CS 60–231
Solution to Assignmemt 3

Fall 2017

Exercises 3.8, 5.

Solution: (Proof by Contradiction)

Suppose the set of all sets exists. i.e. $(\exists X)(X = \{Y|Y\ is\ a\ set\})$.

Then $\mathbb{U} = \{Y|Y\ is\ a\ set\}$ (EI)

$\Rightarrow (\exists Z)(Z = \{x \in \mathbb{U} \mid x \notin x\})$ (Principle of Specification)

$\Rightarrow B = \{x \in \mathbb{U} \mid x \notin x\}$ (EI)

$\Rightarrow B \in \mathbb{U}$ (Because B is a set) . . . (I)

Now, either $B \in B$ or $B \notin B$.

(i) Suppose $B \in B$. . . . (II)

Then $B \in \mathbb{U} \wedge B \notin B$ (because B satisfies the defining property of B)

$\Rightarrow B \notin B$ (E9, I2) . . . (III)

$\Rightarrow B \in B \wedge B \notin B$ ((II),(III), I6)

$\Rightarrow false$ (a contradiction). (E1)

(ii) Suppose $B \notin B$. . . . (IV)

Then $B \in \mathbb{U} \wedge B \notin B$ ((I),(IV), I6)

$\Rightarrow B \in B$ (because B satisfies the defining property of B) . . . (V)

$\Rightarrow B \in B \wedge B \notin B$ ((V),(IV), I6).

$\Rightarrow false$ (a contradiction). (E1)

Hence, the set of all sets does not exist. ■

Exercises 4.9, 5(b)

Solution:

$$\begin{aligned}
&\left(\overline{A} \cap B \cap C\right) \cup \left(\overline{A} \cap C \cap A\right) \cup \left(\overline{A} \cap B \cap C \cap \overline{D \cap E}\right) \\
&= \left(\overline{A} \cap B \cap C\right) \cup \left(C \cap \left(A \cap \overline{A}\right)\right) \cup \left(\overline{A} \cap B \cap C \cap \overline{D \cap E}\right) \quad (\text{Theorems } 4.1.1(iii); 4.2.2(iii)) \\
&= \left(\overline{A} \cap B \cap C\right) \cup \left(C \cap \emptyset\right) \cup \left(\overline{A} \cap B \cap C \cap \overline{D \cap E}\right) \quad (\text{Theorem} 4.3.7(iii)) \\
&= \left(\overline{A} \cap B \cap C\right) \cup \emptyset \cup \left(\overline{A} \cap B \cap C \cap \overline{D \cap E}\right) \quad (\text{Theorem} 4.2.2(i)) \\
&= \left(\overline{A} \cap B \cap C\right) \cup \left(\overline{A} \cap B \cap C \cap \overline{D \cap E}\right) \quad (\text{Theorem} 4.1.1(i)) \\
&= \left(\overline{A} \cap B \cap C \cap \overline{D \cap E}\right) \cup \left(\overline{A} \cap B \cap C\right) \quad (\text{Theorem} 4.1.1(iii)) \quad \cdots (\text{I})
\end{aligned}$$

$\left(\left(\overline{A} \cap B \cap C\right) \cap \overline{D \cap E}\right) \subseteq \left(\overline{A} \cap B \cap C\right)$ (Corollary 4.2.2.1)

$\Rightarrow \left(\left(\overline{A} \cap B \cap C\right) \cap \overline{D \cap E}\right) \cup \left(\overline{A} \cap B \cap C\right) = \left(\overline{A} \cap B \cap C\right)$.(Theorem 4.1.1(v)) . . . (II)

Hence,

$$\left(\overline{A} \cap B \cap C\right) \cup \left(\overline{A} \cap C \cap A\right) \cup \left(\overline{A} \cap B \cap C \cap \overline{D \cap E}\right) = \left(\overline{A} \cap B \cap C\right) \quad (\because (\text{I})(\text{II}), \text{Theorem} 3.2.3(iii))\blacksquare$$

Exercises 4.9, 8(a)

Solution:Solution:

$$\begin{aligned}
A \cap (B - A) &= A \cap \left(B \cap \overline{A}\right) \quad (\text{Theorem } 4.3.7(v)) \\
&= A \cap \left(\overline{A} \cap B\right) \quad (\text{Theorem } 4.2.2(iii)) \\
&= \left(A \cap \overline{A}\right) \cap B \quad (\text{Theorem } 4.2.2(iv)) \\
&= \emptyset \cap B \quad (\text{Theorem } 4.3.7(iii)) \\
&= B \cap \emptyset \quad (\text{Theorem } 4.2.2(iii)) \\
&= \emptyset \quad (\text{ Theorem } 4.2.2(i)) \quad \blacksquare
\end{aligned}$$

Exercises 4.9, 9(a) without using Theorem 4.3.4(*i*).

