Final 12 April Winter 2013, questions and answers

Discrete Mathematics for Computing (University of Ottawa)

Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Discrete Mathematics for Computing MAT1348 A

Final Examination

12 April 2013

Instructor: Nevena Francetić

Instructions:

- This is a three-hour *closed-book* exam; no notes are allowed. Calculators (without graphing or programming function) are allowed.
- The exam consists of 16 questions on 17 pages. Page 17 provides additional work space. Do not detach it.
- Questions 1-8 are multiple-choice. You must enter the letter corresponding to each correct answer in the table at the bottom of this page. No partial marks will be given for other work.
- Question 9 is short-answer. Write the final answer for each of the 5 independent parts into the appropriate answer box. You need not justify your answer, but do show your work to receive partial marks.
- Questions 10-16 are long-answer. You must clearly show all relevant steps in your solution to receive full marks. Clearly indicate the final answer.
- Be sure to read carefully and follow the instructions for the individual problems.
- For rough work, you may use the back pages. Do not use scrap paper of your own.
- Use proper mathematical notation and terminology.
- If you require clarification, raise your hand.
- Good luck!

Last name: – Solutions –

First name: ________

Student number: ________

Signature: ________

Question	1	2	3	4	5	6	7	8
Answer	E	C	B	E	D	F	F	B

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For graders use only:

Question	1 – 8	9	10	11	12	13	14	15	16	Total
Max	16	10	5	5	5	5	4	5	5	60
Marks										

Questions 1–8 are multiple choice. Enter the **letter** *corresponding to each correct answer in the appropriate box on the first page (page 1).*

[2pts] 1. Let G be a graph in which every vertex has degree 3 or 5. If G has 30 vertices and 50 edges, how many of its vertices have degree 3?

A. 20 **B.** 12 **C.** 18 **D.** 15 **E.** 25
F. None of the above

Let x denote the # of vertices with degree 3 and y denote the # of vertices with degree 5.

$x+y=30$
$3x+5y=100=2\cdot e$ (Hand-shake Lemma)

$2y=10$
$y=5$
$\boxed{x=25}$

[2pts] 2. How many binary strings of length 8 contain at least six 0s?

A. 36 **B.** 25 **C.** 37 **D.** 47 **E.** 35 **F.** 28 **G.** 400
H. None of the above

Let P_n denote the number of binary strings of length 8 having exactly n 0s. Then

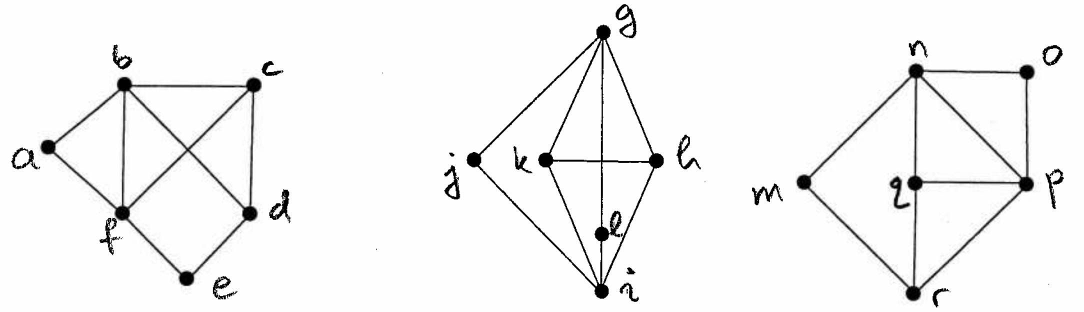
$|P_6|=\binom{8}{6}=\frac{8!}{6!\,2!}=\frac{8\cdot 7}{2}=28$

$|P_7|=\binom{8}{7}=8$

$|P_8|=\binom{8}{8}=1$

$|P_6|+|P_7|+|P_8|=28+8+1=37$

[2pts] 3. Consider the following three graphs:



G H K

Which of the following statements is **true**?

