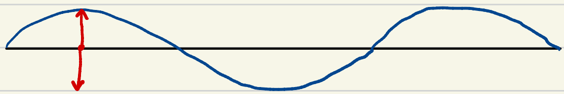
Transverse waves on a string:

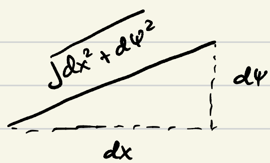


Wave equation:

→ Imagine a string that stretches till infinity on both sides.

→ Let 'T' be its tension & $\mu = m/l$.

at some point:



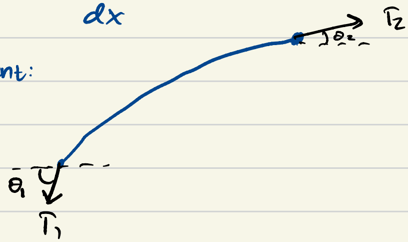
$d\psi$ } assume very small

$$\sqrt{dx^2 + d\psi^2} = dx\left[1 + \left(\frac{d\psi}{dx}\right)^2\right]^{1/2} \approx dx\left[1 + \frac{1}{2}\left(\frac{d\psi}{dx}\right)\right]$$

$$= dx + d\psi \cdot \frac{1}{2}\frac{d\psi}{dx} \approx dx$$

assume: $\frac{d\psi}{dx} \to$ small. (slope is small).

segment:



If the slope is small $\Rightarrow \theta_1$ & θ_2 are small.

Since there is no longitudinal motion: $T_1 \cos\theta_1 = T_2 \cos\theta_2$

$\psi(x,t) \rightarrow$ displacement of each point from its equilibrium position

Hence, $T_1\left[1-\frac{\theta_1^2}{2}+\cdots\right] = T_2\left[1-\frac{\theta_2^2}{2}+\cdots\right]$

$\Rightarrow \quad T_1 \approx T_2 = T \rightarrow$ constant.

F_{net} (transeverase) $\simeq T_2 \sin\theta_2 - T_1 \sin\theta_1$

for small angle $\sin\theta \approx \tan\theta \approx \theta \rightarrow$ slope

Hence, $F_{net} = T_2\,\psi'(x+dx) - T_1\,\psi'(x) = T\,\psi'(x+dx) - T\,\psi'(x)$

$$= T\,dx\left[\frac{\psi'(x+dx) - \psi'(x)}{dx}\right]$$

$$m\,a = T\,dx\,\frac{d^2\psi}{dx^2}$$

$$\Rightarrow m\frac{d^2\psi}{dt^2} = T\,dx\,\frac{d^2\psi}{dx^2}$$

$$\Rightarrow \mu\,\cancel{dx}\,\frac{d^2\psi}{dt^2} = T\cancel{dx}\,\frac{d^2\psi}{dx^2}$$

Hence: $\boxed{\frac{\partial^2\psi}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2\psi}{\partial x^2}} \rightarrow \psi(x,t)$

$\therefore \quad v^2 = \frac{T}{\mu} \quad \Rightarrow \quad v = \sqrt{\frac{T}{\mu}}$

The solution we have is: $\psi(x,t) = Ae^{i(\pm kx \pm \omega t)}$; $\boxed{\frac{\omega}{k} = \sqrt{\frac{T}{\mu}}}$

$\Rightarrow \psi(x,t) = c_1\cos(kx+\omega t) + c_2\sin(kx+\omega t) + c_3\sin(kx-\omega t)$
$+ c_4\cos(\omega t - kx)$.

and, $\psi(x,t) = D_1 \cos(Kx)\cos(\omega t) + D_2 \sin(Kx)\sin(\omega t)$
$+ D_3 \sin(Kx)\cos(\omega t) + D_4 \cos(Kx)\sin(\omega t)$.

