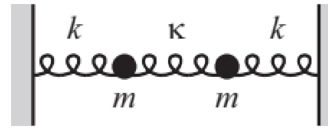
Physics 273 — Homework 7 Solutions

Question 1



Consider the system of two masses and three springs discussed in lecture and in Morin's section 2.1 . Let's explore the solutions we found by considering some different specific cases.

(a) Suppose the middle spring has the same spring constant as the side ones, i.e. $\kappa = k$. Using that with the general results we found for this example system, what are the two normal-mode oscillation frequencies, ω_s and ω_f, in this case?

The slower normal-mode frequency (ω_s) is still $\sqrt{k/m}$ because that doesn't depend on κ, while the faster normal-mode frequency is now $\omega_f = \sqrt{\frac{k+2\kappa}{m}} = \sqrt{\frac{3k}{m}} = \sqrt{3}\,\omega_s$.

(b) Instead, suppose that the two side springs are made weaker and weaker (that is, $k \to 0$) while the strength of the middle spring (κ) is kept fixed. What are the two oscillation frequencies in this limit? Explain what the smaller frequency means and why it makes sense in that limit, and also explain why the larger frequency is not simply $\sqrt{\kappa/m}$.

Plugging in $k = 0$, the two normal mode frequencies are now $\omega_s = 0$ and $\omega_f = \sqrt{2\kappa/m}$. That zero (or essentially zero) oscillation frequency makes sense because the center-of-mass of the pair doesn't oscillate; it can just drift forever with constant velocity (until something hits a wall, I guess). Or you could say that the "oscillation period" goes to infinity, and therefore $\omega = 2\pi/T = 0$. The larger frequency is not simply $\sqrt{\kappa/m}$ because there is not just one mass attached to the spring, there are two masses and they both can move. One way to look at it is to see that the center of the spring is fixed at the center of the pair, so each mass really only has half a spring associated with its motion; and if you cut a spring in half, each part has twice the spring constant of the original. (An alternative way to think of this is to note that the "reduced mass" of the pair is $m/2$, so $\omega = \sqrt{\kappa/(m/2)} = \sqrt{2\kappa/m}$.)

(c) Now consider a different limit: the two side springs have their original strength k, while the middle spring is weaker, but not infinitely weak. That is, $\kappa > 0$ but $\kappa \ll k$. Find the ratio of oscillation frequencies, i.e. ω_f/ω_s, in this limit. Because kappa is small, use the binomial approximation (that is, a Taylor series expansion keeping only the first term to show the lowest-order dependence of this ratio on κ) to express this ratio as $1 + \mathit{something}$.

$\frac{\omega_f}{\omega_s} = \frac{\sqrt{\frac{k+2\kappa}{m}}}{\sqrt{k/m}} = \sqrt{1 + \frac{2\kappa}{k}} \approx 1 + \frac{1}{2}\left(\frac{2\kappa}{k}\right)$ since $(\kappa/k) \ll 1$. So the ratio is $\approx 1 + (\kappa/k)$.

(d) Because the two frequencies are close together, it is natural to talk about "beats" in the general oscillating solution. Morin's section 2.1.4 works this out, using specific initial conditions to set the masses in motion with an equal mixture of the two normal modes. Using those initial conditions and the small-κ approximation, what is the "beat frequency", which is 2ϵ in Morin's notation? – 3 points

Referring to Morin's treatment, we get the fast oscillation frequency $\Omega = \frac{\omega_f+\omega_s}{2} \approx \frac{\omega_s\left(1+\frac{\kappa}{k}\right)+\omega_s}{2} = \omega_s\left(1 + \frac{\kappa}{2k}\right)$ and $\epsilon = \frac{\omega_f-\omega_s}{2} = \frac{\kappa/k}{2}\omega_s = \frac{\kappa}{2k}\omega_s$. The "beat frequency" is the rate of maxima of the envelope "bumps" or "bubbles", which occur twice for each cycle of the ϵ oscillation (the envelope is basically taking the absolute value of the sinusoid), so the beat frequency is $2\epsilon = (\kappa/k)\omega_s$.

