Physics 273 — Homework 6 Solutions

Question 1 (10 points)

An RLC electrical circuit which is NOT driven by an AC voltage source is mathematically equivalent to an un-driven mechanical oscillator.

(a) Write down the standard differential equation for $Q(t)$ for a generic RLC circuit. – no points

We worked this out in class: $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$

(b) Now consider the three electrical properties (resistance, capacitance, inductance) and their reciprocals $\left(\frac{1}{R}, \frac{1}{C}, \frac{1}{L}\right)$. Look at where they appear in your differential equation from part a, and deduce from that how they relate to the properties of a mechanical oscillator. From among those six quantities, – 3 points

(i) Which plays the role of the mass of the oscillator? L, the inductance

(ii) Which plays the role of the spring constant (k) for the oscillator? $\frac{1}{C}$, the reciprocal of the capacitance

(iii) Which plays the role of the damping constant (b) for the oscillator? R, the resistance

(c) Examining the differential equation can also tell you how the units (dimensions) of the electrical properties are related, and how they can be combined. – 3 points

(i) What combination of R and L (e.g. a product or a ratio) has dimensions of time?

All terms in the differential equation must have the same dimensions, and remember that each time derivative has units of $\frac{1}{\text{time}}$. So $\frac{[L]}{(\text{time})^2} = \frac{[R]}{\text{time}} = \frac{1}{[C]}$. From that we can see that $\frac{[L]}{[R]} = \text{time}$, i.e. the ratio $\frac{L}{R}$ has dimensions of time.

(ii) What combination of R and C has dimensions of time?

From $\frac{[R]}{\text{time}} = \frac{1}{[C]}$ above, we see that RC has dimensions of time

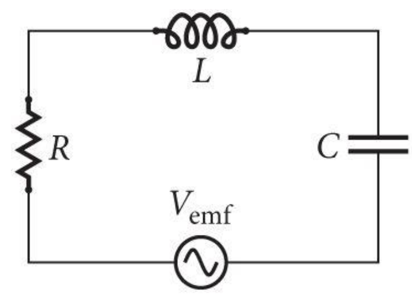
(iii) What combination of L and C has dimensions of time?

From $\frac{[L]}{(\text{time})^2} = \frac{1}{[C]}$, we see that $\sqrt{LC}$ has units of time.

(d) The combinations in part c define time scales, so the dynamics of an un-driven circuit will be determined by those. Each combination can therefore be called a "time constant", but context determines which one is relevant for a given circuit. For example, consider a circuit containing a resistance R and a capacitance C, and no inductor. The combination you found in (c)(ii) above has dimensions of time, so call that combination τ (tau). If you connect the circuit to a constant voltage V, such as a battery, then τ sets the scale for how long it takes the capacitor to charge up. Specifically, it will approach a maximum charge $Q_{max} = CV$ in an exponential fashion: $Q(t) = Q_{max}\left(1 - e^{-t/\tau}\right)$. Now let's put in some numbers to try this out… if the circuit has a 21.0 μF capacitor and a 140 Ω resistor, then what is τ (with units)? And how long will it take the capacitor to charge up to 99.9% of Q_{max} ? (Hint: your answer should be between 0.01 and 0.05 seconds.) – 4 points

$\tau = RC = (140\ \Omega)(21.0 \times 10^{-6}\ \text{F}) = 0.00294$ s. Now, we're looking for the time when $Q(t) = 0.999\ Q_{max}$, which implies that $e^{-t/\tau} = 0.001$. Taking the natural logarithm of both sides, $-t/\tau = -6.908$, so $t = 0.0203$ s.

Question 2 (24 points)



A series RLC circuit has component values $R = 8.9\ \Omega$, $L = 2.12$ H and $C = 12.0\ \mu$F, as well as an oscillating voltage (emf) source of the form $V(t) = (12.0\ \text{V}) \sin(\omega t)$ with a tunable frequency ω.

For the first five parts, suppose the frequency has been set to $\omega = 315$ rad/s.

(a) At this frequency, what is the reactance of the capacitor, and what is the reactance of the inductor? – 2 points

The reactance of a capacitor is $X_C = 1/\omega C$, so here it is $1/(315\ \text{rad/s})(12.0 \times 10^{-6}\ \text{F}) = 265\ \Omega$. The reactance of an inductor is $X_L = \omega L$, so here it is $(315\ \text{rad/s})(2.12\ \text{H}) = 668\ \Omega$.

