Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Introduction to Linear Algebra MAT 1341C

Final Exam

April 16, 2023

You must **sign below** to confirm that you have read, understand, and will follow these **instructions**:

- This is a 3-hour **closed-book** exam; no notes are allowed. **Calculators are not permitted.**
- The exam consists of 8 questions on 13 pages. Page 13 provides additional work space. If you need additional space, you can use the backs of any of the pages. Indicate clearly where your answer can be found if it is not written down immediately after where the question is given.
- Question 1–3 are short-answer questions. No justification is required for these questions.
- Questions 4–8 are long-answer questions. You must show all relevant steps and give appropriate justifications in order to obtain full marks.
- On Page 12, there is a bonus question. Only attempt it after you have finished all the other questions.
- **Cellular phones** and other electronic devices **are not permitted** during this exam. Phones and other devices must be turned off completely and stored out of reach. Do not keep them in your possession, such as in your pockets. If you are caught with such a device, the following may occur: academic fraud allegations will be filed which may result in your obtaining a 0 (zero) for the exam.

LAST NAME:________________

First name:________________

Student Number:________________

Signature: Final Exam Solution

Q. 1	Q. 2	Q. 3	Q. 4	Q. 5	Q. 6	Q. 7	Q. 8	Bonus	Total
$\frac{}{16}$	$\frac{}{6}$	$\frac{}{10}$	$\frac{}{12}$	$\frac{}{12}$	$\frac{}{16}$	$\frac{}{16}$	$\frac{}{12}$	$\frac{}{6}$	$\frac{}{100}$

(1) C1. If B is a subset of $\mathbb{R}^3$ such that $B = \mathrm{Span} B$, then $B = \{\mathbf{0}\}$.

C2. Let A be an $n \times n$ matrix. If the columns of A form an orthogonal set, then A is invertible.

C3. If A and B are two $n \times n$ matrices, then

$$\det(A+B)^2 = \det(A)^2 + 2\det(A)\det(B) + \det(B)^2.$$

C4. If $\{v_1, \ldots, v_m\}$ is a linearly dependent subset of $\mathbb{R}^d$, then $m > d$.

C5. If $\{f, g, h\}$ is a spanning set of in $\mathbb{P}_2$, then it is a basis of $\mathbb{P}_2$.

C6. If A and B have the same RREF, then $\det(A) = \det(B)$.

C7. Let W be a subspace of $\mathbb{R}^3$. If $x \in \mathbb{R}^3$ such that $\mathrm{proj}_W(x) = 0$, then $x \in W^\perp$.

C8. If $T : \mathbb{R}^3 \to \mathbb{R}^5$ is a linear map such that $\ker(T) = \{\mathbf{0}\}$, then the image of T is of dimension 3.

In each box below, circle T if the statement is true or F if the statement if alse. [16]

C1	C2	C3	C4	C5	C6	C7	C8
T/F	T/F	T/F	T/F	T/F	T/F	T/F	T/F

(2) (a) Suppose a system of linear equations is described by the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 3 & 2 & -1 \\ 1 & 4 & 4 & 0 \\ -1 & -1 & e & f \end{array}\right],$$

where $e, f \in \mathbb{R}$ are parameters. What values of e and f cause this system to have infinitely many solutions?

A. $e = -2,\ f = 3$

B. $e = 2,\ f = 3$

C. $e = -2,\ f \neq 3$

D. $e = 2,\ f \neq 3$

E. $e \neq 2,\ f \in \mathbb{R}$

F. $e \neq -2,\ f \in \mathbb{R}$

Answer: B [4]

(b) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & -1 \\ 2 & -4 & 1 \\ -2 & 1 & 1 \end{pmatrix}$. What is the entry on the third row and second column of AB?

A. $c_{3,2} = 10$

B. $c_{3,2} = 2$

C. $c_{3,2} = 8$

D. $c_{3,2} = -10$

E. $c_{3,2} = -2$

F. $c_{3,2} = -8$

Answer: D [2]

(3) Let A be a matrix of dimension 13×15 and C a matrix of dimension 15×9. For each of the following statements, write down the value of b (or the values of b and d) for which the statement is correct:

C1. Row(A) is a subspace of $\mathbb{R}^b$ $b =$ 15 [2]

C2. Rank$(A) + \mathrm{Null}(A) = b$. $b =$ 15 [2]

C3. The matrix A^T is of dimension $b \times d$. $b =$ 15, $d =$ 13 [2]

C4. If T is the linear transformation given by $T(x) = Ax$, then $T : \mathbb{R}^b \to \mathbb{R}^d$.
$b =$ 15 $d =$ 13 [2]

C5. The matrix AC est is of dimension $b \times d$. $b =$ 13 $d =$ 9 [2]

(4) Let W be the set of 2×2 matrices defined by $\{A \in M_{2,2} : A^T = -A\}$.

