Recall:

If x_1 & x_2 are two unknowns, and

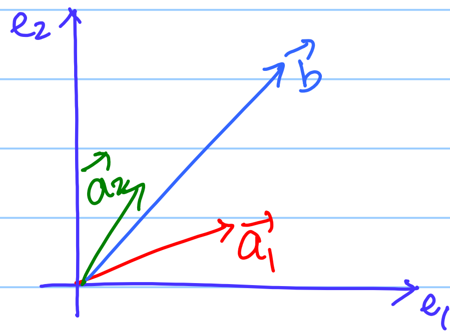
$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Solving for x_1 & x_2 to satisfy the 2 equations simultaneously is the same as finding the scalars x_1 & x_2 st

$$x_1 \overbrace{\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}}^{\vec{a}_1} + x_2 \overbrace{\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}}^{\vec{a}_2} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$A\vec{x} = \vec{b}$$

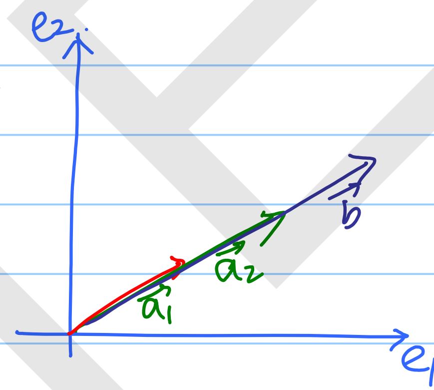
$$A = \begin{array}{cc} \vec{a}_1 & \vec{a}_2 \end{array}$$
$$A = \begin{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} & \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$



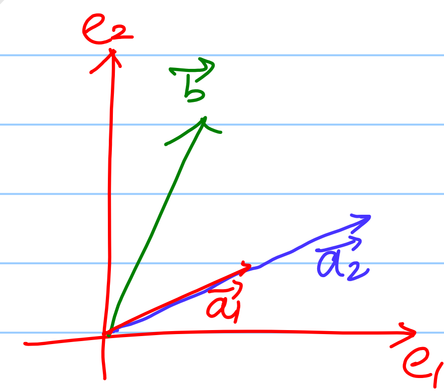
$\vec{b} = x_1(\vec{a}_1) + x_2(\vec{a}_2)$

Unique Soln
for every $\vec{b}$

$\det(A) \neq 0$.



Infinitely many
Soln.



No Solution.

$\longrightarrow$ $\det(A) = 0$.

Suppose we have the following sys. of linear eqns in 3 unknowns

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$$
$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3.$$

Solve for x_1, x_2, x_3

$$A\vec{x} = \vec{b}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

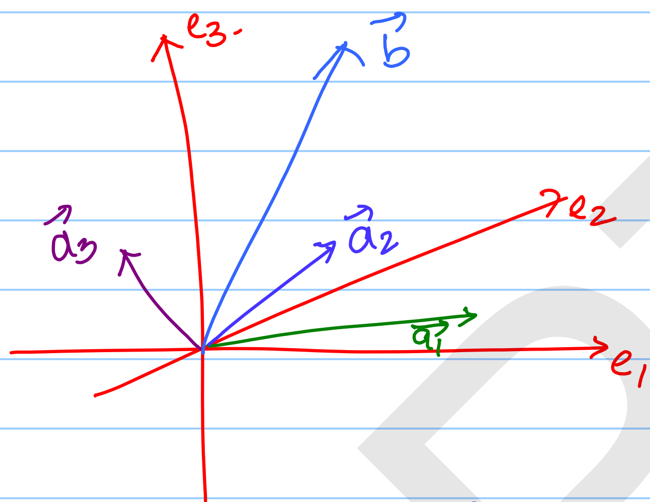
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If $\vec{a_1}$, $\vec{a_2}$ & $\vec{a_3}$ i.e.,

$$\vec{a_1} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \quad \vec{a_2} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \quad \& \quad \vec{a_3} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

are such that they form a tetrahedron, then for every $\vec{b}$ in 3D, we will have a unique soln (x_1, x_2, x_3).