(b) What is the magnitude of the impedance of the whole RLC circuit at this frequency? – 3 points

The complex impedance is $R + i\omega L + \frac{1}{i\omega C} = R + i\left(\omega L - \frac{1}{\omega C}\right) = (8.9\ \Omega) + i(668 - 265\ \Omega) = (8.9 + 403\ i)\ \Omega$. The magnitude of that is $|Z| = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(8.9)^2 + (403)^2} = 403\ \Omega$.

(c) What is the maximum current that flows in the circuit? (In other words, the amplitude of the oscillating current that flows in the circuit.) – 2 points

$I_0 = V_0/|Z| = (12.0\ \text{V})/(403\ \Omega) = 0.0298\ \text{A} = 29.8\ \text{mA}$.

(d) What is the maximum voltage across the resistor as the circuit oscillates? – 2 points

I_0 is the same everywhere in the circuit; the voltage across a component is then given by Ohm's Law using the component's impedance (or reactance, the magnitude of a complex impedance) instead of ordinary resistance. Across the resistor, $\Delta V_{max} = I_0 R = (0.0298\ \text{A})(8.9\ \Omega) = 0.265$ V.

(e) What is the maximum voltage across the capacitor as the circuit oscillates? – 2 points

From part a, the reactance of the capacitor at this frequency is 265 Ω, so $\Delta V_{max} = I_0 X_C = 7.87$ V.

For the remaining parts, the frequency is tuned to the resonant frequency of the circuit.

(f) What is the resonant frequency of the circuit? Express your answer either as an angular frequency or as a cycles-per-second frequency, indicating which you have chosen. (Hint: one is more convenient than the other.) – 3 points

The resonant frequency as an angular frequency is
$\omega_0 = 1/\sqrt{LC} = 1/\sqrt{(2.12\text{ H})(12.0 \times 10^{-6}\text{ F})} = 198.3\text{ rad/s}$.
To get a cycles-per-second frequency, divide by 2π, giving 31.6 Hz.

(g) At this frequency, what is the reactance of the capacitor, and what is the reactance of the inductor? – 2 points

Capacitor: $X_C = 1/\omega C = 1/(198.3\text{ rad/s})(12.0 \times 10^{-6}\text{ F}) = 420.3\ \Omega$.
Inductor: $X_L = \omega L = (198.3\text{ rad/s})(2.12\text{ H}) = 420.3\ \Omega$. Same, as they should be at resonance!

(h) What is the magnitude of the impedance of the whole RLC circuit at this frequency?
And how does it compare to what you got in part b ? – 2 points

Because $X_L = X_C$, the impedance is simply $Z = R = 8.9\ \Omega$. That's a lot smaller than what we got in part b, which makes sense, because at resonance the inductor and capacitor cancel each other out and thus make no net contribution to the impedance.

(i) What is the maximum current that flows in the circuit? (In other words, the amplitude of the oscillating current that flows in the circuit.) How does it compare to what you got in part c ?
– 2 points

$I_0 = V_0/Z = (12.0\text{ V})/(8.9\ \Omega) = 1.35\text{ A}$. This is much larger than in part c !

(j) What is the maximum voltage across the resistor as the circuit oscillates? – 2 points

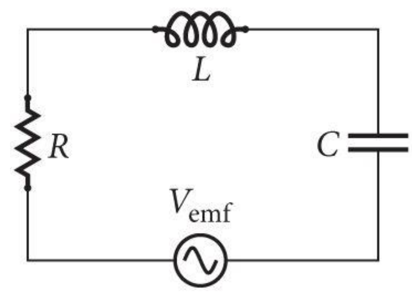
Across the resistor, $\Delta V_{max} = I_0 R = (1.35\text{ A})(8.9\ \Omega) = 12.0\text{ V}$. At resonance, the voltage across the resistor simply follows the oscillating V_{emf} .

(k) What is the maximum voltage across the capacitor as the circuit oscillates? – 2 points

From part g, the reactance of the capacitor at this frequency is $420.3\ \Omega$, so
$\Delta V_{max} = I_0 X_C = 567\text{ V}$!

In part k, you should have found that the amplitude of the voltage across the capacitor is significantly larger than the 12-volt amplitude driving the circuit! This is similar to how a mechanical oscillator can build up a large motion from continuous small "pushing" at the resonant frequency, kind of like "pumping" a playground swing with your legs.

