Calc Midterm Winter 2022 (solutions)

Calculus II (University of Ottawa)

MULTIPLE-CHOICE QUESTIONS Questions 1–4 are **multiple-choice questions** worth 2 points each. Your answers to multiple-choice questions do not need to be justified. When you reach your answer, clearly indicate the question number and write the letter of your response beside the question number:

For example: (*write out your scrap work, but it will not be graded*)

(*clearly indicate your final choice*) $\boxed{\textbf{Q1.}\ [\texttt{letter of your choice}]}$

Q1. The number N of bacteria in a culture satisfies the differential equation $\frac{dN}{dt} = kN$, where t is measured in hours. If there are initially 1000 bacteria and 3 hours later there are 3000 bacteria, find the number of bacteria expected 4 hours after the initial measurement. Round your answer to the nearest integer.

A. 4327 **B.** 4819 **C.** 3711 **D.** 3628 **E.** 4658 **F.** 3226

Solution: $\boxed{\textbf{A}}$

Since $\frac{dN}{dt} = kN$, we have $\int \frac{1}{N}\,dN = \int k\,dt \implies \ln|N| = kt + C \implies N = Ae^{kt}$.

Since $N(0) = 1000$, we find $A = 1000$, so $N(t) = 1000e^{kt}$.

Since $N(3) = 3000$, we find $3 = e^{3k} \implies k = \frac{\ln(3)}{3}$.

Thus, $N(t) = 1000e^{\frac{\ln(3)\cdot t}{3}} = 1000(3)^{t/3}$.

Finally, $N(4) = 1000(3)^{4/3} \approx 4327$

Marking: 2 points all or nothing

Q2. Which one of the following statements is always true?

A. If the series $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ must converge.

B. If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then the series $\displaystyle\sum_{n=1}^{\infty} |a_n|$ must converge.

C. If the series $\displaystyle\sum_{n=1}^{\infty} |a_n|$ diverges, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ must converge.

D. If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then the series $\displaystyle\sum_{n=1}^{\infty} |a_n|$ must converge

Solution: $\boxed{\textbf{A}}$

If the series is absolutely convergent, then, it must be convergent.

Marking: 2 points all or nothing

Q3. If it is convergent, find the sum of the geometric series $\displaystyle\sum_{m=1}^{\infty} \frac{(-2)^m \cdot 3^{m-1}}{4^{m+3}}$.

A. $\frac{1}{320}$ **B.** $-\frac{1}{320}$ **C.** -2 **D.** $\frac{2}{5}$ **E.** $-\frac{1}{64}$ **F.** $\frac{1}{64}$ **G.** It diverges.

Solution: $\boxed{\textbf{G}}$

This geometric series has first term $a = -\frac{2}{4^4}$ and common ratio $r = -\frac{6}{4}$ with $|r| > 1$. Therefore, it diverges.

Marking: 2 points all or nothing

Q4. Fact: for each $m \geq 2$, $\displaystyle\int_m^{\infty} \frac{1}{x(\ln(x))^2}\,dx = \frac{1}{\ln(m)}$ and the series $S = \displaystyle\sum_{m=2}^{\infty} \frac{1}{m(\ln(m))^2}$ converges by virtue of the Integral Test. According to the Remainder Estimate Theorem, what is the smallest value of N for which the error $R_N = S - S_N$ is at most 0.5 ?

Solution: $\boxed{\textbf{F}}$

By the Remainder Estimate Theorem, $R_N \leq \int_N^{\infty} \frac{1}{x(\ln(x))^2}\,dx = \frac{1}{\ln(N)}$, so it suffices to find N such that $\frac{1}{\ln(N)} \leq 0.5$ We find $\frac{1}{0.5} = 2 \leq \ln(N) \implies e^2 \leq N$ Thus, we need $7.389... \leq N$. Since N must be an integer, we need $N \geq 8$.

Marking: 2 points all or nothing

A. $N = 12$ **B.** $N = 11$ **C.** $N = 9$ **D.** $N = 10$ **E.** $N = 7$ **F.** $N = 8$ **G.** $N = 13$

LONG-ANSWER QUESTIONS Questions 5–7 are **long-answer questions** worth a total of 17 points. For long-answer questions, **all of your work must be justified and your steps must be written in a clear and logical order**. Clearly indicate Question numbers.

For example: $\boxed{\textbf{Q5(a).}}$ [`write a fully justified solution`].