This is the example of a non dispersive wave $\rightarrow \forall\ \omega$'s $\Rightarrow V$ is constant. we know that $\psi(x,t) = f(x \pm vt)$; $v = \sqrt{\frac{T}{\mu}} = c$

$\rightarrow$ $f(z) = \int_{-\infty}^{\infty} c(K) e^{iKz}\, dK$, $c(K) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)\, e^{-iKz}\, dz$

A vector $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$.

Similarly, $c(K)$ is the component of a wave.

let $z = x - ct$

$\Rightarrow f(x-ct) = \int c(K)\, e^{iK(x-ct)}\, dK = \int_{-\infty}^{\infty} c(K)\, e^{i(Kx-\omega t)} dK$; $c = \frac{\omega}{K}$

different

$\rightarrow$ $f(x-xt)$ travels in the $+x$ direction & does not change & the shape does not change. A dispersive wave could change with time

<u>Fourier transform in 2D:</u>

$f(x) = \int_{-\infty}^{\infty} c(K)\, e^{iKx} dK$

$\Rightarrow \psi(x,t) = \int_{-\infty}^{\infty} c(K,t)\, e^{iKx} dK$ & $c(K,t) = \int_{-\infty}^{\infty} \beta(K,\omega)\, e^{i\omega t}\, d\omega$

$\therefore \psi(x,t) = \iint \beta(K,\omega)\, e^{i(Kx+\omega t)}\, dK\, d\omega$

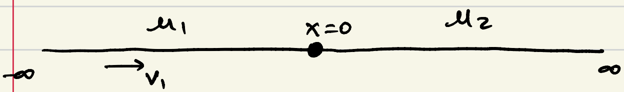
we know: $\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}$.

$$\therefore \iint \beta(k,\omega) \frac{\partial^2}{\partial t^2}\left(e^{i(kx+\omega t)}\right) dk\,d\omega = c^2 \iint \beta(k,\omega) \frac{\partial^2}{\partial x^2}\left(e^{i(kx+\omega t)}\right) dk\,d\omega$$

$$\iint \beta(k,\omega)\left[c^2k^2 - \omega^2\right] e^{i(kx+\omega t)}\, dk\,d\omega = 0$$

$$\therefore \boxed{\omega = \pm ck}$$

Reflection & transmission:



assume that the tension is the same.

$$\psi_i(x,t) = f_i(x - v_1 t) = f_i\left(t - \frac{x}{v_1}\right)$$; $v_1 t = t'$ $\Rightarrow x' = x/v_1$

$$\psi_r(x,t) = f_r\left(t + \frac{x}{v_1}\right) \quad ; \quad v_1 = \sqrt{\frac{T}{\mu_1}}$$

$$\psi_t(x,t) = f_t\left(t - \frac{x}{v_2}\right) \quad ; \quad v_2 = \sqrt{\frac{T}{\mu_2}}$$

at $x=0$, the function should be continuous, ie :

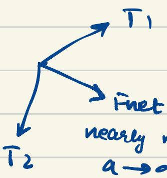
$$\psi_L(x,t) = \psi_i(x,t) + \psi_r(x,t)$$
$$= f_i\left(t - \frac{x}{v_1}\right) + f_r\left(t + \frac{x}{v_1}\right)$$

$$\psi_R(x,t) = \psi_t(x,t) = f_t\left(t - \frac{x}{v_2}\right)$$

$\therefore$ at Boundary: $\psi_2(x,t)\Big|_{x=0} = \psi_R(x,t)\Big|_{x=0}$

$$\therefore \quad \boxed{f_i(t) + f_r(t) = f_t(t)}$$

→ The slope must be continuous too.



nearly massless $a \to \infty$, which is wrong.

$$\therefore \frac{\partial \psi_L}{\partial x}(x=0,t) = \frac{\partial \psi_R}{\partial x}(x=0,t)$$

$$\Rightarrow \frac{-1}{v_1} f_i'(t) + \frac{1}{v_1} f_r'(t) = \frac{-1}{v_2} f_t'(t)$$

$$\therefore v_2 f_i'(t) - v_2 f_r'(t) = v_1 f_t'(t)$$

$$\Rightarrow \boxed{v_2 f_i(t) - v_2 f_r(t) = v_1 f_t(t)}$$

→ Given $f_i(t)$ we can solve for the other 2.

