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1. If m and n are positive integers, prove that the following is true.
$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}.\frac{n!}{n^n}$$

2. Consider the following funciton
$$f(x) = (1 - x + x^2)e^x$$
Prove that every coefficeint of the Taylor expansion about 0 of function f is a rational number and when written in lowest terms has numerator either 1 or a prime number.

3. Consider you have a sack which contains W white balls and B black balls. You also have an additional supply of infinite black and wihte balls. Now you randomly take out two balls from the sac. If they are both of the same colour, you discard the both and put back a white ball in the sack and if the balls taken out are of different colour, you put back the black ball and discard the white one. Given W and B, can you determine the last ball remaining in the sack.

4. Out of boredom, person A and B created a game. They took a pile of N coins and decided that they can either remove 1 coin or the number of coin of the form p^x where p is a prime and x a positive integer. The person to pick last coin wins the game. Person A moves first. Can you determine who will win the game, if both the players play optimally?

5. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose f has infinitely many zeroes but there is no $x \in (a, b)$ such that $f(x) = f'(x) = 0$.
   (a) Prove that $f(a)f(b) = 0$.
   (b) Give an example of such a function.

6. Let n be a positive integer, and denote by $\mathbb{Z}_n$ the ring of modulo n.
$\mathbb{Z}_n$ is defined as follows.
$$\mathbb{Z}_n = \{0, 1, 2, \ldots n - 1\}$$
if $x, y \in \mathbb{Z}_n$ then
$$x + y = (x + y)(\bmod n)$$
and
$$x \times y = (x \times y)(\bmod n)$$
Suppose that there exists a function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ satisfying following properties:-
   (a) $f(x) \neq x$
   (b) $f(f(x)) = x$
   (c) $f(f(f(x + 1) + 1) + 1) = x$

   for all $x \in \mathbb{Z}_n$ Prove that $n \equiv 2(\bmod 4)$