13. Relations

Let A and B be sets. A **(binary) relation from** A **to** B is a subset of $A \times B$.

Let A be a set. A **(binary) relation on a set** A is a relation from A to itself. ie a subset of $A \times A$

Example 13.1. Let $A = \{1, 2, 3\}$ and let $B = \{x, y\}$

$\mathcal{R}_1 = \{(1,x),(1,y),(3,y)\}$ $\quad \mathcal{R}_1 \subseteq A \times B \; \therefore \mathcal{R}_1$ is a relation from A to B

$\mathcal{R}_2 = \{(x,x),(y,x)\}$ $\quad \mathcal{R}_2 \subseteq B \times B \; \therefore \mathcal{R}_2$ is a relation on B

$\mathcal{R}_3 = \{(1,2),(2,1),(2,3),(3,2)\}$ $\quad \mathcal{R}_3 \subseteq A \times A \; \therefore \mathcal{R}_3$ is a relation on A

$\mathcal{R}_4 = \{(x,1),(y,2)\}$ $\quad \mathcal{R}_4 \subseteq B \times A \; \therefore \mathcal{R}_4$ is a relation from B to A

Notation for Relations

- since $(1,x) \in \mathcal{R}_1$ it means "1 is related to x by $\mathcal{R}_1$"
- since $(2,x) \notin \mathcal{R}_1$, it means "2 is not related to x by $\mathcal{R}_1$"

Special Relation Notation:

- for short, we will write $1 \mathcal{R}_1 x$ for "1 is related to x by $\mathcal{R}_1$"
  and $2 \not\mathcal{R}_1 x$ for "2 is not related to x by $\mathcal{R}_1$"

Representing Relations: a finite list of pairs of related elements VS. a rule to relate elements

For a small finite relation, we can simply write it as a list (set) of ordered related pairs of elements:

ex $\mathcal{R}_3 = \{(1,2),(2,1),(2,3),(3,2)\}$

Alternatively, some relations may be described by a rule that tells us exactly when/how two elements are related:

ex for all $u, v \in A = \{1,2,3\}$, $u \mathcal{R}_3 v$ if and only if $u+v$ is odd.

* These notes are solely for the personal use of students registered in MAT1348.

Example 13.2. Friends and Family

Ex for all people A, B, $A \mathcal{R} B$ if and only if A and B share a common ancestor.

Ex for all Facebook users X, Y, $X \mathcal{R} Y$ if and only if X and Y are "friends"

Example 13.3. Graphs

The graph of a function $f: A \to B$ is is the set of coordinate pairs $\mathcal{G} = \{(a, f(a)) : a \in A\}$

$\mathcal{G} \subseteq A \times B \therefore \mathcal{G}$ is a relation from A to B.

rule for $\mathcal{G}$: for all $a \in A, b \in B$, $a \mathcal{G} b$ if and only if $f(a) = b$

Ex the graph of the equation $x^2 + y^2 = 1$ is a subset of $\mathbb{R}^2$ $\therefore$ it's a relation on $\mathbb{R}$

rule: for all $x, y \in \mathbb{R}$, $x \mathcal{R} y$ if and only if $x^2 + y^2 = 1$

Example 13.4. Equality $\boxed{= \text{ is a relation on } \mathbb{R}}$

Let $\mathcal{E}$ be the relation on the set of real numbers defined by the rule

For all $x, y \in \mathbb{R}$, $x \mathcal{E} y$ if and only if $x = y$

Ex $1 \mathcal{E} 1$ because $1 = 1$ so $(1,1) \in \mathcal{E}$

Ex $1 \not\mathcal{E} \pi$ because $1 \neq \pi$ so $(1, \pi) \notin \mathcal{E}$

Example 13.5. Inequality

$\boxed{< \text{ is a relation on } \mathbb{R}}$

Ex 2 is related to 2.1 because $2 < 2.1$
2 is not related to 2 because $2 < 2$ is false.

