Homework 3

Note: In the calculations you're doing in these homework questions, keep at least 4 significant digits for all numerical values, and then round the final answer to 3 significant digits.

Problem 1

You have a harmonic oscillator with a mass of 0.22 kg, a spring with spring constant 17 N/m, and a dashpot damper. You don't know the damping coefficient b to begin with, but you happen to know the Q of the system to be 3.45.

(a) What is ω_0 for this oscillator, with proper units?

(b) From knowing the Q, determine the damping coefficient b in units of kg/s.

(c) If you pull the mass to an initial displacement of 10.0 cm, hold it there a moment and then release it, it will respond with damped oscillations. What is the angular frequency of these oscillations? How does it compare with ω_0 ?

(d) After exactly one oscillation cycle, what will be the mass's displacement in cm? (Hint: remember that the frequency you got in part c is an angular frequency. Your answer should be somewhere between 3 and 7 cm.)

(e) What will be the mass's displacement after one more oscillation cycle, i.e., a total of two cycles from when you let go of it?

(f) Now you attach a drive unit which applies a driving force of the form $F_d cos\,(\omega t)$ with force amplitude $F_d = 1.6$ N. Note that this is not the same as the F in the differential equation we've been working with, $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = Fe^{i\omega t}$. To start with, if $F_d = 1.6$ N, what is the value of F with proper units?

(g) Predict the resonant (angular) frequency – that is, the frequency which will give the largest-amplitude oscillations – for driven oscillations of this system (or from Morin or French). (Hint: the answer is $\approx 8.60\ rad/s$; make sure that is what your calculation gives you!)

(h) You set the drive unit to a low frequency, 0.25 rad/s, and wait for it to settle into its steady-state solution (i.e. you wait for the start-up transient to die away). Using the expression for the amplitude of $x(t)$ as a function of ω, calculate the amplitude of the steady-state oscillations at this frequency. (Include units, of course. Note that the units of the amplitude A are not the same as the units of F.)

(i) You double that frequency, to 0.50 rad/s. Calculate the amplitude of steady-state oscillations at this frequency. How similar is this to what you got in part h?

(j) You turn up the frequency to 8.70 rad/s. Calculate the amplitude of steady-state oscillations at this frequency.

(k) 8.70 rad/s is not exactly the frequency which maximizes the amplitude, but it is pretty close, so the amplitude you found in part j should be nearly the maximum amplitude. Calculate the ratio of that part-j amplitude to the amplitude at very low frequency (part h). This ratio should be approximately equal to the Q of the system – compare and see if it is!

(l) Now you turn up the frequency way up, to 20.0 rad/s. Calculate the amplitude of steady-state oscillations at this high frequency. How does this compare with the low-frequency amplitude you found in part h?

Problem 2

For a driven, damped harmonic oscillator described by $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = Fe^{i\omega t}$, Morin worked out that the maximum amplitude (at resonance) was $A = F/\gamma\omega_0$, but that was an approximation valid in the limit of small γ. Let's do this calculation more carefully.

(a) Go back to the exact expression for the amplitude A, do the math explicitly to show that it is maximized when $\omega = \sqrt{\omega_0^2 - \gamma^2/2}$. (Hint: use the chain rule correctly.)

(b) Use your result from part a to find the true maximum amplitude, not making any approximation.

Problem 3

A certain damped harmonic oscillator has mass 16 grams and spring constant $0.24\ N/cm$. (Careful about the units! The safest thing is to immediately convert those to base SI units.) It also has a damping force constant of $0.060\ kg/s$, and an oscillating driving force given by $(0.75\ N)cos\ (\omega t)$.

(a) What are ω_0 and Q for this oscillator? (Hint: ω_0 is somewhere near $40\ rad/s$.)

(b) That $0.75\ N$ in the description above is a force amplitude in newtons. What is the corresponding F in the notation we've used in Morin and in class?

(c) Use some graphing software (such as desmos, Matlab, etc.) to plot the magnitude and phase of the complex amplitude (what we called C in class) as a function of ω. You can either plot them both on the same plot or make separate plots; in either case, include a screenshot or saved image file in your homework submission. Be sure the horizontal axis range spans $\omega = 0$ to the resonance peak and somewhat beyond, so we can see the magnitude trending down toward zero at higher frequencies. (And negative frequencies are uninteresting, so you might as well set the axis range to start at $\omega = 0$.)

(d) Calculate what you expect the peak amplitude to be based on the properties of the oscillator and the driving force. You may use an approximation if you can say that the damping is small; just be sure to explain your reasoning. Then read the actual peak amplitude off the graph; do they agree?

(e) Calculate the average power dissipated by the damping force, in watts, when $\omega = 8.0\, rad/s$, i.e. far below the resonance frequency. (Hint: first calculate the amplitude of the motion, which Morin calls A but which is equal to $|C|$, at this frequency.)

(f) Calculate the average power dissipated by the damping force, in watts, when $\omega = 40\, rad/s$, i.e. near the resonance frequency.

(g) Finally, calculate the average power dissipated by the damping force, in watts, when $\omega = 200\, rad/s$, i.e. far above the resonance frequency.

Problem 4

Consider an "unconstrained particle": an object with mass m which has no spring and no damping force. It is, however, subjected to a sinusoidal force F_d cos $cos\,(\omega t)$.

(a) Write out the differential equation that comes from Newton's second law for this system.
(b) Express the force as a complex exponential and assume that there is a steady-state solution for the motion of the particle, $x(t) = Ce^{i\omega t}$. Put this into the differential equation, as we usually do, to find out what C must equal.
(c) Rearrange what you got in Q2 to find the **force-to-position transfer function**. Remember that a transfer function (TF) is the ratio $\frac{output}{input}$ for a sinusoidal input; here, that is $\frac{C}{F_d}$

[Since the $output$ coefficient may be complex (and in fact the $input$ coefficient can too), a TF can, in general, be a **complex** function of ω. That would indicate that the output has **a phase shift relative to the input**. However, the answer to Q3 happens to be a real function.]

(d) Compare your result against the TF of a damped, driven harmonic oscillator, which is a complex function of ω. Show that at high-enough frequencies, the damped harmonic oscillator behaves essentially the same as an unconstrained particle.