Randomized Rounding for set cover

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Let x^* be the optimal sol^n to the LP-relaxation of set-cover problem

We want to round it to some $\hat{x}$ (randomly)

High Level Idea:- Interpret the fractional value x_j^* as Probability that $\hat{x}_j$ should be set to 1.

Each subset S_j is included in our solution w.p. x_j^* and m events are independent.

$$X_j = \begin{cases} 1 & \text{if } S_j \text{ in sol}^n \\ 0 & \text{o/w} \end{cases}$$

$$E\left[\sum_{j=1}^{m} w_j X_j\right] = \sum_{j=1}^{m} w_j \Pr[X_j = 1]$$

$$= \sum_{i=1}^{m} w_j x_j^*$$

$$= Z^*_{LP}$$

But May NOT give a SET COVER!

$\Pr[e_i \text{ is not covered}]$

$$= \prod_{j: e_i \in S_j} (1 - x_j^*)$$

$$\leq \prod_{j: e_i \in S_j} e^{-x_j^*} \quad (\because 1 - x \leq e^{-x})$$

$$= e^{-\sum_{j: e_i \in S_j} x_j^*}$$

$$\leq e^{-1} \quad \left(\sum_{j: e_i \in S_j} x_j^* \geq 1\right)$$

How the make failure Probability smaller?

— Toss a coin for each j c ln n times instead of just once

If it comes heads even once, include S_j in sol^n

Now $\Pr[S_j \text{ not included}]$

$$= (1 - x_j^*)^{c \ln n} \quad (c \geq 2)$$

$\Pr[e_i \text{ is not covered}]$

$$= \prod_{j: e_i \in S_j} (1 - x_j^*)^{c \ln n}$$

$$\leq \prod_{j: e_i \in S_j} e^{-x_j^* (c \ln n)}$$

$$= e^{-(c \ln n) \sum_{j: e_i \in S_j} x_j^*}$$

$$\leq \frac{1}{n^c}$$

Thm :- Randomized Rounding Algo is a randomized $O(\ln n)$ approx that produces set cover w.h.p

Proof :- $p_j(x_j^*)$ = prob S_j is included in sol^n

$$p_j(x_j^*) = 1 - (1 - x_j^*)^{c \ln n}$$

If $x_j^* \in [0,1]$ and $c \ln n \geq 1$

$$p_j'(x_j^*) = c \ln n \, (1 - x_j^*)^{(c \ln n) - 1}$$

$$\leq c \ln n$$

$p_j(0) = 0$

Slope of p_j is bounded above by $c \ln n$ on $[0,1]$

$$p_j(x_j^*) \leq (c \ln n) x_j^* \text{ on } [0,1]$$

$$E\left[\sum_{j=1}^{m} w_j X_j\right]$$

$$= \sum_{j=1}^{m} w_j \Pr[X_j = 1]$$

$$\leq \sum_{j=1}^{m} w_j (c \ln n) x_j^*$$

$$= (c \ln n) Z^*_{LP}$$

Not Enough !!

Have to check for set cover

Let F be the event that solution obtained is feasible set cover

$\bar{F}$ is the complement event

$$\boxed{\Pr[F] \geq 1 - \frac{1}{n^{c-1}}}$$

Now

$$E\left[\sum_{j=1}^{m} w_j X_j\right]$$

$$= E\left[\sum_{j=1}^{m} w_j X_j \,\middle|\, F\right] \Pr[F] + E\left[\sum_{j=1}^{m} w_j X_j \,\middle|\, \bar{F}\right] \Pr[\bar{F}]$$

Also $w_j \geq 0 \quad \forall j$

$$\therefore \quad E\left[\sum_{j=1}^{m} w_j X_j \,\middle|\, \bar{F}\right] \geq 0$$

$$\therefore \quad E\left[\sum_{j=1}^{m} w_j X_j \,\middle|\, F\right]$$

$$= \frac{1}{\Pr[F]} \left(E\left[\sum_{j=1}^{m} w_j X_j\right] - E\left[\sum_{j=1}^{m} w_j X_j \,\middle|\, \bar{F}\right] \Pr[\bar{F}] \right)$$

$$\leq \frac{1}{\Pr[F]} \cdot E\left[\sum_{j=1}^{m} w_j X_j\right]$$

$$\leq \frac{(c \ln n) Z^*_{LP}}{1 - \frac{1}{n^{c-1}}}$$

$$\leq 2c (\ln n) Z^*_{LP} \quad \left(\text{for } n \geq 2, \; c \geq 2\right)$$