Numerical Methods and Optimization

25/2/2024 – 3^{rd} Year

ENSIA — Spring 2024

Tutorial 3

- Direct methods for solving systems of linear equations -

Exercise 1: True or False

Are the following statements true or false?

(1) The matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ admits a Cholesky decomposition.

(2) The matrix $B = \begin{pmatrix} 1 & -2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is symmetric positive definite.

(3) The matrix B above admits an LU decomposition.

(4) The matrix $\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$ can be written as C^tC.

(5) The matrix $\begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$ admits a Cholesky decomposition $A = C^tC$ with $C = \begin{pmatrix} -1 & -1 \\ 0 & -2 \end{pmatrix}$

(6) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$

(a) The matrix AA^t admits a Cholesky decomposition.

(b) The matrix A^tA admits a Cholesky decomposition.

Exercise 2: Gaussian elimination

We want to solve the linear system $Ax = b$ with

$$A = \begin{pmatrix} 1 & -2 & 0 & 1 \\ -2 & 13 & 3 & 7 \\ 0 & 3 & 2 & 1 \\ 4 & -8 & -2 & 12 \end{pmatrix} \quad and \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

(1) Give the triangular system $A^{(4)} = b^{(4)}$ obtained by Gauss elimination.

(2) Deduce $det(A)$ and the inverse matrix A^{-1}.

(3) Deduce from question **(1)** the inverse L^{-1} of LU decomposition.

Exercise 3: Gauss elimination pivot

Let's assume numbers are represented in floating-point decimal with 3 significant digits, and the result of operations is rounded to 3 significant digits. Consider the linear system Ax = b with

$$\begin{pmatrix} 10^{-4} & 1 \\ 1 & 1 \end{pmatrix} \quad and \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(1) Solve using the Gauss elimination method by choosing 10^{-4} as the first pivot.

(2) Solve using the Gauss elimination method by choosing the second row as the pivot at the first step.

(3) Conclude.

Exercise 4: About the decomposition LL^t

Let A be a square matrix of order n, symmetric positive definite, and dense. We want to solve the system $Ax = b$.
Two methods are proposed to solve this system:

- Calculate A^2, perform the LL^t decomposition of A^2, solve the system $LL^t x = b$.
- Calculate the LL^t decomposition of A, solve the systems $LL^t y = b$ and $LL^t x = y$.

Calculate the number of elementary operations required for each of the two methods and compare.

Exercise 5: LU and Cholesky decompositions

Let $M = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

(1) Calculate the principal minors of M. Deduce that M admits LU and Cholesky decompositions.

(2) Provide the LU decomposition of M.

(3) Provide the Cholesky decomposition of M.

(4) Solve the system $Ax = b$ using the previous decomposition such that $b = \begin{pmatrix} 8 \\ 13 \\ 5 \end{pmatrix}$