SVD

Sunday, 12 February 2023 8:01 PM

Eigen value Decomposition

A (non zero) vector v of n dimension is an eigen vector of the nxn matrix A if it satisfies

$Av = \lambda v$

λ ← eigen value

If A is a square matrix (nxn) with n linearly independent eigenvectors A can be factorized as

$A = Q \Lambda Q^{-1}$

$$Q = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

Every nxn real symmetric matrix

⇒ eigen values are real & Eigen vectors can be chosen orthonormal

$\therefore A = Q \Lambda Q^T$

Algebraic Multiplicity :- No. of repititions of a particular eigen value is its algebraic multiplicity

Geometric Multiplicity :- No. of LI eigenvectors associated with it i.e. dimension of null space of $A - eI$

$AM(e) \geq GM(e)$

Issues with EVD :-

- Only Applicable to Square Matrices
- May be complex eigenvalues.

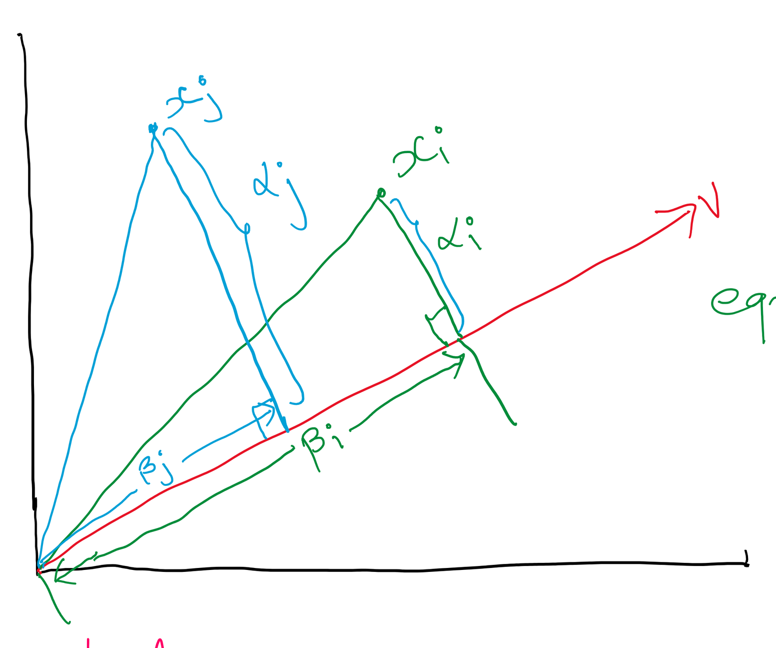
SVD :-

Any matrix A (nxd) can be decomposed as

$A = U \Sigma V^T$

$U_{n\times n}$	$U_{n\times d}$
$\Sigma_{n\times d}$	$\Sigma_{d\times d}$
$V^T_{d\times d}$	$V^T_{d\times d}$

- Columns of U & V are orthonormal
- Σ is diagonal matrix with nonnegative real entries



Min $\Sigma \alpha^2$ equivalent to Max $\Sigma \beta^2$

Singular Vectors:-

Consider rows of A as n points in d dimensions. Consider the best fit line through the origin. Let $\underline{v}$ be unit vector along the line.

The length of projection of a_i (ith row of A) onto v is $|a_i \cdot v|$

$\therefore$ The sum of length squared of projections is $\|Av\|^2$

Maximizing $\|Av\|_2^2$ is best fit line

First singular vector of A

$$v_1 = \underset{\|v\|_2 = 1}{\arg\max} \|Av\|_2$$

$$\sigma_1(A) = \|Av_1\|_2$$

Similarly $v_2 = \underset{v \perp v_1, \|v\|_2 = 1}{\arg\max} \|Av\|_2$

similarly Find $v_3, v_4, \ldots v_d$

Important Result :- Let A be an nxd matrix where $v_1, \ldots v_r$ are singular vectors. For $1 \leq k \leq r$, let V_k be subspace spanned by $v_1, \ldots v_k$, then for each k, V_k is best fit k-dim subspace for A

Frobenius Norm :-

$$\|A\|_F = \sqrt{\sum_{j,k} a_{jk}^2}$$

Imp Result :-

$$\Sigma \sigma_i^2(A) = \|A\|_F^2$$

Define Vectors $u_i = \frac{1}{\sigma_i(A)} Av_i$

$u_1, u_2, \ldots u_d$ are called left singular vectors of A.

$$\boxed{A = \sum_{i=1}^{d} \sigma_i u_i v_i^T}$$

Suppose we consider the Square symmetric matrix

A^TA

Let x be eigenvector of A^TA and λ be its eigenvalue

$A^TAx = \lambda x$

Multiplying by A on both sides

$AA^T(Ax) = \lambda(Ax)$

$A^TA = VDV^T$

$AA^T = UDU^T$

$$A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma^T U^T U \Sigma V^T$$

$$= V\Sigma^T\Sigma V^T$$

$$= V\Sigma^2 V^T$$

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T V \Sigma^T U^T$$

$$= U\Sigma^T\Sigma U^T$$

$$= U\Sigma^2 U^T$$

U – formed by eigenvectors of AA^T

V - formed by eigenvectors of A^TA

Singular Values :- square roots of eigenvalues of A^TA

Geometric Interpretation

$A = U\Sigma V^T$

$\therefore Ax = U\Sigma V^T x$