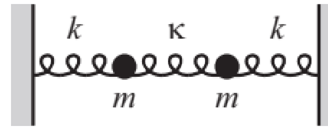
Homework 7

Due on Friday, April 7th (beginning of class)

Problem 1



Consider the system of two masses and three springs discussed in lecture and in Morin's section 2.1 . Let's explore the solutions we found by considering some different specific cases.

(a) Suppose the middle spring has the same spring constant as the side ones, i.e. $\kappa = k$. Using that with the general results we found for this example system, what are the two normal-mode oscillation frequencies, ω_s and ω_f, in this case?

(b) Instead, suppose that the two side springs are made weaker and weaker (that is, $k \to 0$) while the strength of the middle spring (κ) is kept fixed. What are the two oscillation frequencies in this limit? Explain what the smaller frequency means and why it makes sense in that limit, and also explain why the larger frequency is not simply $\sqrt{\kappa/m}$.

(c) Now consider a different limit: the two side springs have their original strength k, while the middle spring is weaker, but not infinitely weak. That is, $\kappa > 0$ but $\kappa \ll k$. Find the ratio of oscillation frequencies, i.e. ω_f/ω_s, in this limit. Because kappa is small, use the binomial approximation (that is, a Taylor series expansion keeping only the first term to show the lowest-order dependence of this ratio on κ) to express this ratio as $1 + something$.

(d) Because the two frequencies are close together, it is natural to talk about "beats" in the general oscillating solution. Morin's section 2.1.4 works this out, using specific initial conditions to set the masses in motion with an equal mixture of the two normal modes. Using those initial conditions and the small-κ approximation, what is the "beat frequency", which is 2ϵ in Morin's notation?

(e) Now consider a more concrete case with $\kappa = k/10$ and set this system into motion with specific initial conditions: $\dot{x}_1(0) = \dot{x}_2(0) = 0$, $x_1(0) = \frac{1}{2}A$, $x_2(0) = A$, where A is a constant. Note that this will produce an *unequal* mixture of the two normal modes. Work out the resulting motion $x_1(t)$ and $x_2(t)$.

(f) Plot the motions you found in part e using the parameter values $m = 1\ kg$, $k = 2\ N/m$, and $A = 0.3\ m$. Specifically, use a computer to plot $x_1(t)$ and $x_2(t)$ over the time scale $t = 0$ to 120 s. You can plot them either on the same plot or on two separate plots. Include a printout with your homework, or send a screenshot to the TA.

Problem 2

Consider a generic system of linear equations with constants p, q, r, s and c:

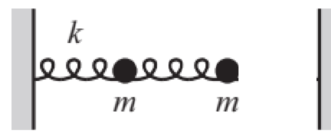
$$pA_1 + qA_2 = cA_1$$
$$rA_1 + sA_2 = cA_2$$

(a) Write this differently, as a matrix equation with the right side equaling zero. The left side should be the product of a 2×2 matrix and a 2×1 matrix, and the right side should just be a column vector of zeros. (This isn't very hard.)

(b) Find the determinant of the 2×2 matrix. When the determinant is equal to zero, there are solutions of the system of equations with A_1 and A_2 not equal to zero.

(c) Now, as an alternative way to solve this, go back to the original pair of equations and use algebra to eliminate one of the A's by substitution. Use that to show that the condition for having solutions with A_1 and A_2 not equal to zero is the same as what you got in part b.

Problem 3



Now consider a case where only one of the side springs has been removed, and $\kappa = k$. This system is no longer symmetric, so we can't use the "method 1" approach to find solutions. We need to assume a solution of the form $x_1(t) = A_1 e^{i\omega t}$ and $x_2(t) = A_2 e^{i\omega t}$ and go through the math to find the two normal-mode frequencies.

(a) Do that to work out the two normal-mode frequencies. (Note: to get ω, you may find yourself taking the square root of a quantity that already contains a square root in one term. It looks funny, but it is correct to have nested square-root signs in the end.)

(b) For the lower-frequency normal-mode, find the relationship that must be satisfied by the amplitudes A_1 and A_2 . Express your answer in the form $A_2 = (some\ factor)\ A_1$. This is telling you the amplitude of the second mass's oscillating motion relative to the first mass's; the amplitudes are not equal for this system.

(c) Repeat part b but for the <u>higher</u>-frequency mode. (Hint: the "some factor" will now be smaller in magnitude, and negative.)

Problem 4

For the 3-mass symmetric system, Morin works out (on pages 9-10) the three normal-mode eigenvectors. Show explicitly that each of these is orthogonal to each of the other two. (There are three pairs.)

Problem 5

A pendulum bob, such as a nut, swinging with a small angle θ experiences a restoring force from the horizontal component of the tension in the string. In this double pendulum setup, the tension in the lower string segment is mg, while the tension in the upper segment is $2mg$ because that part of the string has to support both of the nuts. The horizontal component of the tension from one string has magnitude T sin $sin\ \theta \approx T\theta$ (again, assuming a small angle θ). To go one more step, express that in terms of the horizontal displacement x: for a small angle, $\theta \approx x/l$ where l is the length of the string segment, so the force is $\approx \frac{T}{l}x$. Finally, you need to use the right sign(s) and consider that the upper nut has two strings pulling on it.

(a) Consider a double pendulum, assuming that both string segments have the same length, l, and both nuts have mass m. Sketch this for some nonzero angles, showing x_1 (horizontal displacement of the upper nut) and x_2 (horizontal displacement of the lower nut from its equilibrium position).

(b) Use Newton's second law to write differential equations for the two nuts' positions. The differential equation for $\ddot{x}_1$ should involve both x_1 and x_2 on the right side, while the differential equation for $\ddot{x}_2$ should only involve x_2 on the right side. Then simplify these differential equations by dividing everything by m and by defining $\omega_0 = \sqrt{g/l}$.

(c) Look for a solution of the form $x_1(t) = A_1 e^{i\omega t}$, $x_2(t) = A_2 e^{i\omega t}$: plug those in to get two equations, express those in the form of an eigenvalue problem, and if you have time, find the eigenvalues and eigenvectors.