Problem Sheet 5

1. The spectral density function $f(\omega)$ of a stationary time series model having autocovariance function $\{\gamma_k\}$ $(k = 0, 1, 2, \dots)$ may be written as:

$$f(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k e^{-i\omega k}, \quad \text{for } -\pi < \omega < \pi \tag{1}$$

Stating any general properties of the autocorrelation function that you assume, show that $f(\omega)$ may be written equivalently as:

$$f(\omega) = \frac{\gamma_0}{2\pi} \left[1 + 2 \sum_{k=1}^{\infty} \rho_k \cos(k\omega) \right] \tag{2}$$

where ρ_k is the autocorrelation function of the time series. Deduce that:

$$f(\omega) = f(-\omega), \quad \text{for } -\pi < \omega < \pi \tag{3}$$

2. (a) Consider the simple moving-average filter $L\{W_t\} = \sum_{j=-q}^{q} g_j W_{t-j}$ with weights $g_j = \frac{1}{2q+1}$, $-q \leq j \leq q$. If $W_t = a + \beta t$ where a and β are non-zero constants, show that $L\{W_t\} = W_t$.

   (b) Now let $L\{e_t\} = X_t$, where e_t a zero mean white noise process with variance σ^2. By finding the mean and variance of X_t describe the behaviour of X_t when q is large.

3. Consider the ARMA(p, q) process

$$\phi(B) X_t = \theta(B) e_t \tag{4}$$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$ and $\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$, $\theta_q \neq 0$, $\phi_p \neq 0$.

   (a) Derive the form of the spectral density function $S(f)$ of $\{X_t\}$.

   (b) Hence, or otherwise, identify the stationary and invertible process $\{X_t\}$ having the spectral density

$$S(f) = \frac{17 - 8\cos(2\pi f)}{13 - 12\cos(2\pi f))} \tag{5}$$

   obtaining values for any autoregressive parameters $\{\phi_j\}$, any moving average parameters $\{\theta_j\}$, and σ^2.
   Hint: Start by considering the moving average part (numerator).