Physics 273 — Homework 4 Solutions

Question 1 (8 points)

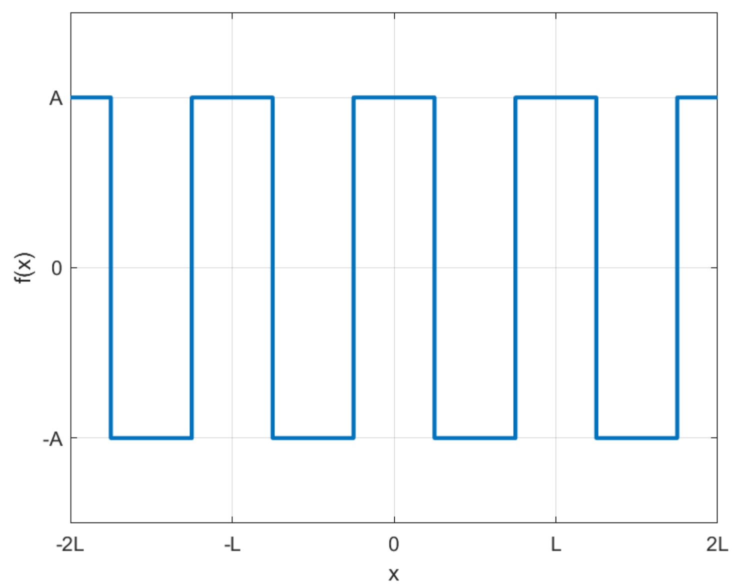
Consider a function defined by:

$$f(x) = \{A \quad for \ 0 \le x < \frac{L}{4} \quad -A \quad for \ \frac{L}{4} \le x < \frac{3L}{4} \quad A \quad for \ \frac{3L}{4} \le x < L$$

(This is similar to, but not the same as, the example Morin works on page 14 of Chapter 3. I recommend that you study Morin's example there, but be careful about the limits of integration.)

(a) Draw (by hand) or plot (with a computer) this function over the interval $-2L \le x \le 2L$. (Hint: remember, this is a periodic function.) — 2 points

I made this with Matlab, and looked up how to use the `xticks`, `xticklabels`, `yticks`, and `yticklabels` functions to make custom non-numeric labels:



(b) Find the values of all the Fourier series coefficients, i.e. a_0, a_n and b_n. In some cases you can provide an argument for why a certain coefficient (or set of coefficients) must equal zero instead of having to do an actual integral, but the integral will turn out that way too, if you do it. The nonzero coefficients can often be expressed as a function of n. — 6 points

Because this function has step discontinuities, we need to break up the integrals into segments based on the discontinuities. However, in some cases we can make a symmetry argument that a particular coefficient is zero.

The a_0 component is given by $a_0 = \frac{1}{L}\int_0^L f(x)dx$. In other words it is the average value of the function over the interval $0 \le x < L$. You can argue that the average value of the function above is zero, or else do the split-up integrals explicitly to find $a_0 = \frac{1}{L}\left[\frac{AL}{4} + \left(-\frac{AL}{2}\right) + \frac{AL}{4}\right] = 0$.

The a_n coefficients are for the cosines, which are even functions. This $f(x)$ is an even function too so we expect at least some of the a_n to be nonzero. Writing out the integral and then splitting it up,

$$a_n = \frac{2}{L}\int_0^L f(x) \cos cos \left(\frac{2\pi n}{L}x\right) dx$$

$$= \frac{2}{L}\left[\int_0^{L/4} A \cos cos \left(\frac{2\pi n}{L}x\right) dx + \int_{L/4}^{3L/4} (-A) \cos cos \left(\frac{2\pi n}{L}x\right) dx + \int_{3L/4}^{L} A \cos cos \left(\frac{2\pi n}{L}x\right) dx\right]$$

Pull out the common factor of A. The remaining integral is the same in all three cases, just with a minus sign for the middle one and with different limits of integration. Since

$\int \cos cos \left(\frac{2\pi n}{L}x\right) dx = \frac{L}{2\pi n} \sin sin \left(\frac{2\pi n}{L}x\right)$, we get

