CS 60–231
Solution to Assignmemt 5

Fall 2017

Exercises 7.8, 2:

Solution: $\forall Y \in \mathcal{P}(A)$, let $\chi_Y : A \to \{0,1\}$ such that $\chi_Y(x) = \begin{cases} 1 & x \in Y \\ 0 & x \notin Y \end{cases}$.

Then $\chi_Y \in 2^A, \forall Y \in \mathcal{P}(A)$.

Now, define $\Phi : \mathcal{P}(A) \to 2^A$ such that $\Phi(Y) = \chi_Y, \forall Y \in \mathcal{P}(A)$.

[**Remark:** We are to prove that Φ is one-to-one and onto.]

(i) Φ is one-to-one: (Direct Proof)

Let $\Phi(Y) = \Phi(Z)$.

Then $\chi_Y = \chi_Z$. (Definition of Φ) . . . . . . (I)

It follow that $x \in Y \Leftrightarrow \chi_Y(x) = 1$ (Definition of χ_Y)

$\Leftrightarrow \chi_Z(x) = 1$ ((I), Lemma 6.1.1)

$\Leftrightarrow x \in Z$ (Definition of χ_Z)

Therefore, $(\forall x)(x \in Y \Leftrightarrow x \in Z)$ (Gen)

$\Rightarrow Y = Z$ (Principle of Extension)

(ii) Φ is onto: (Direct Proof)

Let $f \in 2^A$.

Then $f : A \to \{0,1\}$. (Definition of 2^A)

Define $Y = \{x \in A \mid f(x) = 1\}$.

Then $x \in Y \Rightarrow x \in A \wedge f(x) = 1$ (Definition of *Y*)

$\Rightarrow x \in A$ (I2)

$\Rightarrow (\forall x)(x \in Y \Rightarrow x \in A)$ (Gen)

$\Rightarrow Y \subseteq A$ (Definition of $\subseteq$)

$\Rightarrow Y \in \mathcal{P}(A)$ (Definition of $\mathcal{P}(A)$) $\cdots$ (I)

$\Rightarrow \Phi(Y) = \chi_Y$ (Definition of Φ) $\cdots$ (II)

We shall prove that $\chi_Y = f$.

Let $x \in A$.

Then $x \in Y \Leftrightarrow f(x) = 1$ and $x \notin Y \Leftrightarrow f(x) = 0$ (Definition of *Y*)

But $\chi_Y(x) = 1 \Leftrightarrow x \in Y$ and $\chi_Y(x) = 0 \Leftrightarrow x \notin Y$ (Definition of χ_Y)

$\Rightarrow \chi_Y(x) = 1 \Leftrightarrow f(x) = 1$ and $\chi_Y(x) = 0 \Leftrightarrow f(x) = 0$ (Theorwm 1.7.2(iv))

$\Rightarrow \chi_Y(x) = f(x)$ (Sub$_=$)

We thus have $x \in A \Rightarrow \chi_Y(x) = f(x)$

$\Rightarrow (\forall x)(x \in A \Rightarrow \chi_Y(x) = f(x))$ (Gen)

$\Rightarrow \chi_Y = f$ (Lemma 6.1.1) $\cdots$ (III)

Hence, $Y \in \mathcal{P}(A) \wedge \Phi(Y) = \chi_Y$ ((I), (II), I6)

$\Rightarrow Y \in \mathcal{P}(A) \wedge \Phi(Y) = f$ ((III), Sub$_=$)

$\Rightarrow (\exists Z)(Z \in \mathcal{P}(A) \wedge \Phi(Z) = f)$ (EQ)

$\Rightarrow \Phi$ is onto. (Definition of onto function)

hence, $\Phi : \mathcal{P}(A) \overset{1-1}{\underset{onto}{\to}} 2^A \Rightarrow \mathcal{P}(A) \sim 2^A$. (Definition of $\sim$) ■

Exercises 8.9, 11. [**Remark:** This problem is Example 9 of Section 10.5 presented in a different way.]

Solution: Let the graph be $G = (V, E)$.

Let $u, v \in V$. We shall prove that $\{u, v\} \in E$

G contains no isolated vertex $\Rightarrow deg(u) \geq 1$ and $deg(v) \geq 1$ (Definition of isolated vertex)

$\Rightarrow (\exists x \in V)\{u, x\} \in E$ and $(\exists x \in V)\{v, x\} \in E$ (Definition of degree)

$\Rightarrow a \in V \wedge \{u, a\} \in E,\ b \in V \wedge \{v, b\} \in E.$ (EI)

$\Rightarrow \{u, a\} \in E$ and $\{v, b\} \in E.$ (E9,I2) $\cdots\cdots\cdots\cdots\cdots$ (I)

If $a = v$ or $b = u$, then $\{u, a\} \in E$ or $\{v, b\} \in E \Rightarrow \{u, v\} \in E.$ ((I), $Sub_=$)

Suppose $a \neq v$ and $b \neq u$.

