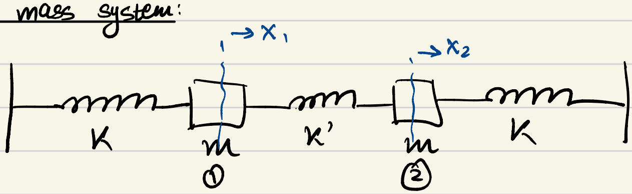
We speak about a multiple mass system.

Two mass system:



→ eqm position is when everything is at rest

① $m\ddot{x}_1 = -Kx_1 - K'(x_1 - x_2)$; if $x_1 < x_2 \Rightarrow x_1 - x_2 < 0$
& $-K(x_1 - x_2) > 0$.

② $m\ddot{x}_2 = -Kx_2 - K'(x_2 - x_1)$

first method:

adding the equations: $m(\ddot{x}_1 + \ddot{x}_2) = -K(x_1 + x_2)$

if $x_s = x_1 + x_2 \Rightarrow m\ddot{x}_s = -Kx_s$

$$\Rightarrow \ddot{x}_s = \frac{-K}{m} x_s = -\omega_s^2 x_s \; ; \; \omega_s^2 = \frac{K}{m}$$

$$\Rightarrow x_s = 2A_s \cos(\omega_s t + \phi_s) = x_1 + x_2$$

taking the difference: $m(\ddot{x}_1 - \ddot{x}_2) = -(K + 2K')(x_1 - x_2)$

if $x_1 - x_2 = x_f \Rightarrow x_f = 2A_f \cos(\omega_f t + \phi_f)$; $\omega_f^2 = \frac{K + 2K'}{m}$

↳ $x_1 - x_2$

$$\therefore \boxed{\begin{aligned} x_1 &= A_s \cos(\omega_s t + \phi_s) + A_f \cos(\omega_f t + \phi_f) \\ x_2 &= A_s \cos(\omega_s t + \phi_s) + A_f \cos(\omega_f t + \phi_f) \end{aligned}}$$

**
<u>Second method</u> :

→ we use complex numbers.

$$m\ddot{x}_1 = -Kx_1 - K'(x_1 - x_2).$$
$$m\ddot{x}_2 = -Kx_2 - K'(x_2 - x_1).$$

Lets Assume : $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{i\omega t}$

$\Rightarrow x_1 = A_1 e^{i\omega t}$; $x_2 = A_2 e^{i\omega t}$

$\therefore -m\omega^2 A_1 e^{i\omega t} = -KA_1 e^{i\omega t} - K'(A_1 - A_2)e^{i\omega t}.$

$\Rightarrow$ $\left\{ \begin{array}{l} -m\omega^2 A_1 = -KA_1 - K'(A_1 - A_2). \\ -m\omega^2 A_2 = -KA_2 - K'(A_2 - A_1). \end{array} \right\}$ &

$\Rightarrow (-m\omega^2 + K + K')A_1 - K'A_2 = 0$
& $-K'A_1 - (m\omega^2 + K + K')A_2 = 0$

$$\therefore \begin{bmatrix} (-m\omega^2 + K + K') & (-K') \\ (-K') & (-m\omega^2 + K + K') \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

If $\det(\quad) = 0$ → $A_1 = A_2 = 0$ → we dont want this.
↳ No Inverse → which we want

$\Rightarrow (-m\omega^2 + K + K') - K'^2 = 0$

$\Rightarrow -m\omega^2 + K + K' = \pm K'$

$\Rightarrow \omega_s^2 = \dfrac{K}{m}$ or $\omega_f^2 = \dfrac{K + 2K'}{m}$

$\Rightarrow \omega_s = \pm\sqrt{\dfrac{K}{m}}$ or $\omega_f = \pm\sqrt{\dfrac{K + 2K'}{m}}$

→ we plug in the value of ω into the matrices.
$\omega_s^2 = k/m$:

$$\therefore \begin{bmatrix} k' & -k' \\ -k' & k' \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

$$\Rightarrow \quad k'A_1 - k'A_2 = 0 \; ; \quad -k'A_1 + k'A_2 = 0$$

hence $A_1 = A_2$

$\omega_f^2 = (k+2k')/m$:

$$\begin{bmatrix} -k' & -k' \\ -k' & -k' \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

$$\Rightarrow \quad A_1 = -A_2 .$$

$$x_1(t) = c_1 e^{i\omega_s t} + c_2 e^{-i\omega_s t} + c_3 e^{i\omega_f t} + c_4 e^{-i\omega_f t}.$$
$$x_2(t) = c_1 e^{i\omega_s t} + c_2 e^{-i\omega_s t} - c_3 e^{i\omega_f t} - c_4 e^{-i\omega_f t}$$

when we take ω_s, $A_1 = A_2$ &
when we take ω_f, $A_1 = -A_2$.

