- Tutorial 7 -

Numerical Methods and Optimization

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Numerical integration

Exercise 1: **Newton-Cotes formulas**

(1)- Let $P_1(x)$ be a polynomial of degree less than or equal to 1 such that: $P_1(x) = ax + b$, and let f a function such that: $f(x) = \frac{1}{x}$.

We want to approximate the function f with $P_1(x)$ in the sense of continuous least squares over $[\frac{1}{2}, 1]$.

We have to find the coefficients a and b such that the error $\int_{\frac{1}{2}}^{1} (f(x) - P_1(x))^2 \, dx$ is minimzed.

The error $g(a,b)$ is given by:

$$g(a,b) = \int_{\frac{1}{2}}^{1} (f(x) - P_1(x))^2\, dx = \int_{\frac{1}{2}}^{1} (f(x) - ax - b)^2\, dx$$

$$= \int_{\frac{1}{2}}^{1} f(x)^2 - 2f(x)(ax+b) + (a^2x^2 + 2abx + b^2)dx$$

$$= \int_{\frac{1}{2}}^{1} f(x)^2 - 2b\int_{\frac{1}{2}}^{1} f(x)dx - 2a\int_{\frac{1}{2}}^{1} xf(x)dx + b(1-\tfrac{1}{2}) + ab(1^2 - \tfrac{1}{2}^2) + \tfrac{1}{3}a^2(1^3 - \tfrac{1}{2}^3)$$

To minimize the error $g(a,b)$ we invoke the conditions

$$\frac{\partial g}{\partial a} = -2\int_{\frac{1}{2}}^{1} xf(x)dx + b(1^2 - \tfrac{1}{2}^2) + \tfrac{2}{3}a(1^3 - \tfrac{1}{2}^3) = 0$$

$$\frac{\partial g}{\partial b} = -2\int_{\frac{1}{2}}^{1} f(x)dx + 2b(1-\tfrac{1}{2}) + a(1^2 - \tfrac{1}{2}^2) = 0$$

We get:

$2b + a = 2\int_{\frac{1}{2}}^{1} \frac{1}{x} dx$

$b + \frac{2}{3}a = 2\int_{\frac{1}{2}}^{1} x\frac{1}{x} dx$

$$\Leftrightarrow \begin{bmatrix} 1 & \frac{3}{4} \\ \frac{3}{4} & \frac{7}{12} \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} -2ln(\frac{1}{2}) \\ 1 \end{bmatrix}$$

Finally:

$a = -1.9068$ and $b = 2.8164$

So $\frac{1}{x} \simeq 2.8164 - 1.9068x$

(2)- Calculate $\int_{\frac{1}{2}}^{1} f(x)$ with two methods.

- $\int_{\frac{1}{2}}^{1} f(x) \simeq \int_{\frac{1}{2}}^{1} P_1(x) = \left[2.8164x - 0.6931x^2\right]_{\frac{1}{2}}^{1} = 0.8884$
- $\int_{\frac{1}{2}}^{1} f(x) = 0.6931$

Exercise 2: **Practice problems**

(1)- Compute the trapezoidal approximation for $\int_0^2 \sqrt{x}\, dx$ using a regular partition with $n = 4$.

$n = 4$ is the number of subintervals, first we find h such that $h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$, and $x_i = a + ih = ih, i = 0, \ldots, 4$.

Then we have $n+1$ points: $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}$ and $x_4 = 2$.

We calculate using the Trapezoidal rule:

$$\int_0^2 \sqrt{x}dx \simeq \frac{1}{2}\left(\frac{f(2)+f(0)}{2} + \sum_{i=1}^{n-1} f(a+ih)\right) = \frac{1}{2}\left(\frac{\sqrt{2}}{2} + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right)\right)$$

$$\simeq \frac{1}{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{6}}{3}\right) = 1.81948$$ The exact value is: $\frac{4\sqrt{2}}{3} = 1.88562$

The approximation underestimates the actual area with an error of 3.51% of the exact value.

(2)- Use Simpsonís rule to approximate $\int_0^2 \sqrt{x}\, dx$ using a regular partition with $n = 4$.
The expression of Simpson's rule is as follows:

$$Area = \frac{h}{3}[y_{first} + 4(y_{odd}) + 2(y_{even}) + y_{last}]$$

$n = 4$ is the number of subintervals, first we find h such that $2h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$ then $h = \frac{1}{4}$, and $x_i = a + ih = ih = \frac{i}{4}, i = 0, ..., 8$.

Then we have $2n + 1 = 9$ points: $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}, x_4 = 1, x_5 = \frac{5}{4}, x_6 = \frac{6}{4}, x_7 = \frac{7}{4}$ and $x_8 = 2$.

We calculate using Simpson's rule:

$$\int_0^2 \sqrt{x}dx \simeq \frac{h}{3}\left(f(2) + f(b) + 2\sum_{i=1}^{n-1} f(a + 2ih) + 4\sum_{i=1}^{n} f(a + (2i - 1)h)\right)$$

$$\simeq \frac{1}{12}\left(\sqrt{2} + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + 2f(x_6) + 4f(x_7)\right)$$

$$\simeq \frac{1}{6}\left(3\sqrt{2} + 2\sqrt{6} + 2\right) = 1.8569367$$

The exact value is: $\frac{4\sqrt{2}}{3} = 1.88562$
The approximation underestimates the actual area with an error of 1.52% of the exact value.

Exercise 3: **Generalized Trapezoidal/Simpson methods**

Let the integral $I = \int_0^{\pi} \sin x \, dx$. **(1)-** Calculate the exact value of I:

$I = \left[-cos(x)\right]_0^{\pi} = -cos(\pi) + cos(0) = 2$

(2)- (a) Calculate I:

- Using the trapezoidal method for $h = \frac{\pi}{4} \Rightarrow n = \frac{\pi - 0}{\frac{\pi}{4}} = 4$.