Q5. [**6 points**] For each integer $k \geq 1$, let $a_k = \sqrt{2k+5} - \sqrt{2k+3}$.

a) Consider the **sequence** $\{a_k\}_{k=1}^{\infty} = \left\{\sqrt{2k+5} - \sqrt{2k+3}\right\}_{k=1}^{\infty}$.

Is this sequence convergent or divergent? If it converges, find its exact value. Show all your work to justify your answer. Be sure to use appropriate mathematical notation.

Solution: We have

$$\begin{aligned}
\lim_{k\to\infty} a_k &= \lim_{k\to\infty} \sqrt{2k+5} - \sqrt{2k+3} \\
&= \lim_{k\to\infty} \left(\sqrt{2k+5} - \sqrt{2k+3}\right)\left(\frac{\sqrt{2k+5} + \sqrt{2k+3}}{\sqrt{2k+5} + \sqrt{2k+3}}\right) \\
&= \lim_{k\to\infty} \left(\frac{(2k+5) - (2k+3)}{\sqrt{2k+5} + \sqrt{2k+3}}\right) \\
&= \lim_{k\to\infty} \left(\frac{5-3}{\sqrt{2k+5} + \sqrt{2k+3}}\right) \\
&= 0
\end{aligned}$$

Thus, the sequence $\{a_k\}_{k=1}^{\infty}$ converges to 0.

Marking: [2 points]
0.5 points for trying to rationalize to evaluate limit
0.5 points for correctly simplifying after rationalization
0.5 points for concluding the sequence converges (provided the limit was attempted in some reasonable way) 0.5 points for finding that it converges to 0 (provided reasonable work was shown)

b) Consider the **telescopic** series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\sqrt{2k+5} - \sqrt{2k+3}\right)$.

Find a simplified expression for the nth partial sum S_n, by cancelling terms telescopically. Show all your work!

Solution:

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \sqrt{2k+5} - \sqrt{2k+3}$$

$$= \sum_{k=1}^{n} \sqrt{2k+5} - \sum_{k=1}^{n} \sqrt{2k+3}$$

$$= \left(\sqrt{7} + \cdots + \sqrt{2n+3} + \sqrt{2n+4} + \sqrt{2n+5}\right)$$
$$- \left(\sqrt{5} + \sqrt{6} + \sqrt{7} + \cdots + \sqrt{2n+3}\right)$$

$$= \sqrt{2n+4} + \sqrt{2n+5} - \sqrt{5} - \sqrt{6}$$

Marking: [2 points]
0.5 points for correct series expression for S_n
0.5 points for attempting telescopic cancellations
1 point for correctly simplified expression for S_n

c) Determine whether the series $\sum_{k=1}^{\infty} \{a_k\}$ converges or diverges. If it converges, find its exact sum; otherwise, briefly explain why it diverges.

Solution: Using the expression from b), we find

$$\sum_{k=1}^{\infty} \{a_k\} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sqrt{2n+4} + \sqrt{2n+5} - \sqrt{5} - \sqrt{6} = \infty$$

Thus, this series diverges because the limit, as $n \to \infty$ of its nth partial sum, does not exist.

Marking: [2 points]
0.5 points for knowing they should find the limit as $n \to \infty$ of the expression for S_n
1 points for correctly determining that limit approaches ∞
0.5 points for correct conclusion (diverges), or for answer that is consistent with their work (if they made a small mistake but the rest of their reasoning makes sense with what they thought the limit approached)

Q6. [**6 points**] Use appropriate series tests to determine whether each of the following series is convergent or divergent.

If you apply a certain test, name it, and state and show your work to verify the conditions needed in order for the test to apply.

Show all your work. Clearly explain your conclusion.

(a) $\displaystyle\sum_{N=1}^{\infty}\left(\frac{-N}{4N+2}\right)^N$ (b) $\displaystyle\sum_{k=1}^{\infty}(-1)^k k e^{-k^2}$

Solution: (a)

Marking: [3 points]
0.5 points for the answer "converges" provided a justification was attempted (give 0 if they write "divergent" with no attempt at justifying their guess).
0.5 points for applying the root test
0.5 points for handling the absolute value correctly
0.5 points for simplifying the root correctly
0.5 points for reaching the correct limit 0.5 for conclusion explained

Solution: (b) This is an alternating series with $b_n = |a_n| = ne^{-n^2}$.

Let $f(x) = xe^{-x^2}$.

