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Improvements and extensions to the Miller–Tucker–Zemlin subtour elimination constraints

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This paper shows how the subtour elimination constraints developed by Miller, Tucker and Zemlin for the traveling salesman problem can be improved and extended to various types of vehicle routing problems.

traveling salesman problem * vehicle routing problem * subtour elimination constraints * lifting * facets

1. Introduction

Consider a complete graph $G = (V, A)$ where $V = \{v_1, \ldots, v_n\}$ is the vertex set and $A = \{(v_i, v_j)\colon v_i, v_j \in V, i \neq j\}$ is the arc set. Let $C = (c_{ij})$ be a matrix of positive costs, distances, or travel times associated with A. The assumption that G is complete is not restrictive since missing arcs can be introduced with an arbitrarily large cost. The *traveling salesman problem* (TSP) consists of determining a least cost Hamiltonian circuit on G. Several types of *vehicle routing problems* (VRPs) can also be defined on G. Let v_1 represent a *depot* at which m identical vehicles are based. VRPs consist of determining m routes of least total cost, starting and ending at the depot, in such a way that every vertex of $V \setminus \{v_1\}$ is visited exactly once, and that some side constraints are satisfied:

(i) In the *capacitated* VRP (CVRP), there is a positive weight q_i at vertex v_i $(i = 2, \ldots, n)$ and all vehicles have the same capacity Q. Then the sum of weights of any route must not exceed Q.

(ii) In the *distance constrained* VRP (DVRP), the total length of any route may not exceed a preset bound L.

(iii) In the VRP *with time windows* (TWVRP), every vertex v_i must be visited within a specified time frame $[a_i, b_i]$ $(i = 2, \ldots, n)$. Note that waiting is allowed, i.e. a vehicle may arrive at v_i before a_i and wait until a_i to actually visit that vertex.

Comprehensive references on the TSP and on several types of VRPs can be found in Lawler et al. [9], Laporte and Nobert [8] and Golden and Assad [3]. For the TWVRP, see Desrochers et al. [2].

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Several formulations have been suggested for the TSP. The following formulation, due to Dantzig, Fulkerson and Johnson (DFJ) [1], associates one variable x_{ij} to every directed arc (v_i, v_j). It is valid for the asymmetric TSP (ATSP) and thus, for the symmetric case. Let x_{ij} be a binary variable equal to 1 if and only if arc (v_i, v_j) is used in the solution. Then the DFJ formulation is

$$\text{(DFJ)} \quad \text{minimize} \quad \sum_{i \neq j} c_{ij} x_{ij} \tag{1}$$

$$\text{subject to} \quad \sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ij} = 1 \quad (i = 1, \ldots, n), \tag{2}$$

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} x_{ij} = 1 \quad (j = 1, \ldots, n), \tag{3}$$

$$\sum_{v_i, v_j \in S} x_{ij} \leqslant |S| - 1 \quad (S \subset V,\ |S| \geqslant 2), \tag{4}$$

$$x_{ij} \in \{0, 1\} \quad (i,\ j = 1, \ldots, n;\ i \neq j). \tag{5}$$

Constraints (2) and (3) are degree constraints, and constraints (4) are subtour elimination constraints. Grötschel and Padberg [4] have proved that the latter are *facets* of P_A^n, the polytope of the convex hull of feasible ATSP solutions defined by (2)–(5).

The Miller, Tucker and Zemlin (MTZ) formulation [10] reduces the number of subtour elimination constraints at the expense of extra free variables u_i $(i = 2, \ldots, n)$. The MTZ subtour elimination constraints can be expressed as

$$u_i - u_j + (n-1) x_{ij} \leqslant n - 2 \quad (i,\ j = 2, \ldots, n;\ i \neq j), \tag{6}$$

$$1 \leqslant u_i \leqslant n - 1 \quad (i = 2, \ldots, n). \tag{7}$$

Constraints (7) ensure that there is only one solution corresponding to a given feasible tour. It is well known that this formulation produces a weak LP relaxation [6,12]. This can be explained by the fact that the MTZ subtour elimination constraints are not facets of the polytope of the convex hull of feasible solutions defined by (2), (3), (5), (6) and (7).

