Physics 273 — Homework 1 Solutions

Question 1 (3 points)

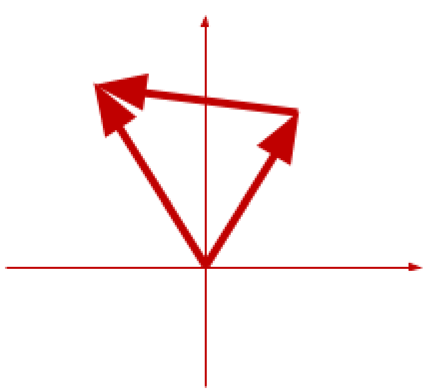
(a) Convert the number $4 + 6i$ to the form $Ae^{i\phi}$, i.e. find the (real-valued) magnitude A and the complex phase ϕ, with the latter measured in radians. (It's not a convenient fractional number of radians in this case, so you'll have to write out a decimal.)

This is equivalent to converting between Cartesian (rectangular) and polar coordinates. Get the magnitude $A = 7.21$ using the Pythagorean theorem, and get the angle ϕ with a little trig: $\tan tan\ \phi = 6/4$ is the most direct way, but having gotten A, you could also get ϕ via $\cos cos\ \phi = 4/7.21$ or $\sin sin\ \phi = 6/7.21$. Any way you do it, $\phi = 0.983$ rad, so the answer can be written $(7.21)e^{i(0.983)}$. (In general you have to be careful about what quadrant you are in when you use inverse trig functions, but in this case we are in the first quadrant so the calculated angles are correct.)

(b) What is $5e^{i(0.7)\pi} - 4e^{i(0.3)\pi}$? (The complex phases of the two numbers are $0.7\ \pi$ radians and $0.3\ \pi$ radians.) Express your answer in the form $Ae^{i\theta}$, i.e. find numerical values of A and θ. (Hint: find numerical values of the real and imaginary parts—keeping several decimal places—and subtract those, then convert to the “polar” form.)

We get $(-\ 2.939 + 4.045\ i) - (2.351 + 3.236\ i) =-\ 5.290 + 0.809\ i$. When converting to polar form like we did in part a, we have to be careful about the phase: $(0.809/5.290) =-\ 0.152$ rad, but that would be a vector in the fourth quadrant (positive x, negative y) and just looking at the complex number we have here, it must be in the second quadrant (negative x, positive y). So we need to add π to that angle, getting 2.990 rad. (If you're puzzled by that, remember $\tan tan\ \theta = \tan tan\ (\theta + \pi)$ in general, so the arctangent function can't tell which angle you want.) We get $(5.352)e^{i(2.990)}$.

(c) Thinking about part b graphically, sketch the two original complex numbers as vectors on a complex number plane (based at the origin), or else use a computer to plot them accurately. Then sketch or plot the complex number you calculated in part b as a vector completing a triangle (i.e., not based at the origin). Does this third vector in your sketch or plot look consistent with the numerical answer (magnitude and phase) you got in part b?



Remember how vector subtraction works: $\vec{A} - \vec{B} = \vec{A} + (-\ \vec{B})$. So the resulting complex number vector should go mostly left and a bit up. (Not right and down.)

(d) Consider two general complex numbers $Ae^{i\alpha}$ and $Be^{i\beta}$. Generalizing what you did in part b and visualized in part c, subtract one from the other and square the magnitude of that to explicitly prove the *law of cosines*: $C^2 = A^2 + B^2 - 2AB \cos cos\, \theta$ where $\theta = \alpha - \beta$ is the angle between the vectors. – 4 points

$Ae^{i\alpha} - Be^{i\beta} = (A \cos cos\, \alpha - B \cos cos\, \beta) + i(A \sin sin\, \alpha - B \sin sin\, \beta)$. The square of the magnitude of that – i.e., the square of the length of the third side of the triangle – is $C^2 = (A \cos cos\, \alpha - B \cos cos\, \beta)^2 + (A \sin sin\, \alpha - B \sin sin\, \beta)^2$. When you carry out the squares, you can combine terms using the standard identity relating sine-squared and cosine-squared, and it becomes $A^2 + B^2 - 2AB(\cos cos\, \alpha \cos cos\, \beta + \sin sin\, \alpha \sin sin\, \beta)$. Using the angle difference trig identity, that last factor is $\cos cos\, (\alpha - \beta)$, and we have what we were trying to prove.

Question 2 (3 points)

Expand $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$ using Euler's formula on both sides. Use your result to prove the standard trig sum formulas: $\sin sin\, (\alpha + \beta) = \ldots$ and $\cos cos\, (\alpha + \beta) = \ldots$.

