Homework (1)

Exercice 1:

$\boxed{1}$ Let R be the set of the form $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, \dots, n\}$

for any two points $x, y \in R$ each component i of the coordinates of x and y satisfy $\begin{cases} \alpha_i \leq x_i \leq \beta_i \\ \text{and} \\ \alpha_i \leq y_i \leq \beta_i \end{cases}$ with $i = 1, \dots, n$

Let's take $\lambda \in [0,1]$

as $\lambda \geq 0$ and $(1-\lambda) \geq 0$ we can whrite :

$$\begin{cases} \lambda \alpha_i \leq \lambda x_i \leq \lambda \beta_i & (1) \\ \text{and} & \\ (1-\lambda)\alpha_i \leq (1-\lambda) y_i \leq (1-\lambda)\beta_i & (2) \end{cases} \quad \text{with } i = 1, \dots, n$$

(1) + (2) gives :

$$(\lambda + 1 - \lambda)\alpha_i \leq \lambda x_i + (1-\lambda) y_i \leq (\lambda + 1 - \lambda)\beta_i \quad \text{with } i = 1, \dots, n$$

$$\Rightarrow \quad \alpha_i \leq \lambda x_i + (1-\lambda) y_i \leq \beta_i \quad \text{with } i = 1, \dots, n$$

which implies that $(\lambda x + (1-\lambda) y) \in R$ for any $\lambda \in [0,1]$

therefore, the set R is convex

$\boxed{2}$ Let S be the set defined by : $S = \{x \in \mathbb{R}^2_+ \mid x_1 x_2 \geq 1\}$

Let's take $x, y \in S$ and $\lambda \in [0,1]$

Let's define $Z = \lambda x + (1-\lambda) y = \begin{pmatrix} \lambda x_1 + (1-\lambda) y_1 \\ \lambda x_2 + (1-\lambda) y_2 \end{pmatrix}$

$$Z_1 . Z_2 = [\lambda x_1 + (1-\lambda) y_1] [\lambda x_2 + (1-\lambda) y_2]$$

So $z_1 . z_2 = \lambda^2 x_1 . x_2 + (1-\lambda)^2 y_1 . y_2 + \lambda(1-\lambda)[x_1 y_2 + x_2 y_1]$

$x, y \in \mathbb{R}^2_+$ which means that $x_1 y_2 \geq 0$ and $x_2 y_1 \geq 0$

Using the AM-GM inequality we can whrite: $\frac{x_1 y_2 + x_2 y_1}{2} \geq \sqrt{x_1 y_2 x_2 y_1}$

which means that: $x_1 y_2 + x_2 y_1 \geq 2\sqrt{x_1 x_2 y_1 y_2} \geq 2\sqrt{1.1}$

So we can write: $\lambda(1-\lambda)[x_1 y_2 + x_2 y_1] \geq \lambda(1-\lambda) . 2$ (1)

In an other hand, we know that $x, y \in S$ which implies:

$$\begin{cases} x_1 x_2 \geq 1 \\ \text{and} \\ y_1 y_2 \geq 1 \end{cases} \quad \text{so} \quad \begin{cases} \lambda^2 x_1 x_2 \geq \lambda^2 & (2) \\ (\lambda-1)^2 x_1 x_2 \geq (1-\lambda)^2 & (3) \end{cases}$$

(1) + (2) + (3) gives: $z_1 . z_2 \geq \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)$

$\Rightarrow \quad z_1 . z_2 \geq \cancel{\lambda^2} + 1 + \cancel{\lambda^2} - \cancel{2\lambda} + \cancel{2\lambda} - \cancel{2\lambda^2}$

$\Rightarrow \quad \boxed{z_1 . z_2 \geq 1}$

which implies that $z \in S$ for any $\lambda \in [0, 1]$

therefore, S is a Convex set.

3 Let A be the set defined by: $A = \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \ \forall y \in S\}$

Where $S \subseteq \mathbb{R}^n$.

$$A = \bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$$

$$= \bigcap_{y \in S} B_y$$

We will try to prove that B_y is a half-space.

