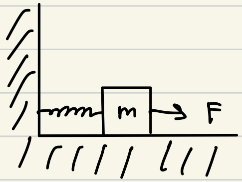
Forced oscillator :

→ Imagine a mass on a spring which is subjected to an external force, which varies.



$$\sum F = ma$$

$$\Rightarrow -kx + F(t) = m\ddot{x}$$

$$\Rightarrow \ddot{x} + \omega_0^2 x = \frac{F(t)}{m}$$

let's assume $F(t)$ varies with $F_0 \cos(\omega_f t)$, where 'ω_f' is the frequency of the motor which creates the force. It can be any real number and can be varied by changing the motor rotation speed.

We can solve this by using complex exponentials. Let's assume a world in which 'x' & 'F' are complex :

$$\therefore \; x \equiv x_r + i x_i .$$

$$\& \; F \equiv F_r + i F_i .$$

$$\therefore \frac{d^2}{dt^2}(x_r + i x_i) + \omega_0^2 (x_r + i x_i) = \frac{F_r + i F_i}{m}$$

$$\Rightarrow \left[\frac{d^2 x_r}{dt^2} + \omega_0^2 x_r - \frac{F_r}{m} \right] + i \left[\frac{d^2 x_i}{dt^2} + \omega_0^2 x_i - \frac{F_i}{m} \right] = 0$$

$$\therefore \; \frac{d^2 x_r}{dt^2} + \omega_0^2 x_r = \frac{F_r}{m} \; ; \; \frac{d^2 x_i}{dt^2} + \omega_0^2 x_i = \frac{F_i}{m}$$

Hence, By solving the "complex" equation of motion, we are solving the true physics equations. It is both the real and complex component.

So, let's assume a driving force which looks like:
$F(t) = F_0 \exp(i\omega_f t)$, which really means $F(t) = F_0 \cos(\omega_f t)$.
Lets just pretend 'x' is a complex variable, Now we solve:

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \exp(i\omega_f t)$$; Recall $\omega_f \neq \omega_0$ in general.

Let's guess the solution for this: trying our old solution:
$$X(t) = A \exp[i(\omega_0 t + \delta)]$$, which really means $x(t) = A\cos(\omega_0 t + \delta)$.

Here, we are guessing that the frequence of the oscillator does not change and still goes at it's old natural frequency ω_0.

$\therefore\ \ddot{x} = -\omega_0^2 x \rightarrow$ Substituting this in our diff eq

$$\therefore -\omega_0^2 x + \omega_0^2 x \stackrel{?}{=} \frac{F_0}{m} \exp(i\omega_f t).$$

$\therefore F_0 = 0$
↳ This means that there is no driving force.

We need a better better guess: Let's assume that it oscillates at the driving frequency 'ω_f'.

$$\therefore\ x(t) = A \exp\left[i(\omega_f t + \delta)\right]$$

$$\& \ \ddot{x} = -\omega_f^2 x = -\omega_f^2 \left[A \exp\left(i(\omega_f t + \delta)\right)\right].$$

$$\Rightarrow -\omega_f^2\left[A\exp\left(i(\omega_f t+\delta)\right)\right] + \omega_0^2\left[A\exp\left(i(\omega_f t+\delta)\right)\right] = F_0\exp(i\omega_f t).$$

$$\Rightarrow A\exp(i\delta)\left(\omega_0^2-\omega_f^2\right) = \frac{F_0}{m}$$

$$\Rightarrow \boxed{A = \frac{F_0\exp(-i\delta)}{m(\omega_0^2-\omega_f^2)}}.$$

A has to be a real number, this is because it is the amplitude of a complex number in polar form.

$\therefore \exp(-i\delta) \rightarrow$ real

$\therefore \cos(\delta) - i\sin(\delta) \rightarrow$ real

$\therefore \sin(\delta) = 0$

$\Rightarrow \delta = 0$

Hence, the phase shift is 0. What does this mean? By assumption, the driving force has a phase of 0: $F(t) = F_0\exp(-i\omega_f t)$

↳ we can observe there is no phase shift.

This means that we have chosen that at $t=0$ the force must be the maximum. However, we have allowed the position of the oscillator to have some phase: $x(t) = A\exp\left[i(\omega_f t+\delta)\right]$

↳ we allow the oscillator to have a non zero phase at $t=0$.

But after we substitute this into the diff eq, we find that for this solution to work, the phase shift of the oscillation has to be zero. Hence, this moves in phase with the force equation, i.e., when the force is max, so is our position. But there is a caveat:

our result is: $A = \frac{F}{m(\omega_0^2 - \omega_f^2)}$. our guessed solution works

but the amplitude is no longer a free parameter determined by initial conditions. Instead, the amplitude is fixed by the magnitude of the driving force, the mass, the driving frequency and the natural frequency

Amplitude vs driving force:

The amplitude depends on how close the driving frequency is to the natural frequency.

