Q1) $X(t) = \mathrm{Re}\left[C(1+i)e^{i\omega t}\right]$

a) $C(1+i)e^{i\omega t} = C(1+i)(\cos\omega t + i\sin\omega t)$

$= C(\cos\omega t + i\sin\omega t + i\cos\omega t - \sin\omega t)$

$\therefore X(t) = \mathrm{Re}\left[C(\cos\omega t - \sin\omega t) + iC(\sin\omega t + \cos\omega t)\right]$

$= C(\cos\omega t - \sin\omega t)$

$= C\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\omega t - \frac{1}{\sqrt{2}}\sin\omega t\right)$

$= C\sqrt{2}\left(\cos\omega t \cos\frac{\pi}{4} - \sin\omega t \sin\frac{\pi}{4}\right)$

$= C\sqrt{2}\cos(\omega t + \frac{\pi}{4})$.

$\therefore$ amplitude of oscillation $= C\sqrt{2}$

b) $\cancel{C\sqrt{2}} = \cancel{C\sqrt{2}}\cos(\omega t + \frac{\pi}{4})$

$\Rightarrow \cos(\omega t + \frac{\pi}{4}) = 1$

$\therefore \omega t + \frac{\pi}{4} = 2n\pi \; ; \forall n \in \mathbb{Z}$

$\Rightarrow \omega t = \pi\left[2n - \frac{1}{4}\right]$

$\therefore t = \frac{\pi}{\omega}\left[2n - \frac{1}{4}\right]$

$C\sqrt{2}\cos(\omega t + \frac{\pi}{4}) = -C\sqrt{2}$

$\Rightarrow \cos(\omega t + \frac{\pi}{4}) = -1$

$\therefore \omega t + \frac{\pi}{4} = (2n+1)\pi \; ; n \in \mathbb{Z}$

$\Rightarrow t = \frac{\pi}{\omega}\left(2n + \frac{3}{4}\right)$

Hence at $t = \frac{\pi}{\omega}\left[2n - \frac{1}{4}\right]$, we have its max value.

Q2) $m = 1\,kg$, $K = 25\,\frac{N}{m}$, $b = 0.5\,\frac{Ns}{m}$, $\omega_0 = 5\,s^{-1}$, $F_0 = 10\,N$

a)

according to newton's second law: $\sum_i F_i = ma = m\ddot{x}$

$\therefore\ m\ddot{x} = -b\dot{x} - Kx + F_0 \sin(\omega t)$

$\Rightarrow\ m\ddot{x} + b\dot{x} + Kx = F_0 \sin(\omega t).$

$\Rightarrow\ \ddot{x} + r\dot{x} + \omega_0^2 x = F'\sin(\omega t).\ ;\ r \equiv \frac{b}{m},\ \omega_0^2 \equiv \frac{K}{m}$ & $F' \equiv \frac{F_0}{m}.$

lets assume the solution is of the form $Ce^{i\omega t}$, $C \equiv Ae^{i\phi}$

$\therefore\ \sin\omega t = \text{Im}(e^{i\omega t})\ ;\ x(t) = \text{Im}(Ce^{i\omega t}).$

$\therefore\ \ddot{x} = -\omega^2 Ce^{i\omega t},\ \dot{x} = i\omega Ce^{i\omega t}$

$\Rightarrow\ -\omega^2 C\cancel{e^{i\omega t}} + ri\omega C\cancel{e^{i\omega t}} + \omega_0^2 C\cancel{e^{i\omega t}} = F\cancel{e^{i\omega t}}$

$\Rightarrow\ C\left[\omega_0^2 - \omega^2 + ir\omega\right] = F'$

$$\therefore\ C = \frac{F'}{\omega_0^2 - \omega^2 + ir\omega} = \frac{F'(\omega_0^2 - \omega^2 - ir\omega)}{(\omega_0^2 - \omega^2)^2 + r^2\omega^2}$$

$$\therefore\ x(t) = \text{Im}(Ce^{i\omega t}) = \text{Im}\left[F'\left(\frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + r^2\omega^2} - i\frac{r\omega}{(\omega_0^2 - \omega^2)^2 + r\omega^2}\right)\left(\cos\omega t + i\sin\omega t\right)\right]$$

$$\boxed{x(t) = \frac{(\omega_0^2 - \omega^2)F'}{(\omega_0^2 - \omega^2)^2 + r^2\omega^2}\sin(\omega t) - \frac{(r\omega)F'}{(\omega_0^2 - \omega^2)^2 + r^2\omega^2}\cos(\omega t).}$$