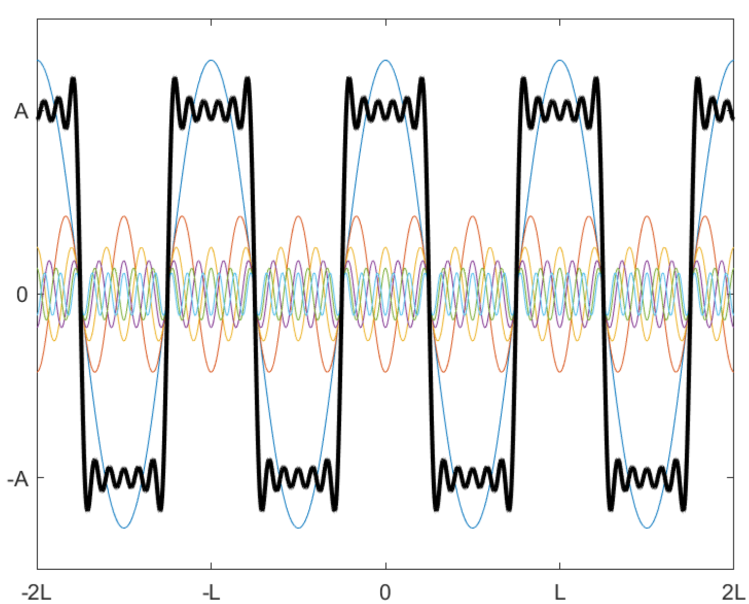
$$a_n = \frac{2A}{L}\left(\frac{L}{2\pi n}\right)\left[\sin sin \left(\frac{2\pi n}{L}x\right)\Big|_0^{L/4} - \sin sin \left(\frac{2\pi n}{L}x\right)\Big|_{L/4}^{3L/4} + \sin sin \left(\frac{2\pi n}{L}x\right)\Big|_{3L/4}^{L}\right]$$

$$= \frac{A}{\pi n}\left[\sin sin \left(\frac{n\pi}{2}\right) - \sin sin (0) - \sin sin \left(n\frac{3\pi}{2}\right) + \sin sin \left(n\frac{\pi}{2}\right) + \sin sin (2\pi n) - \sin sin \left(n\frac{3\pi}{2}\right)\right]$$

The $\sin sin (0)$ and $\sin sin (2\pi n)$ terms are zero. The other terms are in pairs at angles of $\frac{n\pi}{2}$ and $\frac{3n\pi}{2}$. The trickiest part of simplifying this is to see that the angle $\frac{3n\pi}{2}$ is equivalent to the angle $\frac{-n\pi}{2}$, and $\sin sin \left(\frac{-n\pi}{2}\right) = -\sin sin \left(\frac{n\pi}{2}\right)$ since sine is an odd function. So all four nonzero terms inside the brackets above are the same, and $a_n = \frac{A}{\pi n} 4 \sin sin \left(\frac{n\pi}{2}\right)$. This is an acceptable answer. However, we could go a bit farther because that sine is equal either to 0 or to ± 1, depending on the value of n. Specifically, $a_n = 0$ for all even n; $a_n = \frac{4A}{\pi n}$ if n is a multiple of 4 plus 1 (i.e. $n = 1, 5, 9, 13, \ldots$); and $a_n = -\frac{4A}{\pi n}$ if n is a multiple of 4 plus 3 (or equivalently, a multiple of 4 minus 1) (i.e. $n = 3, 7, 11, 15, \ldots$). Whew!

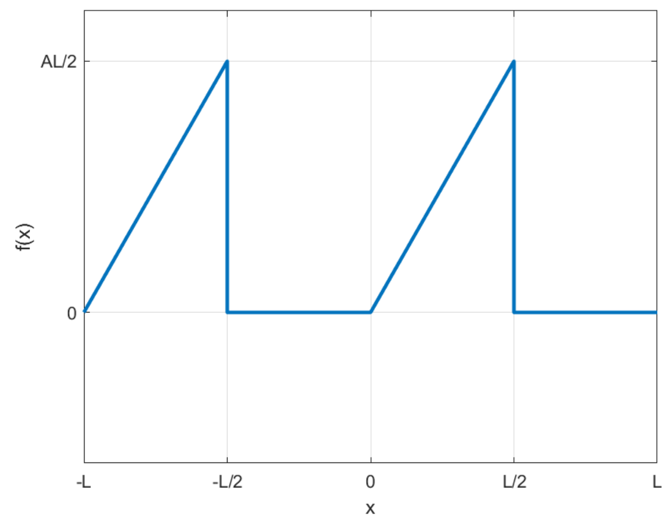
Finally, the b_n coefficients are all zero because those go with odd functions (sines) while $f(x)$ is even.

