Monte Carlo Methods

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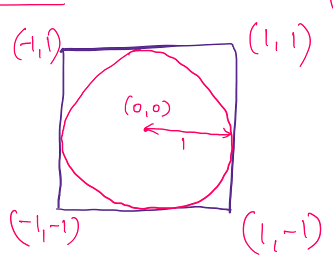
Example 1 :- Value of π



Area of circle $= \pi$

Throw a dart

Let $Z = \begin{cases} 1 & \text{if } \sqrt{x^2+y^2} \le 1 \\ 0 & \text{o/w} \end{cases}$

$Pr(Z=1) = \pi/4$. why??

Run experiment m times

Z_i = value of Z at ith run.

$$W = \sum_{i=1}^{m} Z_i$$

$$E[W] = \sum_{i=1}^{m} E[Z_i] = m\pi/4$$

$$\Rightarrow \pi = W'\left(\frac{4}{m}\right)$$

$$Pr\left(|W'-\pi| \ge \epsilon\pi\right) = Pr\left(\left|W - \frac{m\pi}{4}\right| \ge \frac{\epsilon m \pi}{4}\right)$$

$$= Pr\left(|W - E[W]| \ge \epsilon E[W]\right)$$

$$\le 2e^{-m\pi\epsilon^2/12} \quad \text{(How??)}$$

What is m? for good estimate.

Def :- A randomized algo. gives (ϵ, δ) approximation for value V if output X of algo satisfies

$$Pr(|X-V| \le \epsilon V) \ge 1-\delta$$

Def :- An FPRAS (Fully Polynomial randomized approximation scheme) for a problem is a randomized alg. for which given an input x and parameters ϵ, δ with $0<\epsilon,\delta<1$ alg. outputs (ϵ,δ)-appr. to $V(x)$ in time that is polynomial in $\frac{1}{\epsilon}$, $\ln\frac{1}{\delta}$ and size of input x.

DNF counting Problem :-

DNF : Disjunctive Normal Form

$$C_1 \vee C_2 \vee C_3 \dots \vee C_t$$

$$\downarrow$$

$$(x_1 \wedge \bar{x}_2 \wedge x_3)$$

Need to satisfy only 1 clause

Easier to find satisfying assignment than CNF

Hard to count #. of satisfying assignments for a DNF !!

DNF Counting Alg 1 :-

I/p : DNF Formula with n variables

O/p : Y = an approximation of $C(F)$ (i.e. count of satisfying assign.)

$X \leftarrow 0$

For $k = 1$ to m do :-

i) Generate a random assignment u.a.r from all 2^n assignments

ii) If random assignment satisfies F, then $X \leftarrow X+1$

Return $Y \leftarrow \left(\frac{X}{m}\right)2^n$

$C(F) > 0$, why not 0??

$X_k = 1$ if k^{th} iter assignment is satisfying
$= 0$ else

$$X = \sum_{i=1}^{m} X_k$$

What is probability of $X_k = 1$

$E[Y] = C(F)$. How??

m ??

$$m \ge \frac{3 \cdot 2^n \ln(2/\delta)}{\epsilon^2 C(F)}$$

If $C(F) \ge 2^n/\alpha(n)$ good

but if $C(F) \ll 2^n$, problem

FPRAS for the problem

Let $F = C_1 \vee C_2 \vee \dots \vee C_t$

– No clause includes a variable and its negation

– Need to satisfy atleast 1 clause in F.

→ **For each** clause i, exactly 2^{n-l_i} satisfying assignment where l_i is no. of literals in clause C_i

SC_i = set of assignments satisfying clause i

$$U = \{(i,a) \mid 1 \le i \le t \text{ and } a \in SC_i\}$$

$$|U| = \sum_{i=1}^{t} |SC_i|$$

$|SC_i|$ is computable

$$C(F) = \left|\bigcup_{i=1}^{t} SC_i\right|$$

$C(F) \le |U|$ why??

Use U to get $C(F)$

Define

$$S = \{(i,a) \mid 1 \le i \le t,\ a \in SC_i,\ a \notin SC_j, \text{ for } j<i\}$$

By uniform sampling can get $|S|/|U|$

$|S|/|U| \ge 1/t$ why??

Choose with probability $\frac{|SC_i|}{|U|}$

$$Pr((i,a) \text{ is chosen}) = Pr(i \text{ is chosen}) \cdot Pr(a \text{ is chosen} \mid i \text{ is chosen})$$

$$= \frac{|SC_i|}{|U|} \cdot \frac{1}{|SC_i|}$$

$$= \frac{1}{|U|}$$

Algorithm 2 :-

i/p : A DNF formula F with n variables

o/p : Y = an approximation of $C(F)$

$X \leftarrow 0$

For $k = 1$ to m do

a) w.p. $|SC_i| / \sum_{i=1}^{t} |SC_i|$ choose u.a.r an assignment $a \in SC_i$

b) If a is not in any SC_j, $j<i$, then $X \leftarrow X+1$

Return $Y \leftarrow \left(\frac{X}{m}\right) \sum_{i=1}^{t} |SC_i|$

Claim : Algo 2 is FPRAS for DNF counting problem with $m = \left\lceil (3t/\epsilon^2) \ln(2/\delta) \right\rceil$

Proof :- Step 2a) of alg. chooses an element of U uniformly at random.

Probability that the element belongs to S is at least $1/t$. Fix any $\epsilon > 0$, $\delta > 0$

let $m = \left\lceil \frac{3t}{\epsilon^2} \ln\frac{2}{\delta} \right\rceil$

Why this works ??