A. G and H are isomorphic, but G and K are not isomorphic.

B. G and K are isomorphic, but G and H are not isomorphic.

C. H and K are isomorphic, but G and H are not isomorphic.

D. The three graphs are isomorphic.

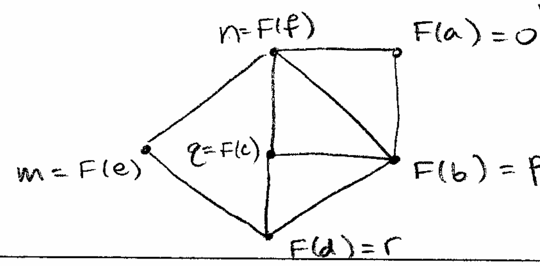
E. None of the above.

G and H have two vertices of degree 4, namly b and f, and g and i. Vertices b and f are adjacent, g and i are not. Hence $G \not\cong H$.

The isomorphism between G and K is the following:

$F: V(G) \to V(K)$

$v \in V(G)$	a	b	c	d	e	f
$F(v) \in V(K)$	o	p	q	r	m	n



[2pts] 4. How many ways can 4 people from a group of 7 be arranged in a row?

A. 680 **B.** 720 **C.** 560 **D.** 1260 **E.** 840 **F.** 35

G. None of the above

$$P(7,4) = \overline{7} \cdot \overline{6} \cdot \overline{5} \cdot \overline{4} = \frac{7!}{3!} = 840$$

[2pts] 5. Let $\mathbb{R}^*$ denote the set of non-zero real numbers. An equivalence relation $\mathcal{R}$ on the set $\mathbb{R}^* \times \mathbb{R}$ is defined as follows:

$$(x,y)\mathcal{R}(a,b) \qquad \text{if and only if} \qquad xb = ya$$

Which of the following elements of $\mathbb{R}^* \times \mathbb{R}$ is in the equivalence class of $(-1,-2)$?

A. $(2,-4)$ **B.** $(-3,6)$ **C.** $(-2,4)$ **D.** $(2,4)$ **E.** $(1,3)$
F. None of the above

$$[(-1,-2)]_{\mathcal{R}} = \{(x,y) \mid (x,y)\mathcal{R}(-1,-2)\} =$$
$$= \{(x,y) \mid -2x = -y\} =$$
$$= \{(x,y) \mid y = 2x\}$$

$(2,4) \in [(-1,-2)]_{\mathcal{R}}$ since $4 = 2 \cdot 2$

[2pts] 6. Which of the following is the coefficient of x^4y^3 in the expansion of $(2x-3y)^7$?

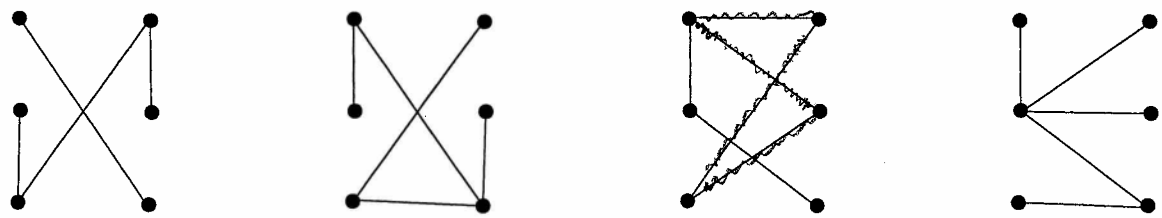
A. 15120 **B.** -6048 **C.** 10206 **D.** 35 **E.** -560 **F.** -15120
G. None of the above

$$(2x-3y)^7 = \sum_{i=0}^{7} \binom{7}{i} (2x)^{7-i} (-3y)^i$$
$$= \sum_{i=0}^{7} \binom{7}{i} 2^{7-i} (-3)^i x^{7-i} y^i$$

Coefficient of x^4y^3 is obtained when $i=3$:

$$\binom{7}{3} 2^{7-3} (-3)^3 = \binom{7}{3} 2^4 (-3)^3 =$$
$$= \frac{7!}{3!4!} \cdot 16 \cdot (-27) = \frac{7 \cdot \cancel{6} \cdot 5}{\cancel{3} \cdot \cancel{2}} \cdot (16)(-27) =$$
$$= -15120$$

[2pts] 7. Which of the following four graphs are **trees**?