In spite of their relative weakness, these constraints offer some advantages: they generalize quite easily to a number of VRP formulations and, in a subtour elimination constraint relaxation algorithm, it is straightforward to identify which of these (generalized) constraints are violated. Note that the DFJ subtour elimination constraints can also easily be extended to the CVRP [7]. However, their generalization to the DVRP is not so easy [7] and no such extension is known for the TWVRP. Another advantage of the MTZ subtour elimination constraints is that they can be incorporated into other types of formulations, in conjunction with stronger constraints. Consider, for example, the VRP with capacity and distance restrictions [7]. When such a problem is solved by branch and bound, violated DFJ capacity constraints imposed at any node of the search tree remain valid throughout the algorithm. The validity of DFJ distance constraints, on the other hand, depends on the fixed variables at the time of their generation and these constraints must therefore be removed when backtracking. A possible approach is then to use the DFJ constraints for capacity violations and the MTZ constraints for distance violations, since these constraints are valid for every subproblem of the branch and bound tree.

The object of this paper is to strengthen the MTZ subtour elimination constraints for the TSP and to generalize these stronger constraints to a variety of VRPs. This is done in Sections 2 and 3, respectively. Computational results are reported in Section 4.

2. Lifting the MTZ subtour elimination constraints for the TSP

The MTZ subtour elimination constraints for the TSP can be strengthened by using a lifting technique [11,12].

Proposition 1. *The constraints*

$$u_i - u_j + (n-1)x_{ij} + (n-3)x_{ji} \leqslant n-2 \quad (i,\ j = 2,\dots,n;\ i \neq j) \tag{8}$$

with $u_i,\ u_j \in [1,\ n-1]$, *are valid inequalities for the TSP.*

Proof. Consider the MTZ constraints

$$u_i - u_j + (n-1)x_{ij} + \alpha_{ji}x_{ji} \leqslant n-2, \tag{9}$$

where currently $\alpha_{ji} = 0$. The lifting process computes the largest possible value for α_{ji} so that (9) remains a valid inequality. There are two cases: $x_{ji} = 0$ and $x_{ji} = 1$.

Case 1. $x_{ji} = 0$. Then (9) is satisfied for any α_{ji}.

Case 2. $x_{ji} = 1$. This implies $x_{ij} = 0$ (for $n > 2$) and $u_j + 1 = u_i$, so that $\alpha_{ji} \leqslant n-3$. □

It should be noted that (8) cannot be lifted further since for any other variable x_{kl}, there is a tight solution to the inequality if $x_{kl} = 0$ or $x_{kl} = 1$. We now derive some properties of P_{M}^n, the polytope of the convex hull of integer solutions defined by (2), (3), (5), (7) and (8).

Lemma 1. *There is a one to one correspondence between the integer points of* P_{M}^n *and those of* P_{A}^n.

Proof. Dropping the u_i variables from an MTZ solution $(x,\ u)$ trivially provides a feasible ATSP solution (x). From an ATSP solution (x), a feasible MTZ solution $(x,\ u)$ is uniquely determined by solving the following system for (u):

$$u_1 = 0,$$

$$1 \leqslant u_i \leqslant n-1 \quad (i = 2,\dots,n),$$

$$u_{i_k} - u_{i_{k+1}} \leqslant -1 \quad \left(i_1 = 1;\ x_{i_k i_{k+1}} = 1;\ k = 1,\dots,n-1\right). \quad \square$$

Lemma 2. *Consider a set of p solutions to MTZ and the corresponding set of ATSP solutions as defined by Lemma* 1. *Then, the p MTZ solutions are affinely independent if and only if the p ATSP solutions are affinely independent.*

Proof. We proceed by proving the contraposition in each direction. First consider p affinely dependent solutions $(\bar{x}^k,\ \bar{u}^k)$ $(k = 1,\dots,p)$ to MTZ. Then, there exists a vector $(\lambda_1,\dots,\lambda_p) \in \mathrm{R}^p$ with at least one non-zero component, and such that

$$\sum_{k=1}^{p} \lambda_k = 0, \quad \sum_{k=1}^{p} \lambda_k \bar{x}^k = 0 \quad \text{and} \quad \sum_{k=1}^{p} \lambda_k \bar{u}^k = 0.$$

