MATH324 Crib Sheet

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1 Properties of Estimators and Statistics

1.1 Biasedness

An estimator $\hat{\theta}$ is biased if $\mathbb{E}(\hat{\theta}) \neq \theta$.
Example:
$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is biased if X_i are i.i.d. with mean μ and variance σ^2.

$$\mathbb{E}(\hat{\theta}) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i) = \frac{1}{n}n\mu = \mu \tag{1}$$

The bias is determined by the equation

$$\mathbb{B}(\hat{\theta}) = \mathbb{E}(\hat{\theta}) - \theta \tag{2}$$

We can generally find an unbiased estimator from a biased estimator by eliminating the constants surrounding the estimator, such that $\mathbb{E}(\hat{\theta}) \to \theta$.

1.2 Consistency

An estimator $\hat{\theta}_n$ is consistent if $\hat{\theta}$ converges in probability to θ as $n \to \infty$.

$$\lim_{n\to\infty} P(|\hat{\theta}_n - \theta| > \epsilon) = 0 \tag{3}$$

$$\lim_{n\to\infty} P(|\hat{\theta}_n - \theta| \leq \epsilon) = 1 \tag{4}$$

This is equivalent to the following:

$$\lim_{n\to\infty} \mathbb{V}(\hat{\theta_n}) = 0 \tag{5}$$

1.3 Asymptotic Normality

An estimator $\hat{\theta}$ is asymptotically normal if $\hat{\theta}$ converges in distribution to a normal distribution as $n \to \infty$.
Example:
$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is asymptotically normal if X_i are independent and identically distributed (i.i.d.) with mean μ and variance σ^2.

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \tag{6}$$

$$\frac{\bar{X} - \mu}{\sqrt{\frac{\sigma^2}{n}}} \sim N(0, 1) \tag{7}$$

1.4 Efficiency

An estimator $\hat{\theta}$ is efficient if $\hat{\theta}$ has the smallest variance among all unbiased estimators of θ.

1.4.1 The Rao Blackwell Theorem

Let $\hat{\theta}$ be an unbiased estimator of θ such that $\mathbb{V}(\hat{\theta}) < \infty$. If U is a sufficient statistic for θ, define $\hat{\theta}^* = \mathbb{E}(\hat{\theta}|U)$. Then $\forall\theta$:

$$\mathbb{E}(\hat{\theta}^*) = \theta \quad \text{and} \quad \mathbb{V}(\hat{\theta}^*) \leq \mathbb{V}(\hat{\theta})$$

Remark: The result of the Rao Blackwell Theorem is the *minimum-variance unbiased estimator* of θ. (MVUE)

1.5 Sufficiency

A statistic T is sufficient for θ if T contains all the information about θ. This implies that θ can be uniquely determined from an estimator based on T without any loss of information.
Example:
$T = \bar{X}$ is sufficient for μ if X_i are i.i.d. with mean μ and variance σ^2.

2 Hypothesis Testing

2.1 The T-test

2.1.1 Large-Sample Hypothesis Testing

2.1.2 Small-Sample Hypothesis Lemma

2.2 The Likelihood Ratio Test

2.2.1 The Likelihood Ratio Test for a Single Parameter

The Neyman-Pearson Lemma

2.2.2 The Likelihood Ratio Test for Multiple Parameters

3 Linear Regression

3.1 Parameters of a Linear Model

3.2 The Least Squares Estimator

3.3 The correlation coefficient

3.4 Hypothesis Testing for Linear Regression

3.4.1 The T-test

3.4.2 The F-test