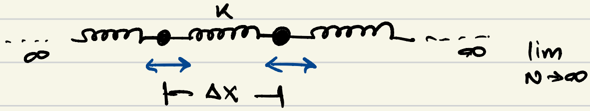
Longitudinal waves:

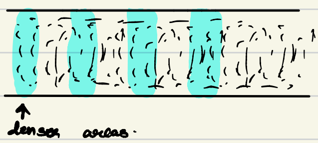
going back to the mass spring system:



the equation for this would be: $\frac{\partial^2 \psi}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 \psi}{\partial x^2}$

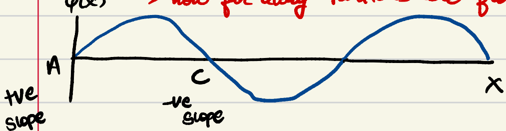
where $E \equiv K\Delta X$

Sound wave:



On a graph:

$\psi(x) \rightarrow$ how far away particles are from their equilibrium position.



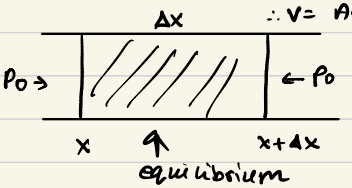
at equilibrium, we have high density → this is because there will be no place for them to move. Point 'C' has high density.
At point 'A', we have very low density → observe on the left & the right

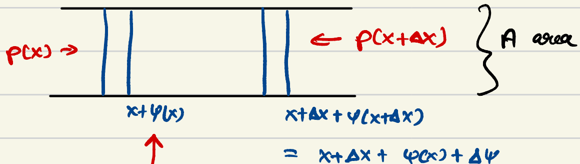
Note: $\rho \propto -\left(\frac{\partial \psi}{\partial x}\right)$ → from above observation.
$\left(\frac{\partial \psi}{\partial x}\right) \propto v_x$
(assuming wave is travelling right).

& $\rho \propto \left(\frac{\partial \psi}{\partial x}\right)$ → assuming wave is travelling left.
$\left(\frac{\partial \psi}{\partial x}\right) \propto -v_x$

Sound wave: longitudinal wave in position, density, pressure

$\therefore V = A\Delta x$





$$\psi(x+\Delta x) = \psi(x) + \Delta\psi$$

$$\Delta\psi \equiv \left(\frac{\partial\psi}{\partial x}\right)\Delta x \quad \text{for small } \Delta x$$

we can also use pressure: $P(x) = P_0 + \psi_p(x)$ — A → change of pressure from equi.

$$P(x+\Delta x) = P_0 + \psi_p(x+\Delta x)$$
$$= P_0 + \psi_p(x) + \Delta\psi_p$$

$$\therefore \Delta\psi_p \equiv \left(\frac{\partial\psi_p}{\partial x}\right)\Delta x$$

Wave equation:

1) we can observe there is a change in volume.

we know: $\Delta V \propto -V\psi_p$ (volume is inversely proportional to volume).

$$\Rightarrow \Delta V = -\underset{\uparrow \text{ compressability}}{K}\, V\psi_p$$

$$\therefore \frac{\Delta V}{V} = -K\psi_p = \frac{\cancel{A}\Delta\psi}{\cancel{A}\Delta x} = \frac{\partial\psi}{\partial x} \quad \text{— A}$$

2) what is 'K'. during the process of this wave traversal, the heat is conserved, hence it is an adiabatic process. Hence, $pV^\gamma = \text{const.}$

$\gamma = 7/5$ for diatomic.

$$\therefore dp\, V^\gamma + p\gamma V^{\gamma-1} dV = 0$$

$$\Rightarrow dp = -p\gamma \frac{dV}{V} \quad \text{or} \quad dV = -\frac{1}{\gamma p} V\, dp$$

↳ very similar to eq (A)