(e) Now consider a more concrete case with $\kappa = k/10$ and set this system into motion with specific initial conditions: $\dot{x}_1(0) = \dot{x}_2(0) = 0$, $x_1(0) = \frac{1}{2}A$, $x_2(0) = A$, where A is a constant. Note that this will produce an *unequal* mixture of the two normal modes. Work out the resulting motion $x_1(t)$ and $x_2(t)$. – 4 points

The general solution has four coefficients. Using the same notation as Morin's equation 20,

$$x_1(t) = a\cos cos\,(\omega_s t) + b\sin sin\,(\omega_s t) + c\cos cos\,(\omega_f t) + d\sin sin\,(\omega_f t)$$
$$x_2(t) = a\cos cos\,(\omega_s t) + b\sin sin\,(\omega_s t) - c\cos cos\left(\omega_f t\right) - d\sin sin\,(\omega_f t)$$

and the derivatives are

$$\dot{x}_1(t) =- a\omega_s\sin sin\left(\omega_s t\right) + b\omega_s\cos cos\left(\omega_s t\right) - c\omega_f\sin sin\left(\omega_f t\right) + d\omega_f\cos cos\,(\omega_f t)$$
$$\dot{x}_2(t) =- a\omega_s\sin sin\left(\omega_s t\right) + b\omega_s\cos cos\left(\omega_s t\right) + c\omega_f\sin sin\left(\omega_f t\right) - d\omega_f\cos cos\,(\omega_f t)$$

The four initial conditions give us four equations to determine the four coefficients: $b\omega_s + d\omega_f = 0$, $b\omega_s - d\omega_f = 0$, $a + c = \frac{1}{2}A$, $a - c = A$. The first two equations can only be satisfied if $b = d = 0$, while solving the last two gives $a = \frac{3}{4}A$, $c =- \frac{1}{4}A$.

So $x_1(t) = \frac{3}{4}A\cos cos\,(\omega_s t) - \frac{1}{4}A\cos cos\,(\omega_f t)$, $x_2(t) = \frac{3}{4}A\cos cos\,(\omega_s t) + \frac{1}{4}A\cos cos\,(\omega_f t)$.

(f) EXTRA CREDIT (optional): Finally, plot the motions you found in part e using the parameter values $m = 1\,kg$, $k = 2\,N/m$, and $A = 0.3\,m$. Specifically, use a computer to plot $x_1(t)$ and $x_2(t)$ over the time scale $t = 0$ to 120 s. You can plot them either on the same plot or on two separate plots. Include a screenshot or printout in your pdf submission, or if you can't do that, attach it to a comment in ELMS after submitting your main pdf.

