→ argmin is

* Eigen value decomposition

- A (non zero) vector v of n dimention is an eigenvector of n×n matrix A if it satisfies:

$$\rightarrow A v = \lambda v$$

eigenvector

eigenvalue

↳ to find eigenvalue $|A - \lambda I| = 0$

→ If A is square matrix (n×n) with n linearly independent eigen vectors A can be factorize as

$$A = Q \lambda Q^{-1}$$ → here possible became linearly independent

$$Q = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

→ Eigen value are real number.

→ eigen vector can be choosen as orthonormal.

$$A = Q \lambda Q^T$$

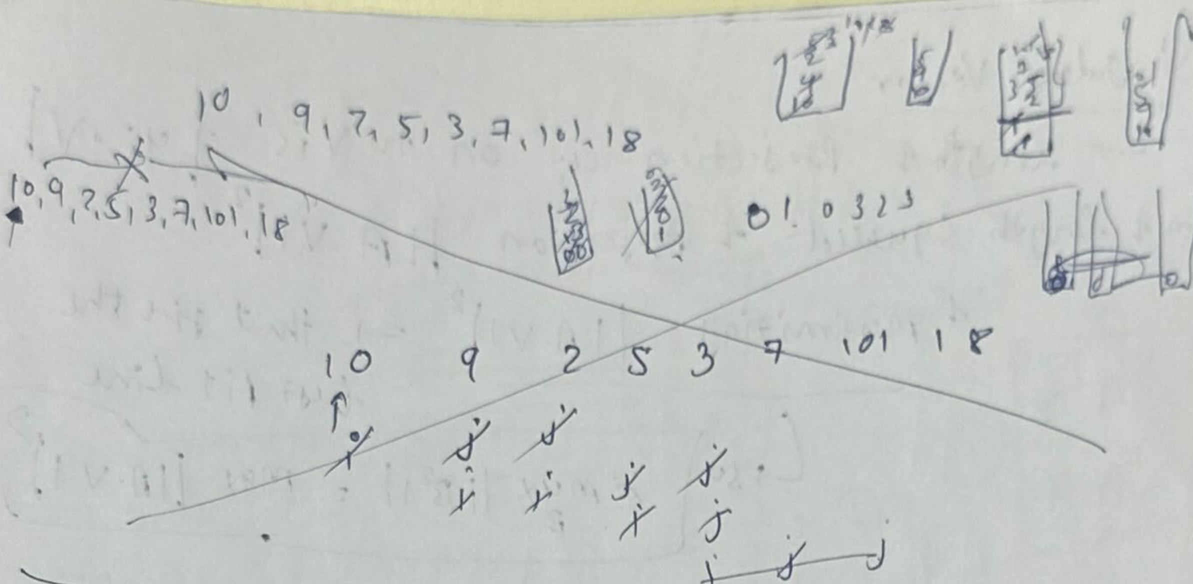
⇒ v_1 & v_2 orthogonal ⇒ $v_1^T v_2 = 0$

orthonormal ⇒ $||v||_2 = 1$, $||v_1|| = 1$

* Algebric Multiplicity

No of repetation of perticular eigen value is AM.

* geomatric M : # of L.I. eigen vector associated with it i.e. dimention of null space of $A - eI$ → $A \cdot x = 0$ all possible values of x that form a Matrix



* Issue with SVD → only applicable to square matrix
- may be complex eigen value.

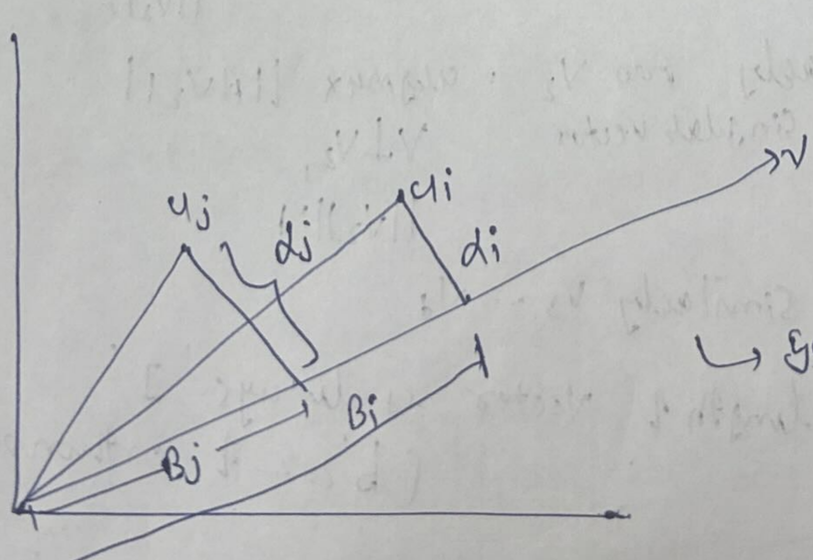
* **SVD**: Any Matrix A (n×d) can be decompose as

