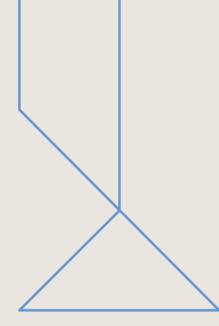


- Tutorial 5 -

Numerical Methods and Optimization April 14, 2024 — ENSIA ensi3



Polynomial Interpolation

Exercise 1: Lagrange's Interpolation Formula

(1)- We have $f(x) = 2sin(\frac{\pi x}{6})$, and 3 given points $x_0 = 0$, $x_1 = 1$, and $x_2 = 3$, then the polynomial is of degree $\le 3 - 1 = 2$.

We have the following data:

The polynomial is found using the following expression:

$$P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$$

Such that:

$$L_{\rm O}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}(x-1)(x-3)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}x(x-3)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}x(x-1)$$

Finally:

$$P_2(x) = OL_0(x) + 1L_1(x) + 2L_2(x) = -\frac{1}{2}x(x-3) + \frac{1}{3}x(x-1)$$

We obtain the following approximation:

$$f(2) = P_2(2) = \frac{5}{3}$$
, $f(2.4) = P_2(2.4) = 1.84$

Relative percent errors:

$$f(2) = 2sin(\frac{\pi}{3}) = 1.732051, \quad e_{rel}\% = \frac{|1.732051 - \frac{5}{3}|}{1.732051} \times 100 = 3.7\%$$

$$f(2.4) = 1.902113$$
, $e_{rel}\% = \frac{|1.902113 - 1.84|}{1.902113} \times 100 = 3.2\%$

(2)- Find the polynomial f(x) by using Lagrange's formula for:

Xi	_		2	
$f(x_i) = y_i$	2	3	12	147

We have 4 given points, then the polynomial $P_3(x)$ is of degree $\leq 4 - 1 = 3$ such that :

$$P_3(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x)$$

With:

$$L_{\rm O}(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} = -\frac{1}{10} \left(x-1\right) \left(x-2\right) \left(x-5\right)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} = \frac{1}{4}x\left(x-2\right)\left(x-5\right)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} = -\frac{1}{6}x(x-1)(x-5)$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} = \frac{1}{60}x(x-1)(x-2)$$

Finally:

$$P_3(x) = -\frac{1}{5}(x-1)(x-2)(x-5) + \frac{3}{4}x(x-2)(x-5) - 2x(x-1)(x-5) + \frac{147}{60}x(x-1)(x-2)$$

And

$$f(3) = 35$$

Exercise 2: Newton's Interpolation Formula

Newton's formula:

$$P_n(x) = C_0 + C_1(x - x_0) + C_2(x - x_0)(x - x_1) + \dots + C_n(x - x_0)(x - x_1) \dots (x - x_{n-1})$$

(1)- We calculate the coefficient of the polynomial using divided differences table:

	C_0 C_1 C_2	C_3
		_
x_i	y_i Δy $\Delta^2 y$ $\Delta^3 y$	/
1960	6652516	
1970	8597133 194461.70 2953.0006 44.2942	
1984	12311805	
2000	18825034 407076.812	

Such that:

For the column
$$\Delta y$$

$$194461.7 = \frac{8597133 - 6652516}{1970 - 1960}$$

$$265333.714 = \frac{12311805 - 8597133}{1984 - 1970}$$

$$407076.812 = \frac{18825034 - 12311805}{2000 - 1984}$$

For the column
$$\Delta^2 y$$

$$2953.0006 = \frac{265333.714 - 194461.7}{1984 - 1960}$$

$$4724.7699 = \frac{407076.812 - 265333.714}{2000 - 1970}$$

For the column
$$\Delta^3 y$$

44.2942 = $\frac{4724.7699-2953.0006}{2000-1960}$

Finally:

$$P_3(x) = 6652516 + 194461.7(x - 1960) + 2953.0006(x - 1960)(x - 1970) + 44.2942(x - 1960)(x - 1970)(x - 1984)$$

The population size during the year 1999 is:

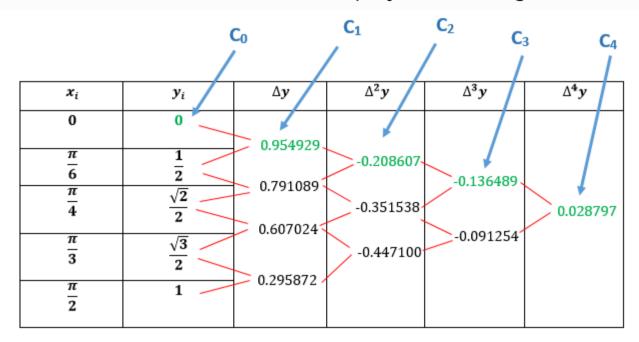
$$P_3(1999) = 6652516 + 7584006.3 + 33399843.68 + 751451.103 = 18327817.1$$

(2)- We have f(x) = sin(x), and 5 given points, then the polynomial is of degree $\le 5 - 1 = 4$.

We have the following data:

X_i	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
y _i	0	1/2	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

We calculate the coefficient of the polynomial using divided differences table:



Finally:

$$P_4(x) = 0 + 0.954929 \ x - 0.208607 \ x \left(x - \frac{\pi}{6} \right) - 0.136489 \ x \left(x - \frac{\pi}{6} \right) \left(x - \frac{\pi}{4} \right) + 0.028797 \ x \left(x - \frac{\pi}{6} \right) \left(x - \frac{\pi}{4} \right) \left(x - \frac{\pi}{3} \right)$$

Evaluate $sin\left(\frac{3\pi}{8}\right)$:

$$P_4(\frac{3\pi}{8}) = 0.923963$$

 $f(\frac{3\pi}{8}) = 0.923879$

Relative percent error e_{rel} % = $\frac{|0.923879 - 0.923963|}{0.923879} \times 100 = 0.009\%$

Exercise 3: Rolle's theorem

We have f(x) = sin(x) is sufficiently differentiable in \mathbb{R} .

The interpolation polynomial P_2 is of degree ≤ 2 .

And we have:
$$x_0 = 0$$
; $x_1 = \frac{\pi}{4}$, $x_2 = \frac{\pi}{2}$ so $[a, b] = [x_0, x_2] = [0, \frac{\pi}{2}]$.

We know that:

$$|f(x) - P_n(x)| \le \frac{M_{n+1}}{(n+1)!} \left| \prod_{i=0}^n (x - x_i) \right|$$
 (1)

And we have n = 2, with

$$M_{n+1} = \max_{x \in [0, \frac{\pi}{2}]} \left| f^{n+1}(x) \right| \Rightarrow M_3 = \max_{x \in [0, \frac{\pi}{2}]} \left| f^3(x) \right| = \max_{x \in [0, \frac{\pi}{2}]} \left| -\cos(x) \right| = 1$$

Using the inequation (1) we get:

$$|f(x) - P_2(x)| \le \frac{1}{3!} |(x - 0)(x - \frac{\pi}{4})(x - \frac{\pi}{2})|$$

For $x = \frac{\pi}{5}$ we have:

$$\left| f\left(\frac{\pi}{5}\right) - P_2\left(\frac{\pi}{5}\right) \right| \le \frac{1}{6} \left| \left(\frac{\pi}{5} - 0\right) \left(\frac{\pi}{5} - \frac{\pi}{4}\right) \left(\frac{\pi}{5} - \frac{\pi}{2}\right) \right| = 0.015503$$

Here we just estimate the error of interpolation without calculating the polynomial.