MAT1320 Final Exam (With Solutions)

Calculus I (University of Ottawa)

Solution to the Final Examination

MAT1320X, Summer 2020

Part I. Multiple-Choice Questions

$3 \times 10 = 30$ points

In all questions, (A) is the right answer.

1.1. The domain of the function $f(x) = \sqrt{1 - \ln(x - e)}$ is

(A) $e < x \le 2e$; (B) $e \le x \le 2e$; (C) $x > 2e$; (D) $x < e$.

Solution. $1 - \ln(x - e) \ge 0$, $\ln(x - e) \le 1$, $0 < x - e \le e$, $e < x \le 2e$.

1.2. The domain of the function $f(x) = \sqrt{1 - \ln(e - x)}$ is

(A) $0 \le x < e$; (B) $0 \le x \le e$; (C) $x < 0$; (D) $x \ge e$.

Solution. $1 - \ln(e - x) \ge 0$, $\ln(e - x) \le 1$, $0 < e - x \le e$, $0 \le x < e$.

1.3. The domain of the function $f(x) = \sqrt{1 - \ln(e + x)}$ is

(A) $-e < x \le 0$; (B) $-e \le x < 0$; (C) $x < -e$; (D) $x \ge 0$.

Solution. $1 - \ln(e + x) \ge 0$, $\ln(e + x) \le 1$, $0 < e + x \le e$. $-e < x \le 0$.

2.1. Some values of functions $f(x)$ and $g(x)$, and their derivatives $f'(x)$ and $g'(x)$ are given in the following table:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	1	2	3
2	1	2	3	5
3	2	4	1	7

Let $z = h(x) = (f \circ g)(x)$, what is $h(1) + h'(1)$?

(A) 7; (B) 14; (C) 8; (D) 22.

Solution. $h(1) = f(g(1)) = f(2) = 1$. $h'(1) = f'(g(1))g'(1) = f'(2)g'(1) = 2 \times 3 = 6$. $h(1) + h'(1) = 7$.

2.2. Some values of functions $f(x)$ and $g(x)$, and their derivatives $f'(x)$ and $g'(x)$ are given in the following table:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	1	1	3	3
2	3	2	1	5
3	2	4	2	7

Let $z = h(x) = (f \circ g)(x)$, what is $h(1) + h'(1)$?

(A) 14; (B) 7; (C) 8; (D) 20.

Solution. $h(1) = f(g(1)) = f(3) = 2$. $h'(1) = f'(g(1))g'(1) = f'(3)g'(1) = 4 \times 3 = 12$. $h(1) + h'(1) = 14$.

2.3. Some values of functions $f(x)$ and $g(x)$, and their derivatives $f'(x)$ and $g'(x)$ are given in the following table:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	1	1	3
2	1	2	3	5
3	2	4	2	7

Let $z = h(x) = (f \circ g)(x)$, what is $h(2) + h'(2)$?

(A) 22; (B) 14; (C) 7; (D) 8.

Solution. $h(2) = f(g(2)) = f(3) = 2$. $h'(2) = f'(g(2))g'(2) = f'(3)g'(2) = 4 \times 5 = 20$. $h(2) + h'(2) = 22$.

3.1. The derivative of the function $f(x) = \dfrac{e^{(x^2)}}{x}$ at $x = 2$ is

(A) $\dfrac{7}{4}e^4$; (B) $\dfrac{4}{3}e^4$; (C) $\dfrac{7}{3}e^4$; (D) $\dfrac{3}{4}e^4$.

Solution. By the quotient rule, $f'(x) = \dfrac{2x^2e^{x^2} - e^{x^2}}{x^2} = \dfrac{e^{x^2}(2x^2 - 1)}{x^2}$. When $x = 2$, $f'(2) = \dfrac{7}{4}e^4$.

3.2. The derivative of the function $f(x) = \dfrac{\sin^2 x}{x}$ at $x = \dfrac{\pi}{4}$ is

(A) $\dfrac{4(\pi - 2)}{\pi^2}$; (B) $\dfrac{4\pi + 2}{\pi^2}$; (C) $\dfrac{4(\pi + 2)}{\pi^2}$; (D) $\dfrac{4\pi - 2}{\pi^2}$.

Solution. (B) By the quotient rule, $f'(x) = \dfrac{2x\sin x\cos x - \sin^2 x}{x^2}$. When $x = \dfrac{\pi}{4}$,

$$f'\left(\frac{\pi}{4}\right) = \frac{\frac{\pi}{2}\times\frac{1}{2}-\frac{1}{2}}{\left(\frac{\pi}{4}\right)^2} = \frac{4(\pi-2)}{\pi^2}.$$

3.3. The derivative of the function $f(x) = \dfrac{(\ln x)^2}{x}$ at $x = e$ is

(A) e^{-2}; (B) e; (C) e^{-1}; (D) e^2.

Solution. (E) Use the quotient rule. $f'(x) = \dfrac{2\frac{\ln x}{x}x - (\ln x)^2}{x^2} = \dfrac{\ln x(2-\ln x)}{x^2}$. When $x = e$, $f'(e)$ $= e^{-2}$.

4.1. Let $f(x) = \dfrac{(11-3x)^{2/3}e^{x^2-1}}{(x+1)\sqrt{3-2x}}$. Then $f'(1) =$

(A) $\dfrac{9}{2}$; (B) $\dfrac{5}{2}$; (C) $-\dfrac{7}{3}$; (D) $-\dfrac{5}{4}$.

Solution. Taking the logarithm on both sides,

$$\ln f(x) = \frac{2}{3}\ln(11-3x) + (x^2-1) - \ln(x+1) - \frac{1}{2}\ln(3-2x).$$

Then take the derivative with respect to x on both sides:

$$\frac{f'(x)}{f(x)} = -\frac{2}{11-3x} + 2x - \frac{1}{x+1} + \frac{1}{3-2x}.$$

$$f'(x) = f(x)\left(-\frac{2}{11-3x} + 2x - \frac{1}{x+1} + \frac{1}{3-2x}\right).$$

When $x = 1$, $f(1) = \dfrac{(11-3)^{2/3}e^0}{2\times\sqrt{1}} = 2$, $f'(1) = 2\left(-\dfrac{2}{8} + 2 - \dfrac{1}{2} + 1\right) = \dfrac{9}{2}$.

4.2. Let $f(x) = \dfrac{(3x+11)^{1/3}e^{x^2-1}}{(x+3)\sqrt{x+5}}$. Then $f'(-1) =$

(A) $-\frac{5}{4}$; (B) $-\frac{7}{3}$; (C) $\frac{5}{2}$; (D) $-\frac{9}{2}$.

