Recall:

A nonempty collection V with $(+, \cdot)$, the rules for carrying out addition of 2 elements of V and for multiplying a real scalar with the element of V respectively and that obeys properties (i) ... (x) is a called as a vector space over $\mathbb{R}$.

Every element of V is called as a vector.

Examples:

(1) $\mathcal{V} = \left\{ \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix}, \quad x_1 \in \mathbb{R} \right\}$.

is a vector space.

Suppose $\vec{u}$ & $\vec{v}$ are elements of $\mathcal{V}$, then

$\vec{u} = \begin{pmatrix} u_1 \\ 2u_1 \end{pmatrix}$ $\quad \vec{v} = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix}$

Let α, β be real scalars.

(i) $\vec{u} + \vec{v} = \begin{pmatrix} u_1 \\ 2u_1 \end{pmatrix} + \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ 2u_1 + 2v_1 \end{pmatrix}$

$= \begin{pmatrix} u_1 + v_1 \\ 2(u_1 + v_1) \end{pmatrix}$

$\vec{u} + \vec{v} \in \mathcal{V} \rightarrow$ closed under VA.

(ii) $\alpha \rightarrow$ real scalar

$\alpha \cdot \vec{u} = \alpha \begin{pmatrix} u_1 \\ 2u_1 \end{pmatrix} = \begin{pmatrix} \alpha u_1 \\ \alpha(2u_1) \end{pmatrix} = \begin{pmatrix} \alpha u_1 \\ 2(\alpha u_1) \end{pmatrix} \in \mathcal{V}$

V is closed under Scalar multiplication.

(iii) $\vec{u} = \begin{pmatrix} u_1 \\ 2u_1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix}$

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_1 \\ 2(u_1 + v_1) \end{pmatrix}$$

$$\vec{v} + \vec{u} = \begin{pmatrix} v_1 + u_1 \\ 2(v_1 + u_1) \end{pmatrix} = \vec{u} + \vec{v}$$

(iv) u, v, w.

$$u + (v + w) = \begin{pmatrix} u_1 \\ 2u_1 \end{pmatrix} + \begin{pmatrix} v_1 + w_1 \\ 2(v_1 + w_1) \end{pmatrix}$$

$$= \begin{pmatrix} u_1 + v_1 + w_1 \\ 2(u_1 + v_1 + w_1) \end{pmatrix}$$

$$(u + v) + w = \begin{pmatrix} u_1 + v_1 \\ 2(u_1 + v_1) \end{pmatrix} + \begin{pmatrix} w_1 \\ 2w_1 \end{pmatrix}$$

$$(u + v) + w = (u + v) + w.$$

(iv) $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2(0) \end{pmatrix} \in V.$

Please check other properties.

$\Rightarrow \quad V = \left\{ \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix}, \ x_1 \in \mathbb{R} \right\}.$

is a vector space.

Ex (2). $V = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$ is a vector space.

Ex (3). Set of all 2×2 diagonal matrices with diagonal elements from $\mathbb{R}$.

$$\mathcal{D}^{2\times 2} = \left\{ \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \ d_1, d_2 \in \mathbb{R} \right\}.$$

is a vector space over $\mathbb{R}$.

Ex: 4: Set of all polynomials of degree ≤ 2 with real coeffecients. Let this set be denoted as P_2.

P_2 Set contains polynomials of the form

$$p = p_0 + p_1 x + p_2 x^2$$

Check if P_2 is a vector space under usual vector addition & scalar multiplication.

Closure under addition:

$a_p : a_0 + a_1 x + a_2 x^2$

$b_p : b_0 + b_1 x + b_2 x^2$

$a_p + b_p = \underbrace{(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2}_{\in P_2}.$

Sum of any 2 polynomials of deg ≤ 2 is also a polynomial of deg ≤ 2.

Scalar Multiplicn:

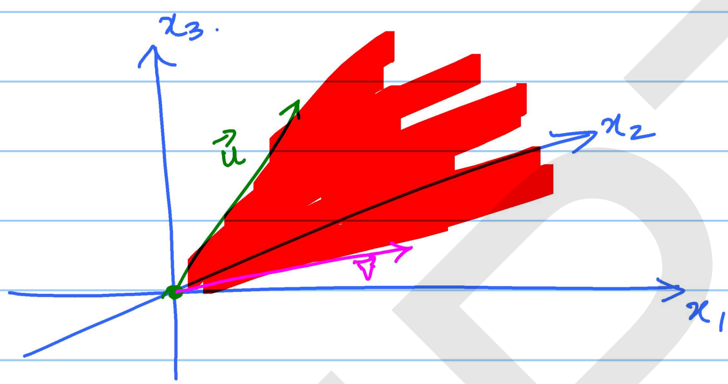
Let α be a real no &

$a_p = a_0 + a_1 x + a_2 x^2.$

$$\alpha a_p = \alpha(a_0 + a_1 x + a_2 x^2)$$
$$= \alpha a_0 + (\alpha a_1)x + (\alpha a_2)x^2 \in P_2.$$

Verify other properties & establish P_2 is a vector space.

Ex 5: A plane in $\mathbb{R}^3$ that passes through the origin.



The initial point of all vectors in $\mathbb{R}^3$ is at the origin.

Let W be the plane passing thro' the origin in 3D.

Let $\vec{x}$ and $\vec{y}$ be any 2 vectors on the plane.

The parallelogram formed by $\vec{x}$ & $\vec{y}$ is entirely in the plane. and

$\vec{x}+\vec{y} \rightarrow$ Diagonal of the parallelogram is also in W.

W is closed under Vector addition

Scalar Multiplicⁿ: Let α be any Scalar & $\vec{u}$ be a vector in W.

$\alpha\vec{u} \rightarrow$ either parallel to $\vec{u}$ or equal to the zero vector. $\in W$.

Since the plane passes thro' the Origin if $\alpha\vec{u} = \vec{0}$, it is in W.

Rest of the properties hold good.

$\therefore$ Any plane passing thro' the Origin in $\mathbb{R}^3$ is a Vector space.

(i) Any line passing thro' the Origin is a vector space.

(ii) Any plane passing thro' the Origin is a vector space.