(i) If $a = b$, then $\{u, a\}, \{v, b\} \in E$ (by (I))

$\Rightarrow \{u, a\}, \{v, a\} \in E$ ($a = b, Sub_=$)

$\Rightarrow \{u, a\}, \{a, v\} \in E_{\langle\{u,a,v\}\rangle}$ (Definition of $\langle\{u, a, v\}\rangle$)

$\Rightarrow \{u, v\} \in E_{\langle\{u,a,v\}\rangle}$ ($\langle\{u, a, v\}\rangle$ has more than two edges)

$\Rightarrow \{u, v\} \in E.$ ($E_{\langle\{u,a,v\}\rangle} \subseteq E$)

(note: $E_{\langle\{u,a,v\}\rangle}$ is the edge set of the subgraph induced by $\{u, a, v\}$)

(ii) if $a \neq b$, let $U = \{u, v, a, b\}$.

Then, $\{u, a\}, \{v, b\} \in E$ (by (I))

$\Rightarrow \{u, a\}, \{v, b\} \in E_{\langle U\rangle}$ (Definition of $\langle U\rangle$)

$\Rightarrow$ at least one of $\{u, v\}, \{u, b\}, \{a, v\}, \{a, b\}$ is in $E_{\langle U\rangle}$. ($\langle U\rangle$ has more than two edges)

We consider the four cases as follows:

- $\{u, v\} \in E_{\langle U\rangle}$: Then $\{u, v\} \in E.$ ($E_{\langle U\rangle} \subseteq E$)
- $\{u, b\} \in E_{\langle U\rangle}$: Then $\{u, b\} \in E_{\langle U\rangle}$ and $\{v, b\} \in E$ (by (I))
  $\Rightarrow \{u, b\} \in E$ and $\{v, b\} \in E$ ($E_{\langle U\rangle} \subseteq E$)
  $\Rightarrow \{u, b\}, \{v, b\} \in E_{\langle\{u,v,b\}\rangle}$ (Definition of $\langle\{u, v, b\}\rangle$)
  $\Rightarrow \{u, v\} \in E_{\langle\{u,v,b\}\rangle}$ ($\langle\{u, v, b\}\rangle$ has moe than two edges)
  $\Rightarrow \{u, v\} \in E.$ ($E_{\langle u,v,b\rangle} \subseteq E$)
- $\{a, v\} \in E_{\langle U\rangle}$: Then $\{a, v\} \in E_{\langle U\rangle}$ and $\{u, a\} \in E$ (by (I))
  $\Rightarrow \{a, v\} \in E$ and $\{u, a\} \in E$ ($E_{\langle U\rangle} \subseteq E$)
  $\Rightarrow \{v, a\}, \{u, a\} \in E_{\langle\{u,v,a\}\rangle}$ (Definition of $\langle\{u, v, a\}\rangle$)
  $\Rightarrow \{u, v\} \in E_{\langle\{u,v,a\}\rangle}$ ($\langle\{u, v, a\}\rangle$ has moe than two edges)
  $\Rightarrow \{u, v\} \in E.$ ($E_{\langle u,v,a\rangle} \subseteq E$)
- $\{a, b\} \in E_{\langle U\rangle}$: Then $\{a, b\} \in E_{\langle U\rangle}$ and $\{u, a\} \in E$ (by (I))
  $\Rightarrow \{a, b\} \in E$ and $\{u, a\} \in E$ ($E_{\langle U\rangle} \subseteq E$)
  $\Rightarrow \{a, b\}, \{u, a\} \in E_{\langle\{u,a,b\}\rangle}$ (Definition of $\langle\{u, a, b\}\rangle$)
  $\Rightarrow \{u, b\} \in E_{\langle\{u,a,b\}\rangle}$ ($\langle\{u, a, b\}\rangle$ has moe than two edges)
  $\Rightarrow \{u, b\} \in E$ ($E_{\langle u,a,b\rangle} \subseteq E$)
  $\Rightarrow \{u, b\} \in E$ and $\{v, b\} \in E$ (by (I))
  $\Rightarrow \{u, b\}, \{v, b\} \in E_{\langle\{u,v,b\}\rangle}$ (Definition of $\langle\{u, v, b\}\rangle$)
  $\Rightarrow \{u, v\} \in E_{\langle\{u,v,b\}\rangle}$ ($\langle\{u, v, b\}\rangle$ has moe than two edges)
  $\Rightarrow \{u, v\} \in E.$ ($E_{\langle u,v,b\rangle} \subseteq E$)

We have thus proven that $\forall u, v \in V, \{u, v\} \in E$ which implis that G is a complete graph. ■

Exercises 9.3, 4.

