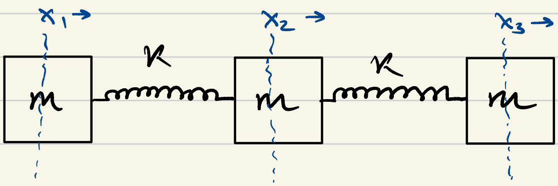

a) for mass 1:

$$m\ddot{x}_1 = -K(x_1 - x_2)$$

for mass 2:

$$m\ddot{x}_2 = -K(x_2 - x_1) - K(x_2 - x_3)$$

for mass 3:

$$m\ddot{x}_3 = -K(x_3 - x_2)$$

b) let's assume:
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} e^{i\omega t}$$

$$\therefore \quad \ddot{x}_n = -A_n \omega^2 e^{i\omega t}$$

the equations of motion then become:

①:

$$-m\omega^2 A_1 = -K(A_1 - A_2)$$

$$\Rightarrow A_1(-m\omega^2 + K) + A_2(-K) + A_3(0) = 0 \quad - (A)$$

②:

$$-m\omega^2 A_2 = -K(A_2 - A_1) - K(A_2 - A_3)$$

$$\Rightarrow A_1(-K) + A_2(-m\omega^2 + 2K) + A_3(-K) = 0 \quad - (B)$$

③

$$-m\omega^2 A_3 = -K(A_3 - A_2)$$

$$\Rightarrow A_1(0) + A_2(-K) + A_3(-m\omega^2 + K) = 0 \quad - (C)$$

Hence from equations (A), (B) & (C):

$$\begin{bmatrix} -m\omega^2+K & -K & 0 \\ -K & -m\omega^2+2K & -K \\ 0 & -K & -m\omega^2+K \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -\omega^2+\omega_0^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & -\omega^2+2\omega_0^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & -\omega^2+\omega_0^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = 0$$

c) $$\therefore \begin{vmatrix} -\omega^2+\omega_0^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & -\omega^2+2\omega_0^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & -\omega^2+\omega_0^2 \end{vmatrix} = 0$$

$$(-\omega^2+\omega_0^2)\left[(-\omega^2+2\omega_0^2)(-\omega^2+\omega_0^2) - \omega_0^4\right] + \omega_0^2\left[-\omega_0^2(-\omega^2+\omega_0^2)\right] = 0$$

$$\Rightarrow (-\omega^2+\omega_0^2)\left[(-\omega^2+2\omega_0^2)(-\omega^2+\omega_0^2) - 2\omega_0^4\right] = 0$$

$$\Rightarrow (-\omega^2+\omega_0^2)\left[\omega^4 - \omega^2\omega_0^2 - 2\omega^2\omega_0^2 + \cancel{2\omega_0^4} - \cancel{2\omega_0^4}\right] = 0$$

$$\Rightarrow \omega^2.(-\omega^2+\omega_0^2).(\omega^2 - 3\omega_0^2) = 0$$

$$\therefore \quad \omega = 0, \quad \omega = \pm\omega_0, \quad \omega = \pm\sqrt{3}\,\omega_0$$

when $\omega = 0$:

$$\therefore \begin{bmatrix} \omega_0^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & \omega_0^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} A_1 - A_2 + 0 \\ -A_1 + 2A_2 - A_3 \\ 0 - A_2 + A_3 \end{bmatrix} = 0$$

$$\therefore A_1 = A_2, \quad A_1 - 2A_2 + A_3 = 0, \quad A_2 = A_3$$

$$\therefore A_3 = A_1$$

Hence, $$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \propto \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

when $\omega^2 = \omega_0^2$:

$$\therefore \begin{bmatrix} 0 & -\omega_0^2 & 0 \\ -\omega_0^2 & \omega_0^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} 0 - A_2 + 0 \\ -A_1 + A_2 - A_3 \\ 0 - A_2 + A_3 \end{bmatrix} = 0$$

$$\therefore A_2 = 0 \quad \& \quad A_3 = -A_1$$

Hence, $\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \propto \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

when $\omega^2 = 3\omega_0^2$:

$$\therefore \begin{bmatrix} -2\omega_0^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & -\omega_0^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & -2\omega_0^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} -2A_1 - A_2 + 0 \\ -A_1 - A_2 - A_3 \\ 0 - A_2 - 2A_3 \end{bmatrix} = 0$$

$\Rightarrow A_2 = -2A_1 \quad \& \quad A_2 = -2A_3$

$\therefore A_1 = A_3$

Hence, $\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \propto \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Hence the 3 normal modes are:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$\omega^2 = 0 \qquad \omega^2 = \omega_0^2 \qquad \omega^2 = 3\omega_0^2$

d) Hence :

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{i\omega_0 t} + c_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-i\omega_0 t}$$

$$+ c_4 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{i\sqrt{3}\omega_0 t} + c_5 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-i\sqrt{3}\omega_0 t}$$

we know that x_1, x_2 & x_3 are real values, Hence :

$$c_2 = c_3^* \quad \& \quad c_4 = c_5^*$$

$$\Rightarrow \quad c_2 = \frac{B}{2} e^{i\alpha} \, , \quad c_3 = \frac{B}{2} e^{-i\alpha}$$

$$c_4 = \frac{c}{2} e^{i\beta} \, , \quad c_5 = \frac{c}{2} e^{-i\beta}$$

Hence:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + B \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cos(\omega_0 t + \alpha) + c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cos(\sqrt{3}\omega_0 t + \beta)$$