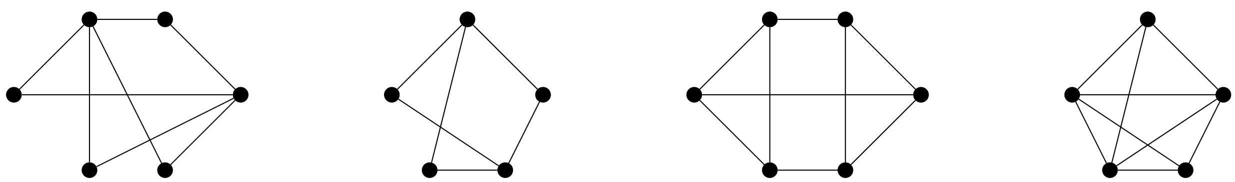
(c) There are no simple graphs with that degree sequence. We are looking for a graph with a vertex of degree 7, but the graph only has 7 vertices. In a simple graph with 7 vertices, the maximum degree is 6. There is, however, a non-simple graph with that degree sequence.



(d) There are no simple graphs with that degree sequence. Suppose, by contradiction, that there is such a simple graph. Therefore, the vertex of degree 6 must be adjacent to all other vertices of the graph, including the vertex of degree 0. This is impossible, since the vertex of degree 0 is not adjacent to any other vertex. There is, however, a non-simple graph with that degree sequence.



QUESTION 13 **(3.3 # 1)**. For each of the following graphs, find an adjacency matrix.



Solution:

(a) $\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$

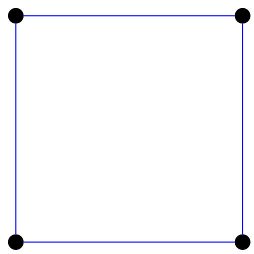
(c) $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

QUESTION 14 **(3.3 # 2)**. Draw a graph whose adjacency matrix is

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

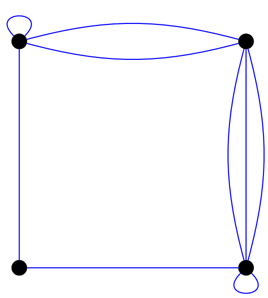
Solution:



QUESTION 15 **(3.3 # 3)**. Draw a graph whose adjacency matrix is

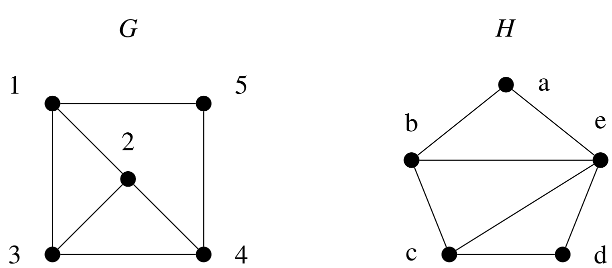
$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

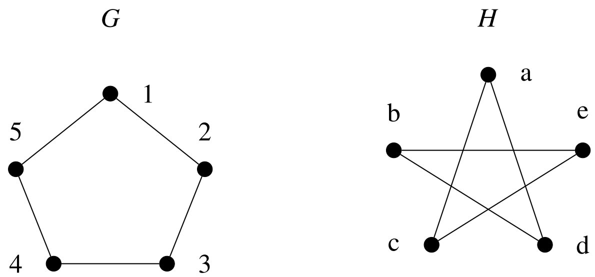
Solution:

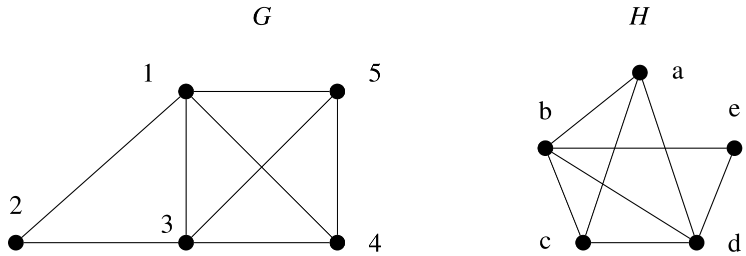


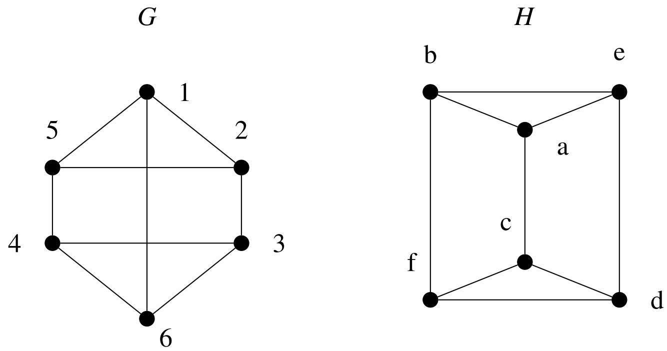
QUESTION 16 **(3.3 # 7)**. For each of the following pairs of graphs, determine if the two graphs are isomorphic. If they are, find an isomorphism. If they are not, explain why.

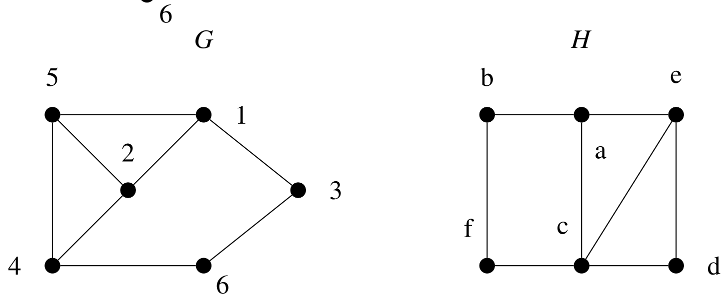
(a)

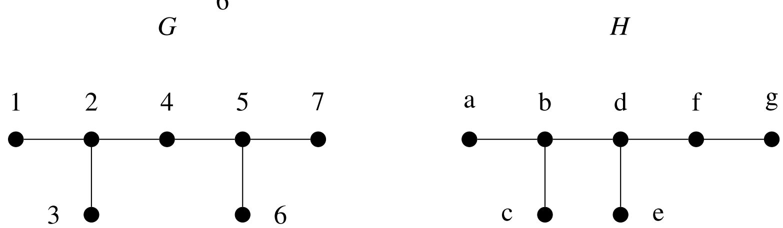


(b) 

(c) 

(d) 

(e) 

(f) 

Solution:

(a) G and H are not isomorphic, since they do not have the same degree sequence. The sequence for G is $(2,3,3,3,3)$ and the sequence for H is $(2,2,3,3,4)$.

(b) The two graphs are isomorphic. Here is an isomorphism: $f : V(G) \to V(H)$: $f(1) = a$, $f(2) = d$, $f(3) = b$, $f(4) = e$ and $f(5) = c$.

(c) The two graphs are isomorphic. Here is an isomorphism: $f : V(G) \to V(H)$: $f(1) = b$, $f(2) = e$, $f(3) = d$, $f(4) = a$ and $f(5) = c$.

(d) The two graphs are isomorphic. Here is an isomorphism: $f : V(G) \to V(H)$: $f(1) = a$, $f(2) = e$, $f(3) = d$, $f(4) = f$, $f(5) = b$ and $f(6) = c$.

(e) G and H are not isomorphic, since they do not have the same degree sequence. The sequence for G is $(2,2,3,3,3,3)$ and the sequence for H is $(2,2,2,3,3,4)$.

(f) G and H are not isomorphic. G has two vertices of degree 3 that are adjacent, while H does not.