- Tutorial 6 -

Numerical Methods and Optimization

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approximation in the sense of Least Squares

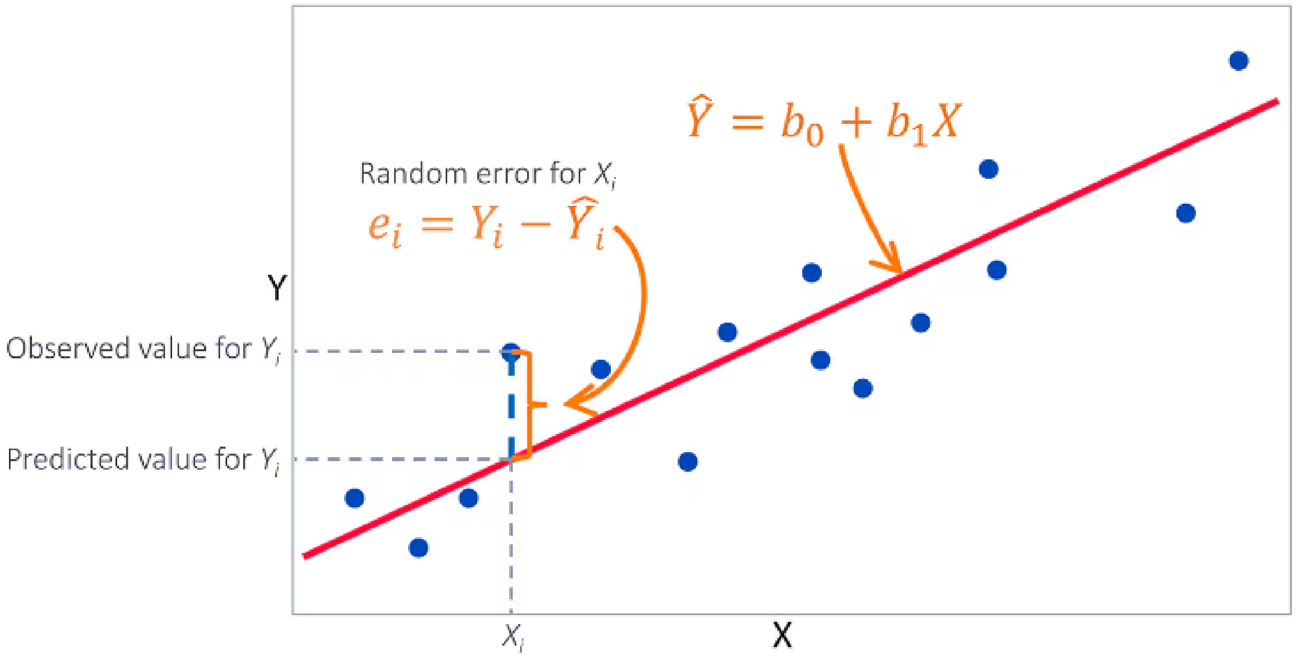
INTRODUCTION

Least squares method: he least square method is the process of finding the best-fitting curve or line of best fit for a set of data points by reducing the sum of the squares of the offsets (residual part) of the points from the curve.

There are two basic categories of least-squares problems:

- Ordinary or linear least squares
- Nonlinear least squares

ILLUSTRATION



Exercise 1: **Descrete Least squares approximation**

(1)- Linear approximation : $y = \alpha x + \beta$ (1)

Step 1: We replace data in the equation (1) and we get a system of equations to find α and β

$$\begin{cases} 1.2 = 0.75\alpha + \beta \\ 1.95 = 2\alpha + \beta \\ 2 = 3\alpha + \beta \\ 2.4 = 4\alpha + \beta \\ 2.4 = 6\alpha + \beta \\ 2.7 = 8\alpha + \beta \\ 2.6 = 8.5\alpha + \beta \end{cases} \Leftrightarrow \begin{bmatrix} 0.75 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 6 & 1 \\ 8 & 1 \\ 8.5 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1.2 \\ 1.95 \\ 2 \\ 2.4 \\ 2.4 \\ 2.7 \\ 2.6 \end{bmatrix}$$

Step 2: We solve $A^tA\ x = A^tb$

$$A^tA = \begin{bmatrix} 202.8125 & 32.25 \\ 32.25 & 7 \end{bmatrix}, \qquad A^tb = \begin{bmatrix} 78.5 \\ 15.25 \end{bmatrix}$$

We solve, we find: $\alpha = 0.1548$ and $\beta = 1.4653$

So: $y = 0.1548x + 1.4653$

(2)- Quadratic approximation : $y = \alpha x^2 + \beta x + \gamma$ (2)

