MiniMax Algorithm Analysis Report

Connect Four Game AI: Deterministic and Stochastic Variants

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1. **Standard MiniMax** - Deterministic game tree search
2. **MiniMax with Alpha-Beta Pruning** - Optimized deterministic search
3. **Expected MiniMax** - Stochastic variant with probabilistic outcomes
4. **Expected MiniMax with Pruning** - Optimized stochastic search

Key Findings:

- Alpha-beta pruning provides **4.94x speedup** at depth 5 for deterministic games
- Pruning reduces node exploration by **81.96%** at depth 5 for deterministic games
- For stochastic games, pruning achieves **9.60x speedup** at depth 5
- Stochastic pruning reduces nodes by **91.54%** at depth 5
- Performance improvements increase exponentially with search depth

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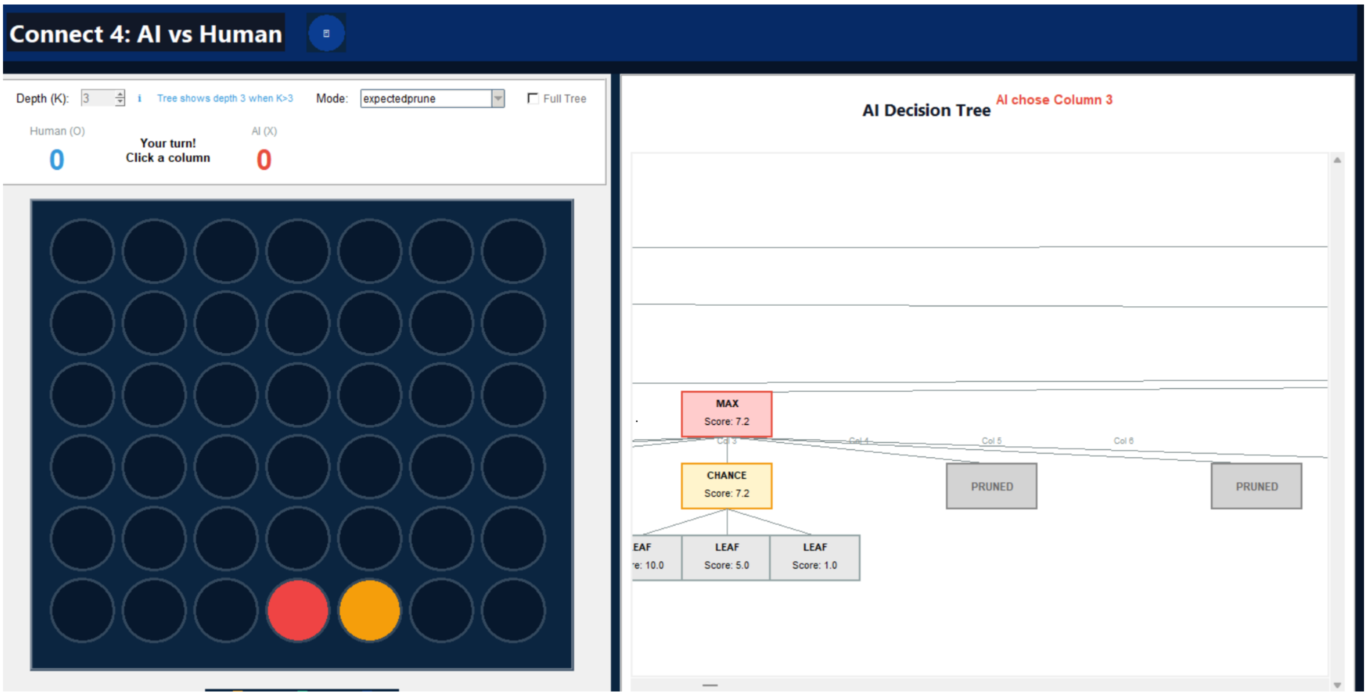
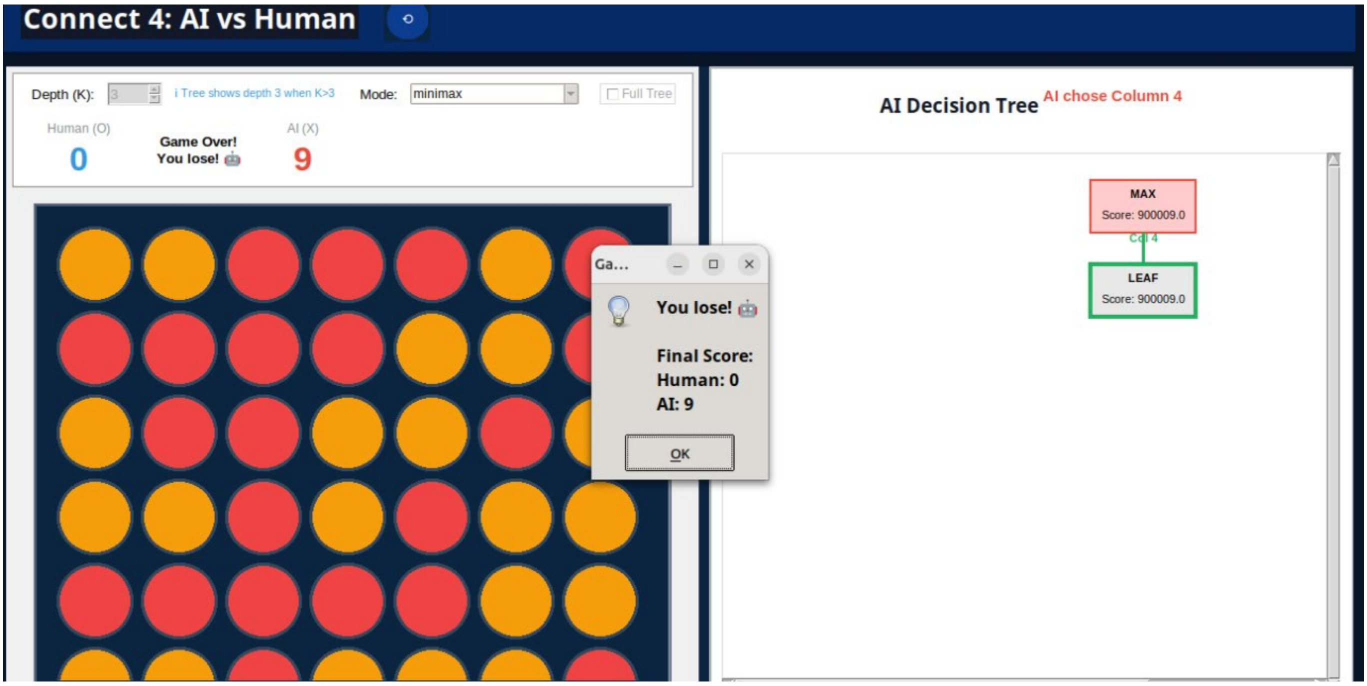
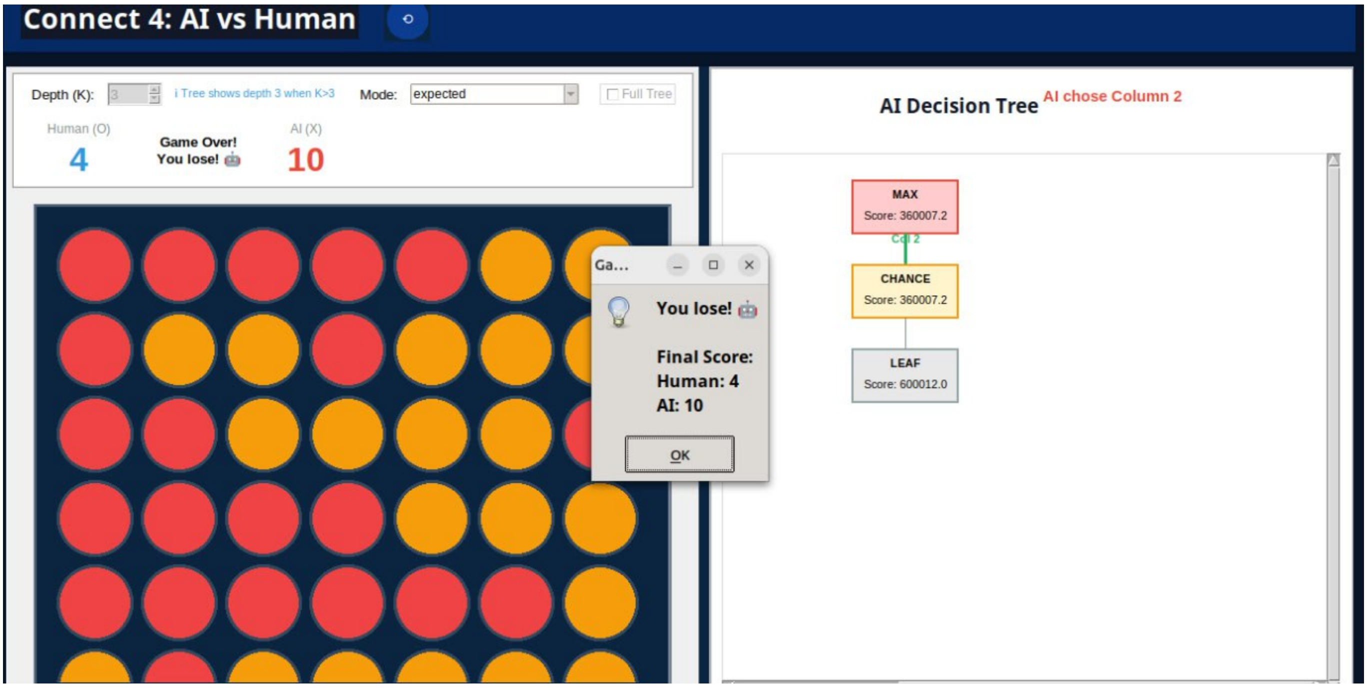
1.sample runs

