Chernoff bound proof and more

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Let $X_1, \dots X_n$ be n independent random variables not necessarily identically distributed, such that X_i takes either value 0 or the value a_i $(0 < a_i \le 1)$.

Then for $X = \sum_{i=1}^{n} X_i$ & $\mu = E[X]$

$L \le \mu \le U$ and $\delta > 0$

$$\Pr[X \ge (1+\delta)U] < \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^U$$

and

$$\Pr[X \le (1-\delta)L] < \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^L$$

Proof:- We prove the first eq.

if $E[X] = 0$, $X = 0$ and bound holds trivially

$\therefore E[X] > 0$ and $E[X_i] > 0$ for some i

Ignore all i with $E[X_i] = 0$

$p_i = \Pr[X_i = a_i]$

Since $E[X_i] > 0$, $p_i > 0$

$\mu = E[X] = \sum_{i=1}^{n} p_i a_i \le U$

For any $t > 0$

$$\Pr[X \ge (1+\delta)U] = \Pr[e^{tX} \ge e^{t(1+\delta)U}]$$

By Markov's inequality

$$\Pr[e^{tX} \ge e^{t(1+\delta)U}] \le \frac{E[e^{tX}]}{e^{t(1+\delta)U}}$$

Now

$$E[e^{tX}] = E\left[e^{t\sum_{i=1}^{n} X_i}\right] = E\left[\prod_{i=1}^{n} e^{tX_i}\right] = \prod_{i=1}^{n} E[e^{tX_i}] \quad (\text{why?})$$

$$E[e^{tX_i}] = (1-p_i) + p_i e^{ta_i} = 1 + p_i(e^{ta_i} - 1)$$

Consider $f(t) = a_i(e^t - 1) - e^{a_i t} - 1$.

$f'(t) = a_i e^t - a_i e^{a_i t} \ge 0$ (why?)

$\Rightarrow f(t)$ non decreasing for $t \ge 0$

$\therefore e^{ta_i} - 1 \le a_i(e^t - 1)$

$\therefore E[e^{tX_i}] \le 1 + p_i a_i (e^t - 1)$

$E[e^{tX_i}] < e^{p_i a_i (e^t - 1)}$ $(\because 1 + x < e^x$ for $x > 0)$

$$\therefore E[e^{tX}] < \prod_{i=1}^{n} e^{p_i a_i (e^t - 1)} = e^{\sum_{i=1}^{n} p_i a_i (e^t - 1)} \le e^{U(e^t - 1)}$$

Let $t = \ln(1+\delta) > 0$

$$\Pr[X \ge (1+\delta)U] \le \frac{E[e^{tX}]}{e^{t(1+\delta)U}} < \frac{e^{U(e^t - 1)}}{e^{t(1+\delta)U}} = \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^U$$

Lemma:- For $0 \le \delta \le 1$

$$\left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^U \le e^{-U\delta^2/3}$$

Proof: Taking log on both sides we want to show

$$U(\delta - (1+\delta)\ln(1+\delta)) \le -U\delta^2/3$$

$\delta = 0$ it holds

If we show derivative of left hand side is no more than right hand side for $0 \le \delta \le 1$, the inequality will hold

$\therefore$ Want to show that

$$-U\ln(1+\delta) \le -2U\delta/3$$

Let $f(\delta) = -U\ln(1+\delta) + 2U\delta/3$

we want to show $f(\delta) \le 0$ on $[0,1]$

$f(0) = 0$, $f(1) \le 0$ (why?)

As long as $f(\delta)$ is convex on $[0,1]$ $f(\delta) \le 0$ on $[0,1]$

$f'(\delta) = -U/(1+\delta) + 2U/3$

$f''(\delta) = U/(1+\delta)^2 \ge 0$ for $\delta \in [0,1]$

Bound on $X \le (1-\delta)L$ when $\delta = 1$

Lemma:- Let $X_1, \dots X_n$ be n independent random variables, not necessarily identically distributed, $X_i = 0$ or a_i for $0 < a_i \le 1$

