Lesson 7 – Applications of Sinusoidal Functions and Their Derivatives

PART A: Applications

Example 1: Find all the points, in the domain $-2\pi \leq \theta \leq 2\pi$, on the curve $y = 5sin\theta$ such that the slope of the tangent line is 5.

① determine derivative

$y = 5\sin\theta$

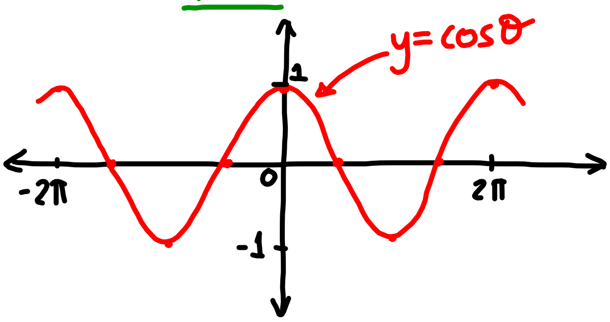
$y' = 5\cos\theta$

② Set derivative equal to 5 and solve for θ

$\frac{5}{5} = \frac{5\cos\theta}{5}$

$1 = \cos\theta$

Recall



③ Write General Solution for θ

$\theta = 0 \pm 2n\pi$

④ Write specific solutions that fit the domain $(-2\pi \leq \theta \leq 2\pi)$

$\theta_1 = 0 + 2(0)\pi = 0$; $n = 0$

$\theta_2 = 0 + 2(-1)\pi = -2\pi$; $n = -1$

$\theta_3 = 0 + 2(1)\pi = 2\pi$; $n = 1$

⑤ Summarize solutions

$\therefore$ For $y = 5\sin\theta$, the slope of the tangent is 5 for

$\theta = -2\pi, 0, 2\pi$ over the interval $-2\pi \leq \theta \leq 2\pi$

Example 2: Determine the values of x, for which the tangent line to $y = sinx + cosx$ is horizontal over the domain $-2\pi \leq x \leq 2\pi$

① Determine derivative:

$y = \sin x + \cos x$

$y' = \cos x - \sin x$

② Set derivative equal to zero $\overbrace{\text{(horizontal tangent)}}^{m = y' = 0}$, solve for x:

$0 = \cos x - \sin x$

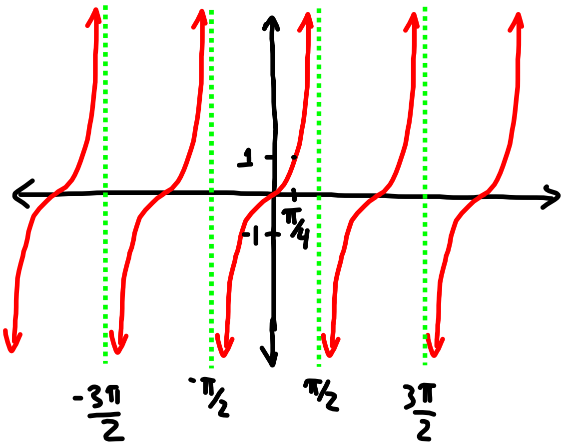
$\sin x = \cos x$

$\frac{\sin x}{\cos x} = \frac{\cancel{\cos x}}{\cancel{\cos x}} \to 1$

$\tan x = 1$

③ provide general solution

$x = \frac{\pi}{4} + n\pi$



④ Provide specific sol^ns that fit domain $(-2\pi \leq x \leq 2\pi)$

$x_1 = \frac{\pi}{4} + (-2)\pi = -\frac{7\pi}{4}$; $n = -2$

$x_2 = \frac{\pi}{4} + (-1)\pi = -\frac{3\pi}{4}$; $n = -1$

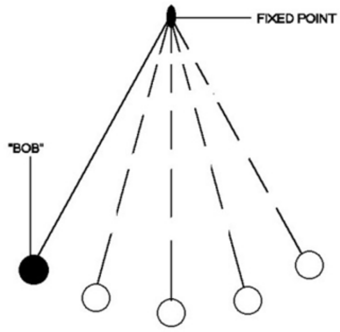
$x_3 = \frac{\pi}{4} + (0)\pi = \frac{\pi}{4}$; $n = 0$

$x_4 = \frac{\pi}{4} + (1)\pi = \frac{5\pi}{4}$; $n = 1$

⑤ Summarize solutions:

The tangent line to $y = \sin x + \cos x$ is horizontal at $x = -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$ over the domain $-2\pi \leq x \leq 2\pi$.

