Homework 5

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PHYS273

1a.) i.) The period of one beat is π.

ii.) The period of $\sin(8t)$ is $\frac{\pi}{4}$ and the period of $\cos(2t)$ is π. The beat period is the same as the larger of those two periods.

1b.) $f(t) = 3\sin(8t)\cos(2t)$

$$= 3\left(\frac{e^{i8t} - e^{-i8t}}{2i}\right)\left(\frac{e^{i2t} + e^{-i2t}}{2}\right)$$

$$= 3\left(\frac{e^{i10t} + e^{i6t} - e^{-i6t} - e^{-i10t}}{4i}\right)$$

$$= \frac{3}{4i}\left(\cancel{\cos(10t)} + i\sin(10t) + \cancel{\cos(6t)} + i\sin(6t) - \cancel{\cos(-6t)} - i\sin(-6t) - \cancel{\cos(-10t)} - i\sin(-10t)\right)$$

$$= \frac{3}{4\cancel{i}}\left(2\cancel{i}\sin(10t) + 2\cancel{i}\sin(6t)\right)$$

$$= \boxed{\frac{3}{2}\sin(10t) + \frac{3}{2}\sin(6t)}$$

1c.) $\tilde{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$

$$= \frac{1}{2\pi}\left(\frac{3}{4i}\right)\int_{-\infty}^{\infty}\left(e^{i10t} - e^{-i10t} + e^{i6t} - e^{-i6t}\right)e^{-i\omega t}\,dt$$

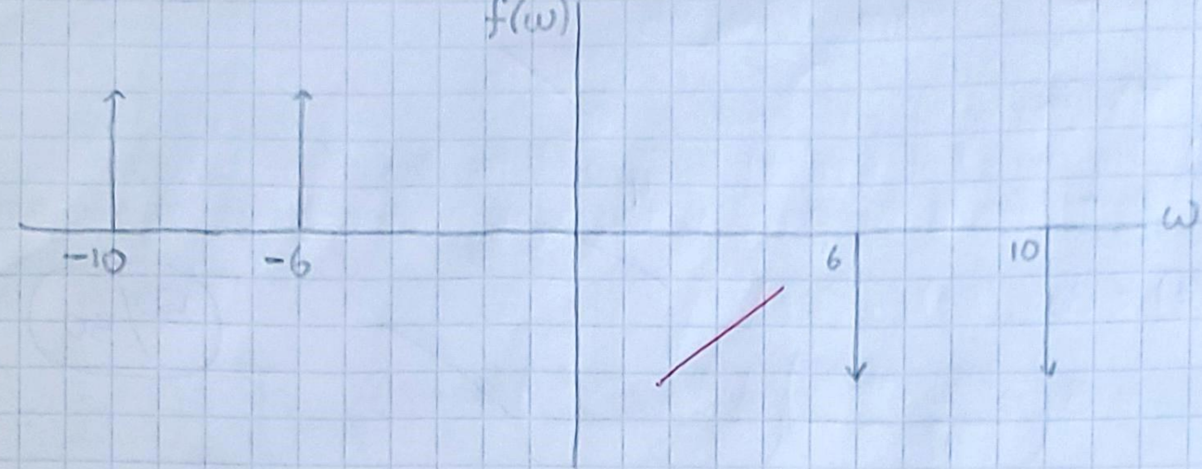
$$= \frac{3}{4i}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{-i(\omega-10)t} + e^{-i(\omega-6)t} - e^{-i(\omega+10)t} - e^{-i(\omega+6)t}\right)dt$$

$$= \frac{3}{4i}\left(\delta(\omega-10) + \delta(\omega-6) - \delta(\omega+10) - \delta(\omega+6)\right)$$

$$= \boxed{-\frac{3i}{4}\delta(\omega-10) - \frac{3i}{4}\delta(\omega-6) + \frac{3i}{4}\delta(\omega+10) + \frac{3i}{4}(\omega+6)}$$

50/50

1d.)



0 everywhere except $\omega = -10, -6, 6, 10$

For $\omega = -10, -6$, the function goes to infinity
$\omega = 6, 10$, the function goes to negative infinity

1e.) $g(t) = 10\sin(3t)\cos(5t)$

$$= 10\left(\frac{e^{i3t} - e^{-i3t}}{2i}\right)\left(\frac{e^{i5t} + e^{-i5t}}{2}\right)$$

$$= \frac{5}{2i}\left(e^{i8t} + e^{-i2t} - e^{i2t} - e^{-i8t}\right)$$

$$\tilde{g}(\omega) = \frac{1}{2\pi}\left(\frac{5}{2i}\right)\int_{-\infty}^{\infty}\left(e^{i8t} + e^{-i2t} - e^{i2t} - e^{-i8t}\right)e^{-i\omega t}\,dt$$

$$= \frac{5}{2i}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{-i(\omega-8)t} + e^{-i(\omega+2)t} - e^{-i(\omega-2)t} - e^{-i(\omega+8)t}\right)dt$$

$$= \frac{5}{2i}\left(\delta(\omega-8) + \delta(\omega+2) - \delta(\omega-2) - \delta(\omega+8)\right)$$

$$= -\frac{5i}{2}\delta(\omega-8) - \frac{5i}{2}\delta(\omega+2) + \frac{5i}{2}\delta(\omega-2) + \frac{5i}{2}\delta(\omega+8)$$

$\omega = -2, -8, 2, 8$ would be nonzero.

