CS 60–231
Solution to Assignmemt #4

Fall 2017

Exercises 5.7, 3(a)

Disprove: If *R* and *S* are asymmetric, then $R \cup S$ is symmetric.
(Proof by Counterexample)
Let $A = \{a, b\}$ and $R = S = \{(a, b)\}$.
Then by inspection, R and S are asymmetric.
Since $R \cup S = \{(a, b)\} \cup \{(a, b)\} = \{(a, b)\}$ and $(a, b) \in R \cup S$ but $(b, a) \notin R \cup S$,
Therefore $R \cup S$ is *not* symmetric.

Disprove: If *R* and *S* are asymmetric, then $R \cap S$ is symmetric.
(Proof by Counterexample)
Let $A = \{a, b\}$ and $R = S = \{(a, b)\}$.
Then by inspection, *R* and *S* are asymmetric.
Since $R \cap S = \{(a, b)\} \cap \{(a, b)\} = \{(a, b)\}$ and $(a, b) \in R \cap S$ but $(b, a) \notin R \cap S$,
Therefore $R \cap S$ is *not* symmetric. ■

Exercises 5.7, 11(b)

(b) (Direct Proof)
Suppose $x \prec y$ and $y \precsim z$. . . . (I)
Then $x \prec y \Rightarrow (x \precsim y) \wedge (x \neq y)$ (Definition of $\prec$)
$\Rightarrow (x \precsim y)$ and $(x \neq y)$ (E9,I2) . . . (II)
$\Rightarrow (x \precsim y) \wedge (y \precsim z)$ ((II),(I), I6)
$\Rightarrow (x \precsim z)$ ($\precsim$ is transitive, common practice) . . . (III)
Now, suppose $x = z$. . . . (IV)
Then $(x \precsim y) \wedge (y \precsim z)$ ((II),(I),I6)
$\Rightarrow (x \precsim y) \wedge (y \precsim x)$ ((IV), $\text{Sub}_=$)
$\Rightarrow x = y$ ($\precsim$ is antisymmetric, common practice) . . . (V)
$\Rightarrow (x = y) \wedge (x \neq y)$ ((V),(II),I6)
$\Rightarrow false$, a contradiction! (E1)
Hence, $x \neq z$. $\cdots\cdots$ (VI)
$\Rightarrow (x \precsim z) \wedge (x \neq z)$ ((III),(VI),I6)
$\Rightarrow (x, z) \in \precsim \ \wedge\ (x, z) \notin X_=$ (Definition of $X_=$)
$\Rightarrow (x, z) \in \precsim - X_=$ (Definition of $-$)
$\Rightarrow x \prec z$ (Definition of $\prec$) ■

Exercises 5.7, 17 (first part)

Solution: First, we shall prove that $(x, z) \in R \circ Y_= \Leftrightarrow (x, z) \in R$.

$\Rightarrow$) (Direct proof)
Let $(x, z) \in R \circ Y_=$.
Then $(\exists y \in Y)((x, y) \in R \wedge (y, z) \in Y_=)$ (Definition of $\circ$)
$\Rightarrow (x, h) \in R \wedge (h, z) \in Y_=$ (EI)
$\Rightarrow (x, h) \in R$ and $(h, z) \in Y_=$ (E9,I2)
$\Rightarrow (x, h) \in R$ and $h = z$ (Definition of $Y_=$)
$\Rightarrow (x, z) \in R$ ($\text{Sub}_=$)
Therefore, $(x, z) \in R \circ Y_= \Rightarrow (x, z) \in R$.

$\Leftarrow$) Let $(x, z) \in R$. $\cdots\cdots$ (I)

Then $R \subseteq X \times Y$ (R is a relation from X to Y))

$\Rightarrow (\forall u)(u \in R \Rightarrow u \in X \times Y)$ (Definition of $\subseteq$)

$\Rightarrow ((x, z) \in R \Rightarrow (x, z) \in X \times Y)$ (EI) $\cdots\cdots$ (II)

$\Rightarrow (x, z) \in X \times Y$ ((I),(II),I3)

$\Rightarrow x \in X \wedge z \in Y$ (Definition of $\times$)

$\Rightarrow z \in Y$ (E9,I2)

$\Rightarrow z \in Y \wedge z \in Y$ (E3)

$\Rightarrow (z, z) \in Y \times Y$ (Definition of $\times$) $\cdots\cdots$ (III)

Since $z = z$, (A1, p.65) $\cdots\cdots$ (IV)

we thus have $(z, z) \in Y \times Y \wedge (z = z)$ ((III),(IV),I6)

$\Rightarrow (z, z) \in Y_{=}$ (Definition of $Y_{=}$) $\cdots\cdots$ (V)

$\Rightarrow (x, z) \in R \wedge (z, z) \in Y_{=}$ ((I),(V),I6)

$\Rightarrow (\exists y \in Y)((x, y) \in R \wedge (y, z) \in Y_{=})$ (EQ)

$\Rightarrow (x, z) \in R \circ Y_{=}$ (Definition of $\circ$)

Therefore, $(x, z) \in R \Rightarrow (x, z) \in R \circ Y_{=}$.

