Numerical Methods and Optimization

Topic 2:

Solution of Nonlinear Equations

Lectures 5-11:

Read Chapters 5 and 6 of the textbook

Lecture 5

Solution of Nonlinear Equations

(Root Finding Problems)

- Definitions
- Classification of Methods
  - Analytical Solutions
  - Graphical Methods
  - Numerical Methods
    - Bracketing Methods
    - Open Methods
- Convergence Notations

Reading Assignment: **Sections 5.1 and 5.2**

Root Finding Problems

Many problems in Science and Engineering are expressed as:

$$\text{Given a continuous function } f(x),$$
$$\text{find the value } r \text{ such that } f(r) = 0$$

These problems are called root finding problems.

Roots of Equations

A number *r* that satisfies an equation is called a root of the equation.

The equation: $x^4 - 3x^3 - 7x^2 + 15x = -18$

has four roots: $-2,\ 3,\ 3,\ and\ -1$.

i.e., $x^4 - 3x^3 - 7x^2 + 15x + 18 = (x+2)(x-3)^2(x+1)$

The equation has two simple roots $(-1$ and $-2)$ and a repeated root (3) with multiplicity = 2.

Zeros of a Function

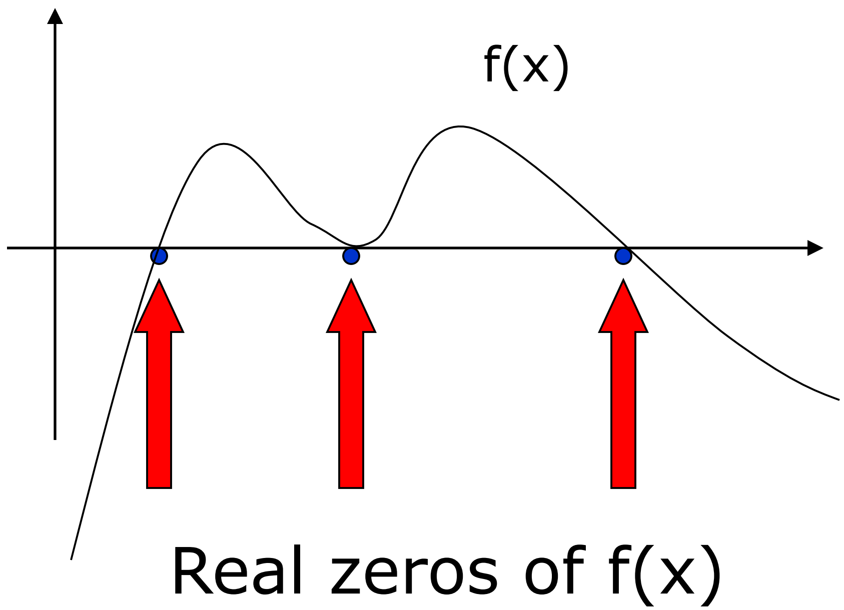
Let $f(x)$ be a real-valued function of a real variable. Any number r for which $f(r)=0$ is called a zero of the function.

Examples:

2 and 3 are zeros of the function $f(x) = (x-2)(x-3)$.

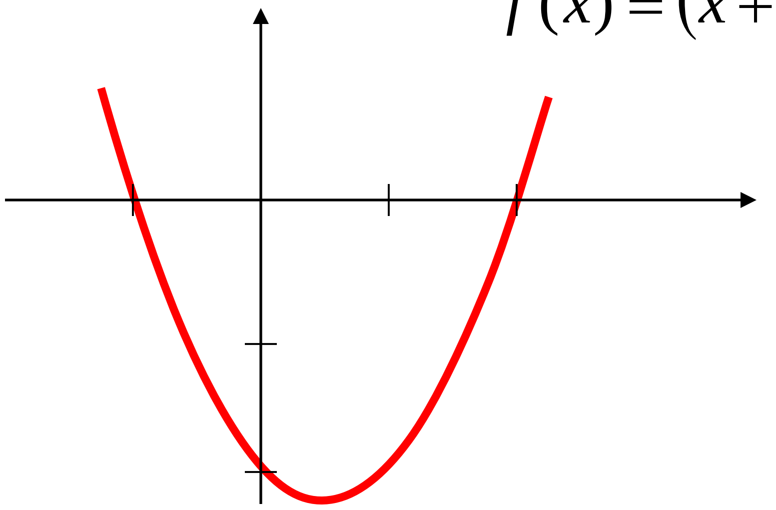
Graphical Interpretation of Zeros

- The real zeros of a function ***f(x)*** are the values of ***x*** at which the graph of the function crosses (or touches) the x-axis.



