Transverse wave:

For example, in a string the wave travels perpendicular to the direction of the string.

Longitudinal wave:

As in sound waves, propagation and motion of each air molecule is in the same direction.

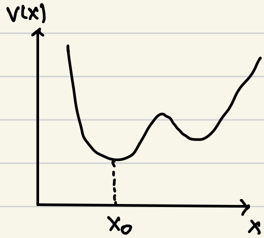
Simple harmonic motion:

From Hooke's law, we know that:

$$F(x) = -kx \quad \& \quad V(x) = \frac{1}{2}kx^2$$

In reality, if we stretch the string long enough, it will deviate from this potential.

Let's consider an arbitrary potential and understand its behaviour around a local min 'x_0'.



$$\therefore V(x) = \sum_{n=0}^{\infty} \frac{V^{(n)}(x_0)\,(x-x_0)^n}{n!} = V(x_0) + V'(x_0)(x-x_0) + \frac{1}{2}V''(x_0)(x-x_0)^2 \cdots$$

We can see that the first term goes to 0 as the physics does not change by shifting the potential by an amount. Furthermore:

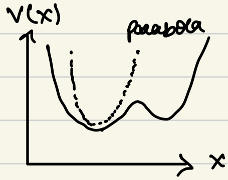
$V'(x_0) = 0$ as we are looking around the minimum.

As $x \to x_0$, $(x-x_0)^3 \to 0$.

$$\text{So, } \boxed{V(x) \approx \frac{1}{2}V''(x_0)(x-x_0)^2}$$

We can observe that we have a potential of the form $\frac{1}{2} K x^2$, where $K = V''(x_0)$, we have shifted our origin by x_0.

So, any potential can be approximated to Hooke's law, as long as we are around a local min:



$$\therefore \boxed{\omega = \sqrt{\frac{V''(x_0)}{m}}}$$

Solving for $x(t)$:

The long way:

$$-Kx = ma = m\left(\frac{v\,dv}{dx}\right)$$

$$\Rightarrow \int -Kx\,dx = \int mv\,dv$$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{constant} \\ \text{of integration.}}}{E} - \frac{1}{2}Kx^2 = \frac{1}{2}mv^2$$

→ which happens to be energy.

$$\therefore \underset{\substack{\uparrow \\ \frac{dx}{dt}}}{v} = \pm\sqrt{\frac{2}{m}}\sqrt{E - \frac{1}{2}Kx^2}$$

$$\therefore \int \frac{dx}{\sqrt{E}\cdot\sqrt{1 - \frac{Kx^2}{2E}}} = \pm\sqrt{\frac{2}{m}}\int dt$$

$$\Rightarrow x(t) = A\cos(\omega t + \phi) \quad \text{where } \omega = \sqrt{\frac{K}{m}}$$

'A' and 'ϕ' are arbitrary constants that are determined by initial conditions. A happens to be $\sqrt{2E/K}$

The short way:

$$-kx = ma = m\frac{d^2x}{dt^2}.$$

$\therefore$ $X(t) = A\cos(\omega t + \phi)$. $x(t)$ can also be approximated by sine or exponential function.

$$\therefore \; -kA\cos(\omega t+\phi) = m\frac{d^2}{dt^2}(A\cos(\omega t+\phi))$$

$$\Rightarrow -k\cancel{A}\cos(\omega t+\phi) = -m\cancel{A}\omega^2\cos(\omega t+\phi)$$

$$\Rightarrow \cos(\omega t+\phi)\,[m\omega^2 - k] = 0$$

$$\therefore \quad \omega = \sqrt{\frac{k}{m}}$$

From the above, we can conclude $\forall$ A & ϕ, the diff eq will be satisfied provided $\omega = \sqrt{\frac{k}{m}}$.

$$A\cos(\omega t+\phi) = A\cos(\phi)\cos(\omega t) - A\sin(\phi)\sin(\omega t).$$

Basically, we have found two solutions: a sine and cosine function with arbitrary coeffecients. The solution to the diff eq can be written as a sum of these two individual solutions. The fact that the sum of two solutions is again a solution is a consequence of Linearity of $F = ma$, i.e., x has a power of 1. The number of derivatives dont matter. Hence, $\left(\frac{d^n x}{dy^n}\right)'$ has n solutions of $x(y)$.