Expanding on both sides,
$\cos cos\, (\alpha + \beta) + i \sin sin\, (\alpha + \beta) = (\cos cos\, \alpha + i \sin sin\, \alpha)(\cos cos\, \beta + i \sin sin\, \beta)$. Using $i^2 = -1$ and equating the (real/imaginary) parts on each side with each other gives the standard angle-sum trig formulas.

Question 3 (3 points)

Re-express the sinusoidal function $x(t) = 6 \cos cos \left(\omega t - \frac{\pi}{3}\right)$ in the form $x(t) = Re\left\{Ce^{i\omega t}\right\}$ such that they are equivalent functions. That is, find the value of the complex constant C, expressed in the form $a + bi$.

We can write $x(t) = Re\{6\, e^{i(\omega t - \pi/3)}\} = Re\{6\, e^{-i\pi/3}\, e^{i\omega t}\}$, so $C = 6\, e^{-i\pi/3}$. Expanding the the complex exponential out in terms of cosine and sine, we get $C = 3.00 - 5.20\, i$. One check you can do is to put in $t = 0$ and make sure you get the same thing from both versions of the function.

Question 4 (3 points)

(a) Let $\theta(t)$ be an arbitrary function of time. If $x(t) = Ae^{i\theta(t)}$, find $\frac{dx}{dt}$, i.e. the velocity $v(t)$. (Hint: use the chain rule.)

Using the chain rule, $\frac{dx}{dt} = Ae^{i\theta(t)} \frac{d}{dt}[i\theta(t)] = Ai\frac{d\theta}{dt}\, e^{i\theta(t)}$. Writing $\frac{d\theta}{dt}$ as $\dot{\theta}$ is fine too.

(b) For the special case that $\theta(t)$ grows linearly with time, i.e. $\theta(t) = \theta_0 + \omega t$, show that $v(t) = i\omega\, x(t)$. (Of course, the *actual* velocity is just the real part of the complex-valued $v(t)$.)
– 3 points

For this function, $\frac{d\theta}{dt}$ is simply ω. Putting that in and identifying the factors that make up $x(t)$ gives us $\frac{dx}{dt} = i\omega\, x(t)$.

Question 6 (3 points)

Morin's "long way" of finding the motion of a simple harmonic oscillator in section 1.1.2 involved getting an expression for v as a function of x (equation 5), then writing v as dx/dt, separating those onto two sides of the equation, and integrating. (However, Morin has a typo: the left side of equation 6 is missing an integral sign.)

(a) Pick it up from there and go step-by-step, i.e. make the key substitution, do the integration, and carry it through to find the result $x(t) = A \cos cos\, (\omega t + \phi)$ with $A = \sqrt{2E/k}$.

Starting from Morin's equation 6 with the missing integral sign added,

$$\frac{1}{\sqrt{E}}\int \frac{dx}{\sqrt{1-\frac{kx^2}{2E}}} = \pm\sqrt{\frac{2}{m}}\int dt$$

Make the substitution $\sqrt{\frac{k}{2E}}x = \cos cos\, \theta$. Then the trig relationship between sine and cosine makes the square-root factor become $\sin sin\, \theta$, and $dx = -\sqrt{\frac{2E}{k}}\sin sin\, \theta\ d\theta$, and the left-side integral simplifies beautifully. The right-side integral is trivial, of course. We get

$$-\sqrt{\frac{2}{k}}\theta = \pm\sqrt{\frac{2}{m}}t + Constant$$

And then

$$\theta = \mp\sqrt{\frac{k}{m}}t + Constant$$

This is not the same constant as the previous line, but still a constant! Actually I will now call that constant ϕ. Take the cosine of both sides of the equation:

$$\sqrt{\frac{k}{2E}}\, x(t) = \cos cos\, \theta\ = \cos cos \left(\sqrt{\frac{k}{m}}t + \phi\right)$$

where I have dropped the $\mp$ because cosine is an even function, so they are equivalent solutions. So finally, $x(t) = A \cos cos\, (\omega t + \phi)$ with $A = \sqrt{2E/k}$ and $\omega = \sqrt{k/m}$.

(b) Where did the ϕ come from? Explain.

From the constant of integration – really the combination of constants of integration – when doing the indefinite integrals.

(c) Starting with the expression for $x(t)$ from above, show explicitly that the total energy of the harmonic oscillator is conserved (i.e., independent of time) and is equal to E. – 4 points

We have $x(t)$ above. Take the derivative of that to get $v(t)$, the velocity as a function of time, remembering to use the chain rule (which brings out a factor of ω). Write the total energy as the sum of potential ($\frac{1}{2}kx^2$) and kinetic ($\frac{1}{2}mv^2$), each of which is a sinusoidal function of time, but when added together, they have the form of a common constant times $(\omega t + \phi) + (\omega t + \phi)$, which is 1. Showing that the constant factor is the same in both terms involves using $\omega^2 = k/m$, and showing that the overall factor is equal to E involves going back to $A = \sqrt{2E/k}$.