Elías Masquil

Convex Optimization Homework 1

Ex 1

1) $\{x \in \mathbb{R}^n \mid \alpha_i \le x_i \le \beta_i \ , i = 1, \dots n\} = C$

Claim: $S = \{x \in \mathbb{R}^n / \alpha_i \le x_i \ , i = 1, \dots n\}$ is convex

Proof: $x_1, x_2 \in S \ , 0 \le \theta \le 1$

$$\theta x_1 + (1-\theta)x_2 \ge \theta \alpha_i + (1-\theta)\alpha_i = \alpha_i$$

$$\Rightarrow \theta x_1 + (1-\theta)x_2 \in S$$

Claim: $S' = \{x \in \mathbb{R}^n / x_i \le \beta_i \quad i = 1, \dots n\}$

Proof

$x_1, x_2 \in S' \ , 0 \le \theta \le 1$

$$\theta x_1 + (1-\theta)x_2 \le \beta_i \theta + (1-\theta)\beta_i = \beta_i$$

$$\Rightarrow \theta x_1 + (1-\theta)x_2 \in S'$$

Thus C is convex because is the intersection of 2 convex sets.

2) $H = \{x \in \mathbb{R}^2_+ \mid x_1 x_2 \ge 1\}$

Applying the definition

$$\begin{pmatrix} \theta x_1 + (1-\theta)y_1 \\ \theta x_2 + (1-\theta)y_2 \end{pmatrix} = \theta X + (1-\theta)Y$$

$$[\theta x_1 + (1-\theta)y_1][\theta x_2 + (1-\theta)y_2] = \theta^2 x_1 x_2 + (1-\theta)^2 y_2 y_1 + \underbrace{\theta(1-\theta)(x_1 y_2 + x_2 y_1)}_{\ge 0 \text{ because } \theta \in [0,1],\ x,y \in \mathbb{R}^2_+}$$

because $x, y \in H$

$$\ge 2\theta^2 + 1 - 2\theta + \theta(1-\theta)(x_1 y_2 + x_2 y_1)$$

AM-GM inequality

$$\ge 2\theta^2 + 1 - 2\theta + \theta(1-\theta) \cdot 2\sqrt{x_1 x_2 y_1 y_2}$$

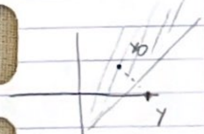
$$\ge 1$$

Then H is convex because $\theta x_1 + (1-\theta)x_2 \in H$

$\forall\ x, y \in H \ . \ \theta \in [0,1]$

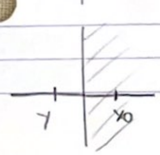
$\square$

3) $C = \{x / \|x - x_0\|_2 \le \|x - y\|_2 \quad \forall\ y \in S\} \ , \ S \subseteq \mathbb{R}^n$



$$C = \bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \le \|x - y\|_2\}$$

If $\{x \mid \|x - x_0\|_2 \le \|x - y\|_2\}$ (the set of points closer to a given point than another point) $\Rightarrow$ C is convex.



$$\|x - x_0\|_2 \leq \|x - y\|_2 \iff (x-x_0)^T(x-x_0) \leq (x-y)^T(x-y)$$

$$\cancel{x^Tx} - 2x_0^T x + x_0^T x_0 \leq \cancel{x^Tx} - 2y^Tx + y^Ty$$

$$(*) \quad 2(y^T - x_0^T)\, x \leq y^Ty - x_0^T x_0$$

For y fixed, $(*)$ is a half-space ($Ax \leq b$)

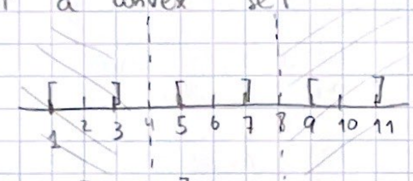
Thus $C = \bigcap_{y \in S} \{x \,|\, \|x - x_0\|_2 \leq \|x-y\|_2\} = \bigcap A_i x \leq b$

is the intersection of half-spaces $\Rightarrow$ C is a convex set

4) $A = \{x \,/\, dist(x, S) \leq dist(x, T)\}$

A is not a convex set

Ex: $\mathbb{R}^1$



$S = [1,3] \cup [9,11]$

$T = [5,7]$

$A = (-\infty, 4] \cup [8, +\infty)$ which is not convex

5) $T = \{x \,|\, x + S_2 \subseteq S_1\}$ $\quad S_1, S_2 \subseteq \mathbb{R}^n$, S_1 convex

