CS 60–231
Solution to Assignmemt #1

Fall 2017

Exercises 1.9, 1(b)

Solution: $(((p \Rightarrow q) \Rightarrow (r \Rightarrow p)) \Rightarrow (r \Rightarrow p))$

p	q	r	$(p \Rightarrow q)$	$(r \Rightarrow p)$	$((p \Rightarrow q) \Rightarrow (r \Rightarrow p))$	$(((p \Rightarrow q) \Rightarrow (r \Rightarrow p)) \Rightarrow (r \Rightarrow p))$
F	F	F	T	T	T	T
F	F	T	T	F	F	T
F	T	F	T	T	T	T
F	T	T	T	F	F	T
T	F	F	F	T	T	T
T	F	T	F	T	T	T
T	T	F	T	T	T	T
T	T	T	T	T	T	T

Exercises 1.9, 3(f): Prove $\vdash (\alpha \Rightarrow \beta) \vee (\beta \Rightarrow \alpha)$..
Solution: You may use 'proof by contradivction'. The following is a proof based on the definition of 'proof'.

1. $(\alpha \vee \sim\alpha)$ axiom
2. $(\alpha \vee \sim\alpha) \vee (\beta \vee \sim\beta)$ 1, I1
3. $\alpha \vee (\sim\alpha \vee (\beta \vee \sim\beta))$ 2, E12
4. $\alpha \vee ((\sim\alpha \vee \beta) \vee \sim\beta)$ 3, E12
5. $((\sim\alpha \vee \beta) \vee \sim\beta) \vee \alpha$ 4, E10
6. $(\sim\alpha \vee \beta) \vee (\sim\beta \vee \alpha)$ 5, E12
7. $(\alpha \Rightarrow \beta) \vee (\sim\beta \vee \alpha)$ 6, E18
8. $(\alpha \Rightarrow \beta) \vee (\beta \Rightarrow \alpha)$ 7, E18

Hence, $\vdash (\alpha \Rightarrow \beta) \vee (\beta \Rightarrow \alpha)$. ■

Exercises 1.9, 3(g): Prove $\vdash (\alpha \Rightarrow \beta) \Rightarrow ((\alpha \wedge \gamma) \Rightarrow (\beta \wedge \gamma))$.
Solution: (Indirect Proof) [Recall that you cannot apply the Deduction Theorem directly.]

1. $\sim ((\alpha \wedge \gamma) \Rightarrow (\beta \wedge \gamma))$ Hypothesis
2. $\sim (\sim (\alpha \wedge \gamma) \vee (\beta \wedge \gamma))$ 1, E18
3. $\sim\sim (\alpha \wedge \gamma) \wedge \sim (\beta \wedge \gamma)$ 2, E17
4. $(\alpha \wedge \gamma) \wedge \sim (\beta \wedge \gamma)$ 3, E15
5. $(\alpha \wedge \gamma) \wedge (\sim \beta \vee \sim \gamma)$ 4, E16
6. $((\alpha \wedge \gamma) \wedge \sim \beta) \vee ((\alpha \wedge \gamma) \wedge \sim \gamma)$ 5, E13
7. $((\alpha \wedge \gamma) \wedge \sim \beta) \vee (\alpha \wedge (\gamma \wedge \sim \gamma))$ 6, E11
8. $((\alpha \wedge \gamma) \wedge \sim \beta) \vee (\alpha \wedge false)$ 7, E1
9. $((\alpha \wedge \gamma) \wedge \sim \beta) \vee false$ 8, E7
10. $((\alpha \wedge \gamma) \wedge \sim \beta)$ 9, E6
11. $(\alpha \wedge (\gamma \wedge \sim \beta))$ 10, E11
12. $(\alpha \wedge (\sim \beta \wedge \gamma))$ 11, E9
13. $((\alpha \wedge \sim \beta) \wedge \gamma)$ 12, E11
14. $(\alpha \wedge \sim \beta)$ 13, I2

15. $(\sim\sim \alpha \wedge \sim \beta)$ 14, E15
16. $\sim (\sim \alpha \vee \beta)$ 15, E17
17. $\sim (\alpha \Rightarrow \beta)$ 16, E18

Hence, $\vdash (\alpha \Rightarrow \beta) \Rightarrow ((\alpha \wedge \gamma) \Rightarrow (\beta \wedge \gamma))$. ■

Exercises 1.9, 4.: Prove Theorem 1.7.2(v): $\vdash (\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Leftrightarrow (\alpha \wedge \beta \Rightarrow \gamma)$.
Solution: (Bidirectional proof)

$\Rightarrow$) (Direct proof)

1. $(\alpha \Rightarrow (\beta \Rightarrow \gamma))$ hypothesis
2. $(\sim \alpha \vee (\beta \Rightarrow \gamma))$ 1, E18
3. $(\sim \alpha \vee (\sim \beta \vee \gamma))$ 2, E18
4. $(\sim \alpha \vee \sim \beta) \vee \gamma$ 3, E12
5. $\sim (\alpha \wedge \beta) \vee \gamma$ 4, E16
6. $(\alpha \wedge \beta) \Rightarrow \gamma$ 5, E18