Solution. Taking the logarithm on both sides,

$$\ln f(x) = \frac{1}{3}\ln(3x+11) + (x^2-1) - \ln(x+3) - \frac{1}{2}\ln(x+5).$$

Then take the derivative with respect to x on both sides:

$$\frac{f'(x)}{f(x)} = \frac{1}{3x+11} + 2x - \frac{1}{x+3} - \frac{1}{2(x+5)}.$$

$$f'(x) = f(x)\left(\frac{1}{3x+11} + 2x - \frac{1}{x+3} - \frac{1}{2(x+5)}\right).$$

When $x=-1, f(-1) = \frac{(-3+11)^{1/3}e^0}{2\times 2} = \frac{1}{2}, f'(-1) = \frac{1}{2}\left(\frac{1}{8} - 2 - \frac{1}{2} - \frac{1}{8}\right) = -\frac{5}{4}$.

4.3. Let $f(x) = \frac{(2x+5)^{2/3}e^{x^2-4}}{(x+4)\sqrt{3x+7}}$. Then $f'(-2) =$

(A) $-\frac{7}{3}$; (B) $\frac{9}{2}$; (C) $\frac{5}{2}$; (D) $-\frac{5}{4}$.

Solution. Taking the logarithm on both sides,

$$\ln f(x) = \frac{2}{3}\ln(2x+5) + (x^2-4) - \ln(x+4) - \frac{1}{2}\ln(3x+7).$$

Then take the derivative with respect to x on both sides:

$$\frac{f'(x)}{f(x)} = \frac{4}{3(2x+5)} + 2x - \frac{1}{x+4} - \frac{3}{2(3x+7)}.$$

$$f'(x) = f(x)\left(\frac{4}{3(2x+5)} + 2x - \frac{1}{x+4} - \frac{3}{2(3x+7)}\right).$$

When $x=-2, f(-2) = \frac{(-4+5)^{2/3}e^0}{2\times\sqrt{1}} = \frac{1}{2}, f'(-1) = \frac{1}{2}\left(\frac{4}{3} - 4 - \frac{1}{2} - \frac{3}{2}\right) = \frac{2}{3} - 3 = -\frac{7}{3}$.

5.1. Let $f(x) = x^{2x}$. Then $f'(1) =$

(A) 2; (B) $\ln 2 + 1$; (C) $2 \ln 2 + 2$; (D) $2 \ln 2 + 1$.

Solution. $\ln f(x) = 2x \ln x$. $\dfrac{f'(x)}{f(x)} = 2\ln x + 2$. $f(1) = 1, f'(1) = 1 \times (2 \ln 1 + 2) = 2$.

5.2. Let $f(x) = (2x)^x$. Then $f'(1) =$

(A) $2 \ln 2 + 2$; (B) $\ln 2 + 1$; (C) 2; (D) $2 \ln 2 + 1$.

Solution. $\ln f(x) = x \ln (2x)$. $\dfrac{f'(x)}{f(x)} = \ln(2x) + 1$. $f(1) = 2, f'(1) = 2 \times (\ln 2 + 1) = 2 \ln 2 + 2$.

5.3. Let $f(x) = (x + 1)^x$. Then $f'(1) =$

(A) $2 \ln 2 + 1$; (B) $2 \ln 2 + 2$; (C) 2; (D) $\ln 2 + 1$.

Solution. $\ln f(x) = x \ln (x + 1)$ $\dfrac{f'(x)}{f(x)} = \ln(x + 1) + \dfrac{x}{x+1}$. $f(1) = 2, f'(1) = 2 \times \left(\ln 2 + \dfrac{1}{2}\right)$
$= 2 \ln 2 + 1$.

6.1. A function $y = f(x)$ is defined implicitly by the equation $x^2y + xy^2 + 2y = 0$ near the point $(2, -3)$. Then $f'(2) =$

(A) $-1/2$; (B) $1/2$; (C) $-3/7$; (D) $-4/3$.

Solution. $2xy + x^2y' + y^2 + 2xyy' + 2y' = 0$. At the point $(2, -3)$, $-12 + 4y' + 9 - 12y' + 2y' = 0$. $-6y' = 3$, $y' = -1/2$.

6.2. A function $y = f(x)$ is defined implicitly by the equation $x^2y + xy^2 - 3y = 0$ near the point $(3, -2)$. Then $f'(3) =$

(A) $-4/3$; (B) $-1/2$; (C) $-5/2$; (D) $-3/7$.

Solution. $2xy + x^2y' + y^2 + 2xyy' - 3y' = 0$. At the point $(3, -2)$, $-12 + 9y' + 4 - 12y' - 3y' = 0$. $-6y' = 8$, $y' = -4/3$.

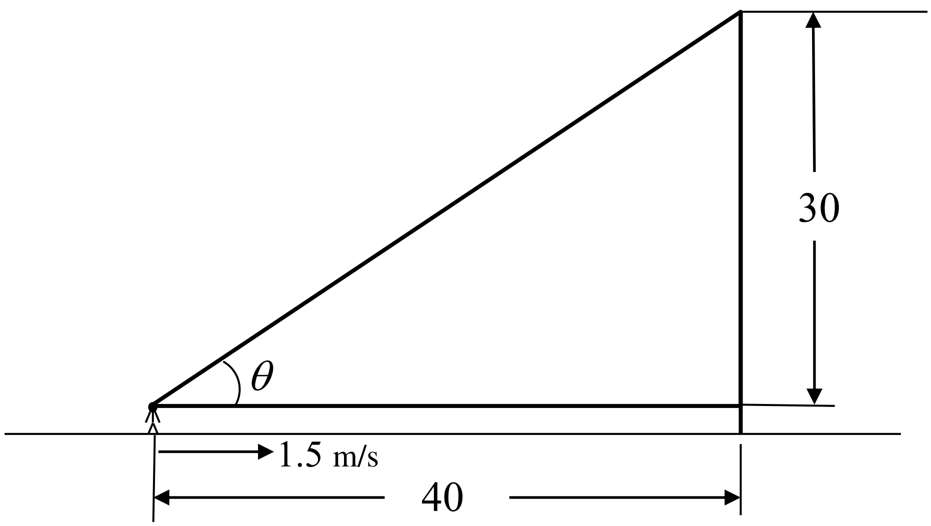
6.3. A function $y = f(x)$ is defined implicitly by the equation $2x^2y - xy^2 - 2y = 0$ near the point $(2, 3)$. Then $f'(2) =$

(A) $5/2$; (B) $-3/7$; (C) $-1/2$; (D) $-4/3$.

Solution. $4xy + 2x^2y' - y^2 - 2xyy' - 2y' = 0$. At the point (2, 3), $24 + 8y' - 9 - 12y' - 2y' = 0$. $6y' = 15$, $y' = 5 / 2$.