The first two conditions ensure that the corresponding ATSP solutions are also affinely dependent. Now consider p affinely dependent ATSP solutions $(\bar{x}^k)$ $(k = 1,\dots,p)$. Without loss of generality, we can express $(\bar{x}^p)$ as an affine combination of $(\bar{x}^1),\dots,(\bar{x}^{p-1})$, i.e. $(\bar{x}^p) = \sum_{k=1}^{p-1}\lambda_k(\bar{x}^k)$, where $\sum_{k=1}^{p-1}\lambda_k = 1$. Let $(\bar{x}^k,\ \bar{u}^k)$ be the MTZ solution corresponding to $(\bar{x}^k)$ $(k = 1,\dots,p)$ as defined in Lemma 1, and let

$$(\bar{x}) = \sum_{k=1}^{p-1} \lambda_k(\bar{x}^k), \qquad (\bar{u}) = \sum_{k=1}^{p-1} \lambda_k(\bar{u}^k).$$

We must show that $(\bar{u}) = (\bar{u}^p)$, or alternatively, that $(\bar{u})$ is feasible for MTZ since there must be a unique feasible solution $(\bar{x}^p, \bar{u}^p)$ corresponding to $(\bar{x}^p)$. Constraints (2), (3), (5), and (7) are trivially satisfied. Substituting $(\bar{x}, \bar{u})$ in (8) yields

$$\begin{aligned}
&\bar{u}_i - \bar{u}_j + (n-1)\bar{x}_{ij} + (n-3)\bar{x}_{ji} \\
&= \sum_{k=1}^{p-1} \lambda_k \bar{u}_i^k - \sum_{k=1}^{p-1} \lambda_k \bar{u}_j^k + (n-1) \sum_{k=1}^{p-1} \lambda_k \bar{x}_{ij}^k + (n-3) \sum_{k=1}^{p-1} \lambda_k \bar{x}_{ji}^k \\
&= \sum_{k=1}^{p-1} \lambda_k \left(\bar{u}_i^k - \bar{u}_j^k + (n-1)\bar{x}_{ij}^k + (n-3)\bar{x}_{ji}^k \right) \\
&\leqslant (n-2) \sum_{k=1}^{p-1} \lambda_k = (n-2) \quad (i,\ j = 2,\dots,n;\ i \neq j). \quad \square
\end{aligned}$$

Proposition 2. (i) *The dimension of* P_{M}^n *is equal to* $n^2 - 3n + 1$ *for* $n \geqslant 3$.

(ii) *Constraints* (8) *define facets of* P_{M}^n *for* $n \geqslant 6$.

Proof. (i) The dimension of P_{A}^n is equal to $n^2 - 3n + 1$ for $n \geqslant 3$ (Theorem 1, [4]) and the result follows directly from Lemma 2.

(ii) Using again Theorem 1 of [4], subtour elimination constraints (4) define facets of P_{A}^n for $|S| = 2$ and $n \geqslant 6$. Now consider an ATSP solution with $x_{ij} + x_{ji} = 1$ for some arc (v_i, v_j), in other words, (4) is tight for $S = \{v_i, v_j\}$. Then in the corresponding MTZ solution, constraint (8) is also tight for (v_i, v_j) and for (v_j, v_i). Using Lemma 1, and the fact that the correspondence between P_{A}^n and P_{A}^n remains one to one when $x_{ij} + x_{ji} = 1$, we can state that the faces defined by constraints (8) have the same dimension as those defined by constraints (4) when $|S| = 2$ and $n \geqslant 6$. The conclusion follows from this observation. $\square$

The lifting technique can also be applied to the bounds on the u_i variables:

$$1 \leqslant u_i \leqslant n-1 \quad (i = 2,\dots,n).$$

The following proposition and most subsequent lifted inequalities will be stated without proof, since their validity can be established by using an argument similar to that of Proposition 1.

Proposition 3. *The constraints*

$$u_i \geqslant 1 + (n-3)x_{i1} + \sum_{\substack{j=2 \\ j \neq i}}^{n} x_{ji} \quad (i = 2,\dots,n) \tag{10}$$

and

$$u_i \leqslant n - 1 - (n-3)x_{1i} - \sum_{\substack{j=2 \\ j \neq i}}^{n} x_{ij} \quad (i = 2,\dots,n) \tag{11}$$

are valid inequalities for the TSP. $\square$

Detecting violated lifted MTZ constraints (8) poses no problem in practice. If $x_{ij} = x_{ji} = 0$ for some arc (v_i, v_j), then (8) is always satisfied. It suffices therefore to restrict the search for violated constraints to arcs (v_i, v_j) and (v_j, v_i) with either $x_{ij} > 0$ or $x_{ji} > 0$.