$T = \{x \,|\, x + y \in S_1 \;\; \forall y \in S_2\}$

$$T = \bigcap_{y \in S_2} \underbrace{\{x \,|\, x + y \in S_1\}}$$

for each y (fixing y)
this set is convex
In fact is just a traslation.

$$T = \bigcap_{y \in S_2} S_1 - y$$

Since T is an intersection of convex sets $\Rightarrow$ T is convex

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Convex Optimization - Homework 1

Ex 2

1) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$

$$\frac{\partial f}{\partial x_1} = x_2 \qquad \frac{\partial f}{\partial x_2} = x_1$$

$$\frac{\partial^2 f}{\partial x_1 x_2} = 1 \qquad \frac{\partial^2 f}{\partial x_1^2} = 0 \qquad \frac{\partial^2 f}{\partial x_2^2} = 0$$

$$\nabla^2 f(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

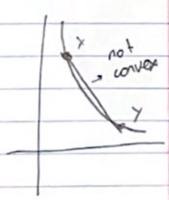
Eigen-values $\lambda^2 - 1 = 0 \iff \begin{matrix}\lambda_1 = 1 \\ \lambda_2 = -1\end{matrix}$

$\Rightarrow \nabla^2 f(x)$ is not positive semi-definite

<u>F is not convex</u>

$\nabla^2 (-f)(x)$ has the same eigen-values $\Rightarrow$ $-f$ is not convex

<u>f is not concave</u>



Consider $S_\alpha = \{ x \in \mathbb{R}^2_{++} / x_1 x_2 \leq \alpha \}$, for $\alpha = 2$

$$S_2 = \{ x \in \mathbb{R}^2_{++} / x_1 x_2 \leq 2 \}$$

Take $x, y \in S_2$ and $z = \theta x + (1-\theta) y$

For ex: $x = (1, 2)^T$, $y = (2, 1)^T$

$$z_1 z_2 = [\theta + 2(1-\theta)][2\theta + (1-\theta)] = (2-\theta)(\theta + 1)$$

$$z_1 z_2 = -\theta^2 + \theta + 2 \qquad \text{choosing } \theta = 1/2$$

$$z_1 z_2 = -\frac{1}{4} + \frac{1}{2} + 2 \geq 2$$

so S_α is not convex $\forall \alpha$ $\Rightarrow$ <u>f is not quasi convex</u>

Consider $S'_\alpha = \{ x \in \mathbb{R}^2_{++} / x_1 x_2 \geq \alpha \}$

Take $x, y \in S'_\alpha$ and $z = \theta x + (1-\theta) y$

From ex 1.2

$z_1 z_2 \geq \theta^2 x_1 x_2 + (1-\theta)^2 y_2 y_1 + \theta(1-\theta)(x_1 y_2 + x_2 y_1)$

AM - GM inequality again

$z_1 z_2 \geq \theta^2 \alpha + (1-\theta)^2 \alpha + 2\theta(1-\theta)\alpha$

$z_1 z_2 \geq \alpha$

$S'_\alpha \{ x \mid f(x) \geq \alpha \}$ is convex $\forall \alpha$

(if $\alpha \leq$ then $S'_\alpha = \mathbb{R}^2_{++}$)

So $\underline{f \text{ is quasiconcave}}$

2) $f(x_1, x_2) = \frac{1}{x_1 x_2}$ on $\mathbb{R}^2_{++}$

$\frac{\partial f}{\partial x_1} = \frac{-1}{x_1^2 x_2}$ $\quad \frac{\partial f}{\partial x_2} = \frac{-1}{x_1 x_2^2}$

$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{1}{x_1^2 x_2^2}$ $\quad \frac{\partial^2 f}{\partial x_1^2} = \frac{2}{x_1^3 x_2}$ $\quad \frac{\partial^2 f}{\partial x_2^2} = \frac{2}{x_2^3 x_1}$

$$\nabla^2_f(x) = \begin{pmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{(x_1x_2)^2} \\ \frac{1}{(x_1x_2)^2} & \frac{2}{x_1 x_2^3} \end{pmatrix} = \frac{1}{x_1 x_2} \begin{pmatrix} \frac{2}{x_1^2} & \frac{1}{x_1 x_2} \\ \frac{1}{x_1 x_2} & \frac{2}{x_2^2} \end{pmatrix}$$