$$\boxed{f_t(t) = \frac{2v_2}{v_1+v_2} f_i(t) \Rightarrow f_t(s) = \frac{2v_2}{v_1+v_2} f_i(s)}$$

and:

$$\boxed{f_r(t) = \frac{v_2 - v_1}{v_1+v_2} f_i(t) \Rightarrow f_r(s) = \frac{v_2 - v_1}{v_1+v_2} f(s)}$$

or: $$f_r\left(t+\frac{x}{v_1}\right) = \frac{v_2-v_1}{v_1+v_2} f_i\left(t+\frac{x}{v_1}\right)$$

$$= \frac{v_2-v_1}{v_1+v_2} f_i\left(t-\frac{-x}{v_1}\right)$$

wave travelling left at (x,t)

↳ moving right at $(-x,t)$

$$\therefore \quad \boxed{\psi_r(x,t) = \frac{v_2-v_1}{v_1+v_2}\psi_i(-x,t)}$$

and, $$f_t\left(t-\frac{x}{v_2}\right) = \frac{2v_2}{v_1+v_2} f_i\left(t-\frac{x}{v_2}\right)$$

$$\Rightarrow \psi_t(x,t) = \frac{2v_2}{v_1+v_2} f_i\left(t-\frac{v_1}{v_2}\left(\frac{x}{v_1}\right)\right)$$

$$= \frac{2v_2}{v_1+v_2} f_i\left(t-\frac{(v_1/v_2\, x)}{v_1}\right)$$

$$\boxed{\psi_t(x,t) = \frac{2v_2}{v_1+v_2}\psi_i\left(\frac{v_1}{v_2}x, t\right)}$$

→ If $v_2 = 3v_1 \Rightarrow \psi_r(x,t) = \frac{1}{2}\psi_i(-x,t)$ → refer TB for graph.

and, $\psi_t(x,t) = \frac{3}{2}\psi_i\left(\frac{1}{3}x, t\right)$

we know: $\psi_r(x,t) = \dfrac{v_2 - v_1}{v_2 + v_1}\,\psi_i(-x,t) = R\,\psi_i(-x,t)$, &

$$\psi_t(x,t) = \frac{2v_2}{v_1+v_2}\,\psi_i\left(\frac{v_1}{v_2}x, t\right) = T\,\psi_i\left(\frac{v_1}{v_2}x,t\right)$$

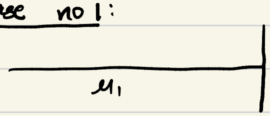
$$\therefore\ R \equiv \frac{v_2 - v_1}{v_2 + v_1} \quad \& \quad T \equiv \frac{2v_2}{v_1 + v_2}$$

$$\therefore\ \boxed{1 + R = T} \qquad v_2 = \sqrt{\frac{T}{\mu_2}},\quad v_1 = \sqrt{\frac{T}{\mu_1}}$$

$$\therefore\ \boxed{R = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}}} \rightarrow \text{reflection coeff.}$$

$$\&\ \boxed{T = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}}} \rightarrow \text{transmission coeff.}$$

Case no 1:



fixed to a wall

$\Rightarrow \mu_2 \to \infty$

$v_2 \to 0$

$$\therefore\ R = \frac{v_2 - v_1}{v_2 + v_1} = -1 \ ,\quad T = 0$$

↳ no wave is transmitted

$\Rightarrow \psi_r(x,t) = -\psi_i(-x,t)$

Case no 2:

→ Light string on the left & heavy string on the right.