It should be observed at this point that the lifted MTZ constraints can still be weaker than the DFJ constraints. In the example depicted in Figure 1, constraints (8) are satisfied for all arcs (i, j), whereas constraints (4) are violated for $S = \{v_1, v_2, v_3, v_4\}$.

In the following section, these improvements will be extended to three types of VRPs.

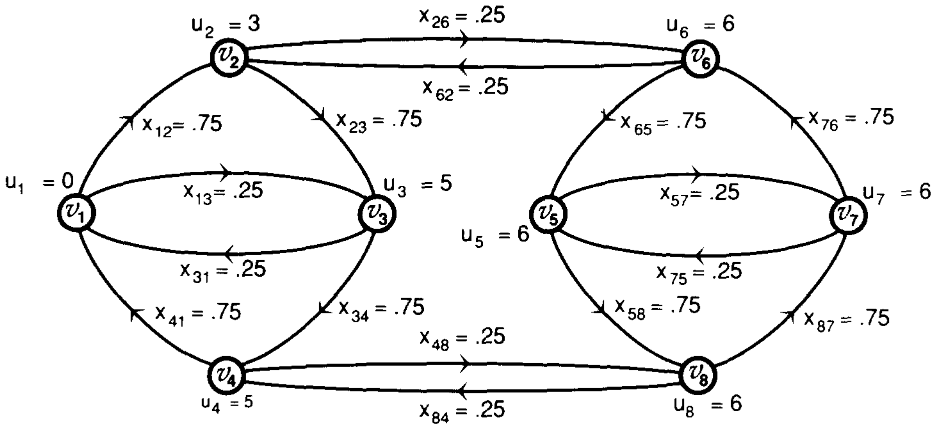

Fig. 1. Counter-example showing that the lifted MTZ constraints can still be weaker than the DFJ constraints.

3. Extending the lifted MTZ constraints to three types of VRPs

3.1. *The capacitated vehicle routing problem*

The original MTZ subtour elimination constraints can easily be adapted to the CVRP (see, for example, Kulkarni and Bhave [5]):

$$u_i - u_j + Qx_{ij} \leqslant Q - q_j \quad (i,\ j = 2, \ldots, n;\ i \neq j;\ q_i + q_j \leqslant Q) \tag{12}$$

and

$$q_i \leqslant u_i \leqslant Q \quad (i = 2, \ldots, n). \tag{13}$$

Applying once more the lifting process, we obtain Proposition 4.

Proposition 4. *The constraints*

$$u_i - u_j + Qx_{ij} + (Q - q_i - q_j)x_{ji} \leqslant Q - q_i \quad (i,\ j = 2, \ldots, n;\ i \neq j;\ q_i + q_j \leqslant Q) \tag{14}$$

are valid inequalities for the CVRP. □

Further improvements are still possible. For example, for any vertex v_k such that $q_i + q_j + q_k > Q$, a stronger constraint of the form

$$u_i - u_j + Qx_{ij} + (Q - q_i - q_j)x_{ji} + \alpha_{ik}x_{ik} + \alpha_{kj}x_{kj} + \alpha_{ki}x_{ki} + \alpha_{jk}x_{jk} \leqslant Q - q_j \tag{15}$$

can be generated. Here, the α coefficients are set equal to $\alpha_{ik} = \alpha_{kj} = q_k$ and $\alpha_{ki} = \alpha_{jk} = 0$. To illustrate, consider α_{ik}. Setting $x_{ik} = 1$ implies $x_{ki} = x_{ji} = x_{ij} = x_{jk} = x_{kj} = 0$, and (15) can be rewritten as

$$\alpha_{ik} \leqslant Q - q_j - u_i + u_j.$$

Since $u_j \geqslant q_j$ and $u_i \leqslant Q - q_k$, it follows that $\alpha_{ik} \leqslant q_k$.

A similar lifting process can be applied to constraints (13).