Question 3 (20 points)



In the circuit shown, the resistance is $4.0\ \Omega$, the capacitance is 17 nF (i.e., 17×10^{-9} farad), and the inductance is 0.30 H. The frequency of the AC voltage source is not yet specified.

a) Calculate the resonant frequency of this circuit in Hz (cycles per second). — 3 points

$\omega_0 = \frac{1}{\sqrt{LC}} = 14003$ rad/s, but we want the cycles-per-second frequency: $f_0 = \frac{\omega_0}{2\pi} = 2229$ Hz.

b) What is the Q of this circuit? — 3 points

First we can calculate a "gamma" for the circuit, remembering the relationships that you worked out in Question 1: gamma is the damping coefficient (resistance) divided by the mass (inductance), i.e. $\gamma = \frac{R}{L} = 13.333$ s^{-1}. Then $Q = \omega_0/\gamma$ (being sure to use the angular frequency) $= 1050$. You could also combine things algebraically: $Q = \frac{\omega_0}{\gamma} = \frac{1}{R}\sqrt{\frac{L}{C}} = 1050$.

c) Suppose you suddenly switch on a driving voltage. How long (in seconds) do you need to wait for the transient component to have died away sufficiently? Explain the reasoning behind your calculation. There is not a single right answer for this, but we are looking for good reasoning supporting a calculated or approximated time. — 3 points

Recall that the transient solution has a multiplicative factor of $e^{-(\gamma/2)t}$ in each of its terms. Thus the "time constant" for the transient component dying away is $2/\gamma$, i.e. after $2/\gamma$ seconds, the amplitude has been reduced by a factor of $1/e$. $2/\gamma = 2L/R$ is equal to 0.15 s. However, to really let the transient die away, you would need to wait several times the time constant, maybe 5 or 10 times – so, something like 0.75 s or 1.5 s. To quantify that, after 5 "time constants" the initial amplitude will have been reduced by $e^{-5} \approx 0.0067$, and after 10 time constants the amplitude will be reduced by $e^{-10} \approx 0.000045$.

Previously, you used dimensional analysis to identify L/R as some sort of a time constant. Although that differs by a factor of 2, it is also good reasoning to figure that you need to wait several times L/R.

d) Calculate the amplitude of the oscillating voltage across the capacitor if the voltage source is set to the circuit's resonant frequency (what you found in part a) with an amplitude of 0.075 V. — 4 points

At the resonant frequency (f_0 or ω_0), the reactances of the capacitor and the inductor cancel, so that the impedance of the circuit is simply the resistance R. Then the AC current flowing in the circuit has amplitude (ignoring phase) $|I_0| = |V_0|/|Z| = V_0/R = 0.01875$ amps. The reactance of the capacitor at this frequency is $1/\omega C = 4201\ \Omega$, so the voltage amplitude across it is equal to that times the current amplitude, 78.77 V.

e) Calculate the amplitude of the oscillating voltage across the capacitor if the voltage source is set to $f = 2500$ Hz with an amplitude of 0.075 V. — 4 points

We need the angular frequency for our impedance calculation: $\omega = 2\pi f = 15708$ rad/s. At this frequency, plugging in values, the complex impedance is $Z = R + i\left(\omega L - \frac{1}{\omega C}\right) = 4.0 + i(4712.4 - 3744.8)\ \Omega = (4.0 + 967.6\, i)\ \Omega$. To get the magnitude of that, use the Pythagorean theorem. Then calculate $|I_0| = \frac{|V_0|}{|Z|} = 7.75 \times 10^{-5}$ amps. Finally, the voltage amplitude across the capacitor is $|I_0| X_c = 0.290$ V.

f) At $f = 2500$ Hz, calculate the phase of the oscillating current flowing in the circuit relative to the driving voltage. That is, if the driving voltage is $V(t) = Re\{V_0 e^{i\omega t}\}$ and $I(t) = Re\{I_0 e^{i\omega t}\}$, find the complex phase of (I_0/V_0) . Express your answer as a number of radians with 4 decimal places. — 3 points

Using complex values instead of just amplitudes, $I_0 = V_0/Z$ so $I_0/V_0 = 1/Z$. Therefore the complex phase of (I_0/V_0) equals the complex phase of $1/Z$, which is the complex phase of Z^*, the complex conjugate of Z. Switching the sign on the imaginary part in what we got in part e, $Z^* = (4.0 - 967.6\, i)\ \Omega$ and the complex phase of that is $\tan^{-1}(-967.6/4.0) = -1.5667$. That is very close to $-\pi/2$ (which is ≈ -1.5708), which makes sense because this is above the resonance frequency of the circuit and the circuit has a large Q. The inductor is the largest contributor to the overall impedance, and for a circuit with just an inductor, the current "follows" the voltage by a quarter cycle, i.e. the current has a relative phase of $-\pi/2$.