

DGD 9

Q1. COUNTING: BINARY STRINGS i. How many **binary strings of length 9** are there?

Procedure:

Build a binary string
of length 9

Task T_i : choose i th entry

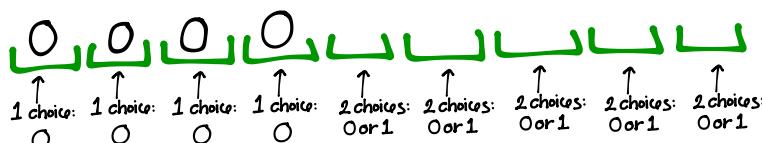


There are $2^9 = 512$ binary strings of length 9.

ii. How many **binary strings of length 9** are there that begin with 4 zeros?

Procedure:

Build a binary string
of length 9 that begins
with four 0's.



Task T_i : choose i th entry

There are $1^4 \cdot 2^5 = 32$ such strings.

iii. How many **binary strings of length 9** are there that contain exactly 4 zeros?

Task T_1 : Choose 4 of the 9 entries in the string to place the zeros

$$(\text{in one of } C(9,4) = \binom{9}{4} = \frac{9!}{4!5!} = \frac{39 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 21 \cdot 6 = 126 \text{ ways})$$

Task T_2 : fill the other 5 entries with ones (in 1 way)

$$\therefore \text{there are } \binom{9}{4} \cdot (1) = 126 \text{ such binary strings.}$$

iv. How many **binary strings of length 9** are there that contain at most 4 zeros?

We will break this up into 5 separate cases:

Case 0: String contains no zeros (hence 9 ones)

There are $\binom{9}{0} = 1$ binary strings of length 9 with no zeros

Case 1: String contains exactly 1 zero (hence 8 ones)

There are $\binom{9}{1} = 9$ binary strings of length 9 with exactly 1 zero

Case 2: String contains exactly 2 zeros (hence 7 ones)

There are $\binom{9}{2} = 36$ binary strings of length 9 with exactly 2 zeros

Case 3: String contains exactly 3 zeros (hence 6 ones)

There are $\binom{9}{3} = 84$ binary strings of length 9 with exactly 3 zeros

Case 4: String contains exactly 4 zeros (hence 5 ones)

There are $\binom{9}{4} = 126$ binary strings of length 9 with exactly 4 zeros

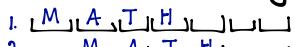
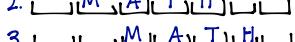
\therefore there are $1 + 9 + 36 + 84 + 126 = 256$ binary strings with at most 4 zeros.

Q2. COUNTING

- i. How many strings consisting of 7 distinct letters (of the English alphabet) contain the substring 'MATH' (i.e. the letters MATH appear consecutively in this order somewhere in the string) ?

Procedure Build such a 7-letter string.

T₁: choose a location in the string for the 4 consecutive letters MATH in one of 4 ways

1. 
2. 
3. 
4. 

T₂: select and arrange 3 letters from the other $26-4=22$ letters of the alphabet in one of $P(22,3)=22 \cdot 21 \cdot 20$ ways (the other 3 letters go in the remaining 3 positions after T₁ has been carried out).

⇒ there are $4 \cdot P(22,3) = 4 \cdot 22 \cdot 21 \cdot 20 = 36960$ ways to create such a string.

- ii. How many strings of 7 distinct letters (of the English alphabet) do not contain the substring 'MATH' ?

$$\begin{aligned} \left(\begin{array}{l} \text{\# strings of 7 distinct} \\ \text{letters that do not contain} \\ \text{the substring 'MATH'} \end{array} \right) &= \left(\begin{array}{l} \text{total \# of strings} \\ \text{of 7 distinct letters} \end{array} \right) - \left(\begin{array}{l} \text{\# strings of 7 distinct} \\ \text{letters that \underline{do} contain} \\ \text{the substring 'MATH'} \end{array} \right) \\ &= P(26,7) \quad - \text{(answer from i.)} \\ &= P(26,7) - 4 \cdot P(22,3) \\ &= 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \quad - 4 \cdot 22 \cdot 21 \cdot 20 \quad \leftarrow \text{without a calculator, you can write your answer like this} \\ &= 3315312000 - 36960 \\ &= 3315275040 \end{aligned}$$

- iii. How many strings of 7 distinct letters (of the English alphabet) contain the letters 'M', 'A', 'T' and 'H', not necessarily consecutively, not in any particular order ?

