Linear and Integer program: Set cover example

Sunday, 12 March 2023 7:24 PM

Linear Program

Given an $c \in \mathbb{Q}^n$ n-dimensional vector, $b \in \mathbb{Q}^m$, m-dimensional vector, $m \times n$ matrix $A = (a_{ij}) \in \mathbb{Q}^{m \times n}$

Canonical Form

$$\text{minimize} \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to} \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad i = 1, \dots m \quad \text{— A}$$

$$x_j \geq 0 \quad j = 1, 2, \dots n \quad \text{— B}$$

In vector notation

$$\min_{x} \; c^T x$$
$$\text{s.t.} \quad Ax \geq b$$
$$x \geq 0$$

x is called variable or x_i's are decision variables

A & B are called constraints

x that satisfies constraints is called feasible solution

Very "efficient" Practical methods to solve Linear Programs (LP)

Integer Programming

Each x_j must be an integer

For e.g. $x_j \in \mathbb{N}$ for $\forall j$

or $x_j \in \{0, 1\}$

Integer Programming is NP-Complete

Revisiting the Set Cover Problem

$E = \{e_1, \dots e_n\}$

$S_1, S_2, \dots S_m$ where each $S_j \subseteq E$

$w_j \geq 0$ for each S_j

Find a minimum weight collection of subsets that covers all of E.

For set cover let

$x_j = 1$ if S_j is a part of sol^n

$= 0$ o/w

To ensure every element e_i is covered

$$\sum_{j: e_i \in S_j} x_j \geq 1 \quad \text{for each } e_i, \; i = 1, \dots n$$

Integer Program for minimum weight-set cover

$$\text{minimize} \sum_{j=1}^{m} w_j x_j$$

$$\text{Subject to} \sum_{j: e_i \in S_j} x_j \geq 1 \quad i = 1, \dots, n$$

$$x_j \in \{0, 1\} \quad j = 1, \dots m$$

Z^*_{IP} be the optimum value of this integer program

$$Z^*_{IP} = OPT$$

However NP-hard to get Z^*_{IP}

Soln: LP's are polynomially time solvable

LP Relaxation

$$\text{minimize} \sum_{j=1}^{m} w_j x_j$$

$$\text{subject to} \sum_{j: e_i \in S_i} x_j \geq 1, \quad i = 1, \dots, n$$

$$x_j \geq 0 \quad j = 1, \dots m$$

Relaxation Means

i) Every Feasible sol^n for original integer program is feasible for this linear program

ii) The value of any feasible sol^n for the integer program has same value in the linear program

Z^*_{LP} is optimal sol^n for the LP

$$Z^*_{LP} \leq Z^*_{IP} = OPT$$

(Get a lower bound in case of min. problem)

A deterministic rounding algorithm

Let x^* be the optimal solution to LP relaxation for the set cover problem

Algorithm:

Given the LP sol^n x^*, include S_j in solution if & only if $x_j^* \geq 1/f$

where f is the max. no. of sets in which any element appears

More formally

$$f_i = |\{j : e_i \in S_j\}|$$
$$f = \max_{i=1,\dots n} f_i$$

Claim :- Collection of subsets S_j, $j \in I$ is a set cover

Proof:- For the solution, an element e_i is covered if solution contains some subset containing e_i.

Want to say each e_i is covered!

x^* is feasible for LP

$$\Rightarrow \sum_{j: e_i \in S_j} x_j^* \geq 1 \quad \text{for } e_i$$

By def. there are $f_i \leq f$ terms in the sum

atleast one term must be atleast $1/f$.

Thus for some j such that $e_i \in S_j$ $\quad x_j^* \geq 1/f \Rightarrow j \in I$

$\Rightarrow e_i$ is covered $\square$

Claim: Rounding Alg is an f-app. for the set cover problem.

Proof:- Alg. runs in poly time

By construction $1 \leq f \cdot x_j^*$ for each $j \in I$

Also $f w_j x_j^*$ is non-negative for $j = 1, \dots m$

$$\sum_{j \in I} w_j \leq \sum_{j=1}^{m} w_j \cdot (f \cdot x_j^*)$$
$$= f \sum_{j=1}^{m} w_j x_j^*$$
$$= f \, Z^*_{LP}$$
$$\leq f \cdot OPT \quad \square$$