Let A 2x2 symmetric

$A \in S^2_{++} \iff \begin{matrix} tr(A) \geq 0 \\ det(A) \geq 0 \end{matrix}$ $\quad \left(\begin{matrix} \lambda\mu \geq 0 \text{ and } \lambda + \mu \geq 0 \\ \iff \lambda, \mu \geq 0 \end{matrix} \right)$

$Tr(\nabla^2 f(x)) = \frac{1}{x_1 x_2}\left(\frac{2}{x_1^2} + \frac{2}{x_2^2}\right) > 0$

$det(\nabla^2 f) = \left(\frac{4}{x_1^2 \cdot x_2^2} - \frac{1}{x_1^2 x_2^2}\right)\frac{1}{x_1 x_2} = \frac{3}{(x_1 x_2)^3} > 0$

$\Rightarrow \nabla^2 f(x) \succeq 0 \Rightarrow \underline{f \text{ is convex}}$

Then, $\underline{f \text{ is quasi convex}}$

Consider $S_\alpha = \{ x \in \mathbb{R}^2_{++} \mid \frac{1}{x_1 x_2} \geq \alpha \}$

$S_\alpha = \{ x \in \mathbb{R}^2_{++} \mid \frac{1}{\alpha} \geq x_1 x_2 \} \rightarrow$ is not convex (ex 2.1)

$\underline{f \text{ is not quasiconcave}}$

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Convex Optimization - Homework 1

Ex 2

2) Lastly, f is not concave

3) $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$

$$\frac{\partial f}{\partial x_1} = \frac{1}{x_2} \qquad \frac{\partial f}{\partial x_2} = \frac{-x_1}{x_2^2}$$

$$\frac{\partial^2 f}{\partial x_1^2} = 0 \qquad \frac{\partial^2 f}{\partial x_2^2} = \frac{2x_1}{x_2^3}$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{-1}{x_2^2}$$

$$\nabla^2 f(x_1, x_2) = \begin{pmatrix} 0 & -1/x_2^2 \\ -1/x_2^2 & \frac{2x_1}{x_2^3} \end{pmatrix}$$

$$\mathrm{tr}(\nabla^2 f) = \frac{2x_1}{x_2^3} \geq 0 \quad \text{on } \mathbb{R}^2_{++}$$

$$\det(\nabla^2 f) = \frac{-1}{x_2^4}$$

$\nabla^2 f \not\succeq 0 \Rightarrow$ f is not convex

Let's consider $-f$

$$\nabla^2 -f = \begin{pmatrix} 0 & 1/x_2^2 \\ 1/x_2^2 & -\frac{2x_1}{x_2^3} \end{pmatrix}$$

$$\mathrm{tr}(\nabla^2 -f) = -\frac{2x_1}{x_2^3} \leq 0$$

$\Rightarrow -f$ is not convex $\Rightarrow$ f is not concave

$$S_\alpha = \{x \in \mathbb{R}^2_{++} \,/\, \frac{x_1}{x_2} \leq \alpha\}$$

$$z = \theta x + (1-\theta) y \quad ; \quad x, y \in S_\alpha$$

$$\frac{z_1}{z_2} = \frac{\theta x_1 + (1-\theta) y_1}{\theta x_2 + (1-\theta) y_2} \underset{(\text{if } \alpha \geq 0)}{\leq} \frac{\theta \alpha x_2 + (1-\theta) \alpha y_2}{\theta x_2 + (1-\theta) y_2} = \alpha$$

And if $\alpha < 0$ $S_\alpha = \emptyset$

So f is quasiconvex

3) Consider now $S'_\alpha = \{ x \in \mathbb{R}^2_{++} \ / \ \frac{x_1}{x_2} \geq \alpha \}$

$$\frac{z_1}{z_2} = \frac{\theta x_1 + (1-\theta) y_1}{\theta x_2 + (1-\theta) y_2} \underset{(\alpha \geq 0)}{\geq} \frac{\theta x_2 \alpha + (1-\theta) \alpha y_2}{\theta x_2 + (1-\theta) y_2} = \alpha$$

And if $\alpha < 0$ $\quad S'_\alpha = \mathbb{R}^2_{++}$

$\Rightarrow$ <u>f is quasiconcave</u>

4) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ on $\mathbb{R}^2_{++}$, $0 \leq \alpha \leq 1$

