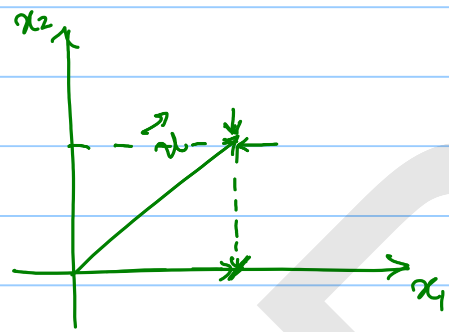
Projection Matrices:



P is called a projection matrix if and only if

a) $P^T = P$ $\longrightarrow$ Symmetric.

b) $P^2 = P$ $\longrightarrow$ Idempotent.

Proof:

Suppose P is a matrix satisfying $P^T = P$ & $P^2 = P$, we need to prove that

$\vec{b} - P\vec{b}$ is orthogonal to Col.sp(P) for all $\vec{b}$.

Any vector in the col. Sp(P) is of the form $P\vec{u}$

$\Rightarrow \underbrace{P\vec{u}}\cdot\underbrace{(b-Pb)} = u^T P^T (b - Pb)$

$= u^T(P^T b - P^T P b) \rightarrow (1)$

But we know that $P^T = P$; $P^T P = P.P = P^2 = P$

$\Rightarrow$

$(1) = u^T(Pb - Pb) = \underline{\underline{0}}.$

<u>Conversely:</u>

Suppose P is projection matrix, we have

$(b - Pb)$ is orthogonal to Pu.

$\Rightarrow 0 = (Pu)\cdot(b-Pb)$
$= (Pu)^T\cdot(b-Pb)$
$= u^T P^T(b - Pb)$
$= u^T(P^T b - P^T P b)$

$$0 = u \cdot (P^T b - P^T P b)$$

The only vector orthogonal to all the vectors $\vec{u}$ is the zero vector.

$$\Rightarrow \quad P^T b - P^T P b = \vec{0}$$

$$\Rightarrow \quad P^T \underset{=}{b} = P^T P \underset{=}{b}.$$

$$\Rightarrow \quad P^T = P^T P.$$

We know that $P^T P$ is Symmetric

$\therefore$ P^T is Symmetric.

$$\Rightarrow \boxed{P^T = P.}$$

$$\Rightarrow \quad P = P^T P = P \cdot P = P^2.$$

$$P^T P = PP = P^2 = P$$

$$\Rightarrow \boxed{P^2 = P}$$

Given vectors u & v, what is the matrix that projects the vector u onto the direction of $\vec{v}$?

$$\text{Proj}_{\vec{v}}\,\vec{u} = \underbrace{\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}}\,\vec{v} = t\,\vec{v}$$

$$\text{Proj}_{\vec{v}}\,\vec{u} = \left(\frac{v^T u}{v^T v}\right)\vec{v}$$

$$\Rightarrow \frac{\vec{v}\,v^T u}{v^T v}$$

$$= \left(\frac{v v^T}{v^T v}\right)\vec{u}$$

$v^T v$ = Dot product of v with itself
= Scalar.

$\vec{v}v^T$:

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Outer product of 2 vectors. $\Rightarrow$ Always results in a matrix of rank 1.

$$\underline{\underline{vv^T}} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \left(v_1 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad v_2 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad v_3 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right) = \begin{pmatrix} v_1^2 & v_1 v_2 & v_1 v_3 \\ v_1 v_2 & v_2^2 & v_2 v_3 \\ v_1 v_3 & v_2 v_3 & v_3^2 \end{pmatrix}$$

Symmetric

Rank 1 matrix

Rank of the $vv^T = 1$

(1) vv^T : Rank 1 $n \times n$ Symmetric matrix

(2) uv^T : If $\vec{u}$ & $\vec{v}$ are of different dimensions, then uv^T results in an $m \times n$ matrix of rank 1.

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$uv^T = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} v_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} & v_2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} & v_3 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix}$$

2×3 matrix of rank 1.

Matrix that projects any vector on to the direction of $\vec{v}$ is given by

$$P = \frac{v v^T}{v^T v}.$$

$$P^T = \frac{1}{v^T v}\left(v v^T\right)^T$$

$$= \frac{1}{v^T v}\left(\left(v^T\right)^T v^T\right)$$

$$= \frac{1}{v^T v}\left(v v^T\right) = P.$$

$$P^2 = \frac{v v^T}{v^T v} \cdot \frac{v v^T}{v^T v}$$

$$\Rightarrow \frac{v\,\cancel{\left(v^T v\right)}\,v^T}{\cancel{\left(v^T v\right)}\left(v^T v\right)}$$

$$\Rightarrow \frac{v v^T}{v^T v} = P.$$

Suppose we want to project a vector onto a plane spanned by orthogonal vectors. $\vec{u}$ & $\vec{v}$

$\vec{u} \cdot \vec{v} = 0$.

Projection Matrix.

Project $\vec{x}$ onto the plane spanned by u, v, $u \perp v$.

$\Rightarrow \mathrm{Proj}_{pl(u,v)}\, x = \left(\frac{uu^T}{u^Tu} + \frac{vv^T}{v^Tv}\right)\vec{x}$.

Check if $\left(\frac{uu^T}{u^Tu} + \frac{vv^T}{v^Tv}\right)$ is symmetric & idempotent!

Recall: We said

$$Ax^+ = \underbrace{A(A^TA)^{-1}A^T}_{\text{Proj. Matrix}} b.$$

$P^T = P$

$$\left(A(A^TA)^{-1}A^T\right)^T$$

$$= (A^T)^T\left[(A^TA)^{-1}\right]^T A^T.$$

$$\Rightarrow \underline{A(A^TA)^{-1}A^T}.$$

$P^2 = P$:

$$A\underbrace{(A^TA)^{-1}A^T.A}_{I}(A^TA)^{-1}A^T$$

$$= A.I.(A^TA)^{-1}A^T$$

$$= A(A^TA)^{-1}A^T.$$

$\therefore$ $A(A^TA)^{-1}A^T$ is a projection matrix