x_1 & x_2 are both real → $x_1 = x_1^*$ & $x_2 = x_2^*$
Hence , $c_1 = c_2^*$ & $c_3 = c_4^*$

So, $c_1 \equiv \frac{A_s}{2} e^{i\phi_s} = c_2^* \Rightarrow c_2 = \frac{A_s}{2} e^{-i\phi_s}$

$c_3 \equiv \frac{A_f}{2} e^{i\phi_f} = c_4^* \Rightarrow c_4 = \frac{A_f}{2} e^{-i\phi_f}$

$$\therefore x_1 = \frac{A_s}{2} e^{i(\omega_s t + \phi_s)} + \frac{A_s}{2} e^{-i(\omega_s t + \phi_s)} + \frac{A_f}{2} e^{i(\omega_f t + \phi_f)} + \frac{A_f}{2} e^{-i(\omega_f t + \phi_f)}$$

$$= A_s \cos(\omega_s t + \phi_s) + A_f \cos(\omega_f t + \phi_f).$$

$$x_2 = A_s \cos(\omega_s t + \phi_s) - A_f \cos(\omega_f t + \phi)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A_s \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_s t + \phi_s) + A_f \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_f t + \phi_f)$$

If $A_f = 0$,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A_s \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_s t + \phi_s).$$

This means that, if the system has frequency ω_s, $x_1 = x_2$.
This is one mode of oscillation (1 normal mode).

If $A_s = 0$,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A_f \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_f t + \phi)$$

This means that, if the system has frequency ω_f, $x_1 = -x_2$
This is 2-normal modes.

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A_s \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_s t + \phi_s) + A_f \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_f t + \phi_f)$$

we define new co-ordinates:

→ $x_1 + x_2 = 2A_s \cos(\omega_s t + \phi_s)$ → Normal co-ordinate associated with the first normal mode. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

→ $x_1 - x_2 = 2A_f \cos(\omega_f t + \phi_f)$ → Normal co-ordinate associated with the second normal mode. $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

In general, $\underset{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{x_1 + x_2}$ & $\underset{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{x_1 - x_2}$ are the normal co-ordinates for this system.

moving together in same direction

moving with same amp. in opposite dir.

Lets assume a system (hypothetical):
the solution is:

$$\begin{bmatrix} x \\ y \end{bmatrix} = B_1 \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{NM_1} \cos(\omega_1 t + \phi_1) + B_2 \underbrace{\begin{bmatrix} 1 \\ -5 \end{bmatrix}}_{NM_2} \cos(\omega_2 t + \phi_2)$$

NM$_1$: They will oscillate in same dir with diff amp

NM$_2$: They will oscillate in diff dir with diff amp.

Normal co-ordinates :

→ $5x + y = 17 B_1 \cos(\omega_1 t + \phi_1)$ → Normal co-ordinate associated with $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

→ $2x - 3y = 17 \cdot B_2 \cos(\omega_2 t + \phi_2)$. → Normal co-ordinate associated with $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$

Beats:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A_s \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_s t + \phi_s) + A_f \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_f t + \phi_f).$$

If we expand:

$$x_1 = a\cos(\omega_s t) + b\sin(\omega_s t) + c\cos(\omega_f t) + d\sin(\omega_f t).$$

$$x_2 = a\cos(\omega_s t) + b\sin(\omega_s t) - c\cos(\omega_f t) - d\sin(\omega_f t).$$