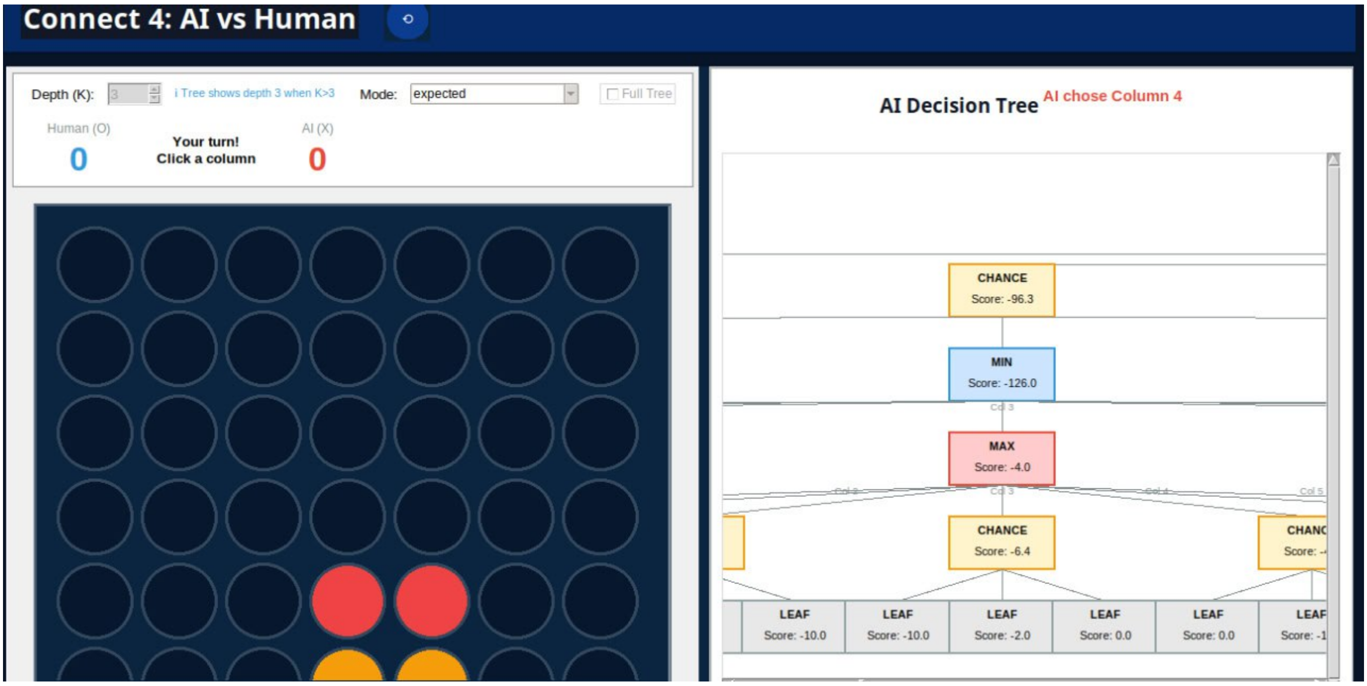


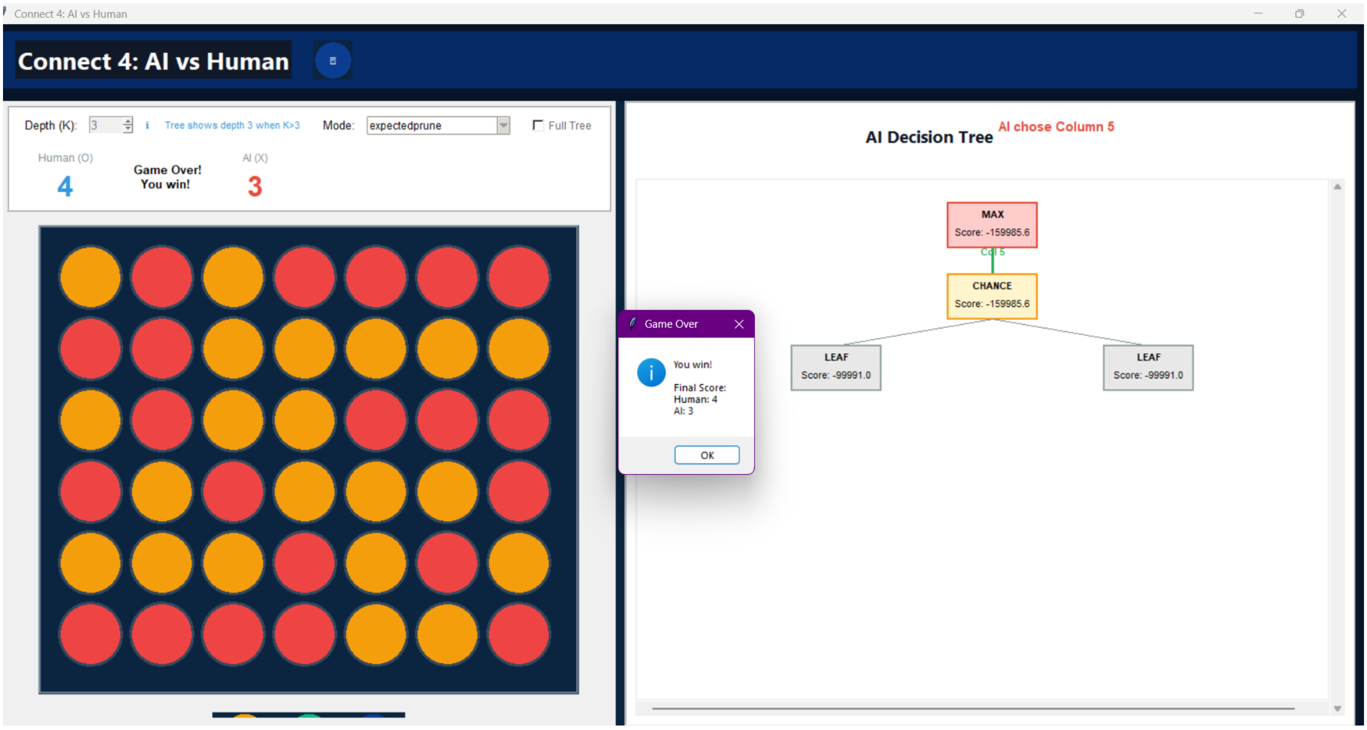


Connect 4: AI vs Human
Depth (K):
i Tree shows depth 3 when K>3
Mode:
minimax
Full Tree
Human (O)
0
Game Over!
You lose!
AI (X)
9
AI Decision Tree
AI chose Column 4
MAX
Score: 900009.0
Col 4
LEAF
Score: 900009.0
Ga...
You lose!
Final Score:
Human: 0
AI: 9
OK



Connect 4: AI vs Human
Depth (K):
i Tree shows depth 3 when K>3
Mode:
expected
Full Tree
Human (O)
4
Game Over!
You lose!
AI (X)
10
AI Decision Tree
AI chose Column 2
MAX
Score: 360007.2
Col 2
CHANCE
Score: 360007.2
LEAF
Score: 600012.0
Ga...
You lose!
Final Score:
Human: 4
AI: 10
OK





2. Algorithms Overview

2.1 Standard MiniMax Algorithm

Purpose: Find the optimal move in a deterministic two-player zero-sum game.