$\therefore \mu_1 < \mu_2 < \infty \Rightarrow v_2 < v_1$

$$\therefore R = \frac{v_2 - v_1}{v_2 + v_1} < 0 \Rightarrow -1 < R < 0$$

$$T = \frac{2v_2}{v_1 + v_2} < 1 \Rightarrow 0 < T < 1$$

→ Similarly if Left → heavy & Right → light $\Rightarrow v_2 > v_1$

$\therefore \quad 1 > R > 0 \quad \& \quad 0 < T < 1$

Case no 3:

→ If $\mu_1 = \mu_2 \Rightarrow v_1 = v_2$

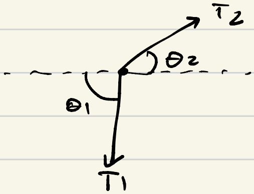
$\therefore \quad R = 0 \quad \& \quad T = 1$

$$v_1 = v_2$$

$$\Rightarrow \sqrt{\frac{T}{\mu_1}} = \sqrt{\frac{T}{\mu_2}}$$ → this is called as 'Impedence matching'.

Impedence:

at some time:



$(T_1 \neq T_2)$

$\therefore (F_{net})_x = T_2 \cos\theta_2 - T_1 \cos\theta_1 + F_{pole} = 0$

↳ preventing motion in x direction.

$(F_{net})_y = T_2 \sin\theta_2 - T_1 \sin\theta_1 = 0$

↳ ring is massless → no acceleration.

$\therefore \quad T_1 \sin\theta_1 = T_2 \sin\theta_2$

for small angles : $T_1 \left.\dfrac{\partial \varphi_L}{\partial x}\right|_{x=0} = T_2 \left.\dfrac{\partial \varphi_R}{\partial x}\right|_{x=0}$

$$\therefore \varphi_L(x,t) = f_i\left(t - \frac{x}{v_1}\right) + f_r\left(t + \frac{x}{v_1}\right)$$

$$\varphi_R = f_t\left(t - \frac{x}{v_2}\right)$$

→ assume : at $x=0$, string is continuous

$$\therefore \boxed{f_i(t) + f_r(t) = f_t(t)}$$

$$\text{eq}_1 \quad \frac{-T_1}{v_1} f_i'(t) + \frac{T_1}{v_1} f_r'(t) = \frac{-T_2}{v_2} f_t'(t)$$

$$\Rightarrow \boxed{-\frac{T_1}{v_1} f_i(t) + \frac{T_1}{v_1} f_r(t) = \frac{-T_2}{v_2} f_t(t)}$$

Solving the simultaneous equations:

$$\therefore f_r(t) = \frac{T_1/v_1 - T_2/v_2}{T_1/v_1 + T_2/v_2} f_i(t)$$

$$f_t(t) = \frac{2\,T_1/v_1}{T_1/v_1 + T_2/v_2} f_i(t)$$

we can do the same as we previously did.

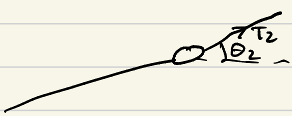
$$\therefore \boxed{\begin{aligned} \varphi_t(x,t) &= \underbrace{\frac{2T_1/v_1}{T_1/v_1 + T_2/v_2}}_{T} \varphi_i\left(\frac{v_1}{v_2}x, t\right) \\ \varphi_r(x,t) &= \underbrace{\frac{T_1/v_1 - T_2/v_2}{T_1/v_1 + T_2/v_2}}_{R} \varphi_i(-x, t) \end{aligned}}$$

let : (Impedence) $Z \equiv \dfrac{T}{v} = \dfrac{T}{\sqrt{T/\mu}} = \sqrt{T\mu}$

$$\therefore \boxed{\begin{aligned} \varphi_t(x,t) &= \frac{2Z_1}{Z_1+Z_2} \varphi_i\left(\frac{v_1}{v_2}x, t\right) \\ \varphi_r(x,t) &= \frac{Z_1 - Z_2}{Z_1+Z_2} \varphi_i(-x, t) \end{aligned}}$$

$$\therefore \boxed{R \equiv \frac{Z_1 - Z_2}{Z_1 + Z_2}} \quad \& \quad \boxed{T = \frac{2Z_1}{Z_1 + Z_2}}$$

