IT584 Approximation Algorithms	February 13, 2024

Lecture 9

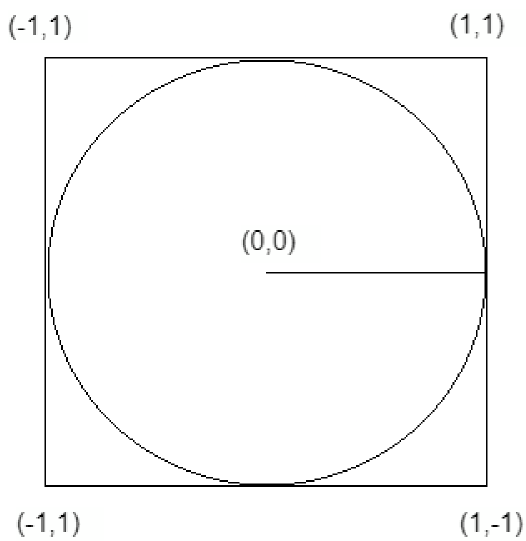
Lecturer: Rachit Chhaya *Scribe/s: Harsh Mistry*

1 Monte Carlo Methods

In Monte Carlo methods, random sampling is used to simulate a large number of possible outcomes, and these outcomes are then aggregated to estimate the desired quantity or solution. The estimation of Pi using Monte Carlo methods is a classic example of how randomness and probability can be leveraged to solve mathematical problems.

1.1 Estimation of Pi

By randomly tossing darts at the dartboard, we generate points distributed across the square. Each dart landing within the square serves as a data point. By tallying how many of these points fall within the quarter circle, we effectively gauge the ratio of successful throws to the total attempts. This ratio, when scaled appropriately, provides an approximation of the ratio of the area of the quarter circle to that of the square. Through this method, we can iteratively refine our estimate of Pi by increasing the number of darts thrown.



The concept involves generating random (x, y) coordinates within a 2-D plane, where the plane is bounded by a square with sides of length 2r and centered at (0,0). Within this square, a circle with radius r is positioned such that it is inscribed within the square. The objective is to determine the ratio of points falling inside the circle to the total number of points generated.

Throw a dart and check if it is falling in circle/square, assuming it will not go outside of square.

$$\mathrm{P(Z = z)} = \begin{cases} 1 & \text{if } \sqrt{x^2 + y^2} \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Lemma 1 $P(z = \frac{\pi}{4})$

Proof Area of square = 4 and area of circle = π

$$\frac{\pi r^2 \text{ (area of circle)}}{(2r)^2 \text{ (Area of Board)}}$$

Run this Experiment m times.

$Z_i =$ Value of Z at i^{th} run.

$W = \sum_{i=1}^{m} z_i$

$E[w] = \sum_{i=1}^{m} E[z_i] = \frac{m\pi}{4}$ ($\because$ Linearity of Expectation)

$\pi = w' \frac{4}{m}$

$P(|w' - \pi| \geq \varepsilon\pi) = P(|w - \pi| \geq \varepsilon\pi)$ ($\because$ epsilon-delta approximation)

$= P(|W - E[W]| \geq \varepsilon E[W])$

$\leq 2e^{-m\pi\varepsilon^2/12}$ ($\because$ Chernoff Bounds)

If m increses, the accuracy of getting value of π will increase.
m is number of throws
w' is number of time it hits in circle.
■

1.2 What should be the value of m for good estimate?

$2e^{-m\varepsilon^2/12} \leq \delta$

$\therefore m\pi\varepsilon^2/12 \geq -\ln(\delta/2)$

$\therefore m\pi\varepsilon^2/12 \geq \ln(2/\delta)$

$\therefore m \geq \frac{12\ln(2/\delta)}{e^2\pi}$

$m = \mathcal{O}(\frac{12\ln(2/\delta)}{e^2\pi})$