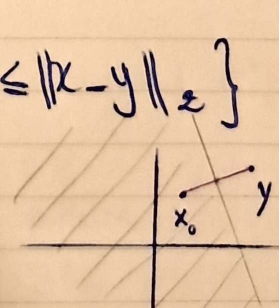


illustration in $\mathbb{R}^2$

For $y \in S$, $x \in B_y$ means that $(x-x_0)^T(x-x_0) \leq (x-y)^T(x-y)$

So we can write: $x^Tx + x_0^Tx_0 - 2x_0^Tx \leq x^Tx + y^Ty - 2y^Tx$

$$\Rightarrow \quad 2(y^T - x_0^T)x \leq \cancel{x^Tx}\ y^Ty - x_0^Tx_0$$

$$\Rightarrow \quad (y^T - x_0)^T . x \leq \frac{\|y\|_2^2 - \|x_0\|_2^2}{2}$$

Let $a = y - x_0$ a vector of $\mathbb{R}^n$ and $b = \frac{\|y\|^2 - \|x_0\|^2}{2}$ a scalar of $\mathbb{R}$.

thus, $B_y = \{ x \in \mathbb{R}^n \mid a^T.x \leq b \}$ is a half space so convex.

Finally, $A = \bigcap_{y \in S} B_y$ is the intersection of convex sets

therefore A is a convex set.

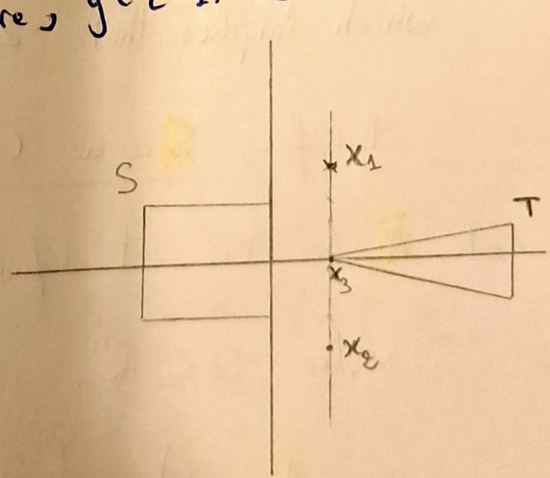
4 Let H be the set defined by: $H = \{ x \mid \text{dist}(x,S) \leq \text{dist}(x,T) \}$

where $S,T \subseteq \mathbb{R}^n$ and $\text{dist}(x,S) = \inf\{ \|x-z\|_2 \mid z \in S \}$

Let's consider x_1 and x_2 of the set H defined in the Figure below:

In the particular case of the figure, $y \in [x_1, x_2]$ but $y \notin H$.

therefore H is not convex.



$\boxed{5}$ Let C be the set defined by: $C = \{x \mid x + S_2 \subseteq S_1\}$

where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.

$$C = \{x \mid x + y \in S_1 \ \forall y \in S_2\}$$
$$= \bigcap_{y \in S_2} \{x \mid x + y \in S_1\}$$

Let f_y be the affine function defined by:

$$f_y : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$z \longmapsto z - y \quad \text{with } y \in S_2 \text{ fixed}$$

$$C = \bigcap_{y \in S_2} \{f_y(x+y) \mid x+y \in S_1\} = \bigcap_{y \in S_2} f_y(S_1)$$
$$= \bigcap_{y \in S_2} \{x \mid x + y \in S_1\}$$

as S_1 is convex $f_y(S_1)$ is convex.
so as $f_y(S_1)$ is convex C is convex as intersection of convex sets.

Exercice (2):

$\boxed{1}$ Let's analyze the function $f(x_1, x_2) = x_1 x_2$ with $\mathrm{dom} f = \mathbb{R}^2_{++}$

- First we note that $\mathrm{dom} f$ is a convex set.
- f is twice differentiable as its domain is open and $\nabla^2 f(x)$ exists $\forall x \in \mathrm{dom}$

we have.
$$\begin{cases} \dfrac{\partial f}{\partial x_1} = x_2 \\[2ex] \dfrac{\partial f}{\partial x_2} = x_1 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial^2 f}{\partial x_1^2} = 0 \\[2ex] \dfrac{\partial^2 f}{\partial x_2^2} = 0 \end{cases} \text{ and } \begin{cases} \dfrac{\partial^2 f}{\partial x_1 \partial x_2} = 1 \\[2ex] \dfrac{\partial^2 f}{\partial x_2 \partial x_1} = 1 \end{cases}$$

$\Rightarrow \quad \nabla^2 f(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Let's compute its eigenvalues:

$$\det(\nabla^2 f - \lambda Id) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

$\Rightarrow$ eigenvalues are $\{-1, 1\}$

$\Rightarrow$ the Hessian $\nabla^2 f$ is not positive semi definite

therefore f is not convex.

