- Tutorial 4 -

Numerical Methods and Optimization

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Iterative Methods

Exercise 1

1. **Method 1 : Splitting**

1.1 Determine the matrix G and the vector c such that $A = M - N$

$$A = M - N \Rightarrow N = M - A = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & -1 \\ 0 & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & 0 \\ -1 & -\frac{1}{2} & 0 & 0 \end{pmatrix}$$

M is diagonal by bloc, then its inverse is obtained by replacing each bloc by its inverse:

$$M_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow M_1^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \Rightarrow M_2^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

So:

$$G = M^{-1}N = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$c = M^{-1}b = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

The iterative process is:

$$x^{(k+1)} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{pmatrix} x^{(k)} + \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

1.2 Calculate: $\|G\|_1$, $\|G\|_2$, $\|G\|_\infty$ and $\rho(G)$

$$\|G\|_1 = \max_{j\in[1,4]} \sum_{i=1}^{4} |g_{ij}| = 1$$

$$\|G\|_\infty = \max_{i\in[1,4]} \sum_{j=1}^{4} |g_{ij}| = 1$$

$$\begin{array}{cc}
\begin{pmatrix} 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{pmatrix} & \begin{matrix} 1 \\ 1/2 \\ 1/2 \\ 1 \end{matrix} \\
\begin{matrix} \frac{1}{2} & \quad 1 & \quad 1 & \quad \frac{1}{2} \end{matrix} &
\end{array}$$

Norm ∞

Norm 1

$$\|G\|_2 = \sqrt{\rho\left(G^t G\right)} = \sqrt{0.654508} = 0,809016$$

$$G^t G = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \lambda_1 = \lambda_2 = 0.095492, \lambda_3 = \lambda_4 = 0.654508$$

$$\rho\left(G^t G\right) = \max_i(\lambda_i) = 0.654508$$

$$\rho(G) = \max_i(\lambda_i) = \max_i\{0.5, -0.5\} = 0.5 < 1$$

So the method converges.

1.3 Given $x^{(0)} = (1, -1, 0, 1)^t$. Calculate $x^{(2)}$

$$x^{(1)} = Gx^{(0)} + c = \begin{pmatrix} \frac{1}{2} \\ -1 \\ -\frac{1}{2} \\ 0 \end{pmatrix}$$

$$x^{(2)} = Gx^{(1)} + c = \begin{pmatrix} \frac{5}{4} \\ -\frac{5}{4} \\ -\frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$$

2. Convergence speed and stopping test
   2.1 we have $\rho(G_{GD} = \frac{5}{8} > 0.5 = \rho(G)$, then Gauss-Seidel is slower to converge than Method 1, so according to the table, the iteration on the top is slower then it corresponds to Gauss-Seidel.
   2.2 Determine if the convergence is achieved

$$k = 5, GS, x^{(5)} = \begin{pmatrix} 1.2062 \\ -1.2062 \\ -0.6031 \\ 0 \end{pmatrix} \Rightarrow b - Ax^{(5)} = \begin{pmatrix} 0.0954 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left\|b - Ax^{(5)}\right\|_2 = \sqrt{0.0954^2} = 0.0954 > 10^{-5}$$

Convergence is not achieved

$$k = 10, Method\ 1, x^{(10)} = \begin{pmatrix} 1.3320 \\ -1.3320 \\ 0.6660 \\ 0 \end{pmatrix}, x^{(9)} = \begin{pmatrix} 1.3320 \\ -1.3320 \\ 0.6641 \\ 0 \end{pmatrix}$$

$$x^{(10)} - x^{(9)} = \begin{pmatrix} 0 \\ 0 \\ 0.0019 \\ 0 \end{pmatrix}, \left\| x^{(10)} - x^{(9)} \right\|_{\infty} = 0.0019, \left\| x^{(10)} \right\|_{\infty} = 1.3320$$

$$So : \frac{\left\| x^{(k)} - x^{(k-1)} \right\|_{\infty}}{\left\| x^{k} \right\|_{\infty}} = 0.014 < 10^{-1}$$

Convergence is achieved

$$k = 4, GS, x^{(4)} = \begin{pmatrix} 1.1299 \\ -1.1299 \\ -0.5649 \\ 0 \end{pmatrix}, x^{(4)} - x^* = \begin{pmatrix} -0.2034 \\ 0.2034 \\ 0.1018 \\ 0 \end{pmatrix}$$

$$\left\| x^{(4)} - x^* \right\|_1 = 0.5086 > 10^{-3}$$

Convergence is not achieved

3.a Jacobi Method

Let the n system of linear equations be Ax = b.

Here,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \cdot & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \; b = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ b_n \end{bmatrix}$$

Let us decompose matrix A into a diagonal component D and remainder R such that A = D + R.

$$D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \cdot & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, R = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}$$

Iteratively the solution will be obtained using the below equation.

$x^{(k+1)} = D^{-1}(b - Rx^{(k)})$

3.a Jacobi Method

Step 1: In this method, we must solve the equations to obtain the values $x_1, x_2, \ldots x_n$.