Initial conditions: $\dot{x}_1(0) = \dot{x}_2(0) = 0$, $x_1(0) = 0$, $x_2(0) = A$

$$\left.\begin{aligned} \dot{x}_1(0) &= b+d = 0 \\ \dot{x}_2(0) &= b-d = 0 \end{aligned}\right\} \Rightarrow b = d = 0$$

$$x_1(t) = a\cos(\omega_s t) + c\cos(\omega_f t).$$

$$x_2(t) = a\cos(\omega_s t) - c\cos(\omega_f t).$$

$$\left.\begin{aligned} x_1(0) &= a + c = 0 \\ x_2(0) &= a - c = A \end{aligned}\right\} \quad a = \frac{A}{2}, \quad c = -\frac{A}{2}$$

$$\left\{\begin{aligned} x_1(t) &= \frac{A}{2}\cos(\omega_s t) - \frac{A}{2}\cos(\omega_f t) \\ x_2(t) &= \frac{A}{2}\cos(\omega_s t) + \frac{A}{2}\cos(\omega_f t). \end{aligned}\right\}$$

$$x_1 = A \sin\left[\frac{(\omega_s + \omega_f)t}{2}\right] \sin\left[\frac{(\omega_f - \omega_s)t}{2}\right]$$

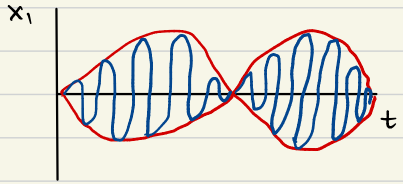
$$x_2 = A \cos\left[\frac{(\omega_s + \omega_f)t}{2}\right] \cos\left[\frac{(\omega_f - \omega_s)t}{2}\right]$$

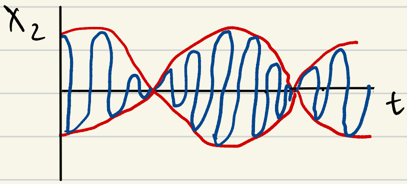
Let $\Omega = \frac{1}{2}(\omega_s + \omega_f)$; $\epsilon = \frac{1}{2}(\omega_f - \omega_s)$

if $\omega_f \to \omega_s \Rightarrow \epsilon \ll \Omega$

↳ beat frequence

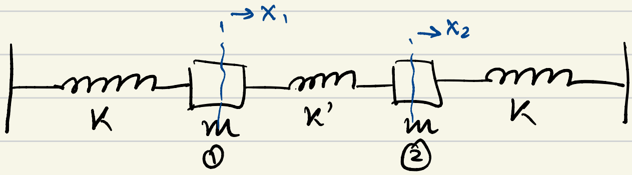
$$\therefore \begin{cases} x_1(t) = A \sin(\Omega t) \sin(\epsilon t) \\ x_2(t) = A \cos(\Omega t) \cos(\epsilon t) \end{cases}$$





Next Page →

driven & damped coupled oscillator:



→ $F_{damping} \propto -\dot{x}$
→ Driving force on mass 1: $F = F_d \cos(\omega t)$

$$\therefore \quad m\ddot{x}_1 = -Kx_1 - K'(x_1 - x_2) - b\dot{x}_1 + F_d \cos\omega t.$$
$$m\ddot{x}_2 = -Kx_2 - K'(x_2 - x_1) - b\dot{x}_2.$$

Adding the 2 equations:
$$m(\ddot{x}_1 + \ddot{x}_2) = -K(x_1 + x_2) - b(\dot{x}_1 + \dot{x}_2) + F_d \cos\omega t.$$
let $z_s = x_1 + x_2$

$$\Rightarrow \ddot{z}_s + \gamma \dot{z}_s + \omega_s^2 z_s = F\cos\omega t \quad ; \quad \gamma \equiv b/m, \quad \omega_s^2 = K/m, \quad F = F_d/m$$

atm, lets concentrate on the steady state solution:

$$\therefore \ z_s = A_s \cos(\omega t + \phi_s) \ ; \ \tan\phi_s = \frac{-\gamma}{\omega_s^2 - \omega^2} \ ; \ A_s = \frac{F}{\sqrt{(\omega_s^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

Subtracting the 2:
$$m(\ddot{x}_1 - \ddot{x}_2) = -(K + 2K')(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) + F_d \cos(\omega t).$$
let $z_f = x_1 - x_2$

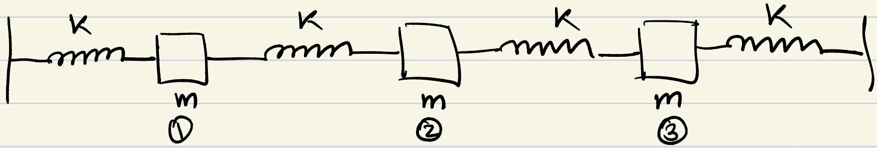
$$\Rightarrow \ddot{z}_f + \gamma \dot{z}_f + \omega_f^2 z_f = F\cos(\omega t) \to \omega_f^2 = \frac{K + 2K'}{m}.$$

the steady state solution: $z_f = A_f \cos(\omega_f t - \phi_f)$.