Using the given parameter values, $\omega_s = \sqrt{k/m} = 1.414\,rad/s$ and

$\omega_f = \omega_s\sqrt{1 + \frac{2\kappa}{k}} = 1.549\,rad/s$, or if you use the small-κ approximation, $\omega_f = 1.556\,rad/s$. I used Matlab:

```
>> omegas=1.414; omegaf=1.549;
>> t=(0:0.01:120);   %-- Array of times
>> x1 = 0.75*cos(omegas*t) - 0.25*cos(omegaf*t);
>> x2 = 0.75*cos(omegas*t) + 0.25*cos(omegaf*t);
>> subplot(2,1,1)
>> plot(t,x1); ylabel('x_1')
>> subplot(2,1,2)
>> plot(t,x2); ylabel('x_2'); xlabel('time (s)');
>> print('HW07_Q2f_plot.png','-dpng','-r300')
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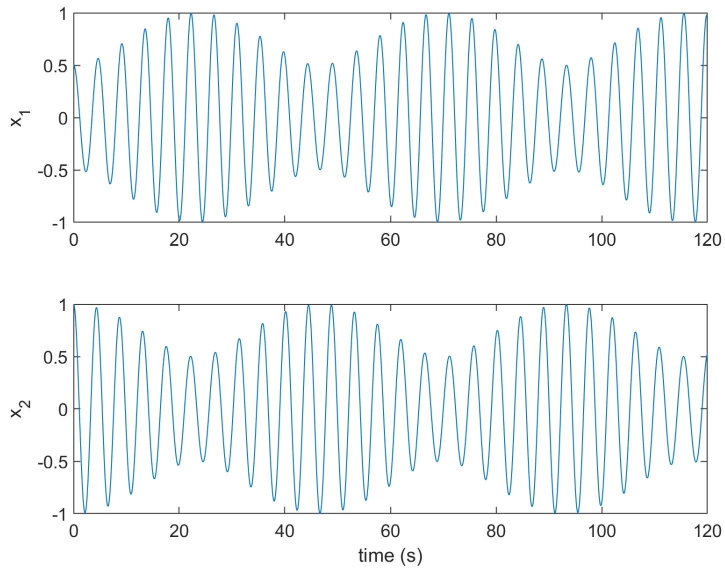


Question 2

Consider a generic system of linear equations with constants p, q, r, s and c:

$$pA_1 + qA_2 = cA_1$$
$$rA_1 + sA_2 = cA_2$$

(a) Write this differently, as a matrix equation with the right side equaling zero. The left side should be the product of a 2×2 matrix and a 2×1 matrix, and the right side should just be a column vector of zeros. (This isn't very hard.)

You can get everything onto the left side of the equation first:

$$(p - c)A_1 + qA_2 = 0$$
$$rA_1 + (s - c)A_2 = 0$$

Then just express that using matrix multiplication:

$$[(p - c)\ q\ r\ (s - c)\ ]\left[A_1\ A_2\right] = [0\ 0\ ]$$

(b) Find the determinant of the 2×2 matrix. When the determinant is equal to zero, there are solutions of the system of equations with A_1 and A_2 not equal to zero. – 2 points

The determinant is $(p - c)(s - c) - qr$. So, solving the equation $(p - c)(s - c) - qr = 0$ for c tells you what values of c make it possible to satisfy this system of equations with nontrivial solutions. I didn't actually ask you to solve for c, but you could do the arithmetic to turn that into a quadratic equation in standard form and then use the quadratic formula.

(c) Now, as an alternative way to solve this, go back to the original pair of equations and use algebra to eliminate one of the A's by substitution. Use that to show that the condition for having solutions with A_1 and A_2 not equal to zero is the same as what you got in part b

We can, for instance, rearrange the second equation to get $A_1 = \frac{c-s}{r} A_2$. Then insert that into the first equation:

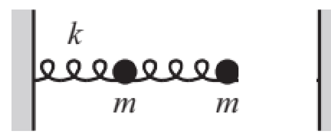
$$(p - c)\frac{c-s}{r} A_2 + qA_2 = 0$$
$$[(p - c)(c - s) + qr]A_2 = 0$$

Or, pulling a negative sign out of the square brackets,

$$[(p - c)(s - c) - qr]\left(- A_2\right) = 0$$

That is satisfied by the trivial solution $A_2 = 0$ (in which case $A_1 = 0$ too), but also is satisfied if $(p - c)(s - c) - qr = 0$, i.e. the same condition we found in part b.

Question 3



Now consider a case where only one of the side springs has been removed, and $\kappa = k$. This system is no longer symmetric, so we can't use the "method 1" approach to find solutions. We need to assume a solution of the form $x_1(t) = A_1 e^{i\omega t}$ and $x_2(t) = A_2 e^{i\omega t}$ and go through the math to find the two normal-mode frequencies.

(a) Do that to work out the two normal-mode frequencies. (Note: to get ω, you may find yourself taking the square root of a quantity that already contains a square root in one term. It looks funny, but it is correct to have nested square-root signs in the end.)