7.1. A man is walking at a speed 1.5 meters/second towards a building. The top of the building is 30 meters above the eyes of the man. The rate of change of the angle θ of elevation of the top of the building when the man is 40 meters away from the building (in radian(s)/second) is

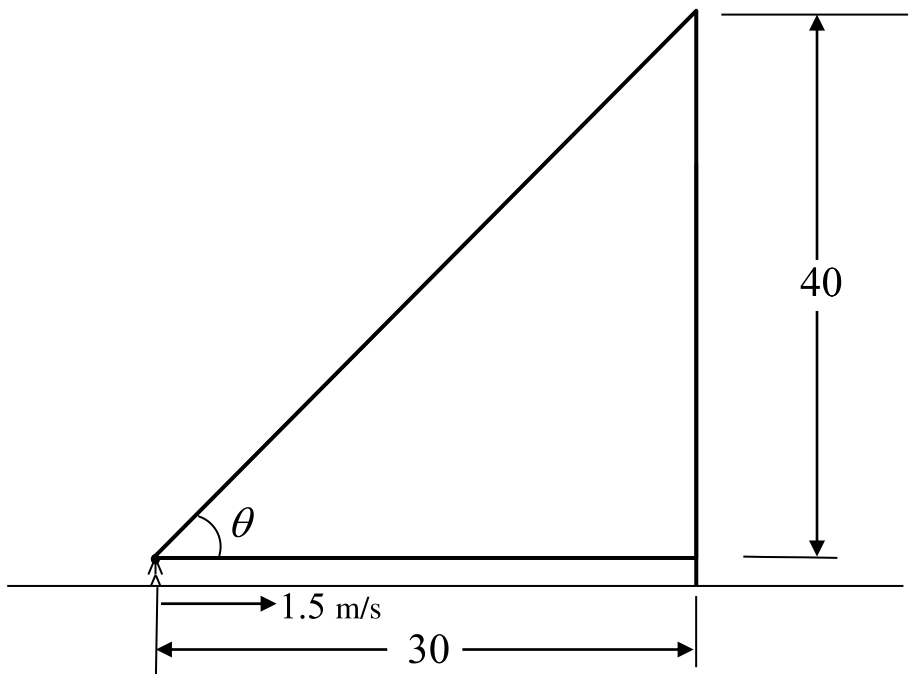
(A) 0.018; (B) 0.012; (C) 0.024; (D) 0.048.



Solution. Let the distance between the man and the building be x. Then $30 = x \tan \theta$. Taking the derivative with respect to t on both sides, $x' \tan \theta + x \sec^2 \theta\, \theta' = 0$. $\theta' = -\dfrac{x' \tan\theta}{x \sec^2 \theta}$. When $x' = -1.5$, $\tan \theta = \dfrac{3}{4}$, and $\sec \theta = \dfrac{4}{5}$. $\theta' = \dfrac{1.5 \times \frac{3}{4} \times \frac{16}{25}}{40} = 0.018$.

7.2. A man is walking at a speed 1.5 meters/second towards a building. The top of the building is 40 meters above the eyes of the man. The rate of change of the angle θ of elevation of the top of the building when the man is 30 meters away from the building (in radian(s)/second) is

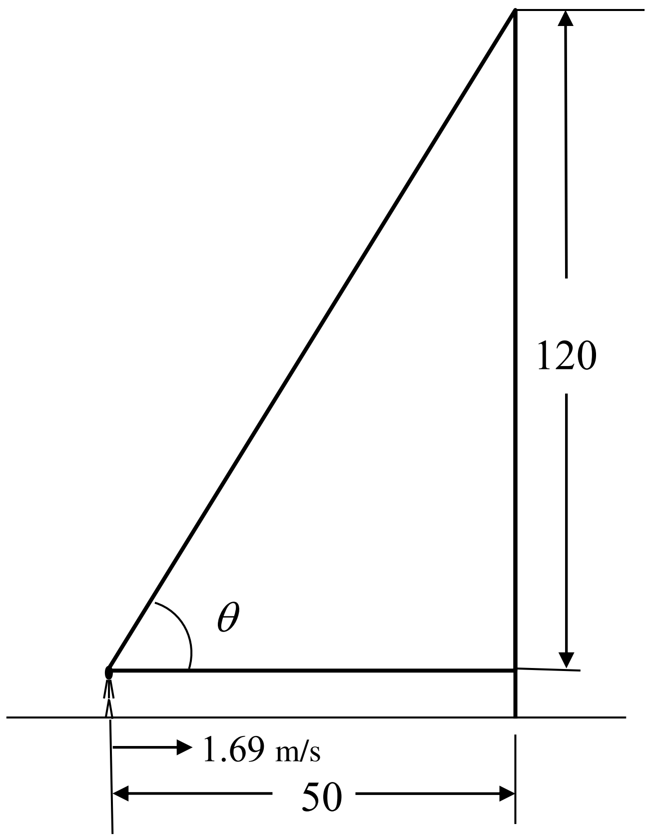
(A) 0.024; (B) 0.018; (C) 0.020; (D) 0.012.



Solution. Let the distance between the man and the building be x. Then $40 = x \tan\theta$. Taking the derivative with respect to t on both sides, $x' \tan\theta + x\sec^2\theta\,\theta' = 0$. $\theta' = -\dfrac{x'\tan\theta}{x\sec^2\theta}$. When $x' = -1.5$, $\tan\theta = \dfrac{4}{3}$, and $\sec\theta = \dfrac{5}{3}$. $\theta' = \dfrac{1.5\times\frac{4}{3}\times\frac{9}{25}}{30} = 0.024$.

7.3. A man is walking at a speed 1.69 meters/second towards a building. The top of the building is 120 meters above the eyes of the man. The rate of change of the angle θ of elevation of the top of the building when the man is 50 meters away from the building (in radian(s)/second) is

(A) 0.012; (B) 0.018; (C) 0.024; (D) 0.048.



Solution. Let the distance between the man and the building be x. Then $50 = x\tan\theta$. Taking the derivative with respect to t on both sides, $x'\tan\theta + x\sec^2\theta\,\theta' = 0$. $\theta' = -\dfrac{x'\tan\theta}{x\sec^2\theta}$. When $x' = -1.69$, $\tan\theta = \dfrac{12}{5}$, and $\sec\theta = \dfrac{13}{5}$. $\theta' = \dfrac{1.69\times\frac{12}{5}\times\frac{25}{169}}{50} = 0.012$.

8.1. Let $f(x) = \int_0^{2x}\sqrt{t^2+t+3}\,dt$. Then $f'(1) =$

(A) 6; (B) 12; (C) −3; (D) −6.

Solution. $f'(x) = 2\sqrt{(2x)^2+(2x)+3}$. $f'(1) = 6$.

8.2. Let $f(x) = \int_0^{3x}\sqrt{t^2+t+4}\,dt$. Then $f'(1) =$

(A) 12; (B) −6; (C) 6; (D) −3.

Solution. $f'(x) = 3\sqrt{(3x)^2+(3x)+4}$. $f'(1) = 12$.

8.3. Let $f(x) = \int_0^{-x}\sqrt{t^2+t+9}\,dt$. Then $f'(1) =$

(A) –3; (B) –6; (C) 6; (D) 12.

Solution. $f'(x) = -\sqrt{(-x)^2+(-x)+9} = -\sqrt{x^2-x+9}$. $f'(1) = -3$.

9.1. Let $f(x) = \dfrac{\sqrt{2-x}}{x}$. The critical number(s) of $f(x)$ is/are

(A) 2 only; (B) 0 and 2 only; (C) 0, 2, and 4 only; (D) 4 only.

