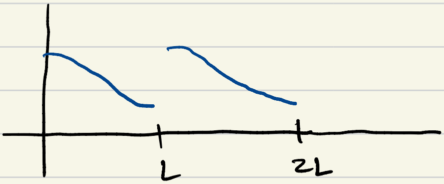
Fourier analysis:

There are 2 ways of expression: Fourier series & Fourier transform.

Fourier series:

any well behaved trig. function can be written as a linear sum of sines & cosines.

consider $f(x)$ as a period of L:



$$\text{Hence, } f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right) \right]$$

lets find:
$$\int_0^L f(x)dx = \int_0^L a_0 dx + \int \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right) \right] dx$$

$$= a_0(L) + \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \cos\left(\frac{2\pi n x}{L}\right) dx}_{\frac{L}{2\pi n} \sin\left(\frac{2\pi n x}{L}\right)\Big|_0^L = 0} + b_n \underbrace{\int_0^L \sin\left(\frac{2\pi n x}{L}\right) dx}_{\frac{-L}{2\pi n} \cos\left(\frac{2\pi n x}{L}\right)\Big|_0^L = 0}$$

$$\therefore \boxed{a_0 = \int_0^L f(x)dx}$$

$$f(x)\cdot \cos\left(\frac{2\pi m x}{L}\right) = a_0 \cos\left(\frac{2\pi m x}{L}\right) + \left(\sum a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right)\right)\cos\left(\frac{2\pi n x}{L}\right)$$

Integrating both sides:

$$= \int_0^L a_0 \cancelto{0}{\cos\left(\frac{2\pi m x}{L}\right)} dx + \sum_n \int_0^L a_n \cos\left(\frac{2\pi n x}{L}\right)\cos\left(\frac{2\pi m x}{L}\right) dx$$

$$+ \int_0^L b_n \sin\left(\frac{2\pi n x}{L}\right)\cos\left(\frac{2\pi m x}{L}\right) dx$$

$$\int \cos\left(\frac{2\pi n x}{L}\right)\cos\left(\frac{2\pi m x}{L}\right) dx = \frac{1}{2}\int\left[\cancelto{0}{\cos\frac{(n+m)2\pi x}{L}} + \underbrace{\cos\frac{(n-m)2\pi x}{L}}\right] dx$$

$$= 0; \quad n \neq m$$
$$= \frac{L}{2}; \quad n = m$$

$$\int_0^L \sin\left(\frac{2\pi n x}{L}\right)\cos\left(\frac{2\pi m x}{L}\right) dx = 0 \quad \text{(orthogonal functions)}$$

Hence, $$\int_0^L f(x)\cos\left(\frac{2\pi m x}{L}\right) dx = \sum_{n=1}^{\infty} a_n \frac{L}{2} \delta_{nm}$$

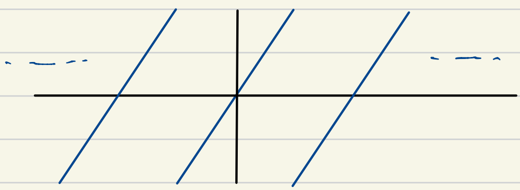
where $$\delta_{nm} = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}$$

$$\Rightarrow \int f(x)\cos\left(\frac{2\pi m x}{L}\right) dx = \frac{L}{2} a_m$$

$$\therefore \boxed{a_m = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi m x}{L}\right)dx}$$

$$\& \boxed{b_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi m x}{L}\right)dx}$$

Example: (Sawtooth function): $f(x) = Ax \; ; \; x \in \left[\frac{-L}{2}, \frac{L}{2}\right]$



$$a_0 = \frac{1}{L}\int_{-L/2}^{L/2} Ax\,dx = \frac{A}{L}(0)$$

(odd . even) = odd

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} \underbrace{Ax}_{\text{odd}} \underbrace{\cos\left(\frac{2\pi n x}{L}\right)}_{\text{even}} dx = 0$$