→ what is the physical meaning of Impedence ?



$$F_y = T_2 \sin\theta_2 \Big|_{x=0}$$

$$= T_2 \frac{\partial \varphi}{\partial x}\Big|_{x=0} = -\frac{T_2}{v_2}\frac{\partial \varphi_R}{\partial t}$$

$$\therefore F_y = -Z_2 v_y$$

↳ This is like a damping force

we know: $R = \dfrac{Z_1 - Z_2}{Z_1 + Z_2}$ & $T = \dfrac{2Z_1}{Z_1 + Z_2}$ → transmission coeff

If there is no reflection, hence $Z_1 = Z_2$
$(R = 0)$ $\Rightarrow \dfrac{T_1}{v_1} = \dfrac{T_2}{v_2}$ → tension

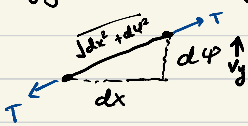
→ This is called as Impedence matching ⇒ maximum transmission.

if $\dfrac{T_1}{v_1} = \dfrac{T_2}{v_2}$

$\Rightarrow \sqrt{T_1 \mu_1} = \sqrt{T_2 \mu_2}$

$$\Rightarrow \boxed{T_1 \mu_1 = T_2 \mu_2}$$

<u>Energy per unit length:</u>



mass = dm $\qquad \therefore dK = \dfrac{1}{2} (dm)\, v_y^2$

$$\therefore dK = \frac{1}{2} (\mu dx) \left(\frac{\partial \psi}{\partial t} \right)^2$$

previously, we derived $\sqrt{dx^2 + d\psi^2} \approx dx + \underbrace{\dfrac{1}{2} dx \left(\dfrac{\partial \psi}{\partial x} \right)^2}_{\text{stretched extra by this much}}$

$$\text{work} = T\,dl = \frac{1}{2} T\, dx \left(\frac{\partial \psi}{\partial x} \right)^2 = U dx$$

$\therefore$ Energy density = Energy per unit length $= \mathcal{E}(x,t)$

$$\therefore \mathcal{E}(x,t) = \frac{dK + U dx}{dx} = \frac{\mu}{2}\left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{T}{2}\left(\frac{\partial\psi}{\partial x}\right)^2$$

$$\therefore \mathcal{E}(x,t) = \frac{\mu}{2}\left[\left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{T}{\mu}\left(\frac{\partial\psi}{\partial x}\right)^2\right]$$

$$\therefore \boxed{\mathcal{E}(x,t) = \frac{\mu}{2}\left[\left(\frac{\partial\psi}{\partial t}\right)^2 + v^2\left(\frac{\partial\psi}{\partial x}\right)^2\right]}$$

for a single traveling wave: $\psi(x,t) = f(x \pm vt)$

we know: $\frac{\partial\psi}{\partial t} = \pm \frac{v\partial\psi}{\partial x}$

$$\Rightarrow \left(\frac{\partial\psi}{\partial t}\right)^2 = v^2\left(\frac{\partial\psi}{\partial x}\right)^2$$

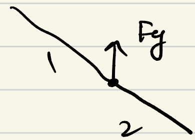
$$\therefore \mathcal{E}(x,t) = \mu\left(\frac{\partial\psi}{\partial t}\right)^2 = \mu v^2\left(\frac{\partial\psi}{\partial x}\right)^2$$

we know: $Z = \sqrt{T\mu} = \frac{T}{v} \Rightarrow v = \sqrt{\frac{T}{\mu}}$

$$\therefore \mathcal{E}(x,t) = \frac{Z}{v}\left(\frac{\partial\psi}{\partial t}\right)^2 \quad \text{or} \quad \mathcal{E}(x,t) = Zv\left(\frac{\partial\psi}{\partial x}\right)^2$$