- the Hessian of $(-f)$ $\nabla^2(-f)$ has the same eigenvalues

$\Rightarrow$ $(-f)$ is not convex

therefore f is not concave.

- Let's define the set $S_\alpha = \{ x \in \mathbb{R}^2_{++} ; f(x) \geq \alpha \}$

$$= \{ (x_1, x_2) \in \mathbb{R}^2_{++} ; x_1 \cdot x_2 \geq \alpha \}$$

Let's take $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $\lambda \in [0,1]$

Let's suppose that $x, y \in S_\alpha$ and $z = \lambda x + (1-\lambda) y$

$$f(z) = [\lambda x_1 + (1-\lambda) y_1] \cdot [\lambda x_2 + (1-\lambda) y_2]$$

$$= \lambda^2 x_1 x_2 + (1-\lambda)^2 y_1 \cdot y_2 + \lambda(1-\lambda)[x_1 y_2 + x_2 y_1]$$

- using the AM-GM inequality we can write:

$$x_1 y_2 + x_2 y_1 \geq 2\sqrt{x_1 y_2 x_2 y_1}$$

and as $\begin{cases} x_1 x_2 \geq \alpha \\ y_1 y_2 \geq \alpha \end{cases}$ $\Rightarrow$ $x_1 y_2 + x_2 y_1 \geq 2\sqrt{\alpha^2}$

if $\alpha \geq 0$ $\rightarrow$ $x_1 y_2 + x_2 y_1 \geq 2\alpha$

if $\alpha \leq 0$ $\longrightarrow$ $S_\alpha = \mathbb{R}^2_{++}$: convex. 1

So in the case where $\alpha \geq 0$

we can write: $f(z) = \lambda^2 x_1 x_2 + (1-\lambda)^2 y_1 y_2 + \lambda(1-\lambda)[x_1 y_2 + x_2 y_1]$

$$\Rightarrow f(z) \geq \lambda^2 \alpha + (1-\lambda)^2 \alpha + 2\lambda(1-\lambda)\alpha$$

$$\Rightarrow f(z) \geq \lambda^2 \alpha + \alpha + \lambda^2 \alpha - 2\lambda\alpha + 2\lambda\alpha - 2\lambda^2 \alpha$$

$$\Rightarrow f(z) \geq \alpha$$

$$\Rightarrow z \in S_\alpha$$

thus, for $\lambda \in [0,1]$ and $x, y \in S_\alpha$ $\lambda x + (1-\lambda) y \in S_\alpha$

therefore S_α is convex for $\alpha \leq 0$. (2)

from (1) and (2) we have that $\forall \alpha$ S_α is convex.

In conclusion f is quasiconcave.

2) Let's consider $f(x_1, x_2) = 1/x_1 x_2$ with $\mathrm{dom} f = \mathbb{R}^2_{++}$

f is twice differentiable on $\mathrm{dom} f$.

$$\begin{cases} \dfrac{\partial f}{\partial x_1} = -\dfrac{1}{x_1^2} \times \dfrac{1}{x_2} \\ \dfrac{\partial f}{\partial x_2} = -\dfrac{1}{x_2^2} \times \dfrac{1}{x_1} \end{cases} \Rightarrow \begin{cases} \dfrac{\partial^2 f}{\partial x_1} = \dfrac{2}{x_2 x_1^3} \\ \dfrac{\partial^2 f}{\partial x_2} = \dfrac{2}{x_2^3 x_1} \end{cases} \text{ and}$$

$$\begin{cases} \dfrac{\partial^2 f}{\partial x_2 \partial x_1} = \dfrac{1}{x_1^2 x_2^2} \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_2} = \dfrac{1}{x_1^2 x_2^2} \end{cases}$$

so we can write

$$\boxed{\nabla^2 f(x) = \begin{pmatrix} \dfrac{2}{x_1^3 x_2} & \dfrac{1}{(x_1 x_2)^2} \\ \dfrac{1}{(x_1 x_2)^2} & \dfrac{2}{x_2^3 x_1} \end{pmatrix}}$$

so $\nabla^2 f(x)$ is symmetric with $\begin{cases} Tr(\nabla^2 f) = \frac{1}{x_1 x_2}\left[\frac{2}{x_1^2} + \frac{2}{x_2^2}\right] > 0 \\ \text{and} \\ \det(\nabla^2 f) = \frac{4}{x_1^4 x_2^4} - \frac{1}{x_2^4 x_1^4} = \frac{3}{x_1^4 x_2^4} > 0 \end{cases}$

thus $\boxed{\nabla^2 f(x) \succeq 0}$

therefore f is convex and then quasiconvex too.