Comments:

→ If the driving force is large compared to ω_0, then A is small & -ve. This means that it is 180° out of phase with the driving force.

→ If driving frequency is similar to the natural frequency ω_0, then the amplitude of oscillation will be large. This is an example of resonance.

Forced Oscillator with damping:

→ A harmonic oscillator with both damping and driving force.

Eq of motion: $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\exp(i\omega_f t)$.

Guessed solution: lets try the solution we previously tried:

$$x(t) = A\exp\left[i(\omega_f t + \delta)\right]$$

$\therefore$ Substituting $\ddot{x} = -\omega_f^2 x$, $\dot{x} = i\omega_f x$

$$\Rightarrow -\omega_f^2 x + i\omega_f \gamma x + \omega_0^2 x = \frac{F_0}{m}\exp(i\omega_f t)$$

$A\exp[i\omega_f t + i\delta]$

$$\Rightarrow \left[-\omega_f^2 + i\omega_f + \omega_0^2\right] A\exp(i\delta) = \frac{F_0}{m}$$

$$\Rightarrow A\left[\omega_0^2 - \omega_f^2\right] + A\left[i\omega_f\right] = \frac{F_0}{m}\exp(-i\delta) = \frac{F_0}{m}\left[\cos\delta - i\sin\delta\right]$$

$$\therefore A\left[\omega_0^2 - \omega_f^2\right] = \frac{F_0}{m}\cos\delta$$

$$\& \quad A\omega_f\gamma = -\frac{F_0}{m}\sin\delta$$

$$\therefore -\tan\delta = \frac{\omega_f\gamma}{\omega_0^2 - \omega_f^2} \Rightarrow \boxed{\delta(\omega_f) = -\tan^{-1}\left(\frac{\omega_f\gamma}{\omega_0^2 - \omega_f^2}\right)}$$

↳ this is a function of ω_f.

We can solve for 'A' by squaring and adding:

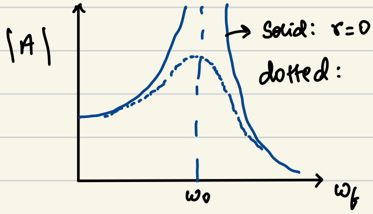
$$\therefore \frac{m^2 A^2}{F_0^2}\left(\omega_0^2 - \omega_f^2\right)^2 + \frac{m^2 A^2 \omega_f^2 r^2}{F_0^2} = 1$$

$$\Rightarrow A^2 = \frac{F_0^2}{m^2\left[(\omega_0^2-\omega_f^2)^2 + (\omega_f r)^2\right]}.$$

$$\therefore \boxed{A(\omega_f) = \frac{F_0}{m\sqrt{(\omega_0^2-\omega_f^2) + (\omega_f r)^2}}.}$$

as a function.

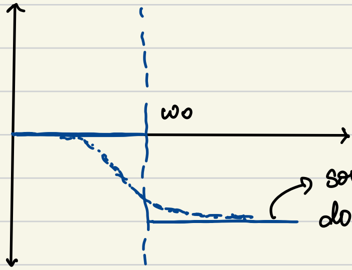
Similarly, we have a frequency where we observe resonance: $\omega_f \to \omega_0$.



solid: $r=0$ (no damping).
dotted: r is non zero (damping exists).

<u>Phase shift</u>: this is the phase shift of the oscillator relative to the driving force.

plot:



solid: $r=0$ (no damping).
dotted: $r \neq 0$ (damping).

The phase shift is negative: the oscillator lags behind the forcing function. When $\omega_f \approx \omega_0$, the phase shift is about -90°

<u>Transient behaviour:</u>

Till now, we have only been studying the long term behaviour of the forced oscillator: the motion of the oscillator once it settles into a repeating pattern. The long term behaviour does not depend on the initial conditions. So, there are no free parameters in the long term solution.

The initial conditions will affect the short term behaviour. This is called the transient solution. Any system with non-zero damping will have its transient behaviour die out as time goes forward, leaving only the long term solution. However if the damping coefficient is small, it may take a long time for the transient solution to go away

How can we study transient behaviour?
The trick is to make the following observation: consider a forced oscillator with damping, and separately a damped oscillator with no forcing.

