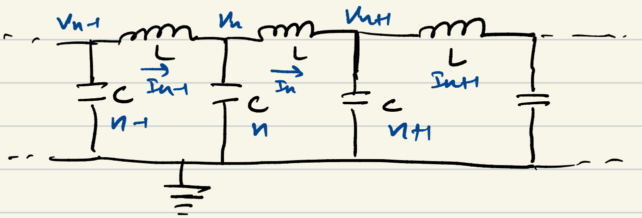
Cable waves: $\rightarrow$ coaxial cable



$\rightarrow$ Charge on any capacitor: $Q = CV$

$\therefore \; q_n = c V_n$

$\rightarrow$ The voltage across an inductor: $V = L \dfrac{dI}{dt}$

$$\therefore \quad V_{n-1} - V_n = L \frac{dI_n}{dt}$$

$\rightarrow$ at a junction: $I_n - I_{n+1} = \dfrac{dq_n}{dt}$

let $L_0 = \dfrac{L}{\Delta x}$, $C_0 = \dfrac{c}{\Delta x}$.

we know: $V_{n-1} - V_n = L \dfrac{dI_n}{dt} \Rightarrow \Delta x \left(-\dfrac{\Delta V}{\Delta x} \right) = L \dfrac{dI_n}{dt}$

$$\therefore \; -\frac{\Delta V}{\Delta x} = \frac{L}{\Delta x} \frac{dI_n}{dt} = L_0 \frac{dI_n}{dt}$$

$$\therefore \; \left(-\frac{1}{L_0} \frac{\partial V}{\partial x} = \frac{\partial I}{\partial t} \right) \frac{\partial}{\partial x}$$

and $I_n - I_{n+1} = \frac{dq_n}{dt} \Rightarrow -\Delta x\left(\frac{\Delta I}{\Delta x}\right) = \frac{dq_n}{dt} = c\,\frac{dV_n}{dt}$

$\therefore\; -\frac{\Delta I}{\Delta x} = C_0\,\frac{dV_n}{dt}$

$\Rightarrow \left(\frac{\partial I}{\partial x} = -C_0\,\frac{\partial V}{\partial t}\right)\frac{\partial}{\partial t}$

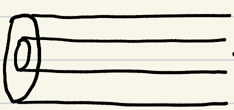
$\therefore\; -\frac{1}{L_0}\,\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 I}{\partial x\,\partial t}$ & $-C_0\,\frac{\partial^2 V}{\partial t^2} = \frac{\partial^2 I}{\partial x\,\partial t}$

$\therefore\; \frac{1}{L_0}\,\frac{\partial^2 V}{\partial x^2} = C_0\,\frac{\partial^2 V}{\partial t^2}$

$\Rightarrow \boxed{\frac{\partial^2 V}{\partial t^2} = \frac{1}{L_0C_0}\,\frac{\partial^2 V}{\partial x^2}} \rightarrow$ non dispersive wave

$\therefore$ speed : $c = \frac{1}{\sqrt{L_0C_0}}$

consider:



$L_0 = \frac{\mu_0}{2\pi}\ln\left(\frac{r_2}{r_1}\right)$, $C_0 = \frac{2\pi\epsilon_0}{\ln(r_2/r_1)}$

$\therefore\; c \approx 3\times 10^8\ m/s$.

Electromagnetic (EM) waves:

Maxwell's equations :

Differential form:

$$\bar{\nabla}\cdot\bar{E} = \frac{\rho_E}{\epsilon_0}$$

$$\bar{\nabla}\cdot\bar{B} = 0 = \rho_B$$

$$\bar{\nabla}\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} + \bar{J}_B$$

$$\bar{\nabla}\times\bar{B} = \mu_0\epsilon_0\frac{\partial\bar{E}}{\partial t} + \mu_0\bar{J}_E$$

Integral form:

$$\int \bar{E}\cdot d\bar{a} = \frac{Q_E}{\epsilon_0}$$

$$\int \bar{B}\cdot d\bar{a} = 0 = Q_B$$

$$\int \bar{E}\cdot d\bar{l} = -\frac{d\Phi_B}{dt} + I_B$$

$$\int \bar{B}\cdot d\bar{l} = \mu_0\epsilon_0\frac{d\Phi_E}{dt} + \mu_0 I_E$$

assume a space with only EM waves : (no charges)

$$\left.\begin{aligned}
\bar{\nabla}\cdot\bar{E} &= 0\\
\bar{\nabla}\cdot\bar{B} &= 0\\
\bar{\nabla}\times\bar{E} &= -\frac{\partial\bar{B}}{\partial t}\\
\bar{\nabla}\times\bar{B} &= \mu_0\epsilon_0\frac{\partial\bar{E}}{\partial t}
\end{aligned}\right\}\ \text{modified maxwell's equations}$$