Solution. The derivative of this function found by the quotient rule:

$$f'(x) = \frac{-\dfrac{x}{2\sqrt{2-x}} - \sqrt{2-x}}{x^2} = \frac{-x-2(2-x)}{2x^2\sqrt{2-x}} = \frac{x-4}{2x^2\sqrt{2-x}}.$$

$f'(x) = 0$, when $x = 4$, and $f'(x)$ does not exist when $x = 0$ or $x = 2$. Since $x = 0$ and $x = 4$ are not in the domain of $f(x)$, the critical numbers of $f(x)$ is $x = 2$ only.

9.2. Let $f(x) = \dfrac{\sqrt{x}}{x+2}$. The critical number(s) of $f(x)$ is/are

(A) 0 and 2 only; (B) –2, 0, and 2 only; (C) 2 only; (D) –2 and 0 only.

Solution. The derivative of this function found by the quotient rule:

$$f'(x) = \frac{\dfrac{x+2}{2\sqrt{x}} - \sqrt{x}}{(x+2)^2} = \frac{2-x}{2(x+2)^2\sqrt{x}}.$$

$f'(x) = 0$, when $x = 2$, and $f'(x)$ does not exist when $x = 0$ or $x = -2$. Since $x = -2$ is not in the domain of $f(x)$, the critical numbers of $f(x)$ are 0 and 2 only.

9.3. Let $f(x) = x\sqrt{x-3}$. The critical number(s) of $f(x)$ is/are

(A) 3 only; (B) 2 and 3 only; (C) 3 only. (D) 0 and 3 only..

Solution. The derivative of this function found by the quotient rule:

$$f'(x) = \sqrt{x-3} + \frac{x}{2\sqrt{x-3}} = \frac{2(x-3)+x}{2\sqrt{x-3}} = \frac{3x-6}{2\sqrt{x-3}}.$$

$f'(x) = 0$, when $x = 2$, and $f'(x)$ does not exist when $x = 3$. Since $x = 2$ is not in the domain of $f(x)$, the critical numbers of $f(x)$ is $x = 3$ only.

10.1. Some values of a function $y = f(x)$ is given in the following table:

x	0	0.5	1	1.5	2
$f(x)$	1.1	1.25	1.4	1.65	1.69

If Simpson's rule with $n = 4$ and the given data are used to estimate definite integral $\int_0^2 f(x)dx$, which one of the following numbers is closest to this estimate?

(A) 2.865; (B) 2.866; (C) 2.867; (D) 2.864.

Solution. (E) With given data and Simpson's rule, $h = 0.5$ and

$$\int_0^2 f(x)dx \approx \frac{0.5}{3}(1.1 + 4 \times 1.25 + 2 \times 1.4 + 4 \times 1.65 + 1.69) = 2.865 .$$

10.2. Some values of a function $y = f(x)$ is given in the following table:

x	0	0.75	1.5	2.25	3
$f(x)$	1.1	1.25	1.4	1.65	2.0

If Simpson's rule with $n = 4$ and the given data are used to estimate definite integral $\int_0^3 f(x)dx$, which one of the following numbers is closest to this estimate?

(A) 4.375; (B) 4.373; (C) 4.377; (D) 4.379.

Solution. (E) With given data and Simpson's rule, $h = 0.5$ and

$$\int_0^3 f(x)dx \approx \frac{0.75}{3}(1.1 + 4 \times 1.25 + 2 \times 1.4 + 4 \times 1.65 + 2.0) = 4.375.$$

10.3. Some values of a function $y = f(x)$ is given in the following table:

x	0	0.25	0.5	0.75	1
$f(x)$	2.1	2.25	2.4	2.65	2.72

If Simpson's rule with $n = 4$ and the given data are used to estimate definite integral $\int_0^1 f(x)dx$, which one of the following numbers is closest to this estimate?

(A) 2.435; (B) 2.866; (C) 2.867; (D) 2.864.

Solution. (E) With given data and Simpson's rule, $h = 0.5$ and

$$\int_0^1 f(x)dx \approx \frac{0.25}{3}(2.1 + 4 \times 2.25 + 2 \times 2.4 + 4 \times 2.65 + 2.72) = 2.435.$$

Part II. Long-answer Questions

30 points

Question 11. 6 points

11.1. Use the definition of the derivative to find the derivative of the function $y = \dfrac{1}{\sqrt{x+1}}$ at $x = 8$.

Solution $y(8) = \dfrac{1}{3}$.

$$y' = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sqrt{9+h}} - \frac{1}{3}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{3-\sqrt{9+h}}{3\sqrt{9+h}}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{\left(3-\sqrt{9+h}\right)\left(3+\sqrt{9+h}\right)}{3\sqrt{9+h}\left(3+\sqrt{9+h}\right)}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{9-(9+h)}{3\sqrt{9+h}\left(3+\sqrt{9+h}\right)}\right) = -\lim_{h\to 0}\frac{1}{3\sqrt{9+h}\left(3+\sqrt{9+h}\right)} = -\frac{1}{54}$$

11.2. Use the definition of the derivative to find the derivative of the function $y = \dfrac{1}{\sqrt{1-2x}}$ at $x = 0$.

Solution. $y(0) = 1$.

$$y' = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sqrt{1-2h}} - 1\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{1-\sqrt{1-2h}}{\sqrt{1-2h}}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{\left(1-\sqrt{1-2h}\right)\left(1+\sqrt{1-2h}\right)}{\sqrt{1-2h}\left(1+\sqrt{1-2h}\right)}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{1-(1-2h)}{\sqrt{1-2h}\left(1+\sqrt{1-2h}\right)}\right) = \lim_{h\to 0}\frac{2}{\sqrt{1-2h}\left(1+\sqrt{1-2h}\right)} = 1.$$

11.3. Use the definition of the derivative to find the derivative of the function $y = \dfrac{1}{\sqrt{x-1}}$ at $x =$ 5.

Solution. $y(5) = \dfrac{1}{2}$.

$$y'(5) = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sqrt{4+h}} - \frac{1}{2}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{2-\sqrt{4+h}}{\sqrt{4+h}}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{\left(2-\sqrt{4+h}\right)\left(2+\sqrt{4+h}\right)}{\sqrt{4+h}\left(2+\sqrt{4+h}\right)}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{4-(4+h)}{\sqrt{4+h}\left(2+\sqrt{4+h}\right)}\right) = -\lim_{h\to 0}\frac{1}{\sqrt{4+h}\left(2+\sqrt{4+h}\right)} = -\frac{1}{16}.$$

Question 12. 6 points

12.1. Find indefinite integral $\int \dfrac{x^2}{(x^2-1)^{5/2}}\,dx$.

Solution. Let $x = \sec u$, $0 \le u \le \pi/2$, or $\pi \le x < 3\pi/2$. Then $x' = \tan u \sec u = \dfrac{\sin u}{\cos^2 u}$, and $\dfrac{x^2}{(x^2-1)^{5/2}} = \dfrac{\sec^2 u}{\tan^5 u}$.