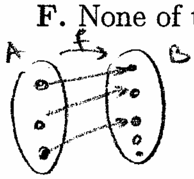
G forest | H tree | K contains a cycle | L tree

A. Only G. **B.** K and L. **C.** G and K
D. All four. **E.** Only H. **F.** H and L.
G. None of the above answers are correct.

[2pts] 8. Let A be a set of cardinality 3 and B a set of cardinality 5. Which of the following statements are **true**?

(i) There is no one-to-one function from A to B. False.
(ii) There is no onto function from A to B. True.
(iii) The number of one-to-one functions from A to B is 15. False.
(iv) The number of binary relations from A to B is 15. False
(v) The cardinality of the power set of A is 9. False

A. (iii) and (v) **B.** only (ii) **C.** (i) and (v) **D.** (ii) and (iv) **E.** only (iii)
F. None of the above answers are correct.

(i) 

(iii) Let $A = \{a, b, c\}$. # of choices for $f(a) = 5$
— " — $f(b) = 4$
— " — $f(c) = 3$
$\Rightarrow 5 \cdot 4 \cdot 3 = 60$
(iv) # of binary relations $= |\mathcal{P}(A \times B)| = 2^{3 \cdot 5} = 2^{15}$
(v) $|\mathcal{P}(A)| = 2^3 = 8$

9. *Short-answer questions — write your final answer in the answer box. You need not justify your answers, but do show your work to receive partial marks.*

(a) In how many ways can a committee be formed in a class of 10 math majors and 15 computer science majors, if the committee is to consist of 4 math majors and 5 computer science majors?

Answer: $\binom{10}{4}\binom{15}{5}$

[2pts]

Order of committee members does not matter and people are distinct and cannot repeat.

\# of ways to choose 4 out of 10 math majors: $\binom{10}{4}$

\# of ways to choose 5 out of 15 CS majors: $\binom{15}{5}$

Both tasks have to be performed.

By product rule:

$$\binom{10}{4}\binom{15}{5}$$

(b) Let x, y, and z be three propositions. If x, y, and z are all true, what is the truth value of the proposition

$$z \to ((\neg x \vee y) \to \neg y)?$$

Answer: F (false)

[2pts]

$\neg x \vee y \equiv F \vee T \equiv T$

$(\neg x \vee y) \to \neg y \equiv T \to F \equiv F$

$z \to ((\neg x \vee y) \to \neg y) \equiv T \to F \equiv F$

(c) How many subsets B of the set $A = \{1, 2, 3, \ldots, 50\}$ have the property that $|B| = 11$ and $\{1, 2, 3, 50\} \subseteq B$?

Answer: $\binom{46}{7}$

[2pts]

Since $\{1,2,3,50\} \subseteq B$, we need to pick remaining $11-4=7$ elements from $A - \{1,2,3,50\}$

Answer: $\binom{46}{7}$

(d) A "word" consists of 3 distinct vowels and 3 distinct consonants such that the vowels and the consonants alternate. How many such "words" are there? *(Note that there are 5 vowels and 21 consonants in the English alphabet.)*

Answer: 957 600

[2pts]

# of ways to arrange 3 out of 5 vowels:

$P(5,3) = \overline{5} \cdot \overline{4} \cdot \overline{3} = 60$ (since no letter repeats)