Proposition 5. *The constraints*

$$u_i \geqslant q_i + \sum_{\substack{j=2 \\ j \neq i}}^{n} q_j x_{ji} \quad (i = 2, \ldots, n) \tag{16}$$

and

$$u_i \leqslant Q - \left(Q - \max_{j \neq i}\{q_j\} - q_i\right)x_{1i} - \sum_{\substack{j=2 \\ j \neq i}}^{n} q_j x_{ji} \quad (i = 2, \ldots, n) \tag{17}$$

are valid inequalities for the CVRP. □

Note that constraints (17) with $Q = n - 1$ and $q_i = 1$ for all $i > 1$ are consistent with (11). The counterpart of (10) cannot be derived for the CVRP. However, constraints (16) are consistent with the following lower bounding constraints, valid for the TSP:

$$u_i \geqslant 1 + \sum_{\substack{j=2 \\ j \neq i}}^{n} x_{ji} \quad (i = 2, \ldots, n).$$

3.2. The distance constrained vehicle routing problem

Now consider the DVRP and define s_i, the length of a shortest path from v_1 to v_i and t_i, the length of a shortest path from v_i to v_1. Observe that $x_{ij} = 0$ whenever $s_i + c_{ij} + t_j > L$. We now prove

Proposition 6. *The following constraint system is valid for the DVRP:*

$$u_i - u_j + Mx_{ij} \leqslant M - c_{ij} \quad (i,\ j = 2, \ldots, n;\ i \neq j), \tag{18}$$

$$u_i \geqslant s_i + (c_{1i} - s_i)x_{1i} \quad (i = 2, \ldots, n), \tag{19}$$

$$u_i \leqslant L - t_i - (c_{i1} - t_i)x_{i1} \quad (i = 2, \ldots, n), \tag{20}$$

where $M \geqslant \max_{i,j}\{L - s_j - t_i + c_{ij}\}$ *in order not to eliminate feasible solutions when* $x_{ij} = 0$.

Proof. We first show that constraints (18) eliminate subtours disconnected from v_1. Consider a subtour $(v_{i_1}, \ldots, v_{i_k}, v_{i_1})$ where $v_{i_1}, \ldots, v_{i_k} \neq v_1$. Summing up the corresponding constraints yields

$$\sum_{l=1}^{k} \left(u_{i_l} - u_{i_{l+1}} + Mx_{i_l i_{l+1}} - M + c_{i_l i_{l+1}}\right) \leqslant 0,$$

where $i_{k+1} \equiv i_1$. This reduces to

$$\sum_{l=1}^{k} \left[M(x_{i_l i_{l+1}} - 1) + c_{i_l i_{l+1}}\right] \leqslant 0$$

and to

$$\sum_{l=1}^{k} c_{i_l i_{l+1}} \leqslant 0,$$

a contradiction, unless G contains subtours of zero length. In order to show that subtours of length greater than L and connected to the depot are eliminated, we derive a similar contradiction. Consider a subtour $(v_{i_1} \equiv v_1, v_{i_2}, \ldots, v_{i_k}, v_{i_1})$. Again, summing up constraints (18) for arcs $(v_{i_l}, v_{i_{l+1}})$, $l = 2, \ldots, k-1$, gives

$$u_{i_2} - u_{i_k} + \sum_{l=2}^{k-1} c_{i_l i_{l+1}} \leqslant 0,$$

which reduces to

$$\sum_{l=1}^{k-1} c_{i_l i_{l+1}} \leqslant u_{i_2} + \sum_{l=2}^{k-1} c_{i_l i_{l+1}} \leqslant u_{i_k}$$

and finally to

$$\sum_{l=1}^{k} c_{i_l i_{l+1}} \leqslant L,$$

since constraints (19) and (20) imply $u_{i_2} \geqslant c_{1 i_2}$ and $u_{i_k} \leqslant L - c_{i_k 1}$, respectively. □

Constraints (18) can be augmented by adding an extra term.