$$A = U \Sigma V^T$$

two ways

if we use this row should be orthonormal →	$U_{n\times n}$	$U_{n\times d}$ ↙ here rows may not be orthonormal
	$\Sigma_{n\times d}$	$\Sigma_{d\times d}$
	$V^T_{d\times d}$	$V_{d\times d}$

two decompose matrices ⇒ Columns of U & V are orthonormal.
⇒ Σ is diagonal matrix with non-negative real entities



↳ so we need to
$\text{Min} \sum d^2$
or
$\text{Max} \sum \beta^2$
for finding best fit line.

* Singular Vector

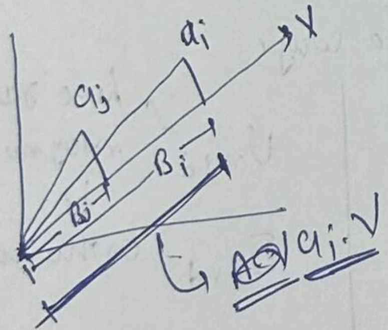
→ length of projection a_i on to V is $|a_i \cdot V|$

→ Sum of length squared of projection $||A.V||^2$.

& maximizing $||A.V||^2$ ⇒ that give the best fit line

↳ so $\boxed{\Sigma \max_{\beta} ||\beta^2|| = \max ||A.V||^2}$

$$\Rightarrow A.V = \begin{bmatrix} — a_1 — \\ — a_2 — \\ \vdots \\ — a_n — \end{bmatrix}^{d} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_d \end{bmatrix}^{d} = \begin{bmatrix} a_1 V \\ \vdots \\ a_n V \end{bmatrix}_n$$



$$= \sum_{i=1}^{n} \beta^2 = \sum_{i=1}^{n} (a_i^T V)^2$$

$$= ||A.V||_2^2$$

* first Singular vector of A

$$V_1 = \underset{||V_2||=1}{\arg\max} ||A.V||_2$$

$$\sigma_1(A) = ||AV||_2 \text{ or } \max_{||V_2||} ||AV||_2$$

* Similarly for $V_2 = \underset{\substack{V \perp V_2, \\ ||V_2||=1}}{\arg\max} ||AV_2||$

↳ second singular vector

Similarly $V_3 \dots V_d$

→ length of vector is always '1'
(b'coz of orthonormal)

$\Rightarrow U^T U = I$, $V^T V = I$ rotation U

$\hookrightarrow$ so. $A^T A = (V \Sigma^T U^T)(U \Sigma V^T)$

stretching

$$\boxed{A^T A = V \Sigma^T \Sigma V^T}$$

same as $\boxed{AA^T = U \Sigma \Sigma^T U^T}$

positive semi definite

$$\boxed{\begin{aligned} A^T A &= V \\ A A^T &= U \end{aligned}}$$

Ex. $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}_{2\times 3}$ $A^T = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}_{3\times 2}$

for $AA^T = U$

$AA^T = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{vmatrix} = 0$

$(11-\lambda)^2 - (1)^2 = 0$

$(11-\lambda-1)(11-\lambda+1) = 0$

$$\boxed{\lambda_1 = 12 \ , \ \lambda_2 = 10}$$

$\lambda_2 = 10$

$\begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 10\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$11x_1 + x_2 = 10x_1$

$x_1 + 11x_2 = 10x_2$

$\Rightarrow \boxed{x_1 = -x_2}$

$\hookrightarrow x_1 = 1$

$x_2 = -1$

$\hookrightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow$ eigen vector

$[A - \lambda I][X] = 0 \Rightarrow \underline{AX = \lambda X}$

$\begin{bmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\lambda = 12$ $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow \cancel{x_1 + x_2 + x_1 - x_2} = 0$

$\lambda = 12$

$\begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 12\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$11x_1 + x_2 = 12x_1$

$x_1 + 11x_2 = 12x_2$

$$\boxed{\begin{aligned} x_2 &= x_1 \\ x_1 &= x_2 \end{aligned}}$$

if $x_1 = 1 \Rightarrow x_2 = 1$

$\Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Pond of the vectors

$$\text{So} = V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{\text{Normalize}}$$

using Gram-Schmidt orthogonalization process

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{matrix} \to \sqrt{1^2+1^2} \\ \to \sqrt{1^2+1^2} \end{matrix}$$

$$\boxed{V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}$$

$\Rightarrow$ for $V = A^T A = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix} \longrightarrow |A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{vmatrix} = 0$$

