Q1) $y(x) = Ax(L-x)$.

$y(x)$ can be written in the form of a Fourier series.

$$\therefore \quad y(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right)\right]$$

$$a_0 = \frac{1}{L}\int_{-\infty}^{\infty} y(x)\,dx = \frac{1}{L}\int_0^L Ax(L-x)\,dx = \frac{A}{L}\int_0^L Lx - x^2\,dx$$

$$= A\int_0^L x\,dx - \frac{A}{L}\int_0^L x^2\,dx = A\left[\frac{1}{2}x^2\right]_0^L - \frac{A}{L}\left[\frac{1}{3}x^3\right]_0^L$$

$$= \frac{A}{2}L^2 - \frac{A}{3}L^2 = \frac{AL^2}{6} \qquad \therefore \boxed{a_0 = \frac{AL^2}{6}}$$

$$a_n = \frac{2}{L}\int_{-\infty}^{\infty} y(x)\cos\left(\frac{2\pi n x}{L}\right)dx = \frac{2}{L}\int_0^L Ax(L-x)\cos\left(\frac{2\pi n x}{L}\right)dx$$

$$= \frac{2A}{\cancel{L}}\int_0^L x\cdot\cancel{L}\cdot\cos\left(\frac{2\pi n x}{L}\right)dx - \frac{2A}{L}\int_0^L x^2\cos\left(\frac{2\pi n x}{L}\right)dx$$

$$= 2A\underbrace{\int_0^L x\cdot\cos\left(\frac{2\pi n x}{L}\right)dx}_{I_1} - \frac{2A}{L}\underbrace{\int_0^L x^2\cdot\cos\left(\frac{2\pi n x}{L}\right)dx}_{I_{11}}$$

$I_1$: $2A\int_0^L x\cdot\cos\left(\frac{2\pi n x}{L}\right)dx \qquad u = x \qquad dv = \cos\left(\frac{2\pi n x}{L}\right)dx$

$du = dx \qquad v = \frac{L}{2\pi n}\sin\left(\frac{2\pi n x}{L}\right)$.

$$\therefore\ I_1 = 2A\frac{Lx}{2\pi n}\sin\left(\frac{2\pi nx}{L}\right) - 2A\int_0^L \frac{L}{2\pi n}\sin\left(\frac{2\pi nx}{L}\right)dx$$

$$= \frac{ALx}{\pi n}\sin\left(\frac{2\pi nx}{L}\right)\Big|_0^L + \frac{AL^2}{2\pi^2 n^2}\cos\left(\frac{2\pi nx}{L}\right)\Big|_0^L$$

$$= 0 + \frac{AL^2}{2\pi^2 n^2}\cos(2\pi n) - \frac{AL^2}{2\pi^2 n^2}$$

$$I_{11} : \frac{2A}{L}\int_0^L x^2 \cdot \cos\left(\frac{2\pi nx}{L}\right)dx \qquad u = x^2 \qquad dv = \cos\left(\frac{2\pi nx}{L}\right)dx$$

$$du = 2x\,dx \qquad v = \frac{L}{2\pi n}\sin\left(\frac{2\pi nx}{L}\right)$$

$$= \frac{2A}{L}\left[\frac{x^2L}{2\pi n}\sin\left(\frac{2\pi nx}{L}\right) - \frac{L}{\pi n}\int_0^L \underbrace{x\cdot\sin\left(\frac{2\pi nx}{L}\right)dx}\right]$$

$$\hookrightarrow u = x \qquad dv = \sin\left(\frac{2\pi nx}{L}\right)dx$$

$$du = dx \qquad v = \frac{-L}{2\pi n}\cos\left(\frac{2\pi nx}{L}\right)$$

$$= \frac{2A}{L}\left[\frac{x^2L}{2\pi n}\sin\left(\frac{2\pi nx}{L}\right) + \frac{L^2x}{2\pi^2 n^2}\cos\left(\frac{2\pi nx}{L}\right) - \frac{L^2}{2\pi^2 n^2}\int_0^L \cos\left(\frac{2\pi nx}{L}\right)dx\right]$$

$$= \frac{2A}{L}\left[\frac{x^2L}{2\pi n}\sin\left(\frac{2\pi nx}{L}\right) + \frac{L^2x}{2\pi^2 n^2}\cos\left(\frac{2\pi nx}{L}\right) - \frac{L^3}{4\pi^3 n^3}\sin\left(\frac{2\pi nx}{L}\right)\right]_0^L$$

