IT584 Approximation Algorithms January 25, 2024

Lecture 8

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1 Chernoff Bounds Proof and More

1.1 Chernoff Bounds

Definition 1 *Chernoff Bounds assert that the sum of n independent 0-1 random variables is highly likely to be situated in proximity to the expected value of the sum.*

Theorem 1 *Consider independent 0-1 random variables $X_1, X_2, ..., X_n$, not necessarily identically distributed, such that X_i takes either value 0 or a_i ($0 < a_i \leq 1$).$\delta > 0$, $X = \sum_{i=1}^{n} x_i$, $L \leq \mu \leq U and \mu = E[X]$, the following inequalities hold:*

$$Pr[X \geq (1+\delta)\mu] < \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$

and

$$Pr[X \leq (1-\delta)L] < \left(\frac{e^{-}\delta}{(1-\delta)^{(1-\delta)}}\right)^{L}$$

1.2 Proof of Chernoff Bounds

Lemma 1 *For any $\delta > 0$,*

$$Pr(X \geq (1+\delta)\mu) \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$

Proof We prove the first equation if $E[X] = 0, X = 0$ and bound holds trivially,

$$\therefore E[X] > 0 \quad and \quad E[X_i] > 0 \; for \; some \; i$$

Ignore all i with $E[X_i] = 0$,

$$P_i = P_r[X_i = a_i]$$

Since $E[X_i] > 0$ and $P_i > 0$

$$\mu = E[X] = \sum_{i=1}^{n} P_i a_i \; \leq \; U$$

For any $t > 0$,

$$P_r[X \geq (1+\delta)\, U] = P_r[e^{tx} \geq e^{t(1+\delta)U}]$$

By Markov's Inquality,

$$P_r[e^{tx} \geq e^{t(1+\delta)U}] \leq \frac{E[e^{tx}]}{e^{t(1+\delta)U}}$$

Now

$$E[e^{tx}] = E[e^{t\sum_{i=1}^{n} X_i}] = \prod_{i=1}^{n} E[e^{tx_i}]$$

$$E[e^{tx_i}] = (1 - P_i) + P_i e^{ta_i} = 1 + P_i \left(e^{ta_i} - 1\right)$$

Consider

$$f\,(t) = a_i \left(e^t - 1\right) - e^{a_i t} - 1$$

$$f'\,(t) = a_i e^t - a_i e^{a_i t} \geq 0$$

$f(t)$ non increasing for $t \geq 0$

$$\therefore e^{ta_i} - 1 \leq a_i \left(e^t - 1\right)$$

$$\therefore E[e^{tX_i}] \leq 1 + P_i a_i \left(e^t - 1\right)$$

$$E[e^{tX_i}] \leq e^{P_i a_i \left(e^t - 1\right)} \quad \left(As\ 1 + X < e^X\ for\ x > 0\right)$$

$$\therefore E[e^{tx}] \leq e^{U\left(e^t - 1\right)}$$

Let $t = ln\,(1 + \delta) > 0$

$$P_r[X \geq (1 + \delta)\,U] \leq \frac{E[e^{tx}]}{e^{t(1+\delta)U}}$$

$$\therefore P_r[X \geq (1 + \delta)\,U] = \left(\frac{e^\delta}{(1 + \delta)^{(1+\delta)}}\right)^U$$

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Lemma 2 *For* $0 < \delta \leq 1$,

$$Pr(X \geq (1+\delta)\mu) \leq \left(e^{-\mu\delta^2/3}\right)$$

Proof Taking *log* on both sides,we want to show

$$U\left(\delta - (1+\delta)\, ln\,(1+\delta)\right) \leq -U\delta^2/3$$

IF we show the derivative of the left-hand side is no more than the right-hand side for $0 \leq \delta \leq 1$, the inequality will hold.

$\therefore$ Want to show that

$$-Uln\,(1+\delta) \leq -2U\delta/3$$

Let

$$f\,(\delta) = -Uln\,(1+\delta) + 2U\delta/3$$

We want to show $f\,(\delta) \leq 0$ on $[0,1]$

As long as $f\,(\delta)$ is convex on $[0,1]$,

$$f\,(\delta) \leq 0 \quad on\ [0,1]$$

$$f'\,(\delta) = \frac{-U}{1+\delta} + 2U/3$$

$$f"\,(\delta) = \frac{U}{(1+\delta)^2} \qquad \geq 0\ for\ \delta \in [0,1]$$

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