T₁: choose the letters M, A, T, and H to go in the string in one way

T₂: choose 3 letters from among the $26-4=22$ other letters in one of $C(22,3)$ ways

T₃: choose an arrangement of the 7 letters chosen in tasks T₁ and T₂ in one of $P(7,7)$ ways

$$\Rightarrow \text{there are } (1) \cdot C(22,3) \cdot P(7,7) = \frac{22!}{3!19!} \cdot 7!$$

$$= \frac{22 \cdot 21 \cdot 20 \cdot 19!}{3! \cdot 19!} \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!$$

$$\begin{aligned} &= 22 \cdot 21 \cdot 20 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \quad (\text{without calculator}) \\ &= 7761600 \\ &\text{such strings} \end{aligned}$$

Q3. COUNTING A 9-member fellowship is to be formed by selecting its members from among a group of 15 people consisting of 1 wizard, 5 hobbits, 5 elves, 2 men, and 2 dwarves.

i. How many different 9-member fellowships could be formed from these people?

• choose 9 out of 15 people (in one of $\binom{15}{9} = \frac{15!}{9!6!} = 5005$ ways)

∴ there are 5005 different 9-member fellowships that could be formed

ii. Suppose one member will be distinguished from the rest as a “ring-bearer”. How many different 9-member fellowships with one ring-bearer can be formed from these people?

T₁: choose the ring-bearer (in one of $\binom{15}{1} = 15$ ways)

T₂: choose 8 other members from among the remaining 14 people
(in one of $\binom{14}{8} = 3003$ ways)

∴ there are $15 \cdot 3003 = 45045$ different 9-member fellowships that could be formed containing one ring-bearer.

iii. How many fellowships are there that include 1 wizard, 1 hobbit as the ring-bearer and 3 other hobbits, 1 elf, 2 men, and 1 dwarf?

T₁: choose 1 wizard (in $\binom{1}{1} = 1$ way)

T₂: choose one ring-bearer from among the 5 hobbits (in one of $\binom{5}{1} = 5$ ways)

T₃: choose 3 other hobbits from among the 4 remaining hobbits
(in one of $\binom{4}{3} = 4$ ways)

T₄: choose an elf (in one of $\binom{5}{1} = 5$ ways)

T₅: choose 2 men (in $\binom{2}{2} = 1$ way)

T₆: choose 1 dwarf (in one of $\binom{2}{1} = 2$ ways).

∴ there are $1 \cdot 5 \cdot 4 \cdot 5 \cdot 1 \cdot 2 = 200$ such fellowships that could be formed.

- iv. After the fellowship has been selected in some way, its nine members will line up for a group photo before they depart on a perilous quest. How many ways can these 9 people be arranged for the photo?

There are $P(9,9) = 9! = 362\,880$ different arrangements (permutations) of the 9 chosen members

- v. How many fellowships are there that include 1 hobbit as the ring-bearer and an equal number of dwarves and elves?

We will break this up into cases where #dwarves = #elves.

Case 1 0 elves & 0 dwarves means we need

9 members from the 1 wizard, 5 hobbits, and 2 men (not possible)

Case 2: 1 dwarf + 1 elf means we need

7 members from the 1 wizard, 5 hobbits, and 2 men

T_0 : choose one hobbit ring-bearer (in one of $\binom{5}{1} = 5$ ways)

T_1 : choose 1 dwarf (in one of $\binom{2}{1} = 2$ ways)

T_2 : choose 1 elf (in one of $\binom{5}{1} = 5$ ways)

T_3 : choose 6 other members from among the 7 others

(1 wizard, 4 hobbits, 2 men)

(in one of $\binom{7}{6} = 7$ ways)

Case 3: 2 dwarves + 2 elves means we need

5 members from the 1 wizard, 5 hobbits, and 2 men

T_0 : choose one hobbit ring-bearer (in one of $\binom{5}{1} = 5$ ways)

T_1 : choose 2 dwarves (in $\binom{2}{2} = 1$ way)

T_2 : choose 2 elves (in one of $\binom{5}{2} = 10$ ways)

T_3 : choose 4 other members from among the 7 others

(1 wizard, 4 hobbits, 2 men)

(in one of $\binom{7}{4} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$ ways)

\therefore there are $5 \cdot 2 \cdot 7 + 5 \cdot 1 \cdot 10 \cdot 35 = 350 + 1750 = 2100$ ways

to form such a fellowship.