Simple Zeros

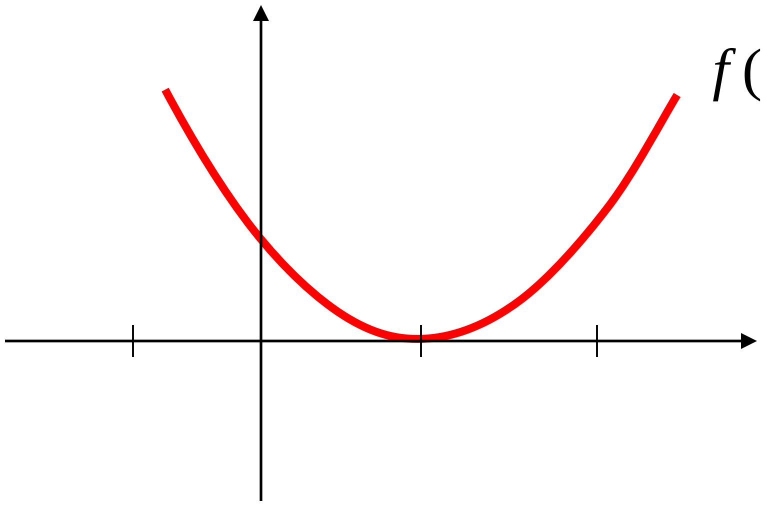
$$f(x) = (x+1)(x-2)$$



$$f(x) = (x+1)(x-2) = x^2 - x - 2$$

has two simple zeros (one at x = 2 and one at x = –1)

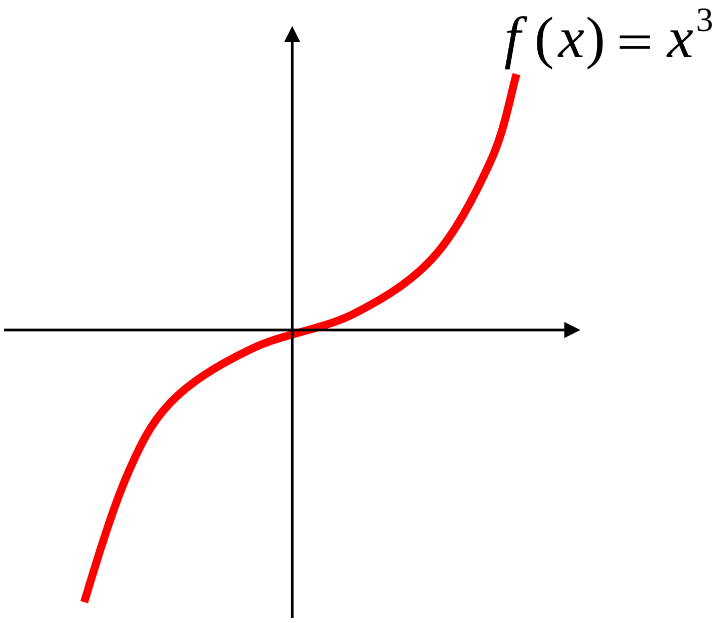
Multiple Zeros



$$f(x) = (x-1)^2 = x^2 - 2x + 1$$

has double zeros (zero with muliplicit y = 2) at x = 1

Multiple Zeros



$$f(x) = x^3$$

$\text{has a zero with muliplicit y} = 3 \text{ at } x = 0$

Facts

- Any n^{th} order polynomial has exactly n zeros (counting real and complex zeros with their multiplicities).
- Any polynomial with an odd order has at least one real zero.
- If a function has a zero at ***x=r*** with multiplicity ***m*** then the function and its first ***(m-1)*** derivatives are zero at ***x=r*** *and the* ***m****th derivative at* ***r*** *is not zero*.

Roots of Equations & Zeros of Function

Given the equation :

$$x^4 - 3x^3 - 7x^2 + 15x = -18$$

Move all terms to one side of the equation :

$$x^4 - 3x^3 - 7x^2 + 15x + 18 = 0$$

Define $f(x)$ as :

$$f(x) = x^4 - 3x^3 - 7x^2 + 15x + 18$$

The zeros of $f(x)$ are the same as the roots of the equation $f(x) = 0$

(Which are -2, 3, 3, and -1)

Solution Methods

Several ways to solve nonlinear equations are possible:

- <u>Analytical Solutions</u>
  - Possible for special equations only
- <u>Graphical Solutions</u>
  - Useful for providing initial guesses for other methods
- <u>Numerical Solutions</u>
  - Open methods
  - Bracketing methods

Analytical Methods

Analytical Solutions are available for special equations only.

Analytical solution of : $a\,x^2 + b\,x + c = 0$

$$roots = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

No analytical solution is available for : $x - e^{-x} = 0$

Graphical Methods

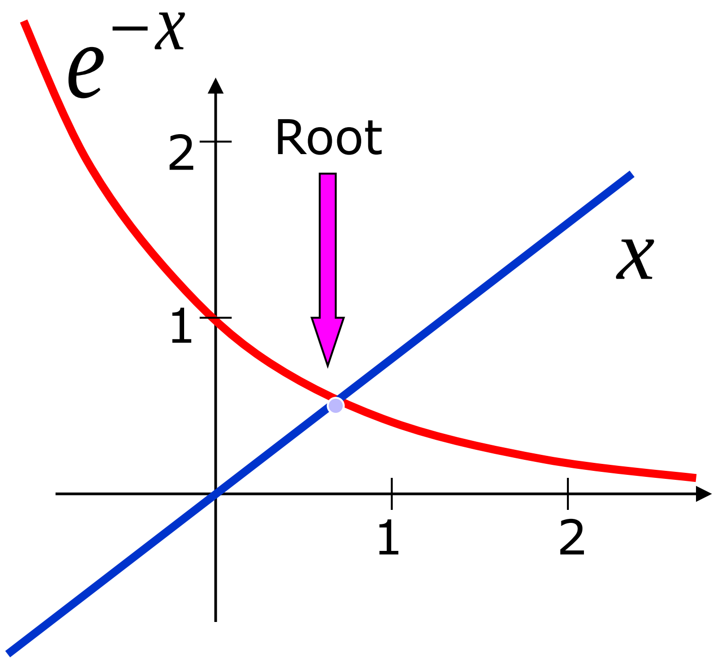
- Graphical methods are useful to provide an initial guess to be used by other methods.

