CS 60–231
Solution to Assignmemt #2

Fall 2017

2.1.(l) Evaluate $(\exists x)((\forall y) \sim Q(y,x) \wedge (P(x) \Rightarrow (\exists z)(Q(z,x) \wedge Q(z,z))))$.

Solution:

Let $x = 1$, we obtain $(\forall y) \sim Q(y,1) \wedge (P(1) \Rightarrow (\exists z)(Q(z,1) \wedge Q(z,z)))$

In $(P(1) \Rightarrow (\exists z)(Q(z,1) \wedge Q(z,z)))$, since $P(1) \equiv \mathrm{T}$, we must evaluate $(\exists z)(Q(z,1) \wedge Q(z,z))$.

Let $z = 1$. We obtain $Q(1,1) \wedge Q(1,1) \equiv \mathrm{F} \wedge \mathrm{F} \equiv \mathrm{F}$,

Let $z = 2$. We obtain $Q(2,1) \wedge Q(2,2) \equiv \mathrm{F} \wedge \mathrm{T} \equiv \mathrm{F}$,

Therefore, $(\exists z)(Q(z,1) \wedge Q(z,z))$ is evaluated to *false*

which implies that $(P(1) \Rightarrow (\exists z)(Q(z,1) \wedge Q(z,z)))$ is evaluated to *false*.

Hence, $(\forall y) \sim Q(y,1) \wedge (P(1) \Rightarrow (\exists z)(Q(z,1) \wedge Q(z,z)))$ is evaluated to *false*.

Let $x = 2$, we obtain $(\forall y) \sim Q(y,2) \wedge (P(2) \Rightarrow (\exists z)(Q(z,2) \wedge Q(z,z)))$.

In $(\forall y) \sim Q(y,2)$, let $y = 1$, we obtain $\sim Q(1,2) \equiv \sim \mathrm{T} \equiv \mathrm{F}$

which implies that $(\forall y) \sim Q(y,2)$ is evaluated to *false*.

Hence, $(\forall y) \sim Q(y,2) \wedge (P(2) \Rightarrow (\exists z)(Q(z,2) \wedge Q(z,z)))$ is evaluated to *false*.

We thus have $(\exists x)((\forall y) \sim Q(y,x) \wedge (P(x) \Rightarrow (\exists z)(Q(z,x) \wedge Q(z,z))))$ is evaluated to *false*. ■

2.2.(e): Prove Prove $\vdash (\forall x)(\alpha(x) \Rightarrow \beta) \Leftrightarrow ((\exists x)\alpha(x) \Rightarrow \beta)$ where *x* is not free in β.

Solution: (Bidirectional Proof)

$\Rightarrow$) (Direct proof)

1. $(\forall x)(\alpha(x) \Rightarrow \beta)$ hypothesis
2. $(\forall x)(\sim \alpha(x) \vee \beta)$ 1, E18
3. $(\forall x){\sim}\alpha(x) \vee \beta$ 2, FE3 (β contains no free occurrence of *x*)
4. $\sim (\exists x)\alpha(x) \vee \beta$ 3, FE8
5. $(\exists x)\alpha(x) \Rightarrow \beta$ 4, E18

$\Leftarrow$) The above proof in reversed order.

2.2.(g): Prove $\vdash (\forall x)(\alpha(x) \vee \beta(x)) \Rightarrow (\forall x)\alpha(x) \vee (\exists x)\beta(x)$..

Solution: (Proof by contradiction)

1. $(\forall x)(\alpha(x) \vee \beta(x)) \wedge \sim ((\forall x)\alpha(x) \vee (\exists x)\beta(x))$ hypothesis
2. $(\forall x)(\alpha(x) \vee \beta(x))$ 1, I2
3. $\sim ((\forall x)\alpha(x) \vee (\exists x)\beta(x)) \wedge (\forall x)(\alpha(x) \vee \beta(x))$ 1,E9
4. $\sim ((\forall x)\alpha(x) \vee (\exists x)\beta(x))$ 3,I2
5. $\sim (\forall x)\alpha(x) \wedge \sim (\exists x)\beta(x)$ 4,E17
6. $\sim (\forall x)\alpha(x)$ 5,I2
7. $\sim (\exists x)\beta(x) \wedge \sim (\forall x)\alpha(x)$ 5,E9
8. $\sim (\exists x)\beta(x)$ 7,I2
9. $(\exists x) \sim \alpha(x)$ 6,FE7
10. $(\forall x) \sim \beta(x)$ 8,FE8
11. $\sim \alpha(k)$ 9,EI, *k* is a constant
12. $\sim \beta(k)$ 10,UI
13. ${\sim}\alpha(k) \wedge {\sim}\beta(k)$ 11,12, I6
14. ${\sim}(\alpha(k) \vee \beta(k))$ 13, E17
15. $(\alpha(k) \vee \beta(k))$ 2,UI
16. $(\alpha(k) \vee \beta(k)) \wedge {\sim}(\alpha(k) \vee \beta(k))$ 14,15, I6

