SC531 FINAL EXAMINATION

INSTRUCTIONS

Form a three digit number from the digits A, C and D of your ID number, as shown below:

Say your student ID number is 2021A1BCD.

Then the three digit number ACD is obtained by concatenating the digits A, C and D. Example: From ID number 20212134<u>5</u>, we get the three digit number ACD = 245.

From the three digit number ACD, calculate the two numbers M and N as shown:

M = mod(ACD, 3) + 2 = remainder(ACD/3) + 2
N = mod(ACD, 4) + 5 = remainder(ACD/4) + 5

SUBSTITUTE THE VALUES OF M AND N IN CALCULATING THE NUMERICAL ANSWERS.

ANSWERS GIVEN IN TERMS OF M AND N WILL GET ZERO MARKS.

KEEP YOUR CAMERA ON UNTIL YOU UPLOAD THE ANSWERS IN GOOGLE FORM.

SC531 PROBABILITY & RANDOM VARIABLES FINAL EXAMINATION

Each question carries 3 marks. Time allowed: 80 minutes.

Use the values of M & N you have calculated earlier.

Q-1. You have 5 bowls containing, respectively, 10, 10, 20, 20 and 20 balls. Some balls in each bowl are white; the rest are black. The number of white balls in the bowls are, respectively, M, 10-M, 11, 4 and 15. A bowl is selected at random, and a single ball is drawn from that bowl at random. It is found that the ball drawn is **black**. Find the probability that the drawn **black** ball came from bowl #2.

From the given data, we get the total number of BLACK balls in the five bowls = 40.

P(bowl-2 | black-drawn)

= P(bowl-2 & black-drawn) / P(black-drawn)

= P(black-drawn | bowl-2) * P(bowl-2) / P(black-drawn)

= [M/10 * 1/5] / [40/80] = **M/25**

Q-2. A fair coin is tossed 10 times. Find the probability that it turns up HEAD either N-1 or N times.

Answer = [$_{10}C_{N-1} + {}_{10}C_N$]/1024

Q-1 M=2

White balls: 2, 8, 11, 4, 15

Black balls: 8, 2, 9, 16, 5

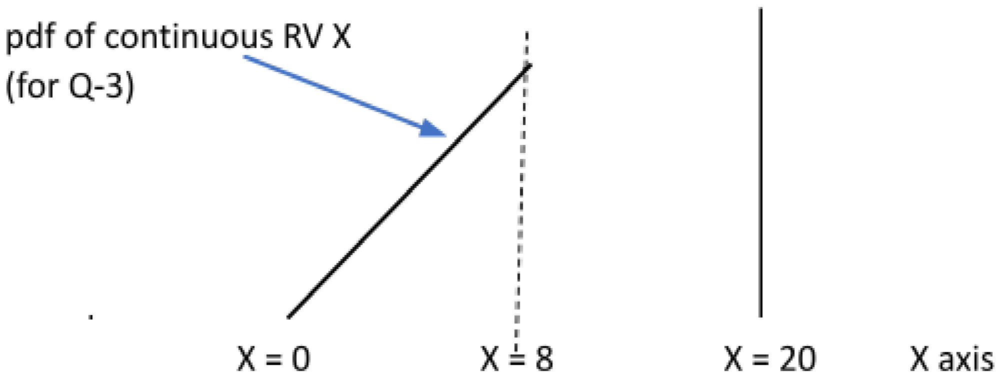
$P(B_i) = \frac{1}{5} \rightarrow P(B_2) = \frac{1}{5}$