$$Solve$$

$$x = e^{-x}$$

$$The \ \ root \in [0,1]$$

$$root \approx 0.6$$



Numerical Methods

Many methods are available to solve nonlinear equations:

- Bisection Method
- Newton's Method
- Secant Method

These will be covered in CISE301

- False position Method
- Muller's Method
- Bairstow's Method
- Fixed point iterations
- ……….

Bracketing Methods

- In bracketing methods, the method starts with an <u>interval</u> that contains the root and a procedure is used to obtain a smaller interval containing the root.

- Examples of bracketing methods:
  - Bisection method
  - False position method

Open Methods

- In the open methods, the method starts with one or more initial guess points. In each iteration, a new guess of the root is obtained.
- Open methods are usually more efficient than bracketing methods.
- They may not converge to a root.

Convergence Notation

A sequence $x_1, x_2, \ldots, x_n, \ldots$ is said to **converge** to x if to every $\varepsilon > 0$ there exists N such that:

$$\left| x_n - x \right| < \varepsilon \quad \forall n > N$$

Convergence Notation

Let $x_1, x_2, \ldots.,$ converge to x.

Linear Convergence: $\dfrac{|x_{n+1} - x|}{|x_n - x|} \le C$

Quadratic Convergence: $\dfrac{|x_{n+1} - x|}{|x_n - x|^2} \le C$

Convergence of order P: $\dfrac{|x_{n+1} - x|}{|x_n - x|^p} \le C$

Speed of Convergence

- We can compare different methods in terms of their convergence rate.
- Quadratic convergence is faster than linear convergence.
- A method with convergence order q converges faster than a method with convergence order p if $q>p$.
- Methods of convergence order $p>1$ are said to have super linear convergence.

Lectures 6-7

Bisection Method

- The Bisection Algorithm
- Convergence Analysis of Bisection Method
- Examples

Reading Assignment: Sections 5.1 and 5.2

Introduction

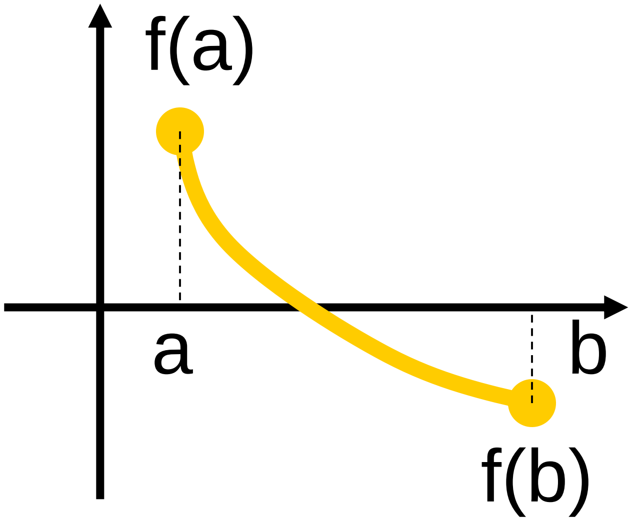
- The **Bisection method** is one of the simplest methods to find a zero of a nonlinear function.
- It is also called **interval halving** method.
- To use the Bisection method, one needs an initial interval that is known to contain a zero of the function.
- The method systematically reduces the interval. It does this by dividing the interval into two equal parts, performs a simple test and based on the result of the test, half of the interval is thrown away.
- The procedure is repeated until the desired interval size is obtained.

Intermediate Value Theorem

- Let f(x) be defined on the interval [a,b].

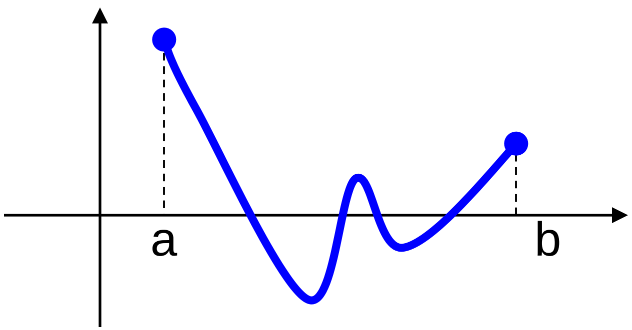
- Intermediate value theorem:

  if a function is continuous and f(a) and f(b) have different signs then the function has at least one zero in the interval [a,b].