(a) Prove that W is a subspace of $M_{2,2}$. [4]

If A is the zero matrix, $A^\top = 0, -A = 0$, so $A \in W$.
If $A, B \in W, A^\top = -A, B^\top = -B$. Then

$$(A+B)^\top = A^\top + B^\top = -A - B = -(A+B)$$

So $A + B \in W$. If $c \in \mathbb{R}, A \in W$, then

$$(cA)^\top = cA^\top = c(-A) = -cA$$

So $cA \in W$. Therefore, the subspace test gives us that W is a subspace of $M_{2,2}$.

(b) Find a spanning set of W. [4]

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then we have

$$A^\top = -A \iff \begin{bmatrix} a & c \\ b & d \end{bmatrix} = -\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

which gives us $a = -a, c = -b, d = -d$. So $a = d = 0$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Therefore, W is spanned by $\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$.

(c) What is the dimension of W? Justify your answer. [4]

Since $W = \text{span}\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \neq 0$. This is a basis.
$\Rightarrow \det(W) = 1$.

(5) Let $A = \begin{pmatrix} 1 & 0 & -1 \\ -2 & 3 & 4 \\ 0 & 1 & 1 \end{pmatrix}$.

(a) Find the determinant of A. [4]

$$\begin{aligned} \det(A) &= \det \begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \det \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} + 0 - \det \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} \\ &= 3 - 4 - (-2) \\ &= 1 \end{aligned}$$

(b) Explain why A is invertible. [2]

Since we know that

$$A \text{ is invertible} \iff \det(A) \neq 0$$

From part (a) we have $\det(A) = 1$, hence A is invertible.

(c) Find the inverse A^{-1} of A. [6]

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ -2 & 3 & 4 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim 2R_1 + R_2 \to R_2 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 3 & 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim$$

$$R_2 \leftrightarrow R_3 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 3 & 2 & 2 & 1 & 0 \end{array}\right] \sim -3R_2 + R_3 \to R_3 \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 & 1 & -3 \end{array}\right] \sim -R_3 \leftrightarrow R_3 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & -1 & 3 \end{array}\right]$$

$$\sim \begin{cases} R_3 + R_1 \to R_1 \\ -R_3 + R_2 \to R_2 \end{cases} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 3 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -2 & -1 & 3 \end{array}\right] = [I_3 \mid A^{-1}]$$

Therefore,

$$A^{-1} = \begin{bmatrix} -1 & -1 & 3 \\ 2 & 1 & -2 \\ -2 & -1 & 3 \end{bmatrix}$$

(6) Let U be a subspace of $\mathbb{R}^4$ and let the set $S = \{v_1,\ v_2,\ v_3\}$ be a basis of U, where

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 1 \\ -1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 4 \\ -1 \\ 0 \end{bmatrix}.$$

(a) Find an orthogonal basis $B = \{u_1, u_2, u_3\}$ of U. [7]

$$u_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} u_2 &= v_2 - \text{proj}_{u_1}(v_2) \\ &= \begin{bmatrix} 4 \\ 1 \\ -1 \\ -1 \end{bmatrix} - \frac{4+1+1}{1+0+1+1} \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 1 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} u_3 &= v_3 - \text{proj}_{u_1}(v_3) - \text{proj}_{u_2}(v_3) \\ &= \begin{bmatrix} 2 \\ 4 \\ -1 \\ 0 \end{bmatrix} - \frac{2+0+1}{1+0+1+1} \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} - \frac{4+4-1}{4+1+1+1} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 4 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 0 \end{bmatrix} \end{aligned}$$

(b) Let $v = \begin{bmatrix} 2 \\ 11 \\ -2 \\ 1 \end{bmatrix}$. Find $c_1, c_2, c_3 \in \mathbb{R}$ such that $v = c_1u_1 + c_2u_2 + c_3u_3$. [4]

We can use the formula:

$$c_i = \frac{v \cdot u_i}{u_i \cdot u_i}$$

Hence

$$c_1 = \frac{2+2-1}{3} = 1,\ c_2 = \frac{4+11-2+1}{4+1+1+1} = 1,\ c_3 = \frac{-2+22+2}{1+9+1} = 3.$$

(Continued on next page)

(c) Let $w = \begin{bmatrix} 1 \\ -3 \\ -10 \\ 11 \end{bmatrix}$. Is $w \in U^{\perp}$? Justify your answer. [3]

Check $w \cdot u_i = 0$ for $i = 1, 2, 3$.

or $w \cdot v_i = 0$ for $i = 1, 2, 3$.

