IT584 Approximation Algorithms February 13, 2024

Lecture 9

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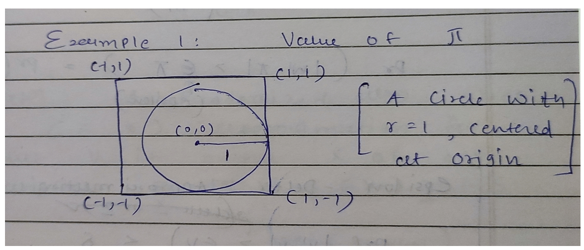
1 Monte Carlo Methods

Monte Carlo methods are a broad class of computational algorithms that rely on repeated random sampling to obtain numerical results. One of the basic examples of getting started with the Monte Carlo algorithm is the estimation of Pi.

1.1 Estimation of Pi

The idea is to simulate random (x, y) points in a 2-D plane with the domain as a square of side 2r units centered on (0,0). Imagine a circle inside the same domain with the same radius, r, and inscribed into the square. We then calculate the ratio of the number of points that lay inside the circle and the total number of generated points. Refer to the image below:



The concept involves generating random (x, y) coordinates within a 2-D plane, where the plane is bounded by a square with sides of length 2r and centered at (0,0). Within this square, a circle with radius r is positioned such that it is inscribed within the square. The objective is to determine the ratio of points falling inside the circle to the total number of points generated.

Throw a dart and check if it is falling in circle/square, assuming it will not go outside of square.

$$\text{P(Z = z)} = \begin{cases} 1 & \text{if } \sqrt{x^2 + y^2} \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Lemma 1 $P(z = \frac{\pi}{4})$

Proof Area of square = 4 and area of circle = π

$$\frac{\pi r^2 \text{ (area of circle)}}{(2r)^2 \text{ (Area of Board)}}$$

Run this Experiment m times.

$Z_i =$ Value of Z at i^{th} run.

$W = \sum_{i=1}^{m} z_i$

$E[w] = \sum_{i=1}^{m} E[z_i] = \frac{m\pi}{4}$ ($\because$ Linearity of Expectation)

$\pi = w' \frac{4}{m}$

$P(|w' - \pi| \geq \varepsilon\pi) = P(|w - \pi| \geq \varepsilon\pi)$ ($\because$ epsilon-delta approximation)

$= P(|W - E[W]| \geq \varepsilon E[W])$

$\leq 2e^{-m\pi\varepsilon^2/12}$ ($\because$ Chernoff Bounds)

If m increses, the accuracy of getting value of π will increase.
m is number of throws
w' is number of time it hits in circle.
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1.2 What should be the value of m for good estimate?

$2e^{-m\varepsilon^2/12} \leq \delta$

$\therefore m\pi\varepsilon^2/12 \geq -\ln(\delta/2)$

$\therefore m\pi\varepsilon^2/12 \geq \ln(2/\delta)$

$\therefore m \geq \frac{12\ln(2/\delta)}{e^2\pi}$

$m = \mathcal{O}(\frac{12\ln(2/\delta)}{e^2\pi})$