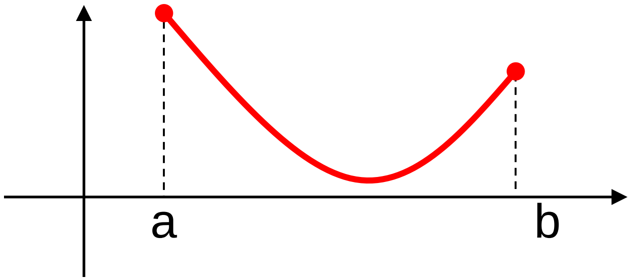
Examples

- If f(a) and f(b) have the same sign, the function may have an even number of real zeros or no real zeros in the interval [a, b].



The function has four real zeros

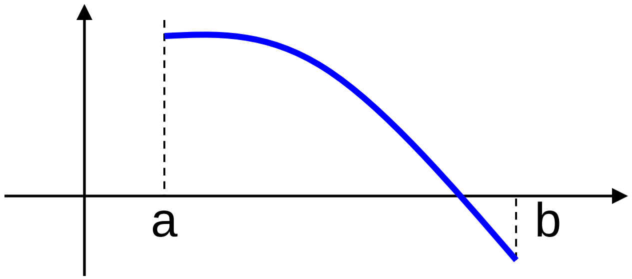
- Bisection method can not be used in these cases.



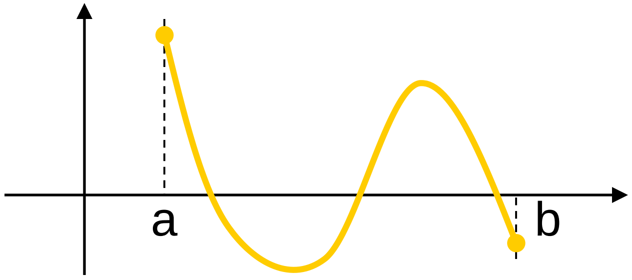
The function has no real zeros

Two More Examples

- If f(a) and f(b) have different signs, the function has at least one real zero.
- Bisection method can be used to find one of the zeros.



The function has one real zero



The function has three real zeros

Bisection Method

- If the function is continuous on [a,b] and f(a) and f(b) have different signs, Bisection method obtains a new interval that is half of the current interval and the sign of the function at the end points of the interval are different.

- This allows us to repeat the Bisection procedure to further reduce the size of the interval.

Bisection Method

Assumptions:

Given an interval [a,b]

$f(x)$ is continuous on [a,b]

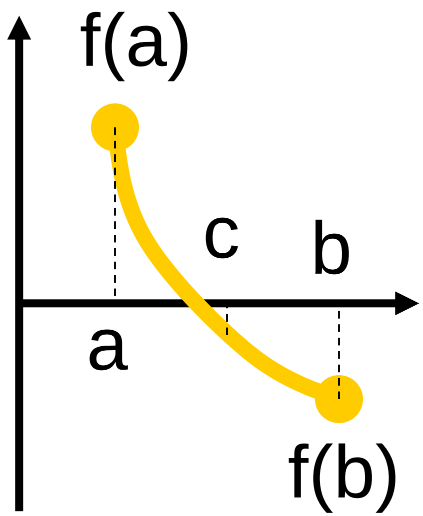
$f(a)$ and $f(b)$ have opposite signs.

These assumptions ensure the existence of at least one zero in the interval [a,b] and the bisection method can be used to obtain a smaller interval that contains the zero.

Bisection Algorithm

Assumptions:

- f(x) is continuous on [a,b]
- f(a) f(b) < 0



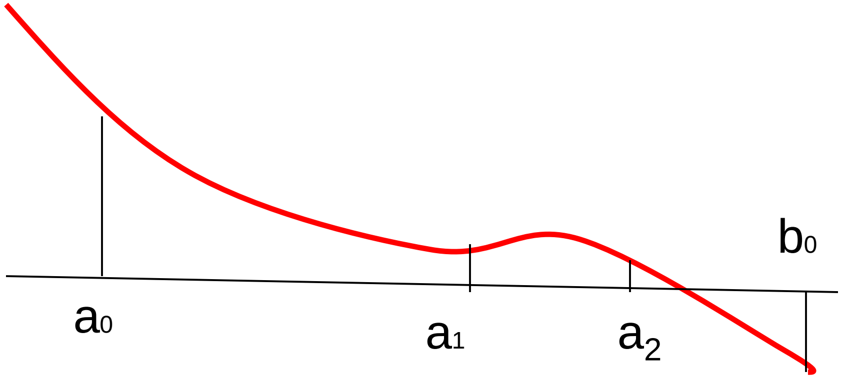
Algorithm:

Loop

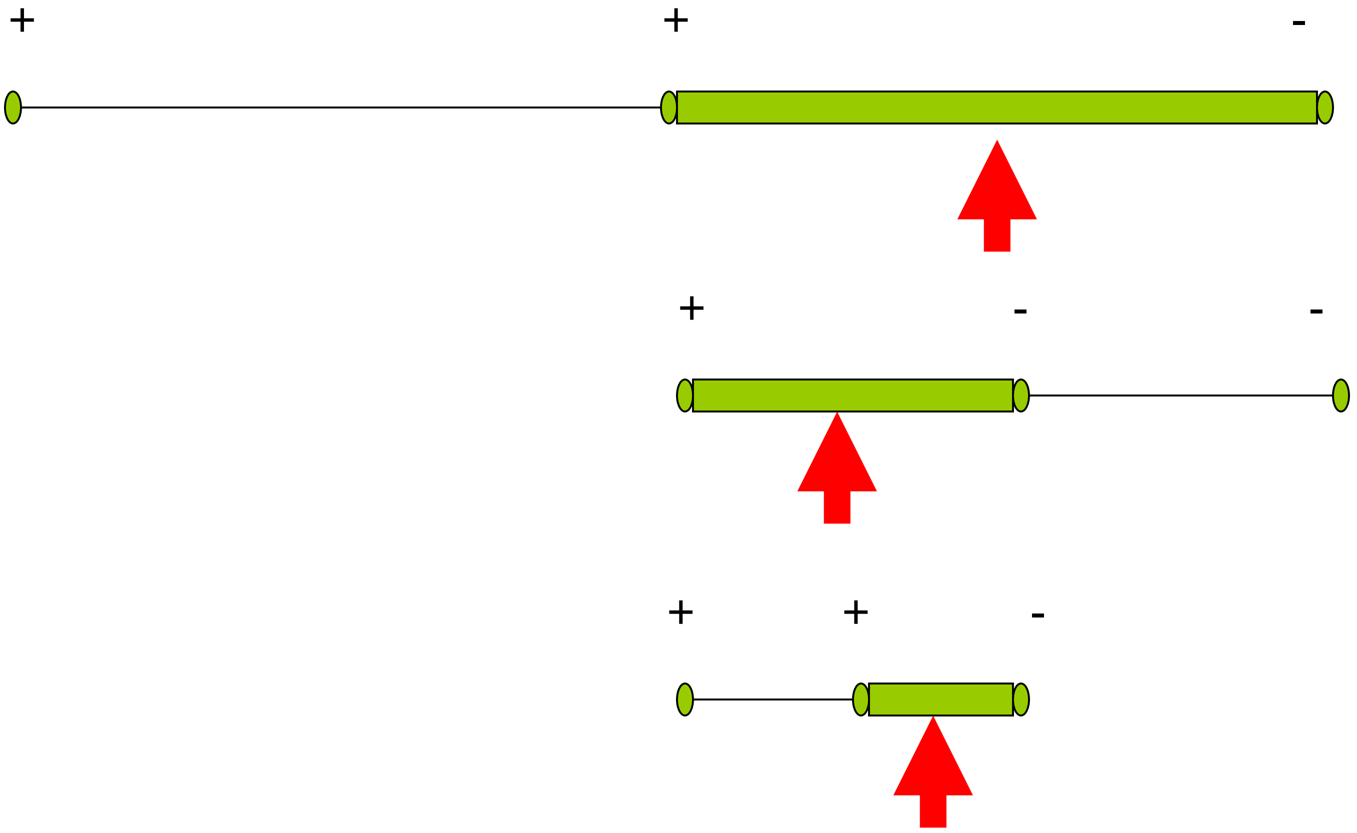
1. Compute the mid point c=(a+b)/2
2. Evaluate f(c)
3. If f(a) f(c) < 0 then new interval [a, c]
   If f(a) f(c) > 0 then new interval [c, b]