3x3

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = \text{trace}(A^T A) = 10 + 10 + 2 = 22$

$S_3 = \det(A^T A) = 0$

$S_2 =$ minor of diagonal matrix

$$\underset{\begin{vmatrix} 10 & 4 \\ 4 & 2 \end{vmatrix}}{4} + \underset{\begin{vmatrix} 10 & 2 \\ 2 & 2 \end{vmatrix}}{16} + \underset{\begin{vmatrix} 10 & 0 \\ 0 & 10 \end{vmatrix}}{100} = 120$$

$\therefore \lambda^3 - 22\lambda^2 + 120\lambda + 0 = 0$

$\therefore \lambda(\lambda^2 - 22\lambda + 120) = 0$

$\therefore \lambda(\lambda - 12)(\lambda - 10) = 0$

~~$\lambda_1 = 0,\ \lambda_2 = 12,\ \lambda_3 = 10$~~

$\lambda_1 = 12,\ \lambda_2 = 10\ \lambda_3 = 0$ → write in descending order

↳ Find eigen vector

$$V = \begin{bmatrix} \cancel{10-12} & \cancel{0} & 2 \\ \cancel{0} & 10-10 & \\ & & \end{bmatrix}$$

$\lambda_1 = 12$ $\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 2 & 4 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

by Cramer's rule

$$\frac{x_1}{20-16} = \frac{-x_2}{0-8} = \frac{x_3}{0-(-4)}$$

$$\therefore \frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

$$1 \quad 2 \quad 1$$

$\lambda_2 = 10$ $= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\frac{x_1}{-16} = \frac{-x_2}{-8} = \frac{x_3}{0}$$

$$2 \quad 1 \quad 0$$

$\lambda_3 = 0$ $\begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\frac{x_1}{4} = \frac{-x_2}{-8} = \frac{x_3}{-20}$$

$$1 \quad 2 \quad -5$$

$$\boxed{V = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & -5 \end{bmatrix}}$$

$\downarrow$

$$\boxed{\begin{array}{cc} & \sqrt{6} \\ V^T = & \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -5 \end{bmatrix} \\ \sqrt{5} & \\ \sqrt{30} & \end{array}}$$

Normalize

$$\boxed{V^T = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 2/\sqrt{5} & 1/\sqrt{5} & 0/\sqrt{5} \\ 1/\sqrt{30} & 2/\sqrt{30} & -5/\sqrt{30} \end{bmatrix}}$$

3 => $A_{2\times3}$: $\Sigma_{2\times3}$

=> Always eigen values of V & U are similar & one of the eigen value is always '0'.

$\lambda_1 = 12$, $\lambda_2 = 10$

$$\boxed{\Sigma = \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}_{2\times3}}$$

always
Place higher value of eigen values

$\lambda_1 > \lambda_2 >$

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}_{2\times 2} \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 1/\sqrt{30} & 2/\sqrt{30} & -5/\sqrt{30} \end{bmatrix}$$

Important Result

$A \to n\times d$ matrix where are singular vector for $1 \leq k \leq r$ let V_k be subspace spanned by $v_1 \dots v_k$ then for each k. V_k is best fit k-dimension subspace by A.

$$\left[\begin{array}{cccccc|c} | & | & | & & | & & | \\ v_1 & v_2 & v_3 & \cdots & v_k & & v_r \\ | & | & | & & | & & | \end{array}\right]$$

$$\text{eg}\quad \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \leftarrow \text{full rank matrix}$$

$$\underbrace{\qquad}_{\text{rank}=2}$$

V_k

$$A = U \Sigma V^T$$

Frobenius form

$$\|A\|_F = \sqrt{\langle A, A\rangle} = \sqrt{Tr(A^TA)} \qquad A^TA = (V\Sigma^T U^T)(U\Sigma V^T) = (V\Sigma^2 V^T)$$

$$= \sqrt{Tr(V\Sigma^2 V^T)}$$

$$= \sqrt{Tr(\Sigma^2)}$$

$$\boxed{\|A\|_F = \sqrt{Tr(\Sigma^2)}}$$

Hence $\boxed{\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2}} = \boxed{\sqrt{\sum_{i=1}^{r} d_i^2}}$

d_i ← diagonal element in Σ

(*) Important Result : $\boxed{\sum \sigma_i^2(A) = \|A\|_F^2}$

singular values

$\sqrt{\lambda_1}$

$\sigma_i(A) = \sqrt{\lambda_i}$

$$\boxed{\underset{n \times d}{A} = \sum_{i=1}^{d} \overset{1\times 1}{\sigma_i} \; \overset{n\times 1}{u_i} \; \overset{1\times d}{v_i^T} = U \Sigma V^T}$$

scaler

but $\boxed{Rank(A) = 1}$

~~Rank~~ here

→ for V $\quad A^TA$ → is $n \times n$ Symmetric PSD matrix

eigenvector $v_1 \ldots v_n$

$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$ $\quad A^TA$ such that $\boxed{A^TAv_i = \sigma_i^2 v_i}$