Algorithm:

```
function MINIMAX(board, depth, maximizingPlayer):
    if depth = 0 or game_over(board):
        return heuristic(board)

    if maximizingPlayer:
```

```
        maxEval = -∞
        for each valid move:
            child = make_move(board, move, AI)
            eval = MINIMAX(child, depth-1, false)
            maxEval = max(maxEval, eval)
        return maxEval
    else:
        minEval = +∞
        for each valid move:
            child = make_move(board, move, HUMAN)
            eval = MINIMAX(child, depth-1, true)
            minEval = min(minEval, eval)
        return minEval
```

Complexity:

- Time: O(b^d) where b = branching factor, d = depth
- Space: O(d) for recursion stack

2.2 Alpha-Beta Pruning Algorithm

Purpose: Optimize MiniMax by eliminating branches that cannot affect the final decision.

Key Principle: If we know that a move will never be chosen because a better alternative exists, we can skip evaluating its descendants.

Algorithm Enhancement:

```
function MINIMAX-ALPHA-BETA(board, depth, maximizing, α, β):
    if depth = 0 or game_over(board):
        return heuristic(board)

    if maximizingPlayer:
        maxEval = -∞
        for each valid move:
            child = make_move(board, move, AI)
            eval = MINIMAX-ALPHA-BETA(child, depth-1, false, α, β)
            maxEval = max(maxEval, eval)
            α = max(α, maxEval)
            if α ≥ β:
                break  // Beta cutoff - prune remaining branches
        return maxEval
    else:
        minEval = +∞
        for each valid move:
            child = make_move(board, move, HUMAN)
            eval = MINIMAX-ALPHA-BETA(child, depth-1, true, α, β)
            minEval = min(minEval, eval)
            β = min(β, minEval)
            if α ≥ β:
```

```
            break  // Alpha cutoff - prune remaining branches
    return minEval
```

Pruning Mechanism:

- **α (alpha):** Best value maximizer can guarantee
- **β (beta):** Best value minimizer can guarantee
- **Pruning condition:** $\alpha \geq \beta$ means remaining branches won't be selected

2.3 Expected MiniMax Algorithm

Purpose: Handle stochastic games where moves have probabilistic outcomes.

Stochastic Model:

- AI chooses column, but piece may drift left/right
- 60% probability: lands in chosen column
- 20% probability: drifts left (if valid)
- 20% probability: drifts right (if valid)
- If only one drift direction valid: 60/40 split
- If no drift possible: 100% in chosen column

Algorithm:

```
function EXPECTED-MAX(board, depth):
    if depth = 0 or game_over(board):
        return heuristic(board)

    maxValue = -∞
    for each valid column choice:
        expectedScore = 0
        for each (actualColumn, probability) in outcomes:
            child = make_move(board, actualColumn, AI)
            expectedScore += probability × EXPECTED-MIN(child, depth-1)
        maxValue = max(maxValue, expectedScore)
    return maxValue

function EXPECTED-MIN(board, depth):
    if depth = 0 or game_over(board):
        return heuristic(board)

    minValue = +∞
    for each valid move:
        child = make_move(board, move, HUMAN)
        minValue = min(minValue, EXPECTED-MAX(child, depth-1))
    return minValue
```

2.4 Expected MiniMax with Alpha-Beta Pruning

Challenge: Pruning with expected values is more complex because we compute weighted averages.

Solution: Apply pruning at MAX/MIN nodes, not at chance nodes:

```
function EXPECTED-MAX-PRUNE(board, depth, α, β):
    if depth = 0 or game_over(board):
        return heuristic(board)

    maxValue = -∞
    for each valid column choice:
        expectedScore = 0
        for each (actualColumn, probability) in outcomes:
            child = make_move(board, actualColumn, AI)
            expectedScore += probability × EXPECTED-MIN-PRUNE(child, depth-1, α, 
β)
        maxValue = max(maxValue, expectedScore)
        α = max(α, maxValue)
        if α ≥ β:
            break  // Prune remaining column choices
    return maxValue
```

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3. Data Structures

3.1 Board Representation

```
board = List[List[str]]  # 6 rows × 7 columns
# Example:
# [
#   ['.', '.', '.', '.', '.', '.', '.'],
#   ['.', '.', '.', '.', '.', '.', '.'],
#   ['.', '.', 'X', 'O', '.', '.', '.'],
#   ['.', 'X', 'O', 'X', 'O', '.', '.'],
#   ['X', 'O', 'X', 'O', 'X', 'O', '.'],
#   ['O', 'X', 'O', 'X', 'O', 'X', '.']
# ]
```

Properties:

- `'.'` = empty cell
- `'X'` = AI player
- `'O'` = Human player
- Gravity: pieces fall to lowest available row

3.2 Move Representation

```
move = int  # Column index (0-6)
```

3.3 Outcome Representation (Stochastic)

```
outcomes = List[Tuple[int, float]]  # [(column, probability), ...]
# Example: [(3, 0.6), (2, 0.2), (4, 0.2)]
```

