Homework 2 (PHYS273)

Muhammad Islam

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1.

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\begin{aligned}
\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0 \\
((A\alpha e^{-\alpha t}(\alpha\cos\omega t + \omega sin(\omega t))) \\
+(-Ae^{-\alpha t}(-\alpha\omega\sin\omega t + \omega^2 cos(\omega t)))) \\
+\gamma(-Ae^{-\alpha t}(\alpha\cos\omega t + \omega\sin(\omega t))) + \omega_0^2(Ae^{-\alpha t}\cos\omega t) = 0
\end{aligned}
$$

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\begin{aligned}
\Rightarrow (Ae^{-\alpha t}(\alpha(\alpha\cos\omega t + \omega\sin(\omega t)) - (-\alpha\omega\sin\omega t + \omega^2 cos(\omega t)))) \\
+\gamma(-Ae^{-\alpha t}(\alpha\cos\omega t + \omega\sin\omega t)) + \omega_0^2(Ae^{-\alpha t}\cos\omega t) = 0
\end{aligned}
$$

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\begin{aligned}
\Rightarrow (Ae^{-\alpha t}((\alpha^2 - \omega^2)\cos\omega t + (2\alpha\omega)\sin\omega t)) \\
+\gamma(-Ae^{-\alpha t}(\alpha\cos\omega t + \omega\sin\omega t)) + \omega_0^2(Ae^{-\alpha t}\cos\omega t) = 0
\end{aligned}
\tag{1}
$$

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\begin{aligned}
\Rightarrow Ae^{-\alpha t}((\alpha^2 - \gamma\alpha - \omega^2 + \omega_0^2)\cos\omega t + (\omega(2\alpha - \gamma))\sin\omega t) = 0 \\
\Rightarrow (\alpha^2 - \gamma\alpha - \omega^2 + \omega_0^2)\cos\omega t + (\omega(2\alpha - \gamma))\sin\omega t = 0 \\
\omega(2\alpha - \gamma) = 0 \\
\boxed{\alpha = \frac{\gamma}{2}}\text{(assuming } \omega \text{ isn't 0)} \\
\alpha^2 - \gamma\alpha - \omega^2 + \omega_0^2 = 0 \\
\frac{\gamma^2}{4} - \frac{\gamma^2}{2} - \omega^2 + \omega_0^2 = 0 \\
\boxed{\omega = \pm\sqrt{-\frac{\gamma^2}{4} + \omega_0^2}}
\end{aligned}
$$

2. (a) $\omega_0 = \sqrt{\frac{12}{3}} = 2$
$\gamma = \frac{1.5}{3} = 0.5$
$(\gamma = 0.5) < (2 * \omega = 4)$ This oscillator is under-damped.

(b) $e^{-\frac{\gamma}{2}t} = \frac{1}{e}$
$-\frac{\gamma}{2}t = -1$
$t = \frac{2}{\gamma} = \boxed{4 \text{ seconds}}$

(c) $e^{-\frac{\gamma}{2}t} = \frac{1}{e}$
$-\frac{\gamma}{2}t = -1$
$\boxed{t = \frac{2}{\gamma} = \frac{2m}{b}}$

(d) $e^{-\frac{\gamma}{2}t} = \frac{1}{2}$
$-\frac{\gamma}{2}t = \ln\frac{1}{2}$
$t = -\frac{2}{\gamma}\ln\frac{1}{2}$
$= -4\ln\frac{1}{2} \approx \boxed{2.77\text{seconds}}$ (< answer to part (b))

3. (a) $\omega = 0.9\omega_0 = \omega_0\sqrt{-\frac{\gamma^2}{4\omega_0^2} + 1}$
$0.9 = \sqrt{-\frac{\gamma^2}{4\omega_0^2} + 1}$
$-0.19 = -\frac{\gamma^2}{4\omega_0^2}$
$\gamma = 2\omega_0\sqrt{0.19} = \boxed{0.87\omega_0}$

(b) $Q = \frac{\omega_0}{\gamma}$
$\Rightarrow = \frac{\omega_0}{0.87\omega_0}$
$\Rightarrow = \boxed{1.15}$

(c) $Q = \frac{\omega_0}{\gamma} = 6$
$\omega = \omega_0\sqrt{-\frac{1}{4}(\frac{\gamma}{\omega_0})^2 + 1}$
$\omega = \omega_0\sqrt{-\frac{1}{4}(\frac{1}{6})^2 + 1}$
$\omega = \omega_0\sqrt{\frac{143}{144}} \approx 0.9965\omega_0$
, differing by about $\boxed{0.35\%}$

