`dmdd` Rate Calculators

The `rate_UV`, `rate_NR`, and `rate_genNR` submodules of `dmdd` each allow calculation of differential and total rates using, respectively, the functions `dRdQ` (with units counts/kg/sec/keV) and `R` (with units counts/kg/sec):

- `dmdd.rate_UV` describes the scattering controlled by a few Lorentz-invariant contact operators
- `dmdd.rate_NR` describes the scattering for arbitrary coefficients in front of the nuclear responses allowed by the appropriate effective field theory
- `dmdd.rate_genNR` describes the scattering for a completely generic coefficients of the full set of nonrelativistic contact operators

These rate calculators differ in the types of cross sections they handle. We describe the physics of these rate calculators here. **We assume throughout that the dark matter is spin-1/2.** We refer to the rest of the documentation for further details on the `dmdd` package.

Before describing how the cross sections are handled by the code, we first mention that all rate calculations described below require, in addition to particle physics cross sections, an assumption about astrophysical parameters. The rate is defined as:

$$\frac{dR}{dE_R}(E_R) = \frac{\rho_\chi}{m_T m_\chi} \int\limits_{v_{\min}}^{v_{\rm esc,lab}} v f_\oplus(\mathbf{v}) \frac{d\sigma_T}{dE_R}(E_R, v) d^3v. \tag{1}$$

We assume that the terrestrial dark matter velocity distribution $f_\oplus(\mathbf{v})$ is related to a Galactic velocity distribution $f(\mathbf{v})$ that is described as a Maxwell-Boltzmann distribution with a smooth velocity cut-off,

$$f(\mathbf{v}) \propto \left[\exp\left(-\frac{\mathbf{v}^2 - v_0^2}{v_{\rm rms}^2}\right) - \exp\left(-\frac{v_{\rm esc}^2 - v_0^2}{v_{\rm rms}^2}\right)\right]. \tag{2}$$

Furthermore, we generically find two classes of scattering cross sections,

$$\frac{d\sigma_T}{dE_R} \propto \left\{ \begin{array}{ll} v^{-2} & \text{standard scattering} \\ v^0 & \text{velocity-suppressed scattering} \end{array} \right. . \tag{3}$$

With these assumptions, the integral in Eq. 1 is analytic, and is denoted $\eta(v_{\min})$ or $\zeta(v_{\min})$ in the standard or velocity-suppressed scenario, respectively. We use the velocity integrals performed in the Appendix of [1].

$\boxed{\texttt{rate\_UV}}$ allows calculation of rates for a variety of Lorentz-invariant models of contact interactions between dark matter and the Standard Model. Each Lagrangian has an overall mass dimension negative two normalization as a free parameter; following conventional practice, we define this parameter to be the cross section for scattering off of a proton, $\sigma^p_{\mathbf{M}}$, except in the case of spin-dependent scattering off neutrons, in which case $\sigma^n_{\tt sd}$ or $\sigma^p_{\tt sd}$ are both valid selections. Each model **M** only contributes to the rate when `sigma_`[[**M**]] is set to be nonzero.

Some models have a second free parameter that sets the ratio of the dark matter couplings to neutrons and protons. If the ratio of the nucleon couplings is a free parameter, this is set by

Table 1: Models available to dmdd.rate_UV. The name of the model $\mathbf{M}$ recognized by dmdd.rate_UV is given in the first column, followed by its more conventional name in the second column. If f_n/f_p is a free parameter, we put a checkmark in the third column. The normalization of $\sigma^p_{\mathbf{M}}$ in the massive and massless mediator cases is given in the last two columns.

$\mathbf{M}$	Colloquial Name	f_n/f_p free?	σ_{massive}	σ_{massless}
si	canonical spin-independent	✓	10^{-47} cm^2	10^{-48} cm^2
sd	canonical spin-dependent	✓	10^{-42} cm^2	10^{-43} cm^2
sd_neutron	canonical spin-dependent (n-only)	✓	10^{-42} cm^2	10^{-43} cm^2
anapole	anapole moment	–	10^{-40} cm^2	10^{-45} cm^2
magdip	magnetic dipole moment	–	10^{-40} cm^2	10^{-41} cm^2
elecdip	electric dipole moment	–	10^{-46} cm^2	10^{-47} cm^2
LS	"$\vec{L}\cdot\vec{S}$" generating	✓	10^{-44} cm^2	10^{-42} cm^2
f1	pseudoscalar-scalar (DM-SM)	✓	10^{-47} cm^2	10^{-48} cm^2
f2	scalar-pseudoscalar (DM-SM)	✓	10^{-42} cm^2	10^{-43} cm^2
f3	pseudoscalar-pseudoscalar	✓	10^{-41} cm^2	10^{-42} cm^2

fnfp_[[$\mathbf{M}$]]. Each model may have _massless appended to it to calculate the same rate assuming a massless mediator. The list of models currently incorporated are summarized in Table 1, along with the overall normalization that sets the scale of the cross section $\sigma^p_{\mathbf{M}}$. These UV-inspired models are calculated in [2, 3].

