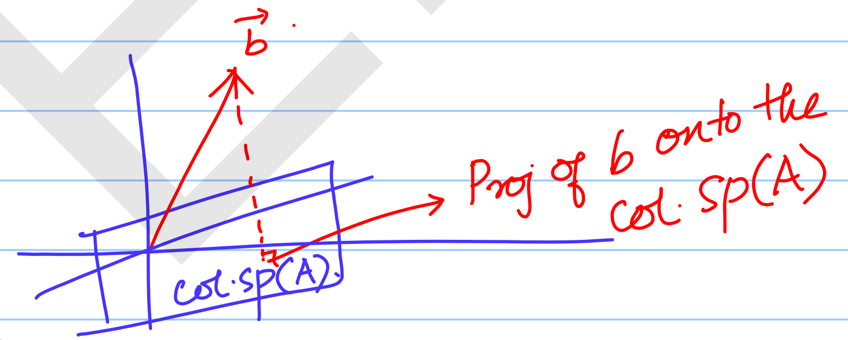
Suppose $A \in \mathbb{R}^{m\times n}$ with rank of $A = n$. i.e., A is a full rank matrix and let 'b' be any vector in $\mathbb{R}^m$.

$Ax = b$ does not have a solution if b is not in the col. space of A.



$$x^{\dagger} = (A^TA)^{-1}A^Tb.$$

$(A^TA)^{-1}A^T$ is called the pseudoinverse of the matrix A. It is denoted by $A^{\dagger}$.

Definition:

$A^\dagger$, the pseudo inverse of the matrix A, is that matrix that satisfies the following:

(i) $A^\dagger b$ is in the col. sp (A^T)
$\Rightarrow A^\dagger b$ is in the rowsp (A)

(ii) $AA^\dagger$ = Projection Matrix that projects vectors in $\mathbb{R}^m$ onto the col. space of A.

$A^\dagger$ is an $n \times m$ matrix, if A is an $m \times n$ matrix.

Properties of $A^\dagger$

What does this product boil down to?

(i) $A A^\dagger A = A$. $\Rightarrow$ $A(A^TA)^{-1}(A^TA) = A$.

(ii) $A^\dagger A A^\dagger = A^\dagger$ $\Rightarrow$ $(A^TA)^{-1}A^T(A)A^\dagger = A^\dagger$.

Pseudo Inverse of certain special matrices.

(1) Suppose A is invertible.

↳ $\text{Rank}(A) = n$.

$A \in \mathbb{R}^{n \times n}$

Col space of A is $\mathbb{R}^n$.

$Ax = b \qquad x \in \mathbb{R}^n, \quad b \in \mathbb{R}^n$.

$\Rightarrow$ The matrix that projects $b \in \mathbb{R}^n$ to the col. Sp(A) which is $\mathbb{R}^n$ itself is the identity matrix I.

$AA^{\dagger} = P$ is the matrix that projects every vector on to the col. sp(A).

$\Rightarrow \quad AA^{\dagger} = I$.

Recall that $A^{\dagger}$ provides the best solution to $Ax = b$.

$\Rightarrow \quad A^{\dagger} = A^{-1}$

If A is invertible, the pseudoinverse of A is the inverse of A itself.

For A invertible, $\boxed{A^{\dagger} = A^{-1}}$.

(2) Suppose A is an $m \times n$ matrix with rank of $A = n$.

$\Rightarrow$ A has linearly indep. cols.

$$\boxed{A^{\dagger} = (A^T A)^{-1} A^T}$$

A has line indep rows:

A is $m \times n$ matrix with linearly indep rows

$\Rightarrow$ rank of $A = m$.

AA^T is an $m \times m$ matrix and invertible.

$\rightarrow$ Rank $= m$.

$\Rightarrow$ To solve $Ax = b$

Example:

$A^{2\times3}$ Rank of $A = 2$.

$$\Rightarrow A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}^{2\times3}$$

$\Rightarrow$ col. sp(A) is $\mathbb{R}^m$.

Thus the col. sp(A) is $\mathbb{R}^m$.

The projection matrix that projects b in $\mathbb{R}^m$ to $\mathbb{R}^m$, is the Identity matrix $I^{m\times m}$.

$$\boxed{A^{+} = A^T(AA^T)^{-1}}$$

if rank of $A = m$.

If A has l.i. rows then

$$A^{\dagger} = A^T(AA^T)^{-1}.$$

Outline of proof:

$$Ax = b.$$

Since rank of A is m,
rank of $A^T = m$,
express $x = A^T y$.
$\hookrightarrow$ col. sp(A^T).

$$A(A^T y) = b$$

$$\Rightarrow y = (AA^T)^{-1} b.$$

$$x^{\dagger} = A^T y.$$

$$x^{\dagger} = A^T(AA^T)^{-1} b$$

$$\boxed{A^{\dagger} = A^T(AA^T)^{-1}}$$

Suppose A itself is a projection matrix.

(i) $A^2 = A$; (ii) $A^T = A$.

Matrix that projects vectors on to col. sp(A) is A itself.

We know that $AA^\dagger = P$

Here

$$AA^\dagger = A$$

$$\Rightarrow A(A^\dagger - A) = 0.$$

$$\Rightarrow A(\underbrace{A^\dagger b - Ab}) = 0 \text{ for any vector } b.$$

$\Rightarrow A^\dagger b - Ab$ is in the null sp(A) for any b.

But this vector is also in the col sp(A^T) because

$A^\dagger b$ and $Ab = A^T b$ are

in the col. sp (A^T).

Since colsp(A^T) & null ~~sp~~(A) are orthog complements,

$\Rightarrow$ $A^\dagger b - Ab = 0$ for all b

$\Rightarrow$ $\boxed{A^\dagger = A}$