17. $false$ 16,E1 ■

2.3 (a).

Solution:

1. $(\exists x)(\forall y)(P(x) \wedge (Q(y) \Rightarrow R(x,y)))$ from Γ
2. $(\forall x)(\forall y)((P(x) \wedge S(y)) \Rightarrow \sim R(x,y))$ from Γ
3. $(\forall y)(P(k) \wedge (Q(y) \Rightarrow R(k,y)))$ 1, EI, k is a constant
4. $P(k) \wedge (Q(y) \Rightarrow R(k,y))$ 3, UI
5. $(\forall y)((P(k) \wedge S(y)) \Rightarrow \sim R(k,y))$ 2, UI
6. $(P(k) \wedge S(y)) \Rightarrow \sim R(k,y)$ 5, UI
7. $P(k)$ 4, I2
8. $(Q(y) \Rightarrow R(k,y)) \wedge P(k)$ 4, E9
9. $Q(y) \Rightarrow R(k,y)$ 8,I2
10. $\sim R(k,y) \Rightarrow \sim Q(y)$ 9, E19
11. $(P(k) \wedge S(y)) \Rightarrow \sim Q(y)$ 6,10, I5
12. $\sim (P(k) \wedge S(y)) \vee \sim Q(y)$ 11, E18
13. $(\sim P(k) \vee \sim S(y)) \vee \sim Q(y)$ 12, E16
14. $\sim P(k) \vee (\sim S(y) \vee \sim Q(y))$ 13, E12
15. $P(k) \Rightarrow (\sim S(y) \vee \sim Q(y))$ 14, E18
16. $(\sim S(y) \vee \sim Q(y))$ 7,15, I3
17. $(\sim Q(y) \vee \sim S(y))$ 16, E10
18. $(Q(y) \Rightarrow \sim S(y))$ 17, E18
19. $(\forall x)(Q(x) \Rightarrow \sim S(x))$ 18, Gen ■

2.3 (d).

Solution:

1. $(\exists x)(P(x) \wedge (\forall y)((R(y) \wedge S(x,y)) \Rightarrow Z(x,y)))$ from Γ
2. $(\forall x)(P(x) \Rightarrow (\exists y)(R(y) \wedge \sim U(x,y) \wedge T(x,y)))$ from Γ
3. $(\forall x)(\forall y)((P(x) \wedge R(y) \wedge T(x,y)) \Rightarrow S(x,y))$ from Γ
4. $P(k) \wedge (\forall y)((R(y) \wedge S(k,y)) \Rightarrow Z(k,y))$ 1, EI, k is a new constant
5. $P(k)$ 4, I2
6. $(\forall y)((R(y) \wedge S(k,y)) \Rightarrow Z(k,y)) \wedge P(k)$ 4,E9
7. $(\forall y)((R(y) \wedge S(k,y)) \Rightarrow Z(k,y))$ 6, I2
8. $P(k) \Rightarrow (\exists y)(R(y) \wedge \sim U(k,y) \wedge T(k,y))$ 2, UI
9. $(\exists y)(R(y) \wedge \sim U(k,y) \wedge T(k,y))$ 5,8, I3
10. $((R(c) \wedge \sim U(k,c)) \wedge T(k,c))$ 9, EI, c is a new constant
11. $(R(c) \wedge S(k,c)) \Rightarrow Z(k,c)$ 7, UI
12. $(\forall y)((P(k) \wedge R(y) \wedge T(k,y)) \Rightarrow S(k,y))$ 3,UI
13. $((P(k) \wedge R(c)) \wedge T(k,c)) \Rightarrow S(k,c)$ 12, UI
14. $(\sim U(k,c) \wedge R(c)) \wedge T(k,c)$ 10, E9
15. $\sim U(k,c) \wedge (R(c) \wedge T(k,c))$ 14, E11
16. $(R(c) \wedge T(k,c)) \wedge \sim U(k,c)$ 15, E9
17. $R(c) \wedge T(k,c)$ 16, I2
18. $P(k) \wedge (R(c) \wedge T(k,c))$ 5,17, I6
19. $(P(k) \wedge R(c)) \wedge T(k,c)$ 18,E11
20. $S(k,c)$ 19,13, I3
21. $R(c)$ 17, I2
22. $R(c) \wedge S(k,c)$ 21,20, I6
23. $Z(k,c)$ 22,11, I3
24. $P(k) \wedge R(c)$ 19, I2
25. $(P(k) \wedge R(c)) \wedge Z(k,c)$ 24,23, I6
26. $\sim U(k,c)$ 15, I2
27. $P(k) \wedge R(c) \wedge Z(k,c) \wedge \sim U(k,c)$ 25,26, I6