Hence, $(x, z) \in R \circ Y_{=} \Leftrightarrow (x, z) \in R$

$\Rightarrow (\forall u) u \in R \circ Y_{=} \Leftrightarrow u \in R$ (Gen)

$\Rightarrow R \circ Y_{=} = R$. (Principle of Extension) ■

Exercises 5.7, 18(d)

Solution:

$\Rightarrow$) (Direct proof) Suppose R is antisymmetric.

We shall first prove that $(a, b) \in R \cap R^{-1} \Rightarrow (a, b) \in X_{=}$.

(Direct proof)

$(a, b) \in R \cap R^{-1}$

$\Rightarrow (a, b) \in R \wedge (a, b) \in R^{-1}$ (Definition of $\cap$)

$\Rightarrow (a, b) \in R \wedge (b, a) \in R$ (Definition of R^{-1}) . . . . . . (I)

R is antisymmetric

$\Rightarrow (a, b) \in R \wedge (b, a) \in R \Rightarrow a = b$ (Definition of antisymmetricity). . . . (II)

$\Rightarrow a = b$ ((I),(II),I3)

$\Rightarrow (a, b) \in X_{=}$ (Definition of $X_{=}$)

Hence, $(a, b) \in R \cap R^{-1} \Rightarrow (a, b) \in X_{=}$

$\Rightarrow (\forall u)\big(u \in R \cap R^{-1} \Rightarrow u \in X_{=}\big)$ (Gen)

$\Rightarrow R \cap R^{-1} \subseteq X_{=}$. (Definition of $\subseteq$

$\Leftarrow$) (Direct proof) Suppose $R \cap R^{-1} \subseteq X_{=}$. . . . (I)

We are to prove that $(a, b) \in R \wedge (b, a) \in R \Rightarrow a = b$.

(Direct proof)

$(a, b) \in R \wedge (b, a) \in R$

$\Rightarrow (a, b) \in R \wedge (a, b) \in R^{-1}$ (Definition of R^{-1})

$\Rightarrow (a, b) \in R \cap R^{-1}$ (Definition of $\cap$) . . . . . . (II)

$R \cap R^{-1} \subseteq X_{=}$ (by (I))

$\Rightarrow (\forall u)\big(u \in R \cap R^{-1} \Rightarrow u \in X_{=}\big)$ (Definition of $\subseteq$)

$\Rightarrow (a, b) \in R \cap R^{-1} \Rightarrow (a, b) \in X_{=}$. (EI) . . . . . . (III)

$\Rightarrow (a,b) \in X_=.$ ((II),(III), I3)

$\Rightarrow a = b$ (Definition of $X_=$)

We thus have: $(a,b) \in R \wedge (b,a) \in R \Rightarrow a = b.$

Hence, R is antisymmetric. ■

Exercises 6.6, 2.

Solution: Let $g((a,b)) = g((c,d))$. Then $2^a 3^b = 2^c 3^d$.

Without loss of generality, we assume $a \geq c$.

Suppose $a > c$.

Then $a - c > 0 \Rightarrow a - c \geq 1 \Rightarrow a - c - 1 \geq 0$ (h.s.a.)

$\Rightarrow 2^{a-c-1} \geq 2^0$ (h.s.a.)

$\Rightarrow 2^{a-c-1} \geq 1$ (h.s.a.)

$\Rightarrow 2^{a-c-1} \in \mathbf{N}$ (h.s.a.) . . . (I)

Moreover, $3 \in \mathbf{N} \Rightarrow 3^b \in \mathbf{N}$ (h.s.a.) . . . (II)

$\Rightarrow 2^{a-c-1} \in \mathbf{N} \wedge 3^b \in \mathbf{N}$ ((I),(II),I6)

$\Rightarrow 2^{a-c-1}3^b \in \mathbf{N}$ (h.s.a.) . . . (III)

It follows that $2^a 3^b = 2^c 3^d$

$\Rightarrow 2^{a-c}3^b = 3^d$ (h.s.a.)

$\Rightarrow 2\big(2^{a-c-1}3^b\big) = 3^d$ (h.s.a.) . . . (IV)

$\Rightarrow \big(2^{a-c-1}3^b \in \mathbf{N}\big) \wedge 2\big(2^{a-c-1}3^b\big) = 3^d$ ((III),(IV),I6)

$\Rightarrow \big(2^{a-c-1}3^b \in \mathbf{N}\big) \wedge 3^d = 2\big(2^{a-c-1}3^b\big)$ (= is symmentric)

$\Rightarrow (\exists k \in \mathbf{N})3^d = 2k$ (EQ)

$\Rightarrow 2 \mid 3^d$ (Definition of $|$)

$\Rightarrow 2 \mid 3$, a contradiction! (h.s.a.)

Hence, $a = c$ which implies that $2^{a-c}3^b = 3^d \Rightarrow 2^0 3^b = 3^d$ (h.s.a.)

$\Rightarrow 3^b = 3^d$ (h.s.a.)

$\Rightarrow 3^{b-d} = 1$ (h.s.a.)