$\boxed{\texttt{rate\_NR}}$ calculates rates for **isolated nuclear responses**, which we label by the index $\mathbf{H}$. We include momentum and velocity dependence up to second order in the effective field theory expansion parameters v^2 and $\vec{q}^{\,2}/m_N^2$. Making some standard substitutions in Eq. 1, the output of dmdd.rate_NR.dRdQ on a target nucleus T is [3, 4]

$$\begin{aligned}\frac{dR_T}{dE_R} = \sum_{\mathbf{H}} \frac{dR_{T|\mathbf{H}}}{dE_R} &= \sum_{\mathbf{H}} \sum_{N,N'} \frac{\rho_\chi}{m_T m_\chi} \int\limits_{v_{\min}}^{v_{\text{esc,lab}}} d^3v \frac{f_\oplus(v)}{v} \left(\frac{\vec{q}^{\,2}}{m_N^2}\right)^{\beta(\mathbf{H})} \frac{d\sigma_{\mathbf{H}}}{dE_R} R_{\mathbf{H}}(v^2, \vec{q}^{\,2}, \dots) \widetilde{W}^{(N,N')}_{\mathbf{H}|T}(\vec{q}^{\,2}) \\ &= \frac{\rho_\chi \sigma^p_{\mathbf{H}}}{2\mu_T^2 m_\chi} \sum_{\mathbf{H}} \Bigg[\left(R^{(0)}_{\mathbf{H}} + \frac{\vec{q}^{\,2}}{m_N^2} R^{(q^2)}_{\mathbf{H}} + \frac{\vec{q}^{\,4}}{m_N^4} R^{(q^4)}_{\mathbf{H}} \right) \eta(v_{\min}) + \\ &\qquad + \left(R^{(v^2)}_{\mathbf{H}} + \frac{\vec{q}^{\,2}}{m_N^2} R^{(v^2 q^2)}_{\mathbf{H}} \right) \zeta(v_{\min}) \Bigg] \sum_{N,N'} \widetilde{W}^{(N,N')}_{\mathbf{H}|T}(\vec{q}^{\,2})/f_p^2,\end{aligned} \tag{4}$$

where E_R is the nuclear recoil energy; ρ_χ and m_χ are the local dark matter density and mass; v is the relative speed between the dark matter particle and the detector; $f_\oplus(v)$ is the local speed distribution, obtained from Eq. 2; N, N' each represent a neutron or a proton; $\beta(\mathbf{H}) = 0\,(1)$ if the response $\mathbf{H}$ is "standard" ("novel"); and in the second line we factor out the coupling of dark matter to the proton, f_p, from all of the response functions. The functions η and ζ are the integrals of the velocity distribution defined below Eq. 3. The response functions $R^{(\dots)}_{\mathbf{H}}$ are dimensionless coefficients that in principle are entirely free. The functions $\widetilde{W}^{(N,N')}_{X|T}$ are fixed upon choosing N, N', T, and E_R. The overall normalization $\sigma^p_{\mathbf{H}}$ is free, just as $\sigma^p_{\mathbf{M}}$ is free in rate_UV.

Each response is turned on or off with sigma_[[$\mathbf{H}$]], all of which have units of 10^{-47} cm^2. The ratio of proton to neutron couplings can be set independently for each response by specifying fnfp_[[$\mathbf{H}$]]. The $R^{(\dots)}_{\mathbf{H}}$ are set as follows. The three standard responses have "standard coefficients" with no velocity- or momentum-dependence that are invoked by `stdco_[[H]]`. All responses have higher-order coefficients called by `v2co_[[H]]`, `q2co_[[H]]`, `v2q2co_[[H]]`, and `q4co_[[H]]`. By default, the leading coefficient for each response is set to be nonzero and the rest are set to vanish.