Using coordinates x_1 and x_2 measured from the equilibrium positions and directed toward the right, the stretching of the left spring is given by x_1 while the stretch of the spring connecting the masses is given by $\left(x_2 - x_1\right)$. Using Newton's second law, the equations of motion are:

$m\ddot{x}_1 = - kx_1 + k\left(x_2 - x_1\right)$ which simplifies to $m\ddot{x}_1 = - 2kx_1 + kx_2$

$m\ddot{x}_2 = - k\left(x_2 - x_1\right)$ which can be written $m\ddot{x}_2 = kx_1 - kx_2$

Now assume a solution of the form given in the question, with complex exponentials and (possibly complex) coefficients A_1 and A_2. The second time derivative just brings down a factor of $(i\omega)^2$, i.e. $\ddot{x}_1 = A_1(i\omega)^2 e^{i\omega t} = - \omega^2 A_1 e^{i\omega t}$. The x_1 and x_2 terms also contain $e^{i\omega t}$ – every term does in both equations, in fact – so we cancel those. We are left with

$$- m\omega^2 A_1 = - 2kA_1 + kA_2$$

$$- m\omega^2 A_2 = kA_1 - kA_2$$

With a little algebra they can be written so that each relates A_1 and A_2 more directly:

$kA_2 = \left(2k - m\omega^2\right) A_1$ or $A_2 = \left(2 - \frac{m}{k}\omega^2\right) A_1$

$$kA_1 = \left(k - m\omega^2\right) A_2 \qquad\qquad \text{or } A_1 = \left(1 - \tfrac{m}{k}\omega^2\right) A_2$$

These two equations can be simultaneously satisfied only for certain values of ω. We can eliminate one variable to get a quadratic equation to solve, or else follow the matrix approach that Morin uses on page 3-4 to do the same thing in a more linear-algebra way.

To show the "eliminate one variable" approach, let me first make the substitution $z \equiv \frac{m}{k}\omega^2$. Combining the equations, $A_1 = (1 - z)(2 - z)A_1$. Therefore, solutions must satisfy the quadratic equation $1 = 2 - 3z + z^2$ or equivalently $z^2 - 3z + 1 = 0$. The solutions are $z = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2} = \frac{3 \pm \sqrt{5}}{2}$. Converting z back to omega, $\omega = \pm\sqrt{\frac{k}{m}z} = \pm\sqrt{\frac{k}{m}\left(\frac{3\pm\sqrt{5}}{2}\right)}$.

As a final step, define $\omega_0 \equiv \sqrt{k/m}$, so that $\omega = \pm\sqrt{\frac{3\pm\sqrt{5}}{2}}\,\omega_0$. The negative frequencies are practically equivalent to positive frequencies (both describe sinusoids), so you can omit the ±.

To show the matrix approach, rearrange the equations above into matrix form:

$$\left[m\omega^2 - 2k \;\; k \;\; k \;\; m\omega^2 - k\right]\left[A_1 \;\; A_2\right] = [0 \; 0]$$

The allowed values of ω^2 are those which make the determinant of the matrix zero:

$$\left(m\omega^2 - 2k\right)\left(m\omega^2 - k\right) - k^2 = 0$$
$$m^2\left(\omega^2\right)^2 - 3mk\omega^2 + k^2 = 0$$

This would be the same quadratic equation as above if I made the z substitution, but this time let's just plug it into the quadratic formula as-is:

$$\omega^2 = \frac{3mk \pm \sqrt{(3mk)^2 - 4m^2k^2}}{2m^2} = \frac{3k}{2m} \pm \frac{\sqrt{5m^2k^2}}{2m^2} = \frac{3}{2}\frac{k}{m} \pm \frac{\sqrt{5}}{2}\frac{k}{m} = \left(\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right)\omega_0^2$$

