Solving the differential eq:

Now, let us consider the damped oscillator. In addition, we now have a damping force:

$$F_{damping} = -b\dot{x}.$$

↳ this is not due to friction.
It can be thought of as something moving through a fluid.

$$\therefore \quad F_{spring} + F_{damping} = m\ddot{x}$$

$$\Rightarrow \quad -Kx - b\dot{x} = m\ddot{x}$$

$$\Rightarrow \quad m\ddot{x} + b\dot{x} + Kx = 0$$

$$\Rightarrow \quad \ddot{x} + \frac{b}{m}\dot{x} + \frac{K}{m}x = 0$$

$$\Rightarrow \quad \ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \quad ; \quad \omega_0 \equiv \sqrt{\frac{K}{m}}, \quad \gamma \equiv \frac{b}{m}.$$

$- A$

The above is a linear differential equation. Hence, the solution is exponential.

$$\therefore \quad x(t) = C\exp(\alpha t).$$

$- B$

plugging 'B' in 'A'

$$\Rightarrow \quad \alpha^2 \cancel{C\exp(\alpha t)} + \gamma\alpha \cancel{C\exp(\alpha t)} + \omega_0^2 \cancel{C\exp(\alpha t)} = 0$$

$$\Rightarrow \quad \alpha^2 + \gamma\alpha + \omega_0^2 = 0$$

$$\therefore \boxed{\alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2}}$$

we will now consider three cases based on the sign of the discriminant.

1) Underdamping:

→ occurs when $\gamma < 2\omega_0$. In this case, $D < 0$ & hence we have imaginary α.

lets define $\underbrace{\omega_u}_{u \,\equiv\, \text{underdamped}} \equiv \frac{1}{2}\sqrt{4\omega_0^2 - \gamma^2} \Rightarrow \omega_u = \omega_0 \sqrt{1 - \left(\frac{\gamma}{2\omega_0}\right)^2}$.

$$\Rightarrow \alpha = -\frac{\gamma}{2} \pm \omega_u \cdot i$$

$$\therefore x_1(t) = c_1 \exp\left[\left(-\frac{\gamma}{2} + \omega_u i\right) t\right]$$

$$x_2(t) = c_2 \exp\left[\left(-\frac{\gamma}{2} - \omega_u i\right) t\right]$$

$$\therefore x(t) = \exp\left(-\frac{\gamma t}{2}\right)\left[c_1 \exp(\omega_u i t) + c_2 \exp(-\omega_u i t)\right]$$

we have to impose the condition that $x(t)$ has to be real. Hence, the two solutions have to be the conjugate of each other.

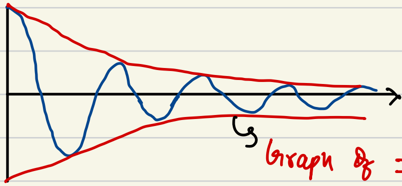
$$\therefore c_2 = c_1^* \quad ; \quad c_1 \equiv c\exp(i\phi)$$

$$\Rightarrow c_2 = c\exp(-i\phi).$$

$$\therefore x(t) = \exp\left(-\frac{\gamma t}{2}\right) \cdot c \cdot \left[\exp\left(i(\omega_u t + \phi)\right) + \exp\left(-i(\omega_u t + \phi)\right)\right]$$

$$\boxed{x_u(t) = A \cdot \exp\left(-\frac{\gamma t}{2}\right) \cdot \cos(\omega_u t + \phi)} \quad ; \quad A \equiv 2c$$

Graph of $x(t)$:



Graph of $\pm A\exp(-\frac{\gamma t}{2})$.

Very light damping:

Occurs when $\gamma \ll \omega_0$. from taylor series: $\sqrt{1+\epsilon} \approx 1 + \frac{1}{2}\epsilon$.

$$\therefore \omega_v = \omega_0\left(1 - \frac{1}{2}\left(\frac{\gamma}{2\omega_0}\right)^2\right) = \omega_0\left(1 - \frac{1}{8}\frac{\gamma^2}{\omega_0^2}\right)$$

$$\Rightarrow \omega_v = \omega_0 - \frac{1}{8}\frac{\gamma^2}{\omega_0}$$

ω_v essentially equals ω_0 for very small values of ω_0

Energy:

The energy of the damped oscillator is given by:

$$E = \frac{1}{2}Kx + \frac{1}{2}m\dot{x}^2.$$

$$\dot{x} = \frac{d}{dt}\left(A\exp\left(-\frac{\gamma t}{2}\right)\cos(\omega_v t)\right)$$

$$= A\exp\left(-\frac{\gamma t}{2}\right)\left[-\frac{\gamma}{2}\cos(\omega_v t) - \omega_v\sin(\omega_v t)\right].$$

$$E = \frac{1}{2}mA^2\exp\left(-\frac{\gamma t}{2}\right)\left(\frac{\gamma}{2}\cos(\omega_v t) + \omega_v\sin(\omega_v t)\right)^2 + \frac{1}{2}KA^2\exp(-\gamma t)\cos^2(\omega_v t)$$

Using the defination of ω_v & $K = m\omega_0^2$:

we get: $E = \frac{1}{2} m A^2 \exp(-\gamma t) \left[\frac{\gamma^2}{4} \cos(2\omega_0 t) + \frac{\gamma\omega_0}{2} \sin(2\omega_0 t) + \omega_0^2 \right]$

As a double check, when $\gamma = 0$; $E = \frac{1}{2} m \omega_0^2 A^2$.

$\forall \exp(-\frac{\gamma t}{2}) \neq 0$, the loss in energy keeps on reducing. This lost energy is converted into the heat that is generated due to the damp. The energy has an oscillation frequency of $2\omega_0$. This is due to both the forward and backward motion of the object.

Let's take the case where 'γ' is very small. Hence the cos & sin term vanish. Hence, $\exp(-\gamma t)$ decays slowly.

$$\therefore \langle E \rangle = \frac{1}{2} m \omega_0^2 A^2 \exp(-\gamma t) = \frac{1}{2} k A^2 \exp(-\gamma t).$$

<u>Energy decay</u>:

It is defined at $\frac{d\langle E \rangle}{dt}$.

$$\therefore \frac{d}{dt}\langle E \rangle = \frac{d}{dt}\left(\frac{1}{2} m \omega_0^2 A^2 \exp(-\gamma t) \right)$$

$$= -\gamma \langle E \rangle.$$

This tells us that the rate of fractional change of the energy is 'γ', ie, in one unit of time we loose a fraction of γ of its value. This result however holds true only for small γ and only for average cases. Let's take a look at exact values now.

We know: $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$

$$\Rightarrow \frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(m\ddot{x} + kx)$$

and: $m\ddot{x} + b\dot{x} + kx = 0$

$$\Rightarrow m\ddot{x} + kx = -b\dot{x}$$

$\therefore \frac{dE}{dt} = -b\dot{x}^2$; $\forall\ x \in \mathbb{R}$, $\frac{dE}{dt} < 0 \Rightarrow$ energy is lost.

at $b = 0$; $\frac{dE}{dt} = 0 \Rightarrow$ no damping, which makes sense.

$$\therefore \frac{d\langle E\rangle}{dt} = -b\langle\dot{x}^2\rangle.$$

both the values are equal.

lets calculate $\langle E\rangle = \langle T\rangle + \langle V\rangle$

$$= \frac{1}{2}m\langle\dot{x}^2\rangle + \frac{1}{2}m\langle\dot{x}\rangle^2 = m\langle\dot{x}^2\rangle.$$

$$\therefore \langle\dot{x}^2\rangle = \frac{\langle E\rangle}{m}$$

$$\therefore \frac{d\langle E\rangle}{dt} = -\gamma\langle E\rangle$$

↳ Same result.

Note:

$Q \equiv \frac{\gamma}{\omega_0}$ or $\frac{\omega_0}{\gamma}$.

for very small dampening ($\gamma \ll \omega_0$): the time to complete Q cycles: $Q \cdot 2\pi = \omega_0 t \Rightarrow t = \frac{2\pi Q}{\omega_0} = \frac{2\pi\,\cancel{\omega_0}}{\gamma\,\cancel{\omega_0}}$ $\therefore e^{-\gamma t/2} = e^{-\pi} \approx \underline{\underline{0.043}}$

II) overdampening :

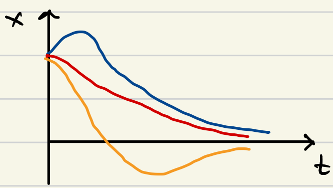
→ Occurs when $\gamma > 2\omega_0$. If so, we can say that α is a real quantity. let's define μ_1 & μ_2:

$$\mu_1 \equiv \frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \quad ; \quad \mu_2 \equiv \frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

$$\therefore \boxed{X_{overdamped}(t) = c_1 \exp(-\mu_1 t) + c_2 \exp(-\mu_2 t).}$$

Here c_1 and c_2 are determined by initial conditions. Both these terms are positive and hence both the solutions undergo decay. we can observe that $\mu_1 > \mu_2$ and hence for large values of t, $x(t) \approx c_2 \exp(-\mu_2 t)$.