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{E}) = \bar{\nabla}\times\frac{\partial\bar{B}}{\partial t} = \frac{\partial}{\partial t}(\bar{\nabla}\times\bar{B}) = -\mu_0\epsilon_0\frac{\partial^2 E}{\partial t^2}$$

$$\Rightarrow \bar{\nabla}(\cancelto{0}{\bar{\nabla}\cdot\bar{E}}) - (\bar{\nabla}\cdot\bar{\nabla})\bar{E} = -\mu_0\epsilon_0\frac{\partial^2 E}{\partial t^2}$$

$$\therefore \boxed{\bar{\nabla}^2\bar{E} = \mu_0\epsilon_0\frac{\partial^2\bar{E}}{\partial t^2}} \quad ; \quad \bar{\nabla}^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

in 1D : $\frac{\partial^2 \bar{E}}{\partial t^2} = c^2 \frac{\partial^2 \bar{E}}{\partial x^2}$ or $\frac{\partial^2 \bar{B}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \bar{\nabla}^2 \bar{B}$

where $c \equiv \frac{1}{\sqrt{\mu_0 \epsilon_0}} \sim 3 \times 10^8$ m/s

if we have a dielectric: $v = \frac{1}{\sqrt{\mu \epsilon}} = \frac{c}{n}$; $n \rightarrow$ refractive index

$\therefore n = \frac{c}{v} = \sqrt{\frac{\mu \epsilon}{\mu_0 \epsilon_0}}$

for most dielectrics: $\mu \approx \mu_0$, $\therefore n = \sqrt{\frac{\epsilon}{\epsilon_0}}$

taking the electric field equation:

$$\frac{\partial^2 E_i}{\partial t^2} = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E_i \quad ; \; i = x, y, z$$

$\therefore \bar{E} = \bar{E}_0 e^{i(\bar{k} \cdot \bar{r} - \omega t)}$, $\bar{B} = \bar{B}_0 e^{i(\bar{k} \cdot \bar{r} - \omega t)}$

$|\bar{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{2\pi}{\lambda}$

$E_x = E_{0x} e^{i(k_x x + k_y y + k_z z - \omega t)}$ $\rightarrow$ plugging into the diff eq.

$\therefore -\omega^2 = -c^2 \left[k_x^2 + k_y^2 + k_z^2 \right]$

$\Rightarrow \omega^2 = |\bar{k}|^2 c^2 \Rightarrow \omega = c|\bar{k}|$

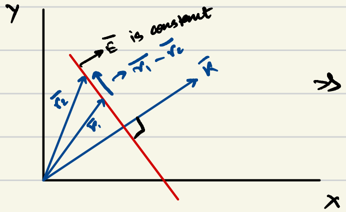
↳ dispersionless wave

Hence, as long as the wave is in 1 medium, dispersion does not occur.

we know:

$\bar{E} = \bar{E}_0 \, e^{i(\bar{k}\cdot\bar{r} - \omega t)}$

at some time 't', we will have constant $\bar{E}$ when $\bar{k}\cdot\bar{r} = c$



$\bar{k}\cdot\bar{r}_1 = \bar{k}\cdot\bar{r}_2$

$\Rightarrow \bar{k}\cdot(\bar{r}_1 - \bar{r}_2) = 0$

we know: $\bar{\nabla}\cdot\bar{E} = 0$

$$\Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

$$\Rightarrow ik_x E_x + ik_y E_y + ik_z E_z = 0$$

$$\Rightarrow \boxed{\bar{k}\cdot\bar{E} = 0} \rightarrow \bar{k} \perp \bar{E} \; *$$

Similarly, $\boxed{\bar{k}\cdot\bar{B} = 0} \rightarrow \bar{k} \perp \bar{B} \; *$