Solution:

(a) Let $G_1, G_2 \in \mathcal{G}_{p,q}$ such that $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ and $G_1 \cong G_2$.

Then $G_1, G_2 \in \mathcal{G}_{p,q} \Rightarrow |V_1| = |V_2| = p$ and $|E_1| = |E_2| = q$. (Definition of $\mathcal{G}_{p,q}$) $\cdots\cdots\cdots$ (I)

Let $\overline{G_1} = (\overline{V_1}, \overline{E_1})$ and $\overline{G_2} = (\overline{V_2}, \overline{E_2})$.

Then $G_1 \cong G_2 \Rightarrow \overline{G_1} \cong \overline{G_2}$. (Theorem 9.1.4) $\cdots\cdots\cdots$ (A)

Moreover, $\overline{V_1} = V_1$ and $\overline{V_2} = V_2$ (Definition of the complement of a graph)

$\Rightarrow |\overline{V_1}| = |V_1|$ and $|\overline{V_2}| = |V_2|$ (Definition of $|\ |$)

$\Rightarrow |\overline{V_1}| = p$ and $|\overline{V_2}| = p$ ((I),$Sub_=$) $\cdots\cdots\cdots$ (B)

It remains to prove that $|\overline{E_1}| = |\overline{E_2}| = \binom{p}{2} - q$.

By Lemma 8.6.6(i), $E_1 \cap \overline{E_1} = \emptyset$. $\cdots\cdots\cdots$ (II)

By Lemma 8.6.6(ii), $G_1 \cup \overline{G_1} = K_p$

$\Rightarrow E_{G_1 \cup \overline{G_1}} = E_{K_p}$ (Definition of identical graphs)

$\Rightarrow E_{G_1} \cup E_{\overline{G_1}} = E_{K_p}$ (Definition of union of graphs)

$\Rightarrow E_1 \cup \overline{E_1} = E_{K_p}$ ($E_{G_1} = E_1; \overline{E_{G_1}} = \overline{E_1}$)

$\Rightarrow |E_1 \cup \overline{E_1}| = |E_{K_p}|$. (Definition of $|\ |$)

$\Rightarrow |E_1| + |\overline{E_1}| = |E_{K_p}|$ ((II),Theorem 7.2.3)

$\Rightarrow q + |\overline{E_1}| = |E_{K_p}|$ ((I),$Sub_=$)

$\Rightarrow q + |\overline{E_1}| = \binom{p}{2}$ (§8.8,Ex.6)

$\Rightarrow |\overline{E_1}| = \binom{p}{2} - q$. $\cdots\cdots\cdots$ (III)

$\overline{G_1} \cong \overline{G_2} \Rightarrow |\overline{E_1}| = |\overline{E_2}|$ (Theorem 9.1.2.)

$\Rightarrow |\overline{E_2}| = \binom{p}{2} - q$. ((IV),$Sub_=$) $\cdots\cdots\cdots$ (IV)

$\Rightarrow |\overline{E_1}| = \binom{p}{2} - q$ and $|\overline{E_2}| = \binom{p}{2} - q$. ((III),(IV),I6) $\cdots\cdots\cdots$ (C)

Hence, $|\overline{V_1}| = |\overline{V_2}| = p, |\overline{E_1}| = |\overline{E_2}| = \binom{p}{2} - q$ and $\overline{G_1} \cong \overline{G_2}$ ((A),(B),(C))

$\Rightarrow \overline{G_1}, \overline{G_2} \in \mathcal{G}_{p,\binom{p}{2}-q}$. (Definition of $\mathcal{G}_{p,\binom{p}{2}-q}$) $\square$

(c) Since $|E| = 4$, by Thoerem 8.4.1, $\sum_{v \in V} deg(v) = 2|E| \Rightarrow \sum_{v \in V} deg(v) = 2 \cdot 4 = 8$.

Since G is a simple graph and $|V| = 4, |E| = 4$, $deg(v) \leq 3, \forall v \in V$.

It follows that the possible degree sequences of G are:

(i) 3, 3, 2, 0

(ii) 3, 3, 1, 1

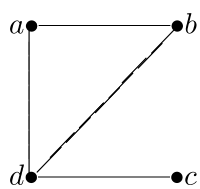
(iii) 3, 2, 2, 1

(iv) 2, 2, 2, 2

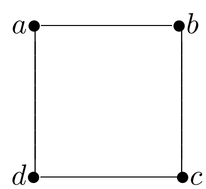
By inspection, sequences (i) and (ii) are degree sequences of non-simple graphs.

