Q1)

a) The group velocity in a dispersion relation is given by '$\frac{d\omega}{dK}$'. when $\omega = \omega_{max}$, we can observe the $\omega'(K)$ or $\frac{d\omega}{dK}$ has to be zero. Hence, when $\omega = \omega_{max}$, the group velocity is zero.

b) For any value to the left of a maximum, the graph will be increasing, ie, the first derivative is positive. For any value to the right of the maximum, the graph will decreasing, ie, the first derivative will have a negative value.
Hence, $\forall$ $K > K_m$, ie, $\forall$ K on the right of K_m, the value of the first derivative, ie, the group velocity will be negative.

Q2)

a)
$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \left[\frac{\partial^2 \psi}{\partial x^2} + L^2 \frac{\partial^4 \psi}{\partial x^4} \right]$$

$$\psi(x,t) = A e^{i(Kx \pm \omega(K) t)}$$

$$\frac{\partial \psi}{\partial t} = \pm i \omega(K) \psi(x,t) \quad ; \quad \frac{\partial^2 \psi}{\partial t^2} = -\omega^2(K) \psi(x,t)$$

$$\frac{\partial \psi}{\partial x} = iK \cdot \psi(x,t) \quad ; \quad \frac{\partial^2 \psi}{\partial x^2} = -K^2 \cdot \psi(x,t)$$

$$\therefore \frac{\partial^4 \psi}{\partial x^4} = K^4 \psi(x,t)$$

$$\therefore -\omega^2(K) = v^2 \left(-K^2 + L^2 K^4\right)$$

$$\Rightarrow \omega(K) = \sqrt{v^2 K^2 - L^2 v^2 K^4} \quad //$$

b) ω will have a max when $v^2k^2 - L^2v^2k^4$ is max

$$\therefore \frac{d}{dk}\left(v^2k^2 - L^2v^2k^4\right) = 0$$

$$\Rightarrow \quad 2v^2k - 4L^2v^2k^3 = 0$$

$$\Rightarrow \quad 2v^2k\left(1 - 2L^2k^2\right) = 0$$

$$\therefore \quad 2L^2k^2 = 1$$

$$\Rightarrow \quad k^2 = \frac{1}{2L^2} \quad \Rightarrow \quad k = \frac{1}{L\sqrt{2}}$$

$$\therefore \frac{d^2}{dk^2}\left(v^2k^2 - L^2v^2k^4\right) = 2v^2 - 12L^2v^2k^2$$

at $k = \frac{1}{L\sqrt{2}}$ $\rightarrow$ $2v^2 - \overset{6}{\cancel{12}}\cancel{L^2}v^2 \cdot \frac{1}{\cancel{2}\cancel{L^2}} = -4v^2$

the second derivative at $k = \frac{1}{L\sqrt{2}}$ is negative, hence we have a maxima at $k = \frac{1}{L\sqrt{2}}$

$$\therefore \omega_{max} = \sqrt{v^2 \cdot \frac{1}{2L^2} - \cancel{L^2} \cdot v^2 \cdot \frac{1}{4L^{\cancel{4}\,2}}} = \sqrt{\frac{v^2}{2L^2} - \frac{v^2}{4L^2}}$$

$$\therefore \quad \omega_{max} = \sqrt{\frac{v^2}{4L^2}} = \frac{v}{2L}$$