$\boxed{\leq \text{ is a relation on } \mathbb{R}}$

Ex 2 is related to 2 because $2 \leq 2$
2 is related to 1000.1 because $2 \leq 1000.1$
2 is not related to 0 because $2 \not\leq 0$

Example 13.6. Divides $\boxed{\mid \text{ ("divides") is a relation on } \mathbb{Z}}$

for all $m, n \in \mathbb{Z}, m \neq 0$, $m \mid n$ if and only if $n = km$ for some integer k.

Ex $5 \mid 100$ $\quad 100 \nmid 5$ $\quad 3 \mid 99$ ← these are not fractions! Never forget: "divides" is a relation on $\mathbb{Z}$, not the arithmetic operation of division!

Example 13.7. Logical Equivalence

$\equiv$ is a relation on the set of all compound proposions

rule: for all propositions P, Q, $P \equiv Q$ if and only if $P \leftrightarrow Q$ is a tautology

[**Reflexive**] A relation $\mathcal{R}$ on a set A is called **reflexive** if the implication $(x \in A) \to ((x,x) \in \mathcal{R})$ is true.

Equivalently, $(x \in A) \to (x \mathcal{R} x)$

[**Symmetric**] A relation $\mathcal{R}$ on a set A is called **symmetric** if for all $x, y \in A$, the implication $((x,y) \in \mathcal{R}) \to ((y,x) \in \mathcal{R})$ is true.

Equivalently, $(x \mathcal{R} y) \to (y \mathcal{R} x)$

[**Antisymmetric**] A relation $\mathcal{R}$ on a set A is called **antisymmetric** if for all $x, y \in A$, the implication $((x,y) \in \mathcal{R} \text{ and } (y,x) \in \mathcal{R}) \to (x = y)$ is true.

Equivalently, $(x \mathcal{R} y \text{ and } y \mathcal{R} x) \to (x = y)$

this should remind you of $(x \le y \text{ and } y \le x) \to (x = y)$

contrapositive form: $(x \ne y) \to ((x,y) \notin \mathcal{R} \text{ or } (y,x) \notin \mathcal{R})$

[**Transitive**] A relation $\mathcal{R}$ on a set A is called **transitive** if for all $x, y, z \in A$,

the implication $((x,y) \in \mathcal{R} \text{ and } (y,z) \in \mathcal{R}) \to ((x,z) \in \mathcal{R})$ is true.

Equivalently, $(x \mathcal{R} y \text{ and } y \mathcal{R} z) \to (x \mathcal{R} z)$

this should remind you of $(x \le y \text{ and } y \le z) \to (x \le z)$

EQUIVALENCE RELATIONS

A relation $\mathcal{R}$ on a set A is called **an equivalence relation** if

$\mathcal{R}$ is reflexive, symmetric, and transitive.

Example 13.8. Let $\mathcal{R}_2$ be a relation on $\mathbb{Z}$ defined by the rule

$$\text{for all } a, b \in \mathbb{Z}, \qquad (a, b) \in \mathcal{R}_2 \quad \text{if and only if} \quad a + b \quad \text{is even}$$

Examples $(0,0) \in \mathcal{R}_2$ because $0+0=0$ is even

$(2,-23) \notin \mathcal{R}_2$ because $2+(-23)=-21$ is not even

$5 \mathcal{R}_2 -55$ (meaning $(5,-55) \in \mathcal{R}_2$) because $5+(-55)=-50$ is even

$2 \not\mathcal{R}_2 3$ (meaning $(2,3) \notin \mathcal{R}_2$) because $2+3=5$ is not even

PROPERTIES

[reflexive] To prove $\mathcal{R}_2$ is reflexive, we must prove $(a \in \mathbb{Z}) \to (a \mathcal{R}_2 a)$

Let $a \in \mathbb{Z}$. Then $a+a=2a$ and $a \in \mathbb{Z}$

$\therefore$ $a+a$ is even. (def of even)

$\therefore$ $a \mathcal{R}_2 a$ (by the rule for $\mathcal{R}_2$) $\therefore$ $\mathcal{R}_2$ is reflexive.