We use the seven responses compatible with the symmetries of the EFT that can be UV completed (including two interference terms) [4]. The three "standard" responses are:

- the M nuclear response, controlled by `sigma_M`, `fnfp_M`, etc. The leading coefficient corresponding to $R_M^{(0)}$ is `stdco_M`. This response controls canonical spin-independent scattering.
- the Σ' nuclear response, controlled by `sigma_SigP`, `fnfp_SigP`, etc. The leading coefficient corresponding to $R_{\Sigma'}^{(0)}$ is `stdco_SigP`. This response partially contributes to the canonical spin-dependent scattering.
- the Σ'' nuclear response, controlled by `sigma_SigPP`, `fnfp_SigPP`, etc. The leading coefficient corresponding to $R_{\Sigma''}^{(0)}$ is `stdco_SigPP`. This response partially contributes to the canonical spin-dependent scattering.

Four novel responses are also compatible with the symmetries of direct detection scattering. These are

- the Φ'' nuclear response, controlled by `sigma_PhiPP`, `fnfp_PhiPP`, etc. Because `stdco_PhiPP` does not exist, the leading coefficient corresponding to $R_{\Phi''}^{(q^2)}$ is `q2co_PhiPP`.
- the Δ nuclear response, controlled by `sigma_Delta`, `fnfp_Delta`, etc. Because `stdco_Delta` does not exist, the leading coefficient corresponding to $R_{\Delta}^{(q^2)}$ is `q2co_Delta`.
- the $M - \Phi''$ interference term, controlled by `sigma_MPhiPP`, `fnfp_MPhiPP`, etc. Because `stdco_MPhiPP` does not exist, the leading coefficient corresponding to $R_{M-\Phi''}^{(q^2)}$ is `q2co_MPhiPP`. For certain choices of the coefficients for the coefficients of the M and Φ'' responses, this interference term may be necessary to ensure a positive rate.
- the $\Sigma' - \Delta$ interference term, controlled by `sigma_SigPDelta`, `fnfp_SigPDelta`, etc. Because `stdco_SigPDelta` does not exist, the leading coefficient corresponding to $R_{\Delta-\Sigma'}^{(q^2)}$ is `q2co_SigPDelta`. For certain choices of the coefficients for the coefficients of the Σ' and Δ responses, this interference term may be necessary to ensure a positive rate.

It is possible that a rate calculated using `dmdd.rate_NR.dRdQ` will return negative values: this is a signal that the coefficients chosen are **unphysical**. Because the coefficients that enter the rate are chosen by the user, arbitrary choices of the parameters in `rate_NR` **may** provide meaningless rates.

$\boxed{\texttt{rate\_genNR}}$ allows calculations of rates for a nonrelativistic Lagrangian with arbitrary coefficients. `dmdd.rate_genNR.dRdQ` and `dmdd.rate_genNR.R` *automatically* calculate the correct momentum and velocity dependence *in the rate* up to second order in v^2 and $\vec{q}^2/m_N^2$. We use the naming convention of [5] for the dimensionful coefficients, so there are twenty-eight free parameters characterizing these coefficients. To allow for novel momentum dependence *in the coefficients*, each coefficient must be entered as a `numpy` array. The (Pythonically numbered) n^{th} element of this array multiplies $\left(\vec{q}^2/m_{\text{DM}}^2\right)^n$, so the first (second) [third] array elements correspond to terms in the calculation that carry no momentum-dependence (multiply $\vec{q}^2/m_{\text{DM}}^2$) [multiply $\vec{q}^4/m_{\text{DM}}^4$]. Thus, for the nonrelativistic Lagrangian

$$\mathcal{L}_{\rm NR} = \sum_{X\in\{1,3,4,5,...15\}} \sum_{N=p,n} c_{XN}\mathcal{O}_{XN} \tag{5}$$

$$= \bar{c} \sum_{X\in\{1,3,4,5,...15\}} \sum_{N=p,n} \left[\tilde{c}^{(0)}_{XN} + \frac{\vec{q}^{\,2}}{m_{\rm DM}^2}\tilde{c}^{(2)}_{XN} + \frac{\vec{q}^{\,4}}{m_{\rm DM}^4}\tilde{c}^{(4)}_{XN}\right]\mathcal{O}_{XN} \tag{6}$$

the corresponding free parameters are

- `c_scale` $\equiv 1/\sqrt{\bar{c}}$, which sets the order of magnitude for the cross section. Because $\mathcal{O}_{XN}$ is a dimension six contact operator, $\bar{c}$ is of mass dimension negative two. In `rate_genNR`, the corresponding free parameter is a **mass scale** that has units GeV and is set to 500 by default. There is *only one scale per rate.*

