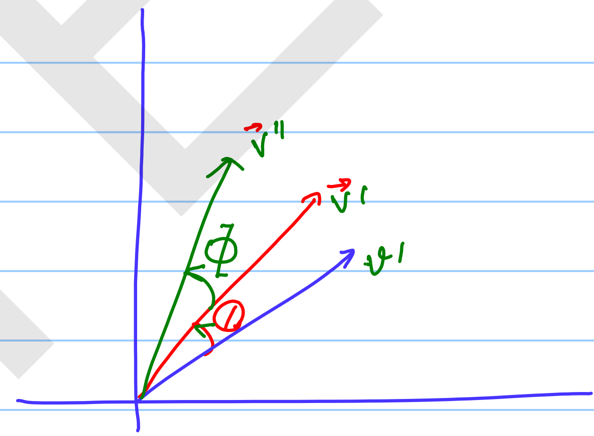
Rotations in 3D.

Rotation in 2D - Recall

$$R_\theta := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

rotates every vector by an angle θ in the counterclockwise direction.



$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\vec{v}' = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix}$$

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$$R_{\hat{\phi}} = \begin{bmatrix} \cos\hat{\phi} & -\sin\hat{\phi} \\ \sin\hat{\phi} & \cos\hat{\phi} \end{bmatrix}$$

$$\vec{v}'' = R_{\bar{\phi}}(\vec{v}')$$

$$= \begin{bmatrix} \cos\bar{\phi} & -\sin\bar{\phi} \\ \sin\bar{\phi} & \cos\bar{\phi} \end{bmatrix} \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\bar{\phi}\,(x_1\cos\theta - x_2\sin\theta) - \sin\bar{\phi}\,[x_1\sin\theta + x_2\cos\theta] \\ \sin\bar{\phi}\,(x_1\cos\theta - x_2\sin\theta) + \cos\bar{\phi}\,[x_1\sin\theta + x_2\cos\theta] \end{bmatrix}$$

$$= \begin{bmatrix} x_1\cos\theta\cos\bar{\phi} - x_2\sin\theta\cos\bar{\phi} - x_1\sin\theta\sin\bar{\phi} - x_2\cos\theta\sin\bar{\phi} \\ x_1\cos\theta\sin\bar{\phi} - x_2\sin\theta\sin\bar{\phi} + x_1\sin\theta\cos\bar{\phi} + x_2\cos\theta\cos\bar{\phi} \end{bmatrix}$$

$$v'' = \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ \cos\theta\sin\phi + \sin\theta\cos\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

$$v'' = R_\phi \left\{ R_\theta \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\} = v'' = R_{\theta+\phi}$$

$$\Rightarrow \boxed{R_\theta . R_\phi = R_\phi . R_\theta = R_{\theta+\phi}}$$

Rotation in 2D is commutative

$AB = BA$.

$A^{2\times2} B^{2\times2} = B^{2\times2} A^{2\times2}$ for $A = R_\theta$, $B = R_\phi$.

Rotation in 3D:

R_θ about e_1 axis

$$(R_\theta)_{e_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

R_θ about e_2 axis

$$(R_\theta)_{e_2} : \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

R_θ about e_3 axis

$$(R_\theta)_{e_3} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Suppose $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Rotate $\vec{x}$ by an angle ϕ about the e_1 axis

$$(R_\phi)_{e_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$(R_\phi)_{e_1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \vec{u}.$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} 1 \\ 2\cos\phi - 3\sin\phi \\ 2\sin\phi + 3\cos\phi \end{bmatrix}$$

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Rotate vector $\vec{u}$ about e_3-axis by an angle $\bar{\phi}$

$$\left(R_{\bar{\phi}}\right)_{e_3} = \begin{bmatrix} \cos\bar{\phi} & -\sin\bar{\phi} & 0 \\ \sin\bar{\phi} & \cos\bar{\phi} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{\vec{v}}} = \left(R_{\bar{\phi}}\right)_{e_3} \vec{u} = \begin{bmatrix} \cos\bar{\phi} & -\sin\bar{\phi} & 0 \\ \sin\bar{\phi} & \cos\bar{\phi} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2\cos\theta - 3\sin\theta \\ 2\sin\theta + 3\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\bar{\phi} - \sin\bar{\phi}\,(2\cos\theta - 3\sin\theta) \\ \sin\bar{\phi} + \cos\bar{\phi}\,(2\cos\theta - 3\sin\theta) \\ 2\sin\theta + 3\cos\theta \end{bmatrix}$$

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$$\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Rotate $\vec{x}$ by an angle $\bar{\phi}$ about x_3 axis followed by rotation by an angle θ about x_1 axis

$$R_{\bar{\phi}} : \begin{bmatrix} \cos\bar{\phi} & -\sin\bar{\phi} & 0 \\ \sin\bar{\phi} & \cos\bar{\phi} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\vec{u}' = \begin{bmatrix} \cos\bar{\phi} - 2\sin\bar{\phi} \\ \sin\bar{\phi} + 2\cos\bar{\phi} \\ 3 \end{bmatrix}$$

R_θ on $\vec{u}'$ we get v'

$$v' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\bar{\phi} - 2\sin\bar{\phi} \\ \sin\bar{\phi} + 2\cos\bar{\phi} \\ 3 \end{bmatrix}$$

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$$(R_\theta)_{e_1}\left[(R_\phi)_{e_3}\right]$$

$$\vec{v}\,' = \begin{bmatrix} \cos\phi - 2\sin\phi \\ \cos\theta(\sin\phi + 2\cos\phi) - 3\sin\theta \\ \sin\theta(\sin\phi + 2\cos\phi) + 3\cos\theta \end{bmatrix}$$

$$(R_\phi)_{e_3}(R_\theta)_{e_1}$$

$$\vec{v} = \begin{bmatrix} \cos\phi - \sin\phi(2\cos\theta - 3\sin\theta) \\ \sin\phi + \cos\phi(2\cos\theta - 3\sin\theta) \\ 2\sin\theta + 3\cos\theta \end{bmatrix}$$

$$\boxed{(R_\phi)_{e_3}\cdot(R_\theta)_{e_1} \neq (R_\theta)_{e_1}\cdot(R_\phi)_{e_3}}$$

Rotation about 2 diff axes do not commute.