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} Ax\sin\left(\frac{2\pi n x}{L}\right)dx = \frac{AL}{\pi n}(-1)^{n+1}$$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} \frac{4L}{\pi n} (-1)^{n+1} \sin\left(\frac{2\pi n x}{L}\right)$$

$$= \frac{4L}{\pi} \left[\sin\left(\frac{2\pi x}{L}\right) - \frac{1}{2} \sin\left(\frac{4\pi x}{L}\right) + \frac{1}{3} \sin\left(\frac{6\pi x}{L}\right) \cdots \right]$$

if $x = \frac{L}{4}$:

$$\frac{4L}{4} = \frac{4L}{\pi} \left[1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} \cdots \right]$$

Hence: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$

<u>Fourier exponential series:</u>

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{i2\pi n x/L} \quad ; \quad c_n = \int_0^L f(x) \, e^{-i2\pi n x/L} \, dx}$$

we know $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ & $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$

Proof:

$$\int_0^L f(x) \cdot e^{-i2\pi m x/L} \, dx = \int_0^L \sum_{n=-\infty}^{\infty} c_n e^{i2\pi n x/L} \cdot e^{-i2\pi m x/L} \, dx$$

$$= \sum_{n=-\infty}^{\infty} c_n \int_0^L dx \, e^{i2\pi n x/L} \cdot e^{-i2\pi m x/L}$$

Orthogonality of functions: $\int_{\mathbb{R}} f(x)\, g(x)\, dx = \begin{cases} 0 \; ; & f \neq g \\ 1 \; ; & f = g \end{cases}$

then f & g are orthogonal to each other.

Hence: $$\int_0^L e^{i 2\pi n x/L} \cdot e^{-i 2\pi m x/L}\, dx = \delta_{mn} L$$

$$\therefore \int_0^L f(x)\, e^{-i 2\pi m x/L}\, dx = \sum_{n=-\infty}^{\infty} c_n L\, \delta_{mn}$$

$$\therefore \boxed{c_m = \frac{1}{L} \int_0^L f(x)\, e^{-i 2\pi n x/L}\, dx}$$

Example: (sawtooth function)

$$f(x) = Ax \quad , \quad x \in \left[-\frac{L}{2}, \frac{L}{2}\right]$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i 2\pi n x/L} \quad ; \quad c_n = \frac{1}{L} \int_{-L/2}^{L/2} Ax\, e^{-i 2\pi n x/L}\, dx$$

$$\therefore c_n = \frac{A}{L} \int_{-L/2}^{L/2} e^{-i 2\pi n x/L}\, x\, dx = -\frac{AL}{i 4\pi n}\left(e^{-i\pi n} + e^{i\pi n}\right)$$

$$= \frac{iAL}{2\pi n} \cos(n\pi) = \frac{iAL}{2\pi n}(-1)^n$$

$(n \neq 0)$

if $n=0$: $\quad c_n = \dfrac{1}{L}\displaystyle\int_{-L/2}^{L/2} Ax\,dx = 0$

$$\therefore f(x) = \sum_{n\neq 0} (-1)^n \frac{iAL}{2\pi n} e^{i2\pi n x/L}$$

Now, is this the same as we found before?

$$f(x) = \sum_{n\neq 0} (-1)^n \frac{iAL}{2\pi n}\left\{ \underbrace{\cos\left(\frac{2\pi n x}{L}\right)}_{\text{even}=0} + i\sin\left(\frac{2\pi n x}{L}\right)\right\}$$