Step 1: We replace data in the equation (2) and we get a system of equations to find α, β and γ

$$\begin{cases} 1.2 = 0.5625\alpha + 0.75\beta + \gamma \\ 1.95 = 4\alpha + 2\beta + \gamma \\ 2 = 9\alpha + 3\beta + \gamma \\ 2.4 = 16\alpha + 4\beta + \gamma \\ 2.4 = 36\alpha + 6\beta + \gamma \\ 2.7 = 64\alpha + 8\beta + \gamma \\ 2.6 = 72.25\alpha + 8.5\beta + \gamma \end{cases} \Leftrightarrow \begin{bmatrix} 0.5625 & 0.75 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \\ 36 & 6 & 1 \\ 64 & 8 & 1 \\ 72.25 & 8.5 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1.2 \\ 1.95 \\ 2 \\ 2.4 \\ 2.4 \\ 2.7 \\ 2.6 \end{bmatrix}$$

Step 2: We solve $A^tA\ x = A^tb$

$$A^tA = \begin{bmatrix} 10965 & 1441.5 & 201.81 \\ 1441.5 & 201.81 & 32.25 \\ 201.81 & 32.25 & 7 \end{bmatrix}, \qquad A^tb = \begin{bmatrix} 526.375 \\ 80.200 \\ 15.45 \end{bmatrix}$$

We solve, we find: $\alpha = -0.0254$, $\beta = 0.4137$ and $\gamma = 1.0336$

So: $y = -0.0254x^2 + 0.4137x + 1.0336$

Exercise 2: **Linear Least Squares approximation**

(1)- Linear approximation : $y = a_0 + a_1x_1 + a_2x_2$ (3)

Step 1: We replace data in the equation (3) and we get a system of equations to find a_0, a_1 and a_2

$$\begin{cases} 0.8 = a_0 + 5a_1 + 3a_2 \\ 0.8 = a_0 + 3a_1 + 4a_2 \\ 0.6 = a_0 + a_1 + 5a_2 \\ 0.4 = a_0 + 2a_1 + a_2 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 5 & 3 \\ 1 & 3 & 4 \\ 1 & 1 & 5 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.8 \\ 0.6 \\ 0.4 \end{bmatrix}$$

Step 2: We solve $A^tA\,x = A^tb$

$$A^tA = \begin{bmatrix} 4 & 11 & 13 \\ 11 & 39 & 34 \\ 13 & 34 & 51 \end{bmatrix}, \qquad A^tb = \begin{bmatrix} 2.6 \\ 2.8 \\ 9 \end{bmatrix}$$

We solve, we find: $a_0 = 0.1381$, $a_1 = 0.0905$ and $a_2 = 0.0810$

So: $y = 0.1381 + 0.0905x_1 + 0.0810x_2$

Exercise 3: **Continuous Least Squares approximation**

Let $L_1(x)$ be a polynomial of degree less than or equal to 1 such that: $L_1(x) = ax + b$, and let f a function such that: $f(x) = e^x$.

We want to approximate the function f with $L_1(x)$ in the sense of continuous least squares over $[0, 1]$.

We have to find the coefficients a and b such that the error $\int_0^1 (f(x) - L_1(x))^2\, dx$ is minimzed.

The error $g(a,b)$ is given by:

$$\begin{aligned} g(a,b) &= \int_0^1 (f(x) - L_1(x))^2\,dx = \int_0^1 (f(x) - ax - b)^2\,dx \\ &= \int_0^1 f(x)^2 - 2f(x)(ax+b) + (a^2x^2 + 2abx + b^2)dx \\ &= \int_0^1 f(x)^2 - 2b\int_0^1 f(x)dx - 2a\int_0^1 xf(x)dx + b(1-0) + ab(1^2 - 0^2) + \tfrac{1}{3}a^2(1^3 - 0^3) \end{aligned}$$

To minimize the error $g(a,b)$ we invoke the conditions

$\frac{\partial g}{\partial a} = -2\int_0^1 xf(x)dx + b(1^2 - 0^2) + \frac{2}{3}a(1^3 - 0^3) = 0$

$\frac{\partial g}{\partial b} = -2\int_0^1 f(x)dx + 2b(1-0) + a(1^2 - 0^2) = 0$

We get:

$2b + a = 2\int_0^1 e^x dx$

$b + \frac{2}{3}a = 2\int_0^1 xe^x dx$

$$\Leftrightarrow \begin{bmatrix} 2 & 1 \\ 1 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 2e - 2 \\ 2 \end{bmatrix}$$

Finally:

$a = 18 \times 10^{-6}$ and $b = 4 \times 10^{-10}$

So $e^x \simeq (18 \times 10^{-6})x + (4 \times 10^{-10})$