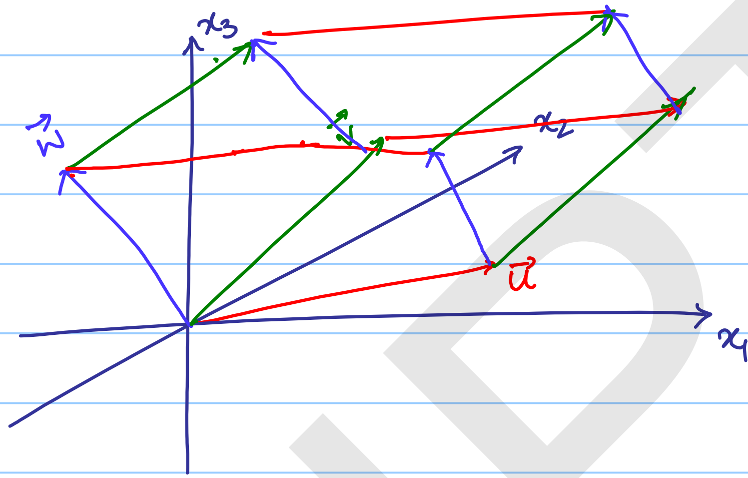
Let $\vec{u}, \vec{v}$ and $\vec{w}$ be three l.i. vectors in $\mathbb{R}^3$



Since $\vec{u}$ & $\vec{v}$ are l.i., then the distance of the vector $\vec{w}$ to the plane determined by $\vec{u}$ & $\vec{v}$ equals the length of the projection of $\vec{w}$ along $\vec{u} \times \vec{v}$ since $\vec{u} \times \vec{v}$ is orthog. to the plane

$$\text{Distance} = \frac{|\vec{w} \cdot (\vec{u} \times \vec{v})|}{\|\vec{u} \times \vec{v}\|}$$

Volume of the parallelopiped

$= \text{Base Area} * \text{height}$

$= \|\cancel{\vec{u} \times \vec{v}}\| * \dfrac{|\vec{w}\cdot(\vec{u}\times\vec{v})|}{\|\cancel{\vec{u}\times\vec{v}}\|}$

$$\boxed{\text{Vol} = |\vec{w}\cdot(\vec{u}\times\vec{v})|} \leftarrow$$

If $\vec{u}, \vec{v}$ & $\vec{w}$ are linearly dep, the parallelopiped is contained in the plane and hence the vol $= 0$.

$\Rightarrow \left[\vec{w}\cdot(\vec{u}\times\vec{v})\right) = 0$.

$\rightarrow$ Scalar triple product

Lines in 3D space

Recall the parametric equation of a line 2D

$$\ell(t) = \vec{p} + t\vec{v}$$

$\vec{p}$: point

t: scalar

$\vec{v}$: vector along whose direction the line is.

In 3D also the representation is similar.

$$\ell(t) = \vec{p} + t\vec{v}$$

$$\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

In 2D

let $\ell(t) = \vec{p} + t\vec{v}$

$\ell(s) = \vec{q} + s\vec{w}$

The point of intersection is found by taking 2 scalars $\hat{t}$ & $\hat{s}$ s.t

$$\vec{p} + \hat{t}\vec{v} = \vec{q} + \hat{s}\vec{w}$$

$$\Rightarrow \hat{t}\vec{v} - \hat{s}\vec{w} = \vec{q} - \vec{p}$$

$$\hat{t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \hat{s}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \end{pmatrix}$$

2 eqns in 2 unknowns namely $\hat{t}$ & $\hat{s}$.