Sequences (iii) and (iv) are degree sequences of the following simple graphs:

3,2,2,1



2,2,2,2



By inspection, every two graphs of order 4 and size 4 with degree sequence 3, 2, 2, 1 are isomorphic, and every two graphs of order 4 and size 4 with degree sequence 2, 2, 2, 2 are isomorphic.

Hence, there are two equivalence classes in $[\mathcal{G}_{4,4}]/\cong$.

By the Pigeon-hole principle, among three graphs of order 4 and size 4, at least $\lceil \frac{3}{2} \rceil (= 2)$ of them are isomorphic. $\blacksquare$

Exercises 10.6, 10.

Solution: [Note: By definition, C_3 is a cycle of length 3 and P_4 is a path of length 3.]

First, we shall prove that G is bipartite.

(Proof by contradiction)

Suppose G is not bipartite. Then G contains a cycle of odd length. (Theorem 10.3.10)

Let $C : w_1 w_2 \ldots w_k w_1, w_i \neq w_j, 1 \leq i < j \leq k$, be a *shortest* cycle of *odd* length in G.

Since G contains no C_3 as an induced subgraph and k is odd, $k \neq 3 \Rightarrow k \geq 5$.

It follows that the path $w_1 w_2 w_3 w_4, w_i \neq w_j, 1 \leq i < j \leq 4$, exists which is a P_4 subgraph of the cycle C and hence of G.

Since G contains no P_4 as an induced subgraph, this P_4 subgraph of G is not the subgraph induced by $\{w_1, w_2, w_3, w_4\}$. As a result, one of the edges $\{w_1, w_3\}, \{w_2, w_4\}, \{w_1, w_4\}$ must exist in G.

In the first two cases, G contains $w_1 w_2 w_3 w_1$ and $w_2 w_3 w_4 w_2$, respectively, as an C_3 subgraph. Since G is simple, the C_3 subgraph is an induced subgraph of G, contradicting G contains no C_3 as induced subgraph.

In the last case, $w_1 w_4 w_5 \ldots w_k w_1$ is a cycle of length $k - 3 + 1 = k - 2$.

Since k is odd, $k - 2$ is odd. As a result the cycle $w_1 w_4 w_5 \ldots w_k w_1$ is a cycle of odd length that is shorter than cycle C, contradicting C is a shortest cycle of odd length in G.

Hence, G is a bipartite graph.

Next, we shall prove that G is a complete bipartite graph.

(Proof by contradiction)

Suppose G is not a complete bipartite graph.

Let $\{X, Y\}$ be a bipartition of G. (G is bipartite)

Then $\exists u \in X, v \in Y$ such that $\{u, v\} \notin E$. (G is not a complete bipartite graph) $\cdots\cdots\cdots$ (I)

Since G is connected, $u \sim v$ (Definition of connected graph)

$\Rightarrow$ there is an $u - v$ path in G. (Definition of $\sim$)

Let $P : (u =) x_1 x_2 \ldots x_k (= v), x_i \neq x_j, 1 \leq i < j \leq k$, be a *shortest* $u - v$ path in G.

Then $u = x_1 \Rightarrow x_1 \in X$ ((I),$Sub_=$)

$\Rightarrow x_2 \in Y$ and $x_3 \in X$ ($\{X, Y\}$ is a bipartition of G) $\cdots\cdots\cdots$ (II)

$v \in Y \Rightarrow v \neq x_2$; otherwise, $\{x_1, x_2\} \in E \Rightarrow \{u, v\} \in E$, contradicting (I)

$v \in Y \Rightarrow v \neq x_3$; otherwise, by (II), $x_3 \in X \Rightarrow v \in X$, contradicting $v \in Y$.

Hence, $v = v_k$, for some $k \geq 4$

$\Rightarrow$ the path $x_1 x_2 x_3 x_4$ exists which is a P_4 subgraph of P and hence of G.

Since G contains no P_4 as an induced subgraph, this P_4 subgraph of G is not the subgraph induced by $\{x_1, x_2, x_3, x_4\}$, i.e $\langle \{x_1, x_2, x_3, x_4\} \rangle$. As a result, one of the edges $\{x_1, x_3\}, \{x_2, x_4\}$ and $\{x_1, x_4\}$ must exist in $\langle \{x_1, x_2, x_3, x_4\} \rangle$ and hence in G.

Since $x_1, x_3 \in X, \{x_1, x_3\} \notin E$. ($\{X, Y\}$ is bipartition of G)

Likewise, $x_2, x_4 \in Y \Rightarrow \{x_2, x_4\} \notin E$. ($\{X, Y\}$ is bipartition of G)

It follows that $\{x_1, x_4\} \in E \Rightarrow x_1 x_4 \dots x_k$ is an $u - v$ path of length at most $k - 2$, contradicting P is a shortest $u - v$ path.

Hence G is a complete bipartite graph. $\blacksquare$