Observation:

$$X\left(t+\frac{2\pi}{\omega}\right) = A\cos\left[\omega\left(t+\frac{2\pi}{\omega}\right)+\phi\right] = A\cos(\omega t+\phi+2\pi) = A\cos(\omega t+\phi).$$

$\therefore$ we can observe the system repeats itself after time = '$\frac{2\pi}{\omega}$'

$$v(t) = x'(t)$$
$$= -A\omega \sin(\omega t + \phi)$$

Similarly, $v(t + \omega/2\pi) = v(t)$. Hence both position and velocity repeat after certain time period $T \equiv 2\pi/\omega$.

$\sqrt{k/m}$

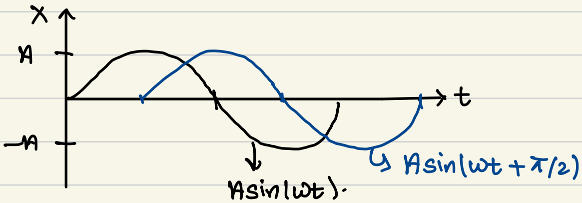
$$\therefore \boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

$$\boxed{f \equiv \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

Note:

- → ω is 'angular frequency'.
- → f is 'frequency'.
- → A is 'amplitude'.
- → ϕ is 'phase angle'.

How much the graph is shifted by.



Various ways to write $x(t)$:

We found that the position can be expressed as:

$$x(t) = A\cos(\omega t + \phi)$$
$$= A\sin(\omega t + \phi') \qquad \phi' = \phi + \pi/2.$$
$$= B_c\cos(\omega t) + B_s\sin(\omega t) \qquad B_c = A\cos\phi\ ;\ B_s = -A\sin\phi.$$
$$= C\exp(i\omega t) + C^*\exp(-i\omega t)$$
$$= \mathrm{Re}\left[D\exp(i\omega t)\right]$$

where A, B_c & $B_s \in \mathbb{R}$, C & $D \in \mathbb{C}$.

We can observe that in the above equations there are two parameters: 'A' & 'ϕ'. This is consistent with the fact that there are two initial conditions that must be satisfied.

↳ position & velocity.

Linearity:

As stated earlier, linear differential equations have the property that the sum of the solutions is also a solution. This is also consistent with our solutions to Hooke's law ($x(t)$).

Proof:

lets assume: $A\ddot{x} + B\dot{x} + Cx = 0$

let $x_1(t)$ & $x_2(t)$ be the solutions to the above.

$$\therefore \quad A\ddot{x}_1 + B\dot{x}_1 + Cx_1 = 0 \text{, and}$$

$$A\ddot{x}_2 + B\dot{x}_2 + Cx_2 = 0.$$

let $x_3 = x_1 + x_2$

$$\Rightarrow A \cdot \frac{d^2}{dt^2}(x_3) + B\frac{d}{dt}(x_3) + Cx_3$$

$$= A\frac{d^2}{dt^2}(x_1+x_2) + B\frac{d}{dt}(x_1+x_2) + C(x_1+x_2)$$

$$= 0$$

hence x_3 is also a solution.

Now lets suppose that we have a diff eq:

$$A\ddot{x} + B\dot{x}^2 + Cx$$

If 'x_1' & 'x_2' are the solutions then if we add the diff eq applied to each of the above, we get:

$$A\frac{d^2}{dt^2}(x_1+x_2) + B\left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2\right] + C(x_1+x_2) = 0$$

which clearly is not same as:

$$A\frac{d^2}{dt^2}(x_1+x_2) + B\left[\frac{d}{dt}(x_1+x_2)\right]^2 + C(x_1+x_2) = 0.$$

So, $x_3 = x_1 + x_2$ is not a solution.