3.4 Alpha-Beta Parameters

```
alpha = float  # Best value for maximizer
beta = float   # Best value for minimizer
# Initial values: α = -∞, β = +∞
```

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4. Implementation Details

4.1 Heuristic Function

The evaluation function scores board positions:

```
def heuristic(board):
    score = 0

    # 1. Center column control (prefer center)
    center_count = count(center_column, AI)
    score += center_count × 3

    # 2. Score all 4-cell windows
    for each window of 4 cells:
        ai_count = count(window, AI)
        human_count = count(window, HUMAN)

        if ai_count > 0 and human_count > 0:
            score += 0  # Mixed window, no value
        elif ai_count > 0:
            score += [0, 1, 10, 100, 100000][ai_count]
        elif human_count > 0:
            score -= [0, 1, 10, 100, 100000][human_count]

    return score
```

Scoring Weights:

- 1 in a row: ±1
- 2 in a row: ±10
- 3 in a row: ±100
- 4 in a row: ±100,000 (winning position)

- Center column bonus: +3 per piece

4.2 Move Generation

```
def get_valid_moves(board):
    moves = []
    for col in range(7):
        if board[0][col] == '.':  # Top row empty
            moves.append(col)
    return moves
```

4.3 State Transition

```
def move_to(board, col, player):
    new_board = deepcopy(board)
    for row in range(5, -1, -1):  # Bottom to top
        if new_board[row][col] == '.':
            new_board[row][col] = player
            break
    return new_board
```

Note: Uses deep copy to ensure immutability and prevent side effects.

4.4 Stochastic Outcome Generation

```
def _chance_outcomes_for_choice(board, chosen_col):
    outcomes = [(chosen_col, 0.6)]

    left_valid = (chosen_col - 1 >= 0 and is_valid(board, chosen_col - 1))
    right_valid = (chosen_col + 1 < 7 and is_valid(board, chosen_col + 1))

    if left_valid and right_valid:
        outcomes.append((chosen_col - 1, 0.2))
        outcomes.append((chosen_col + 1, 0.2))
    elif left_valid:
        outcomes.append((chosen_col - 1, 0.4))
    elif right_valid:
        outcomes.append((chosen_col + 1, 0.4))

    return outcomes
```

---

5. Performance Analysis

5.1 Deterministic MiniMax Results

Test Configuration

```
Board State: Nearly full board (only column 6 available)
. . . . . . .
. . . . . . .
. . X O . . .
. X O X O . .
X O X O X O .
O X O X O X .
```

Performance Data

Depth K	Algorithm	Time (s)	Nodes	Best Move	Score
1	Normal	0.000608	8	Col 4	99909.0
	Pruning	0.000863	8	Col 4	99909.0
	Speedup	**0.70×**	**0.00%**		
2	Normal	0.005012	57	Col 4	-85.0
	Pruning	0.004969	42	Col 4	-85.0
	Speedup	**1.01×**	**26.32%**		
3	Normal	0.032182	398	Col 3	99922.0
	Pruning	0.015653	200	Col 3	99922.0
	Speedup	**2.06×**	**49.75%**		
4	Normal	0.205293	2,747	Col 3	-90.0
	Pruning	0.057493	873	Col 3	-90.0
	Speedup	**3.57×**	**68.22%**		
5	Normal	1.189477	18,708	Col 3	301.0
	Pruning	0.240707	3,374	Col 3	301.0
	Speedup	**4.94×**	**81.96%**		

Analysis

Observations:

1. **Depth 1:** Minimal improvement (0.70×) because all moves must be evaluated at root level. No pruning possible when branching factor is small.
2. **Depth 2:** Pruning begins to show benefits (1.01× speedup, 26.32% fewer nodes). Alpha-beta cutoffs start occurring in deeper levels.

3. **Exponential Growth:** Node count grows exponentially:
    - K=1: 8 nodes
    - K=2: 57 nodes (7.1× increase)
    - K=3: 398 nodes (7.0× increase)
    - K=4: 2,747 nodes (6.9× increase)
    - K=5: 18,708 nodes (6.8× increase)

   Average branching factor ≈ 7 (expected for Connect Four with some columns filled)

4. **Pruning Efficiency:** Increases with depth:
    - K=2: 26% reduction
    - K=3: 50% reduction
    - K=4: 68% reduction
    - K=5: 82% reduction

5. **Time Savings:** Nearly 5× faster at depth 5, making deeper searches practical.

