**The National Higher School of Artificial Intelligence ENSIA**

**Stochastic Modeling and Simulation Semester 5 2023/2024**

# Worksheet 1

**Exercice 1** *Suppose that the weather from one day to the next is described by a Markov chain on the* $0, 1, 2$ ($0$ *for sunny,* $1$ *for cloudy,* $2$ *for rainy*) *whose transition matrix is given by*

$$\begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

*It's Tuesday and cloudy.*

1. *Draw the representative graph of this chain.*
2. *Calculate the probability that the next three days will be sunny.*
3. *Calculate the probability that next Friday will be sunny.*

**Exercice 2** *A Markov chain on three states, say* $0, 1$ *and* $2$*, has as transition matrix*

$$P = \begin{pmatrix} 1/4 & 3/4 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}.$$

1. *Draw the representative graph of this chain and determine the types of the different states;*
2. *Determine the* $n$*-step transition matrix;*
3. *Determine the limit of this matrix when* $n$ *goes to infinity.*

**Exercice 3** *The successive results of a player's chess games against chess software follow a Markov chain on the states* $V$ *for win,* $D$ *for loss and* $N$ *for draw with the corresponding transition matrix given by*

$$P = \begin{pmatrix} 3/4 & 0 & 1/4 \\ 0 & 3/4 & 1/4 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}.$$

*Determine*

1. *The average proportion of long-term wins for this player;*
2. *The expected number of games from one win to the next.*

**Exercice 4** *A Markov chain over a finite number of states with transition matrix* $P$ *is said to be regular if there exists an integer* $N \geq 1$ *such that matrix* $P^N$ *is strictly positive (i.e. all its entries are greater than* $0$*). Show that the Markov chain on states* $0, 1, 2, 3$ *and* $4$ *with the transition matrix below is regular and find its stationary distribution:*

$$\begin{pmatrix} 3/4 & 1/4 & 0 & 0 & 0 \\ 3/4 & 0 & 1/4 & 0 & 0 \\ 3/4 & 0 & 0 & 1/4 & 0 \\ 3/4 & 0 & 0 & 0 & 1/4 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**Exercice 5** *A slot machine in a casino gives you* 30 *DZD if you win a game, but you have to pay* 20 *DZD per game to play. The machine advertises that there is always at least a* 50/50 *chance of winning each game. In fact, the probability of success in a game is equal to* $\frac{k+1}{k+2}$*, where* $k$ *represents the number of successes in the two previous games. Is this slot machine profitable for the casino in the long term?*

**Exercice 6** *An electronics store keeps a maximum of* 3 *units of a product in inventory, so each day the demand is* $0, 1, 2$ *and* 3 *with corresponding probabilities* $\frac{3}{10}, \frac{4}{10}, \frac{2}{10}$ *and* $\frac{1}{10}$. *It renews the inventory at* 3 *units for the next morning only if the number of units of the product is less than or equal to* 1 *at the end of the day.*

*Determine*

1. *The transition matrix for the number of units at the end of one day, from one day to the next;*
2. *The long-term average net profit per day if the profit on a unit sold is* 120 *DZD and the cost of keeping a unit in inventory overnight is* 20 *DZD. Compare this profit with the one obtained in the case where the inventory is always renewed at* 3 *units for the next morning.*

**Exercice 7** *Consider a four-state Markov chain* $S = \{1, 2, 3, 4\}$ *whose transition matrix is*

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

1. *Draw the representative graph associated with this Markov chain.*
2. *Show that states* 3 *and* 4 *are absorbing. Are the others transient or recurrent?*
3. *Calculate the probabilities of the following trajectories as a function of the initial probabilities of each state* $(p_0, q_0, r_0, s_0)$*:*

   $(X_0 = 1, \forall n \geq 1 \quad X_n = 3), (X_0 = 1, X_1 = 2, \forall n \geq 2 \quad X_n = 4)$,

   $(X_n = 1$ *if* $n$ *is even and* $X_n = 2$ *if* $n$ *is odd).*
4. *Show that the path* $(X_n = 1$ *if* $n$ *is even and* $X_n = 2$ *if* $n$ *is odd) has probability zero.*
5. *Assume that the distribution between the four states is uniform at initial time* $t = 0$. *Calculate the distribution at time* $t = 1$ *and then at time* $t = 2$. *Same question if we start from an initial distribution* $\pi_0 = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$.
6. *Show that any initial distribution of the form* $\pi_0 = (0, 0, r_0, s_0)$ *with* $r_0 + s_0 = 1$ *is a stationary distribution. Are there any others?*