28. $(\exists y)(P(k) \wedge R(y) \wedge Z(k,y) \wedge \sim U(k,y))$ 27, EQ
29. $(\exists x)(\exists y)(P(x) \wedge R(y) \wedge Z(x,y) \wedge \sim U(x,y))$ 28, EQ ■

2.4 (h). Let U be the set of all animals. .

$K(x)$: x kicks;
$E(x)$: x is excitable;
$D(x)$: x is a donkey;
$B(x)$: x is a buffalo;
$H(x)$: x has horns;
$T(x)$: x can toss one over a gate;
$S(x)$: x is easy to swallow.

P1: $(\forall x)(\sim K(x) \Rightarrow \sim E(x))$
P2: $(\forall x)(D(x) \Rightarrow \sim H(x))$
P3: $(\forall x)(B(x) \Rightarrow T(x))$
P4: $\sim (\exists x)(K(x) \wedge S(x))$
P5: $\sim (\exists x)(\sim H(x) \wedge T(x))$
P6: $(\forall x)(\sim B(x) \Leftrightarrow E(x))$
C: $(\forall x)(D(x) \Rightarrow \sim S(x))$

(Direct Proof)

1. $(\forall x)(\sim K(x) \Rightarrow \sim E(x))$ from Γ
2. $(\forall x)(D(x) \Rightarrow \sim H(x))$ from Γ
3. $(\forall x)(B(x) \Rightarrow T(x))$ from Γ
4. $\sim (\exists x)(K(x) \wedge S(x))$ from Γ
5. $\sim (\exists x)(\sim H(x) \wedge T(x))$ from Γ
6. $(\forall x)(\sim B(x) \Leftrightarrow E(x))$ from Γ
7. $(\forall x) \sim (K(x) \wedge S(x))$ 4, FE8
8. $\sim (K(x) \wedge S(x))$ 7, UI
9. $(\sim K(x) \vee \sim S(x))$ 8, E16
10. $K(x) \Rightarrow \sim S(x)$ 9, E18
11. $(\forall x) \sim (\sim H(x) \wedge T(x))$ 5, FE8
12. $\sim (\sim H(x) \wedge T(x))$ 11, UI
13. $\sim\sim H(x) \vee \sim T(x)$ 12, E16
14. $\sim H(x) \Rightarrow \sim T(x)$ 13, E18
15. $(D(x) \Rightarrow \sim H(x))$ 2, UI
16. $(D(x) \Rightarrow \sim T(x))$ 15,14, I5
17. $(B(x) \Rightarrow T(x))$ 3, UI
18. $(\sim T(x) \Rightarrow \sim B(x))$ 17, E19
19. $(D(x) \Rightarrow \sim B(x))$ 16,18, I5
20. $(\sim B(x) \Leftrightarrow E(x))$ 6, UI
21. $(\sim B(x) \Rightarrow E(x)) \wedge (E(x) \Rightarrow \sim B(x))$ 20, E20
22. $(\sim B(x) \Rightarrow E(x))$ 21, I2
23. $(D(x) \Rightarrow E(x))$ 19,22, I5
24. $(\sim K(x) \Rightarrow \sim E(x))$ 1, UI
25. $(E(x) \Rightarrow K(x))$ 24, E19
26. $(D(x) \Rightarrow K(x))$ 23,25, I5
27. $(D(x) \Rightarrow \sim S(x))$ 26,10, I5
28. $(\forall x)(D(x) \Rightarrow \sim S(x))$ 27, Gen ■