$$\tan\phi_f = \frac{-r\omega}{\omega_f^2 - \omega^2}, \quad A_f = \frac{F}{\sqrt{(\omega_f^2-\omega^2)^2 + r^2\omega^2}}$$

$$\therefore \; x_1(t) = C_s \cos(\omega_s t + \phi_s) + C_f \cos(\omega_f t + \phi_f).$$
$$x_2(t) = C_s \cos(\omega_s t + \phi_s) - C_f \cos(\omega_f t + \phi_f).$$
$$C_s \equiv \frac{A_s}{2}, \quad C_f \equiv \frac{A_f}{2}$$

<u>Three mass system:</u>



$$m\ddot{x}_1 = -Kx_1 - K(x_1 - x_2).$$
$$m\ddot{x}_2 = -K(x_2 - x_1) - K(x_2 - x_3).$$
$$m\ddot{x}_3 = -K(x_3 - x_2) - Kx_3.$$

Assume: $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} e^{i\omega t}$$

$$\therefore \; (-m\omega^2 + 2K)A_1 - KA_2 + 0 = 0$$
$$-KA_1 + (-m\omega^2 + 2K)A_2 - KA_3 = 0$$
$$0 - KA_2 + (-m\omega^2 + 2K)A_3 = 0$$

$$\begin{bmatrix} -m\omega^2 + 2K & -K & 0 \\ -K & -m\omega^2 + 2K & -K \\ 0 & -K & -m\omega^2 + 2K \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = 0$$

$\therefore \det(\ ) = 0$

$$\begin{vmatrix} -\omega^2 + 2\omega_0^2 & -\omega^2 & 0 \\ -\omega_0^2 & -\omega^2 + 2\omega_0^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & -\omega^2 + 2\omega^2 \end{vmatrix} = 0$$

$\Rightarrow \omega = \pm \sqrt{2}\, \omega_0$ or $\omega = \pm \sqrt{2 + \sqrt{2}}\, \omega_0$
or $\omega = \pm \sqrt{2 - \sqrt{2}}\, \omega_0$

if we plug in $\omega = 2\omega_0^2$ in original matrix:

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \propto \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Similarly $\omega^2 = (2 + \sqrt{2})\omega_0^2$:

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

Similarly for $\omega^2 = (2 - \sqrt{2})\,\omega_0^2$

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

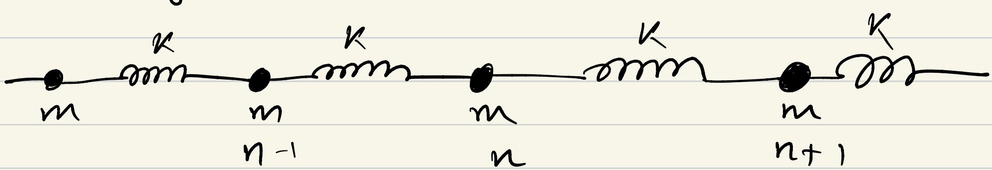
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{i\sqrt{2}\omega_0 t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-i\sqrt{2}\omega_0 t} +$$

$$c_3 \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} e^{i\left(\sqrt{2+\sqrt{2}}\right)\omega_0 t} + c_4 \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} e^{-i\left(\sqrt{2+\sqrt{2}}\right)\omega_0 t}$$

$$+ c_5 \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} e^{i\left(\sqrt{2-\sqrt{2}}\right)\omega_0 t} + c_6 \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} e^{-i\left(\sqrt{2-\sqrt{2}}\right)\omega_0 t}$$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow$ must be real, hence: $c_1 = c_2^*$, $c_3 = c_4^*$, $c_5 = c_6^*$.

'N' mass system:



$\rightarrow$ for the n^{th} mass: $F_n = -K(x_n - x_{n-1}) - K(x_n - x_{n+1})$.

$$= Kx_{n-1} - 2Kx_n + Kx_{n+1}$$

$\therefore$ from Newton's 2^{nd} law: $\ddot{x}_n = K \cdot x_{n-1} - 2K \cdot x_n + K \cdot x_{n+1}$

$\therefore$ $\ddot{x}_n = \omega_0^2 \cdot x_{n-1} - 2\omega_0^2 \cdot x_n + \omega_0^2 \cdot x_{n+1}$.

