Span:

Let S be a nonempty subset of a vector space V.

Def: Span(S) is the set of all possible combinations of vectors in S.

Ex: $V = \mathbb{R}^3$ $\quad S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

For α_1, α_2 real, the set of all possible l.c. of vectors in S correspond to

$$\text{Span}(S) = \left\{ \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \,\middle|\, \alpha_1, \alpha_2 \in \mathbb{R} \right\}.$$

$$\text{Span}(S) = \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} \middle| \ \alpha_1, \alpha_2 \in \mathbb{R} \right\}.$$

Ex: 2: $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$$\text{Span}(S) = \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \middle| \ \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}$$

$\Rightarrow$ Span(S) is entire $\mathbb{R}^3$.

Suppose $S = \{\vec{u}_1, \vec{u}_2 \dots \vec{u}_k\}$ & let $V = \mathbb{R}^n$.

$$\text{Span}(S) = \left\{ \sum_{i=1}^{k} \alpha_i u_i, \ \alpha_i \in \mathbb{R} \ \ i = 1 \dots k \right\}.$$

* Span of a single non-zero vector

$S = \left\{ \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \right\}$. $\Rightarrow$ Span$(S) = \{\alpha \vec{u}\}$ $\alpha \in \mathbb{R}$.

Line passing thro' the Origin

Span of any 2 l.i. vectors in $\mathbb{R}^n$ is a plane passing thro' the origin.

Note: Any subspace of a vector space V, that contains $\vec{u}_1, \vec{u}_2 \dots \vec{u}_k$ must contain all possible l.c. of $\vec{u}_1, \dots \vec{u}_k \rightarrow \text{Span}(\vec{u}_1, \vec{u}_2 \dots \vec{u}_k)$

Span(S) $\rightarrow$ Smallest subsp. containing S.

S: nonempty subset of vector space V

Let S be a subset of a V.S V

Let $\vec{u}$ be any vector in V.

Q1: Does there exist a l.c in S that is equal to $\vec{u}$?

Q2: If such a l.c. exists, is the l.c. unique?

$\Rightarrow$ Is S a l.i. set?

BASIS for a vector space.

Let V be a VS and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2 \dots \vec{b}_k\}$ be a subset of V.

$\mathcal{B}$ is a basis for V if and only if

a) $\mathcal{B}$ spans V

b) $\mathcal{B}$ is a linearly indep set.

Examples:

① $V = \mathbb{R}^2$, $\boxed{\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}}$ is a basis for $\mathbb{R}^2$.

② $V = \mathbb{R}^2$ $\mathcal{B}_2 = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$.

③ $V = \mathbb{R}^n$

$$\boxed{\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_n, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_n \cdots \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}_n \right\}}$$

Standard Ordered basis for $\mathbb{R}^2$ & $\mathbb{R}^n$ respectively.

$P_2(x)$: Vector Space of polynomials of degree ≤ 2. $\{(a_0+a_1x+a_2x^2) \mid a_0, a_1, a_2 \in \mathbb{R}\}$

$\mathcal{B}_1 = \{1, x, x^2\}$

$\mathcal{B}_2 = \{x^2, \underbrace{x-1, x+1}\}$.

Coordinate vectors are $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Corresp to x^2.

$x-1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $\quad x+1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

1. Suppose if $\mathcal{B} = \{\vec{b}_1, \vec{b}_2 \cdots \vec{b}_n\}$ is a basis for $\mathbb{R}^n$,

any vector $\vec{u} \in \mathbb{R}^n$ has a basis expansion

$\vec{u} = \alpha_1\vec{b}_1 + \alpha_2\vec{b}_2 + \cdots + \alpha_n\vec{b}_n$

Ex: Let $u = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ $V = \mathbb{R}^2$

$$\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

$$\vec{u} = \alpha_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$\alpha_1 = 1$, & $\alpha_2 = 2$

$$1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

Try this: $V = \mathbb{R}^2$.

$$\vec{u} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad \mathcal{B}_2 = \left\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$$

Verify that

$$\boxed{\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \frac{13}{10} \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \frac{1}{10} \begin{pmatrix} -1 \\ 2 \end{pmatrix}}$$

ORTHONORMAL SET OF VECTORS.

Consider $\mathcal{B} = \{\vec{\varphi}_1, \vec{\varphi}_2 \ldots \vec{\varphi}_n\}$

Set of Orthonormal vectors

ie., $\hat{\varphi}_i \cdot \hat{\varphi}_j = \begin{cases} 0 & i \neq j \rightarrow \text{ORTHOGONAL} \\ 1 & i = j \rightarrow \text{Length} = 1. \end{cases}$

A set of Orthonormal vectors is a l.i. Set.

$(\alpha_1 \vec{\varphi}_1 + \alpha_2 \vec{\varphi}_2 + \ldots + \alpha_n \vec{\varphi}_n = \vec{0})$ (*)

(for α_i's real) $\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0.$

Take the dot product of $\vec{\varphi}_1$ with (*)

$\vec{\varphi}_1 \cdot (\underbrace{\alpha_1 \vec{\varphi}_1} + \alpha_2 \vec{\varphi}_2 + \ldots + \alpha_n \vec{\varphi}_n) = \cancel{\vec{\varphi}_1} \cdot 0.$

$\alpha_1 + \cancel{(\vec{\varphi}_2 \cdot \vec{\varphi}_1)}\alpha_2 + \ldots + \alpha_n \cancel{(\vec{\varphi}_n \cdot \vec{\varphi}_1)} = \vec{0}$

$\Rightarrow \alpha_1 = 0.$

Similarly we can establish all the α_i's $= 0$

Thus a set of orthonormal vectors is a l.i. set.

If $\vec{\varphi}_1, \vec{\varphi}_2 \cdots \vec{\varphi}_n$ is set of n-orthonormal vectors in $\mathbb{R}^n$, they form a basis for $\mathbb{R}^n$.

Orthonormal Basis expansion for $\vec{u} \in \mathbb{R}^n$

$$\vec{u} = \alpha_1 \vec{\varphi}_1 + \cdots + \alpha_n \vec{\varphi}_n$$

$$\vec{u} \cdot \vec{\varphi}_1 = \alpha_1 \underbrace{\vec{\varphi}_1 \cdot \vec{\varphi}_1}_{1}$$

$$\Rightarrow \alpha_1 = \vec{u} \cdot \vec{\varphi}_1$$

$$\alpha_2 = \vec{u} \cdot \vec{\varphi}_2$$

$$\vdots$$

$$\vec{u} = (\vec{u} \cdot \vec{\varphi}_1)\vec{\varphi}_1 + (\vec{u} \cdot \vec{\varphi}_2)\vec{\varphi}_2 + \cdots + (\vec{u} \cdot \vec{\varphi}_n)\vec{\varphi}_n$$

FOURIER EXPANSION

Dimension of a vector Subspace:

The number of elements is a basis for a vector space V is called the dimension of V.

For ex: $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

$V = \mathbb{R}^2$

Suppose W is a subspace of a k-dim Vector space V, with basis for W having d-elements, then

W is a d-dim Subsp. of V

$W = \{\vec{0}\}$. $V = \mathbb{R}^n$.

$\Rightarrow$ Basis for $W = \{\vec{0}\}$ is the empty set.

Hence $W = \{\vec{0}\}$ is called a 0-Dim Subspace of $\mathbb{R}^n$

Any line passing thro' the Origin is 1D Subspace of $\mathbb{R}^n$

Any plane passing thro' the Origin is a 2D Subsp. of $\mathbb{R}^n$.