Homework 6

Due at the beginning of class on Friday, March 31st

Problem 1

An RLC electrical circuit which is NOT driven by an AC voltage source is mathematically equivalent to an un-driven mechanical oscillator.

(a) Write down the standard differential equation for $Q(t)$ for a generic RLC circuit.

(b) Now consider the three electrical properties (resistance, capacitance, inductance) and their reciprocals $(\frac{1}{R}, \frac{1}{C}, \frac{1}{L})$. Look at where they appear in your differential equation from part a, and deduce from that how they relate to the properties of a mechanical oscillator. From among those six quantities,

(i) Which plays the role of the mass of the oscillator?

(ii) Which plays the role of the spring constant (k) for the oscillator?

(iii) Which plays the role of the damping constant (b) for the oscillator?

(c) Examining the differential equation can also tell you how the units (dimensions) of the electrical properties are related, and how they can be combined.

(i) What combination of R and L (e.g. a product or a ratio) has dimensions of time?

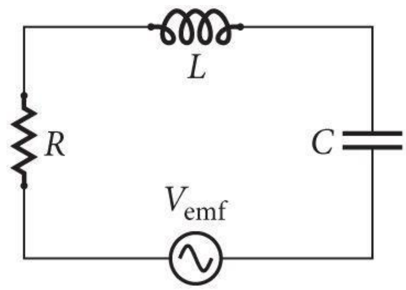
(ii) What combination of R and C has dimensions of time?

(iii) What combination of L and C has dimensions of time?

(d) The combinations in part c define time scales, so the dynamics of an undriven circuit will be determined by those. Each combination can therefore be called a "time constant", but context determines which one is relevant for a given circuit. For example, consider a circuit containing a resistance R and a capacitance C, and no inductor. The combination you found in (c)(ii) above has dimensions of time, so call that combination τ (tau). If you connect the circuit to a constant voltage V, such as a battery, then τ sets the scale for how long it takes the capacitor to charge up. Specifically, it will approach a maximum charge $Q_{max} = CV$ in an exponential fashion: $Q(t) = Q_{max}\left(1 - e^{-t/\tau}\right)$. Now let's put in some numbers to try this out… if the circuit has a $21.0\ \mu F$ capacitor and a 140 Ω resistor, then what is τ (with units)? And how long will it take the capacitor to charge up to 99.9% of Q_{max}? (Hint: your answer should be between 0.01 and 0.05 seconds.)

Problem 2

A series RLC circuit has component values $R = 8.9\ \Omega$, $L = 2.12$ H and $C = 12.0$ μF, as well as an oscillating voltage (emf) source of the form $V(t) = (12.0\ V)\sin sin\,(\omega t)$ with a tunable frequency ω.



For the first five parts, suppose the frequency has been set to $\omega = 315$ rad/s.

(a) At this frequency, what is the reactance of the capacitor, and what is the reactance of the inductor?

(b) What is the magnitude of the impedance of the whole RLC circuit at this frequency?

(c) What is the maximum current that flows in the circuit? (In other words, the amplitude of the oscillating current that flows in the circuit.)

(d) What is the maximum voltage across the resistor as the circuit oscillates?

(e) What is the maximum voltage across the capacitor as the circuit oscillates?

For the remaining parts, the frequency is tuned to the resonant frequency of the circuit.

(f) What is the resonant frequency of the circuit? Express your answer either as an angular frequency or as a cycles-per-second frequency, indicating which you have chosen. (Hint: one is more convenient than the other.)

(g) At this frequency, what is the reactance of the capacitor, and what is the reactance of the inductor?

(h) What is the magnitude of the impedance of the whole RLC circuit at this frequency? And how does it compare to what you got in part b ?

(i) What is the maximum current that flows in the circuit? (In other words, the amplitude of the oscillating current that flows in the circuit.) How does it compare to what you got in part c ?

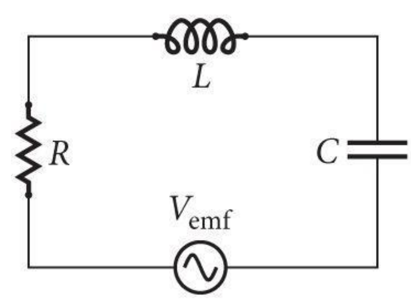
(j) What is the maximum voltage across the resistor as the circuit oscillates?

(k) What is the maximum voltage across the capacitor as the circuit oscillates?

In part k, you should have found that the amplitude of the voltage across the capacitor is significantly larger than the 12-volt amplitude driving the circuit! This is similar to how a mechanical oscillator can build up a large motion from continuous small "pushing" at the resonant frequency, kind of like "pumping" a playground swing with your legs.

Problem 3

In the circuit shown, the resistance is $4.0\ \Omega$, the capacitance is 17 nF (i.e., 17×10^{-9} farad), and the inductance is 0.30 H. The frequency of the AC voltage source is not yet specified.



a) Calculate the resonant frequency of this circuit in Hz (cycles per second).

b) What is the Q of this circuit?

c) Suppose you suddenly switch on a driving voltage. How long (in seconds) do you need to wait for the transient component to have died away sufficiently? Explain the reasoning behind your calculation. There is not a single right answer for this, but we are looking for good reasoning supporting a calculated or approximated time.

d) Calculate the amplitude of the oscillating voltage across the capacitor if the voltage source is set to the circuit's resonant frequency (what you found in part a) with an amplitude of 0.075 V.

e) Calculate the amplitude of the oscillating voltage across the capacitor if the voltage source is set to $f = 2500$ Hz with an amplitude of 0.075 V.

f) At $f = 2500$ Hz, calculate the phase of the oscillating current flowing in the circuit relative to the driving voltage. That is, if the driving voltage is $V(t) = Re\left\{V_0 e^{i\omega t}\right\}$ and $I(t) = Re\left\{I_0 e^{i\omega t}\right\}$, find the complex phase of (I_0/V_0). Express your answer as a number of radians with 4 decimal places.