End loop

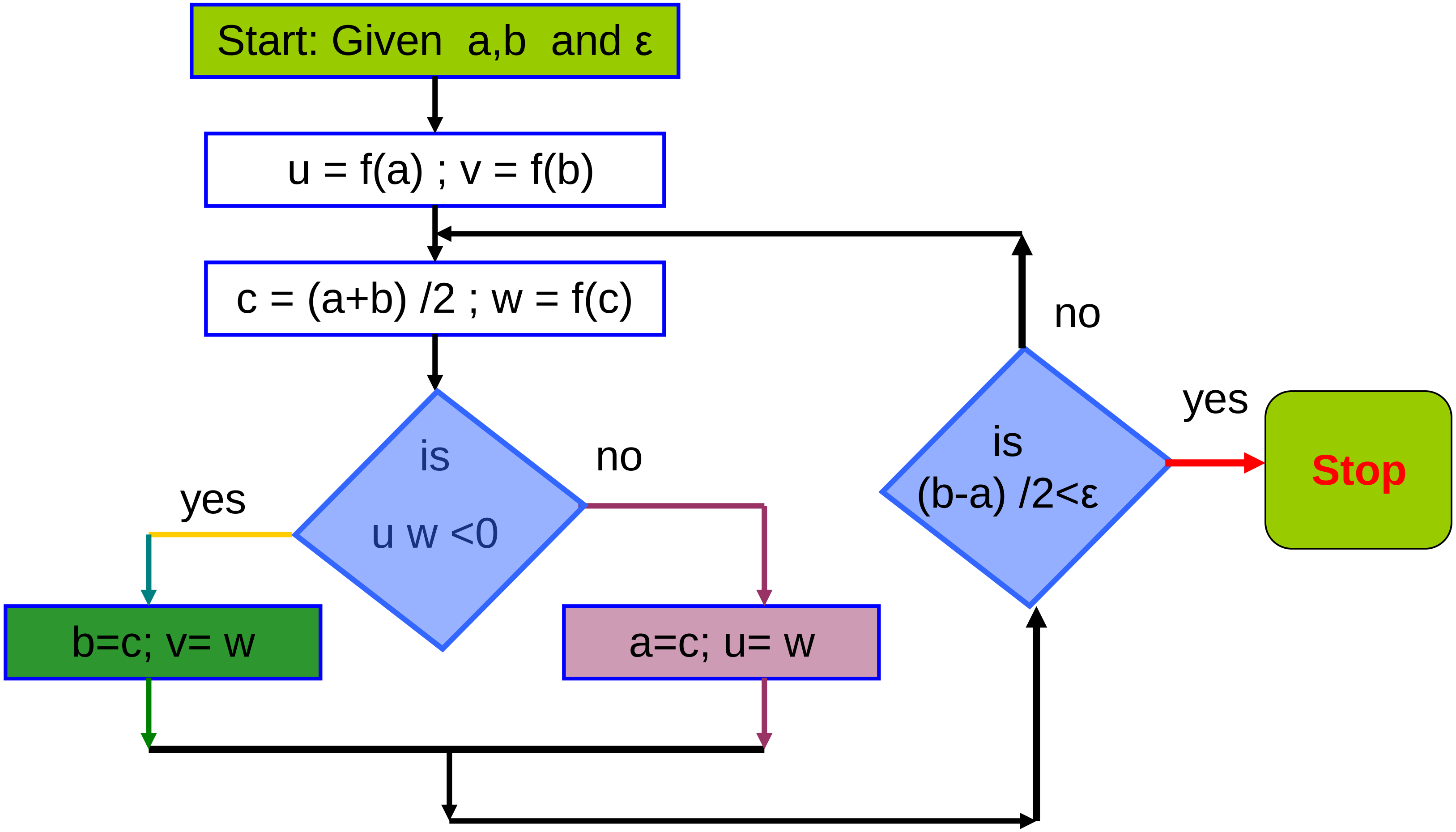
Bisection Method



Example



Flow Chart of Bisection Method



Example

Can you use Bisection method to find a zero of :

$f(x) = x^3 - 3x + 1$ in the interval $[0,2]$?

Answer:

$f(x)$ is continuous on $[0,2]$

and $\mathrm{f}(0) * \mathrm{f}(2) = (1)(3) = 3 > 0$

$\Rightarrow$ Assumptions are not satisfied

$\Rightarrow$ Bisection method can not be used

Example

Can you use Bisection method to find a zero of :

$f(x) = x^3 - 3x + 1$ in the interval [0,1]?

Answer:

$f(x)$ is continuous on [0,1]

and $\mathrm{f}(0) * \mathrm{f}(1) = (1)(-1) = -1 < 0$

$\Rightarrow$ Assumptions are satisfied

$\Rightarrow$ Bisection method can be used

Best Estimate and Error Level

Bisection method obtains an interval that is guaranteed to contain a zero of the function.

Questions:

- What is the best estimate of the zero of ***f(x)***?
- What is the error level in the obtained estimate?

Best Estimate and Error Level

The best estimate of the zero of the function ***f(x)*** after the first iteration of the Bisection method is the mid point of the initial interval:

$$\text{Estimate of the zero}: \ r = \frac{b+a}{2}$$

$$\text{Error} \leq \frac{b-a}{2}$$

Stopping Criteria

Two common stopping criteria

1. Stop after a fixed number of iterations
2. Stop when the absolute error is less than a specified value

How are these criteria related?

Stopping Criteria

c_n : is the midpoint of the interval at the n[th] iteration

(c_n is usually used as the estimate of the root).

r : is the zero of the function.

After n iterations :

$$|error| = |r - c_n| \leq E_a^n = \frac{b-a}{2^n} = \frac{\Delta x^0}{2^n}$$

Convergence Analysis

$Given \quad f(x), a, \ b, and \ \ \varepsilon$

How many iterations are needed such that: $|x - r| \le \varepsilon$

where r is the zero of $f(x)$ and x is the

bisection estimate (i.e., $x = c_k$)?

$$\boxed{n \ge \frac{\log(b-a) - \log(\varepsilon)}{\log(2)}}$$

Convergence Analysis – Alternative Form

$$n \geq \frac{\log(b-a) - \log(\varepsilon)}{\log(2)}$$

$$n \geq \log_2\left(\frac{\text{width of initial interval}}{\text{desired error}}\right) = \log_2\left(\frac{b-a}{\varepsilon}\right)$$

Example

$a = 6, \; b = 7, \; \varepsilon = 0.0005$

How many iterations are needed such that: $|x - r| \le \varepsilon$?

