# - Tutorial 5 -

Numerical Methods and Optimization

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# Polynomial Interpolation

# Exercise 1: **Lagrange's Interpolation Formula**

**(1)-** We have $f(x) = 2sin(\frac{\pi x}{6})$, and 3 given points $x_0 = 0$, $x_1 = 1$, and $x_2 = 3$, then the polynomial is of degree $\leq 3 - 1 = 2$.
We have the following data:

| $x_i$ | 0 | 1 | 3 |
|---|---|---|---|
| $y_i$ | 0 | 1 | 2 |

The polynomial is found using the following expression:
$P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$

Such that:

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}(x-1)(x-3)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}x(x-3)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}x(x-1)$$

Finally:

$P_2(x) = 0L_0(x) + 1L_1(x) + 2L_2(x) = -\frac{1}{2}x(x-3) + \frac{1}{3}x(x-1)$

We obtain the following approximation:

$f(2) = P_2(2) = \frac{5}{3}, \quad f(2.4) = P_2(2.4) = 1.84$

Relative percent errors:

$f(2) = 2sin(\frac{\pi}{3}) = 1.732051, \quad e_{rel}\% = \frac{|1.732051 - \frac{5}{3}|}{1.732051} \times 100 = 3.7\%$

$f(2.4) = 1.902113, \quad e_{rel}\% = \frac{|1.902113 - 1.84|}{1.902113} \times 100 = 3.2\%$

**(2)-** Find the polynomial $f(x)$ by using Lagrange's formula for:

| $x_i$ | 0 | 1 | 2 | 5 |
|---|---|---|---|---|
| $f(x_i) = y_i$ | 2 | 3 | 12 | 147 |

We have 4 given points, then the polynomial $P_3(x)$ is of degree $\leq 4 - 1 = 3$ such that :

$$P_3(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x)$$

With:

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} = -\frac{1}{10}(x-1)(x-2)(x-5)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} = \frac{1}{4}x\,(x-2)\,(x-5)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} = -\frac{1}{6}x\,(x-1)\,(x-5)$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} = \frac{1}{60}x\,(x-1)\,(x-2)$$

Finally:

$$P_3(x) = -\frac{1}{5}(x-1)(x-2)(x-5) + \frac{3}{4}x(x-2)(x-5) - 2x(x-1)(x-5) + \frac{147}{60}x(x-1)(x-2)$$

And

$f(3) = 35$

# Exercise 2: **Newton's Interpolation Formula**

Newton's formula:

$$P_n(x) = C_0 + C_1(x - x_0) + C_2(x - x_0)(x - x_1) + \dots + C_n(x - x_0)(x - x_1)\dots(x - x_{n-1})$$

**(1)-** We calculate the coefficient of the polynomial using divided differences table:

$C_0$ $C_1$ $C_2$ $C_3$

| $x_i$ | $y_i$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|
| 1960 | 6652516 | | | |
| | | 194461.70 | | |
| 1970 | 8597133 | | 2953.0006 | |
| | | 265333.714 | | 44.2942 |
| 1984 | 12311805 | | 4724.7699 | |
| | | 407076.812 | | |
| 2000 | 18825034 | | | |

Such that:

For the column $\Delta y$

$$194461.7 = \frac{8597133-6652516}{1970-1960}$$

$$265333.714 = \frac{12311805-8597133}{1984-1970}$$

$$407076.812 = \frac{18825034-12311805}{2000-1984}$$

For the column $\Delta^2 y$

$$2953.0006 = \frac{265333.714-194461.7}{1984-1960}$$

$$4724.7699 = \frac{407076.812-265333.714}{2000-1970}$$

For the column $\Delta^3 y$

$44.2942 = \frac{4724.7699 - 2953.0006}{2000 - 1960}$

Finally:

$P_3(x) = 6652516 + 194461.7(x - 1960) + 2953.0006(x - 1960)(x - 1970) + 44.2942(x - 1960)(x - 1970)(x - 1984)$

The population size during the year 1999 is:

$P_3(1999) = 6652516 + 7584006.3 + 33399843.68 + 751451.103 = 18327817.1$

**(2)-** We have $f(x) = sin(x)$, and $5$ given points, then the polynomial is of degree $\leq 5 - 1 = 4$.

We have the following data:

| | | | | | |
|---|---|---|---|---|---|
| $x_i$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
| $y_i$ | $0$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $1$ |

We calculate the coefficient of the polynomial using divided differences table:

| | $C_0$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|---|
| $x_i$ | $y_i$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
| $0$ | $0$ | | | | |
| | | 0.954929 | | | |
| $\frac{\pi}{6}$ | $\frac{1}{2}$ | | -0.208607 | | |
| | | 0.791089 | | -0.136489 | |
| $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | | -0.351538 | | 0.028797 |
| | | 0.607024 | | -0.091254 | |
| $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | | -0.447100 | | |
| | | 0.295872 | | | |
| $\frac{\pi}{2}$ | $1$ | | | | |

Finally:

$$P_4(x) = 0 + 0.954929\, x - 0.208607\, x\left(x - \frac{\pi}{6}\right) - 0.136489\, x\left(x - \frac{\pi}{6}\right)\left(x - \frac{\pi}{4}\right) + 0.028797\, x\left(x - \frac{\pi}{6}\right)\left(x - \frac{\pi}{4}\right)\left(x - \frac{\pi}{3}\right)$$

Evaluate $sin\left(\frac{3\pi}{8}\right)$:

$$P_4\left(\frac{3\pi}{8}\right) = 0.923963$$

$$f\left(\frac{3\pi}{8}\right) = 0.923879$$

Relative percent error $e_{rel}\% = \frac{|0.923879-0.923963|}{0.923879} \times 100 = 0.009\%$

# Exercise 3: **Rolle's theorem**

We have $f(x) = sin(x)$ is sufficiently differentiable in $\mathbb{R}$.
The interpolation polynomial $P_2$ is of degree $\leq 2$.
And we have: $x_0 = 0; x_1 = \frac{\pi}{4}, x_2 = \frac{\pi}{2}$ so $[a, b] = [x_0, x_2] = [0, \frac{\pi}{2}]$.
We know that:

$$\left|f(x) - P_n(x)\right| \leq \frac{M_{n+1}}{(n+1)!} \left|\prod_{i=0}^{n}(x - x_i)\right| \qquad (1)$$

And we have $n = 2$, with

$$M_{n+1} = \max_{x\in[0,\frac{\pi}{2}]} \left|f^{n+1}(x)\right| \Rightarrow M_3 = \max_{x\in[0,\frac{\pi}{2}]} \left|f^3(x)\right| = \max_{x\in[0,\frac{\pi}{2}]} \left|-cos(x)\right| = 1$$

Using the inequation (1) we get:

$$\left|f(x) - P_2(x)\right| \leq \frac{1}{3!}\left|(x-0)\left(x-\frac{\pi}{4}\right)\left(x-\frac{\pi}{2}\right)\right|$$

For $x = \frac{\pi}{5}$ we have:

$$\left|f\left(\frac{\pi}{5}\right) - P_2\left(\frac{\pi}{5}\right)\right| \leq \frac{1}{6}\left|\left(\frac{\pi}{5}-0\right)\left(\frac{\pi}{5}-\frac{\pi}{4}\right)\left(\frac{\pi}{5}-\frac{\pi}{2}\right)\right| = 0.015503$$

Here we just estimate the error of interpolation without calculating the polynomial.