$\Rightarrow b - d = 0$ (h.s.a.)

$\Rightarrow b = d.$ (h.s.a.)

We thus have $(a = c) \wedge (b = d) \Rightarrow (a,b) = (c,d)$ (Lemma 4.4.9)

Therefore, g is one-to-one.

The function g is *not* onto:

(Proof by contradiction) Suppose to the contrary that g is onto.

Then $(\forall z \in \mathbf{N})((\exists (i,j) \in \mathbf{N} \times \mathbf{N})g((i,j)) = z)$ (Definition of ontoness) . . . (I)

Since $5 \in \mathbf{N}$, . . . (II)

$5 \in \mathbf{N} \Rightarrow ((\exists (i,j) \in \mathbf{N} \times \mathbf{N})g((i,j)) = 5)$ ((I),UI) . . . (III)

$\Rightarrow ((\exists (i,j) \in \mathbf{N} \times \mathbf{N})g((i,j)) = 5)$ ((II),(III),I3)

$\Rightarrow (a,b) \in \mathbf{N} \times \mathbf{N} \wedge g((a,b)) = 5$ (EI)

$\Rightarrow (a,b) \in \mathbf{N} \times \mathbf{N}$ and $g((a,b)) = 5$ (E9,I2)

$\Rightarrow a \in \mathbf{N} \wedge b \in \mathbf{N}$ and $g((a,b)) = 5$ (Definition of $\times$)

$\Rightarrow (a \geq 1) \wedge (b \geq 1)$ and $g((a,b)) = 5$ (Definition of $\mathbf{N}$)

$\Rightarrow 2^{a-1}3^b \in \mathbf{N}$ and $g((a,b)) = 5$ (h.s.a)

$\Rightarrow 2^{a-1}3^b \in \mathbf{N}$ and $2^a 3^b = 5$ (Definition of g)

$\Rightarrow 2^{a-1}3^b \in \mathbf{N}$ and $5 = 2\big(2^{a-1}3^b\big)$ (h.s.a)

$\Rightarrow 2^{a-1}3^b \in \mathbf{N} \wedge 5 = 2\big(2^{a-1}3^b\big)$ (I6)

$\Rightarrow (\exists k \in \mathbf{N})(5 = 2 \cdot k)$ (EQ)

$\Rightarrow 2 \mid 5$ (Definition of $|$), . . . (V)
But $2 \nmid 5$ (h.s.a) . . . (VI)
$\Rightarrow 2 \mid 5 \wedge 2 \nmid 5$ ((V),(VI),I6)
$\Rightarrow false$, a contradiction!
Hence, g is *not* onto. ■

Exercises 6.6, 14(b)

(i) $g = h$:

First, we shall prove that $\forall y \in Y, g(y) = h(y)$
(Direct proof) Let $y \in Y$.
Then $g(y) = g(I_Y(y))$ (Definition of I_Y)
$= g((h \circ f)(y))$ ($I_Y = h \circ f$)
$= ((h \circ f) \circ g)(y)$ (equivalent notations; see the remark on p.153)
$= (h \circ (f \circ g))(y)$ (Theorem 5.6.5(iii))
$= (h \circ I_X)(y)$ ($I_X = f \circ g$)
$= I_X(h(y))$ (equivalent notations)
$= h(y)$ (Definition of I_X)
We thus have: $y \in Y \Rightarrow g(y) = h(y)$
$\Rightarrow (\forall y)(y \in Y \Rightarrow g(y) = h(y))$ (Gen)
$\Rightarrow g = h$ (Theorem 6.1.1)

(ii) $f^{-1} = g$:

First, we shall prove that $\forall y \in Y, f^{-1}(y) = g(y)$
(Direct proof) Let $y \in Y$.
Then $f^{-1}(y) = x$ $\cdots$ (I)
$\Rightarrow (y, x) \in f^{-1}$ (equivalent notations)
$\Rightarrow (x, y) \in f$ (Definition of f^{-1})
$\Rightarrow f(x) = y$ (equivalent notations)
$\Rightarrow g(f(x)) = g(y)$ ($g : Y \to X$, and $f(x), y \in Y$)
$\Rightarrow (f \circ g)(x) = g(y)$ (equivalent notations)
$\Rightarrow I_X(x) = g(y)$ ($I_X = f \circ g$)
$\Rightarrow x = g(y)$ (Definition of I_X) $\cdots$ (II)
$\Rightarrow f^{-1}(y) = g(y)$ ((I),(II), $\text{Sub}_=$)
We thus have: $y \in Y \Rightarrow f^{-1}(y) = g(y)$
$\Rightarrow (\forall y)\big(y \in Y \Rightarrow f^{-1}(y) = g(y)\big)$ (Gen)
$\Rightarrow f^{-1} = g$ (Theorem 6.1.1)

(iii) $f^{-1} = h$:

$g = h \wedge f^{-1} = g$ (Parts (i) and (ii), I6)
$\Rightarrow f^{-1} = g \wedge g = h$ (E9)
$\Rightarrow f^{-1} = h$ (= is transitive)

Hence, $g = h = f^{-1}$. ■