$$\int \frac{x^2}{(x^2-1)^{5/2}}\,dx = \int \frac{1}{\cos^2 u}\frac{\cos^5 u}{\sin^5 u}\frac{\sin u}{\cos^2 u}\,du = \int \frac{\cos u}{\sin^4 u}\,du = -\frac{1}{3\sin^3 u} + C = -\frac{x^3}{3(x^2-1)^{3/2}} + C.$$

12.2. Find indefinite integral $\int \dfrac{1}{(1+x^2)^{5/2}}\,dx$.

Solution. Let $x = \tan u$, $-\pi/2 < u < \pi/2$. Then $x' = \sec^2 u = \dfrac{1}{\cos^2 u}$, and $\dfrac{1}{(1+x^2)^{5/2}} = \dfrac{1}{\sec^5 u}$.

$$\int \frac{1}{(1+x^2)^{5/2}}\,dx = \int \cos^5 u\frac{1}{\cos^2 u}\,du = \int \cos^3 u\,du = \int (1-\sin^2 u)\,d\sin u = \sin u - \frac{1}{3}\sin^3 u + C$$

$$\frac{x}{\sqrt{1+x^2}} - \frac{x^3}{3(1+x^2)^{3/2}} + C.$$

12.3. Find indefinite integral $\int \dfrac{x^2}{(1+x^2)^{5/2}}\,dx$.

Solution. Let $x = \tan u$, $-\pi/2 < u < \pi/2$. Then $x' = \sec^2 u = \dfrac{1}{\cos^2 u}$, and $\dfrac{x^2}{(1+x^2)^{5/2}} = \dfrac{\tan^2 u}{\sec^5 u}$.

$$\int \frac{x^2}{(1+x^2)^{5/2}}dx = \int \frac{\tan^2 u}{\sec^5 u\cos^2 u}du = \int \frac{\sin^2 u}{\cos^2 u}\cos^5 u\frac{1}{\cos^2 u}du = \int \sin^2 u\cos u du$$
$$= \frac{1}{3}\sin^3 u + C = \frac{x^3}{3(1+x^2)^{3/2}} + C.$$

Question 13. 11 points

13.1. Consider function $f(x) = \dfrac{27 - x^3}{15x}$. $f'(x) = -\dfrac{2x^3 + 27}{15x^2}$, and $f''(x) = -\dfrac{2(x^3 - 27)}{15x^3}$.

(a) (2 point) Find the interval(s) where $f(x)$ is increasing, and the interval(s) where $f(x)$ is decreasing.

(b) (1 point) Find all local maxima/minima of $f(x)$, if any.

(c) (2 point) Find the intervals where the graph of $f(x)$ is concave up, and the interval(s) where the graph of $f(x)$ is concave down.

(d) (1 point) Find all inflection point(s), if any.

(e) (1 point) Find all vertical / horizontal asymptote(s) of $f(x)$, if any.

(f) (4 points) Sketch the graph of $f(x)$ in the range $-10 \le x \le 10$, $-10 \le y \le 10$.

Solution. (a) The function is defined for all $x \ne 0$. Let $f'(x) = -\dfrac{2x^3 + 27}{15x^2} = 0$,

$x = x_0 = -\left(\dfrac{27}{2}\right)^{1/3} = -\dfrac{3}{2^{1/3}} \approx -2.4$. $f'(x) > 0$ when $x < x_0$, and $f'(x) < 0$ for $x_0 < x < 0$, or $0 < x$.

Hence, $f(x)$ increase in interval $x < x_0$, and it decreases in intervals $x_0 < x < 0$, and $x > 0$.

(b) Function $f(x)$ attains a local maximum at $x = x_0$, $f(x_0) \approx -1.13$.

(c) Let $f''(x) = -\dfrac{2(x^3 - 27)}{15x^3} = 0$. $x = 3$.

When $x < 0$ or $x > 3$, $f''(x) < 0$, and when $0 < x < 3$, $f''(x) > 0$. The graph of $f(x)$ is concave down in intervals $x < 0$ and $x > 3$, and it is concave up in interval $0 < x < 3$.

(d) The graph of $f(x)$ has an inflection point (3, 0).

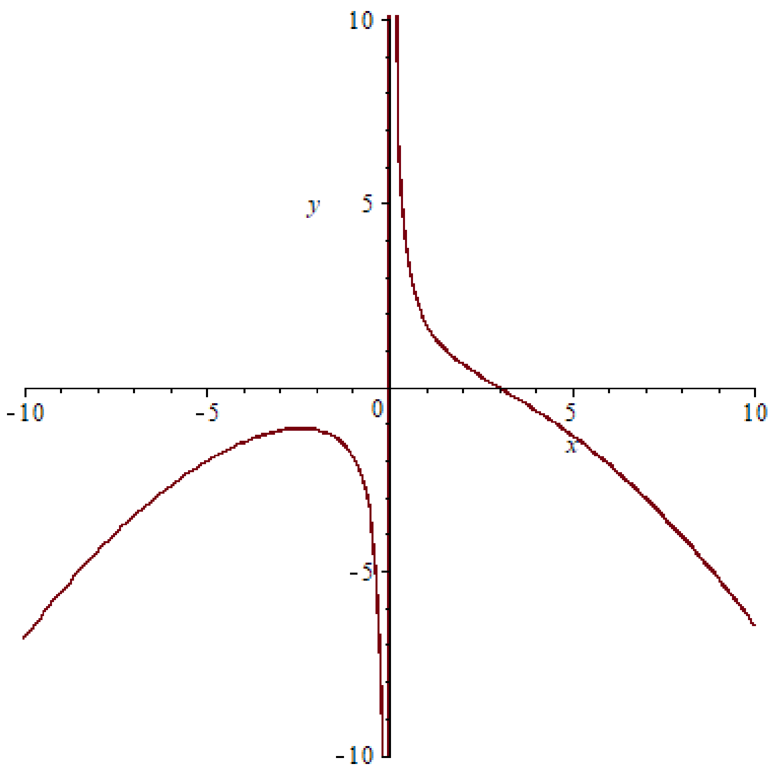
(e) Since $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = -\infty$, the graph of $f(x)$ does not have a horizontal asymptote.

Since $\lim_{x\to 0^-} f(x) = -\infty, \lim_{x\to 0^+} f(x) = \infty$, $x = 0$ is a vertical asymptote of the graph of $f(x)$.

(f) Find some particular values of $f(x)$ as follows:

$f(-10) \approx -6.85$, $f(x_0) \approx -1.13$, $f(-1) \approx -1.87$, $f(0.5) \approx 3.58$, $f(3) = 0$, $f(5) \approx -1.31$, $f(10) \approx -6.49$.

The graph of $f(x)$ looks like the following:



13.2. Consider function $f(x) = \dfrac{27 + x^3}{15x}$. $f'(x) = \dfrac{2x^3 - 27}{15x^2}$, and $f''(x) = \dfrac{2(x^3 + 27)}{15x^3}$.

