IT584 Approximation Algorithms	January 25, 2024

Lecture 8

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1 Proof of Chernoff Bound and Beyond

Consider a set of independent random variables, $x_1, \ldots, x_n$, which may not be identically distributed. Let x_i take values either 0 or a_i where $0 < a_i \leq 1$.

For the random variable $X = \sum_{i=1}^{n} x_i$ and expected value $\mu = \mathbb{E}[X]$, and given $L \leq U$ and $\delta > 0$:

$$Pr[X \geq (1+\delta)U] < \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{U} \tag{1}$$

$$Pr[X \leq (1-\delta)L] < \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{L} \tag{2}$$

Proof

We will establish the equation (1). If $\mathbb{E}[X] = 0$, the bound trivially holds since $X = 0$. Therefore, we assume $\mathbb{E}[X] > 0$ and $\mathbb{E}\left[X_i\right] > 0$ for some i.

Exclude all i with $\mathbb{E}[X] = 0$.

Let $p_i = \Pr\left[X_i = a_i\right]$. Since $E\left[X_i\right] > 0$, we have $p_i > 0$.

$$\mu = \mathbb{E}[X] = \sum_{i=1}^{n} p_i a_i \leq U$$

For any $t > 0$,

$$\Pr[X \geq (1+\delta)U] = \Pr\left[e^{tX} \geq e^{t(1+\delta)U}\right]$$

By Markov's inequality,

$$\Pr\left[e^{tX} \geq e^{t(1+\delta)U}\right] \leq \frac{E\left[e^{tX}\right]}{e^{t(1+\delta)U}}$$

Now,

$$\begin{aligned}
\mathbb{E}[e^{tX}] &= \mathbb{E}\left[e^{t\sum_{i=1}^{n} x_i}\right] \\
&= \mathbb{E}\left[\prod_{i=1}^{n} e^{tx_i}\right] \\
&= \prod_{i=1}^{n} \mathbb{E}[e^{tx_i}]
\end{aligned} \tag{3}$$

$$\begin{aligned}
\mathbb{E}[e^{tx_i}] &= (1-p_i) + p_i e^{ta_i} \\
&= 1 + p_i(e^{ta_i} - 1)
\end{aligned} \tag{4}$$

Consider,

$$f(t) = a_i(e^{t-1}) - (e^{ta_i} - 1)$$

$$f'(t) = a_i e^t - a_i e^{ta_i} \geq 0$$

$f(t)$ is non-decreasing for $t \geq 0$.

$$e^{ta_i} - 1 \leq a_i(e^t - 1)$$

$$\mathbb{E}[e^{tx_i}] \leq 1 + p_i a_i(e^t - 1)$$

$$\mathbb{E}[e^{tx_i}] \leq e^{p_i a_i (e^t - 1)} \tag{5}$$

Now, substituting equation (5) in equation (3),

$$\begin{aligned}
\mathbb{E}[e^{tX}] &< \prod_{i=1}^{n} e^{p_i a_i (e^{t-1})} \\
&= e^{\sum_{i=1}^{n} p_i a_i (e^{t-1})} \\
&\leq e^{U(e^{t-1})}
\end{aligned} \tag{6}$$

Let $t = \ln(1+\delta) > 0$,

$$\begin{aligned}
\Pr(X \geq (1+\delta)U) &\leq \frac{\mathbb{E}[e^{tX}]}{e^{t(1+\delta)U}} \\
&< \frac{e^{\varepsilon e^{-1} U}}{e^{t(1+\delta)U}} \quad \text{(From equation (6))} \\
&= \frac{\varepsilon e^{\delta U}}{(1+\delta)(1+\delta)}
\end{aligned}$$

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