If $\alpha = 0$ or $\alpha = 1$, f is concave and convex

$$\frac{\partial f}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{1-\alpha} \qquad \frac{\partial f}{\partial x_2} = (1-\alpha) x_1^\alpha x_2^{-\alpha}$$

$$\frac{\partial^2 f}{\partial x_1^2} = \alpha(\alpha-1) x_1^{\alpha-2} x_2^{-\alpha} \qquad \frac{\partial^2 f}{\partial x_2^2} = -\alpha(1-\alpha)^2 x_1^\alpha x_2^{-\alpha-1}$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha}$$

$$\nabla^2 f = \begin{pmatrix} \alpha(\alpha-1) x_1^{\alpha-2} x_2^{1-\alpha} & \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} \\ \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} & -\alpha(1-\alpha) x_1^\alpha x_2^{-\alpha-1} \end{pmatrix}$$

$$= \alpha(1-\alpha) x_1^\alpha x_2^{1-\alpha} \begin{pmatrix} -1/x_1^2 & 1/x_1x_2 \\ 1/x_1x_2 & -1/x_2^2 \end{pmatrix}$$

$\mathrm{Tr}(\nabla^2 f) = -\frac{1}{x_1^2} - \frac{1}{x_2^2} < 0 \Rightarrow$ <u>f is not convex</u>

Let's review $-f$

$$\mathrm{Tr}(\nabla^2 -f) = \frac{1}{x_1^2} + \frac{1}{x_2^2} > 0$$

$$\det(\nabla^2 -f) = \frac{1}{x_1^2 x_2^2} - \frac{1}{x_1^2 x_2^2} = 0$$

$\Rightarrow -f$ is convex $\Rightarrow$ <u>f is concave</u>

Then <u>f is quasi concave</u>

Let's choose $\alpha = 1/2$ and consider $S_\beta = \{ x \in \mathbb{R}^2_{++} \ / \ f(x) \leq \beta \}$

$S_\beta = \{ x \in \mathbb{R}^2_{++} \ / \ \sqrt{x_1 x_2} \leq \beta \}$, for $\beta > 0$

$S_\beta = \{ x \in \mathbb{R}^2_{++} \ / \ x_1 x_2 \leq \beta^2 \}$ which is not convex (ex 2.1) $\Rightarrow$ <u>f not quasiconvex</u>

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Convex Optimization - Homework 1

Ex 3

1) $f(X) = \mathrm{Tr}(X^{-1})$ on S^n_{++}

dom f is convex ✓

Let's consider f across a line

$$g(t) = \mathrm{Tr}\left((X + tV)^{-1} \right)$$

$$= \mathrm{Tr}\left[\left(X^{1/2} \left(I + t X^{-1/2} V X^{-1/2} \right) X^{1/2} \right)^{-1} \right]$$

$$= \mathrm{Tr}\left[X^{-1/2} \left((I + t X^{-1/2} V X^{-1/2}) X^{1/2} \right)^{-1} \right]$$

$$= \mathrm{Tr}\left[X^{-1} \left(I + t \underbrace{X^{-1/2} V X^{-1/2}}_{\in S_n^{++}} \right)^{-1} \right]$$

$\Rightarrow \exists\, Q$ orthogonal $/ \ X^{-1/2} V X^{-1/2} = Q \Lambda Q^T$

Λ diagonal

$$= \mathrm{Tr}\left(X^{-1} (I + t Q \Lambda Q^T)^{-1} \right)$$

$$= \mathrm{Tr}\left(X^{-1} (Q Q^T + t Q \Lambda Q^T)^{-1} \right)$$

$$= \mathrm{Tr}\left(X^{-1} [Q (Q^T + t \Lambda Q^T)]^{-1} \right)$$

$$= \mathrm{Tr}\left(X^{-1} [(I + t \Lambda) Q^T]^{-1} Q^T \right)$$

$$= \mathrm{Tr}\left(X^{-1} Q (I + t \Lambda)^{-1} Q^T \right)$$

$$\overset{\mathrm{Tr}(AB) = \mathrm{Tr}(BA)}{=} \mathrm{Tr}\left(Q^T X^{-1} Q (I + t \Lambda)^{-1} \right)$$

$$= \sum_i \underbrace{[Q^T X^{-1} Q]_{ii}}_{\substack{\geq 0 \\ \text{because} \\ Q^T X^{-1} Q \succeq 0}} \ \underbrace{\frac{1}{1 + \lambda_i t}}_{\substack{\text{convex} \\ \text{in} \\ t}}$$

λ_i eigen values of X

It's a positive sum of convex functions

$f(X)$ is convex

2) $f(X,y) = y^T X^{-1} y$ on $S^n_{++} \times \mathbb{R}^n$

$$f(X,y) = \sup_z \underbrace{\left(y^T z - 1/2\ z^T X z \right)}_{g(X,y,z)}$$

$g(X,y) = U(y) + V(X)$ with z fixed.