Example 3: A Simple Pendulum



For small amplitudes, and ignoring the effects of friction, a pendulum is an example of simple harmonic motion. Simple harmonic motion is motion that can be modelled by a sinusoidal function, and the graph of a function modelling simple harmonic motion, has a constant amplitude.
The period of a simple pendulum depends only on its length and can be found using the relation

Period: $\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$

$\frac{l}{g} \quad \frac{[m]}{[m/s^2]} = m \div \frac{m}{s^2} = \cancel{m} \times \frac{s^2}{\cancel{m}}$

where,

T is the period, in seconds
l is the length of the pendulum, in metres,
g is the acceleration due to gravity.

(On or near the surface of the Earth, g has a constant value of 9.8 m/s^2.

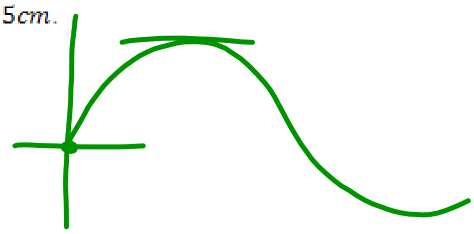
Under these conditions, the horizontal position of the bob as a function of time can be described by the function

Horizontal Position: $\boxed{h(t) = A\cos\left(\frac{2\pi t}{T}\right)}$

where,

A is the amplitude in cm
t is the time, in seconds
T is the period of the pendulum, in seconds.

Find the maximum speed of the bob and the time at which that speed first occurs for a pendulum having a length of $1.0m$ and an amplitude of $5cm$.



$s(t)$ position
$s'(t)$ velocity
$s''(t)$ acc.

Find the maximum speed of the bob and the time at which that speed first occurs for a pendulum having a length of $1.0m$ and an amplitude of $5cm$.

① Determine period T given length

$T = 2\pi\sqrt{l/g}$

$= 2\pi\sqrt{1/9.8}$

$\boxed{T \doteq 2\ s}$

Note: to find max vel. we need to find the zeros of the acc. function (2nd derivative of position func^n)

② To determine max speed we need to find zeros of acc. function (2nd derivative of position function)

∴ find $h''(t)$, set to zero and solve:

$h(t) = A\cos\left(\frac{2\pi t}{T}\right)$; $A = 5$, $T = 2$

$= 5\cos\left(\frac{\cancel{2}\pi t}{\cancel{2}}\right)$

$= 5\cos \pi t$

$h'(t) = -5\sin\pi t \cdot \pi = -5\pi\sin\pi t$

$h''(t) = -5\pi\cos\pi t \cdot \pi = -5\pi^2\cos\pi t$

$\therefore a(t) = -5\pi^2\cos\pi t$

$0 = -5\pi^2\cos\pi t$

③ General Solution

let $\theta = \pi t$

$0 = \cos\theta$

$\theta = \frac{\pi}{2} + 2n\pi$, replace θ with πt, solve for t:

$\frac{\pi t}{\pi} = \frac{\frac{\pi}{2}}{\pi} + \frac{2n\pi}{\pi}$

$t = \frac{1}{2} + 2n$

④ specific solutions that fit domain (first occurence)

$t = \frac{1}{2} + 2(0) = \frac{1}{2}$

⑤ determine speed at $t = 1/2$ to find max speed:

$h'(t) = v(t) = -5\pi\sin\pi t$

$v(1/2) = -5\pi\sin(\pi/2)$

$v(1/2) \doteq -15.7\ cm/s$

Example 4: An AC-DC Coupled Circuit

A power supply delivers a voltage signal that consists of an alternating current (AC) component and a direct current (DC) component. The signal is modeled by the function:

$V(t) = 5\sin t + 12$

where, t is the time in seconds V is the voltage in volts

using calculus!!

a) Find the maximum and minimum voltages. At which times do these values occur?
b) Determine the period, T, in seconds, frequency, f, in hertz, and the amplitude, A, in volts for this signal.

① need to find zeros of first derivative $(V'(t)=0)$

$V(t) = 5\sin t + 12$

$V'(t) = 5\cos t$

$0 = 5\cos t$

$t = \frac{\pi}{2} + 2n\pi \quad , \quad t = \frac{3\pi}{2} + 2n\pi$

$t_1 = \frac{\pi}{2} \; ; n=0 \quad , \quad t_2 = \frac{3\pi}{2} \; ; n=0$

$V(t_1) = V(\pi/2) = 5\sin\left(\frac{\pi}{2}\right) + 12$

$= 17$ Volts

$V(t_2) = V\left(\frac{3\pi}{2}\right) = 5\sin\left(\frac{3\pi}{2}\right) + 12$

$= 7$ volts

$\therefore$ Max Voltage is 17 Volts

Min Voltage is 7 volts

b). $T = 2\pi$

$f = \frac{1}{T} = \frac{1}{2\pi}$ Hz

$A = \frac{\text{max} - \text{min}}{2} = \frac{17-7}{2} = 5$