# of ways to arrange 3 out of 21 consonants:

$P(21,3) = 21 \cdot 20 \cdot 19 = 7980$

Having $V_1 V_2 V_3$ and $C_1 C_2 C_3$ we can arrange them in a word to alternate in 2 ways:

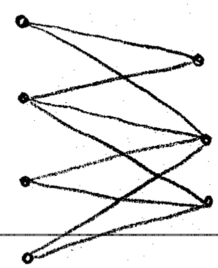
$V_1 C_1 V_2 C_2 V_3 C_3$ or

$C_1 V_1 C_2 V_2 C_3 V_3$

Answer: $60 \cdot 7980 \cdot 2 = 957\,600$

(e) Give an example of a bipartite simple graph with 7 vertices and 9 edges. *A figure suffices.*

Answer:



(Note, the solution is not unique.)

[2pts]

Questions 10–16 are long-answer questions. You must clearly show all relevant steps in your solution to receive full marks. Clearly indicate the final answer.

[5pts] 10. How many integers between 200 and 985 (inclusive) are divisible by 5 or 7 (or both)?

Let $A = \{n \mid 200 \leq n \leq 985 \text{ and } 5 \mid n\}$
$B = \{n \mid 200 \leq n \leq 985 \text{ and } 7 \mid n\}$.

We want to compute $|A \cup B|$. By inclusion-exclusion principle, $|A \cup B| = |A| + |B| - |A \cap B|$. We shall compute $|A|$, $|B|$ and $|A \cap B|$ in two ways.

Method I	Method II

If $n \in A$ then
$200 \leq n \leq 985$
$\frac{200}{5} \leq \frac{n}{5} \leq \frac{985}{5}$ and $\frac{n}{5} \in \mathbb{Z}$
$40 \leq \frac{n}{5} \leq 197$
There are $197 - 40 + 1 = 158$ integers between 40 and 197, inclusive, so $|A| = 158$
If $n \in B$ then
$200 \leq n \leq 985$
$\frac{200}{7} \leq \frac{n}{7} \leq \frac{985}{7}$ and $\frac{n}{7} \in \mathbb{Z}$
$28.57 \leq \frac{n}{7} \leq 140.71$
There are $140 - 28 = 112 = \lvert B \rvert$ integers between 28.57 and 140.71.
Finally, $n \in A \cap B$ then $35 \mid n$, so $\frac{n}{35} \in \mathbb{Z}$ and $200 \leq n \leq 985$.
$5.71 \leq \frac{n}{35} \leq 28.14$.
$\lvert A \cap B \rvert = 28 - 5 = 23$
	$A = \{n \mid 1 \leq n \leq 985,\ 5 \mid n\} - \{n \mid 1 \leq n \leq 199,\ 5 \mid n\}$
	So, $\lvert A \rvert = \lfloor \frac{985}{5} \rfloor - \lfloor \frac{199}{5} \rfloor =$
	$= 197 - 39 = 158$
	$B = \{n \mid 1 \leq n \leq 985,\ 7 \mid n\} - \{n \mid 1 \leq n \leq 199,\ 7 \mid n\}$.
	So,
	$\lvert B \rvert = \lfloor \frac{985}{7} \rfloor - \lfloor \frac{199}{7} \rfloor = 140 - 28 = 112$
	Finally,
	$A \cap B = \{n \mid 1 \leq n \leq 985,\ 35 \mid n\} - \{n \mid 1 \leq n \leq 199,\ 35 \mid n\}$
	$\lvert A \cap B \rvert = \lfloor \frac{985}{35} \rfloor - \lfloor \frac{199}{35} \rfloor =$
	$= 28 - 5 = 23$.

$|A \cup B| = |A| + |B| - |A \cap B| = 158 + 112 - 23 = 247$

[5pts] 11. Consider a sequence $a_1, a_2, a_3, \ldots$ defined by

$$\begin{aligned} a_1 &= 3, \\ a_2 &= 5, \\ a_n &= 3a_{n-1} - 2a_{n-2} \text{ for all } n \geq 3. \end{aligned}$$

Use **Mathematical Induction** to prove that $a_n = 2^n + 1$ for all integers $n \geq 1$.

