Samasya

Samasya is a mathematics discussion and problem solving club. We discuss a variety of mathematical topics and solve problems as well. We encourage participants to have a look at these problems before the meeting. Discussion, however, will not be limited to these problems. Participants can bring their own problems or mathematical ideas they wish to discuss.

Date: Friday, 4$^{\text{th}}$ September
Time: 9 p.m.
Venue: OPB LAN Room

Problem 1. An algebraic number is a number which is the root of a polynomial with integer coefficients. For example, all rational number are algebraic because they are the root of polynomials of the form $mx - n$, where m and n are integers and $m \neq 0$. Some irrational numbers are also algebraic, e.g. $\sqrt{2}$ is algebraic because it is the root of $x^2 - 2$. Show that there exists a bijection between the set of algebraic numbers in $\mathbb{R}$ and the natural numbers $\mathbb{N}$. By doing so, prove that there exist some real numbers which are not algebraic. Such numbers are called transcendental.

Problem 2. Let $\mathbb{C}^\times$ denote the complex numbers excluding 0. Let f be a function from $\mathbb{C}^\times$ to $\mathbb{C}^\times$ such that $f(uf(v)) = \frac{f(u)}{v}$ for all u and v in $\mathbb{C}^\times$. Show that this function is bijective and $f(ab) = f(a)f(b)$ for all a and b in $\mathbb{C}^\times$. Construct an example of such a function, if possible, otherwise prove no such function exists.

Does such a function still exist when $\mathbb{C}^\times$ is replaced with $\mathbb{Q}^+$, the positive rationals?

Problem 3. Show that one cannot tile a 28×17 rectangle with rectangles of dimension 4×7. What about the more general case of tiling an $m \times n$ rectangle with rectangles of dimension $a \times b$? Find the necessary and sufficient conditions for a tiling to be possible.

Problem 4. Let a be any real number. Show that for all values of $\epsilon > 0$, there exist positive integers m and n such that $|ma - n| < \epsilon$.

Problem 5. Prove that for any finite sequence of digits, there exists a power of two that starts with these digits.

Problem 6. For any natural number n, define $E(n)$ to be the highest exponent of a prime that divides it. For example, $E(27) = 3$ and $E(18) = 2$. Show that the following limit exists:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=2}^{N} E(n)$$