	Forced w/ damping	Damped.
Eq:	$\ddot{x}_f + r\dot{x}_f + \omega_0^2 x_f = \frac{F_0}{m}\exp(i\omega_f t)$	$\ddot{x}_d + \gamma\dot{x}_d + \omega_0^2 x_d = 0$
Solution:	$x_f = \underbrace{A(\omega_f)}_{\text{amp. function}} \cdot \exp\left[i(\omega_f t + \underbrace{\delta_f}_{\text{phase shift}})\right]$	$x_d = B\exp(-\frac{\gamma t}{2})\exp\left[i(\omega_d t + \delta_d)\right]$
no. of parameters:	Zero	Two: B & δ_d.

→ Notice that if we add the forced solution to the damped solution, we get a new solution which also satisfies the forced equation of motion:

∴ General solution of forced oscillator with damping :

$$x(t) = \underbrace{x_f(t)}_{\text{forced solution}} + \underbrace{x_d(t)}_{\text{damped solution}}$$

$$= A(\omega_f)e^{i(\omega_f t + \delta_f)} + \underbrace{Be^{-i\gamma t/2}e^{i(\omega_d t + \delta_d)}}_{\text{two free parameters: } B,\ \delta_d}$$

<u>Lets show this works:</u>

Eq of motion: $\underbrace{\ddot{x}}_{(\ddot{x}_f + \ddot{x}_d)} + \gamma \underbrace{\dot{x}}_{(\dot{x}_f + \dot{x}_d)} + \omega_0^2 \underbrace{x}_{(x_f + x_d)} \overset{?}{=} \frac{F_0}{m}\exp(i\omega_f t)$.

$$\therefore \underbrace{(\ddot{x}_d + \gamma \dot{x}_d + \omega_0^2 x_d)}_{\text{this qty goes to zero.}} + (\ddot{x}_f + r\dot{x}_f + \omega_0^2 x_f) \overset{?}{=} \frac{F_0}{m}\exp(i\omega_f t).$$

$$\therefore \ddot{x}_f + r\dot{x}_f + \omega_0^2 x_f \overset{?}{=} \frac{F_0}{m}\exp(i\omega_f t)$$

→ Hence we know that the above is true (this is just the forced oscillator). Solution: x_f.

∴ generally:

$$\boxed{x(t) = A(\omega_f)e^{i(\omega_f t + \delta_f)} + Be^{-rt/2}e^{i(\omega_d t + \delta_d)}}$$

→ 'B' & 'δ_d' should be chosen so that the initial conditions are satisfied.

→ Notice that the second term dies out exponentially as time goes forward. This is the transient part of the solution.

→ After the transient part dies out, we are just left with the long term solution, ie, the first part.

<u>Simple example</u>: Suppose we start with: $x(t=0)=0$ & $\dot{x}(0)=0$ and $\omega_f = \omega_0$.

lets assume: $\omega_d = \sqrt{\omega_0^2 - \frac{r^2}{4}} \approx \omega_0$

$$\therefore A(\omega_f) = A(\omega_0) = \frac{F_0}{m\omega_0 r}$$

$$\tan(\delta_f(\omega_f)) = -\frac{\omega_0 r}{\omega_f^2 - \omega_0^2} = -\infty$$

$$\therefore \delta_f = -\frac{\pi}{2}$$

$$\therefore x(t) = \frac{F_0}{m\omega_0 r}\cos\left(\omega_0 t - \frac{\pi}{2}\right) + Be^{-rt/2}\cos(\omega t + \delta_d)$$

$$= \frac{F_0}{m\omega_0 r}\sin(\omega_0 t) + Be^{-rt/2}\cos(\omega t + \delta_d).$$

$$\therefore x(t=0) = 0 \Rightarrow Be^{-rt/2}\cos(\delta_d) = 0$$

$$\Rightarrow \underline{\underline{\delta_d = \pi/2}}$$

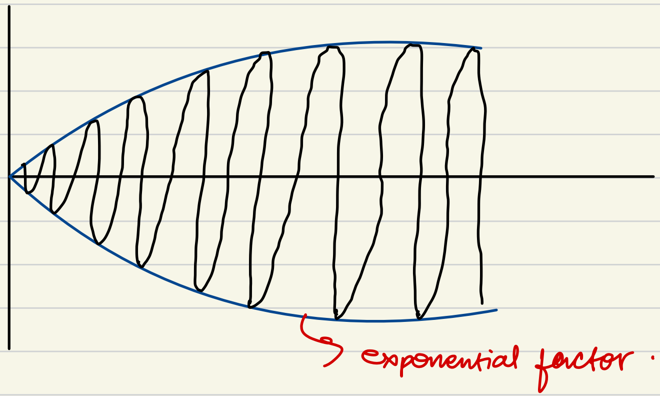
$$\dot{x}(0) = 0 = \frac{F_0}{mr}\cos(\omega t) + B\left(-\omega_0 \sin(\underset{0}{\omega_0 t} + \underset{\pi/2}{\delta_d})e^{-rt/2} - \frac{r}{2}\cos(\underset{0}{\omega_0 t} + \underset{\pi/2}{\delta_d})e^{-rt/2}\right)$$

$$\Rightarrow B\omega_0 = \frac{F_0}{mr} \Rightarrow \underline{\underline{B = \frac{F_0}{m\omega_0 r}}}$$

$\therefore\ x(t) = \frac{F_0}{m\omega_0\gamma}\sin(\omega_0 t) + \frac{F_0}{m\omega_0\gamma}e^{-\gamma t/2}\underbrace{\cos\left(\omega_0 t + \frac{\pi}{2}\right)}_{\hookrightarrow\ -\sin(\omega_0 t)}$.