Assume : $x_n = A_n e^{i\omega t}$

$\Rightarrow \quad \ddot{x} = -\omega^2 A_n e^{i\omega t} = -\omega^2 x_n$

$\therefore \quad -\omega^2 A_n = \omega_0^2 \left[A_{n-1} - 2A_n + A_{n+1} \right]$

$\Rightarrow \quad A_n (2\omega_0^2 - \omega^2) = \omega_0^2 (A_{n-1} + A_{n+1})$

$\Rightarrow \quad \boxed{\dfrac{2\omega_0^2 - \omega^2}{\omega_0^2} = \dfrac{A_{n-1} + A_{n+1}}{A_n}} = \text{const.}$

↳ doesn't depend on n.

↳ we can find the values of recursive A's

Claim:

If $\omega \leq 2\omega_0$, then $A_n = B\cos(n\theta) + C\sin(n\theta)$.

Proof: define $\cos\theta \equiv \dfrac{A_{n-1} + A_{n+1}}{2A_n} \Rightarrow \cos\theta = \dfrac{2\omega_0^2 - \omega^2}{2\omega_0^2}$

$\therefore \quad -1 \leq \dfrac{2\omega_0^2 - \omega^2}{2\omega_0^2} \leq 1$

$\Rightarrow \quad -4\omega_0^2 \leq -\omega^2 \leq 0$

$\Rightarrow \quad 0 \leq \omega^2 \leq 4\omega_0^2$

$\Rightarrow \quad \omega \leq 2\omega_0$

$\therefore$ If $\omega \leq 2\omega_0$, we define $\cos\theta = \dfrac{2\omega_0^2 - \omega^2}{2\omega_0^2} = \dfrac{A_{n-1} + A_{n+1}}{2A_n}$

Now, we will prove $A_n = B\cos(n\theta) + C\sin(n\theta)$ using induction

→ If A_0 & A_1 are given: $A_0 = B\cos 0 + C\sin 0$

$\Rightarrow A_0 = B$

& $A_1 = B\cos\theta + C\sin\theta$ → now we have C

So, we assume: $A_n = B\cos(n\theta) + C\sin(n\theta)$

& prove: $A_{n+1} = B\cos[(n+1)\theta] + C\sin[(n+1)\theta]$

we know: $\cos\theta = \dfrac{A_{n-1} + A_{n+1}}{2A_n} \Rightarrow 2A_n \cdot \cos\theta = A_{n+1} + A_{n-1}$

$$\therefore A_{n+1} = 2A_n \cdot \cos\theta - A_{n-1}.$$

$$= 2[B\cos(n\theta) + C\sin(n\theta)]\cos\theta - [B\cos[(n-1)\theta] + C\sin[(n-1)\theta]]$$

$$= B[2\cos(n\theta)\cos\theta - \cos[(n-1)\theta]] + C[2\sin(n\theta)\cos\theta + \sin[(n+1)\theta]]$$

↳ expand   ↓ expand

$\therefore \cos(n\theta - \theta) = \cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta$

& we prove the above

$$\therefore X_n = A_n e^{i\omega t} = [B\cos(n\theta) + C\sin(n\theta)]e^{i\omega t} \quad -①$$

also, $X_n = A_n e^{-i\omega t} = [D\cos(n\theta) + E\sin(n\theta)]e^{-i\omega t} \quad -②$

generally we can say:

$$X_n = ① + ②$$

$$= \cos(n\theta)[Be^{i\omega t} + De^{i\omega t}] + \sin(n\theta)[Ce^{i\omega t} + Ee^{-i\omega t}].$$

We know that $X_n \in \mathbb{R} \Rightarrow B = D^*, \quad C = E^*$

let $B = Fe^{i\beta}$, $C = Ge^{i\gamma}$

$\Rightarrow$ $D = Fe^{-i\beta}$, $E = Ge^{-i\gamma}$

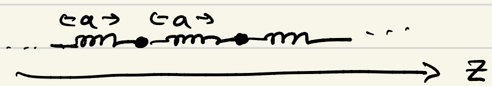
$$\therefore \; x_n = F\cos(n\theta)\underbrace{\cos(\omega t+\beta)}_{\cos(\omega t)\cos\beta - \sin(\omega t)\sin\beta} + G\sin(n\theta)\underbrace{\cos(\omega t+\gamma)}_{''\;''}.$$

or $x_n = c_1\cos(n\theta)\cos\omega t + c_2\cos(n\theta)\sin\omega t + c_3\sin(n\theta)\cos\omega t$
$+ c_4\sin(n\theta)\sin(\omega t)$.

where $\theta \equiv \cos^{-1}\left(\dfrac{2\omega_0^2 - \omega^2}{2\omega_0^2}\right)$

c_1, c_2, c_3 & c_4 will be determined by four initial conditions.

