Lecture 2 set cover definition

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The Set Cover Problem

We have $E = \{e_1, e_2, \dots e_n\}$

$S_1, S_2, \dots\dots S_m$ each $S_j \subseteq E$

Non-negative weights $w_j \geq 0$ for each S_j

Goal: Find a minimum-weight collection of subsets that covers all E.

i.e Find $I \subseteq \{1, 2, \dots m\}$ that minimizes $\sum_{j \in I} w_j$

s.t. $\bigcup_{j \in I} S_j = E$

Unweighted set cover $w_j = 1 \quad \forall j$

Lecture 2 set cover algorithm

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The ALGO

$I \leftarrow \phi$

$\hat{S}_j \leftarrow S_j, \quad \forall j$

while I is not setcover do

$$\ell \leftarrow \underset{j : \hat{S}_j \neq \phi}{\operatorname{argmin}} \frac{w_j}{|\hat{S}_j|}$$

$I \leftarrow I \cup \{\ell\}$

$\hat{S}_j \leftarrow \hat{S}_j - S_\ell \quad \forall j$

Lecture 2 set cover analysis

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k^{th} Harmonic number

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{k}$$

Useful Fact :-

For positive numbers $a_1, \dots a_k$ & $b_1, \dots b_k$

$$\min_{i \in [k]} \frac{a_i}{b_i} \leq \frac{\sum_{i=1}^{k} a_i}{\sum_{i=1}^{k} b_i} \leq \max_{i \in [k]} \frac{a_i}{b_i}$$

Lecture 2 set cover proof

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PROOF

Lemma 1 : In the kth iter

$$\min_{j : \hat{S}_j \neq \phi} \frac{w_j}{|\hat{S}_j|} \leq \frac{OPT}{n_k}$$

n_k = # elements uncovered at start of iter k

proof : O – indices of set in optimal sol$\underline{n}$

Claim :-

$$\min_{j : \hat{S}_j \neq \phi} \frac{w_j}{|\hat{S}_j|} \leq \frac{\sum_{j \in O} w_j}{\sum_{j \in O} |\hat{S}_j|}$$

$$= \frac{OPT}{\sum_{j \in O} |\hat{S}_j|} = \frac{OPT}{n_k}$$

Lecture2 proof continued

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Lemma 2 :- For set j chosen in k^{th} iteration

$$w_j \leq \frac{n_k - n_{k+1}}{n_k} OPT$$

Proof :- j minimizes ratio.

So $$\frac{w_j}{|\hat{S}_j|} \leq \frac{OPT}{n_k}$$

$$w_j \leq \frac{|\hat{S}_j| \, OPT}{n_k} = \frac{n_k - n_{k+1}}{n_k} OPT$$

$\because \left(n_{k+1} = n_k - |\hat{S}_j| \right)$

Lecture2 Final part of proof

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$$\sum_{j \in I} w_j \leq \sum_{k=1}^{l} \frac{n_k - n_{k+1}}{n_k} \, OPT$$

$$\leq OPT \cdot \sum_{k=1}^{l} \left(\frac{1}{n_k} + \frac{1}{n_k - 1} + \dots + \frac{1}{n_{k+1} + 1} \right)$$

$$= OPT \cdot \sum_{i=1}^{n} \frac{1}{i}$$

$$= H_n \cdot OPT \qquad \boxtimes$$