Samasya

Samasya is a mathematics discussion and problem solving club. We discuss a variety of mathematical topics and solve problems as well. We encourage participants to have a look at these problemsbefore the meeting. Discussion, however, will not be limited to these problems. Participants can bring their own problems or mathematical ideas they wish to discuss.

Date: Friday, 14^{th} August
Time: 9 p.m.
Venue: OPB LAN Room

Problem 1. Consider an equivalence relation on the set $(0, +\infty)$: $x \sim y$ if $\frac{x}{y} \in \mathbb{Q}$. The relation creates equivalence classes on the set. Show that the cardinality of the set of the equivalence classes is the cardinality of $\mathbb{R}$. Also, show that any such equivalence class intersects every open interval in $(0, +\infty)$.

Problem 2. Consider a square of side length n, and take any $(n+1)^2$ points inside it. Show that three of these points form a triangle whose area is less than or equal to $\frac{1}{2}$.

Problem 3. For a given prime number q, find all polynomials $P(x)$ with integer coefficients such that $P(x) \mid x^q - 1$ for infinitely many $q \in \mathbb{Z}$.

Problem 4. Show that the cardinality of the set of continuous functions on $\mathbb{R}$ is $\aleph_1$, i.e. the cardinality of the real numbers.

Problem 5. Consider a function $f : [0, 1] \to \mathbb{R}$ such that $f(3x) = 2f(x)$ and $f(x) + f(1 - x) = 1$ for all $x \in [0, 1]$. Does such a function actually exist? If it does, is it unique? What if we impose monotonicity? What if we impose continuity?

Problem 6. Consider the ring of real continuous functions on $[0, 1]$. Show that every maximal ideal of the ring is of the form

$$I_\alpha = \{f \mid f(\alpha) = 0\}$$

for a given $\alpha \in [0, 1]$. In other words, show that that there exists an injective map between the set of maximal ideals of the ring and the set of points in $[0, 1]$. Is this still true if we take the space $(0, 1)$ instead?

Problem 7. A cover of a topological space X is a collection of subsets $\{C_i\}$ of X such that $\bigcup C_i = X$. Each C_i inherits the subspace topology from X. A cover C_i is fundamental if: a subset U of X is open in X is equivalent to saying that $U \cap C_i$ is open in C_i for all C_i. A cover is locally finite if every point in X has a neighbourhood that is contained in finitely many C_i. Prove the following:

- Every open cover is fundamental.
- A finite closed cover is fundamental.
- Every locally finite closed cover is fundamental.
- If f is a function on X, and the restriction of f on C_i for all i is continuous, then f is continuous.