Rounding a dual solution for set cover

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Constructing the dual:

Suppose some element e_i can be covered by a low-weight subset S_j while some other element requires a high-weight subset S_r to be included to get covered

Lets say we give weights y_i to each element e_i accordingly

Sum of Prices of elements in subset S_j cannot be more than weight of the set

$$\therefore \forall j \quad \sum_{i: e_i \in S_j} y_i \le w_j$$

Highest total price for elements

$$\text{maximize} \quad \sum_{i=1}^{n} y_i$$

$$\text{Subject to} \quad \sum_{i: e_i \in S_j} y_i \le w_j \qquad j = 1, \dots, m$$

$$y_i \ge 0 \qquad i = 1, \dots n$$

Dual of Set Cover LP relaxation

Now $\sum_{i=1}^{n} y_i$ (for y which is dual feasible)

$$\le \sum_{i=1}^{n} y_i \sum_{j: e_i \in S_j} x_j \quad (x \text{ is feasible for primal})$$

Now

$$\sum_{i=1}^{n} y_i \sum_{j: e_i \in S_j} x_j = \sum_{j=1}^{m} x_j \sum_{i: e_i \in S_j} y_i$$

$$\le \sum_{j=1}^{m} x_j w_j$$

(Weak Duality)

$$\therefore \sum_{i=1}^{n} y_i \le Z_{LP}^{*} \le OPT$$

By Strong Duality if x^* is primal optimal & y^* is dual optimal

$$\sum_{j=1}^{m} w_j x_j^* = \sum_{i=1}^{n} y_i^*$$

Algorithm

Let y^* be an optimal solution to the dual LP relaxation.

Consider subsets for which dual inequality is tight

i.e. $\sum_{i: e_i \in S_j} y_i^* = w_j$.

I' is the indices of subsets in the solution.

Return I'.

Claim :- set of S_j's where $j \in I'$ is set cover

Proof: Suppose there is an uncovered element e_k.

Then for each subset S_j containing e_k

$$\sum_{i: e_i \in S_j} y_i^* < w_j$$

Let $\epsilon = \min_{j: e_k \in S_j} \left(w_j - \sum_{i: e_i \in S_j} y_i^* \right)$

Now $\epsilon > 0$

Consider new dual sol^n^ y' in which

$$y_k' = y_k^* + \epsilon$$

and every other component of y' is same as y^*

y' is dual feasible sol^n^

[$\because$ for each j such that $e_k \in S_j$

$$\sum_{i: e_i \in S_j} y_i' = \sum_{i: e_i \in S_j} y_i^* + \epsilon \le w_j]$$

For each j such that $e_k \notin S_j$

$$\sum_{i: e_i \in S_j} y_i' = \sum_{i: e_i \in S_j} y_i^* \le w_j$$

Also $\sum_{i=1}^{n} y_i' > \sum_{i=1}^{n} y_i^*$!!

CONTRADICTION

Thm :- Dual - Rounding gives f - approximation for set cover

$$\sum_{j \in I'} w_j = \sum_{j \in I'} \sum_{i: e_i \in S_j} y_i^*$$

$$= \sum_{i=1}^{n} \left| \{ j \in I' : e_i \in S_j \} \right| \cdot y_i^*$$

$$\le \sum_{i=1}^{n} f_i \, y_i^*$$

$$\le f \sum_{i=1}^{n} y_i^*$$

$$\le f \cdot OPT \qquad \square$$