3 Let's consider the function $f(x_1, x_2) = \frac{x_1}{x_2}$ with $\text{dom} f = \mathbb{R}^2_{++}$

f is twice differentiable on dom f and:

$$\begin{cases} \frac{\partial f}{\partial x_1} = \frac{1}{x_2} \\ \frac{\partial f}{\partial x_2} = \frac{-x_1}{x_2^2} \end{cases} \rightarrow \begin{cases} \frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{-1}{x_2^2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} = -\frac{1}{x_2^2} \end{cases} \text{ and } \begin{cases} \frac{\partial^2 f}{\partial^2 x_1} = 0 \\ \frac{\partial^2 f}{\partial^2 x_2} = \frac{2x_1}{x_2^3} \end{cases}$$

$$\Rightarrow \nabla^2 f(x) = \begin{pmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{pmatrix}$$

so $Tr(\nabla^2 f(x)) = \frac{2x_1}{x_2^3} \geq 0$ on $\mathbb{R}^2_{++}$

$\det(\nabla^2 f(x)) = -\frac{1}{x_2^4} < 0$

so f is not convex

$Tr(\nabla^2 -f) = -\frac{2x_1}{x_2^3} \leq 0$

$\det(\nabla^2 -f) = \frac{-1}{x_2^4}$

so $(-f)$ is not convex then f is not concave.

- Now let's consider the set $S_\alpha = \left\{ x \in \mathbb{R}^2_{++} \mid \frac{x_1}{x_2} \leq \alpha \right\}$

Let's take $x, y \in S_\alpha$ and $\lambda \in [0,1]$ and define $z = \lambda x + (1-\lambda) y$

$$\frac{z_1}{z_2} = \frac{\lambda x_1 + (1-\lambda) y_1}{\lambda x_2 + (1-\lambda) y_2} \underset{(\alpha \geq 0)}{\leq} \frac{\lambda \alpha x_2 + (1-\lambda) \cdot \alpha y_2}{\lambda x_2 + (1-\lambda) y_2} = \alpha$$

if $\alpha < 0 \xrightarrow{\text{then}} S_\alpha = \phi$

then S_α is convex for every α.

therefore f is quasiconvex.

4 Let's consider the function $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ with $0 \leq \alpha \leq 1$ and $\mathrm{dom}\, f = \mathbb{R}^2_{++}$.

$$\left.\begin{array}{ll} \text{if } \alpha = 0 & f(x_1, x_2) = x_2 \\ \text{if } \alpha = 1 & f(x_1, x_2) = x_1 \end{array}\right\} f \text{ is concave and convex.}$$

f is twice differentiable on $\mathrm{dom}\, f$ with:

$$\left\{\begin{array}{l} \frac{\partial f}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{1-\alpha} \\ \frac{\partial f}{\partial x_2} = (1-\alpha) x_1^{\alpha} x_2^{-\alpha} \end{array}\right. \longrightarrow \left\{\begin{array}{l} \frac{\partial^2 f}{\partial^2 x_1} = \alpha(\alpha-1) x_1^{\alpha-2} x_2^{-\alpha} \\ \text{and} \\ \frac{\partial f}{\partial^2 x_2} = -\alpha(1-\alpha) x_1^{\alpha} x_2^{-\alpha-1} \end{array}\right. \left\{\begin{array}{l} \frac{\partial^2 f}{\partial x_2 \partial x_1} = \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} \\ \frac{\partial f}{\partial x_1 \partial x_2} = \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} \end{array}\right.$$

$$\nabla^2 f = \alpha(1-\alpha) x_1^{\alpha} x_2^{1-\alpha} \begin{pmatrix} -1/x_1^2 & \frac{1}{x_1 x_2} \\ 1/x_1 \cdot x_2 & -\frac{1}{x_2^2} \end{pmatrix}$$

so $Tr(\nabla f) = -\frac{1}{x_2^2} - \frac{1}{x_2^2} < 0$ so f is not convex

and $\left\{\begin{array}{l} Tr(\nabla - f) = \frac{1}{x_1^2} + \frac{1}{x_2^2} > 0 \\ \det(\nabla - f) = \frac{1}{x_1^2 x_2^2} - \frac{1}{x_1^2 x_2^2} = 0 \end{array}\right.$ so f is concave and quasiconcave