$\omega_0 = 5, \quad r = 0.5$

$$\Rightarrow x(t) = \frac{(25-\omega^2)10}{(25-\omega^2)^2 + 0.25\omega^2}\sin(\omega t) - \frac{(0.5\,\omega)10}{(25-\omega^2)^2 + 0.25\omega^2}\cos\omega t$$

b) I previously defined $C \equiv Ae^{i\phi}$, hence $A = \sqrt{C \cdot C^*}$

$$\therefore A = \sqrt{F'^2\left(\frac{\omega_0^2 - \omega^2 + ir\omega}{(\omega_0^2-\omega^2)^2 + r^2\omega^2}\right)\left(\frac{\omega_0^2 - \omega^2 - ir\omega}{(\omega_0^2-\omega^2)^2 + r^2\omega^2}\right)}$$

$$= \sqrt{F'^2\left(\frac{(\omega_0^2-\omega^2)^2 + r^2\omega^2}{((\omega_0^2-\omega^2)^2 + r^2\omega^2)^2}\right)} = \sqrt{\frac{100}{(25-\omega^2)^2 + 0.25\omega^2}}$$

$$\boxed{A(\omega) = \frac{10}{\sqrt{(25-\omega^2)^2 + 0.25\omega^2}}}$$

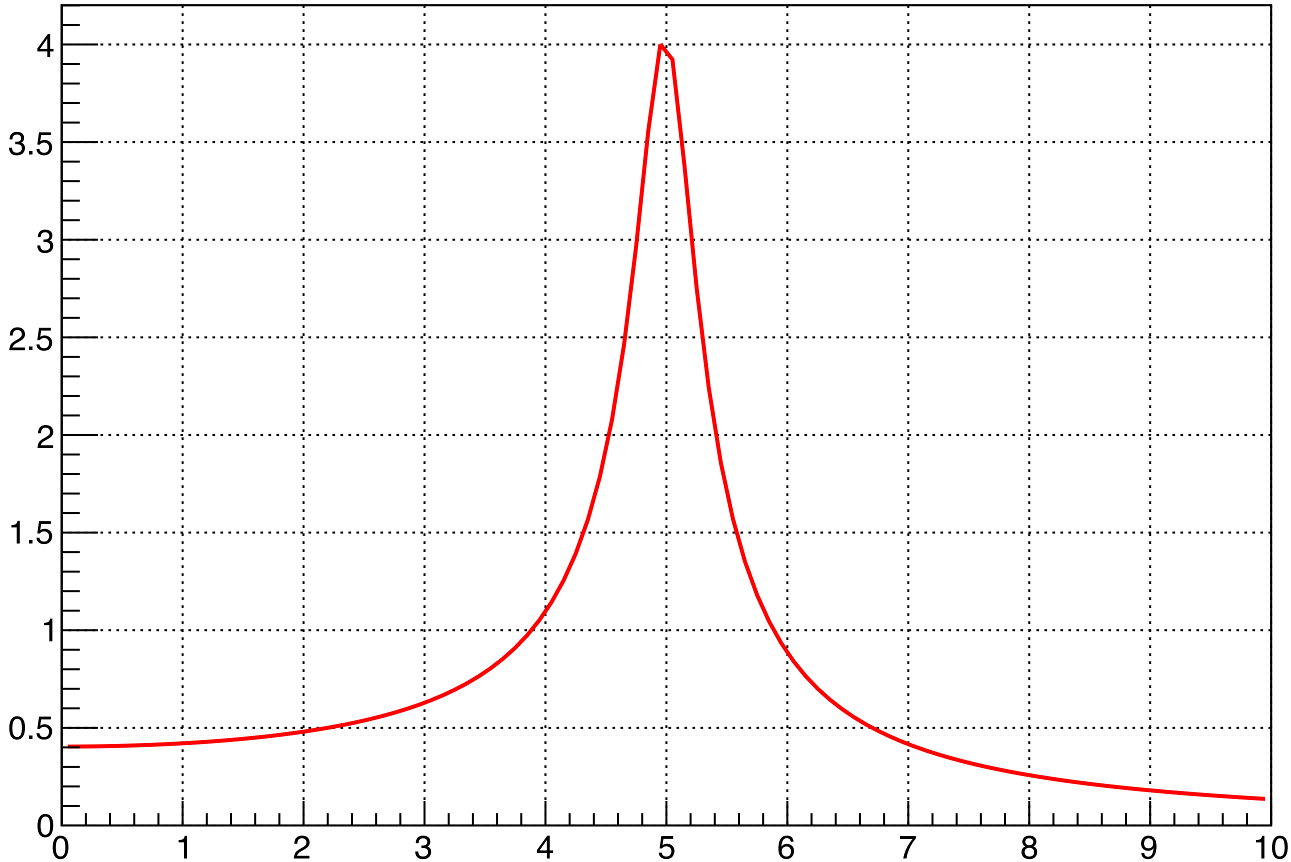
Amplitude as a function of angular frequency.