Power:

$$p(x,t) = \frac{dW}{dt}$$



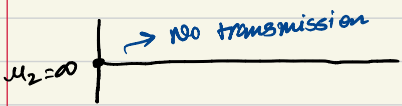
$$= F_y \left(\frac{\partial \varphi}{\partial t}\right)$$

$$= \left(-T\frac{\partial \varphi}{\partial x}\right)\left(\frac{\partial \varphi}{\partial t}\right)$$

$$= \mp \frac{T}{V}\left(\frac{\partial \varphi}{\partial t}\right)^2$$

$$\boxed{p(x,t) = \mp Z\left(\frac{\partial \varphi}{\partial t}\right)^2 = \mp V\,\varepsilon(x,t)}$$

Standing waves:



$\leftarrow \varphi_i = A\cos(kx + \omega t + \phi)$

$V = \frac{\omega}{k} = \sqrt{\frac{T}{\mu}}$

$\therefore \varphi_r(x,t) = R\,\varphi_i(-x,t)$

$R = \frac{Z_1 - Z_2}{Z_1 + Z_2}$ $\quad\therefore Z_2 = \sqrt{T\mu_2} \to \infty$

$\Rightarrow R = \lim_{Z_2 \to \infty} \frac{Z_1/Z_2 - 1}{Z_1/Z_1 + 1} = -1$

$\therefore \varphi_r(x,t) = -A\cos(\omega t - kx + \phi)$

$$\therefore \quad \psi(x,t) = \psi_i(x,t) + \psi_r(x,t)$$

$$= A\cos(\omega t + Kx + \phi) - A\cos(\omega t - Kx + \phi)$$

$$= -2A\sin(Kx)\sin(\omega t + \phi)$$

↳ different points have different amplitudes.

This is what is called a standing wave.

Some points when $\psi(x,t) = 0$

$$\Rightarrow \sin(Kx) = 0$$

$$\Rightarrow Kx = n\pi \quad ; \quad \forall n \in \mathbb{Z}^+$$

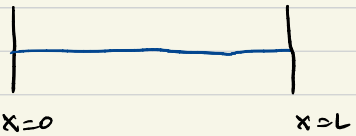
$$\Rightarrow x = \frac{n\pi}{K}$$

These points are called as "nodes"

→ distance between neighbouring nodes $= \dfrac{\pi}{K} = \dfrac{\lambda}{2}$

→ mid point of nodes is anti-nodes.

<u>Finite string</u>:



$$\psi(x,t) \propto \sin(Kx)\sin(\omega t + \phi) \quad ; \quad v = \frac{\omega}{K} = \sqrt{\frac{T}{\mu}}$$

we know: $\psi(x=L, t) = 0$

$\therefore \sin(kL) = 0$

$\Rightarrow kL = n\pi \quad ; \forall \ n \in \mathbb{Z}^+$

$\Rightarrow \frac{2\pi}{\lambda} L = n\pi$

$\therefore \boxed{\lambda_n = \frac{2L}{n}}$

These are called as normal modes of vibration.

we know: $V = f_n \lambda_n \Rightarrow f_n = \frac{V}{\lambda_n} = \frac{V}{2L} n$

frequence

$f_1 = \frac{V}{2L} \rightarrow$ fundamental

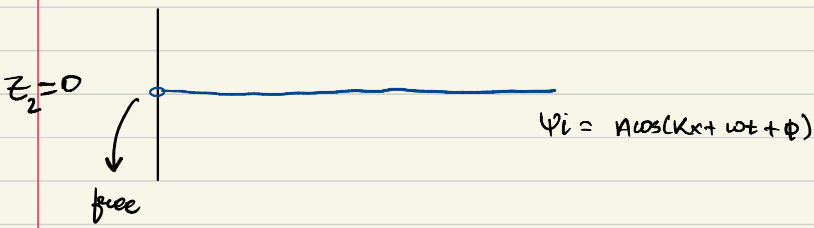
$f_2 = 2f_1$

$f_3 = 3f_1$

$\therefore \boxed{f_{n+1} - f_n = f_1}$

$$\therefore \boxed{\psi(x,t) = \sum_{n=1}^{\infty} B_n \sin(k_n x) \sin(\omega_n t + \phi_n)}$$

