IT584 Approximation Algorithms	February 29, 2024
Lecture 12	
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Integer Programming

In Set Cover problem, we first try to decide the subjects S_j that should be used in the solution and the decision variable x_j representing the choices made . So there will be two cases i.e. x_j will be 1 if the subject S_j is included and 0 if not.

So, the constraints will be $x_j \geq 1$ for all subjects S_j and $x_j \leq 0$ for all subjects S_j. However, this is not enough to conclude that $x_j \in \{0, 1\}$ as it may contain the fractional parts being a part of linear program.

Thus we have to introduce Integer Programming here. This will make it clear that the decision variable x_j is an integer such that $x_j \in \{0, 1\}$ that it will be either 1 or 0.

Integer Programming is NP-complete.

Revisiting the Set Cover problem

$$E = e_1, e_2, ..., e_n$$

, where e_i denotes the elements

$$S_1, S_2, ..., , S_m$$

, where S_i denotes Subject.

To guarantee comprehensive coverage of every element e_i, it must be stated that there is at least one subject S_j that encompasses element e_i. This is written as

$$\sum_{j:e_i \in S_j} x_j \geq 1$$

for each e_i, i = 1, ... , n.

Now, we have to find the set cover with minimum weight, so here our subject will be the weight of a set cover, i.e. $\sum_{j=1}^{m} w_j x_j$. And our objective will be to minimize $\sum_{j=1}^{m} w_j x_j$

So the integer program for minimum weight set cover will be

$$minimize \sum_{j=1}^{m} w_j x_j$$

$$subjectto \quad \sum_{j:e_i \in S_j} x_j \geq 1, \quad i = 1, \ldots, n$$

$$x_j \in \{0, 1\}, \quad j = 1, \ldots, m$$

Z^*_{IP} represents the optimum value of this Integer program for an instance of the set cover problem.As integer programming models the problem correctly we have

$$Z^*_{IP} = OPT$$

Generally, Integer programs are hard to solve in polynomial time because set cover problem is a NP-hard problem, so solving the above integer program for any set cover in polynomial time will give P=NP. However, Linear programs do get solved in polynomial time, because here we don't have the constraint that the decision variable should be an integer.

So if we replace the above integer program constarint $x_j \in \{0,1\}$ with $x_j \geq 0$ it becomes a linear program that is solvable in polynomial time.

$$minimize \sum_{j=1}^{m} w_j x_j$$

$$subjectto \quad \sum_{j:e_i \in S_j} x_j \geq 1, \quad i = 1, \ldots, n$$

$$x_j \geq 0, \quad j = 1, \ldots, m$$

This linear program is called LP relaxation as it is a relaxation of the IP.
Relaxation means:

- Every feasible solution for original Integer Program is feasible for this Linear Program.
- The value any feasible solution for Integer program has same value in Linear Program.

To determine whether the linear program is a relaxation, any solution of the integer program such that $x_j \in \{0,1\}$ for each j=1,...,m and $\sum_{j:e_i \in S_j} x_j^* \geq 1$ for each i=1,...,m will satisfy all the constraints of linear program.

Z_{LP}^* denotes the optimal solution for the linear program. If there exist an optimal solution to IP then it is feasible for LP as well with the value Z_{IP}^*.Thus, any optimal solution to the linear program will have value $Z_{LP}^* \leq Z_{IP}^* =$ OPT., as this linear program finds a feasible solution of the lowest possible value.In case of maximization problem it will be vice-versa.

A Deterministic Rounding Algorithm

Let x^* represent the optimal solution to LP relaxation for the set cover problem.Here's an easy way to obtain a solution of set cover problem:

Given the LP solution x^*, we will consider subset S_j if and only if $x_j^* \geq 1/f$, where f represents the maximum number of sets in which an element occurs.

More formally,

$$f_i = |\{j : e_i \in S_j\}|$$

$$f = \max_{i=1,\ldots,n} f_i$$

Let I represents the indices j of the subsets in this solution.Here, if we round of the solution x^* to integers $\hat{x}$ by setting $\hat{x_j}$ to be 1 if $x_j^* \geq 1/f$ and 0 otherwise. Thus we can clearly state that $\hat{x}$ is a feasible solution to the integer program and I indexes a set cover.

Lemma 1 *The collection of subsets S_j $j \in I$, is a set cover.*

Proof For the solution, an element e_i is covered if solution contains some subset containing e_i.

We show that each element e_i is covered. Because the optimal solution x^* is a feasible solution for linear program. $\sum_{j:e_i \in S_j} x_j^* \geq 1$ for element e_i.

According to definition of f_i and f, we have $f_i \leq f$ terms in the sum such that at least one term must be at least 1/f. So for some j such that $e_i \in S_j$, $x_j^* \geq 1/f$. Thus j $\in$ I, and element e_i is covered.

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Reference

The Design of Approximation Algorithm by David P. Williamson,David B. Shmoys