$\rightarrow$ $\bar{\nabla}\times\bar{E} = -\dfrac{\partial B}{\partial t}$

$\Rightarrow \bar{k}\times\bar{E} = \omega\bar{B}$

Hence, $\bar{B}$ is perpendicular to both $\bar{k}$ & $\bar{E}$

$\therefore \bar{E} \perp \bar{B} \; *$

Since, $\bar{K} \perp \bar{E} \Rightarrow |\bar{K} \times \bar{E}| = |\bar{K}||\bar{E}| \sin\frac{\pi}{2} = \omega B$

$$\therefore \bar{E} = \frac{\omega \bar{B}}{|\bar{K}|}$$

$$\Rightarrow \boxed{\bar{E} = c\bar{B}}$$

→ $\bar{\nabla} \times \bar{B} = \frac{1}{c^2} \frac{\partial \bar{E}}{\partial t}$

$\Rightarrow \bar{K} \times \bar{B} = -\frac{1}{c^2} \omega \bar{E}$ → which proves nothing new.

$$\therefore \boxed{\begin{array}{l} \bar{E} \perp \bar{K} \\ \bar{B} \perp \bar{K} \\ \bar{E} \perp \bar{B} \\ E = cB \end{array}}$$

→ Hence, the wave is traveling in the 'k' direction. It is known as the 'wave vector'.

<u>Standing waves:</u>

$\longrightarrow$ $\longleftarrow$ z axis

$\bar{E}_1 = \hat{i} E_0 \cos(kz - \omega t)$ $\bar{E}_2 = \hat{i} E_0 \cos(kz + \omega t)$

$\vec{B}_1 = \hat{j} B_0 \cos(kz - \omega t)$ $\bar{B}_2 = -\hat{j} B_0 \cos(kz + \omega t)$

$\therefore \bar{E}_{standing} = \hat{i}\,(2E_0) \cos kz \cos \omega t$

*

$\bar{B}_{standing} = \hat{j}\left(\frac{2E_0}{c}\right) \sin kz \sin \omega t$

} $\bar{E}$ & $\bar{B}$ are perp.

* $(\bar{E}_{stan})\ max = (\bar{B}_{stand})\ min$

Hence, $\bar{E}$ & $\bar{B}$ are out of phase by $\pi/2$ for standing waves & in phase for traveling waves

<u>Energy:</u>

$\mathcal{E}_E = \frac{1}{2}\epsilon_0 E^2$ (energy density)

$\mathcal{E}_B = \frac{1}{2\mu_0} B^2$ " "

$\therefore\ \mathcal{E} = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2$ $(\bar{E}^2 = \bar{E}\cdot\bar{E})$

$$\frac{\partial \mathcal{E}}{\partial t} = \epsilon_0 \bar{E}\cdot\frac{\partial \bar{E}}{\partial t} + \frac{1}{\mu_0}\bar{B}\cdot\frac{\partial \bar{B}}{\partial t}$$

$$= \epsilon_0 \bar{E}\cdot\left[\frac{1}{\mu_0\epsilon_0}\bar{\nabla}\times\bar{B}\right] - \frac{1}{\mu_0}\bar{B}\cdot(-\bar{\nabla}\times\bar{E})$$

$$= \frac{1}{\mu_0}\left[\bar{E}\cdot(\bar{\nabla}\times\bar{B}) - \bar{B}\cdot(\bar{\nabla}\times\bar{E})\right]$$

$$= \frac{1}{\mu_0}\left[\bar{\nabla}\cdot(\bar{B}\times\bar{E})\right]$$

$$\therefore \int_V \frac{\partial \mathcal{E}}{\partial t}\,dV = \int \frac{1}{\mu_0}\bar{\nabla}\cdot(\bar{B}\times\bar{E})\,dV$$

$$\Rightarrow \frac{\partial}{\partial t}(W_V) = \frac{1}{\mu_0}\int \underset{\substack{\downarrow \\ \text{energy going out}}}{(\bar{B}\times\bar{E})\cdot d\bar{A}} = \frac{1}{\mu_0}\int \underset{\substack{\downarrow \\ \text{energy going in}}}{(\bar{E}\times\bar{B})\cdot d\bar{A}_{in}}$$

$$\therefore \frac{\partial W_V}{\partial t} = \int \bar{S}\cdot d\bar{A}_{in} \ ; \quad S \equiv \frac{1}{\mu_0}\bar{E}\times\bar{B} \rightarrow \text{poynting vector.}$$