$$n \ge \frac{\log(b-a) - \log(\varepsilon)}{\log(2)} = \frac{\log(1) - \log(0.0005)}{\log(2)} = 10.9658$$

$$\Rightarrow n \ge 11$$

Example

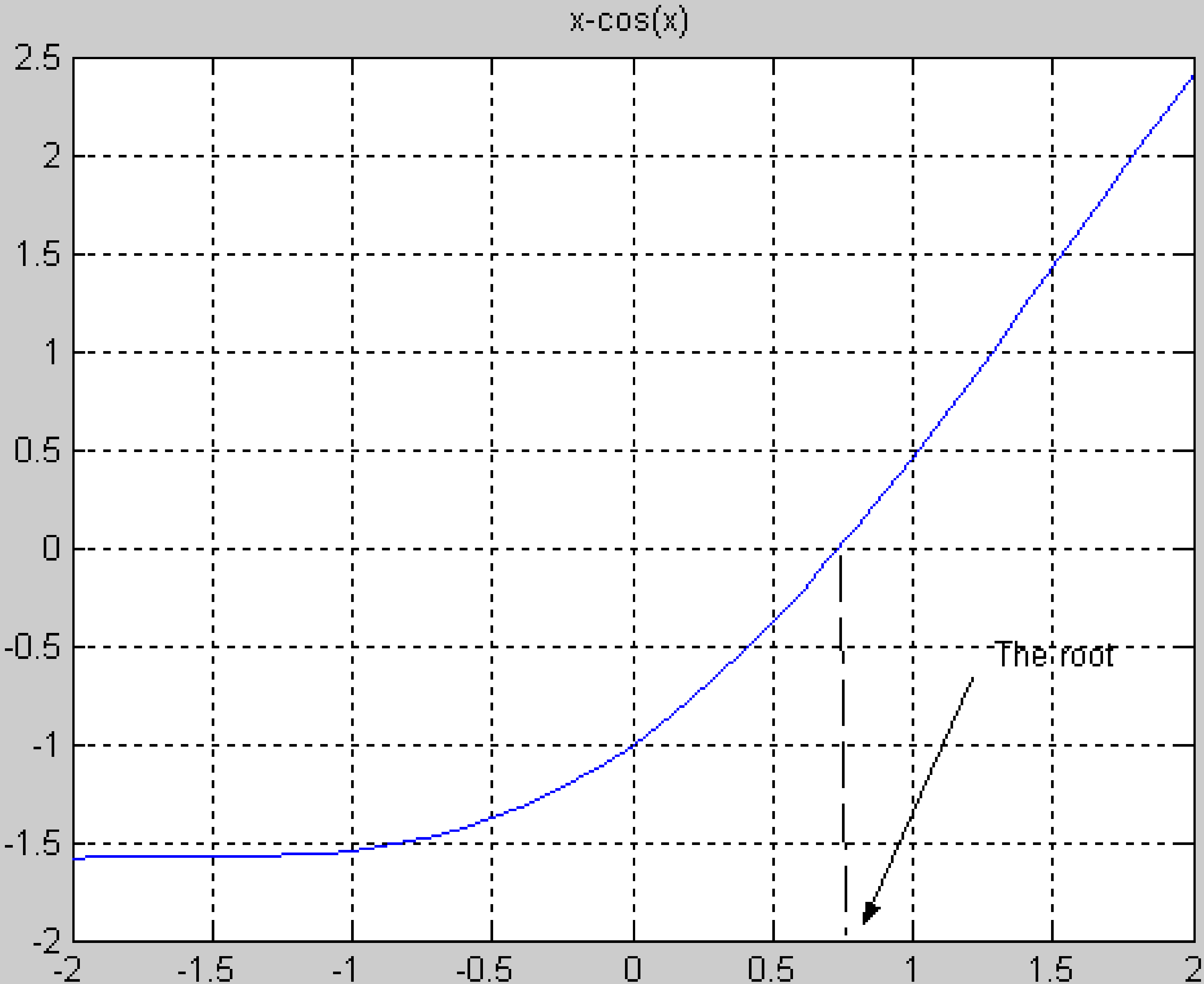
- Use Bisection method to find a root of the equation $x = \cos(x)$ with absolute error <0.02
  (assume the initial interval [0.5, 0.9])

Question 1: What is $f(x)$?

Question 2: Are the assumptions satisfied ?

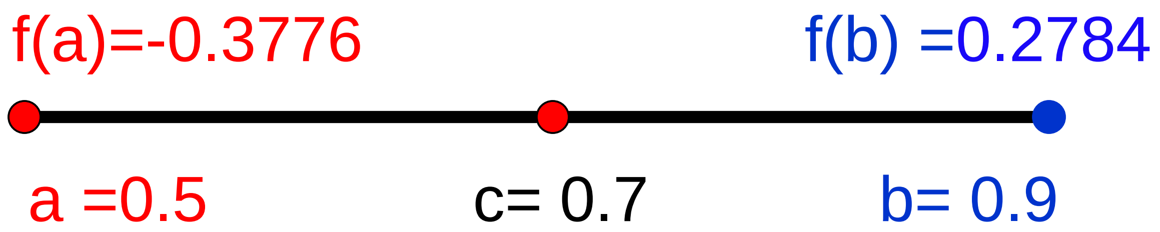
Question 3: How many iterations are needed ?