$\therefore \quad K = \frac{1}{\gamma P_0} \approx \frac{5}{7 P_0}$

3) Calculate the difference in pressure $\Delta \varphi_p$:

$$\therefore \frac{\partial \varphi}{\partial x} = -K \varphi_p$$

$$\Rightarrow \frac{\partial^2 \varphi}{\partial x^2} = -K \frac{\partial \varphi_p}{\partial x}$$

$$\Rightarrow \frac{\partial \varphi_p}{\partial x} = \frac{-1}{K} \frac{\partial^2 \varphi}{\partial x^2}$$

$$\text{or} \quad \Delta \varphi_p = \frac{-1}{K} \frac{\partial^2 \varphi}{\partial x^2} \Delta x$$

4) Newton's eq.

$$\therefore F_{net} = A\left[P(x) - P(x+\Delta x)\right] = A(-\Delta \varphi_p) = ma$$

$$\therefore -A \Delta \varphi_p = \underbrace{(\rho A \Delta x)}_{m} \underbrace{\frac{\partial^2 \varphi}{\partial t^2}}_{a}$$

$$\therefore \cancel{A}\left[\frac{1}{K} \frac{\partial^2 \varphi}{\partial x^2} \cancel{\Delta x}\right] = (\rho \cancel{A} \cancel{\Delta x}) \frac{\partial^2 \varphi}{\partial t^2}$$

$$\frac{\partial^2 \varphi}{\partial t^2} = \frac{1}{K\rho} \frac{\partial^2 \varphi}{\partial x^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 \varphi}{\partial t^2} = \frac{\gamma P}{\rho} \frac{\partial^2 \varphi}{\partial x^2}} \qquad \therefore \text{speed}^2 = c^2 = \frac{\gamma P_0}{\rho}$$

φ is $\varphi(x,t)$.

$P_0 = 10^5 \ N/m^2 \qquad \rho \approx 1.3 \ kg/m^3 \qquad \gamma = \frac{7}{5}$

$$C = \sqrt{\frac{7/5 \times 10^5}{1.3}} \approx 328 \ m/s$$

we know: $\frac{\partial \psi}{\partial x} = -K \psi_p$

$$\frac{\partial}{\partial x}\left[\frac{\partial^2 \psi}{\partial t^2} = \frac{\gamma P_0}{\rho} \frac{\partial^2 \psi}{\partial x^2}\right]$$

$$\Rightarrow \frac{\partial^2}{\partial t^2} \frac{\partial \psi}{\partial x} = \frac{\gamma P_0}{\rho} \frac{\partial^2}{\partial x^2} \frac{\partial \psi}{\partial x}$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}(-K \psi_p) = \frac{\gamma P_0}{\rho} \frac{\partial^2}{\partial x^2}(-K \psi_p)$$

$$\therefore \boxed{\frac{\partial^2 \psi_p}{\partial t^2} = \frac{\gamma P_0}{\rho} \frac{\partial^2 \psi_p}{\partial x^2}}$$

↳ wave equation of pressure

Impedence:

$$\rightarrow Z = \frac{T}{v} = \sqrt{T\mu} = \frac{F}{v}$$

we know $F = A \psi_p = A\left(-\frac{1}{K} \frac{\partial \psi}{\partial x}\right)$

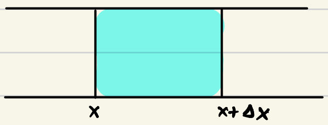
we know $\frac{\partial \psi}{\partial x} = \mp \frac{1}{c} \frac{\partial \psi}{\partial t}$

or $\frac{Z}{A} = \frac{1}{KC} = \sqrt{\gamma \rho P_0}$

$$\therefore F = A\left(\pm \frac{1}{KC} \frac{\partial \psi}{\partial t}\right) \Rightarrow \boxed{Z = \frac{F}{\frac{\partial \psi}{\partial t}} = \frac{A}{KC}}$$

→ Relations between 2 media remain the same as in transeverse waves.