$U(y) = y^T z$ is convex in y $\forall z \in \mathbb{R}^n$ (affine)

$V(X) = -\frac{1}{2} z^T X z$ is convex in X $\forall z \in \mathbb{R}^n$ (affine)

$\Rightarrow$ $g(x,y)$ positive sum of convex functions is convex in X, y $\forall z \in \mathbb{R}^n$

$\Rightarrow$ $\sup_z \left(y^T z - \frac{1}{2} z^T X z \right) = \frac{y^T X^{-1} y}{2}$ is convex

$\Rightarrow$ $y^T X^{-1} y$ is convex

3) $f(x) = \sum_i^n \sigma_i(x)$ σ_i singular values of X

Let's conside U orthogonal

$|Tr(UX)| = |Tr(UQ\Sigma V)|$ where $Q\Sigma V$ is the SVD decomposition of X

$|Tr(UX)| = |Tr(VUQ\Sigma)| = \sum_i |c_{ii}|\ \sigma_i(x)$

$|Tr(UX)| \leq \sum_i \sigma_i(x)$ because $|c_{ii}| \leq 1$ since they are the entries of a product of orthogonal matrices

And when $U = V^T Q^T$, the equality holds

$\Rightarrow$ $f(x) = \sum_i \sigma_i(x) = \sup_{U \text{ orthogonal}} |Tr(UX)|$

$f(x) = \sup_{U \text{ orthogonal}} g(U,X)$

- $Tr(U,X)$ is convex in X $\forall U$, because is affine
- Absolute value is convex in $\mathbb{R}$

$\Rightarrow$ Composition with an affine matrix is convex

Then $g(U,X)$ is convex in X $\forall U$

$\Rightarrow$ $f(x) = \sum_i \sigma_i(x)$ is convex

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Convex Optimization - Homework 1

Ex 4

$$K_{m+} = \{x \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \dots \geq x_n \geq 0\}$$

1) Prove that K_{m+} is a proper cone

K_{m+} is a cone:

$x, y \in K_{m+}$

$z = \theta_1 x + \theta_2 y \qquad \theta_1, \theta_2 \geq 0$

$z \in K_{m+}$ because: $\theta_1 x_1 + \theta_2 y_1 \geq \theta_1 x_2 + \theta_2 y_2 \geq \dots \geq \theta_1 x_n + \theta_2 y_n$

≥ 0

K_{m+} is solid:

K_{m+} interior is not empty:

$x = (1, 1/2, \dots, 1/n)$ is on the interior of K_{m+}

K_{m+} is pointed

If $x \in K_{m+} \Rightarrow x_1 \geq x_2 \geq \dots \geq x_n \geq 0$

Then $-x \mid 0 \geq x_n \geq \dots \geq x_1 \Rightarrow -x \in K_{m+}$

$\Leftrightarrow x = 0$

K_{m+} is closed

Since the cone is defined by n inequalities (which are not strict), it's clear that it contains its boundary (points where the equalities hold).

2) $K_{m+}^* = \{y \mid y^T x \geq 0 \quad \forall x \in K_{m+}\}$

$$\sum_i^n x_i y_i = (x_1 - x_2) y_1 + (x_2 - x_3)(y_1 + y_2) + (x_3 - x_4)(y_1 + y_2 + y_3) + \dots + (x_{n-1} - x_n)(y_1 + \dots + y_{n-1}) + x_n (y_1 + \dots + y_n)$$

$$y^T x = \sum_i x_i y_i \geq 0 \iff \underbrace{\sum_{i=1}^{k} y_i \geq 0 \quad \forall k\ 1, \dots, n}_{K_{m+}^*}$$

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Convex Optimization - Homework 1

Ex 5

1) $f(x) = \max_{i=1,\dots,n} x_i$ on $\mathbb{R}^n$

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \left(y^T x - \max_i x_i \right)$$

