Physics 273 — Homework 1

Question 1

(a) Convert the number $4 + 6i$ to the form $Ae^{i\phi}$, i.e. find the (real-valued) magnitude A and the complex phase ϕ, with the latter measured in radians. (It's not a convenient fractional number of radians in this case, so you'll have to write out a decimal.)

(b) What is $5e^{i(0.7)\pi} - 4e^{i(0.3)\pi}$? (The complex phases of the two numbers are $0.7\,\pi$ radians and $0.3\,\pi$ radians.) Express your answer in the form $Ae^{i\theta}$, i.e. find numerical values of A and θ. (Hint: find numerical values of the real and imaginary parts—keeping several decimal places—and subtract those, then convert to the "polar" form.)

(c) Thinking about part b graphically, sketch the two original complex numbers as vectors on a complex number plane (based at the origin), or else use a computer to plot them accurately. Then sketch or plot the complex number you calculated in part b as a vector completing a triangle (i.e., not based at the origin). In other words, the third vector is the difference between the two original vectors, which is why there's a minus sign in part b. So, does this third vector in your sketch or plot look consistent with the numerical answer (magnitude and phase) you got in part b?

(d) Consider two general complex numbers $Ae^{i\alpha}$ and $Be^{i\beta}$. Generalizing what you did in part b and visualized in part c, subtract one from the other and square the magnitude of that to explicitly prove the *law of cosines*: $C^2 = A^2 + B^2 - 2Acos\,\theta$ where $\theta = \alpha - \beta$ is the angle between the vectors and C is the length (magnitude) of the third vector.

Question 2 (3 points)

Expand $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$ using Euler's formula on both sides. Use your result to prove the standard trig sum formulas: $sin\,(\alpha + \beta) = ...$ and $cos\,(\alpha + \beta) = ...$.

Question 3 (3 points)

Re-express the sinusoidal function $x(t) = 6cos\left(\omega t - \frac{\pi}{3}\right)$ in the form $x(t) = Re\left\{Ce^{i\omega t}\right\}$ such that they are equivalent functions. That is, find the value of the complex constant C, expressed in the form $a + bi$.

Question 4

(a) Let $\theta(t)$ be an arbitrary function of time. If $x(t) = Ae^{i\theta(t)}$, find $\frac{dx}{dt}$, i.e. the velocity $v(t)$. (Hint: use the chain rule.)

(b) For the special case that $\theta(t)$ grows linearly with time, i.e. $\theta(t) = \theta_0 + \omega t$, show that $v(t) = i\omega\, x(t)$. (Of course, the *actual* velocity is just the real part of the complex-valued $v(t)$.)

Question 5

Morin's "long way" of finding the motion of a simple harmonic oscillator in section 1.1.2 involved getting an expression for v as a function of x (equation 5), then writing v as dx/dt, separating those onto two sides of the equation, and integrating. (Morin has a typo: the left side of equation 6 is missing an integral sign.)

(a) Pick it up from there and go step-by-step, i.e. make the key trig substitution, do the integration, and carry it through to find the result of the form $x(t) = A\cos\,(\omega t + \phi)$ with $A = \sqrt{2E/k}$.

(b) Where did the ϕ come from? Explain.

(c) Starting with the expression for $x(t)$ from above, show explicitly that the total energy of the harmonic oscillator is conserved (i.e., independent of time) and is equal to E.