(a) Find the interval(s) where $f(x)$ is increasing, and the interval(s) where $f(x)$ is decreasing.

(b) Find all local maxima/minima of $f(x)$, if any.

(c) Find the intervals where the graph of $f(x)$ is concave up, and the interval(s) where the graph of $f(x)$ is concave down.

(d) Find all inflection point(s), if any.

(e) Find all vertical / horizontal asymptote(s) of $f(x)$, if any.

(f) Sketch the graph of $f(x)$ in the range $-10 \le x \le 10$, $-10 \le y \le 10$.

Solution. (a) The function is defined for all $x \ne 0$. Let $f'(x) = \dfrac{2x^3 - 27}{15x^2} = 0$,

$x = x_0 = \left(\dfrac{27}{2}\right)^{1/3} = \dfrac{3}{2^{1/3}} \approx 2.4$. $f'(x) > 0$ when $x > x_0$, and $f'(x) < 0$ for $0 < x < x_0$, or $x < 0$.

Hence, $f(x)$ increase in interval $x > x_0$, and it decreases in intervals $0 < x < x_0$, and $x < 0$.

(b) Function $f(x)$ attains a local minimum at $x = x_0$. $f(x_0) \approx 1.13$.

(c) Let $f''(x) = \dfrac{2(x^3 + 27)}{15x^3} = 0$. $x = -3$.

When $x > 0$ or $x < -3$, $f''(x) > 0$, and when $-3 < x < 0$, $f''(x) < 0$. The graph of $f(x)$ is concave up in intervals $x > 0$ and $x < -3$, and it is concave down in interval $-3 < x < 0$.

(d) The graph of $f(x)$ has an inflection point $(-3, 0)$.

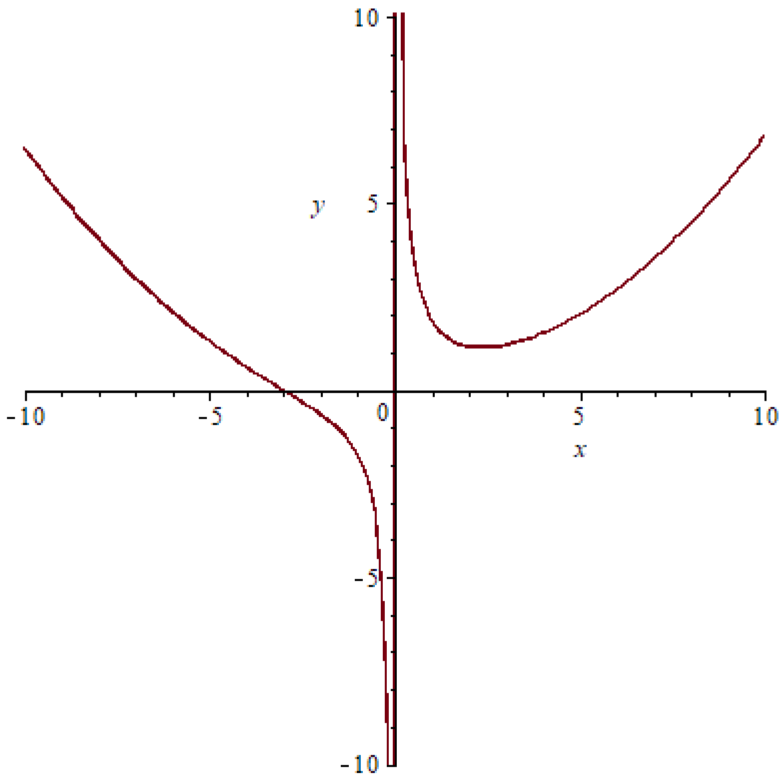
(e) Since $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$, the graph of $f(x)$ does not have a horizontal asymptote.

Since $\lim_{x\to 0^-} f(x) = -\infty$, $\lim_{x\to 0^+} f(x) = \infty$, $x = 0$ is a vertical asymptote of the graph of $f(x)$.

(f) Find some particular values of $f(x)$ as follows:

$f(10) \approx 6.85$, $f(x_0) \approx 1.13$, $f(1) \approx 1.87$, $f(-0.5) \approx -3.58$, $f(-3) = 0$, $f(-5) \approx 1.31$, $f(-10) \approx 6.49$.

The graph of $f(x)$ looks like the following:



13.3. Consider function $f(x) = \dfrac{x^3 - 27}{15x}$. $f'(x) = \dfrac{2x^3 + 27}{15x^2}$, and $f''(x) = \dfrac{2(x^3 - 27)}{15x^3}$.

(a) Find the interval(s) where $f(x)$ is increasing, and the interval(s) where $f(x)$ is decreasing.

(b) Find all local maxima/minima of $f(x)$, if any.

(c) Find the intervals where the graph of $f(x)$ is concave up, and the interval(s) where the graph of $f(x)$ is concave down.

(d) Find all inflection point(s), if any.

(e) Find all vertical / horizontal asymptote(s) of $f(x)$, if any.

(f) Sketch the graph of $f(x)$ in the range $-10 \le x \le 10$, $-10 \le y \le 10$.

Solution. (a) The function is defined for all $x \neq 0$. Let $f'(x) = \dfrac{2x^3 + 27}{15x^2} = 0$,

$x = x_0 = -\left(\dfrac{27}{2}\right)^{1/3} = -\dfrac{3}{2^{1/3}} \approx -2.4$. $f'(x) < 0$ when $x < x_0$, and $f'(x) > 0$ for $x_0 < x < 0$, or $0 < x$.

Hence, $f(x)$ decrease in interval $x < x_0$, and it increases in intervals $x_0 < x < 0$, and $x > 0$.

(b) Function $f(x)$ attains a local minimum at $x = x_0$. $f(x_0) \approx 1.13$.

(c) Let $f''(x) = \dfrac{2(x^3 - 27)}{15x^3} = 0$. $x = 3$.

When $x < 0$ or $x > 3$, $f''(x) > 0$, and when $0 < x < 3$, $f''(x) < 0$. The graph of $f(x)$ is concave up in intervals $x < 0$ and $x > 3$, and it is concave down in interval $0 < x < 3$.

(d) The graph of $f(x)$ has an inflection point (3, 0).