The proof is by strong induction.

$P(n)$: "$a_n = 2^n + 1$" for $n \geq 1$

Basis: Check if $P(1)$ is true:

$a_1 = 2^1 + 1 = 2 + 1 = 3$. True.

Hypothesis: Let $k \geq 1$. For every i, $1 \leq i \leq k$, $P(i)$ is true, i.e. $a_i = 2^i + 1$.

Induction step: Using inductive hypothesis, prove that $P(k+1)$ is true, i.e. $a_{k+1} = 2^{k+1} + 1$.

- if $k+1 < 3$, then $k+1 = 2$ since $k \geq 1$.
  $a_2 = 5 = 2^2 + 1 = 4 + 1$. True.

- if $k+1 \geq 3$, then
  $a_{k+1} = 3a_k - 2a_{k-1}$. By induction hypothesis,
  (note, $k-1 \geq 1$ since $k+1 \geq 3$).

$$\begin{aligned} a_{k+1} &= 3(2^k + 1) - 2(2^{k-1} + 1) = \\ &= 3 \cdot 2^k + 3 - 2^k - 2 = \\ &= (3-1) \cdot 2^k + (3-2) = \\ &= 2 \cdot 2^k + 1 = \\ &= 2^{k+1} + 1. \end{aligned}$$

By strong induction, $a_n = 2^n + 1$ for all $n \geq 1$.

12. Use any method you know to determine whether or not the argument below is valid. If you claim that the argument is not valid, give a **counterexample**. *Fully justify your answer.*

[5pts]

H_1: $\neg E \to (\neg A \to D) \equiv \neg E \to (A \vee D) \equiv E \vee A \vee D \equiv (E \vee A) \vee D$
H_2: $\neg D \wedge \neg E$
H_3: $\neg(\neg C \to D) \to A \equiv \neg(C \vee D) \to A \equiv C \vee D \vee A \equiv (C \vee D) \vee A$

C: $\therefore A \wedge \neg C$

The argument is not valid. We will prove it using truth tree.

H_1: $(E \vee A) \vee D$ ✓
H_2: $\neg D \wedge \neg E$ ✓
H_3: $(C \vee D) \vee A$ ✓
$\neg C$: $\neg(A \wedge \neg C) \equiv \neg A \vee C$ ✓

- $E \vee A$ ✓, $\neg D$, $\neg E$
  - $C \vee D$ ✓
    - $\neg A$
      - E ✗
      - A ✗
    - C
      - E ✗
      - A
        - C
        - D ✗
  - A
    - $\neg A$ ✗
    - C
      - E ✗
      - A
- D, $\neg D$, $\neg E$ ✗

$A \equiv T$
$C \equiv T$
$D \equiv F$
$E \equiv F$

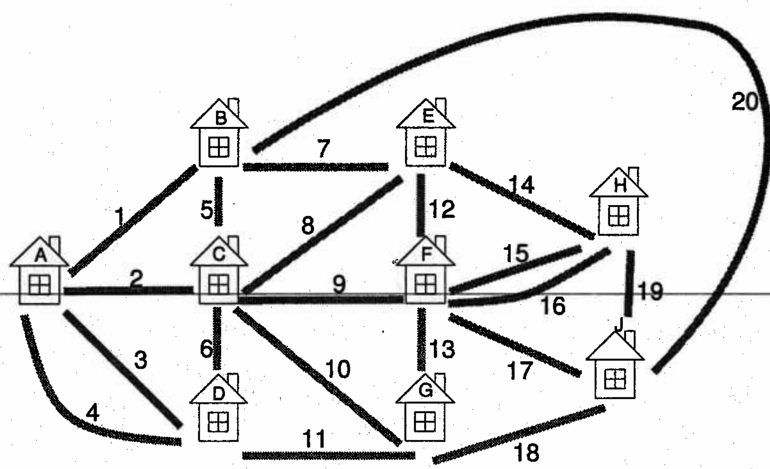
Counter-example:
$A \equiv T, C \equiv T, D \equiv F, E \equiv F$