→ lets say in equilibrium, the distance between the masses is 'a'



$\therefore$ $z = na \Rightarrow n = \dfrac{z}{a}$

$$\therefore \; x_n = c_1\cos\left(\frac{z\theta}{a}\right)\cos\omega t + c_2\cos\left(\frac{z\theta}{a}\right)\sin\omega t + c_3\sin\left(\frac{z\theta}{a}\right)\cos\omega t$$
$$+ c_4\sin\left(\frac{z\theta}{a}\right)\sin\omega t$$

Boundary conditions:

→ Fixed walls ⇒ $x_0(t) = 0$; $\forall\, t \in \mathbb{R}$ & $x_{N+1}(t) = 0$; $\forall\, t \in \mathbb{R}$

↳ the 0^{th} mass & $(N+1)^{th}$ the wall

$x_0(t) = F\cos(\omega t + \beta) = 0$; $\forall\, t \in \mathbb{R}$

$\Rightarrow F = 0$

$\therefore\ x_n(t)$ (for fixed walls) $= G\sin(n\theta)\sin(\omega t + r)$

→ $x_{N+1} = G\sin\left[(N+1)\theta\right]\sin(\omega t + r) = 0 \qquad (x_{N+1} = 0)$

$\therefore\ (N+1)\theta = m\pi \Rightarrow \theta = \dfrac{m\pi}{N+1}\ \ \forall\ m \in \mathbb{Z}^+$

$$\therefore\ x_n(t) = \underbrace{G\sin\left(\frac{n m\pi}{N+1}\right)}_{A_n = G\sin\left(\frac{nm\pi}{N+1}\right)}\cos(\omega t + r)$$

We know: $\cos\theta = \dfrac{2\omega_0^2 - \omega^2}{2\omega_0^2}$

$\Rightarrow 2\omega_0^2\cos\theta = 2\omega_0^2 - \omega^2$

$\Rightarrow 2\omega_0^2\left[1 - \cos\theta\right] = \omega^2$

$\Rightarrow \omega^2 = 2\omega_0^2 \cdot 2\sin^2\dfrac{\theta}{2}$

$\Rightarrow \omega = 2\omega_0\sin\dfrac{\theta}{2} = 2\omega_0\sin\left(\dfrac{m\pi}{2(N+1)}\right)$; $\forall\ m \in \mathbb{Z}^+, [1, N]$

$$\boxed{\begin{array}{l} x_n(t) = A_n\cos(\omega t + r) \\ A_n = G\sin\left(\dfrac{m\pi}{N+1}\right), \quad \omega = 2\omega_0\sin\left(\dfrac{m\pi}{2(N+1)}\right) \end{array}}$$

**

$N=2$:

$$\omega = 2\omega_0 \sin\left(\frac{m\pi}{2\cdot 3}\right) = 2\omega_0 \sin\left(\frac{m\pi}{6}\right), \quad m \in [1,2] \; m \in \mathbb{Z}^+$$

when $m=1$, $\omega = 2\omega_0 \sin\left(\frac{\pi}{6}\right) = \omega_0$

& $A = G \sin\left(\frac{n\pi}{3}\right)$, if $n=1 \rightarrow$ first mass $\Rightarrow A_1 \propto \sin(\pi/3)$

$n=2 \rightarrow$ second mass $\Rightarrow A_2 \propto \sin(2\pi/3)$

$$\therefore \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

when $m=2$, $\omega = 2\omega_0 \sin(\pi/3) = \sqrt{3}\,\omega_0$

& $A_n = G \sin\left(\frac{2n\pi}{3}\right)$

$\therefore A_1 \propto \sin(2\pi/3)$

$A_2 \propto \sin(4\pi/3)$

$$\therefore \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \propto \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$N=3$:

$$\omega = 2\omega_0 \sin\left(\frac{m\pi}{8}\right)$$

$m=1 \Rightarrow A_n \propto \sin\left(\frac{n\pi}{4}\right)$, $\omega = 2\omega_0 \sin(\pi/8) = \sqrt{2-\sqrt{2}}\,\omega_0$

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \propto \begin{bmatrix} \sin(\pi/4) \\ \sin(2\pi/4) \\ \sin(3\pi/4) \end{bmatrix} \propto \begin{bmatrix} 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \end{bmatrix} \propto \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

we can do the same for when $m=2$ & $m=3$. we will get the same values as previously derived.