$P(B/B_2) = \frac{2}{10}$

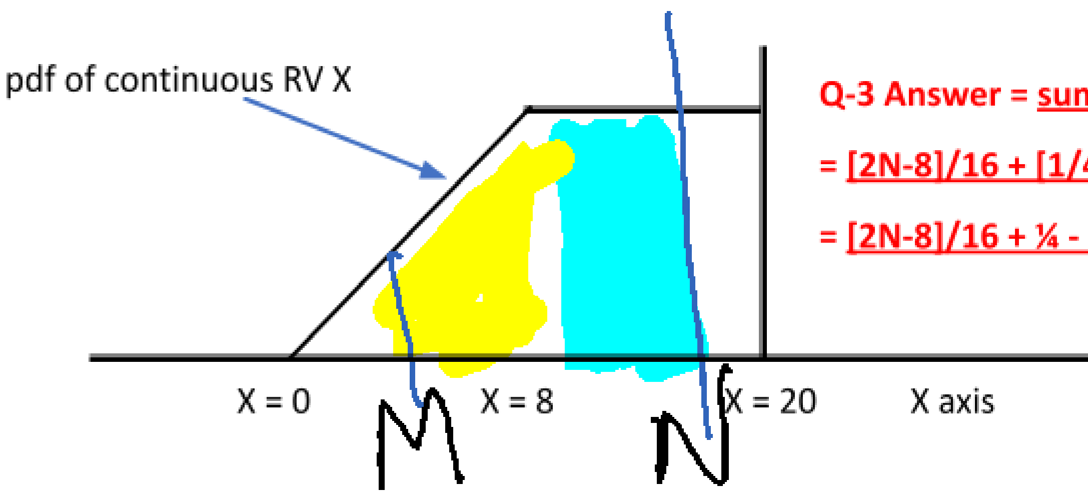
$P(B_2/B) = \frac{P(B/B_2) \cdot P(B_2)}{P(B)}$

$= \frac{\frac{2}{10} * \frac{1}{5}}{\frac{(8+2+9+16+5)}{80}} = \frac{\frac{2}{10} \cdot \frac{1}{5}}{\frac{4\cancel{0}}{8\cancel{0}}} = \frac{2}{25}$

Q-2 prob that it turns up head N-1 or N times $= \left(\frac{1}{2}\right)^{10} \left({}^{10}C_{N-1} + {}^{10}C_N \right)$



Q-3. Given the probability density function shown in the diagram above, find Prob[M <= X <= 2*N].

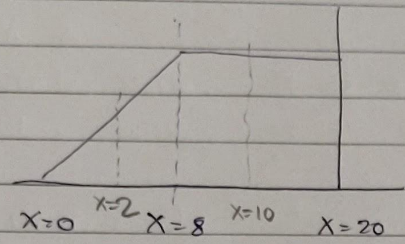


Q-3 Answer = <u>sum of two shades areas</u>

= <u>[2N-8]/16 + [1/4 – (1/2)*M*(M/128)]</u>

= <u>[2N-8]/16 + ¼ - M^2/256</u>

Q-3



$P(M <= X <= 2^* N) = ?$ $M=2, N=5$

$P(2 \le X \le 10) = ?$

Total area = 1

triangle area + rectangle area = 1

$\frac{1}{2} \times h \times l + h \times l = 1$

$h \left[\frac{8}{2} + (20-8) \right] = 1$

$\therefore h = \frac{1}{16}$ m for triangle $= \frac{1/16 - 0}{8-0} = \frac{1}{128}$

Rectangle area = $h \times l$

$= \frac{1}{16} \times 2 = \frac{1}{8} = 0.125$

Triangle area = $\int_2^8 mx \cdot dx$

$= \frac{1}{128} \left[\frac{x^2}{2} \right]_2^8 = \frac{64-4}{256} = 0.2343$

Total area = 0.125 + 0.2343

= 0.359375

Q-4. At a router, the number of packets arriving per millisecond, denoted by P, follows Poisson distribution, with mean rate of 2 packets per millisecond. Find Prob[M ≤ P ≤ M+3]. Refer to the table below.

Four entries in the table must be added, for count = M, M+1, M+2, M+3

Poisson distribution, rate = 2

count, x	**p(x)**
0	0.1353
1	0.2707
2	0.2707
3	0.1804
4	0.0902
5	0.0361
6	0.0120
7	0.0034
8	0.0009
9	0.0002
10	0.0000

Q-4

$\mu = 2,\ M = 2$

$$P(M \le P \le M+3) = P(2 \le P \le 5) =$$

$$= \frac{e^{-2} 2^{x}}{x!}$$

$$= e^{-2}\left[\frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!}\right]$$

$$= 0.5774$$

Q-5. The average working life of a certain power supply is claimed to be 10000 hours, with standard deviation of 400*M hours. We test a sample of size 25 of the power supplies, and calculate the sample mean. Find the probability that the sample mean is between 9600 and 10400 hours.