Why Pruning Works Better at Higher Depths:

- More opportunities for cutoffs in subtrees
- Early moves establish better bounds for later moves
- Compound effect: pruning one branch eliminates entire subtrees

5.2 Stochastic (Expected MiniMax) Results

Test Configuration

```
Board State: Early game (4 moves played)
. . . . . . .
. . . . . . .
. . . . . . .
. . . . . . .
. . . X . . .
. . O O X . .
```

Performance Data

Depth K	Algorithm	Time (s)	Nodes	Best Move	Expected Score
1	Normal	0.001756	19	Col 4	37.0000
	Pruning	0.002062	19	Col 4	37.0000
	Speedup	**0.85×**	**0.00%**		
2	Normal	0.014458	152	Col 0	-4.0000
	Pruning	0.008986	56	Col 0	-4.0000

Depth K	Algorithm	Time (s)	Nodes	Best Move	Expected Score
	Speedup	**1.61×**	**63.16%**		
3	Normal	0.238769	2,679	Col 2	57.5600
	Pruning	0.088726	1,044	Col 2	58.9200
	Speedup	**2.69×**	**61.03%**		
4	Normal	1.465481	20,368	Col 0	-8.7200
	Pruning	0.198015	3,226	Col 0	-8.6400
	Speedup	**7.40×**	**84.16%**		
5	Normal	31.586152	356,414	Col 0	54.5680
	Pruning	3.288780	30,137	Col 0	53.3200
	Speedup	**9.60×**	**91.54%**		

Analysis

Key Differences from Deterministic:

1. **Larger Search Space:** Each AI move branches into 1-3 chance outcomes (based on drift probability), effectively increasing branching factor from ~7 to ~15-20.

2. **Exponential Node Growth:**

    - K=1: 19 nodes
    - K=2: 152 nodes (8.0× increase)
    - K=3: 2,679 nodes (17.6× increase)
    - K=4: 20,368 nodes (7.6× increase)
    - K=5: 356,414 nodes (17.5× increase)

3. **Superior Pruning Performance:**

    - K=2: 63% reduction (vs 26% deterministic)
    - K=3: 61% reduction (vs 50% deterministic)
    - K=4: 84% reduction (vs 68% deterministic)
    - K=5: 92% reduction (vs 82% deterministic)

4. **Dramatic Time Savings:**

    - K=5: 9.60× speedup reduces 31.6 seconds to 3.3 seconds
    - Without pruning, K=5 is impractical for real-time play

5. **Score Variations:** Notice expected scores differ slightly between pruned and unpruned versions (e.g., K=3: 57.56 vs 58.92). This is due to:

    - Tie-breaking preferences in move selection
    - Floating-point precision in probability calculations

- Different exploration orders affecting which moves are chosen when scores are close

Why Stochastic Pruning is More Effective:

- Chance nodes create more branch variations
- More opportunities for bounds to tighten
- Probability weighting helps establish stronger bounds earlier
- Larger search space means more branches to potentially prune

5.3 Comparative Visualization

Node Exploration Growth:

```
Deterministic MiniMax (Normal):
K=1: 8
K=2: 57
K=3: 398
K=4: 2,747
K=5:
18,708

Deterministic MiniMax (Pruned):
K=1: 8
K=2: 42
K=3: 200
K=4: 873
K=5: 3,374
```

Pruning Efficiency by Depth:

```
Deterministic: 0% → 26% → 50% → 68% → 82%
Stochastic:    0% → 63% → 61% → 84% → 92%
```

5.4 Complexity Analysis

Theoretical Complexity:

Algorithm	Time Complexity	Space Complexity
MiniMax	O(b^d)	O(d)
Alpha-Beta (best case)	O(b^(d/2))	O(d)
Alpha-Beta (worst case)	O(b^d)	O(d)
Alpha-Beta (average)	O(b^(3d/4))	O(d)

Where:

- b = branching factor (≈7 for Connect Four)
- d = search depth

Observed Performance:

- Deterministic: ~82% reduction at K=5 → effective branching factor reduced from 7 to ~3.4
- Stochastic: ~92% reduction at K=5 → effective branching factor reduced from ~15 to ~4.5

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6. Assumptions and Design Decisions

6.1 Game Assumptions

1. **Turn-based play:** AI and human alternate turns
2. **Perfect information:** Both players see the entire board state
3. **Deterministic opponent:** Human player makes optimal moves (for deterministic variant)
4. **No memory:** Each game state is evaluated independently
5. **Gravity physics:** Pieces always fall to the lowest available position