(d) Find $\dim U^{\perp}$. [2]

$\dim(U) + \dim(U^{\perp}) = 4$, and $\dim(U) = 3$

$\Rightarrow \dim(U^{\perp}) = 1$

(7) Let $A = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix}$.

(a) Find the characteristic polynomial of A. [5]

$$\begin{aligned}
\text{char. poly. of } A = \det(A - \lambda I_3) &= \begin{bmatrix} -1-\lambda & 3 & 0 \\ 0 & 2-\lambda & 0 \\ 1 & -1 & 2-\lambda \end{bmatrix} \\
&= (2-\lambda)\det \begin{bmatrix} -1-\lambda & 3 \\ 0 & 2-\lambda \end{bmatrix} \\
&= (2-\lambda)(-1-\lambda)(2-\lambda) \\
&= -(\lambda-2)^2(\lambda+1)
\end{aligned}$$

(b) Using the characteristic polynomial, explain why the eigenvalues of A are -1 and 2. [2]

The zeros of the characteristic polynomial are 2 and -1.

(Continued on next page)

(c) Find a basis for the eigenspace $E_{-1} = \{\mathbf{v} \in \mathbb{R}^3 \mid A\mathbf{v} = -\mathbf{v}\}$. [3]

$$\text{Null}(A+I) = \text{Null}\begin{bmatrix} 0 & 3 & 0 \\ 0 & 3 & 0 \\ 1 & -1 & 3 \end{bmatrix}$$

Then

$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 3 & 0 \\ 1 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $y = 0,\ x = -3z$.

Which means $\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for E_{-1}.

(d) Find a basis for the eigenspace $E_2 = \{\mathbf{v} \in \mathbb{R}^3 \mid A\mathbf{v} = 2\mathbf{v}\}$. [4]

$$\text{Null}(A-2I) = \text{Null}\begin{bmatrix} -3 & 3 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

so $x = y$, and z is free variable.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Which means $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for E_2.

(e) Write down an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. [2]

Answer: $P = \begin{bmatrix} -3 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $\qquad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(8) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 6x_1 + 2x_2 - 2x_3, 3x_1 + x_2 - x_3)$$

(a) Find the standard matrix of T. Justify your answer. [3]

$T(1,0,0) = (1,6,3),\ T(0,1,0) = (1,2,1),\ T(0,0,1) = (1,-2,-1).$

or directly write down

$$T(x_1, x_2, x_3) = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 2 & -2 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(b) Find $\mathrm{Ker}(T)$. Justify your answer. [5]

We know that $\text{Ker}(T) = \text{Null}(A)$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 2 & -2 \\ 3 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

so $x = z,\ y = -2z.$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Therefore, $\text{Ker}(T) = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$

(c) Find $\mathrm{Im}(T)$. Justify your answer. [4]

$\text{Im}(T) = \text{Col}(A)$. The pivots of the RREF of A are in the 1st and 2nd columns.

Hence, we have $\text{Im}(T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}.$

Bonus: Let A be an $n \times n$ matrix such that $A = A^T$ and let $\mu \neq \lambda$ be two distinct eigenvalues of A. Prove that the corresponding eigenspaces E_λ and E_μ are orthogonal (that is, for all $u \in V_\mu$ and all $v \in V_\lambda$, we have $u \cdot v = 0$). [6]

The eigenspace $E_\lambda = \{\mathbf{u} \in \mathbb{R}^n \mid A\mathbf{u} = \lambda\mathbf{u}\},\ E_\mu = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = \mu\mathbf{v}\}$.

Then we have

$$\begin{aligned}
(A\mathbf{u})^\top = (\lambda\mathbf{u})^\top &\iff \mathbf{u}^\top A^\top = \lambda\mathbf{u}^\top \\
&\iff \mathbf{u}^\top A = \lambda\mathbf{u}^\top \qquad A = A^\top \\
&\iff \mathbf{u}^\top A\mathbf{v} = \lambda\mathbf{u}^\top\mathbf{v} \\
&\iff \mathbf{u}^\top(\mu\mathbf{v}) = \lambda\mathbf{u}^\top\mathbf{v} \qquad Av = \mu v \\
&\iff \mu\mathbf{u}\cdot\mathbf{v} = \lambda\mathbf{u}\cdot\mathbf{v}
\end{aligned}$$

since $\mu \neq \lambda$, thus $\mathbf{u} \cdot \mathbf{v} = 0$.