$N \to \infty$:

we will get the wave equation in this system.

we know, $m\ddot{x}_n = K x_{n-1} - 2K x_n + K x_{n+1}$. upto now, x_n is the position of the n^{th} mass, now we will change that:

x_n is the equilibrium position of the n^{th} mass

ε_n is the displacement of the n^{th} mass from its equilibrium.

→ position of n^{th} mass : $x_n + \varepsilon_n$

→ acceleration $= a = \ddot{\varepsilon}_n$

$$\therefore \quad m\ddot{\varepsilon}_n = K\varepsilon_{n-1} - 2K\varepsilon_n + K\varepsilon_{n+1}$$

let $\Delta x \equiv x_n - x_{n-1} \to$ constant.

$$\therefore \; m\ddot{\varepsilon}(x_n) = K\varepsilon(x_n - \Delta x) - 2K\varepsilon(x_n) + K\varepsilon(x_n + \Delta x)$$

as a function

$$\Rightarrow \frac{m}{\Delta x} \cdot \frac{d^2\varepsilon(x)}{dt^2} = \frac{K\varepsilon(x-\Delta x)}{\Delta x} - \frac{2K\varepsilon(x)}{\Delta x} + \frac{K\varepsilon(x+\Delta x)}{\Delta x}$$

$$= K\left[\left(\frac{\varepsilon(x+\Delta x) - \varepsilon(x)}{\Delta x}\right) - \left(\frac{\varepsilon(x) - \varepsilon(x-\Delta x)}{\Delta x}\right)\right]$$

for infinite continuous mass, $\Delta x \to 0$ (they are very close)

$$= K\left[\varepsilon'(x) - \varepsilon'(x-\Delta x)\right]$$

dividing by Δx, $\frac{m}{\Delta x} \equiv \rho$ (mass/length), $K\Delta x \equiv E$ (elastic modulus)

$$\Rightarrow \rho \frac{d^2 \mathcal{E}(x)}{dt^2} = E\, \mathcal{E}''(x)$$

$$\therefore \boxed{\rho \frac{\partial^2 (x,t)}{\partial t^2} = E \frac{\partial^2 \mathcal{E}(x,t)}{\partial x^2}} \rightarrow \text{wave equation !!}$$

This is a wave on a string.

Guess:

$$\begin{bmatrix} \vdots \\ \mathcal{E}_{n-1} \\ \mathcal{E}_n \\ \mathcal{E}_{n+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ a_{n-1} \\ a_n \\ a_{n+1} \\ \vdots \end{bmatrix} e^{\pm i\omega t}$$

$$\therefore \rho\, a(x)\,(-\omega^2) e^{i\omega t} = E e^{i\omega t} \frac{\partial^2 a(x)}{\partial x^2} = E e^{i\omega t} \frac{d^2 a(x)}{dx^2}$$

$$\therefore E \frac{d^2 a(x)}{dx^2} = -\rho \omega^2 a(x)$$

$$\Rightarrow \frac{d^2 a}{dx^2} = -\frac{\rho \omega^2}{E} a$$ → like a 1 mass attached to the spring.

$$= -k^2 a \; ; \; k \equiv \sqrt{\frac{\rho \omega^2}{E}} = \omega \sqrt{\frac{\rho}{E}}$$

→ wave number

$$\therefore a(x) = e^{\pm ikx}$$

$$\therefore \mathcal{E}(x,t) \propto e^{\pm i(kx \pm \omega t)}$$

$$\mathcal{E}(x,t) = A_1 e^{i(kx+\omega t)} + A_2 e^{i(-kx-\omega t)} + A_3 e^{i(kx-\omega t)} + A_4 e^{i(-kx+\omega t)}$$

The wave repeats after every K becomes 2π.

hence, $K\lambda = 2\pi \Rightarrow \lambda = \dfrac{2\pi}{K}$

$\mathcal{E}(x,t)$ has to be real, hence $A_1 = A_2^*$ & $A_3 = A_4^*$

$\therefore \; A_1 = \dfrac{B_1}{2} e^{i\phi_1}$; $A_3 = \dfrac{B_2}{2} e^{i\phi_2}$

