Numerical Methods and Optimization

3/3/2024 – 3^{rd} Year

ENSIA — Spring 2024

Tutorial 4

- Iterative methods for solving systems of linear equations -

Exercise 1: Iterative methods

Use the given matrices to answer the following questions:

$$A = \begin{pmatrix} 2 & 1 & \frac{1}{2} & 1 \\ 1 & 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & 1 \\ 1 & \frac{1}{2} & 1 & 2 \end{pmatrix} \qquad M = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(1) Method 1: Splitting

(a) Use the splitting $A = M - N$ to determine the matrix G and the vector c that allow for the calculation of the solution of $Ax = b$ through the iterative process $x^{(k+1)} = Gx^{(k)} + c$.

(b) Calculate: $\|G\|_1, \|G\|_2, \|G\|_\infty$ and $\rho(G)$. Does the method converge?

(c) Given $x^{(0)} = (1, -1, 0, 1)^t$. Calculate $x^{(2)}$.

(2) Convergence Speed and Stopping Test:

If the Gauss-Seidel method is used to solve the system $Ax = b$, we obtain an iteration matrix G_{GS} with a spectral radius of $\frac{5}{8}$. The following table provides the iterations performed by the two considered iterative methods.

k	0	1	2	3	4	5	...	10	...	Exacte
	0	0.5000	0.8125	1.0078	1.1299	1.2062		1.3212		1.3333
	0	-0.5000	-0.8125	-1.0078	-1.1299	-1.2062		-1.3212		-1.3333
$\mathbf{x}^{(k)}$	0	-0.2500	-0.4063	-0.5039	-0.5649	-0.6031		-0.6606		-0.6667
	0	0.	0.	0.	0.	0.		0.		0.
	0	1.	1.	1.2500	1.2500	1.3125		1.3320		1.3333
	0	-1.	-1.	-1.2500	-1.2500	-1.3125		-1.3320		-1.3333
$\mathbf{x}^{(k)}$	0	0.	-0.5000	-0.5000	-0.6250	-0.6250		-0.6660		-0.6667
	0	0.	0.	0.	0.	0.		0.		0.

(a) Associate each calculation with an iterative method.

(b) Determine if convergence is achieved in each of the following cases:

(i) $\left\|b - Ax^{(k)}\right\| \leq \epsilon$, GS method, $k = 5$, Norm 2, $\epsilon = 10^{-5}$.

(ii) $\frac{\left\|x^{(k)} - x^{(k-1)}\right\|}{\left\|x^{k}\right\|} \leq \epsilon$, Method 1, $k = 5$, Norm ∞, $\epsilon = 10^{-1}$.

(iii) $\left\|x^{(k)} - x^{*}\right\| \leq \epsilon$, GS method, $k = 4$, Norm 1, $\epsilon = 10^{-3}$.

(3) Perform 2 iterations with Jacobi and Gauss-Seidel such that $x^0 = (0, 0, 0, 0)^t$. Estimate the error committed for each method using $\|.\|_1$.

Exercise 2: Sufficient conditions for the convergence of J and GS

We consider the matrix $A = \begin{pmatrix} 1 & 0 & \beta \\ \alpha & 1 & \beta \\ -\beta & \beta & 1 \end{pmatrix}$

(1) Determine the domain $D_1 = \left\{(\alpha, \beta) \in \mathbb{R}^2 \mid A \textit{ is Strictly Diagonally Dominant SDD}\right\}$. What can you conclude concerning the convergence of the Jacobi and Gauss-Seidel methods associated with the system $Ax = b$?

(2) Determine the domain D_2 (resp. D_3) which gives the set of pairs $(\alpha, \beta) \in \mathbb{R}^2$ for which the Jacobi method (resp. Gauss-Seidel) converges. Compare D_1 and D_2 (resp. D_3).