The various ways we can have the graph of $x(t)$ depending on the way the object is released is as follows:



In any of the ways, the object crosses the origin at most once.

$$\therefore c_1 \exp(-\mu_1 t) + c_2 \exp(-\mu_2 t) = 0 \Rightarrow \exp[(\mu_1 - \mu_2)t] = -\frac{c_1}{c_2}$$

$$\therefore t = \frac{1}{\mu_1 - \mu_2} \ln\left(-\frac{c_1}{c_2}\right)$$

we have hence found only one solution for t.

→ if $-c_1/c_2 > 0 \rightarrow t > 0 \rightarrow$ crosses origin once.

→ if $-c_1/c_2 = 1 \rightarrow t = 0 \rightarrow$ never crosses the origin.

→ if $0 < -c_1/c_2 < 1 \rightarrow t < 0 \rightarrow$ impossible.

→ if $-c_1/c_2 < 0 \rightarrow t \in \mathbb{C} \rightarrow$ impossible.

Very heavy damping:

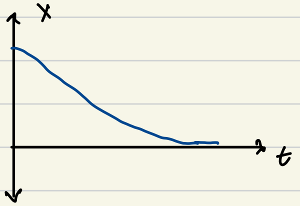
Occurs when $\gamma >> \omega_0$. This corresponds to a very weak spring immersed into a very thick fluid.

$$\therefore \mu_1 \approx \gamma$$

and $$\mu_2 = \frac{\gamma}{2} - \frac{\gamma}{2}\sqrt{1 - \frac{4\omega_0^2}{\gamma^2}} = \frac{\gamma}{2} - \frac{\gamma}{2}\left(1 - \frac{2\omega_0^2}{\gamma^2}\right) = \frac{\omega_0^2}{\gamma}$$

Observation: $\mu_1 >> \mu_2$ which means that $\exp(-\mu_1 t) \to 0$

$$\therefore x(t) \approx c_2 \exp(-\mu_2 t) = c_2 \exp\left(-\frac{\omega_0^2}{\gamma} t\right) = c_2 \exp(-t/\tau) \; ; \; \tau \equiv \frac{\gamma}{\omega_0^2}.$$



we can define $T \equiv \frac{\gamma}{\omega_0^2}$ as "relaxation time". The displacement increases by a factor of $1/e$ for every increase in T.

Observation: $\gamma/\omega_0^2 >> \underbrace{1/\omega_0}$ $\Rightarrow$ T (relax) $>>$ T (time period).

time period of a spring.

$$T = \gamma/\omega_0^2 = \frac{b/m}{k/m} = b/k \qquad \therefore x(t) = c_2 \exp\left(-\frac{kt}{b}\right).$$

$$F_{damping} = -b\dot{x} = -b\left(c_2 \cdot -\frac{k}{b} \exp\left(-\frac{kt}{b}\right)\right) = k \exp\left(-\frac{kt}{b}\right) = kx$$

$\therefore$ $F_{damping} = -F_{spring}$.

III) <u>Critical damping</u>:

→ Occurs when $\gamma = 2\omega_0$. By substituting this we get only one solution to our differential equation. We would need another one to satisfy our initial conditions of $x(0)$ & $v(0)$. From the theory of differential equations, another solution to this diff eq is $t\cdot\exp(-\omega_0 t)$.

<u>Verification:</u>

we have: $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$

$\Rightarrow \ddot{x} + 2\omega_0\dot{x} + \omega_0^2 x = 0$

$x(t) = t\exp(-\omega_0 t)$

$\Rightarrow \dot{x} = -\omega_0 t\exp(-\omega_0 t)$

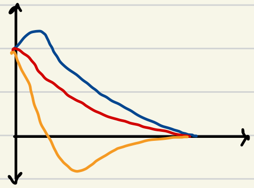
$\ddot{x} = t\,\omega_0^2\exp(-\omega_0 t)$

$\therefore\ t\cdot\omega_0^2\cdot\exp(-\omega_0 t) + 2\omega_0\cdot(-\omega_0)t\exp(-\omega_0 t) + t\omega_0^2\exp(-\omega_0 t)$
$= 0$

We could have also derived the above result by taking $\lim_{\gamma\to 2\omega_0}$ of either the under or overdamped case.

$$\therefore\ x_{critical}(t) = (A+Bt)\exp(-\omega_0 t).$$

Graph:



This looks very similar to the case of overdamping, but we can observe that the graph converges to the origin much quickly in this case.