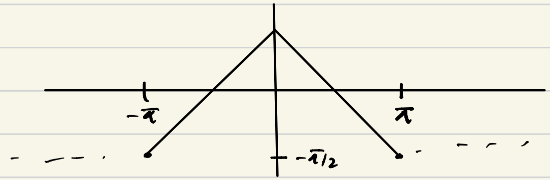
$$= \sum_{n\neq 0} (-1)^n \frac{iAL}{2\pi n}\, i\sin\left(\frac{2\pi n x}{L}\right)$$

$$= \sum_{n\neq 0} (-1)^{n+1} \frac{AL}{2\pi n} \sin\left(\frac{2\pi n x}{L}\right) \rightarrow \text{same as what we got in the previous section.}$$

$$= \sum_{n=1}^{\infty} \frac{AL}{\pi} \sin\left(\frac{2\pi n x}{L}\right)$$

Second example:

$$f(x) = \begin{cases} \pi/2 + x \; ; & x \in [-\pi, 0] \\ \pi/2 - x \; ; & x \in [0, \pi] \end{cases}$$



$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right)$$

$L = 2\pi$ in this case,

$$\therefore \quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} \frac{\pi}{2} + x\,dx + \frac{1}{2\pi}\int_{0}^{\pi} \frac{\pi}{2} - x\,dx$$

$$\therefore \quad a_0 = 0$$

$$a_n = \frac{2}{L}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{2\pi n x}{L}\right)dx = \frac{2}{2\pi}\int_{-\pi}^{0}\left(\frac{\pi}{2}+x\right)\cos(nx)\,dx$$

$$+ \frac{2}{2\pi}\int_{0}^{\pi}\left(\frac{\pi}{2}-x\right)\cos(nx)\,dx$$

$$\therefore \quad a_n = \frac{1}{\pi n^2}\left[(1-\cos n\pi) - (\cos n\pi - 1)\right]$$

$$= \frac{2}{\pi n^2}(1-\cos n\pi) = \frac{2}{\pi n^2}\left(1-(-1)^n\right)$$

$$= \frac{4\pi}{n^2} \quad \text{when } n \to \text{odd}$$

$$\hookrightarrow \quad 0 \quad \text{when } n \to \text{even}$$

$$\therefore f(x) = \sum_{n=\text{odd}} \frac{4}{\pi n^2}\cos nx = \sum_{K=0}^{\infty} \frac{4}{\pi} \frac{\cos[(2K+1)x]}{(2K+1)^2}$$

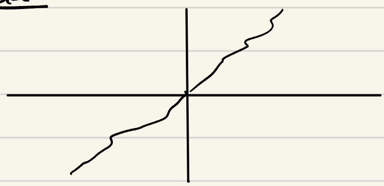
the b_n terms all go to zero, this is because $f(x)$ is an even function. If a function is even, then all its odd components are zero

Note:

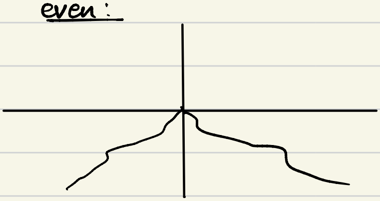
if $f(x) \to$ odd $\Rightarrow$ $a_0 = 0$ & $a_n = 0$

$f(x) \to$ even $\Rightarrow$ $b_n = 0$

odd:



even:



Fourier transform:

→ Fourier series work well for periodic functions.

→ what if the function is not periodic?

Previously:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi n x/L} \quad \text{where} \quad c_n = \frac{1}{L}\int_0^L f(x) e^{-i2\pi n x/L} dx$$

→ not periodic $\Rightarrow$ period $= L \to \infty$

define : $k_n \equiv \frac{2\pi n}{L}$ $\Rightarrow$ $dk_n = \frac{2\pi}{L} dn$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} dn = \sum_{k_n} c_n \frac{L}{2\pi} dk_n e^{ik_n x}$$

$$f(x) = \int_{-\infty}^{\infty} c_n \left(\frac{L}{2\pi}\right) e^{ikx}\, dk = \int_{-\infty}^{\infty} c(k) \cdot e^{ikx}\, dk \quad ; \quad c(k) = \frac{cL}{2\pi}$$