For $X = \sum_{i=1}^{n} X_i$ and $\mu = E[X]$, $L \le \mu$

$$\Pr[X = 0] < e^{-L}$$

Proof:- Assume $\mu = E[X] > 0$ else $X = 0$, $L \le \mu = 0$

Let $p_i = \Pr[X_i = a_i]$

$\mu = \sum_{i=1}^{n} a_i p_i$ and

$\Pr[X = 0] = \prod_{i=1}^{n} (1 - p_i)$

(Use $\left(\prod_{i=1}^{k} a_i\right)^{1/k} \le \frac{1}{k}\sum_{i=1}^{k} a_i$ for non negative a_i)

$$\prod_{i=1}^{n}(1-p_i) \le \left[\frac{1}{n}\sum_{i=1}^{n}(1-p_i)\right]^n = \left[1 - \frac{1}{n}\sum_{i=1}^{n} p_i\right]^n$$

Each $a_i \le 1$

$$\therefore \left[1 - \frac{1}{n}\sum_{i=1}^{n} p_i\right]^n \le \left[1 - \frac{1}{n}\sum_{i=1}^{n} a_i p_i\right]^n = \left[1 - \frac{1}{n}\mu\right]^n$$

Now $1 - x < e^{-x}$ for $x > 0$

$$\therefore \left[1 - \frac{1}{n}\mu\right]^n < e^{-\mu} \le e^{-L}$$

Integer Multicommodity Flows

Input: Undirected graph $G(V, E)$

k pairs of vertices $s_i, t_i \in V$

Goal:- For each $i = 1, \dots k$

Find a single simple path from s_i to t_i so as to minimize the maximum number of paths containing the same edge.

Integer Program Formulation:-

P_i - set of all possible simple paths P in G from s_i to t_i

P is set of edges in the path.

$x_P = 1$ when P from s_i-t_i is used

0 o/w

Total number of paths using an edge $e \in E$ is $\sum_{P: e \in P} x_P$

W - denotes maximum number of paths using an edge

IP:- minimize W

subject to $\sum_{P \in P_i} x_P = 1$ $\quad i = 1, \dots k$

$\sum_{P: e \in P} x_P \le W$ $\quad e \in E$

$x_P \in \{0, 1\}$ $\quad \forall P \in P_i$, $i = 1, \dots k$.

Solve LP relaxation

Apply Randomized Rounding

For $i = 1, \dots k$ we choose exactly one path $P \in P_i$ according to the probability distribution x_P^* on paths $P \in P_i$

x^* is optimal solution of value W^*.

Thm:- If $W^* \ge c\ln n$ for some constant c, then w.h.p. the total number of paths using any edge is at most $W^* + \sqrt{cW^* \ln n}$

Proof:- For each $e \in E$ define

$X_e^i = 1$ if s_i-t_i has edge e

0 o/w

No. of paths using edge e is $Y_e = \sum_{i=1}^{k} X_e^i$

Want to bound $\max_{e \in E} Y_e$

$$E[Y_e] = \sum_{i=1}^{k} \sum_{P \in P_i : e \in P} x_P^* = \sum_{P: e \in P} x_P^* \le W^*$$

For fixed edge e, X_e^i are independent

Apply Chernoff Bound

Let $\delta = \sqrt{(c\ln n)/W^*}$

Now $W^* \ge c\ln n$ (by Assumption)

$\therefore \delta \le 1$

Putting $U = W^*$ and applying the Chernoff bound

$$\Pr[Y_e \ge (1+\delta)W^*] \le e^{-W^*\delta^2/3} = e^{-(c\ln n)/3} = \frac{1}{n^{c/3}}$$

Also $(1+\delta)W^* = W^* + \sqrt{cW^* \ln n}$

Total no. of edges $\le n^2$

$$\Pr\left[\max_{e \in E} Y_e \ge (1+\delta)W^*\right] \le \sum_{e \in E} P[Y_e \ge (1+\delta)W^*] \le n^2 \cdot \frac{1}{n^{c/3}} = n^{2 - c/3}$$

For $c \ge 12$, theorem statement fails w.p. atmost $\frac{1}{n^2}$

Observation:- $W^* \ge c\ln n$

Solution produced by Randomized Algo is no more than $2W^* \le 2OPT$

But for $W^* \gg c\ln n$, solution much better.