$$= \left[\frac{Ax^2}{\pi n}\sin\left(\frac{2\pi nx}{L}\right) + \frac{ALx}{\pi^2 n^2}\cos\left(\frac{2\pi nx}{L}\right) - \frac{AL^2}{2\pi^3 n^3}\sin\left(\frac{2\pi nx}{L}\right)\right]_0^L$$

$$= \left[0 + \frac{AL^2}{\pi^2 n^2}\cos(2\pi n) - \frac{AL^2}{2\pi^3 n^3}\sin(2\pi n)\right]$$

$$a_n = I_I - I_{II}$$

$$= \frac{AL^2}{2\pi^2 n^2}\cos(2\pi n) - \frac{AL^2}{2\pi^2 n^2} - \left[\frac{AL^2}{\pi^2 n^2}\cos(2\pi n) - \cancelto{0}{\frac{AL^2}{2\pi^3 n^3}\sin(2\pi n)}\right]$$

$$\Rightarrow \quad -\frac{AL^2}{2\pi^2 n^2} - \frac{AL^2}{2\pi^2 n^2} = -\frac{AL^2}{\pi^2 n^2}$$

$$\therefore \quad \boxed{a_n = \frac{-AL^2}{\pi^2 n^2}}$$

$$b_n = \frac{2}{L}\int_{-\infty}^{\infty} y(x)\sin\left(\frac{2\pi n x}{L}\right)dx = \frac{2}{L}\int_0^L Ax(L-x)\sin\left(\frac{2\pi n x}{L}\right)dx$$

$$= 2A\underbrace{\int_0^L x\cdot\sin\left(\frac{2\pi n x}{L}\right)dx}_{I_I} - \frac{2A}{L}\underbrace{\int_0^L x^2\cdot\sin\left(\frac{2\pi n x}{L}\right)dx}_{I_{II}}$$

$I_I$: $2A\int_0^L x\cdot\sin\left(\frac{2\pi n x}{L}\right)dx$ $\qquad u = x \qquad dv = \sin\left(\frac{2\pi n x}{L}\right)dx$

$du = dx \qquad v = \frac{-L}{2\pi n}\cos\left(\frac{2\pi n x}{L}\right)$

$$= -\frac{AxL}{\pi n}\cos\left(\frac{2\pi n x}{L}\right) + \frac{AL}{\pi n}\int_0^L \cos\left(\frac{2\pi n x}{L}\right)dx$$

$$= \frac{-AxL}{\pi n} \cos\left(\frac{2\pi n x}{L}\right)\Big|_0^L + \frac{AL^2}{2\pi^2 n^2} \sin\left(\frac{2\pi n x}{L}\right)\Big|_0^L$$

$$= \frac{-AL^2}{\pi n} \cos(2\pi n) + 0 = \frac{-AL^2}{\pi n} \cos(2\pi n)$$

$$I_{11} : \frac{2A}{L} \int_0^L x^2 \cdot \sin\left(\frac{2\pi n x}{L}\right) dx \qquad u = x^2 \qquad dv = \sin\left(\frac{2\pi n x}{L}\right)$$

$$du = 2x\,dx \qquad v = \frac{-L}{2\pi n} \cos\left(\frac{2\pi n x}{L}\right)$$

$$= \frac{2A}{L} \left[ \frac{-Lx^2}{2\pi n} \cos\left(\frac{2\pi n x}{L}\right) + \frac{L}{\pi n} \underbrace{\int_0^L x \cdot \cos\left(\frac{2\pi n x}{L}\right) dx} \right]$$

$$u = x \qquad dv = \cos\left(\frac{2\pi n x}{L}\right)$$

$$du = dx \qquad v = \frac{L}{2\pi n} \sin\left(\frac{2\pi n x}{L}\right)$$

$$= \frac{2A}{L} \left[ \frac{-Lx^2}{2\pi n} \cos\left(\frac{2\pi n x}{L}\right) + \frac{xL^2}{2\pi^2 n^2} \sin\left(\frac{2\pi n x}{L}\right) + \frac{L^2}{2\pi^2 n^2} \int_0^L \sin\left(\frac{2\pi n x}{L}\right) dx \right]$$

$$= \frac{2A}{L} \left[ -\frac{Lx^2}{2\pi n} \cos\left(\frac{2\pi n x}{L}\right) + \frac{xL^2}{2\pi^2 n^2} \sin\left(\frac{2\pi n x}{L}\right) + \frac{L^3}{4\pi^3 n^3} \cos\left(\frac{2\pi n x}{L}\right) \right]$$