Exercice 3:

1) Let f be the function defined by: $f(X) = \mathrm{Tr}(X^{-1})$ with $\mathrm{dom} f = S^n_{++}$

- $\mathrm{dom} f$ is convex

Let's define $g(t) = \mathrm{Tr}\left((X + tV)^{-1} \right)$

$$= \mathrm{Tr}\left[X^{1/2} (I + t X^{-\frac{1}{2}} V X^{-\frac{1}{2}}) X^{\frac{1}{2}})^{-1} \right]$$

$$= \mathrm{Tr}\left[X^{-1/2} \left((I + t X^{-\frac{1}{2}} V X^{-\frac{1}{2}}) X^{-\frac{1}{2}} \right)^{-1} \right]$$

$$= \mathrm{Tr}\left[X^{-1} \left((I + \underbrace{t X^{-\frac{1}{2}} V X^{-\frac{1}{2}}}_{\in S_n^{++}} \right)^{-1} \right]$$

So there exists P orthogonal such that $X^{-\frac{1}{2}} V X^{-\frac{1}{2}} = P \Lambda P^{-1}$ with Λ diagonal.

which:

$$g(t) = \mathrm{Tr}\left(X^{-1} (I + t P \Lambda P^T)^{-1} \right]$$

$$= \mathrm{Tr}\left(X^{-1} (P P^T + t P \Lambda P^T)^{-1} \right]$$

$$= \mathrm{Tr}\left(X^{-1} \left[P (P^T + t \Lambda P^T \right]^{-1} \right]$$

$$= \mathrm{Tr}\left(X^{-1} P (I + t \Lambda) P^T \right]^{-1} P^T \right]$$

$$= \mathrm{Tr}\left(P^T X^{-1} P (I + t \Lambda)^{-1} \right] \quad \text{as } \mathrm{Tr}(AB) = \mathrm{Tr}(BA)$$

$$= \sum_i \underbrace{[P^T X^{-1} P]_i}_{\geq 0} \frac{1}{1 + \lambda_i t} \quad \text{with } \lambda_i \text{ eigen values of } X.$$

~~$P^T X^{-1} P \geq 0$~~ as $P^T X^{-1} P \geq 0$, and $\frac{1}{1 + \lambda_i t}$ is convex in t

thus, f is convex as a sum of ~~positive~~ convex functions.

2) Let's consider f the function defined by:

$f(x,y) = y^T X^{-1} y$ on $\mathrm{dom} f = S^n_{++} \times \mathbb{R}^n$

$$f(x,y) = \sup_{z \in \mathbb{R}^n} \underbrace{(y^T z - \tfrac{1}{2} z^T X z)}_{g(x,y,z)}$$

$g(x,y) = u(y) + V(x)$ with z fixed

with
- $u(y) = y^T z$ is convex in y for every $z \in \mathbb{R}^n$ as it is affine.
- $V(x) = -\frac{1}{2} z^T X z$ is convex in X for every $z \in \mathbb{R}^n$ as it is affine

therefore $g(x,y)$ is convex in x, y for every $z \in \mathbb{R}^n$ as a sum of convex functions.

thus $\sup_z g(x,y)$ is convex which means f is convex.

3) Let's define the function $f(X) = \sum_{i=1}^{n} \sigma_i(X)$ on $\mathrm{dom} f = S^n$

with $\sigma_1(X), \ldots, \sigma_n(X)$: singular values of a matrix $X \in \mathbb{R}^{n \times n}$.