Another way:



$$\therefore R = \frac{Z_1 - Z_2}{Z_1 + Z_2} = 1$$

$$\psi_r(x,t) = R\,\psi_i(-x,t)$$

$$\therefore \psi_r(x,t) = \psi_i(-x,t) = A\cos(\omega t - Kx + \phi).$$

$$\therefore \psi(x,t) = \psi_i(x,t) + \psi_r(x,t)$$

$$= A\cos(Kx + \omega t + \phi) + A\cos(-Kx + \omega t + \phi)$$

$$= 2A\cos(Kx)\cos(\omega t + \phi)$$

↳ amplitude of each point is different.

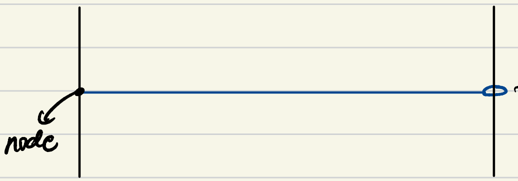
$$\therefore \text{nodes: } Kx = (2n+1)\frac{\pi}{2} \quad ; \quad \forall n \in \mathbb{Z}^+$$

$$\therefore x = (2n+1)\frac{\pi}{2K}$$

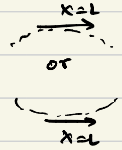
$$\therefore \boxed{x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2} = (2n+1)\frac{\lambda}{4}}$$

$$\therefore \text{nodes: } \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \ldots$$

Finite end on one side & free on the other:



φ at $x = L$:



↳ Vertical component of force $= 0$

$$\therefore \left.\frac{\partial \varphi}{\partial x}\right|_{x=L} = 0$$

$$\varphi(x,t) = -2A\sin(kx)\sin(\omega t + \phi)$$

$$\left.\frac{\partial \varphi}{\partial x}\right|_{x=L} = 0$$

$$\Rightarrow \cos(kx) = 0$$

$$\Rightarrow kL = (n+\tfrac{1}{2})\pi$$

$$\therefore k = (n+\tfrac{1}{2})\frac{\pi}{L}$$

$$\Rightarrow \lambda = \frac{2L}{n+1/2} \quad ; \quad n \in \mathbb{Z}^+$$

$$\therefore \lambda_0 = 4L \rightarrow \text{fundamental}$$

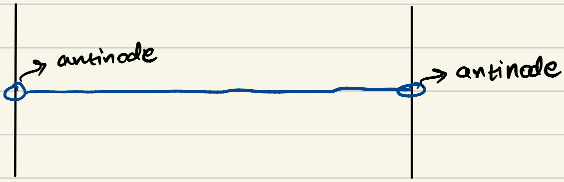
$$\lambda_1 = \frac{4L}{3}$$

$$\therefore \varphi(x,t) = \sum_{n=0}^{\infty} B_n \sin(k_n x)\sin(\omega_n t + \phi_n)$$

$$\lambda_n f_n = v \qquad (v = \sqrt{T/\mu})$$

$$f_n = \frac{v}{\lambda_n} \qquad \therefore \omega_n = \frac{2\pi v}{\lambda_n}$$

Both sides are free:



$$\psi(x,t) \propto \cos(kx)\cos(\omega t+\phi)$$

$$\therefore \left.\frac{\partial \psi}{\partial x}\right|_{x=L} = 0 \propto \sin(kx)$$

$$\therefore \quad kL = n\pi \qquad n \in [1, \infty),\ n \in \mathbb{Z}$$

$$\Rightarrow k = \frac{n\pi}{L}$$

$$\therefore \lambda = \frac{2L}{n}$$

$$\therefore \psi(x,t) = \sum_{n=1}^{\infty} B_n \cos(k_n x)\cos(\omega_n t + \phi_n)$$