$\therefore\ x(t) = \frac{F_0}{m\omega_0\gamma}\sin(\omega_0 t)\left(1 - e^{-\gamma t/2}\right)$

Graph:



Power :

$dW = F\,dx$

$\Rightarrow P = \frac{dW}{dt} = F\frac{dx}{dt} = FV$

damping force: $P_{damp} = (-b\dot{x})(\dot{x})$

$= -b\dot{x}^2$

$= -b(\omega A)^2\sin^2(\omega t + \phi)$

$\therefore\ \langle P_{damp}\rangle = \frac{1}{2}b(\omega A)^2$; $\langle \sin^2\theta\rangle = \frac{1}{2}$

driving force: $P_{driving} = F_{driving} \cdot V$

$$= (F_d \cos\omega_f t)(-\omega_f A \sin(\omega t + \phi))$$

$$\therefore P_{driving} = -F_d \omega_f A \underbrace{\cos\omega_f t \sin\omega_f t}_{1/2 \sin(2\omega_f t)} \cos\phi - F_d \omega_f A \sin\phi \cos^2\omega_f t$$

$$\therefore \langle P_{driving} \rangle = -\frac{1}{2} F_d \omega_f A \sin\phi = \frac{1}{2} b (\omega_f A \omega_f))^2$$

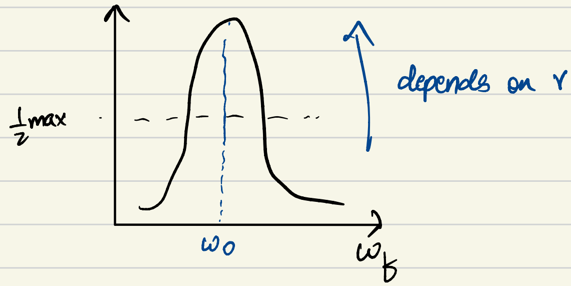
$$= -\frac{1}{2} F_d \omega_f A \left(-\frac{\gamma \omega_f A}{F_d}\right)$$

$$= \frac{1}{2} (\gamma m) \omega_f^2 \frac{(F_d/m)^2}{[(\omega_0^2 - \omega_f^2)^2 + \gamma^2 \omega_f^2]}$$

$$= \frac{\frac{1}{2}(\gamma m) F_d^2}{\gamma^2 m^2} \frac{\gamma^2 m^2}{[(\omega_0^2 - \omega_f^2)^2 + \gamma^2 \omega_f^2]}$$

$$= \frac{F_d^2}{2\gamma m} f(\omega_f)$$

$\therefore \frac{f(\omega_f)}{\gamma}$ in units of $\frac{F_d^2}{2m}$: $f(\omega_f) = \dfrac{1}{\frac{(\omega_0^2 - \omega_f^2)^2}{\gamma^2 \omega_f^2} + 1}$



f is max at $\omega_f = \omega_0$

when $(\omega_0^2 - \omega_f^2)^2 = \gamma^2 \omega_f^2 \Rightarrow \omega_f = \omega_1 , \ \omega_f = \omega_2$

$\Rightarrow \omega_0^2 - \omega_f^2 = \pm \gamma \omega_f$

$\Rightarrow \omega_0^2 - \omega_1^2 = \gamma \omega_1$

& $\omega_0^2 - \omega_2^2 = -\gamma \omega_2$

$\omega_2^2 - \omega_1^2 = \gamma(\omega_1 + \omega_2)$

$\Rightarrow \boxed{\omega_2 - \omega_1 = \gamma}$ → width of the graph.

Q value:

$Q \equiv \frac{\omega_0}{\gamma}$ → with no driving force: the number of cycles in which the amplitude decreases by $e^{-\pi} \approx 4.3\%$

when $\omega = \omega_0 \Rightarrow f$ is max $\Rightarrow \langle P_{driving} \rangle \rightarrow$ max. This frequency is known as "resonance frequency".

$$\frac{A_{resonance}}{A_{\omega_f \approx 0}} = \frac{\omega_0}{\gamma} = Q.$$