$$\therefore \boxed{f(x) = \int_{-\infty}^{\infty} c(k) e^{ikx}\, dk \quad ; \quad c(k) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx}$$

for any function $f(x)$, we can define an even & odd function:

$$f_e = \frac{f(x) + f(-x)}{2} \quad \& \quad f_o(x) = \frac{f(x) - f(-x)}{2}, \text{ and}$$

$$f(x) = f_e(x) + f_o(x).$$

$$\therefore c(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (f_e(x) + f_o(x))\, e^{-ikx}\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} (f_e(x) + f(-x))\,(\cos kx - i \sin kx)\, dx$$

$$= \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} f_e(x) \cos(kx)\, dx}_{c_e(k)} - \underbrace{\frac{i}{2\pi} \int_{-\infty}^{\infty} f_o(x) \sin(kx)\, dx}_{c_o(k)}$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} f_e(x) (-i \sin kx)\, dx \xrightarrow{} 0 \quad + \frac{1}{2\pi} \int_{-\infty}^{\infty} f_o(x) \cos(kx)\, dx \xrightarrow{} 0$$

$$\therefore c(k) = c_e(k) + c_o(k).$$

$$c_e(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_e(x) \cos(kx)\, dx$$, the same for $c_o \rightarrow$ odd func.

↑ even function of k.

→ If $f(x)$ is even & real, $C(K)$ is also even & real.
→ If $f(x)$ is odd & real, $C(K)$ is odd & imaginary.

Example:

$$f(x) = \begin{cases} +A & ; \quad x \in [0, L/2] \\ -A & ; \quad x \in [-L/2, 0] \end{cases}$$

$$C(K) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-iKx} dx = \frac{1}{2\pi}\int_{-L/2}^{0} (-A) e^{-iKx} dx + \frac{1}{2\pi}\int_{0}^{L/2} A e^{-iKx} dx$$

$$= \frac{-A}{2\pi}\left[\frac{-1}{iK} A e^{-iKx}\right]_{-L/2}^{0} + \frac{A}{2\pi}\left[\frac{-1}{iK} e^{-iKx}\right]_{0}^{L/2}$$

$$= \frac{A}{2\pi i K}\left[0 - e^{iKL/2}\right] - \frac{A}{2\pi i K}\left[e^{-iKL/2} - 1\right]$$

$$= \frac{-Ai}{\pi K}\left[1 - \cos\left(\frac{KL}{2}\right)\right]$$

Example:

$f(x) = Ae^{-ax^2}$ → Gaussian.

$$C(K) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Ae^{-ax^2} e^{-iKx} dx = \frac{A}{2\pi}\int_{-\infty}^{\infty} \exp\left[-a\left(x^2 + \frac{iKx}{a} - \frac{K^2}{4a^2} + \frac{K^2}{4a^2}\right)\right] dx$$

$$= \frac{A}{2\pi}\int_{-\infty}^{\infty} \exp\left[-a\left(x + \frac{iK}{2a}\right)^2\right] dx \cdot e^{-K^2/4a^2} dx$$

$$= \frac{Ae^{-K^2/4a^2}}{2\pi}\int_{-\infty}^{\infty} \exp\left[-a\left(x + \frac{iK}{2a}\right)^2\right] dx$$

$$\int_{-\infty}^{\infty} e^{-ax^2}dx = I = \int_{-\infty}^{\infty} e^{-ay^2}dy \quad ;$$

Hence : $I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)}\,dx\,dy \rightarrow$ converting to polar

$$\therefore I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-ar^2}\,r\,dr\,d\theta = 2\pi\int_0^{\infty} e^{-ar^2}\,r\,dr$$

let $u = ar^2$
$\Rightarrow du = 2ar\,dr$

$$= 2\pi\int_0^{\infty} e^{-u}\frac{du}{2a}$$

$\therefore I^2 = \frac{\pi}{a}$; Hence $I = \sqrt{\frac{\pi}{a}}$

Hence , $\frac{Ae^{-k^2/4a^2}}{2\pi}\int_{-\infty}^{\infty} \underbrace{e^{-a(x+ik/2a)^2}\,dx}_{\sqrt{\pi/a}} = \frac{A}{2\sqrt{\pi a}}\,e^{-k^2/4a^2}$, which is also gaussian