Solving n^{th}-order linear differential equations:

from algebra, a polynomial:

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_0$$

can be factored into:

$$a_n (z-r_1)(z-r_2)(z-r_3)\cdots(z-r_n)$$

Similarly:

$$a_n\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots a_1\frac{dx}{dt} + a_0 = 0$$

$$\Rightarrow a_n\left(\frac{d}{dt}-r_1\right)\left(\frac{d}{dt}-r_2\right)\cdots\left(\frac{d}{dt}-r_n\right)x = 0$$

$$\Rightarrow a_n\prod_{i=1}^{n}\left(\frac{d}{dt}-r_i\right)x = 0$$

$$\therefore \left(\frac{d}{dt}-r_i\right)x = 0$$

$$\Rightarrow \frac{dx}{dt} - r_i x = 0$$

$$\Rightarrow \frac{dx}{dt} = r_i x$$

$$\Rightarrow x(t) = A\exp(r_i t).$$

taking the real part:

in the solution to Hooke's law, we guessed:

$x(t) = A\cos(\omega t + \phi)$. But, anything that is trignometric can be expressed exponentially.

$$\exp(i\theta) = \cos\theta + i\sin\theta.$$

let $\theta \to -\theta$ — (A)

$$\therefore \exp(-i\theta) = \cos\theta - i\sin\theta. \quad \text{— (B)}$$

Solving for $\cos\theta$ & $\sin\theta$ from (A) & (B), we get:

$$\cos\theta = \frac{\exp(i\theta) + \exp(-i\theta)}{2} \quad ; \quad \sin\theta = \frac{\exp(i\theta) - \exp(-i\theta)}{2i}$$

Now:

$$Kx = -m\ddot{x}$$

$\Rightarrow$ let $x(t) = C\exp(\alpha t)$.

$$\therefore K\cancel{C}\exp(\alpha t) = -m\alpha^2 \cancel{C}\exp(\alpha t).$$

$$\therefore \alpha^2 = -\frac{K}{m}$$

$$\Rightarrow \alpha = \pm\sqrt{\frac{K}{m}}\, i = \pm i\omega$$

$$\therefore x(t) = \underbrace{C_1 \exp(i\omega t)}_{T_1} + \underbrace{C_2 \exp(-i\omega t)}_{T_2}.$$

we know in the world we observe, $x(t)$ is real, hence: $T_1^* = T_2$

$$\Rightarrow C_1^* = C_2$$

lets choose $c_1 = c_0 \exp(i\phi) \Rightarrow c_2 = c_0 \exp(-i\phi)$.

$$\therefore \; x(t) = c_0 \exp[i(\omega t+\phi)] + c_0 \exp[-i(\omega t+\phi)] .$$
$$= 2c_0 \cos(\omega t+\phi) .$$

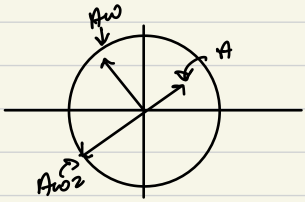
Remark:

we previously put down a condition that $x(t)$ has to be real and hence that bought down 4 variable (c_1, c_2, A, ϕ) to only two (A, ϕ). without this, we have to assume $x(t)$ & $v(t)$ are both real, ie, $x(0) = x_0 + 0 \cdot i$, $v(0) = v_0 + 0 \cdot i$.

Phasor equations:

$$x(t) = A\cos(\omega t+\phi)$$
$$\Rightarrow v(t) = -A\omega \sin(\omega t+\phi) = A\omega\cos(\omega t+\phi+\tfrac{\pi}{2}) .$$
$$\Rightarrow a(t) = -A\omega^2\cos(\omega t+\phi) = A\omega^2\cos(\omega t+\phi+\pi) .$$

phasor:



Initial conditions:

The initial conditions are nothing but the initial position and velocity.

