DGD 6

Q1. PROPERTIES OF FUNCTIONS: INJECTIVE & SURJECTIVE

For each of the following functions, determine whether it is injective and/or surjective. If it is, give a proof; otherwise, provide a concrete numerical counterexample. Let f, g, and h be three functions defined as follows:

$f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ $\quad f(x) = (x, 5 - x)$

$g : \mathbb{Q}^+ \times \mathbb{Q}^+ \to \mathbb{Q}^+$ $\quad g(r, s) = rs$

$h : \mathbb{Z} \to \mathbb{N}$ $\quad h(n) = 2n^2 + 1$

f is injective.

proof. Let $a, b \in \mathbb{Z}$. Assume $f(a) = f(b)$. Then $(a, 5-a) = (b, 5-b)$

$\Rightarrow a = b$ and $5 - a = 5 - b$

$\Rightarrow a = b$ ∎

f is not surjective:

counterexample: $(0,0) \in \mathbb{Z} \times \mathbb{Z}$ but there is no $x \in \mathbb{Z}$ such that $f(x) = (0,0)$ because $(x, 5-x) = (0,0) \Leftrightarrow x = 0$ and $x = 5$ ↯

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g is not injective.

counterexample: $(3,2)$ and $(1.5, 4) \in \mathbb{Q}^+ \times \mathbb{Q}^+$, $(3,2) \neq (1.5, 4)$

but $g(3,2) = 3 \cdot 2 = 6 = 1.5 \cdot 4 = g(1.5, 4)$

g is surjective.

proof Let $q \in \mathbb{Q}^+$. Then $q = 1 \cdot q = g(1, q)$.

Hence any element q of the codomain of g is the image of $(1, q)$. ∎

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h is not injective.

counterexample: -5 and $5 \in \mathbb{Z}$. $-5 \neq 5$ but $h(-5) = h(5)$.

h is not surjective.

counterexample: $2 \in \mathbb{N}$ but there is no integer $n \in \mathbb{Z}$ such that $2 = 2n^2 + 1$

Q2. BIJECTIONS

Definition. A **bijection** is a function that is both injective and surjective.

Prove that the function $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by $g(m, n) = (1 - n, m + 5)$ is a bijection.

[injective]

Let $(m_1, n_1), (m_2, n_2) \in \mathbb{Z} \times \mathbb{Z}$ (arbitrary elements of g's domain).

Assume $g(m_1, n_1) = g(m_2, n_2)$.

(goal is to prove $(m_1, n_1) = (m_2, n_2)$)

Then $(1 - n_1, m_1 + 5) = (1 - n_2, m_2 + 5)$

$\Rightarrow 1 - n_1 = 1 - n_2$ and $m_1 + 5 = m_2 + 5$

$\Rightarrow n_1 = n_2$ and $m_1 = m_2$

$\Rightarrow (m_1, n_1) = (m_2, n_2)$

$\therefore g$ is injective (1-1).

[surjective]

Let $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ (arbitrary elements of g's codomain)

*goal is to find $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ (domain) such that

$$g(m,n) = (p,q)$$

Well, $g(m,n) = (p,q) \Leftrightarrow (1-n, m+5) = (p,q)$

(we want to reverse-engineer what (m,n) should equal <u>in terms of p and q</u>, in order to get $g(m,n) = (p,q)$).

$\Rightarrow 1 - n = p$ and $m + 5 = q$

$\Rightarrow n = 1 - p$ and $m = q - 5$

So $(m,n) = (q-5, 1-p)$ works and

Since $q \in \mathbb{Z}$, so is $q - 5$
Since $p \in \mathbb{Z}$, so is $1 - p$
$\therefore (q-5, 1-p) \in \mathbb{Z} \times \mathbb{Z}$.
and $g(q-5, 1-p) = (p,q)$

$\therefore g$ is surjective (onto)

Since g is both injective and surjective, g is a bijection!