Method II: Make a truth table to evaluate the hypotheses and the conclusion. In the table, row having $A \equiv T$, $C \equiv T$, $D \equiv F$, $E \equiv F$ has

$H_1 \equiv (T \to (F \to F)) \equiv T$
$H_2 \equiv (T \wedge T) \equiv T$
$H_3 \equiv (\neg(F \to F) \to T) \equiv (F \to T) \equiv T$, but
$C \equiv (T \wedge F) \equiv F$.

$\therefore$ All hypotheses are true, but conclusion is false, hence $(H_1 \wedge H_2 \wedge H_3) \to C \equiv (F \to T) \equiv F$ and argument is not valid.

13. [5pts] The picture below shows a village where Lenny's friends live. The gray lines show the 20 roads you can take to get from one house to another.



Use appropriate theorems from graph theory to justify your answers below.

(a) One Sunday afternoon, Lenny decides to visit all his friends. He would like to start his trip at one house, visit all his friends' houses, and traverse each of the 20 roads exactly once, before finishing at the same house he started at. Is this possible? (Note that each house may be visited more than once.)

- We can make a graph representation of the street map in which houses are represented by vertices and streets leading from a house are incident edges.
- Then, every vertex has an even degree, and the graph has an Euler tour, which is solution to our problem.

(b) The following Sunday, Lenny decides to start his trip at one house, traverse each of the 20 roads exactly once, and finish at a different house.

(i) Is such a trip possible?

(ii) If the answer to (i) is yes, is such a trip possible no matter where the trip starts and ends?

(i) No, since all degrees are even, traversing all edges imply that we are making a closed, not open, trail.

(c) A month later, Lenny finds that road number 20 is closed for construction work. Answer questions (a) and (b) above for this case.

c-a) No, there are two vertices, namly B and J, having odd degrees. Hence, an Euler trail is not possible.

c-b) i) Yes, there are exactly two vertices of odd degree. Hence, there exists an Euler trail which traverses each edge exactly once.

ii) No. The trail has to start and/or finish at the two vertices of odd degree, B and J.

14. A subset A of the set $S = \{1, 2, 3, \ldots, 100\}$ is chosen at random. What is the smallest cardinality that A should have in order to guarantee that two elements from A will have their product equal to 24? *Fully justify your answer.*

[4pts]

Pairs of elements in $\{1,2,3,\ldots,100\}$ whose product equals 24:

$1 \times 24 = 24$

$2 \times 12 = 24$

$3 \times 8 = 24$

$4 \times 6 = 24$

By pigeon-hole principle, the smallest subset of $\{1,2,3,4,6,8,12,24\}$ chosen at random which certaily contains a pair of elements whose product is 24 is $4+1=5$. Indeed, given 4 boxes labelled by pairs $(1,24)$, $(2,12)$, $(3,8)$ and $(4,6)$ and 5 elements, PHP guaranties that there is a box containing at least 2 elements.

In the worst case, set A contains all elements in $\{1,2,\ldots,100\} - \{1,2,3,4,6,8,12,24\}$ and some 5 elements from $\{1,2,3,4,6,8,12,24\}$.

Answer: $(100-8)+5 = 97$.

[5pts] 15. Define a binary relation $\mathcal{R}$ on the set

$$P = \{-5, -4, -3, -1, 0, 1, 4, 6, 8, 11, 18, 20, 21\}$$

as follows:

$$x\mathcal{R}y \quad \text{if and only if} \quad x - y \text{ is divisible by } 3.$$

(a) Prove that $\mathcal{R}$ is an equivalence relation on P.

(b) Determine the partition of P into equivalence classes of $\mathcal{R}$.