$\therefore \; x(0) = x_0, \quad v(0) = v_0 .$

$$x(t) = B_c\cos(\omega t) + B_s\sin(\omega t) .$$
$$\Rightarrow v(t) = -B_c\omega\sin(\omega t) + B_s\omega\cos(\omega t) .$$

$\therefore x(0) = B_c = x_0$

$\Rightarrow B_c = x_0$

$v(0) = B_s\,\omega = v_0$

$\Rightarrow B_s = \frac{v_0}{\omega}.$

$$\therefore \boxed{x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t)}$$

if we want to write this in the form: $x(t) = A\cos(\omega t + \phi)$.

then :

$A\cos(\phi) = x_0$ —(i) ; $-A\sin\phi = \frac{v_0}{\omega}$ —(ii)

(ii) $\div$ (i)

$\Rightarrow \tan\phi = \frac{-v_0}{\omega x_0} \Rightarrow \phi = \tan^{-1}\left(\frac{-v_0}{\omega x_0}\right).$

$(i)^2 + (ii)^2$

$\Rightarrow A^2\cos^2\phi + A^2\sin^2\phi = x_0^2 + \frac{v_0^2}{\omega^2}$

$\Rightarrow A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}$ → we can also take –ve, our final ans will change by a factor of 'π'.

$$\therefore \boxed{x(t) = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}} \cdot \cos\left[\omega t + \tan^{-1}\left(\frac{-v_0}{\omega x_0}\right)\right]}$$

Note:

If we have less / more initial conditions, we will not be able to find the particular solutions. Hence, this follows with the linearity of $F = ma$, is a second order differential equation.

Energy:

($F = -kx$) is a conservative force: we can quickly show this by calculating the work done: $W = \int_{x_1}^{x_2} (-kx)\,dx = \underbrace{-\frac{1}{2}kx_2^2 + \frac{1}{2}kx_1^2}_{}$.

↳ depends only on final & initial position.

Hence, energy is conserved.

$$\therefore \; E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

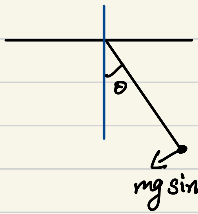
$$= \frac{1}{2}k\left[A\cos(\omega t+\phi)\right]^2 + \frac{1}{2}m\left[A\omega\sin(\omega t+\phi)\right]^2$$

$$= \frac{1}{2}kA^2\left[\cos^2(\omega t+\phi) + \sin^2(\omega t+\phi)\right]$$

$$= \frac{1}{2}kA^2. \rightarrow \text{constant.}$$

Examples:

Simple pendulum:



$$F = ma = m l \alpha$$

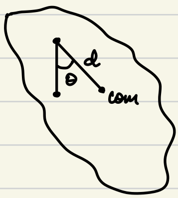
$$\Rightarrow -mg\sin\theta = m(l\ddot{\theta})$$

$\sin\theta \rightarrow \theta$ (small θ)

$$\therefore \; \ddot{\theta} + \omega^2\theta = 0 \quad ; \quad \omega \equiv \sqrt{\frac{g}{l}}$$

$$\therefore \; \theta(t) = A\cos(\omega t + \phi).$$

Physical pendulum:



$$\tau = I\alpha$$

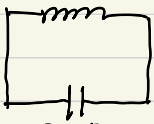
$$\Rightarrow (-mg\underbrace{\sin\theta}_{\hookrightarrow \theta})d = I\ddot{\theta}$$

$$\Rightarrow I\ddot{\theta} + mgd\theta = 0$$

$$\Rightarrow \ddot{\theta} + \frac{mgd}{I}\theta = 0$$

let $\omega \equiv \sqrt{\frac{mgd}{I}}$.

LC circuits:



from Kirchoff's law:

$$-L\frac{di}{dt} + \frac{Q}{C} = 0$$

$$\Rightarrow L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0 \qquad \left(\frac{di}{dt} = -\frac{d^2Q}{dt}\right)$$

↳ charge in cap. is ↓.

$$\Rightarrow \ddot{Q} + \frac{1}{LC}Q = 0$$

$$\therefore \omega \equiv \frac{1}{\sqrt{LC}}$$

$$\therefore Q = Q_0 \cos(\omega t + \phi)$$