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \sum_i y_i x_i - \max_i x_i$$

The domain of f^* consists of y / $y^T x - \max_i x_i$ is bounded for $x \in \mathbb{R}^n$. Let's find it

Consider y / $y_i < 0$ for a fixed i and x / $x_i = -K$ and 0 for the rest of coordinates

$$f^*(y) = \sup_{x \in \mathbb{R}^n} -y_i K - \underbrace{\max_j x_j}_{0} = \sup_x -y_i K \xrightarrow[K \to \infty]{} \infty$$

$\Rightarrow y \preceq 0 \notin \operatorname{dom} f^*(y)$

Now, consider $y \succeq 0$ and $x = K\vec{1}$

$$f^*(y) = \sup_{x \in \mathbb{R}^n} K \sum y_i - K = K\left(\sum_i y_i - 1\right)$$

If $\sum y_i > 1 \Rightarrow f(y) \longrightarrow \infty$ if we grow K to the infinity

If $\sum y_i < 1 \Rightarrow f^*(y) \longrightarrow \infty$ if we grow K to the $-$infinity

Then $\operatorname{dom} f^*(y) = y \succeq 0 \,/\, \sum y_i = 1$

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \sum_i y_i x_i - \max_i x_i$$

For $y \succeq 0$

$$\sum_i y_i x_i - \max_i x_i \leq \sum_i y_i \max_i x_i - \max_i x_i = \max_i x_i \left(\sum_i y_i - 1\right)$$

$$\sum_i y_i x_i - \max_i x_i \leq 0 \quad \forall \; x \in \mathbb{R}^n \quad \text{if } \sum_i y_i = 1$$

Now, consider again $x = K\vec{1}$

$$\sum_i x_i y_i - \max_i x_i = 0$$

$$\Rightarrow \sup_{x\in\mathbb{R}^n} y^T x - \max_i x_i = 0$$

$$f^*(y) = \begin{cases} 0 & y \geq 0 \quad \sum y_i = 1 \\ \infty & \text{otherwise} \end{cases}$$

2) $f(x) = \sum_{i=1}^{r} x_{[i]}$ on $\mathbb{R}^n$ (largest elements)

$$\sup_{x\in\mathbb{R}^n} \sum y_i x_i - \sum^r x_{[i]}$$

Consider $y \,/\, y_j < 0$ and $x \,/\, x_j = -\alpha$ $(\alpha > 0)$
and 0 for the other coordinates

$$f^*(y) = \sup_{x\in\mathbb{R}^n} -y_j\alpha - \underbrace{\sum^r x_{[i]}}_{0} = -y_j\alpha$$

$$\xrightarrow[\text{when } \alpha\to\infty]{} \infty$$

So $y < 0 \notin \operatorname{dom} f$

Consider $y \,/\, \exists_j \; y_j > 1$ and $x \,/\, x_j = \alpha$ and 0 otherwise

$$f^*(y) = \sup_{x\in\mathbb{R}^n} y_j x_j - x_j = \sup_{x\in\mathbb{R}^n} \alpha\underbrace{(y_j - 1)}_{>0} \xrightarrow[\alpha\to\infty]{} \infty$$

(*) The analogous case is $\sum y_i > r$

Consider $y \,/\, \sum y_i < r$ (*) and $x = \vec{1}\alpha$

$$\alpha\sum y_i - r\alpha = \alpha\underbrace{\left(\sum y_i - r\right)}_{\neq 0} \xrightarrow[\alpha\to\infty]{} \infty$$

Finally: $\sum y_i x_i - \sum^r x_{[i]} \leq \sum y_i \max_i x_i - r\max_i x_i$

$$\sum y_i x_i - \sum^r x_{[i]} \leq \max_i x_i \left(\sum y_i - r\right)$$

If $\sum y_i = r \Rightarrow \sum x_i y_i - \sum^r x_{[i]} \leq 0$

Let's consider $x = \vec{1}\alpha$

$$\sum x_i y_i - \sum^r x_{[i]} = \alpha\big(\underbrace{\sum y_i}_{r} - r\big) = 0$$

$$\Rightarrow \sup_{x\in\mathbb{R}^n} y^T x - \sum^r x_i = 0$$

$$f^*(y) = \begin{cases} 0 & \text{if } y \geq 0, \sum y_i = r \text{ and } y_i < 1 \;\forall i \\ \infty & \text{otherwise} \end{cases}$$