$\bar{E}\times\bar{B}$ points in the $\bar{k}$ direction, hence $\bar{S}$ points in the $\bar{k}$ direction

→ Traveling wave:

$$\mathcal{E} = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2 \quad , \quad B = \frac{E}{c}$$

$$\therefore \quad \mathcal{E} = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}\frac{E^2}{c^2}$$

$\rightarrow \epsilon_0 E^2$

$$= \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\epsilon_0 E^2$$ → energy density is same for both E, B

$$\therefore \bar{S} = \frac{1}{\mu_0}\bar{E}\times\bar{B} = \frac{1}{\mu_0}EB\,\hat{k} = \frac{1}{\mu_0}E\cdot\frac{E}{c}\,\hat{k} = \frac{1}{\mu_0}\frac{E^2}{c}\,\hat{k}$$

↳ wave vector

$$\therefore \bar{S} = c\epsilon_0 E^2\,\hat{k} = c\,\mathcal{E}\,\hat{k} \qquad (\mathcal{E} = \epsilon_0 E^2)$$

$$\therefore \quad |\bar{S}| = c\,\mathcal{E}$$

we know:
$$\bar{E} = \hat{i}\,E_0\cos(kz-\omega t)$$
$$\bar{B} = \hat{j}\,\frac{E_0}{c}\cos(kz-\omega t)$$

$$\therefore \quad \mathcal{E} = \epsilon_0 E_0^2\cos^2(kz-\omega t)$$

$$\therefore \quad \mathcal{E}_{avg} = \langle \mathcal{E} \rangle = \frac{1}{2}\epsilon_0 E_0^2$$

$$\bar{S} = \frac{1}{\mu_0}\bar{E}\times\bar{B} = c\epsilon_0\cos^2(kz-\omega t)\,\hat{k}$$

$$\therefore \quad |S_{avg}| = \frac{1}{2}c\epsilon_0 E_0^2$$

↳ intensity of the EM wave

→ standing wave:

$$\bar{E} = \hat{i}\, A \cos kz \cos\omega t$$

$$\bar{B} = \hat{j}\, \frac{A}{c} \sin kz \sin\omega t$$

$$\therefore \quad \mathcal{E} = \frac{1}{2}\epsilon_0 A^2 \cos^2 kz \cos^2\omega t + \frac{1}{2\mu_0}\frac{A^2}{c^2}\sin^2 kz \sin^2\omega t$$

$$\therefore \quad \langle \mathcal{E} \rangle_{time} = \frac{1}{4}\epsilon_0 A^2 \cos^2 kz + \frac{1}{4\mu_0}\frac{A^2}{c^2}\sin^2 kz$$

$$= \frac{1}{4}\epsilon_0 A^2(\cos^2 kz + \sin^2 kz)$$

$$= \frac{1}{4}\epsilon_0 A^2$$

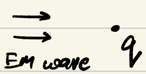
$$\therefore \quad \bar{S} = \frac{1}{\mu_0}\bar{E}\times\bar{B} = \frac{A^2}{c\mu_0}\cos(kz)\sin kz \underbrace{\cos\omega t \sin\omega t}_{\frac{1}{2}\sin(2\omega t)} \hat{k}$$

$$\therefore \quad |\bar{S}_{avg}| = 0$$

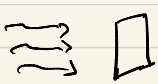
Since energy density is constant, the average energy going into the space has to be zero.

Momentum:

— EM waves carry momentum



→ experiences non zero force



→ $\frac{\text{Force}}{\text{Area}}$ = radiation pressure

polarization:

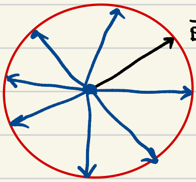
→ say we have a wave in z-direction

$\bar{E} = \hat{i}\, E_0 \cos(kz - \omega t)$, $\bar{B} = \hat{j}\, \frac{E_0}{c} \cos(kz - \omega t)$

This is an example when $\bar{E}$ always points along the x axes. This is a 'linearly polarized' wave.