↳ where

$\sigma_i = \sqrt{\lambda_i}$

eigen values

for $\underset{m \times m}{U}$: $\boxed{AA^Tu_i = \sigma_i^2 u_i}$

$i = 1 \ldots m$

$U = \begin{bmatrix} u_1 & u_2 & u \cdots & u_m \\ | & | & | & | \end{bmatrix}$

Σ = A is $m \times n$ matrix $\quad rank(A) = r$

$r \leq \min(m, n)$

→ For Calculating U → Use this

$A v_i = \sigma_i u_i$

$$\boxed{u_i = \frac{A v_i}{\sigma_i}}$$

~~$E(x) = E(x^2) + (E(x))^2$~~

$\Rightarrow u_1, u_2 \dots u_d \rightarrow$ left singular vectors of A.

$$\boxed{A = \sum_{i=1}^{d} \sigma_i u_i v_i^T}$$

$\rightarrow$ Suppose we consider square symm. matrix A^TA & let x be eigen~~value~~ vector of A^TA & λ be it's eigen value.

$$A^TAx = \lambda x \longrightarrow \text{char. eqn } (\because AX = \lambda X)$$

Multiply with A both side

$$AA^T(Ax) = \lambda(Ax)$$

$$\boxed{A^TA = VDV^T}$$
$$\boxed{AA^T = UDU^T}$$

$(\because A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = (V\Sigma^TU^T)(U\Sigma V^T)$

$$= V\Sigma^T\underbrace{U^TU}_{I}\Sigma V^T$$

$I \rightarrow$ because the orthonormal

$$\boxed{A^TA = V\Sigma^T\Sigma V^T}$$

$$\boxed{A^TA = VDV^T}$$

$\rightarrow D$ value's are squared value.

$$\boxed{D = \Sigma^T\Sigma}$$

$(AA^T = UDU^T$

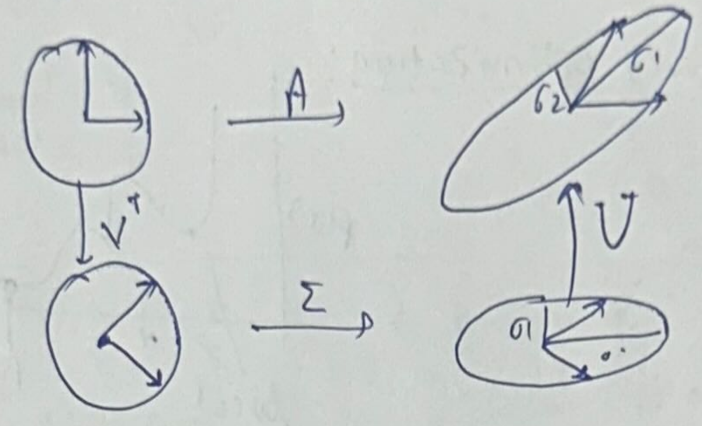
(U = formed by eigenvector of AA^T

V = " " " A^TA)

(① singular values : square root of eigenvalue of A^TA)

* Geometrix Interpritation

$$A = U\Sigma V^T \Rightarrow Ax = U\Sigma V^T x$$



* $\|V_{d\times d}\, x_{d\times 1}\|_2 = \|x\|_2 \;\forall\; x_{(d\times 1)}$

$\Rightarrow$ $\cancel{V^T x}$ $\cancel{\quad}$

$$V^T x = x$$

If
$$\begin{bmatrix} 1 & 0 & 0\;0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\|V^T x\|_2^2 = (V^T x)^T (V^T x)$$
$$= x^T V V^T x = x^T x = \|x\|_2^2$$

$$\boxed{V^T x = x \;\forall}$$

$\Rightarrow$ Singular values cannot be -ve. it is ≥ 0

$\Rightarrow$ but eigenvalue is can be -ve, +ve, 0

* $(A^T A)^{-1} A^T b = (V\Sigma^2 V^T)^{-1} V \Sigma U^T b \quad (\Sigma^T = \Sigma)$

$$= V\Sigma^{-2} \underbrace{V^T V}_{I} \Sigma U^T b \qquad (V^T = V^{-1})$$

$$\boxed{= V\Sigma^{-2} U^T b}$$

vector form

(length of this is min length solution)

when th $(A^T A)^{-1}$ not possible)