4. (a) $m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} \Rightarrow \frac{d^2x}{dt^2} + \underbrace{\frac{b}{m}}_{\gamma}\frac{dx}{dt} + \underbrace{\frac{k}{m}}_{\omega_0^2}x = 0$

(b) $\omega = \omega_0 * \frac{\sqrt{3}}{2} = \omega_0\sqrt{-\frac{b^2}{4\omega_0^2}+1}$

$\frac{\sqrt{3}}{2} = \sqrt{-\frac{\gamma^2}{4\omega_0^2}+1}$

$\frac{3}{4} = -\frac{\gamma^2}{4\omega_0^2}+1$

$\frac{\gamma^2}{4\omega_0^2} = \frac{1}{4}$

$\gamma = \frac{b}{m} = \omega_0 = \sqrt{\frac{k}{m}}$

$\boxed{b = m\sqrt{\frac{k}{m}} = 0.2\sqrt{\frac{80}{0.2}} = 4}$ $\to$ only γ

(c) $Q = \frac{\omega_0}{\gamma} = 1$ since $\gamma = \omega_0$

$Ae^{-\frac{b}{2}t} = Ae^{-\frac{b}{2}(\frac{10*2\pi}{\omega})} = Ae^{-\frac{4}{2}(\frac{10*2\pi}{4\frac{\sqrt{3}}{2}})} = Ae^{-(\frac{20\pi}{\sqrt{3}})} \approx \boxed{1.76 * 10^{-16}A}$

5. (a) $\frac{dx}{dt} = 2\pi v A\cos 2\pi vt$

$\frac{d^2x}{dt^2} = a = -4\pi^2 v^2 A\sin 2\pi vt$

$\frac{dE}{dt} = -\frac{Ke^2a^2}{c^3} = -\frac{Ke^2(-4\pi^2v^2A\sin 2\pi vt)^2}{c^3}$

$E\Big|_0^{\frac{1}{v}} = \int_0^{\frac{1}{v}} -\frac{Ke^2(-4\pi^2v^2A\sin 2\pi vt)^2}{c^3}$ $E\Big|_0^{\frac{1}{v}} = -\frac{16A^2Ke^2\pi^4v^4}{c^3}\int_0^{\frac{1}{v}}\sin^2 2\pi vt$

$E\Big|_0^{\frac{1}{v}} = -\frac{16A^2Ke^2\pi^4v^4}{c^3}\int_0^{\frac{1}{v}}\frac{1-\cos 4\pi vt}{2}$

$E\Big|_0^{\frac{1}{v}} = -\frac{16A^2Ke^2\pi^4v^4}{c^3}(\frac{1}{2}t-\frac{\sin 4\pi vt}{8\pi v})\Big|_0^{\frac{1}{v}}$ $E\Big|_0^{\frac{1}{v}} = -\frac{16A^2Ke^2\pi^4v^4}{c^3}(\frac{1}{2v}-\frac{\sin 4\pi}{8\pi v}) =$

$\boxed{-\frac{8A^2Ke^2\pi^4v^3}{c^3}\text{Joules}}$

(b) $Q = 2\pi\frac{E_{total}}{E_{displacedpercycle}}$

$Q = 2\pi\frac{\frac{1}{2}m\omega_0^2A^2}{\frac{8A^2Ke^2\pi^4v^3}{c^3}}$ (for small γ, which applies in this problem since the motion is close to $A\cos(2\pi vt)$)

$Q = \frac{mc^3\omega_0^2}{8Ke^2\pi^3v^3}$

$Q = \frac{mc^3(2\pi v)^2}{8Ke^2\pi^3v^3} = \boxed{\frac{mc^3}{2Ke^2\pi v}}$

(c) Let $v = 610 * 10^{12}$ (frequency of cyan light)

$Q = \frac{mc^3}{2Ke^2\pi v} = \frac{mc^3}{2Ke^2\pi(610*10^{12})} \approx \boxed{41694576.47}$

$\gamma = \frac{\omega_0}{Q} = \frac{2\pi*(610*10^{12})}{41694576.47} \approx 91924258.78$

$e^{-\frac{\gamma}{2}t} = 0.5$
$t = \frac{-2\ln 0.5}{\gamma} \approx \boxed{1.51 * 10^{-8} seconds}$