Let's write the SVD decomposition of $X = Q \Sigma V$ (Q orthogonal, Σ diag, V orthogonal)

$$\text{so } \mathrm{Tr}(V^T Q^T X) = \mathrm{Tr}(V^T Q^T Q \Sigma V)$$
$$= \mathrm{Tr}(V V^T Q^T Q \Sigma)$$
$$= \sum_{i=1}^{n} \sigma_i(X)$$

$\Rightarrow \mathrm{Tr}(V^T Q^T X) = f(X)$ (1)

- for A orthogonal matrix $\mathrm{Tr}(AX) = \mathrm{Tr}(VAQ\Sigma)$
$$= \sum_i a_{ii} \sigma_i(X)$$

with $|a_{ii}| \leq 1$ as it comes from a product of orthogonal matrixes.

So, $$|Tr(AX)| \leq \sum_i \sigma_i(X) \quad (2)$$

From (1) and (2) we can write $f(X) = \sup_{A \text{ orthogonal}} |Tr(AX)|$

$$= \sup_{A \in O_n} |Tr(AX)|$$

$\forall A \in O_n$ $Tr(AX)$ is convex in X as it is affine

and as absolute value is convex in $\mathbb{R}$

(in X $\forall A \in O_n$)

$\Rightarrow$ $|Tr(AX)|$: is convex as composition of a convex fct with an affine one.

therefore $\underline{f \text{ is convex}}$ as supremum of convex function.

Exercice 4 :

1] Let K_{m+} be the cone defined by : $K_{m+} = \{x \in \mathbb{R}^n \mid x_1 \geq \dots x_n \geq 0\}$

First let's check if K_{m+} is a cone :

Let $x, y \in K_{m+}$ and $Z = \lambda_1 x + \lambda_2 y$ with $\lambda_1, \lambda_2 \geq 0$

$\Rightarrow$ $\lambda_1 x_1 + \lambda_2 y_1 \geq \lambda_1 x_2 + \lambda_2 y_2 \geq \dots \geq \lambda_1 x_n + \lambda_2 y_n \geq 0$

$\Rightarrow Z \in K_{m+}$ so K_{m+} is a cone.

- $(1, \frac{1}{2}, \dots \frac{1}{n}) \in$ interior(K_{m+}) $\longrightarrow$ K_{m+} is solid

- for $x \in K_{m+}$ if $-x \in K_{n+}$ $\Rightarrow$ $\begin{cases} 0 \geq x_n \geq \dots \geq x_1 \\ x_1 \geq \dots \geq x_n \geq 0 \end{cases} \Leftrightarrow X = 0$

so $\boxed{K_{m+} \text{ is pointed}}$

- K_{m+} contains its boundary : points when the equality holds

$\Rightarrow$ K_{m+} is closed

So K_{m+} is a proper cone.

2. Dual cone of K_{m+}:

$K_m^{*} = \{ y \mid y^T x \geq 0 \ \forall x \in K_{m+} \} = \{ y \mid \sum_{i=1}^{n} y_i x_i \geq 0 \ \forall x \in K_{m+} \}$

" $\Rightarrow$ "

So for $X_k = \left\{ \begin{matrix} 1 \\ 1 \\ 0 \end{matrix} \right\}$ k elements $X_k \in K_{m+}$

and $y^T X_k = \boxed{\sum_{i=1}^{k} y_i \geq 0}$ $\forall k \in 1, \dots, n$

" $\Leftarrow$ "

if $\sum_{i=1}^{k} y_i \geq 0 \ \forall k \in 1, \dots n$

$$y^T x = \sum_{i=1}^{n} x_i y_i = \sum_{k=1}^{n-1} \underbrace{(x_k - x_{k+1})}_{\geq 0} \underbrace{\sum_{i=1}^{k} y_i}_{\geq 0} + x_n \underbrace{\sum_{i=1}^{n} y_i}_{\geq 0} \geq 0$$

So $y \in K^*_{m+}$

therefore: $K_m^{*} = \{ \sum_{i=1}^{k} y_i \geq 0$ for $k = 1, \dots n \}$.

Exercice 5.8

1. $f(x) = \max_{i=1,\dots n} X_i$ on $\mathbb{R}^n$

$$f^*(y) = \sup_{X \in \mathbb{R}^n} (y^T x - \max_i X_i)$$

$$f^*(y) = \sup_{X \in \mathbb{R}^n} \sum_i y_i x_i - \max_i x_i$$

$\Rightarrow \operatorname{dom}(f^*) = y \mid y^T x - \max_i x_i$ bounded for $X \in \mathbb{R}^n$.