$A_2 = \dfrac{B_1}{2} e^{-i\phi_1}$; $A_4 = \dfrac{B_2}{2} e^{-i\phi_2}$

$$\therefore \; \mathcal{E}(x,t) = B_1 \underbrace{\cos(Kx + \omega t + \phi_1)}_{-x \text{ direction.}} + B_2 \underbrace{\cos(Kx - \omega t + \phi_2)}_{+x \text{ direction.}}$$

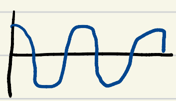
$$\mathcal{E}(x,t) = \underset{\uparrow \atop B_1\cos(\phi_1)}{C_1} \cos(Kx + \omega t) + C_2 \sin(Kx + \omega t) + C_3 \cos(Kx - \omega t) + C_4 \sin(Kx - \omega t)$$

$$\mathcal{E}(x,t) = D_1 \cos(Kx)\cos(\omega t) + D_2 \sin(Kx)\sin\omega t + D_3 \sin Kx \cos(\omega t) + D_4 \cos(Kx)\sin\omega t$$

↳ these are standing waves.

lets observe the wave: $\cos(Kx - \omega t)$.

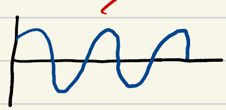
at $t=0 \Rightarrow \cos(Kx) \rightarrow$



at $t = \Delta t \rightarrow \cos(Kx - \omega\Delta t)$

max when $Kx = \omega\Delta t$

$\Rightarrow x_1 = \dfrac{\omega\Delta t}{K} \rightarrow$



this wave has moved x_1 in time Δt

Hence speed of the wave = $\frac{x_1}{\Delta t} = \frac{\omega}{k}$

$$\boxed{V = \frac{\omega}{k} = \frac{\lambda}{T}}$$

we know $k = \sqrt{\frac{\omega^2 \rho}{E}} = \omega \sqrt{\frac{\rho}{E}}$

$$\therefore \frac{\omega}{k} = \sqrt{\frac{E}{\rho}} = V$$

$$\therefore v^2 = \frac{E}{\rho}$$

The wave equation can hence be written as:

$$\frac{\partial^2 \xi}{dt^2} = v^2 \frac{\partial^2 \xi}{dx^2}$$

Note:

$f(x \pm vt)$ is a solution to this wave equation.

Can be proved by taking derivatives.

$f(x+vt) \rightarrow -x$ direction.

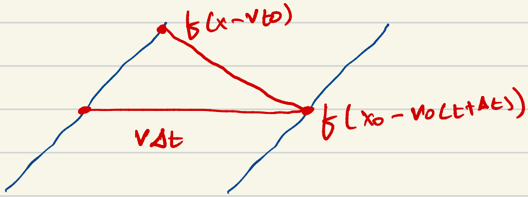
$f(x-vt) \rightarrow +x$ direction.

lets evaluate: $f(x_0 - v t_0)$

$$= f(x_0 + v\Delta t - v\Delta t - v t_0)$$

$$= f[(x_0 + v\Delta t) - v(t_0 + \Delta t)]$$

$\therefore$ the function is the same at both x_0, t_0 & $t_0 + \Delta t$. Hence, this wave has moved to the right



$$\text{Slope} = \frac{-\Delta f}{v\Delta t}.$$

Hence, $$\frac{\partial f}{\partial x} = \frac{-\frac{\partial f}{\partial t}\Delta t}{v\Delta t} = -\frac{1}{v}\frac{\partial f}{\partial t}$$

Hence another wave equation: $\frac{\partial f}{\partial x} = -\frac{1}{v}\frac{\partial f}{\partial t}$.

$$\therefore \frac{\partial}{\partial t}\left[\frac{\partial f}{\partial t}\right] = -v\frac{\partial}{\partial t}\left[\frac{\partial f}{\partial x}\right]$$

$$\Rightarrow \frac{\partial^2 f}{\partial t^2} = -v\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t}\right) = -v\frac{\partial}{\partial x}\left(-v\frac{\partial f}{\partial x}\right) = v^2\frac{\partial^2 f}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial t^2} = v^2\frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial t^2} - v^2 \frac{\partial^2 f}{\partial x^2} = 0 \Rightarrow \left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) f = 0$$

well behaved function*

the reason that we can write $f(x \pm vt)$ as a wave is because it can be written as a series (integral) of sines & cosines.