Tools for Analysis of Randomized Algorithms

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→ **Markov's Inequality**

Let X be a random variable that assumes only non-negative values. Then for all $a>0$

$$Pr(X \geq a) \leq \frac{E[X]}{a}$$

Ex: n coin flips (fair)

What is the probability of obtaining more than $\frac{3n}{4}$ heads?

→ Def:- k^{th} moment of a random variable is defined as $E[X^k]$

→ Def:- $Var[X] =$

$$E\left[(X - E(X))^2\right]$$

$$= E[X^2] - [E(X)]^2$$

→ For 2 random variables X & Y

$$Cov(X,Y) = E\left[(X-E(X))(Y-E(Y))\right]$$

&

$$Var[X+Y] = Var[X] + Var[Y] + 2Cov(X,Y)$$

→ **Chebysev's Inequality**:-

For any $a>0$

$$Pr(|X - E[X]| \geq a) \leq \frac{Var[X]}{a^2}$$

Another form:-

For any $t>1$

$$Pr(|X - E[X]| \geq t \cdot \sigma[X]) \leq \frac{1}{t^2}$$

Ex:- What happens now to the $\frac{3n}{4}$ heads example?

Chernoff Bounds:-

Let $X_1, X_2, \ldots, X_n$ be independent 0–1 random variables such that $Pr(X_i = 1) = p_i$

Let $X = \sum_{i=1}^{n} X_i$ & $\mu = E[X]$

Following Chernoff bounds hold:

1. For any $\delta > 0$

$$Pr(X \geq (1+\delta)\mu) \leq \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu$$

2. For $0 < \delta \leq 1$

$$Pr(X \geq (1+\delta)\mu) \leq e^{-\mu\delta^2/3}$$

THE OTHER SIDE :- For $0<\delta<1$

1. $$Pr(X \leq (1-\delta)\mu) \leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^\mu$$

2. $$Pr(X \leq (1-\delta)\mu) \leq e^{-\mu\delta^2/2}$$

The TWO SIDED RESULT

For $0 < \delta < 1$

$$Pr(|X - \mu| \geq \delta\mu) \leq 2e^{-\mu\delta^2/3}$$

For the coin example

No more than $n/4$ heads or no fewer than $3n/4$ heads in seq. of n independent coin flips

$$Pr\left(\left|X - \frac{n}{2}\right| \geq \frac{n}{4}\right) \leq 2\exp\left\{-\frac{1}{3}\frac{n}{2}\frac{1}{4}\right\}$$

$$\leq 2e^{-n/24}$$