- Let y be such that for a given i $y_i \lneq 0$ and $X_i = -K$ and $\forall j \neq i$ $Y_j = 0$ and $X_j = 0$.

$\Rightarrow f^*(y) = \sup_{x\in\mathbb{R}^n} -y_i k - \underbrace{\max_j x_j}_{=0} = \sup_x -y_i k \xrightarrow[k\to\infty]{} \infty$

$\Rightarrow y \leq 0 \notin \operatorname{dom} f^*(y)$.

- Now, let's consider $y \geq 0$ and $x = k\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$

$$f^*(y) = \sup_{x\in\mathbb{R}^n} k\sum y_i - k = k\left(\sum_i y_i - 1\right)$$

- if $\sum y_i > 1 \Rightarrow f(y) \xrightarrow[k\to+\infty]{} +\infty$
- if $\sum y_i < 1 \Rightarrow f^*(y) \xrightarrow[k\to-\infty]{} +\infty$

$\Rightarrow \boxed{\operatorname{dom} f^*(y) = y \geq 0 \text{ such that } \sum y_i = 1}$

- $f^*(y) = \sup_{x\in\mathbb{R}^n} \sum_i y_i x_i - \max_i x_i$

for $y \geq 0$ $\sum_i y_i x_i - \max_i x_i \leq \underbrace{\sum_i y_i \max x_i - \max_i x_i}_{= \max x_i\left(\sum_i y_i - 1\right)}$

$\rightarrow \sum_i y_i x_i - \max_i x_i \leq 0 \quad \forall x\in\mathbb{R}^n$ if $\sum y_i = 1$

- if $x = k\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} \Rightarrow \sum_i x_i y_i - \max_i x_i = 0$

$\Rightarrow \sup_{x\in\mathbb{R}^n} y^T x - \max_i x_i = 0$

$$\Rightarrow f^*(y) = \begin{cases} 0 & \text{if } y \geq 0 \text{ and } \sum y_i = 1 \\ \infty & \text{otherwise.} \end{cases}$$

2

$f(x) = \sum_{i=1}^{r} x_{[i]}$ on $\mathbb{R}^n$

$\sup_{x \in \mathbb{R}^n} \sum y_i x_i - \sum^r x_{[i]}$

Let consider $y \,/\, y_j < 0$ and $x \,/\, x_j = -\lambda$ $(\lambda > 0)$

and $y_k = 0$ and $x_k = 0$ $\forall j \neq k$

$f^*(y) = \sup_{x \in \mathbb{R}} -y_j \lambda - \sum^r x_{[i]} = -y_j \lambda$

$f^*(y) \xrightarrow[\lambda \to \infty]{} \infty$

$\Rightarrow y < 0 \notin \operatorname{dom} f$

- Let's consider $y \,/\, \exists\, y_i > 1$ and $x \,/\, x_j = \lambda$ and $x_h = 0 \;\forall h \neq j$

$f^*(y) = \sup_{x \in \mathbb{R}^n} y_i x_j - x_j = \sup_{x \in \mathbb{R}^n} \lambda \underbrace{(y_j - 1)}_{> 0} \xrightarrow[\lambda \to \infty]{} \infty$

- Let's consider $y \,/\, \sum y_i < r$ and $x = \lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$\lambda \sum y_i - r\lambda = \lambda \underbrace{(\sum y_i - r)}_{\neq 0} \xrightarrow[\lambda \to \lambda]{} \infty$

$\Rightarrow \sum y_i x_i - \sum^r x_{[i]} \leq \sum y_i \max_i x_i - r \max_i x_i$

$\sum y_i x_i - \sum^r x_{[i]} \leq \max_i x_i (\sum y_i - r)$

if $\sum y_i = r \Rightarrow \sum x_i y_i - \sum^r x_{[i]} \leq 0$

for $x = \lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ $\sum x_i y_i - \sum^r x_{[i]} = \lambda (\underbrace{\sum y_i}_{} - r) = 0$

$\Rightarrow \sup_{x \in \mathbb{R}^n} y^T x - \sum^r x_i = 0 \Rightarrow f^*(y) = \begin{cases} 0 & \text{if } y \geq 0 \;\; \sum y_i = r \;\; y_i \leq 1 \;\forall i \\ \infty & \text{otherwise} \end{cases}$