In 3D:

$\ell(t) = \vec{p} + t\vec{v}$ & $\ell(s) = \vec{q} + s\vec{w}$

for some scalar $\hat{t}$ & $\hat{s}$.

$$\vec{p} + \hat{t}\vec{v} = \vec{q} + \hat{s}\vec{w} \longrightarrow (1)$$

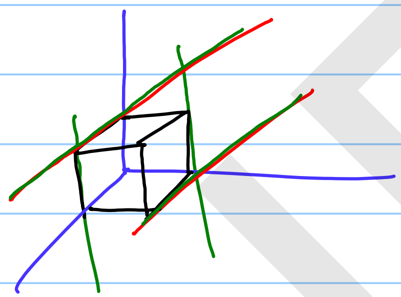
$\vec{p} : \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ & $\vec{q} : \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$ $\vec{v} : \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$\vec{w} : \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

$$\Rightarrow \ \underline{\underline{\hat{t}}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} - \underline{\underline{\hat{s}}} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{pmatrix}$$

$\Rightarrow$ Overdetermined system

$\Rightarrow$ 3 equations in 2 unknowns.



SKEWED lines.

$\Downarrow$

lines in parallel planes.

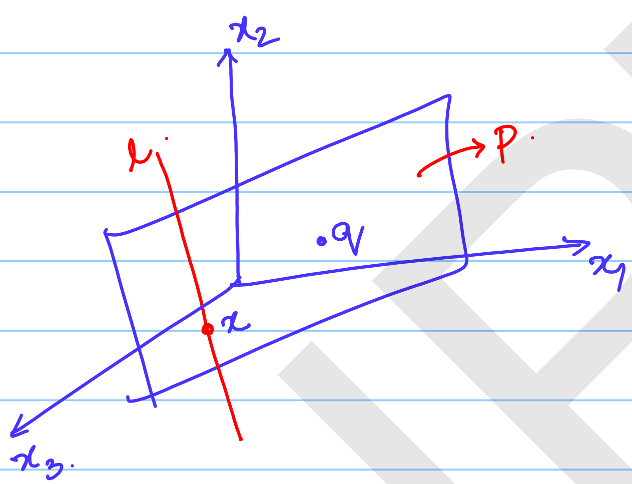
Plane in 3D:

Let P_π be the plane passing thro' the origin & perpendicular to $\vec{u}$.

The eqn of the plane is given by

$\vec{y} \cdot \vec{u} = 0$ where $\vec{y}$ is any vector on P_π.

Intersection of line and a plane.



Given a plane P and a line l what is their point of intersection?

Let $\vec{q}$ be a point on the plane. $\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$

Let $\vec{n}$ to a vector orthogonal to the plane.

Let $\vec{x}$ be the point of intersection of the line & the plane

$\vec{x}$ must be on the plane.

Since $\vec{q}$ is on the plane & $\vec{x}$ is also on the plane, it must satisfy

$$(\vec{x}-\vec{q})\cdot\vec{n} = 0 \longrightarrow (1)$$

By defn, $\vec{x}$ is also on the line given by a point $\vec{p}$ on the line and a vector $\vec{v}$

$$\vec{x} = \vec{p} + t\vec{v} \longrightarrow (2)$$

(2) in (1) we get

$$(\vec{p}+t\vec{v}-\vec{q})\cdot\vec{n} = 0$$

$$= (\vec{p}-\vec{q})\cdot\vec{n} + t\,\vec{v}\cdot\vec{n} = 0$$

$$\Rightarrow t = \frac{(\vec{q}-\vec{p})\cdot\vec{n}}{\vec{v}\cdot\vec{n}}$$

Once t is known, we can find the point of intersection

$$\boxed{\vec{x} = \vec{p} + \frac{(\vec{q}-\vec{p})\cdot\vec{n}}{\vec{v}\cdot\vec{n}}}$$

To Summarize:

* Vector in 3D
* Projection of vector onto another
* Equation of plane passing thro origin and orthogonal to a given vector
* Distance from a point to the plane passing thro' the origin & orthogonal to a given vector
* Distance from a point to a non zero vector

* Area of a ||gm in 3D
  → * Cross product
  Properties of cross prod.
  linearly dep/indep of 2 vectors in 3D.
* Vol. of parallelopiped
* Skewed lines
* Intersection of a plane & a line.