→ elliptical (circular) polarization:

assume:



also imaginary

we know: $\bar{E}(\bar{r}, t) = \bar{E}_0\, e^{i(\bar{k}\cdot\bar{r} - \omega t)}$

$\therefore$ actual $\bar{E} = \mathrm{Re}[\bar{E}(\bar{r}, t)]$

$= \bar{E}_0 \cos(\bar{k}\cdot\bar{r} - \omega t)$ & $\bar{B}_0 \cos(\bar{k}\cdot\bar{r} - \omega t)$

If we take a wave traveling in the z direction $(\vec{k} = k\hat{k})$

ex: $\bar{E} = E_x \hat{i} \Rightarrow \bar{B} = B_y \hat{j}$

$= E_0 \cos(kz - \omega t)\hat{i}$ or $B_0 \cos(kz - \omega t)\hat{j}$

This is linear polarization.

→ This wave will have components in both x & y dir.

$\therefore E_x = E_{0x}\, e^{i(kz - \omega t)}$, $E_y = E_{0y}\, e^{i(kz - \omega t)}$

$E_{0x} \equiv |E_{0x}|\, e^{i\phi_x}$, $E_{0y} \equiv |E_{0y}|\, e^{i\phi_y}$

→ choose x-axis such that $E_{0x} = |E_{0x}|$, hence,
$E_{0y} = |E_{0y}| e^{i\phi}$

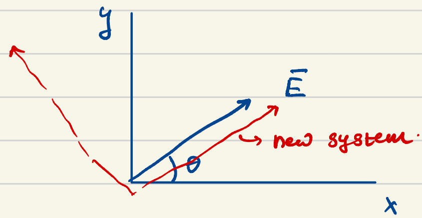
$$\therefore \quad E_x = |E_{0x}| e^{i(kz-\omega t)}$$
$$E_y = |E_{0y}| e^{i(kz-\omega t+\phi)}$$

$$\therefore \quad \bar{E} = |E_0| \cos(kz-\omega t)\,\hat{i} + |E_{0y}| \cos(kz-\omega t+\phi)\,\hat{j}$$

if $\phi = 0$, hence: $\bar{E} = |E_{0x}| \cos(kz-\omega t)\hat{i} + |E_{0y}| \cos(kz-\omega t)\hat{j}$

$$= \cos(kz-\omega t)\,[\underbrace{|E_{0x}|\hat{i} + |E_{0y}|\hat{j}}]$$

Hence, for $\phi = 0$, we get lineary polarized light



if $\phi = \frac{\pi}{2}$:

$$\therefore \quad \bar{E} = \hat{i}\,|E_{0x}| \cos(kz-\omega t) + \hat{j}\,|E_{0y}| \cos(kz-\omega t+\tfrac{\pi}{2})$$

$$= \hat{i}\,|E_{0x}| \cos(kz-\omega t) - \hat{j}\,|E_{0y}| \sin(kz-\omega t)$$

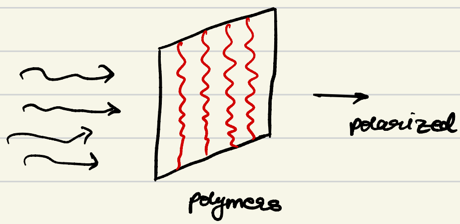
at $z = 0$: $\bar{E} = \hat{i}\,|E_{0x}| \cos(\omega t) + \hat{j}\,|E_{0y}| \sin(\omega t)$
$$= \hat{i}\,E_0 \cos\omega t + \hat{j}\,E_0 \sin\omega t$$

we can observe that this vector is rotating, hence this is known as 'circulary polarized' wave if $\phi = \frac{\pi}{3}$:

$\bar{E} = \hat{i}\, E_0 \cos(kz - \omega t) + \hat{j}\, E_0 \cos(kz - \omega t + \frac{\pi}{3})$

at $z = 0$: $\bar{E} = \hat{i}\, E_0 \cos(\omega t) + \hat{j}\, E_0 \cos(-\omega t + \frac{\pi}{3})$

this is an example of 'Elliptical polarization'.