$\Leftarrow$) (The above proof in reversed order)
Hence, $\vdash (\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Leftrightarrow (\alpha \wedge \beta \Rightarrow \gamma)$. ■

Exercises 1.9, 6.(d):
Solution: (Proof by contradiction)

1. $\sim\sim p$ hypothesis
2. $r \Rightarrow s$ from Γ
3. $p \Rightarrow (q \wedge r)$ from Γ
4. $\sim (q \wedge s)$ from Γ
5. p 1,E15
6. $q \wedge r$ 3,5,I3
7. q 6,I2
8. $r \wedge q$ 6,E9
9. r 8,I2
10. s 9,2,I3
11. $q \wedge s$ 7,10,I6
12. $(q \wedge s) \wedge \sim (q \wedge s)$ 11,4,I6
13. $false$ 12,E1

Hence, $\mathrm{P1}, \mathrm{P2}, \mathrm{P3} \vdash \sim p$ ■

Exercises 1.9, 6.(i):

Solution:

1. $q \Rightarrow \sim p$ from Γ
2. $\sim q \wedge \sim s \Rightarrow \sim r$ from Γ
3. $\sim r \wedge \sim u \Rightarrow \sim t$ from Γ
4. $s \Rightarrow q$ from Γ
5. $\sim (\sim r \wedge \sim u) \vee \sim t$ 3,E18
6. $(\sim\sim r \vee \sim\sim u) \vee \sim t$ 5,E16
7. $\sim\sim r \vee (\sim\sim u \vee \sim t)$ 6,E12
8. $\sim r \Rightarrow (\sim\sim u \vee \sim t)$ 7,E18
9. $(\sim q \wedge \sim s) \Rightarrow (\sim\sim u \vee \sim t)$ 2,8,I5
10. $\sim(\sim q \wedge \sim s) \vee (\sim\sim u \vee \sim t)$ 9,E18
11. $(\sim\sim u \vee \sim t) \vee \sim (\sim q \wedge \sim s)$ 10,E10
12. $(\sim\sim u \vee \sim t) \vee (\sim\sim q \vee \sim\sim s)$ 11,E16

13. $(u \vee \sim t) \vee (\sim\sim q \vee \sim\sim s)$ 12,E15
14. $(u \vee \sim t) \vee (q \vee \sim\sim s)$ 13,E15
15. $(u \vee \sim t) \vee (q \vee s)$ 14,E15
16. $((u \vee \sim t) \vee q) \vee s$ 15,E12
17. $\sim\sim((u \vee \sim t) \vee q) \vee s$ 16,E15
18. $\sim ((u \vee \sim t) \vee q) \Rightarrow s$ 17,E18
19. $\sim ((u \vee \sim t) \vee q) \Rightarrow q$ 18,4,I5
20. $\sim\sim ((u \vee \sim t) \vee q) \vee q$ 19,E18
21. $((u \vee \sim t) \vee q) \vee q$ 20,E15
22. $(u \vee \sim t) \vee (q \vee q)$ 21,E12
23. $(u \vee \sim t) \vee q$ 22,E4
24. $\sim\sim(u \vee \sim t) \vee q$ 23,E15
25. $\sim (u \vee \sim t) \Rightarrow q$ 24,E18
26. $\sim (u \vee \sim t) \Rightarrow \sim p$ 25,1,I5
27. $\sim\sim (u \vee \sim t) \vee \sim p$ 26,E18
28. $(u \vee \sim t) \vee \sim p$ 27,E15
29. $(\sim t \vee u) \vee \sim p$ 28,E10
30. $\sim t \vee (u \vee \sim p)$ 29,E12
31. $\sim t \vee (\sim p \vee u)$ 30,E10
32. $\sim t \vee (p \Rightarrow u)$ 31,E18

Hence, $\mathrm{P1}, \mathrm{P2}, \mathrm{P3}, \mathrm{P4} \vdash \sim t \vee (p \Rightarrow u)$ ■

Exercises 1.9, 7.(d).

Solution: Let *C* denote the casino is shut down;
T denote a gambling tax is imposed;
D denote tourism will decline;
S denote the city will suffer;
L denote the city will be a safer place to live.

P1: $(C \vee T) \Rightarrow (D \wedge S)$
P2: $D \Rightarrow L$
P3: $\sim L$
C: $\sim C$

(Direct proof)

1. $(C \vee T) \Rightarrow (D \wedge S)$ from Γ
2. $D \Rightarrow L$ from Γ
3. $\sim L$ from Γ
4. $\sim D$ 3,2, I4
5. $\sim D \vee \sim S$ 4, I1
6. $\sim (D \wedge S)$ 5, E16
7. $\sim (C \vee T)$ 6,1, I4
8. $\sim C \wedge \sim T$ 7, E16
9. $\sim C$ 8, I2 ■