$$= \left[ -\frac{Ax^2}{\pi n} \cos\left(\frac{2\pi n x}{L}\right) + \frac{AxL}{2\pi^2 n^2} \sin\left(\frac{2\pi n x}{L}\right) + \frac{L^2}{2\pi^3 n^3} \cos\left(\frac{2\pi n x}{L}\right) \right]_0^L$$

$$= \left[ \frac{-AL^2}{\pi n} \cos(2\pi n) + 0 + \overbrace{\frac{L^2}{2\pi^3 n^3} \cos(2\pi n) - \frac{L^2}{2\pi^3 n^3}}^{0} \right]$$

$$\therefore \quad b_n = I_1 - I_{11}$$

$$\Rightarrow \quad b_n = \frac{-AL^2}{\pi n} + \frac{AL^2}{\pi n} = 0$$

$$\therefore \quad y(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right)\right]$$

$$\therefore \quad \boxed{y(x) = \frac{AL^2}{6} + \sum_{n=1}^{\infty} \frac{-AL^2}{\pi^2 n^2} \cos\left(\frac{2\pi n x}{L}\right)}$$

Q2)

$$f(x) = \begin{cases} 1 - |x| , & |x| < 1 \Rightarrow x \in [-1,1] \\ 0 , & \text{otherwise} \end{cases}$$

$$C(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \frac{1}{2\pi}\int_{-1}^{1}(1-|x|)\,e^{-ikx}dx$$

$$= \frac{1}{2\pi}\underbrace{\int_{-1}^{0}(1+x)e^{-ikx}dx}_{I_1} + \frac{1}{2\pi}\underbrace{\int_{0}^{1}(1-x)e^{-ikx}dx}_{I_{11}}$$

$$I_1 : \frac{1}{2\pi}\int_{-1}^{0}(1+x)e^{-ikx}dx \qquad u = (1+x) \qquad dv = e^{-ikx}dx$$

$$du = dx \qquad v = \frac{i}{k}e^{-ikx}$$

$$= \frac{1}{2\pi}\left[\frac{i}{k}(1+x)e^{-ikx} - \frac{i}{k}\int_{-1}^{0} e^{-ikx}dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{i}{K}(1+x)e^{-iKx} + \frac{1}{K^2}e^{-iKx}\right]_{-1}^{0}$$

$$= \frac{1}{2\pi}\left[\frac{i}{K} + \frac{1}{K^2} - \left(0 + \frac{1}{K^2}e^{iK}\right)\right]_{-1}^{0}$$

$$= \frac{1}{2\pi}\left[\frac{i}{K} + \frac{1}{K^2} - \frac{1}{K^2}e^{iK}\right]$$

$I_{11}:$ $\frac{1}{2\pi}\int_0^1 (1-x)e^{-iKx}\,dx$ $\qquad U = 1-x \quad dU = -dx \qquad dv = e^{-iKx}dx \quad v = \frac{i}{K}e^{-iKx}$

$$= \frac{1}{2\pi}\left[\frac{i}{K}(1-x)e^{-iKx} + \frac{i}{K}\int_0^1 e^{-iKx}dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{i}{K}(1-x)e^{-iKx} - \frac{1}{K^2}e^{-iKx}\right]_0^1$$

$$= \frac{1}{2\pi}\left[\frac{-1}{K^2}e^{-iK} - \left(\frac{i}{K} - \frac{1}{K^2}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{-1e^{-iK}}{K^2} - \frac{i}{K} + \frac{1}{K^2}\right]$$

$$\therefore \quad C(K) = I_{I} + I_{II}$$

$$= \frac{1}{2\pi}\left[\frac{i}{K} + \frac{1}{K^2} - \frac{1}{K^2}e^{iK}\right] + \frac{1}{2\pi}\left[\frac{-1e^{-iK}}{K^2} - \frac{i}{K} + \frac{1}{K^2}\right]$$

$$= \frac{1}{2\pi}\left[\cancel{\frac{i}{K}} + \frac{1}{K^2} - \frac{1}{K^2}e^{iK} - \frac{1}{K^2}e^{-iK} - \cancel{\frac{i}{K}} + \frac{1}{K^2}\right]$$

$$= \frac{1}{\cancel{2}\pi}\left[\frac{\cancel{2} - \cancel{2}\cos(K)}{K^2}\right]$$

$$= \frac{2\sin^2(K/2)}{\pi K^2}$$

$$\therefore \quad \boxed{C(K) = \frac{2\sin^2(K/2)}{\pi K^2}}$$