Proposition 7. *The constraints*

$$u_i - u_j + Mx_{ij} + (M - c_{ij} - c_{ji})x_{ji} \leqslant M - c_{ij} \quad (i,\ j = 2, \ldots, n;\ i \neq j), \tag{21}$$

where $M \geqslant \max_{i,j}\{L - s_j - t_i + c_{ij},\ c_{ij} + c_{ji}\}$ *are valid inequalities for the DVRP.* □

Further improvements can still be obtained by using the fact that some arc combinations are infeasible and, as in Laporte et al. [7], several cases are possible. For the sake of brevity, we provide only one example. Consider two arcs (v_k, v_i) and (v_i, v_j) such that $s_k + c_{ki} + c_{ij} + t_j > L$. The lifted constraints then become

$$u_i - u_j + Mx_{ij} + (M - c_{ij} - c_{ji})x_{ji} + \alpha_{ki}x_{ki} \leqslant M - c_{ij} \quad (i,\ j = 2, \ldots, n;\ i \neq j), \tag{22}$$

where $\alpha_{ki} \leqslant M - L + s_j + t_i - c_{ij}$.

As above, the lifting process can be applied to the lower and upper bound constraints on u_i.

Proposition 8. *The constraints*

$$u_i \geqslant s_i + \sum_{\substack{j=1\\ j\neq i}}^{n} (s_j + c_{ji} - s_i)x_{ji} \quad (i = 2, \ldots, n) \tag{23}$$

and

$$u_i \leqslant L - t_i - \sum_{\substack{j=1\\ j\neq i}}^{n} (t_j + c_{ij} - t_i)x_{ij} - \left(L - c_{1i} - \max_{\substack{j\neq i\\ j\neq 1}}\{t_j + c_{ij}\}\right)x_{1i} \quad (i = 2, \ldots, n) \tag{24}$$

are valid inequalities for the DVRP. □

3.3. The vehicle routing problem with time windows

Subtour elimination constraints for the TWVRP can be written as

$$u_i - u_j + Mx_{ij} \leqslant M - c_{ij} \quad (i,\ j = 2, \ldots, n;\ i \neq j) \tag{25}$$

and

$$a_i \leqslant u_i \leqslant b_i \quad (i = 2, \ldots, n), \tag{26}$$

where $[a_i, b_i]$ is the time window for vertex v_i and $M \geqslant \max_{i,j}\{b_i - a_j + c_{ij}\}$.

Because of time window constraints, some arcs (v_i, v_j) are infeasible. The following result only applies if (v_i, v_j) and (v_j, v_i) are both defined.

Proposition 9. *The constraints*

$$u_i - u_j + Mx_{ij} + (M - c_{ij} + \min\{-c_{ji},\ b_j - a_i\})x_{ji} \leqslant M - c_{ij} \quad (i,\ j = 2, \ldots, n;\ i \neq j) \tag{27}$$

are valid inequalities for the TWVRP.

Proof. Consider the general constraints

$$u_i - u_j + Mx_{ij} + \alpha_{ji}x_{ji} \leqslant M - c_{ij}.$$

Several cases must be considered for the determination of α_{ji}:

Case 1. $a_j + c_{ji} \geqslant a_i$. Then,

$$\alpha_{ji} \leqslant M - c_{ij} - c_{ji}.$$

Case 2. $a_j + c_{ji} < a_i$ and $b_j + c_{ji} \geqslant a_i$. Then,

$$\alpha_{ji} \leqslant M - c_{ij} + \min\{-c_{ji}, b_j - a_i\}.$$

*Case*3. $b_j + c_{ji} < a_i$. Then, as in Case 2,

$$\alpha_{ji} \leqslant M - c_{ij} + \min\{-c_{ji}, b_j - a_i\}.$$

It should be noted that by definition of Case 1, we have $-c_{ji} \leqslant a_j - a_i \leqslant b_j - a_i$ and thus, in general,

$$\alpha_{ji} \leqslant M - c_{ij} + \min\{-c_{ji}, b_j - a_i\}. \qquad \square$$

As in the previous two subsections, it is possible to further improve constraints (27) by taking into account infeasible arc combinations. Again, for the sake of brevity, only one example is provided. Consider two arcs (v_k, v_i) and (v_i, v_j) such that $a_k + c_{ki} + c_{ij} > b_j$. Then,

$$u_i - u_j + Mx_{ij} + \left(M - c_{ij} + \min\{-c_{ji}, b_j - a_i\}\right)x_{ji} + \alpha_{ki}x_{ki}$$
$$\leqslant M - c_{ij} \quad (i,\ j = 2,\dots,n;\ i \neq j),$$

where $\alpha_{ki} \leqslant M - c_{ij} - \min\{b_k + c_{ki}, b_i\} + a_j$ and again, $M \geqslant \max_{i,j}\{c_{ij} + c_{ji}\}$.

