A primer or primal dual for LPs

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Notion of Duality

Consider the example

min $6x_1 + 4x_2 + 2x_3$

s.t. $4x_1 + 2x_2 + x_3 \geq 5$

$x_1 + x_2 \geq 3$

$x_2 + x_3 \geq 4$

$x_i \geq 0$ for $i = 1, 2, 3$

$6x_1 + 4x_2 + 2x_3 \geq 4x_1 + 2x_2 + x_3$

Also $6x_1 + 4x_2 + 2x_3 \geq (4x_1 + 2x_2 + x_3) + (x_1 + x_2) + (x_2 + x_3)$

Idea: Use constraints to obtain lower bounds

$$(6x_1 + 4x_2 + 2x_3) \geq y_1(4x_1 + 2x_2 + x_3) + y_2(x_1 + x_2) + y_3(x_2 + x_3)$$

y_i's ≥ 0

Dual

Maximize $5y_1 + 3y_2 + 4y_3$

subject to $4y_1 + y_2 \leq 6$

$2y_1 + y_2 + y_3 \leq 4$

$y_1 + y_3 \leq 2$

$y_i \geq 0$

For P $\xRightarrow{\text{Get}}$ D

minimize $\sum_{j=1}^{n} c_j x_j$ | maximize $\sum_{i=1}^{m} b_i y_i$

subject to $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$, $i = 1, \dots, m$ | subject to $\sum_{i=1}^{m} a_{ij} y_i \leq c_j$ for $j = 1, \dots, n$

$x_j \geq 0$, $j = 1, \dots, n$ | $y_i \geq 0$ for $i = 1, \dots, m$

Weak Duality :-

If x is feasible to LP(P) and y a feasible soln to LP(D)

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$$

Proof :-

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i$$

$$\geq \sum_{i=1}^{m} b_i y_i$$

Strong Duality :-

If LPs (P) and (D) are feasible then for any optimal solutions x^* to (P) and y^* to (D) $\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$

Complementary Slackness :-

Let $\bar{x}$ and $\bar{y}$ be feasible soln to P & D respectively

$\bar{x}$ & $\bar{y}$ obey complementary slackness if $\sum_{i=1}^{m} a_{ij} \bar{y}_i = c_j$ for each j such that $\bar{x}_j > 0$ and if $\sum_{j=1}^{n} a_{ij} \bar{x}_j = b_i$ for each i such that $\bar{y}_i > 0$

$\bar{x}$ & $\bar{y}$ obey complementary slackness $\Longleftrightarrow$ $\bar{x}$ & $\bar{y}$ are optimal

4 Possibilities

	Primal	Dual
i)	Feasible	Feasible
ii)	Infeasible	Unbounded
iii)	Unbounded	Infeasible
iv)	Infeasible	Infeasible