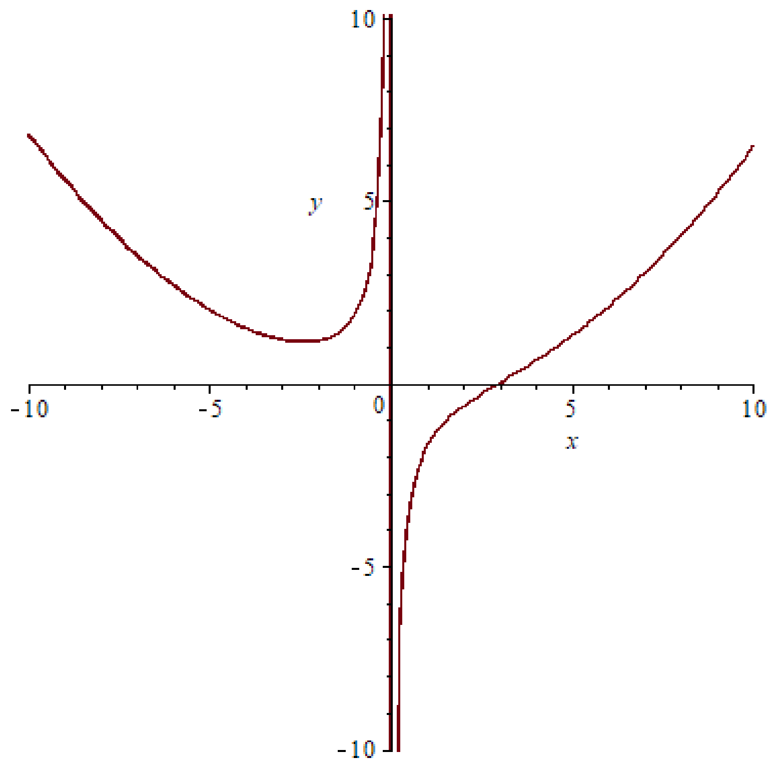
(e) Since $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$, the graph of $f(x)$ does not have a horizontal asymptote.

Since $\lim_{x\to0^-} f(x) = \infty, \lim_{x\to0^+} f(x) = -\infty$, $x = 0$ is a vertical asymptote of the graph of $f(x)$.

(f) Find some particular values of $f(x)$ as follows:

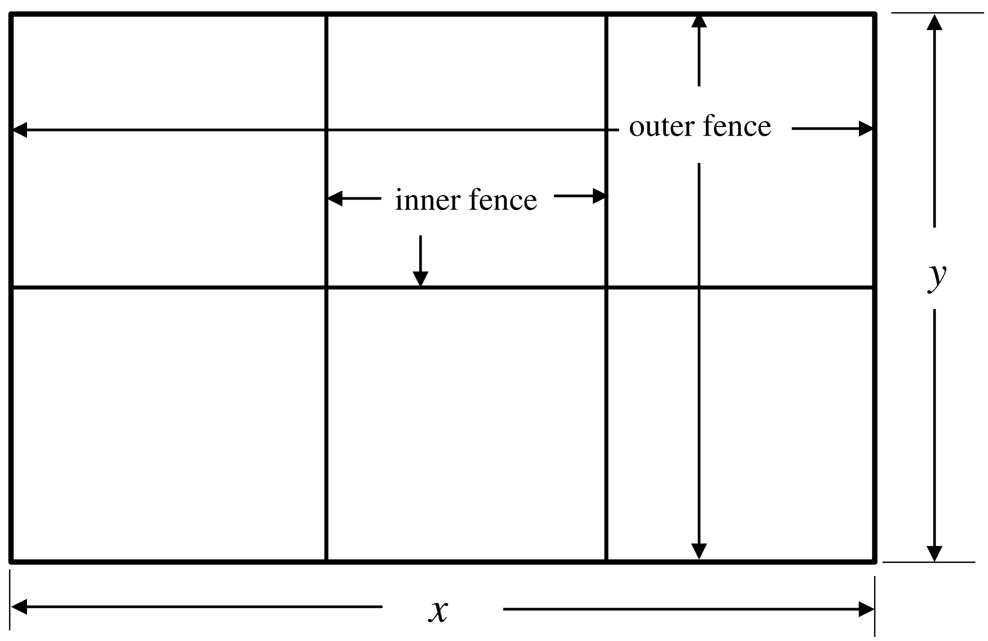
$f(-10) \approx 6.85$, $f(x_0) \approx 1.13$, $f(1) \approx -1.73$, $f(-0.5) \approx 3.61$, $f(3) = 0$, $f(5) \approx -1.31$, $f(10) \approx 6.49$.

The graph of $f(x)$ looks like the following:



Question 14. 7 points

14.1. A farmer wants to use fence to enclose a rectangular region, and divide this region into six equal rectangular parts with inner fences, as shown in the figure:



The outer fence costs \$50 per meter, and the inner fence costs \$25 per meter. If the farmer want spend \$12000 for the outer fence and the inner fence, find the dimensions x and y of the region to maximize the area of the region.

(a) (2 points) Express the area of the region A as a single-variable function of x.

(b) (1 point) Find the domain of function $A(x)$.

(c) (1 point) Find the critical number(s) of function $A(x)$.

(d) (1 point) Find a local maximum of $A(x)$.

(e) (1 point) Justify that this local maximum is the global maximum of $A(x)$.

(f) (1 point) Find the dimensions of this region that maximizes the area, and find the maximum area of this region.

Solution. (a) The area is $A = xy$. Since the total cost is $50(2x + 2y) + 25(x + 2y) = 125x + 150y = 12000$, $y = (12000 - 125x) / 150$.

Hence, $A = x(12000 - 125x) / 150$.

(b) Since $y \geq 0$, $125x \leq 12000$, $x \leq 96$. The domain of $A(x)$ is $0 \leq x \leq 96$.

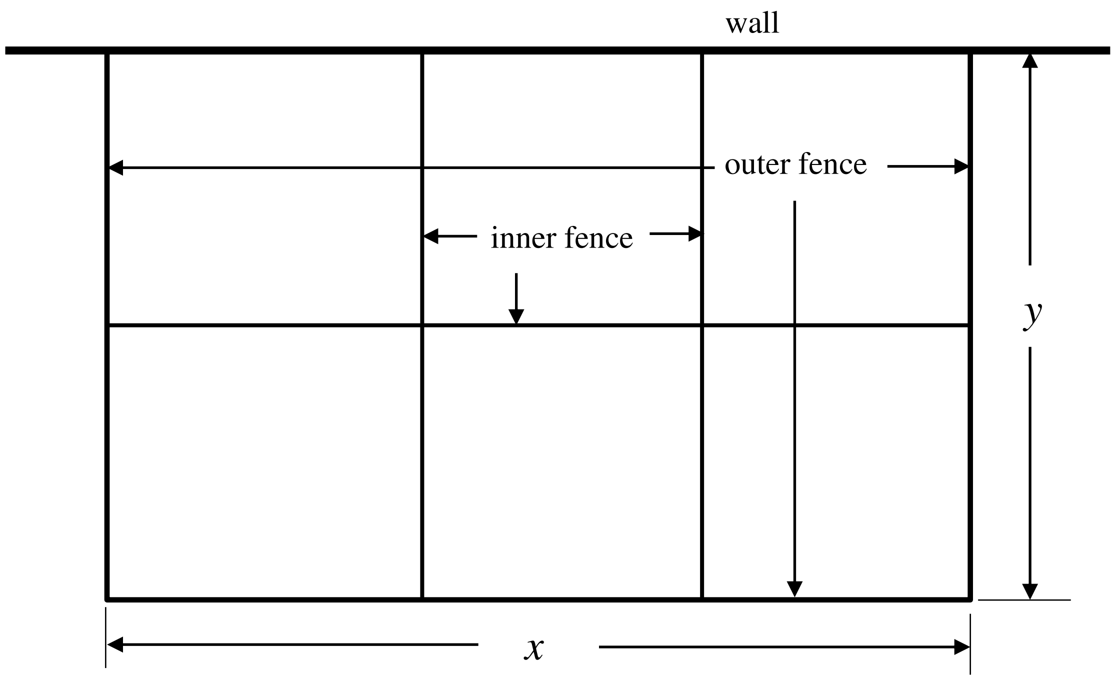
(c) Let $A' = 12000 - 250x = 0$. $x = 12000 / 250 = 48$ is a critical number.