Question 4: How to compute the new estimate ?


x-cos(x)
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
The root

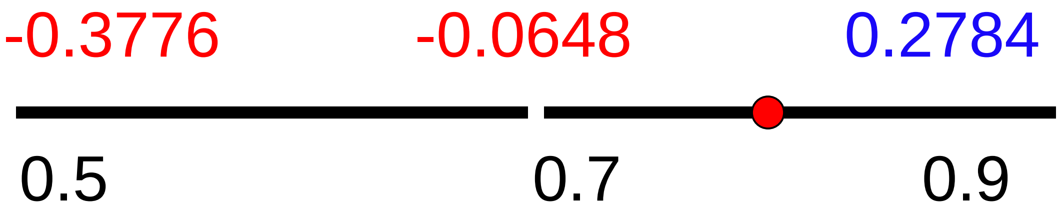
Bisection Method

Initial Interval

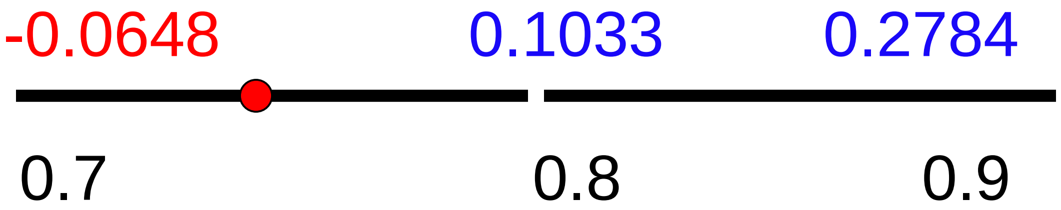


Error < 0.2

Bisection Method

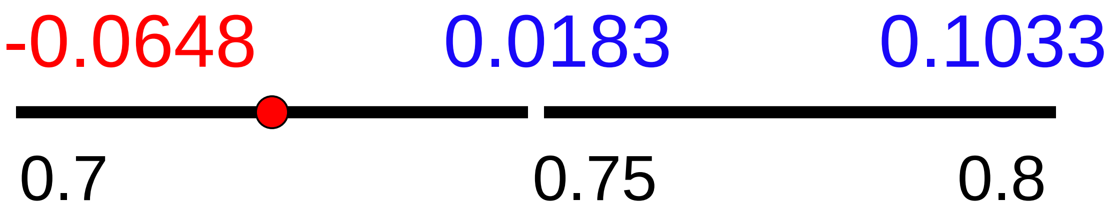


Error < 0.1

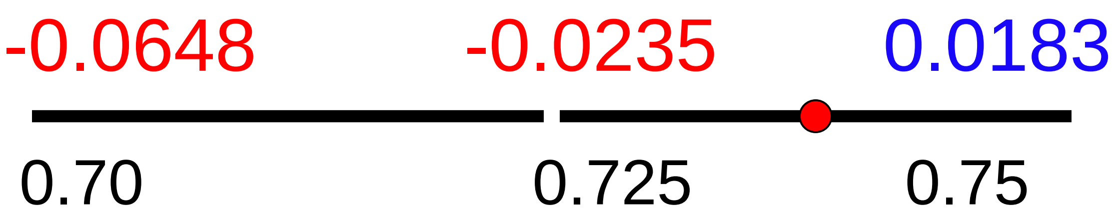


Error < 0.05

Bisection Method



Error < 0.025



Error < .0125

Summary

- Initial interval containing the root: [0.5,0.9]

- After 5 iterations:
  - Interval containing the root: [0.725, 0.75]
  - Best estimate of the root is 0.7375
  - | Error | < 0.0125

A Matlab Program of Bisection Method

```
a=.5; b=.9;
u=a-cos(a);
v=b-cos(b);
    for i=1:5
        c=(a+b)/2
        fc=c-cos(c)
        if u*fc<0
            b=c ; v=fc;
        else
            a=c; u=fc;
        end
    end
```

```
c =
    0.7000
fc =
   -0.0648
c =
    0.8000
fc =
    0.1033
c =
    0.7500
fc =
    0.0183
c =
    0.7250
fc =
  -0.0235
```

Example

Find the root of:

$$f(x) = x^3 - 3x + 1 \text{ in the interval }:[0,1]$$

$*\quad f(x)$ is continuous

$*\quad f(0) = 1,\ \ f(1) = -1 \Rightarrow\ f(a)\ f(b) < 0$

$\Rightarrow$ Bisection method can be used to find the root

Example

Iteration	a	b	c= $\frac{(a+b)}{2}$	f(c)	$\frac{(b-a)}{2}$
1	0	1	0.5	-0.375	0.5
2	0	0.5	0.25	0.266	0.25
3	0.25	0.5	.375	-7.23E-3	0.125
4	0.25	0.375	0.3125	9.30E-2	0.0625
5	0.3125	0.375	0.34375	9.37E-3	0.03125

Bisection Method

Advantages

- Simple and easy to implement
- One function evaluation per iteration
- The size of the interval containing the zero is reduced by 50% after each iteration
- The number of iterations can be determined a priori
- No knowledge of the derivative is needed
- The function does not have to be differentiable

Disadvantage

- Slow to converge
- Good intermediate approximations may be discarded