Energy:



$\Delta V = A \Delta x$

$\Delta m = \rho \Delta V$

$= \rho A \Delta x$

$$\frac{KE}{V} = \frac{\frac{1}{2}(\Delta m)(\partial \psi/\partial t)^2}{V} = \frac{1}{2}\rho\left(\frac{\partial \psi}{\partial t}\right)^2$$

$$\therefore \frac{KE}{\Delta x} = \frac{1}{2}\rho A\left(\frac{\partial \psi}{\partial t}\right)^2$$

$$\frac{PE}{\Delta x} = \frac{1}{2}A\gamma P_0\left(\frac{\partial \psi}{\partial x}\right)^2$$

$$\therefore \mathcal{E}(x,t) = \frac{1}{2}A\rho\left[\left(\frac{\partial \psi}{\partial t}\right)^2 + c^2\left(\frac{\partial \psi}{\partial x}\right)^2\right] \; ; \quad c^2 \equiv \frac{\gamma P_0}{\rho}$$

we know $\frac{\partial \psi}{\partial t} = \pm c \frac{\partial \psi}{\partial x}$ } travelling wave

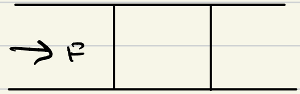
$$\therefore \mathcal{E}(x,t) = A\rho\left(\frac{\partial \psi}{\partial t}\right)^2$$

$$\frac{\mathcal{E}(x,t)}{A} = \rho\left(\frac{\partial \psi}{\partial t}\right)^2$$ → energy per density volume.

power:

$dW = F\,d\psi$

$$P = \frac{dW}{dt} = F\frac{\partial \psi}{\partial t}$$



we know: $F = A(P_0 + \psi_p)$

$$\therefore \quad P = (P_0 + \psi_p)A\frac{\partial \psi}{\partial t}$$

$$= P_0 A\frac{\partial \psi}{\partial t} + \psi_p A\frac{\partial \psi}{\partial t}$$

$$\therefore \quad \langle P \rangle = \psi_p A \frac{\partial \psi}{\partial x}$$

we know: $\psi_p = -\frac{1}{K}\frac{\partial \psi}{\partial x}$

$$= \pm \frac{1}{Kc}\frac{\partial \psi}{\partial t}$$

$$\therefore \quad \langle P \rangle = \pm A\left(\frac{1}{Kc}\right)\left(\frac{\partial \psi}{\partial t}\right)^2$$

$$= \pm A\rho c\left(\frac{\partial \psi}{\partial t}\right)^2 = \pm c\,\mathcal{E}$$

$$= \pm Z\left(\frac{\partial \psi}{\partial t}\right)^2$$

→ The relationship we have for sound waves is the same we had for transverse waves

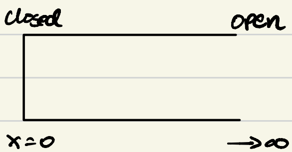
Standing waves:

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}$$; ψ is the displacement of the points in the medium.

$\therefore \psi(x,t) = c_1 \cos(kx + \omega t) + c_2 \sin(kx + \omega t) + c_3 \cos(kx - \omega t) + c_4 \sin(kx - \omega t)$

or, $= D_1 \cos(kx) \sin \omega t + D_2 \sin kx \sin \omega t + D_3 \sin kx \cos \omega t + D_4 \cos kx \cos \omega t$

lets consider a pipe:



→ $\psi(x=0, t) = 0$

$\therefore D_1 \sin \omega t + D_4 \sin \omega t = 0$ → true $\forall$ $t \in R$

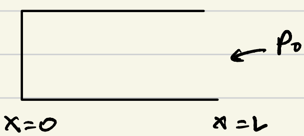
$\therefore D_1 = D_4 = 0$

now, $\psi(x,t) = D_2 \sin kx \sin \omega t + D_3 \sin kx \cos \omega t$

$= \sin kx \, (D_2 \sin \omega t + D_3 \cos \omega t)$.