Finally, the lower and upper bound constraints on u_i can also be lifted.

Proposition 10. *The constraints*

$$u_i \geqslant a_i + \sum_{\substack{j=1\\ j\neq i}}^{n} \left(\max\{0,\ a_j - a_i + c_{ji}\}\right)x_{ji} \quad (i = 2,\dots,n) \tag{28}$$

and

$$u_i \leqslant b_i - \sum_{\substack{j=1\\ j\neq i}}^{n} \left(\max\{0,\ b_i - b_j + c_{ij}\}\right)x_{ij} \quad (i = 2,\dots,n) \tag{29}$$

are valid inequalities for the TWVRP. $\square$

4. Computational results

To gain some insight into whether or not the lifted MTZ formulation provides some improvements over the original MTZ formulation or over the assignment relaxation, we studied TSPs, CVRPs, DVRPs and TWVRPs for three distance classes: Asymmetric random (AR), Symmetric random (SR) and Symmetric Euclidean (SE). Distance matrices rounded to the nearest tenth were generated as follows:

$$\text{AR} : c_{ij} \sim U[0, 100], \quad i \neq j.$$

$$\text{SR} : c_{ij} \sim U[0, 100], \quad i < j.$$

$$\text{SE} : c_{ij} = \left[(x_i - x_j)^2 + (y_i - y_j)^2\right]^{1/2}, \quad \text{with } (x_i, y_i) \sim U[0, 50]^2,\ i < j.$$

Parameters for the VRPs were defined in the following manner:

$$\text{CVRP} : m = 6, \quad Q = 100, \quad q_i \sim \text{Poisson}(10).$$

$$\text{DVRP} : m = 3, \quad L = 100 \text{ for AR and SR}, \quad L = 125 \text{ for SE}.$$

$$\text{TWVRP} : m = 7, \quad a_i = c_{1i} + (200 - c_{1i} - c_{i1})T_1, \quad b_i = a_i + (200 - a_i - c_{i1})T_2,$$
$$\text{where } T_1, T_2 \sim U[0, 1].$$

Table 1
Summary of computational results

		TSP	CVRP	DVRP	TWVRP
AR	MTZ/ASS	1.000	1.002	1.000	1.004
	MTZ$^+$/ASS	1.006	1.006	1.007	1.009
	improvement	0.006	0.004	0.007	0.005
SR	MTZ/ASS	1.009	1.030	1.006	1.003
	MTZ$^+$/ASS	1.274	1.170	1.243	1.050
	improvement	0.265	0.140	0.237	0.047
SE	MTZ/ASS	1.007	1.026	1.011	1.006
	MTZ$^+$/ASS	1.211	1.139	1.181	1.039
	improvement	0.204	0.113	0.170	0.033

For each problem type and each distance class, ten 50-city problems were generated and in each case, three values were computed:

(1) ASS: the value of the assignment bound.

(2) MTZ: the solution value obtained by incorporating all violated MTZ constraints.

(3) MTZ$^+$: the solution value obtained by incorporating all violated lifted MTZ constraints.

The ratios MTZ/ASS and MTZ$^+$/ASS were computed and averaged over the 10 problems. The reported *improvement* is the difference between these two averages (see Table 1).

In the case of asymmetric random problems, low improvements were obtained. At least for the TSP, this was to be expected since the assignment lower bound is already close to the optimum. Better results were achieved in the symmetric case, in particular for random distances where the average improvement is in the region of 15–25% for the first three problem types. The smaller improvements observed in TWVRPs may be due again to the high quality of the assignment bound, since several arcs are initially eliminated because of time window feasibility considerations.

In addition, in the case of the TSP, we have computed DFJ, the solution value obtained by incorporating all DFJ constraints. The DFJ/ASS ratios for AR, SR and SE are 1.012, 1.279 and 1.229, respectively. The corresponding MTZ$^+$/ASS ratios are only slightly lower than these values, which is a clear indication of the relative strength of the lifted MTZ constraints for the TSP.

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