Code :

```
TCanvas *c1 = new TCanvas();

TF1 *func = new TF1("func","[0]/sqrt(([1]-x^2)^2 + [2]*x^2)",0,10);

func->SetTitle("Amplitude as a function of Angular Frequency");
c1->SetGridx();
c1->SetGridy();

func->SetParameter(0,10.1);
func->SetParameter(1,25.0);
func->SetParameter(2,0.25);

func->Draw();
c1->Draw();
c1->Print("273G1.pdf");
```

Graph on next page:

# Amplitude as a function of Angular Frequency



c) we know that the power that is dissipated is:

$$P = F \cdot v$$
$$F = -b\dot{x}, \quad v = \dot{x}$$

$$\Rightarrow P = -b(\dot{x})(\dot{x}) = -b(\dot{x})^2.$$

$$x = A\sin(\omega t + \phi)$$
$$\therefore \dot{x} = A\omega\cos(\omega t + \phi)$$

$$\therefore P = -b(A\omega)^2 \sin^2(\omega t + \phi)$$
$$\Rightarrow \langle P \rangle = -\frac{1}{2} b (A\omega)^2$$ → average of $\cos^2\theta$ in 1 cycle is $\frac{1}{2}$.

$$\therefore \langle P \rangle = -\frac{1}{2} rm\,\omega^2 \cdot \frac{F_0^2/m^2}{(\omega_0^2 - \omega^2)^2 + r^2\omega^2}$$

$$= -\frac{F_0^2}{2rm} \frac{r^2\omega^2}{(\omega_0^2-\omega^2)^2 + r^2\omega^2} = -\frac{F_0^2}{2rm} \frac{1}{\left(\frac{\omega_0^2-\omega^2}{r\omega}\right)^2 + 1} = -100 \cdot \frac{1}{\left(\frac{25-\omega^2}{0.5\omega}\right)^2 + 1}$$

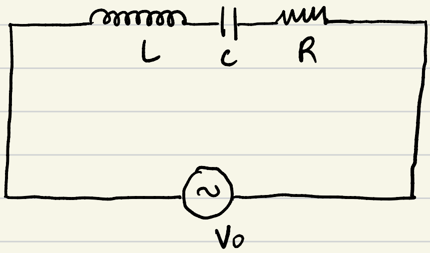
$$= -\frac{F_0^2}{2rm} \cdot f(\omega) \quad ; \quad f(\omega) \equiv \frac{1}{\left(\frac{25-\omega^2}{0.5\omega}\right)^2 + 1}$$

the average dissipative force would be maximum when the denominator of $f(\omega)$ is 1

$$\therefore \frac{25-\omega^2}{0.5\omega} = 0$$

$$\Rightarrow \omega = \omega_0 = 5s^{-1}$$

Q3)




$L = 2\times10^{-2}\,H$
$C = 8\times10^{-6}\,F$
$R = 1.6\times10^{-6}\,\Omega$
$V_0 = 0.2\cos(\omega t)$.

a) from Kirchoff's loop rule: $L\frac{di}{dt} + iR + \frac{Q}{C} = V_0$

we know, $\frac{d^2Q}{dt^2} = \frac{di}{dt}$, $i = \frac{dQ}{dt}$

$$\Rightarrow \frac{d^2Q}{dt^2} + \gamma\frac{dQ}{dt} + \omega_0^2 Q = V\cos\omega t$$

where $\gamma \equiv \frac{R}{L}$, $\omega_0^2 \equiv \frac{1}{LC}$, $V \equiv \frac{V_0}{L}$.