Solution: $A \cap \overline{B} = \emptyset \Leftrightarrow \sim (\exists x)\big(x \in A \cap \overline{B}\big)$ (Theorem 3.5.8)

$\Leftrightarrow (\forall x) \sim \big(x \in A \cap \overline{B}\big)$ (FE8)

$\Leftrightarrow (\forall x) \sim \big(x \in A \wedge x \in \overline{B}\big)$ (Definition of $\cap$)

$\Leftrightarrow (\forall x)\big(x \notin A \vee \sim x \in \overline{B}\big)$ (E16)

$\Leftrightarrow (\forall x)(x \notin A \vee \sim x \notin B)$ (Lemma 4.3.5)

$\Leftrightarrow (\forall x)(x \notin A \vee x \in B)$ (E15)

$\Leftrightarrow (\forall x)(x \in A \Rightarrow x \in B)$ (E18)

$\Leftrightarrow A \subseteq B$ (Definition of $\subseteq$) ■

Exercises 4.9, 11(c)

Solution: (Bidirection Proof)

$X \in \mathcal{P}(A \cap B)$

$\Leftrightarrow X \subseteq (A \cap B)$ (Definition of Power set)

$\Leftrightarrow (\forall x)(x \in X \Rightarrow x \in (A \cap B))$ (Definition of $\subseteq$)

$\Leftrightarrow (\forall x)(x \in X \Rightarrow x \in A \wedge x \in B)$ (Definition of $\cap$)

$\Leftrightarrow (\forall x)(\sim x \in X \vee (x \in A \wedge x \in B))$ (E18)

$\Leftrightarrow (\forall x)((\sim x \in X \vee x \in A) \wedge (\sim x \in X \vee x \in B))$ (E14)

$\Leftrightarrow (\forall x)((x \in X \Rightarrow x \in A) \wedge (x \in X \Rightarrow x \in B))$ (E18)

$\Leftrightarrow (\forall x)(x \in X \Rightarrow x \in A) \wedge (\forall x)(x \in X \Rightarrow x \in B)$ (FE9)

$\Leftrightarrow (X \subseteq A) \wedge (X \subseteq B)$ (Definition of $\subseteq$)

$\Leftrightarrow (X \in \mathcal{P}(A)) \wedge (X \in \mathcal{P}(B))$ (Definition of poer set)

$\Leftrightarrow X \in \mathcal{P}(A) \cap \mathcal{P}(B)$ (Definition of $\cap$)

Therefore, $X \in \mathcal{P}(A \cap B) \Leftrightarrow X \in \mathcal{P}(A) \cap \mathcal{P}(B)$

$\Rightarrow (\forall X)(X \in \mathcal{P}(A \cap B) \Leftrightarrow X \in \mathcal{P}(A) \cap \mathcal{P}(B))$ (Gen)

$\Rightarrow \mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$ (Principle of Extension) ■

Exercises 4.9, 12

Solution: (Direct proof) Suppose $A \subseteq B$. . . . (A)

Let $C = A$. . . . (I)

Then $C \subseteq (C \cup B)$ (Corollary 4.2.2.1)

$\Rightarrow C \subseteq (B \cup C)$ (Theorem 4.1.1(iii))

$\Rightarrow (C \cup C) \subseteq (B \cup C)$ (Theorem 4.1.1(ii))

$\Rightarrow (A \cup C) \subseteq (B \cup C)$ ((I), $\mathrm{Sub}_{=}$)

Next, $(A \cap B) = A$ ((A), Theorem 4.2.2(v)) . . . (II)

$A = (A \cap A)$ (Theorem 4.2.2(ii)) . . . (III)

Then $A \subseteq A$ (Theorem 3.2.3(i))

$\Rightarrow (A \cap A) \subseteq (A \cap B)$ ((II),(III), $\mathrm{Sub}_{=}$)

$\Rightarrow (A \cap A) \subseteq (B \cap A)$ (Theorem 4.2.2(iii))

$\Rightarrow (A \cap C) \subseteq (B \cap C)$ ((I), $\mathrm{Sub}_{=}$)

Hence, A is the set C such that $(A \cup C) \subseteq (B \cup C)$ and $(A \cap C) \subseteq (B \cap C)$. ■

Exercises 4.9, 14(a)

Solution: First, we shall prove: $x \in \bigcup_{X \in \{A\}} X \Leftrightarrow x \in A$.

$\Rightarrow$) (Direct proof)

$x \in \bigcup_{X \in \{A\}} X \Rightarrow (\exists X)(X \in \{A\} \wedge x \in X)$ (Definition of $\bigcup$)

$\Rightarrow K \in \{A\} \wedge x \in K$ (EI)

$\Rightarrow K \in \{A\}$ and $x \in K$ (I2,E9)

$\Rightarrow K = A$ and $x \in K$ (Definition of $\{A\}$)

$\Rightarrow x \in A$ (Sub$_=$)

$\Leftarrow$) (Direct proof)

Let $x \in A$. . . . (I)

Then $A = A$ (Lemma 3.2.1)

$\Rightarrow A \in \{A\}$ (Definition of $\{A\}$) . . . (II)

$\Rightarrow A \in \{A\} \wedge x \in A$ ((II),(I), I6)

$\Rightarrow (\exists X)(X \in \{A\} \wedge x \in X)$ (EQ)

$\Rightarrow x \in \bigcup_{X \in \{A\}} X$ (Definition of $\bigcup$)

We thus have $x \in \bigcup_{X \in \{A\}} X \Leftrightarrow x \in A$

$\Rightarrow (\forall x)\left(x \in \bigcup_{X \in \{A\}} X \Leftrightarrow x \in A\right)$ (Gen)

$\Rightarrow \bigcup_{X \in \{A\}} X = A$ (Principle of Extension) ■