(d) Since $A' > 0$ when $x < 48$, and $A' < 0$ when $x > 48$, $A(x)$ attains a local maximum at $x = 48$. $A(48) = 1920$.

(e) Since $A(0) = A(96) = 0$, this maximum is the absolute maximum.

(f) When $x = 48$, $y = (12000 - 125 \times 48) / 150 = 40$, and the maximum area of the region is $A(48) = 40 \times 48 = 1920$ m^2.

14.2. A farmer wants to use fence to enclose a rectangular region against a wall, and divide this region into six equal rectangular parts with inner fences, as shown in the figure:



The outer fence costs \$50 per meter, and the inner fence costs \$25 per meter. If the farmer want spend \$9000 for the outer fence and the inner fence, find the dimensions x and y of the region to maximize the area of the region.

(a) (2 points) Express the area of the region A as a single-variable function of x.

(b) (1 point) Find the domain of function $A(x)$.

(c) (1 point) Find the critical number(s) of function $A(x)$.

(d) (1 point) Find a local maximum of $A(x)$.

(e) (1 point) Justify that this local maximum is the global maximum of $A(x)$.

(f) (1 point) Find the dimensions of this region that maximizes the area, and find the maximum area of this region.

Solution. (a) The area is $A = xy$. Since the total cost is $50(x + 2y) + 25(x + 2y) = 75x + 150y = 6000$, $y = (9000 - 75x) / 150$.

Hence, $A = x(9000 - 75x) / 150$.

(b) Since $y \geq 0$, $75x \leq 9000$, $x \leq 120$. The domain of $A(x)$ is $0 \leq x \leq 120$.

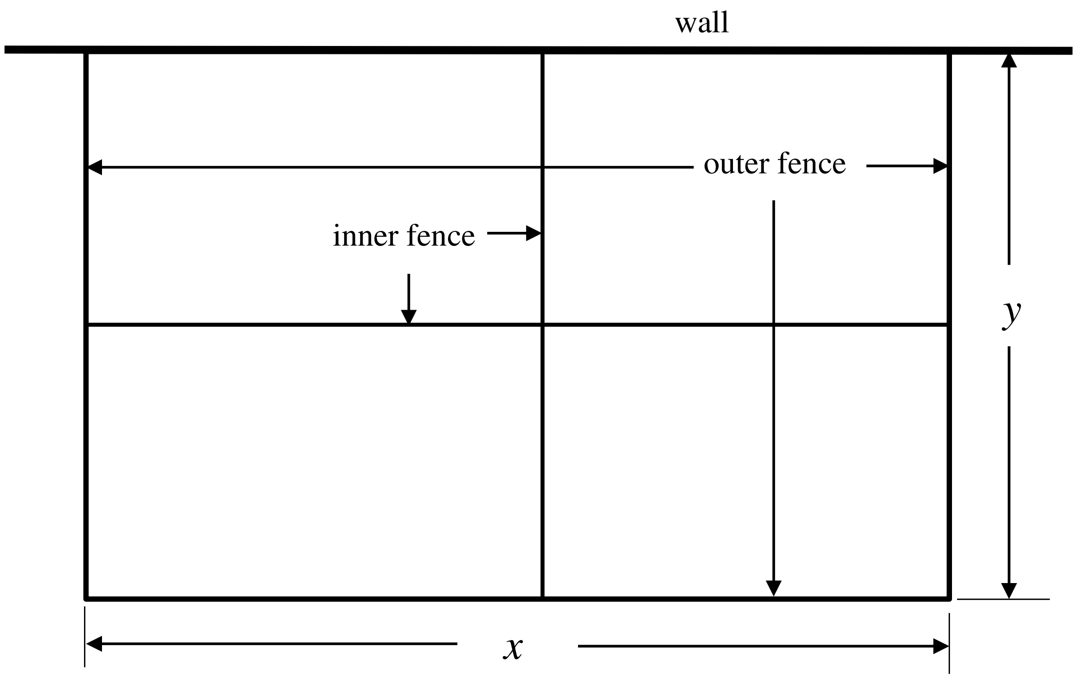
(c) Let $A' = 9000 - 150x = 0$. $x = 9000 / 150 = 60$ is a critical number.

(d) Since $A' > 0$ when $x < 60$, and $A' < 0$ when $x > 60$, $A(x)$ attains a local maximum at $x = 60$. $A(60) = 1800$.

(e) Since $A(0) = A(120) = 0$, this maximum is the absolute maximum.

(f) When $x = 60$, $y = (9000 - 75 \times 60) / 150 = 30$, and the maximum area of the region is $A(40) = 60 \times 30 = 1800\ \text{m}^2$.

14.3. A farmer wants to use fence to enclose a rectangular region against a wall, and divide this region into four equal rectangular parts with inner fences, as shown in the figure:



The outer fence costs \$50 per meter, and the inner fence costs \$25 per meter. If the farmer want spend \$6000 for the outer fence and the inner fence, find the dimensions x and y of the region to maximize the area of the region.

(a) (2 points) Express the area of the region A as a single-variable function of x.

(b) (1 point) Find the domain of function $A(x)$.

(c) (1 point) Find the critical number(s) of function $A(x)$.

(d) (1 point) Find a local maximum of $A(x)$.

(e) (1 point) Justify that this local maximum is the global maximum of $A(x)$.

(f) (1 point) Find the dimensions of this region that maximizes the area, and find the maximum area of this region.

Solution. (a) The area is $A = xy$. Since the total cost is $50(x + 2y) + 25(x + y) = 75x + 125y = 6000$, $y = (6000 - 75x) / 125$.

Hence, $A = x(6000 - 75x) / 125$.

(b) Since $y \geq 0$, $75x \leq 6000$, $x \leq 80$. The domain of $A(x)$ is $0 \leq x \leq 80$.

(c) Let $A' = 6000 - 150x = 0$. $x = 6000 / 150 = 40$ is a critical number.

(d) Since $A' > 0$ when $x < 40$, and $A' < 0$ when $x > 40$, $A(x)$ attains a local maximum at $x = 40$. $A(40) = 960$.

(e) Since $A(0) = A(80) = 0$, this maximum is the absolute maximum.

(f) When $x = 40$, $y = (6000 - 75 \times 40) / 125 = 24$, and the maximum area of the region is $A(40) = 40 \times 24 = 960$ m^2.