$$\Rightarrow \boxed{\frac{d^2Q}{dt^2} + (8\times10^{-5})\frac{dQ}{dt} + (6.25\times10^{6})Q = 10\cos\omega t}$$

b) the differential equation that we have above is analoguous to the damped & driven osillator:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = F_d\cos(\omega t).$$

Hence, the oscillation of charge is underdamped in this case.

lets assume $q(t) = Ce^{i\omega t}$, $C = Ae^{i\varphi}$ & $\cos(\omega t) = Re(e^{i\omega t})$

$$\dot{q} = Ci\omega e^{i\omega t}, \quad \ddot{q} = -\omega^2 Ce^{i\omega t}.$$

$$\therefore \; -\omega^2 \cancel{Ce^{i\omega t}} + ri\omega \cancel{Ce^{i\omega t}} + \omega_0^2 \cancel{Ce^{i\omega t}} = V \cancel{e^{i\omega t}}$$

$$\Rightarrow \quad C = \frac{V}{\omega_0^2 - \omega^2 + ri\omega} = \frac{V(\omega_0^2 - \omega^2 - ir\omega)}{(\omega_0^2 - \omega^2)^2 + r^2\omega^2}$$

$$\Rightarrow \; q(t) = Re\left[Ce^{i\omega t}\right]$$

$$q(t) = \frac{V(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + r^2\omega^2} \cos(\omega t) + \frac{r\omega V}{(\omega_0^2 - \omega^2)^2 + r^2\omega^2} \sin(\omega t)$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{16 \times 10^{-8}}} = \frac{10^4}{4} = 2.5 \times 10^3 \, s^{-1}$$

$$r = \frac{R}{L} = \frac{1.6 \times 10^{-6}}{2 \times 10^{-2}} = 0.8 \times 10^{-4} = 8 \times 10^{-5} \; \frac{\Omega}{H}$$

$$V = \frac{V_0}{L} = \frac{0.2}{2 \times 10^{-2}} = 10$$

$$\therefore \; \boxed{q(t) = \frac{10(6.25 \times 10^6 - \omega^2)}{(6.25 \times 10^6 - \omega^2)^2 + 6.4 \times 10^{-9}\omega^2} \cos \omega t + \frac{8 \times 10^{-4}\omega}{(6.25 \times 10^6 - \omega^2)^2 + 6.4 \times 10^{-9}\omega^2} \sin \omega t}$$

c) we know: $\langle P_{driving} \rangle = \frac{1}{2} b(\omega A)^2 = \frac{1}{2} R(\omega A)^2$

$$= \frac{1}{2} R(\omega)^2 \frac{(V_0/L)^2}{(\omega_0^2 - \omega^2)^2 + r^2\omega^2}$$

$$= \frac{V_0^2 \, rL}{2r^2L^2} \; \frac{r^2\omega^2}{(\omega_0^2 - \omega^2) + r^2\omega^2}$$

$$\therefore \quad P = \frac{V_0^2}{2rL} \frac{r^2\omega^2}{(\omega_0^2-\omega^2)^2 + r^2\omega^2}$$

$$= \frac{0.04}{3.2 \times 10^{-6}} \frac{6.4 \times 10^{-9}\omega^2}{(6.25\times10^{6}-\omega^2)^2 + 6.4\times10^{-9}\omega^2}$$

$\therefore$ max power when $\omega = \sqrt{6.25 \times 10^6}$

$= 2.5 \times 10^3 \ s^{-1}$

max power $= \frac{0.04}{3.2} \times 10^6 = 1.25 \times 10^4$ watts.

e) let $f(\omega) = \dfrac{1}{\left(\dfrac{6.25\times10^6 - \omega^2}{8\times10^{-5}\cdot\omega}\right)^2 + 1}$

the denominator has to be 2 to get $\frac{P_{max}}{2}$

$\therefore \ (6.25 \times 10^6 - \omega^2)^2 = (8\times10^{-5}\omega)^2$

$\Rightarrow \ 6.25\times10^6 - \omega^2 = \pm 8\times10^{-5}\omega$

let $\omega_1$ & $\omega_2$ be the solutions to this equation

$\therefore \ \omega_1 \approx 2499.99996 \ s^{-1}$

$\omega_2 \approx 2500.00004 \ s^{-1}$

e) we find that $\omega_2 - \omega_1 \approx 8 \times 10^{-5} \ s^{-1} \approx r$

f) $Q = \frac{\omega_0}{r} = \frac{2500}{8\times10^{-5}} = 3.125 \times 10^7$