<u>Example</u>: (Lorentzian)

$f(x) = Ae^{-b|x|}$

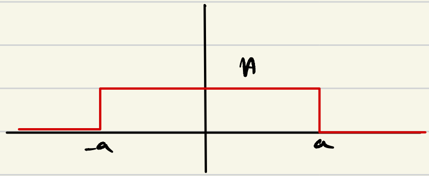
$$C(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Ae^{-b|x|}e^{-ikx}dx = \frac{A}{2\pi}\int_{-\infty}^{0} e^{bx}e^{-ikx}dx + \frac{A}{2\pi}\int_0^{\infty} e^{-bx}e^{-ikx}dx$$

$$= \frac{A}{2\pi}\int_{-\infty}^{0} e^{(b-ik)x}dx + \frac{A}{2\pi}\int_0^{\infty} e^{-(b+ik)x}dx$$

$$\therefore\ C(K) = \frac{A}{2\pi}\left[\frac{2b}{K^2+b^2}\right] = \frac{Ab}{K^2+b^2}$$

<u>Example</u>: (square wave)

$$f(x) = \begin{cases} A & ; \ x \in [-a, a] \\ 0 & \text{otherwise} \end{cases}$$



↳ Area = $2Aa$

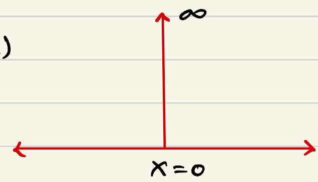
(if $A = \frac{1}{a}$, area $= 2$)

$$C(K) = \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}dx = \frac{A}{i2\pi K}\left[e^{ika} - e^{-ika}\right]$$

$$= \frac{A}{K\pi}\left[\frac{e^{ika} - e^{-ika}}{2i}\right] = \frac{A}{\pi}\frac{\sin(Ka)}{K} \rightarrow \text{sinc function}$$

what if $A = \frac{1}{a} \Rightarrow$ area $= 2$, if $x \in [-a/2, a/2] \Rightarrow$ area $= 1$

→ if $a \rightarrow 0 \Rightarrow A \rightarrow \infty$: $f(x) = \delta(x)$



$$\delta(x) \equiv \begin{cases} \infty, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

→ $$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

→ $$\int_{-\infty}^{\infty} \delta(x) f(x) dx = \int_{-\infty}^{\infty} \delta(x) f(0) dx = f(0)\int_{-\infty}^{\infty}\delta(x)dx = f(0).$$

In general: $\int_{-\infty}^{\infty} \delta(x-x_0) f(x)\,dx = f(x_0)$.

properties: $\delta(-x) = \delta(x) \rightarrow$ even function

$\delta(ax) = \frac{1}{|a|}\delta(x)$.

Hence: $\int_{-\infty}^{\infty} \delta(k) e^{ikx}\,dk = 1$

↳ inverse fourier of 1

$\therefore \quad \delta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\,dx$

→ for a gaussian: $f(x) = e^{-ax^2}$, Hence $c(k) = \frac{e^{-k^2/4a^2}}{2\sqrt{\pi a}}$

$\therefore \quad \delta(k) = \lim_{a\to\infty} \frac{e^{-k^2/4a^2}}{2\sqrt{\pi a}}$

→ for a lorentzian: $\delta(k) = \lim_{b\to 0} \frac{b}{\pi(b^2+k^2)}$

→ for a sinc function: $\delta(k) = \lim_{a\to 0} \frac{\sin(ka)}{\pi k}$