I decided to check my answer by plotting the first six nonzero cosine (a_n) terms, plus their sum (thicker black line): Looks about right.



<u>Question 4</u> (7 points)

Consider a sort of "half sawtooth" function, periodic with period L and defined by:



$$f(x) = \{Ax \quad for\ \ 0 \le x < \frac{L}{2}\ 0 \quad for\ \frac{L}{2} \le x < L$$

Two periods of this function are shown at right. Your job now is to calculate all of the Fourier series coefficients.

(Hint: remember that $\sin sin\ (n\pi)\ = \ 0$ for any n. Note that this is somewhat related to Morin's example on page 6 – so you may want to study that – but it is not the same. Some integral terms that were zero in Morin's example are nonzero for this function.)

The a_0 component is given by $a_0 = \frac{1}{L}\int_0^L f(x)dx$. In other words it is the average value of the function over the interval $0 \le x < L$. You can argue from simple geometry that the average value of the function above is clearly $\frac{1}{4}A$, or else do the integral explicitly:

$$a_0 = \frac{1}{L}\left[\int_0^{L/2} Ax\, dx\ +\ 0\right] = \frac{1}{L}\left[\frac{1}{2}Ax^2\right]_0^{L/2} = \frac{A}{2L}\left(\frac{L}{2}\right)^2 = \frac{AL}{8}.$$

The interval $L/2$ to L doesn't contribute anything in any of the integrals, so the a_n are given by:

$$a_n = \frac{2}{L}\int_0^{L/2} Ax \cos cos\left(\frac{2\pi n}{L}x\right) dx$$

This can be solved using integration by parts. Following Morin's example on page 6 but now for cosine, $\int x \cos cos\ (rx)\ dx\ = \frac{x}{r}\sin sin\ (rx)\ \ + \frac{1}{r^2}\cos cos\ (rx)$. So

$$a_n = \frac{2A}{L}\left[\frac{L}{2\pi n}x \sin sin\left(\frac{2\pi n}{L}x\right)\ +\left(\frac{L}{2\pi n}\right)^2 \cos cos\left(\frac{2\pi n}{L}x\right)\right]_0^{L/2}$$

$$= \frac{2A}{L}\left[\frac{L}{2\pi n}\frac{L}{2}\sin sin\,(\pi n) + \left(\frac{L}{2\pi n}\right)^2 \cos cos\,(\pi n) - 0 \sin sin\,(0) - \left(\frac{L}{2\pi n}\right)^2 \cos cos\,(0)\right]$$

Since $\sin sin\,(n\pi) = 0$, that simplifies to

$a_n = \frac{2A}{L}\left(\frac{L}{2\pi n}\right)^2(\cos cos\,(\pi n) - 1) = \frac{AL}{\pi^2 n^2}(\cos cos\,(\pi n) - 1)$. This is zero when n is even and equals $-AL/\pi^2 n^2$ when n is odd.

Similarly, the b_n are given by :

$$b_n = \frac{2}{L}\int_0^{L/2} Ax \sin sin\left(\frac{2\pi n}{L}x\right)dx$$

And using Morin's integration by parts,

$$b_n = \frac{2A}{L}\left[\frac{-L}{2\pi n}x \cos cos\left(\frac{2\pi n}{L}x\right) + \left(\frac{L}{2\pi n}\right)^2 \sin sin\left(\frac{2\pi n}{L}x\right)\right]_0^{L/2}$$

$$= \frac{2A}{L}\left[\frac{-L}{2\pi n}\frac{L}{2}\cos cos\,(\pi n) + \left(\frac{L}{2\pi n}\right)^2 \sin sin\,(\pi n) - 0 \sin sin\,(0) - \left(\frac{L}{2\pi n}\right)^2 \sin sin\,(0)\right]$$

Since $\sin sin\,(n\pi) = 0$, only one of those four terms is nonzero. It simplifies to $b_n = -\frac{AL}{2\pi n}\cos cos\,(\pi n)$ or, expressed another way, $b_n = -\frac{AL}{2\pi n}(-1)^n = \frac{AL}{2\pi n}(-1)^{n+1}$. Note that this is equal to half the b_n in Morin's example on page 6, which makes sense because here we have half of the sawtooth wave that was in that example and the two halves contributed equally to the odd (sine) component.