MAT2322 C-W20-Midterm 1-solution

Calculus III for Engineers (University of Ottawa)

University of Ottawa
Department of Mathematics and Statistics

MAT 2322C-Winter 2020-Feb 6th - Midterm 1

Professor: Xinhou Hua

Family Name ____________________ First Name ______________________________

Student # ____________________

- You have 80 minutes, 5 questions. You must show your work.
- Closed book exam. No notes. No scratch paper. Non-programmer, non-graphing calculators are allowed.

- Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam, academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

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Question	1	2	3	4	5	Total
Max Mark	4	4	4	4	4	20
Your Mark						

1. (4 points) Find the curvature $\kappa(t)$ of the vector function $\vec{r}(t) = (4t, 3t, t^2)$.

Solution:

$$\vec{r}\,'(t) = (4, 3, 2t), \vec{r}\,''(t) = (0, 0, 2) \Rightarrow \vec{r}\,' \times \vec{r}\,'' = (6, -8, 0).$$
$$|\vec{r}\,' \times \vec{r}\,''| = |(6, -8, 0)| = 10, |\vec{r}\,'| = \sqrt{25 + 4t^2},$$
$$\kappa(t) = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{10}{\sqrt{25 + 4t^2}^3}.$$

2. (4 points) Let S be the surface $z = x^2y + y^3x - 5$. Find the equation of the **tangent plane** of the surface at the point $(1, 1, -3)$.

Solution:

$$z_x = 2xy + y^3 \Rightarrow z_x(1,1) = 3.$$
$$z_y = x^2 + 3y^2x \Rightarrow z_y(1,1) = 4.$$

Thus the equation of the tangent plane at the point $(1, 1, -3)$ is

$$z = -3 + 3(x-1) + 4(y-1), \Rightarrow 3x + 4y - z = 10.$$

3. (4 points) Let $f(x, y) = y^3 + x^2y + 6y^2 + x^2$. Find all critical points and classify them.

Solution:
$f_x(x, y) = 2xy + 2x,\ f_y(x, y) = 3y^2 + x^2 + 12y.$
Setting $f_x = 0$ and $f_y = 0$:
$2xy + 2x = 0,$ (1)
$3y^2 + x^2 + 12y = 0.$ (2)
By (1), $x = 0$ or $y = -1$.
If $x = 0$, by (2), $3y^2 + 12y = 0$, $y = 0$ or -4.
If $y = -1$, by (2), $x^2 - 9 = 0$, $x = \pm 3$.
Thus critical points are:

$$(0, 0),\ (0, -4),\ (3, -1),\ (-3, -1).$$

$$f_{xx}(x, y) = 2y + 2, \quad f_{yy}(x, y) = 6y + 12, \quad f_{xy}(x, y) = 2x.$$
$$d(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}(x, y)^2.$$

$d(0, 0) = 24 > 0$, $f_{xx}(0, 0) = 2 > 0$, thus $(0, 0)$ is a local min point;
$d(0, -4) = 72 > 0$, $f_{xx}(0, -4) = -6 < 0$, thus $(0, -4)$ is a local max point;
$d(\pm 3, -1) = -36 < 0$, $(\pm 3, -1)$ are saddle points.

4. (4 points) Find the absolute minimum and maximum values of $f(x, y) = 4x - 2x^2 - y^2$ on the closed region $x^2 + y^2 \leq 36$.

Solution: **Step 1. Critical points Inside .**

$$f_x(x, y) = 4 - 4x, \quad f_y(x, y) = -2y.$$

Setting $f_x = 0$ and $f_y = 0$: We imply that $(x, y) = (1, 0)$, critical point inside.

$$f(1, 0) = 2.$$

Step 2. The max and min of f On the boundary: $x^2 + y^2 = 36$.

Method 1. From $y^2 = 36 - x^2$,

$$f(x, y) = 4x - 2x^2 - (36 - x^2) = 4x - x^2 - 36.$$

Let $g(x) = 4x - x^2 - 36$, $-6 \leq x \leq 6$.

$$g'(x) = 4 - 2x = 0, \ x = 2$$

which is a critical point.

$$g(2) = -32, \ g(-6) = -96, \ g(6) = -48.$$

Thus on the boundary, the max=-32, min=-96.

Method 2. Let

$$x = 6\cos t, y = 6\sin t.$$

Then

$$g(t) = f(x, y) = 24\cos t - 72\cos^2 t - 36\sin^2 t = 24\cos t - 36\cos^2 t - 36.$$

$$g'(t) = -24\sin t + 72\sin t\cos t = 24\sin t(3\cos t - 1).$$

$g'(t) = 0$ implies that $\sin t = 0$, *or* $\cos t = \frac{1}{3}$.
When $\sin t = 0$, $t = 0, \pi, 2\pi$, $g(0) = g(2\pi) = -48$, $g(\pi) = -96$.
When $\cos t = \frac{1}{3}$, $g(t) = -32$.
Thus on the boundary, the max=-32, min=-96.

Step 3. Comparing values of f in Step 1 and Step 2, the absolute max= $f(1, 0) = 2$, absolute min $= g(\pi) = f(-6, 0) = -96$.

5. (4 points) Use the method of Lagrange Multipliers to find the global maximum and global minimum of the function $f(x, y) = 5x - 12y$, subject to the constraint $x^2 + y^2 = 9$.

Solution: Let $g(x, y) = x^2 + y^2 - 9$. Then

$$\nabla f = (5, -12), \quad \nabla g = (2x, 2y).$$

By

$$\nabla f = \lambda \nabla g, \quad g(x, y) = 0,$$

we imply that

$$\begin{cases} 5 = \lambda 2x & (1) \\ -12 = \lambda 2y & (2) \\ x^2 + y^2 = 9, & (3) \end{cases}$$

By (1) and (2) we see that $\lambda \neq 0$, $x \neq 0$, $y \neq 0$.

(1)/(2): $x = -\frac{5}{12}y$. Substitute this into (3):

$$\left(-\frac{5}{12}y\right)^2 + y^2 = 9, \to y = \pm\frac{36}{13}.$$

Thus critical points are

$$(x, y) = \left(-\frac{15}{13}, \frac{36}{13}\right), \quad \left(\frac{15}{13}, -\frac{36}{13}\right).$$

$f\left(-\frac{15}{13}, \frac{36}{13}\right) = -39$, which is min.

$f\left(\frac{15}{13}, -\frac{36}{13}\right) = 39$, which is max.

Page for rough work.