Answer: 400*M/5 = **80*M** □ standard deviation of sample mean
So number of standard deviations on either side = 400/(80*M) = 5/M

So answer = **2*F(5/M) – 1, where F(z) is the standard normal cdf**

Q-5. $\mu = 10000$, $\sigma = 400 \times 4$ (for M=4)

$n = 25$

$x_1 = 9600$, $x_2 = 10400$

$P(x_1 < x < x_2)$

$$z_1 = \frac{x_1 - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{9600 - 10000}{\frac{400 \times 4}{\sqrt{25}}} = -\frac{5}{4}$$

$$z_2 = \frac{x_2 - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{10400 - 10000}{\frac{400 \times 2}{\sqrt{25}}} = +\frac{5}{4}$$

$$P\left(-\frac{5}{4} < Z < +\frac{5}{4}\right) = \cancel{P(-2.5 < Z < 2.5)}$$

$$P(-1.25 < Z < 1.25)$$

$$= P(1.25) - P(-1.25)$$

$$= 0.8944 - 0.1056$$

$$= 0.7888$$

Standard normal distribution -- cumulative

Index	z	F(z)	Index	z	F(z)
1	0.00	0.5000	11	1.00	0.8413
2	0.10	0.5398	12	1.10	0.8643
3	0.20	0.5793	13	1.20	0.8849
4	0.30	0.6179	14	1.30	0.9032
5	0.40	0.6554	15	1.40	0.9192
6	0.50	0.6915	16	1.50	0.9332

7	0.60	0.7257	17	1.60	0.9452
8	0.70	0.7580	18	1.70	0.9554
9	0.80	0.7881	19	1.80	0.9641
10	0.90	0.8159	20	1.90	0.9713
			21	2.00	0.9772

Q-6. Five pairs of values of random variables X and Y are tabulated below. Find the COVARIANCE of X & Y.

X	1	2	3	4	5
Y	-3M	-4M	0	3M	4M

Sum of ΔX*ΔY product = 21M ⇒ answer = 21M/5 = 4.2*M

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$E(X) = \frac{1+2+3+4+5}{5} = 3$$

$$E(Y) = \frac{-3m-4m+0+3m+4m}{5} = 0$$

If $M = 1$.

$$E(XY) = \frac{-3-8+0+12+20}{5} = 4.2$$

$$Cov(X,Y) = 4.2 - 3*0 = 4.2$$

Q-7. Trucks arrive at a toll booth at the average rate of 12 arrivals per hour, and the arrivals define a Poisson process. What is the probability that the time interval T between two successive arrivals, measured in minutes, satisfies M $\le$ T $\le$ N?

Answer: $e^{-M/5} - e^{-N/5}$, since the rate of arrivals is 1/5 per minute.

Q-7 $\mu = 12$, $M = 2$, $N = 5$.

per hour $\Rightarrow \mu = \frac{12}{60 \text{ mins}} \Rightarrow \mu = \frac{1}{5}$.

$$P(2 \le T \le 5) = P(T \le 5) - P(T \le 2)$$
$$= 1 - e^{-\lambda t} - (1 - e^{-\lambda t})$$
$$= e^{-\lambda 2} - e^{-\lambda \cdot 5}$$
$$= e^{-2/5} - e^{-1}$$
$$= 0.670 - 0.3678$$
$$= 0.3024$$

Q-8. Recall the Markov process defined as "random walk with reflecting barriers". The four states of the process are 1, 2, 3 and 4. The transition probability matrix is as given below, with $\alpha = M/10$. The initial probability distribution over states is (1/4, 1/4, 1/4, 1/4). What is the probability that the process is in state 1 after two time steps?

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & \beta & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Pre-multiply TWICE above matrix with the probability matrix. After one pre-multiplication, probability distribution = (a/4 (1+a)/4 (1+b)/4 b/4). After the second round, only the first element of the row is needed, so only one column multiplication is needed.

So answer = **$\alpha(1+\alpha)/4 = (M/10)*(1+M/10)/4 = M*(M+10)/400$**

Q-8 $\quad p^0 = \left[\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4}\right]$

$$TPM = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2/10 & 0 & 1-\frac{2}{10} & 0 \\ 0 & 2/10 & 0 & 1-\frac{2}{10} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$p^1 = p^0 \cdot TPM$$

$$= \left[\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4}\right] \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.2 & 0 & 0.8 & 0 \\ 0 & 0.2 & 0 & 0.8 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \left[0.05 \quad 0.3 \quad 0.45 \quad 0.2\right]$$

$$p^2 = p^1 \, TPM$$

$$= \left[0.05 \quad 0.3 \quad 0.45 \quad 0.2\right] \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.2 & 0 & 0.8 & 0 \\ 0 & 0.2 & 0 & 0.8 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

state 1

state 1 $\Rightarrow 0.3 * 0.2$

$= 0.06$