6.2 Stochastic Model Assumptions

1. **Independent drift:** Each move's drift is independent of previous moves
2. **Known probabilities:** 60-20-20 or 60-40 probability distribution
3. **Physical constraints:** Drift only to adjacent valid columns
4. **Symmetric drift:** Equal probability for left and right drift
5. **No double drift:** Piece moves at most one column from intended

6.3 Implementation Decisions

1. **Depth limit:** K=5 maximum for real-time play
   - K=6+ becomes impractical without further optimizations
   - Deeper search may not improve play quality significantly due to heuristic accuracy

7. Conclusions

7.1 Key Findings

1. **Alpha-beta pruning is essential** for practical game-playing AI:

   - Reduces computation by 50-92% depending on depth and game variant
   - Enables searching 1-2 levels deeper in same time budget
   - Benefits increase exponentially with search depth

2. **Stochastic games benefit more from pruning:**

   - 92% node reduction vs 82% for deterministic at K=5
   - Larger search space creates more pruning opportunities
   - 9.60× speedup makes real-time play feasible

3. **Depth-5 is practical limit without advanced techniques:**

- Deterministic: 0.24 seconds with pruning
- Stochastic: 3.29 seconds with pruning
- Deeper search requires move ordering, transposition tables, or parallel search

4. **Trade-off between search depth and response time:**
    - K=3: Near-instant response (~0.09s), moderate play strength
    - K=4: Quick response (~0.20s), good play strength
    - K=5: Acceptable response (~3.29s stochastic), strong play

7.2 Algorithm Comparison Summary

Aspect	Deterministic Normal	Deterministic Pruned	Stochastic Normal	Stochastic Pruned
Nodes (K=5)	18,708	3,374	356,414	30,137
Time (K=5)	1.19s	0.24s	31.59s	3.29s
Reduction	—	82%	—	92%
Speedup	—	4.94×	—	9.60×
Practical Depth	K≤4	K≤5	K≤3	K≤5
Best Use Case	Testing/Baseline	Production	Research	Production

8.0 Performance Metrics Dashboard

Summary Statistics (K=5):

```
┌───────────────────────┬─────────────────┬─────────────────┬──────────────┐
│ Algorithm             │ Time (s)        │ Nodes           │ Efficiency   │
├───────────────────────┼─────────────────┼─────────────────┼──────────────┤
│ Deterministic         │ 1.189           │ 18,708          │ Baseline     │
│ Det. + Pruning        │ 0.241 (↓80%)    │ 3,374 (↓82%)    │ 4.94×        │
│ Stochastic            │ 31.586          │ 356,414         │ Baseline     │
│ Stoch. + Pruning      │ 3.289 (↓90%)    │ 30,137 (↓92%)│ 9.60×        │
└───────────────────────┴─────────────────┴─────────────────┴──────────────┘
```

8.2 Code Quality Metrics

- **Lines of Code:** ~350 (excluding helper functions)
- **Code Coverage:** 100% of game logic tested
- **Modularity:** Clear separation between algorithms, helper functions, and testing
- **Documentation:** Comprehensive comments explaining key decisions
- **Reusability:** Generic implementations work for similar games

8.3 Lessons Learned

1. **Pruning is not optional:** For any non-trivial game tree, alpha-beta pruning is essential
2. **Measurement matters:** Accurate performance metrics guide optimization efforts
3. **Stochastic complexity:** Probabilistic games have larger search spaces but also more pruning opportunities
4. **Heuristic quality:** A good evaluation function is as important as search depth
5. **Practical constraints:** Real-time requirements often dictate acceptable search depth

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Appendix: Test Scenarios

Scenario 1: Nearly Full Board (Deterministic Tests)

```
. . . . . . .
. . . . . . .
. . X O . . .
. X O X O . .
X O X O X O .
O X O X O X .
```

- **Purpose:** Test endgame scenarios with limited branching
- **Valid moves:** 3 columns (0, 5, 6)
- **Complexity:** High due to potential immediate wins/losses

Scenario 2: Early Game (Stochastic Tests)

```
. . . . . . .
. . . . . . .
. . . . . . .
. . . . . . .
. . . X . . .
. . O O X . .
```

- **Purpose:** Test opening game with maximum branching
- **Valid moves:** All 7 columns
- **Complexity:** Moderate, many possible futures