[Symmetric] To prove $\mathcal{R}_2$ is symmetric, we must prove $a \mathcal{R}_2 b \to b \mathcal{R} a$.

Let $a, b \in \mathbb{Z}$ be arbitrary elements of the set $\mathbb{Z}$.

Assume $a \mathcal{R}_2 b$. (goal is to prove $b \mathcal{R}_2 a$.)

Then $a+b$ is even (by the rule for $\mathcal{R}_2$)

$\Rightarrow$ $b+a$ is even (since $a+b=b+a$)

$\Rightarrow$ $b \mathcal{R}_2 a$ (by the rule for $\mathcal{R}_2$) $\therefore$ $\mathcal{R}_2$ is symmetric

[antisymmetric] To prove $\mathcal{R}_2$ is antisymmetric, we must prove

$$[(a \mathcal{R}_2 b) \wedge (b \mathcal{R}_2 a)] \to [a=b] \text{ for all } a, b \in \mathbb{Z}$$

Wait! This is not true for all $a, b \in \mathbb{Z}$

Here is a counterexample:

$3, 7 \in \mathbb{Z}$

$3+7$ is even and $7+3$ is even

$\Rightarrow$ both $(3,7) \in \mathcal{R}_2$ and $(7,3) \in \mathcal{R}_2$, but $3 \neq 7$. $\therefore$ $\mathcal{R}_2$ is not antisymmetric

[transitive]

To prove $\mathcal{R}_2$ is transitive, we must prove $[(a\mathcal{R}_2 b)\wedge(b\mathcal{R}_2 c)] \rightarrow [a\mathcal{R}_2 c]$

Let $a,b,c \in \mathbb{Z}$ be arbitrary elements of the set $\mathbb{Z}$.

Assume $a\mathcal{R}_2 b$ and $b\mathcal{R}_2 c$. (goal is to prove $a\mathcal{R}_2 c$)

Then $a+b$ is even and $b+c$ is even. (by $\mathcal{R}_2$'s rule)

$\Rightarrow a+b = 2k$ for some $k \in \mathbb{Z}$

and $b+c = 2\ell$ for some $\ell \in \mathbb{Z}$ (by def. of even).

$\Longrightarrow a = 2k-b$ and $c = 2\ell - b$

Thus, $a+c = (2k-b)+(2\ell-b)$

$= 2k + 2\ell - 2b$

$= 2(k+\ell-b)$

$= 2j$ where $j = k+\ell-b$ so $j \in \mathbb{Z}$

$\therefore$ $a+c$ is even.

$\therefore$ $a\mathcal{R}_2 c$ (by $\mathcal{R}_2$'s rule) $\therefore$ $\mathcal{R}_2$ is transitive

[equivalence relation]

To prove $\mathcal{R}_2$ is an equivalence relation on $\mathbb{Z}$, we must prove that $\mathcal{R}_2$ is reflexive, symmetric, and transitive ← we did prove these

✓ ✓ ✓

$\therefore$ $\mathcal{R}_2$ is an equivalence relation on $\mathbb{Z}$.

Study Guide

Important terms and concepts:

☐ relation from A to B $\mathcal{R} \subseteq A \times B$	☐ relation on A $\mathcal{R} \subseteq A \times A$	x is related to y by $\mathcal{R}$ $(x,y) \in \mathcal{R}$ $\quad x \,\mathcal{R}\, y$
important relations on $\mathbb{Z}$ $= \quad \leq \quad < \quad \mid$	**properties of relations:**	reflexive symmetric antisymmetric transitive

Exercises — Sup.Ex. §6 # 1, 2, 3, 4, 5, 6, 7, 9; Rosen §9.1 # 1, 3, 5, 6, 7, 8, 9, 10, 44, 45, 46, 47a