Q3. THE FLOOR FUNCTION AND THE CEILING FUNCTION

Definitions. The **floor function** $\lfloor\cdot\rfloor : \mathbb{R} \to \mathbb{Z}$ and the **ceiling function** $\lceil\cdot\rceil : \mathbb{R} \to \mathbb{Z}$ are functions defined by the following rules, respectively:

$$\lfloor x\rfloor = \max\{n \in \mathbb{Z} : n \leq x\} \qquad \text{and} \qquad \lceil x\rceil = \min\{n \in \mathbb{Z} : n \geq x\}$$

i. Find the values of each of the following:

$\lfloor 1.8\rfloor = \max\{n \in \mathbb{Z} : n \leq 1.8\}$
$= 1$

$\lfloor -1.8\rfloor = \max\{n \in \mathbb{Z} : n \leq -1.8\}$
$= -2$

$\lfloor -1\rfloor = \max\{n \in \mathbb{Z} : n \leq -1\}$
$= -1$

$\lfloor -5.01\rfloor = \max\{n \in \mathbb{Z} : n \leq -5.01\}$
$= -6$

$\lfloor \pi\rfloor = \max\{n \in \mathbb{Z} : n \leq \pi\}$
$= 3$

$\lfloor \sqrt{2}\rfloor = \max\{n \in \mathbb{Z} : n \leq \sqrt{2}\}$
$= 1$

$\lfloor 1 + \frac{1}{k}\rfloor$ where k is some positive integer.

$= \max\{n \in \mathbb{Z} : n \leq 1 + \frac{1}{k}\}$
$= 1$

$\lfloor 1 - \frac{1}{k}\rfloor$ where k is some positive integer.

$= \max\{n \in \mathbb{Z} : n \leq 1 - \frac{1}{k}\}$
$= 0$

$\lceil 1.8\rceil = \min\{n \in \mathbb{Z} : n \geq 1.8\}$
$= 2$

$\lceil -1.8\rceil = \min\{n \in \mathbb{Z} : n \geq -1.8\}$
$= -1$

$\lceil -1\rceil = -1$

$\lceil -5.01\rceil = -5$

$\lceil \pi\rceil = 4$

$\lceil \sqrt{2}\rceil = 2$

$\lceil 1 + \frac{1}{k}\rceil$ where k is some positive integer.

$= 2$

$\lceil 1 - \frac{1}{k}\rceil$ where k is some positive integer.

$= 1$

ii. Let $g : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $g(k) = \lfloor k/2 \rfloor$.

Is g injective? If so, prove it; otherwise give a counterexample and explanation.

No g is not injective

Counterexample (there are many other counterexamples)

$4, 5 \in \mathbb{Z}$ (domain of g)

$g(4) = \lfloor \frac{4}{2} \rfloor = \lfloor 2 \rfloor = 2$ and $g(5) = \lfloor \frac{5}{2} \rfloor = \lfloor 2.5 \rfloor = 2$

$\therefore g(4) = g(5)$ but $4 \neq 5$ $\therefore$ g is not injective

Is g surjective? If so, prove it; otherwise give a counterexample and explanation.

Yes!

<u>proof</u> Let $y \in \mathbb{Z}$ be an arbitrary element of the codomain of g.

(goal: find $k \in \mathbb{Z}$ (domain) such that $g(k) = y$)

Since $y \in \mathbb{Z}$, $2y \in \mathbb{Z}$.

We have $g(2y) = \lfloor \frac{2y}{2} \rfloor = \lfloor y \rfloor = y$ since $y \in \mathbb{Z} \Longrightarrow \lfloor y \rfloor = y$

$\therefore 2y \in g^{-1}(y)$

$\therefore g^{-1}(y) \neq \emptyset$

Since y was an arbitrary element of g's codomain, this proves that g is surjective.

Q4. PROPERTIES OF FUNCTIONS *there are many other possible examples

i. Give an example of a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ that is neither injective nor surjective.

Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(r,s) = (|r|, |s|)$

f is neither injective nor surjective.

$f(1,-1) = f(-1,-1) = (1,1)$ but $(1,-1) \neq (-1,-1)$ $\therefore$ f is not injective.

$(-5,-8) \in \mathbb{R}^2$ but there is no $(r,s) \in \mathbb{R}^2$ such that $(|r|,|s|) = (-5,-8)$ $\therefore$ f is not surjective

ii. Give an example of a function from $\mathbb{R}$ to $\mathbb{R}$ that is injective but not surjective.

Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = e^x$.

Then g is injective. proof. Let $x_1, x_2 \in \mathbb{R}$.

Assume $g(x_1) = g(x_2)$. Then $e^{x_1} = e^{x_2} \Rightarrow \ln(e^{x_1}) = \ln(e^{x_2}) \Rightarrow x_1 = x_2$. ∎

g is not surjective since there is no $x \in \mathbb{R}$ such that $g(x) = 0$ because $e^x > 0$ $\forall x \in \mathbb{R}$.
(or any negative value...)

iii. Give an example of a function from $\mathbb{R}$ to $\mathbb{R}$ that is surjective but not injective.

Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = x^3 - x$.

h is not injective. Counterex. $0, 1 \in \mathbb{R}$, $0 \neq 1$, but $h(0) = h(1)$

h is surjective since h is a continuous function with domain $\mathbb{R}$ and we can verify that its range is all real #s $\lim_{x \to \infty} h(x) = \infty$ and $\lim_{x \to -\infty} h(x) = -\infty$ ...

iv. Give an example of a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ that is a bijection but is **not** the identity function $\mathrm{id}_{\mathbb{R}^2}$. Compute an expression for the inverse of your function.

Define $p : \mathbb{R}^2 \to \mathbb{R}^2$ by

p is a bijection (verify!)
(for proof, see next page)

$\varphi(r,s)=(r+s,r-s)$ ← clearly, φ is not the identity function because $\varphi(r,s)\neq(r,s)$ $\forall (r,s)\in\mathbb{R}^2$.

<u>proof that φ is injective.</u> Let $(a,b),(c,d)\in\mathbb{R}^2$ (φ's domain)

Assume $\varphi(a,b)=\varphi(c,d)$. (goal: prove $(a,b)=(c,d)$)

Then $(a+b, a-b)=(c+d, c-d)$

$\Rightarrow a+b=c+d$ ① and $a-b=c-d$ ②

$\Rightarrow a=c+d-b$ ③ $\Rightarrow (c+d-b)-b=c-d$ (plug ③ into ②)

$\Rightarrow 2d=2b$

$\Rightarrow d=b$ ④

$\Rightarrow a=c+d-d=c$ (plug ④ into ③)

$\therefore d=b$ and $a=c$ so $(a,b)=(c,d)$ ✓ ▨

<u>proof that φ is surjective</u>

Let $(r,s)\in\mathbb{R}^2$ be an arbitrary element of φ's codomain.

goal: To show φ is surjective, we need to provide an element (a,b) from φ's domain such that $\varphi(a,b)=(r,s)$ (and we need to write what a and b must equal in terms of r and s).

$\varphi(a,b)=(r,s)\Leftrightarrow (a+b,a-b)=(r,s)$

$\Leftrightarrow a+b=r$ and $a-b=s$

$\Leftrightarrow a=r-b$ ① and $b=a-s$ ②

$\therefore a=r-(a-s)$ (plug ② into ①)

$\Rightarrow 2a=r+s$

$\Rightarrow a=\frac{r+s}{2}$ ③ $\Rightarrow b=\frac{r+s}{2}-s=\frac{r-s}{2}$ (plug ③ into ②)

$\therefore$ we need $(a,b)=\left(\frac{r+s}{2},\frac{r-s}{2}\right)$.

Note $\frac{r+s}{2}\in\mathbb{R}$ and $\frac{r-s}{2}\in\mathbb{R}$, since r and r are real numbers. $\therefore \left(\frac{r+s}{2},\frac{r-s}{2}\right)\in\mathbb{R}^2$

Moreover, $\varphi\left(\frac{r+s}{2},\frac{r-s}{2}\right)=\left(\left(\frac{r+s}{2}\right)+\left(\frac{r-s}{2}\right), \left(\frac{r+s}{2}\right)-\left(\frac{r-s}{2}\right)\right)=(r,s)$ (goal!)

$\therefore \varphi$ is surjective ▨

Since φ is both injective and surjective, it's a bijection.