(a) <u>Reflexivity:</u> $x \mathcal{R} x \iff x-x$ is divisibly by 3
$\iff 0$ is divisibly by 3

Since $0 = 0 \cdot 3$, $\mathcal{R}$ is reflexive.

<u>Symmetry:</u> Verify that $(x\mathcal{R}y) \rightarrow (y\mathcal{R}x)$ is true.
Assume that $x\mathcal{R}y$. Then 3 divides $x-y$, i.e. $x-y=3m$, for some $m \in \mathbb{Z}$. Then $y-x=-3m$, so 3 divides $y-x$, and hence $y\mathcal{R}x$.

<u>Transitivity:</u> Verify that $[(x\mathcal{R}y) \wedge (y\mathcal{R}z)] \longrightarrow (x\mathcal{R}z)$.
Assume that $x\mathcal{R}y$ and $y\mathcal{R}z$. Then
3 divides $x-y$ and 3 divides $y-z$, i.e.
$x-y=3m$, for some $m \in \mathbb{Z}$ and $y-z=3l$, for some $l \in \mathbb{Z}$.
Then $x-z=(x-y)+(y-z)=3m+3l=3(m+l)$, $m+l \in \mathbb{Z}$.
Since 3 divides $x-z$, $x\mathcal{R}z$.

(b) $[0]_{\mathcal{R}} = \{x \in P \mid x\mathcal{R}0\} = \{x \in P \mid 3 \text{ divides } x-0=x\} =$
$= \{x \in P \mid 3 \text{ divides } x\} = \{-3, 0, 6, 18, 21\}$

$[1]_{\mathcal{R}} = \{x \in P \mid x\mathcal{R}1\} = \{x \in P \mid 3 \text{ divides } x-1\} =$
$= \{-5, 1, 4\}$

$[-4]_{\mathcal{R}} = \{x \in P \mid x\mathcal{R}-4\} = \{x \in P \mid 3 \text{ divides } x-(-4)\}$
$= \{x \in P \mid 3 \text{ divides } x+4\} =$
$= \{-4, -1, 8, 11, 20\}$

Note, class representatives are not unique. However, the partition of P is unique.

16. (a) Let A and B be two sets, and $f : A \to B$ be a function. Give a precise definition of the following terms:

[5pts]

(i) f is one-to-one if and only if
if $f(a_1) = f(a_2)$ for some $a_1, a_2 \in A$,
then $a_1 = a_2$.

(ii) f is onto if and only if
for every $b \in B$, there exists $a \in A$ such that
$f(a) = b$.

(b) Let a function $f : \mathbb{R}_x \times \mathbb{R}_y \to \mathbb{R}_a \times \mathbb{R}_b$ be defined by

$$f(x, y) = (-y, x).$$

Is f one-to-one? Is it onto? *Fully justify your answer.*

- Assume that for some (x_1, y_1) and (x_2, y_2) we have that $f(x_1, y_1) = f(x_2, y_2)$. Then,

$$(-y_1, x_1) = (-y_2, x_2)$$
$$\Leftrightarrow \quad -y_1 = -y_2 \text{ and } x_1 = x_2$$
$$\Leftrightarrow \quad y_1 = y_2 \text{ and } x_1 = x_2$$
$$\Leftrightarrow \quad (x_1, y_1) = (x_2, y_2).$$

$\therefore f$ is one-to-one.

- Let $(a, b) \in \mathbb{R} \times \mathbb{R}$. To verify if f is onto, we need to find $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $f(x, y) = (a, b)$.
Then $(-y, x) = (a, b) \Leftrightarrow$

$$\Leftrightarrow \quad -y = a \text{ and } x = b$$
$$\Leftrightarrow \quad x = b \text{ and } y = -a \Leftrightarrow (x, y) = (b, -a).$$

$\therefore f(b, -a) = (a, b)$.

f is onto.

Additional work space. Do not detach this page.