Determinant of a 3X3 matrix A.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

We define the

$$\det(A) = a_{11} \cdot \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & a_{33} \\ a_{31} & a_{23} \end{pmatrix} + a_{13} \det\begin{pmatrix} a_{21} & a_{32} \\ a_{31} & a_{32} \end{pmatrix}$$

$$\det(A) = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

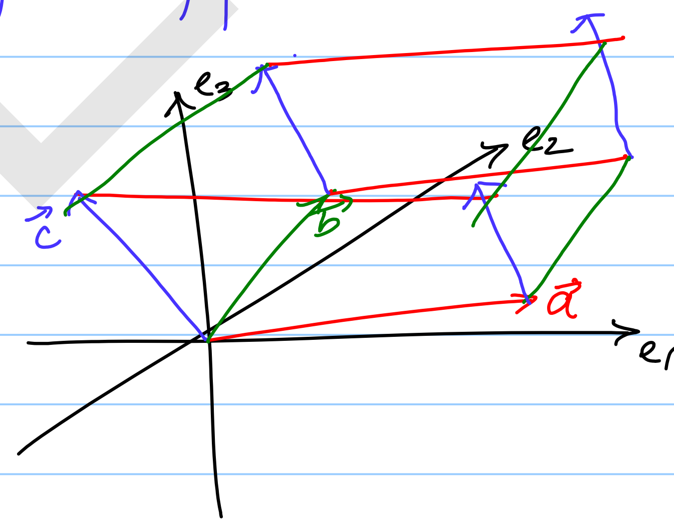
$$= \underset{\Downarrow \atop \vec{a}}{\begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix}} \cdot \underbrace{\begin{pmatrix} a_{22}a_{33} - a_{32}a_{23} \\ a_{31}a_{23} - a_{21}a_{33} \\ a_{21}a_{32} - a_{31}a_{22} \end{pmatrix}}_{(\vec{b} \times \vec{c})}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c})$$

det (A) = Volume of the parallelopiped with sides along the 3 row vectors of A.

$\det(A) = \vec{a} \cdot (\vec{b} \times \vec{c})$ = vol. of the parallelopiped

$\vec{a} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}^T = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix}$



$\vec{b} = \begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \end{bmatrix}$ $\vec{c} = \begin{bmatrix} a_{31} \\ a_{32} \\ a_{33} \end{bmatrix}$

Some properties of the det of A.

1. $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$

We know that $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

2). If two rows of a matrix are interchanged

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \xRightarrow{\text{Changing rows}} A' = \begin{bmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$\begin{aligned} \det \text{ of } A' &= \vec{b}\cdot(\vec{a}\times\vec{c}) \\ &= \vec{b}\cdot -(\vec{c}\times\vec{a}) \\ &= -(\vec{b}\cdot(\vec{c}\times\vec{a})) \\ &= -\det A. \end{aligned}$$

3). The determinant of A^T and A are the same.

$$\det(A^T) = \det(A)$$

4) Suppose we multiply a col. of a matrix by a scalar 'k'

i.e., $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

$= A' = \begin{bmatrix} ka_1 & a_2 & a_3 \\ kb_1 & b_2 & b_3 \\ kc_1 & c_2 & c_3 \end{bmatrix}$

$\det(A') = k \det(A)$

$$\det(A') = ka_1(b_2c_3 - c_2b_3) - a_2(kb_1c_3 - kc_1b_3) + a_3(kb_1c_2 - kc_1b_2)$$

$$\Rightarrow k\left[a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - b_2c_1)\right]$$

$$= k \det(A)$$

$\det[k\,\text{col}1 \;\; \text{col}2 \;\; \text{col}3] = k \det(A).$

(5) $\det(kA) = k^3 \det(A)$.

(6) Suppose we have A with identical rows. Then $\det(A) = 0$.

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

If the 2nd row & 3rd row are the same,

$\det = \vec{a} \cdot (\vec{b} \times \vec{b}) = \vec{a} \cdot \vec{0} = 0$

Let 1st row & 3rd be the same

$\det(A) = \vec{a} \cdot (\underbrace{\vec{b} \times \vec{a}}_{})$

vector $\vec{u}$ orthogonal to both $\vec{a}$ & $\vec{b}$

$\det(A) = \vec{a} \cdot \vec{u} = 0$

7) If one of the rows of A is all zeros, $\det(A) = 0$.

8) The sum of det $\neq$ det of the sum. $\Rightarrow \det(A) + \det(B) \neq \det(A+B)$

9) Product of det = det of prod.

$\Rightarrow \det(A)\det(B) = \det(AB)$.

10) If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$