To get the value of x_1, solve the first equation using the formula given below:

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n) \ldots\ldots (1)$$

To get the value of x_2, solve the second equation using the formulas as:

$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_2 - a_{23}x_3 - \ldots - a_{2n}x_n) \ldots\ldots (2)$$

Similarly, to find the value of x_n, solve the nth equation.

$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_2 - a_{n2}x_3 - \ldots - a_{n,n-1}x_{n-1}) \ldots\ldots (n)$$

Step 2: Now, we have to make the initial guess of the solution as:

$$x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \ldots, x_n^{(0)})$$

3. Jacobi Method

Step 3: Substitute the values obtained in the previous step in equation (1), i.e., into the right hand side the of the rewritten equations in step (1) to obtain the first approximation as:

$$(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_n^{(1)})$$

Step 4: In the same way as done in the previous step, compute

$$x^k = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \ldots, x_n^{(k)});\ k = 1, 2, 3 \ldots.$$

3.a Jacobi Method - Application -

$$\begin{cases} x_1^{(k+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - a_{14}x_4^{(k)}\right) \\ x_2^{(k+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)} - a_{24}x_4^{(k)}\right) \\ x_3^{(k+1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)} - a_{34}x_4^{(k)}\right) \\ x_4^{(k+1)} = \frac{1}{a_{44}}\left(b_4 - a_{41}x_1^{(k)} - a_{42}x_2^{(k)} - a_{43}x_3^{(k)}\right) \end{cases}$$

we substitute:

$$\begin{cases} x_1^{(k+1)} = \frac{1}{2}\left(1 - x_2^{(k)} - \frac{1}{2}x_3^{(k)} - x_4^{(k)}\right) \\ x_2^{(k+1)} = -x_1^{(k)} - \frac{1}{2}x_4^{(k)} \\ x_3^{(k+1)} = -\frac{1}{2}x_1^{(k)} - x_4^{(k)} \\ x_4^{(k+1)} = \frac{1}{2}\left(-x_1^{(k)} - \frac{1}{2}x_2^{(k)} - x_3^{(k)}\right) \end{cases}$$

3.a Jacobi Method - Application -

We find:

$$x^{(1)} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, x^{(2)} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix}$$

The error committed using norm 1:

$$x^{(2)} - x^* = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} - \begin{pmatrix} 1.3333 \\ -1.3333 \\ -0.6667 \\ 0 \end{pmatrix} = \begin{pmatrix} -0,8333 \\ 0,8333 \\ 0,4167 \\ -0.25 \end{pmatrix}, \|x^{(2)} - x^*\|_1 = 2.3333$$

3.b Gauss-Seidel Method

Main idea of Gauss-Seidel

It uses the same expressions to find $x^{(k)}$ as the Jacobi method, the difference is that with the Jacobi method, the values of $x_i^{(k)}$ obtained in the k^{th} iteration remain unchanged until the entire $k+1^{th}$ iteration has been calculated, but with the Gauss-Seidel method, we use the new values $x_i^{(k+1)}$ as soon as they are known. For example, once we have computed $x_1^{(k+1)}$ from the first equation, its value is then used in the second equation to obtain the new $x_2^{(k+1)}$ and so on.

3.b Gauss-Seidel Method - Application -

$$\begin{cases} x_1^{(k+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - a_{14}x_4^{(k)}\right) \\ x_2^{(k+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} - a_{24}x_4^{(k)}\right) \\ x_3^{(k+1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)} - a_{34}x_4^{(k)}\right) \\ x_4^{(k+1)} = \frac{1}{a_{44}}\left(b_4 - a_{41}x_1^{(k+1)} - a_{42}x_2^{(k+1)} - a_{43}x_3^{(k+1)}\right) \end{cases}$$

we substitute:

$$\begin{cases} x_1^{(k+1)} = \frac{1}{2}\left(1 - x_2^{(k)} - \frac{1}{2}x_3^{(k)} - x_4^{(k)}\right) \\ x_2^{(k+1)} = -x_1^{(k+1)} - \frac{1}{2}x_4^{(k)} \\ x_3^{(k+1)} = -\frac{1}{2}x_1^{(k+1)} - x_4^{(k+1)} \\ x_4^{(k+1)} = \frac{1}{2}\left(-x_1^{(k+1)} - \frac{1}{2}x_2^{(k+1)} - x_3^{(k+1)}\right) \end{cases}$$

3.b Gauss-Seidel Method - Application -

We find:

$$x^{(1)} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{4} \\ 0 \end{pmatrix}, x^{(2)} = \begin{pmatrix} \frac{13}{16} \\ -\frac{13}{16} \\ -\frac{13}{32} \\ 0 \end{pmatrix}$$

The error committed using norm 1:

$$\mathrm{x}^{(2)} - x^{*} == \begin{pmatrix} \frac{13}{16} \\ -\frac{13}{16} \\ -\frac{13}{32} \\ 0 \end{pmatrix} - \begin{pmatrix} 1.3333 \\ -1.3333 \\ -0.6667 \\ 0 \end{pmatrix} = \begin{pmatrix} -0,5208 \\ 0,5208 \\ 0,2605 \\ 0 \end{pmatrix}, \left\|x^{(2)} - x^{*}\right\|_1 = 1.3021$$

If we want to estimate the error at iteration n (we don't have the value of x^*, the expression is (B is the matrix of Jacobi or Gauss-Seidel):

$$\|x^n - \bar{x}\|_1 \leq \frac{(\|B\|_1)^n}{1 - \|B\|_1} \times \|x^1 - x^0\|_1$$

Exercise 2

What is a strictly diagonally dominant matrix?