$= A \sin kx \cos(\omega t + \phi)$.

Now, terminate the right side to $x = L$



$$\varphi_p = \frac{-1}{\kappa}\frac{\partial \varphi}{\partial x} = \frac{-AK}{\kappa} \cos Kx \cos(\omega t + \phi)$$

$\therefore$ $\varphi_p(x = L, t) = 0$ $\rightarrow$ pressure at $x = L$ is constant, hence no excess pressure

$\therefore$ $\cos KL = 0$

$$\Rightarrow \frac{2\pi}{\lambda} L = (2n-1)\frac{\pi}{2}$$

$$\Rightarrow \lambda = \frac{4L}{2n-1}$$

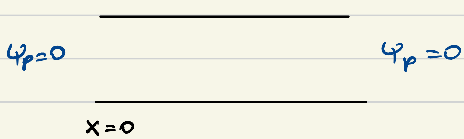
if $n = 1 \Rightarrow \lambda = 4L$ $\rightarrow$ first mode

$n = 2 \Rightarrow \lambda = \frac{4L}{3}$ $\rightarrow$ second mode

$$\therefore f_n = \frac{c}{\lambda} = \frac{c}{4L}(2n-1) \qquad \boxed{f_{n+1} - f_n = 2f_1}$$

$\rightarrow$ Refer TB for modes of vibration diagrams.

Now consider the following tube:



$$\varphi(x,t) = D_1 \cos Kx \sin\omega t + D_2 \sin Kx \sin\omega t + D_3 \sin Kx \cos\omega t + D_4 \cos Kx \cos\omega t$$

$\varphi_p = \frac{-1}{\kappa} \frac{\partial \varphi}{\partial x}$ → same as φ with different coeffecients.

$\therefore \varphi_p(x=0, t) = 0 \Rightarrow D_1 = D_4 = 0$

now $\varphi_p(x,t) = B \sin kx \cos(\omega t + \phi)$.

$\varphi_p(x = L, t) = 0 = B \sin(kL) \cos(\omega t + \phi)$

$\therefore \sin(kL) = 0$

$\Rightarrow kL = n\pi, \quad \forall n \in \mathbb{Z}^+$

$\therefore \frac{2\pi}{\lambda} L = n\pi$

$\lambda = \frac{2L}{n}$

$\therefore \lambda_1 = 2L$ → 1st mode → fundamental

$\lambda_2 = L$

$\lambda_3 = \frac{2L}{3}$

$\therefore f_n = \frac{c}{\lambda_n} = \frac{c}{2L} n$

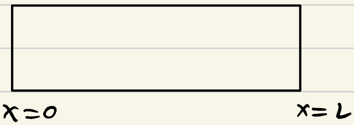
$$\boxed{f_{n+1} - f_n = f_1}$$

$$\psi_p(x,t) = B \sin kx \cos(\omega t + \phi)$$

Since, $\psi_p = -\dfrac{1}{\kappa}\dfrac{\partial \psi}{\partial x}$

$$\therefore \boxed{\psi(x,t) = \frac{B\kappa}{k}\cos(kx)\cos(\omega t + \phi)}$$

Tubes closed on both sides:



$$\psi(x,t) = D_1 \cos kx \sin \omega t + D_2 \sin kx \sin \omega t + D_3 \sin kx \cos \omega t + D_4 \cos kx \cos \omega t$$

$$\psi(x=0,t) = 0 \Rightarrow D_1 = D_4 = 0$$

now, $\psi(x,t) = A \sin(kx)\cos(\omega t + \phi)$

$$\psi(x=L,t) = 0$$

$$\Rightarrow kL = n\pi$$

$$\Rightarrow \frac{2\pi}{\lambda} L = n\pi$$

$$\therefore \lambda_n = \frac{2L}{n}, \quad f_n = \frac{c}{2L} n$$

$$\boxed{f_{n+1} - f_n = f_1}$$