where I defined $\omega_0 \equiv \sqrt{k/m}$. Finally, remember to take the square root:

$\omega = (\pm)\sqrt{\frac{3}{2} \pm \frac{\sqrt{5}}{2}}\,\omega_0$.

(b) For the lower-frequency normal-mode, find the relationship that must be satisfied by the amplitudes A_1 and A_2. Express your answer in the form $A_2 = (some\ factor)\, A_1$. This is telling you the amplitude of the second mass's oscillating motion relative to the first mass's; the amplitudes are not equal for this system.

The lower-frequency mode has $\omega = \sqrt{\frac{3-\sqrt{5}}{2}}\,\omega_0$, or $\omega^2 = \left(\frac{3-\sqrt{5}}{2}\right)\frac{k}{m}$. Stick that back into one of the equations we had earlier that related A_1 and A_2; any such equation will work. To pick one:

$kA_2 = \left(2k - m\omega^2\right) A_1 \qquad \rightarrow \qquad kA_2 = \left[2k - m\left(\frac{3-\sqrt{5}}{2}\right)\frac{k}{m}\right]A_1$

Cancel common factor of k $\quad \rightarrow \qquad A_2 = \left[2 - \frac{3-\sqrt{5}}{2}\right]A_1 = \left(\frac{1+\sqrt{5}}{2}\right)A_1$.

To check whether that is reasonable, the numeric value of the factor $\left(\frac{1+\sqrt{5}}{2}\right)$ is about 1.618 . You may recall that is similar to what you observed with your double-pendulum normal mode: the lower nut swung to a somewhat larger angle than the upper nut.

By the way, $\left(\frac{1+\sqrt{5}}{2}\right)$ happens to be the famous *golden ratio*.

(c) Repeat part b but for the higher-frequency mode. (Hint: the "some factor" will now be smaller in magnitude, and negative.)

The higher-frequency mode has $\omega = \sqrt{\frac{3+\sqrt{5}}{2}}\,\omega_0$, or $\omega^2 = \left(\frac{3+\sqrt{5}}{2}\right)\frac{k}{m}$. Stick that back into one of the equations relating A_1 and A_2 from part a; either equation will work. For instance, $A_2 = \left(2 - \frac{m}{k}\omega^2\right)A_1 = \left(2 - \frac{m}{k}\left(\frac{3+\sqrt{5}}{2}\right)\frac{k}{m}\right)A_1 = \left(\frac{4-(3+\sqrt{5})}{2}\right)A_1 = \left(\frac{1-\sqrt{5}}{2}\right)A_1$. Numerically, that is about $A_2 = (-0.618)\,A_1$.

Question 4

For the 3-mass symmetric system, Morin works out (on pages 9-10) the three normal-mode eigenvectors. Show explicitly that each of these is orthogonal to each of the other two. (There are three pairs.) – 4 points

The three eigenvectors found by Morin, from page 10, are $(1\ 0\ -1)$, $\left(1\ -\sqrt{2}\ 1\right)$, $\left(1\ \sqrt{2}\ 1\right)$. The dot products are:
$(1\ 0\ -1)\cdot\left(1\ -\sqrt{2}\ 1\right) = (1)(1) + (0)\left(-\sqrt{2}\right) + (-1)(1) = 1 + 0 - 1 = 0$,
$(1\ 0\ -1)\cdot\left(1\ \sqrt{2}\ 1\right) = (1)(1) + (0)\left(\sqrt{2}\right) + (-1)(1) = 1 + 0 - 1 = 0$,
$\left(1\ -\sqrt{2}\ 1\right)\cdot\left(1\ \sqrt{2}\ 1\right) = (1)(1) + \left(-\sqrt{2}\right)\left(\sqrt{2}\right) + (1)(1) = 1 - 2 + 1 = 0$. Since these dot products are all zero, the three eigenvectors are all orthogonal. Note that you do have to check all three dot products, because having two vectors orthogonal to the same vector does NOT guarantee that they are orthogonal to each other.