  Then we have $n + 1 = 5$ points: $x_0 = 0, x_1 = \frac{\pi}{4}, x_2 = \frac{\pi}{2}, x_3 = \frac{3\pi}{4}$ and $x_4 = \pi$.

  $$\int_0^{\pi} \sin x dx \simeq \frac{\pi}{4}\left(\frac{f(\pi)+f(0)}{2} + f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}\left(0 + \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2}\right) = 1.89611$$

- Using Simpson's method for $h = \frac{\pi}{4}$. (Here we have $n = 2m \Rightarrow n = 4, m = 2$).
  Then we have the points: $x_0 = 0, x_1 = \frac{\pi}{4}, x_2 = \frac{\pi}{2}, x_3 = \frac{3\pi}{4}$ and $x_4 = \pi$.

$$\int_0^{\pi} \sin x dx \simeq \frac{\pi}{12}\left(f(\pi) + f(0) + 4f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{2}\right) + 4f\left(\frac{3\pi}{4}\right)\right)$$

$$\simeq \frac{\pi}{12}\left(0 + 0 + 2\sqrt{2} + 2 + 2\sqrt{2}\right) = 2.00456$$

(2)- (b) Estimate the error:

We know that the bound of the error e_n for n subintervals is given by:

$e_n \leq \frac{(b-a)^3}{12n^2} \max_{[a,b]} \left|f''(x)\right|$ For the trapezoidal rule.

And:

$e_n \leq \frac{(b-a)^5}{180(n)^4} \max_{[a,b]} \left|f^{(4)}(x)\right|$ For Simpson's rule.

So for the trapezoidal method we have $n = \frac{b-a}{h} = 4$ subintervals we get:

$e_n \leq \frac{\pi^3}{192} \max_{[a,b]} (sin(x))$

$e_n \leq 0.1614$

For Simpson's method ($n = 2m$) we have $m = \frac{b-a}{2h} = 2 \Rightarrow n = 4$ subintervals we get:

$e_n \leq \frac{\pi^5}{46080} \max_{[a,b]} (sin(x))$

$e_n \leq 0.006641$

(2)- (b) Evaluate the error:

So for the trapezoidal method, we have:

$$e_n = \frac{|2-1.89611|}{2} = 0.05194 < 0.1614$$

For Simpson's method, we have:

$$e_n = \frac{|2-2.00455|}{2} = 0.0023 < 0.006641$$

(3)- Provide the value of step h and the number of subdivisions of the interval $[0, \pi]$ so that the error obtained by the generalized trapezoidal method (resp. Simpson's) is less than 5×10^{-4}

- For the Trapezoidal rule, we have to find n such that:

$$\frac{(b-a)^3}{12n^2} \max_{[a,b]} \left|f''(x)\right| \leq 5 \times 10^{-4} \Rightarrow \frac{pi^3}{12n^2} \leq 5 \times 10^{-4} \Rightarrow \frac{12n^2}{\pi^3} \geq \frac{1}{5\times10^{-4}} \Rightarrow n \geq \sqrt{\frac{\pi^3}{60\times10^{-4}}}$$

$\Rightarrow n \geq 71.88 \Rightarrow n = 72$

- For Simpson's rule, we have to find n such that:

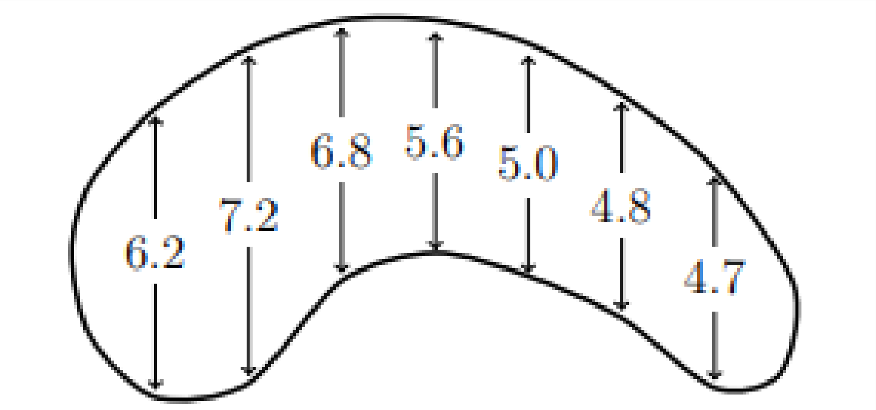
$$\frac{(b-a)^5}{180(n)^4} \max_{[a,b]} \left|f^{(4)}(x)\right| \leq 5 \times 10^{-4} \Rightarrow 180n^4 \geq \frac{\pi^5}{5\times10^{-4}} \Rightarrow n \geq \sqrt[4]{\frac{\pi^5}{0.09}}$$

$\Rightarrow n \geq 7.636 \Rightarrow n = 8$ and $m = 4$.

Exercise 4: **A problem**

If we know the width of the pool at every point, we could get the area as the integral of the width.

The pool is 16 meters long, we divide it into $n = 8$ parts of length $n = 2m$



According to the Simpson's rule, an estimation of the area is:

$$A = \frac{2}{3}(1 \times 0 + 4 \times 6.2 + 2 \times 7.2 + 4 \times 6.8 + 2 \times 5.6 + 4 \times 5 + 2 \times 4.8 + 4 \times 4.7 + 1 \times 0) = 84$$

Such that we have 8 points from x_0 to x_7.
Note: We need to include the two ends of the pool where the width is zero.