$\Rightarrow$ If $\det(A) \neq 0$ then we have unique soln (x_1, x_2, x_3) corresponding to every $\vec{b}$ in 3D.



$$\vec{b} = x_1(\vec{a_1}) + x_2(\vec{a_2}) + x_3\vec{a_3}$$

Unique soln to $A^{3\times3}x^{3\times1} = b^{3\times1}$

What is it to say $A\vec{x} = \vec{b}$ has infinitely many solutions,

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$\det(A) = 0$

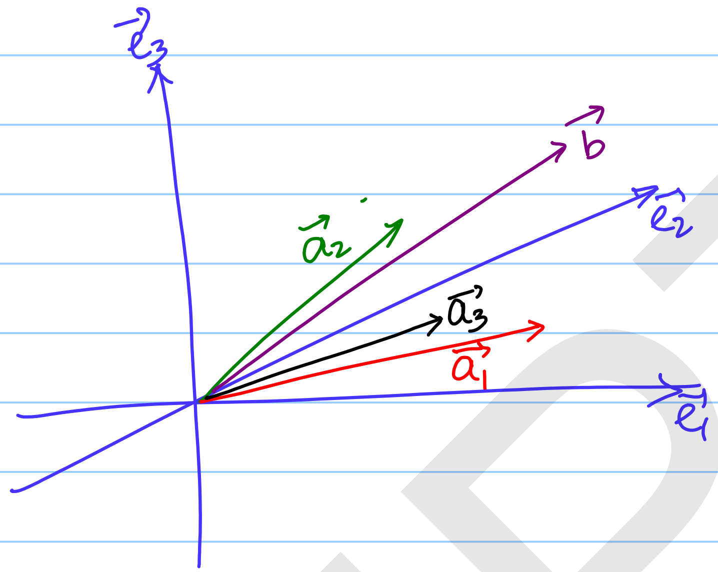
Let $\vec{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}$ $\vec{a}_3 = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$

$\det(A) = 0$ if

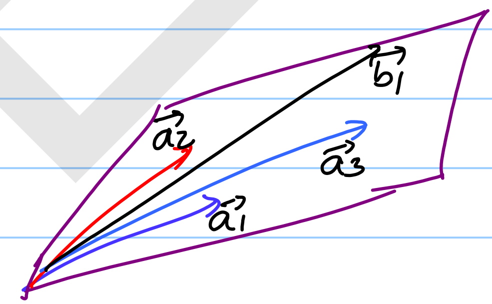
(i) $\vec{a}_1 = \vec{a}_2 = \vec{a}_3 = \vec{0}$, (atleast one is a zero vec)

(ii) $\vec{a}_1 = k\vec{a}_2$; $\vec{a}_1 = l\vec{a}_3$

(iii) $\vec{a}_3 = \alpha\vec{a}_1 + \beta\vec{a}_2$



$\Longrightarrow$

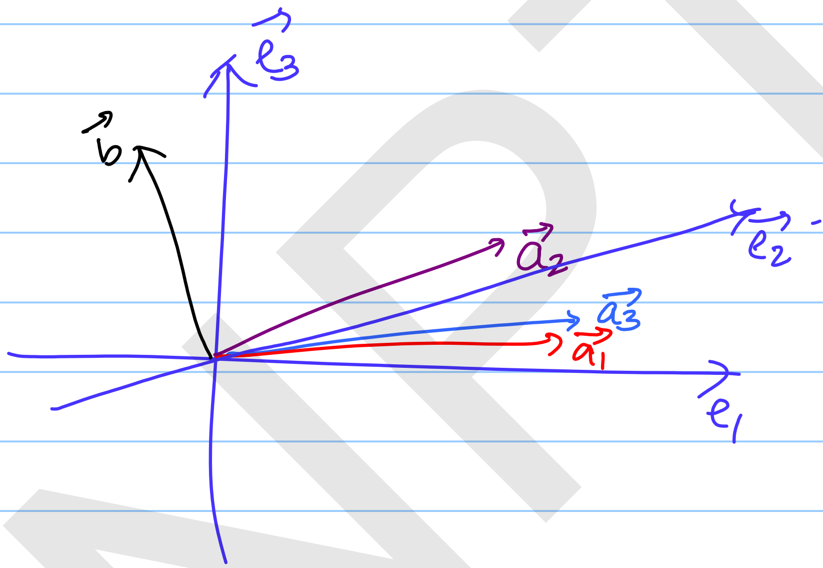


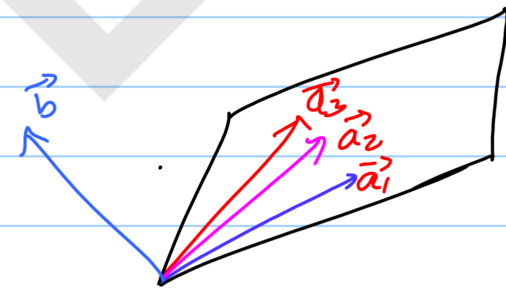
Infinitely many solns.

$\Rightarrow$ All the 4 vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ and $\vec{b}$ are all in the same plane.

No Solutions Case:

$\det(A) = 0$





For example: If $\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

$\vec{a}_3 = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Inconsistent System.

HSE in 3D.

Solving using Gauss elimination

→ LU ←

→ Least Sq. Soln

GSO

→ QR

→ eigenval & eigenvectors.