A $n \times n$ square matrix A is a strictly diagonally dominant matrix if:

$$|a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ij}| \quad for \quad i = 1, 2, ..., n$$

that is, for each row, the absolute value of the diagonal element is strictly greater than the sum of the absolute values of the rest of the elements of that row.

We consider the matrix $A = \begin{pmatrix} 1 & 0 & \beta \\ \alpha & 1 & \beta \\ -\beta & \beta & 1 \end{pmatrix}, (\alpha, \beta) \in \mathbb{R}^2, \alpha > 0$

1. Determine the domain D_1: A is strictly diagonally dominant if :
   $\begin{cases} |\beta| < 1 \\ |\beta| + |\alpha| < 1 \\ 2\,|\beta| < 1 \end{cases} \Rightarrow \begin{cases} |\beta| < \frac{1}{2} \\ \alpha < 1 - |\beta| \end{cases}$
   we obtain : $D_1 = \left\{(\alpha, \beta) \in \mathbb{R}^2 | \beta \in \left]-\frac{1}{2}, \frac{1}{2}\right[\ , \alpha \in]0, 1 - |\beta|[\ \right\}$
2. When $(\alpha, \beta) \in D_1$, matrix A is SDD. A satisfies a sufficient condition for convergence of the Jacobi and Gauss-Seidel methods, so both methods converge for any initial vector $x^{(0)}$.

2. Determine the domain D_2 (resp. D_3) which gives the set of pairs $(\alpha, \beta) \in \mathbb{R}^2$ for which the Jacobi method (resp. Gauss-Seidel) converges.
Let $A, D, E, F \in M_n(\mathbb{R})$ such that $A = D - E - F$. We denote by J_A the Jacobi matrix associated with matrix A and by L_A the Gauss-Seidel matrix associated with matrix A such that:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E + F = D - A = \begin{pmatrix} 0 & 0 & -\beta \\ -\alpha & 0 & -\beta \\ \beta & -\beta & 0 \end{pmatrix}$$

$$J_A = D^{-1}(E + F) = \begin{pmatrix} 0 & 0 & -\beta \\ -\alpha & 0 & -\beta \\ \beta & -\beta & 0 \end{pmatrix}$$

The necessary and sufficient condition for convergence of the Jacobi process is:
$\rho(J_A) < 1$, we calculate $\rho(J_A)$:

$$det(\lambda I - J_A) = \begin{vmatrix} \lambda & 0 & \beta \\ \alpha & \lambda & \beta \\ -\beta & \beta & \lambda \end{vmatrix} = 0 \Rightarrow \lambda\left(\lambda^2 - \beta^2\right) + \beta^2(\alpha + \lambda) = 0 \Rightarrow \lambda^3 + \alpha\beta^2 = 0$$

Thus $\rho(J_A) = \sqrt[3]{\alpha\beta^2}$
$\rho(J_A) < 1 \Leftrightarrow \sqrt[3]{\alpha\beta^2} < 1 \Leftrightarrow \alpha\beta^2 < 1, D_2 = \left\{(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R} \middle| \alpha\beta^2 < 1\right\}$

$$L_A = (D - E)^{-1} F = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ -\beta & \beta & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\beta \\ 0 & 0 & (\alpha - 1)\beta \\ 0 & 0 & -\beta^2(1+\alpha) \end{pmatrix}$$

$$det(\lambda I - L_A) = 0 \Leftrightarrow \lambda^2 \left(\lambda + \beta^2 (1 + \alpha)\right) = 0$$

$$\rho(L_A) < 1 \Leftrightarrow \beta^2 (1 + \alpha) < 1, D_3 = \left\{(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R} \,|\, \beta^2 (1 + \alpha) < 1\right\}$$

Compare:

If $(\alpha, \beta) \in D_1 \Rightarrow \begin{cases} \alpha < 1 \\ |\beta| < 1 \Rightarrow |\beta|^2 < 1 \end{cases} \Rightarrow \alpha \times \beta^2 < 1 \Rightarrow (\alpha, \beta) \in D_2$ so $D_1 \subset D_2$

If $(\alpha, \beta) \in D_1 \Rightarrow \begin{cases} \alpha < 1 \Rightarrow \alpha + 1 < 2 \\ |\beta| < \frac{1}{2} \Rightarrow |\beta|^2 < \frac{1}{4} \end{cases} \Rightarrow (\alpha + 1) \times \beta^2 < \frac{2}{4} < 1 \Rightarrow (\alpha, \beta) \in D_3$ so $D_1 \subset D_3$