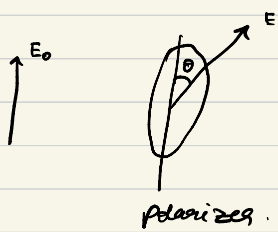


→ Read the textbook to understand the methods of polarization

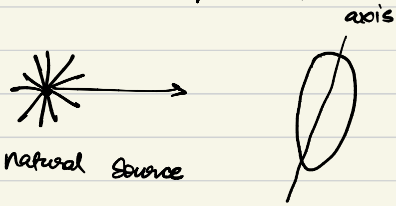
we know $I \propto E^2$
↳ intensity

$\therefore \quad E = E_0 \cos\theta$

$\therefore \quad I_{after} = \cos^2\theta \; I_{befor}$



$\therefore \quad \underline{\underline{I_{af} = I_{in} \cos^2\theta}}$



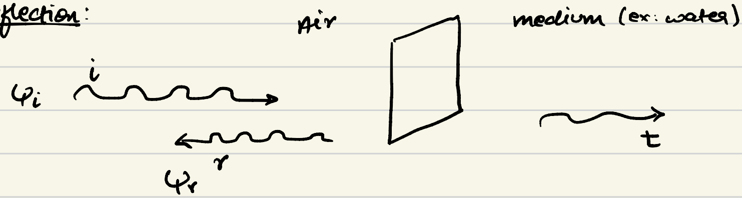
there is randomness

$I_{af} = I_{in} \cos^2\theta$
↳ random angle

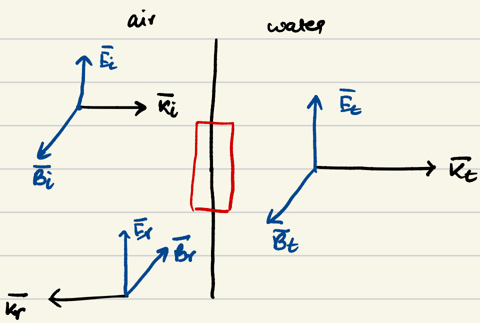
$\therefore$ on average: $\langle I_{af} \rangle = \frac{1}{2} I_{in}$

This is Malus' law

Reflection:



$\varphi_1 = \varphi_i + \varphi_r$ $\qquad$ $\varphi_2 = \varphi_t$



Use:
$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$\bar{E}_1 = \bar{E}_i + \bar{E}_r$ $\qquad$ $\bar{E}_2 = \bar{E}_t$

Consider the red area: $\int_S \bar{\nabla} \times \bar{E} \cdot d\bar{A} = -\underbrace{\int \frac{\partial \bar{B}}{\partial t} \cdot dt}$

$\hookrightarrow -\frac{\partial \Phi_B}{\partial t} \sim 0$

↳ assume no flux

$\therefore \int_S (\bar{\nabla} \times \bar{E}) \cdot d\bar{A} = \oint \bar{E} \cdot d\bar{l} = 0$

$\uparrow$ Stokes'

$\hookrightarrow E_1^{\parallel} = E_2^{\parallel}$

$\rightarrow \quad \bar{\nabla} \times \bar{H} = \frac{\partial \bar{D}}{\partial t}, \quad \bar{H} \equiv \frac{\bar{B}}{\mu}, \quad \bar{D} = \epsilon \bar{E}$

$\therefore \quad H_1^{\parallel} = H_2^{\parallel}$

$\Rightarrow \quad \frac{B_1^{\parallel}}{\mu_1} = \frac{B_2^{\parallel}}{\mu_2}$

we know: $E_i + E_r = E_t$

$\Rightarrow \quad H_i^{\parallel} - H_r^{\parallel} = H_t^{\parallel}$

define impedence: $Z = \frac{E}{H} = \frac{E}{B/\mu} = \mu \frac{E}{B} = \mu v = \sqrt{\frac{\mu}{\epsilon}}$

$\therefore \quad E_i + E_r = E_t$

& $\quad \frac{E_i}{Z_1} - \frac{E_r}{Z_1} = \frac{E_t}{Z_2}$

$\therefore \quad \boxed{E_t = \frac{2Z_2}{Z_1 + Z_2} E_i}$

& $\quad \boxed{E_r = \frac{Z_2 - Z_1}{Z_1 + Z_2}}$

$\boxed{H_r = \frac{Z_2 - Z_1}{Z_1 + Z_2} H_i}$ & $\boxed{H_t = \frac{2Z_1}{Z_1 + Z_2} H_i}$

$Z \propto \frac{1}{n} \quad \therefore \quad E_r = \frac{n_1 - n_2}{n_1 + n_2} E_i$ & $E_t = \frac{2n_1}{n_1 + n_2}$

Refraction