- `cXN=np.array([c0XN,c2XN,c4XN])`, where `X` $\in \{1,3,4,5,\dots 15\}$ and `N` $= p, n$. The first, second, and third elements, respectively, are numbers that multiply 1, $\vec{q}^{\,2}/m_{\rm DM}^2$, and $\vec{q}^{\,4}/m_{\rm DM}^4$, corresponding to $\tilde{c}^{(0)}_{XN}$, $\tilde{c}^{(2)}_{XN}$, and $\tilde{c}^{(4)}_{XN}$ in Eq. 6. Thus, there are 28 independent `cXN` parameters that can contain up to 84 dimensionless coefficients. However, because some of the coefficients only enter the responses at higher order in the EFT expansions parameters, some of these 84 coefficients will not be called: for example, c_{14} enters the responses accompanied by factors $v^2\vec{q}^{\,2}/m_N^2$ and $\vec{q}^{\,4}/m_N^4$ [5], so only the first element of `c14N` is used by the rate calculator. Nonetheless, `c14n` and `c14p` must be entered as numpy arrays with at least one entry.

Because these coefficients are combined inside the rate calculator, it should be impossible to get a negative rate. From the UV perspective, completely generic values for the couplings may be attainable only by exquisite fine-tunings between Lorentz-invariant coefficients. This may be unnatural, but from the low-energy point of view it is not *a priori* unphysical.

The rate calculator accepts the lists of EFT coefficients and evaluates the rate numerically. To find a parametric form of the cross section for a given set of EFT coefficients, we refer to the discussion in [5].

$\boxed{\textbf{Example}}$ We show three ways to calculate the rate given by the $\vec{L}\cdot\vec{S}$ operator [3] scattering off of an iodine target at 0.1, 1, and 10 keVNR, taking the coupling to neutrons to be ten times greater than the coupling to protons:

- ```
  dmdd.rate_UV.dRdQ(np.array([.1,1.,10.]), sigma_LS=1.,
  fnfp_LS=10., element='iodine')
  ```

- ```
  dmdd.rate_NR.dRdQ(np.array([.1,1.,10.]), sigma_M=8.8036e4,
  fnfp_M=10., stdco_M=0., q4co_M=(0.1/0.938272)**4.,
  sigma_MPhiPP=8.8036e4, fnfp_MPhiPP=10.,
  q2co_MPhiPP=0., q4co_MPhiPP=4.*(0.1/0.938272)**4.,
  sigma_PhiPP=8.8036e4, fnfp_PhiPP=10., q2co_PhiPP=0.,
  q4co_PhiPP=4.*(0.1/0.938272)**4., sigma_SigP=8.8036e4,
  fnfp_SigP=10., stdco_SigP=0., v2q2co_SigP=2.*0.1**2/0.938272**2.,
  q4co_SigP=0.1**4./(50.**2.*0.938272**2.)  -
  0.5*0.1**4./0.938272**2.* (dmdd.constants.ELEMENT_INFO['iodine']['weight']
  *0.938272 +50.)**2./ (dmdd.constants.ELEMENT_INFO['iodine']['weight']*0.938272
  *50.)**2., element='iodine')
  ```

- `dmdd.rate_genNR.dRdQ(np.array([.1,1.,10.]),`
  `c1p=np.array([0.,-(50./0.938272)**2./4.,0.]),`
  `c1n=np.array([0.,-10.*(50./0.938272)**2./4.,0.]),`
  `c3p=np.array([-1.,0.,0.]), c3n=np.array([-10.,0.,0.]),`
  `c4p=np.array([0.,50./0.938272,0.]), c4n=np.array([0.,10.*50./0.938272,0.]),`
  `c6p=np.array([-0.938272/50.,0.,0.]), c6n=np.array([-10.*0.938272/50.,0.,0.]),`
  `c_scale=92.933, element='iodine')`

using the calculators in `rate_UV`, `rate_NR`, and `rate_genNR`, respectively.

References

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[2] V. Gluscevic, M. I. Gresham, S. D. McDermott, A. H. G. Peter and K. M. Zurek, [arXiv:1506.04454 [hep-ph]].

[3] M. I. Gresham and K. M. Zurek, Phys. Rev. D **89**, no. 12, 123521 (2014) [arXiv:1401.3739 [hep-ph]].

[4] A. L. Fitzpatrick, W. Haxton, E. Katz, N. Lubbers and Y. Xu, JCAP **1302**, 004 (2013) [arXiv:1203.3542 [hep-ph]].

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