Physics 273 — Homework 8 Solutions

Question 1 (10 points)

If we have a function with two independent variables (or more), we can take the derivative with respect to either variable, but we have to be a bit careful about what is assumed about the other variable. Here, let's practice with the function $f(r,s) = \cos cos\,(r^2 s)$.

(a) The **partial** derivative assumes that the other variable is constant (or all other variables are constant, if there are more than two). With that in mind, calculate $\partial f/\partial r$ and $\partial f/\partial s$. (This is fairly easy, but remember to use the chain rule.) – 3 points

$$\frac{\partial f}{\partial r} =- \sin sin\left(r^2 s\right)\ 2rs\,, \qquad \frac{\partial f}{\partial s} =- \sin sin\left(r^2 s\right)\ r^2$$

(b) However, if one of the variables actually depends on the other, then we might really want to calculate the **total** derivative, which represents the effect of changing that variable both directly and indirectly (through the other variable). For example, suppose that s is a function of r, specifically: $s(r) = r^4$. First, just substitute that into the cosine argument in our definition of f so that the cosine argument only involves r. This makes f effectively just a function of one variable, r. Calculate the total derivative, df/dr . (This is pretty simple. Note that it should not, in general, be equal to $\partial f/\partial r$.) – 3 points

Substitute: $\cos cos\left(r^2 s\right) = \cos cos\left(r^2 r^4\right) = \cos cos\left(r^6\right)$. Then $\frac{df}{dr} =- \sin sin\left(r^6\right)\ 6r^5$.

(c) Alternatively, go back to $f(r,s) = \cos cos\,(r^2 s)$ and use the chain rule for the total derivative: $\frac{df}{dr} = \frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial r}$. Evaluate df/dr this way and show, by substituting and/or simplifying as needed, that you get the same answer as in part b. – 4 points

$$\frac{df}{dr} = \frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial r} =- \sin sin\left(r^2 s\right)\ 2rs + \left[- \sin sin\left(r^2 s\right)\ r^2\right]\left[4r^3\right]$$

$$=- \sin sin\left(r^2 s\right)\ \left(2rs + 4r^5\right)$$

To put this on the same footing as part b, substitute $s = r^4$ now:

$$\frac{df}{dr} =- \sin sin\left(r^2 r^4\right)\ \left(2rr^4 + 4r^5\right) =- \sin sin\left(r^6\right)\ 6r^5$$

Question 2 (10 points)

In class, I first introduced the wave equation using a model for a longitudinal wave, and then we switched to talking about transverse waves. The math has a similar form and the wave equations end up looking very similar. However, the actual nature of the waves differs, and different properties of the wave medium are present as factors in the wave equations.

A metal cylinder that is very much longer than its thickness (i.e., the diameter of the cylinder) could be described as a "string" or "wire", while a cylinder that is somewhat thicker could be described as a “rod”. We normally think of transverse waves on a string/wire and longitudinal waves in a rod, but any cylinder can do either of those, in principle. It’s just a question of how the cylinder is supported and how the wave is initiated.

(a) A wave may “travel”, but any given bit of the medium just moves back and forth (oscillates) around its equilibrium position, and does not have any net displacement after the wave has passed. Think about one specific atom in the cylinder (wire or rod). In words, describe which direction that atom moves back and forth in the case of a longitudinal wave, and then describe which direction that atom moves back and forth in the case of a transverse wave. – 2 points

In a longitudinal wave, the atom moves back and forth parallel to the direction of wave travel.
In a transverse wave, the atom moves sideways (side-to-side) relative to the direction of wave travel.

(b) Suppose that the cylinder is made of copper, with a length of 2.3 m, a cross-sectional area of 1.0 mm^2 (careful: that is $10^{-6}\ m^2$, not $10^{-3}\ m^2$), and is pulled at the ends so as to have a tension of 10.0 N. What is the speed of a longitudinal wave under these conditions, and what is the speed of a transverse wave? (You will need to look up some properties of copper. Write what your source is and what values you are using. Different sources will list slightly different values because it depends on how the copper has been treated in processing and “drawing out” into a long wire or rod, but any you find should be OK for this calculation.) – 3 points

Some values I found online were a density of $8.935\ g/cm^3 = 8935\ kg/m^3$ and Young’s modulus of $127\ GPa$ ($127\times10^9\ N/m^2$). The speed of a longitudinal wave is $v_L = \sqrt{Y/\rho} = 3770$ m/s. The speed of a transverse wave is $v_T = \sqrt{T/\mu}$ where the mass-per-length $\mu = \rho A = 8.935\times10^{-3}\ kg/m$ for this wire. Putting that in with the given tension, $v_T = 33.5$ m/s.

(c) The tension is increased by a factor of 9, i.e. to 90.0 N. What is the speed of a longitudinal wave under these conditions, and what is the speed of a transverse wave? How do these compare with what you found in part b? (Include that comparison in your written answer.) – 3 points

A longitudinal wave doesn’t depend on tension, so its speed is still 3770 m/s.
For the transverse wave, recalculate $v_T = \sqrt{T/\mu}$ with the new tension, getting 100.4 m/s.

It is also acceptable to explain that the speed of a transverse wave scales with tension as $\sqrt{T}$ and then simply multiply the answer from part b by $\sqrt{9} = 3$.

For the comparison: the speed of the longitudinal wave is the same, while the speed of the transverse wave is larger (by a factor of 3).

(d) One end is now un-clamped, so that the tension drops to zero. What speed do you calculate for a transverse wave now? How do you interpret that? – 2 points

The speed of the transverse wave is now calculated to be zero. A transverse wave can’t propagate without tension.