We see that $f'(x) = e^{-x^2} + x(-2xe^{-x^2} = e^{-x^2}(1-2x^2) < 0 \iff 1-2x^2 < 0$. Thus, $f'(x) < 0$ for all x such that $x^2 > \frac{1}{2}$. Therefore, $f(x)$ is decreasing for all $x \geq 1$.

Consequently, $b_{n+1} \leq b_n$ for all $n \geq 1$.

Furthermore

$$\begin{aligned}
\lim_{n\to\infty} b_n &= \lim_{n\to\infty} ne^{-n^2} && \text{indet. form } \infty\cdot 0\\
&= \lim_{n\to\infty}\frac{n}{e^{n^2}} && \text{indet. form } \infty/\infty\\
&= 0 && \text{since } \lim_{n\to\infty}\frac{1}{2ne^{n^2}} = 0\text{, l'Hospital's rule applies}
\end{aligned}$$

Thus, this series converges by virtue of the Alternating Series Test.

Marking: [3 points]
0.5 points for the answer "convergent", provided a justification was attempted (give 0 if they write "convergent" with no attempt at justifying their guess)

1 point for verifying that $f(x)$ is decreasing. Deduct 0.5 per small mistake or omission. If they merely claim $b_{n+1} \leq b_n$ without justification, then give 0 for this part.

1.5 points for limit work. Deduct 0.5 per small mistake or omission. (they must explicitly apply l'Hopital's rule to get 1.5/1.5 for this part)

Q7. [**5 points**] Consider the power series: $\displaystyle\sum_{n=0}^{\infty}\frac{(x+2)^n}{3^{2n}(5n+1)}$

(a) Find its radius of convergence. Show your work.

Solution: We use the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x+2)^{n+1}}{3^{2(n+1)}(5(n+1)+1)}}{\frac{(x+2)^n}{3^{2n}(5n+1)}}\right| = \lim_{n\to\infty}\left|\frac{(x+2)(5n+1)}{9(5n+6)}\right| = \frac{|x+2|}{9} \qquad \text{since } \lim_{n\to\infty}\tfrac{5n+1}{5n+6}=1.$$

By the Ratio Test, this power series converges when $\frac{|x+2|}{9} < 1$, hence for $|x+2| < 9$. We conclude that the radius of convergence is $R = 9$.

Marking: [2 points]
1 point for correctly evaluating limit for Ratio Test. Deduct 0.5 per mistake.

0.5 points for "solving the inequality".
0.5 points for identifying the radius.

(b) Find its interval of convergence. Show your work.

Solution: From (a), we know this power series converges for $-9 < x+1 < 9$. We need to check convergence at its endpoints.

When $x = -11$, we have

$$\sum_{n=0}^{\infty}\frac{(-11+2)^n}{3^{2n}(5n+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{5n+1}$$

This is an alternating series with $b_n = \frac{1}{5n+1}$. We have $b_{n+1} = \frac{1}{5(n+1)+1} < \frac{1}{5n+1} = b_n$ and $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}\dfrac{1}{5n+1} = 0$.

By the Alternating Series Test, this series converges.

When $x = 7$, we have

$$\sum_{n=0}^{\infty}\frac{(7+2)^n}{3^{2n}(5n+1)} = \sum_{n=0}^{\infty}\frac{1}{5n+1}$$

This series has positive terms and we can apply the Comparison Test or the Limit Comparison Test (or even the Integral Test if desired).

For instance, we have $0 \le \frac{1}{6n} = \frac{1}{5n+n} \le \frac{1}{5n+1}$ for all $n \ge 1$. Since $\sum \frac{1}{n}$ is a p-series with $p = 1$ (the harmonic series), the smaller series diverges. We conclude that the larger series must also diverge.

Therefore, the interval of convergence of the above power series is $-11 \le x < 7$ or $[-11, 7)$.

Marking: [3 points]
0.5 points for having the correct endpoints from their answer in (a), e.g. whatever radius R they got in (a), they should be analysing their endpoints $-R$ and R.

1 point for the work to check convergence at $-R$ (0.5 for an appropriate series test and 0.5 for correct conclusion)

1 point for the work to check convergence at R (0.5 for an appropriate series test and 0.5 for correct conclusion)

0.5 for correct final answer for the interval.

If student got wrong endpoints from (a), but are doing the right things to check convergence/divergence in (b), you can give up to 1.5/3 overall for reasonable efforts.

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end of exam!