1f.) Fourier transform is:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(t-4)e^{-ikt}dt$$

We know that

$$\int_{-\infty}^{\infty} \delta(t-t_0)f(t)dt = \int_{-\infty}^{\infty} \delta(t-t_0)f(t_0)dt = f(t_0)$$

So fourier transform becomes

$$\boxed{\frac{1}{2\pi}e^{-i4k}}$$

2.) $f(x) = \begin{cases} e^{-bx}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$$\tilde{f}(\omega) = \frac{1}{2\pi}\int_0^\infty e^{-bx} e^{-ikx}\, dx$$

$$= \frac{1}{2\pi}\int_0^\infty e^{-(b+ik)x}\, dx$$

$$= \frac{1}{2\pi} \lim_{t\to\infty} \int_0^t e^{-(b+ik)x}\, dx$$

$$= \frac{1}{2\pi} \lim_{t\to\infty} \left[-\frac{e^{-(b+ik)x}}{b+ik} \right]_0^t$$

$$= \frac{1}{2\pi} \lim_{t\to\infty} \left[-\frac{1}{(b+ik)e^{(b+ik)t}} + \frac{1}{(b+ik)e^0} \right]$$

$$= \frac{1}{2\pi}\left(\frac{1}{b+ik}\right) \cdot \frac{b-ik}{b-ik}$$

$$= \boxed{\frac{b-ik}{2\pi(b^2+k^2)}}$$

3a.) $f(x)$ can be written as a sum of even and odd functions:

$f(x) = f_e(x) + f_o(x)$

$$\text{so } C(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} (f_e(x) + f_o(x))(\cos(-kx) + i\sin(-kx))\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_e(x)\cos(kx)\, dx - \frac{i}{2\pi}\int_{-\infty}^{\infty} f_o(x)\sin(kx)\, dx$$

Now, if $f(x)$ is purely odd, we are left with:

$$C(k) = -\frac{i}{2\pi}\int_{-\infty}^{\infty} f_o(x)\sin(kx)\, dx$$

And if f(x) is real as well, we see C(k) must be purely imaginary.

3b.) Fourier transform of $g(t-s)$ is:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(t-s)e^{-i\omega t}dt$$

Let $u = t-s$ $\quad t = u+s$

$du = dt$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(u)e^{-i\omega(u+s)}du$$

$$= e^{-i\omega s}\frac{1}{2\pi}\int_{-\infty}^{\infty} g(u)e^{-i\omega u}du$$

$$= \boxed{e^{-i\omega s}\tilde{g}(\omega)}$$

This result shows that time-shifting a function changes the phase of its fourier transform, but not the magnitude.

4.) $$a_0 = \frac{1}{L}\int_0^L f(x)dx$$

$$= \frac{1}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)dx$$

$$= -\frac{1}{\cancel{L}}\left(\frac{\cancel{L}}{\pi}\right)\cos\left(\frac{\pi x}{L}\right)\Big|_0^L$$

$$= -\frac{1}{\pi}(\cos(\pi)-\cos 0)$$

$$= -\frac{1}{\pi}(-1-1)$$

$$= \boxed{\frac{2}{\pi}}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi n x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{2\pi n x}{L}\right)dx$$

$$= \frac{2}{L}\left(\frac{1}{2}\right)\int_0^L \left[\sin\left(\frac{(\pi+2\pi n)x}{L}\right) + \sin\left(\frac{(\pi-2\pi n)x}{L}\right)\right]dx$$

$$= \frac{1}{\cancel{L}}\left[-\frac{\cancel{L}}{\pi+2\pi n}\cos\left(\frac{\pi+2\pi n}{L}x\right) - \frac{\cancel{L}}{\pi-2\pi n}\cos\left(\frac{\pi-2\pi n}{L}x\right)\right]_0^L$$

$$= -\frac{\cos(\pi+2\pi n)}{\pi+2\pi n} - \frac{\cos(\pi-2\pi n)}{\pi-2\pi n} + \frac{\cos 0}{\pi+2\pi n} + \frac{\cos 0}{\pi-2\pi n}$$

$$= \frac{1+1}{\pi+2\pi n} + \frac{1+1}{\pi-2\pi n}$$

$$= \frac{2(\pi-2\pi n)+2(\pi+2\pi n)}{(\pi+2\pi n)(\pi-2\pi n)}$$

$$= \frac{4\pi}{\pi^2-4\pi^2 n^2}$$

$$= \frac{4\cancel{\pi}}{\pi^{\cancel{2}}(1-4n^2)}$$

$$= -\frac{4}{\pi(4n^2-1)}$$

Even function, so $b_n = 0$

$$\boxed{f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty}\left(-\frac{4}{\pi(4n^2-1)}\cos\left(\frac{2\pi n x}{L}\right)\right)}$$