Lines in 2D - Parametric equation of lines

Linear Maps in 2D

Revisit linear Systems (2x2)

Eigenvalues and eigenvectors in 2D.

Point P & point Q

$$Q - P = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \end{pmatrix}$$

$$\vec{r} = Q - P$$

To move from P to Q, we

$$p + k\vec{r}$$

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + k\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

Two elements of 2D geometry defines a line

(i) Two points

(ii) A point and a vector parallel to the line

(iii) A point & a vector perpendicular to the line.

Parametric Eqn of a line.

$\ell(t) = p + t\vec{v}$ ⟶

t: Scalar

t: parameter.

Recall $\vec{v} = Q - P$

$\ell(t) = (1-t)p + tq$ —

$0 < t < 1 \rightarrow$ Convex combination

$t < 0,\ t > 1 \rightarrow$ Extrapolation

Ex: $2x_1 + x_2 = 3$
$4x_1 + 2x_2 = 6$.

Homegeneous Sys:

$2x_1 + x_2 = 0$
$4x_1 + 2x_2 = 0$

The solution to this lies along the line through the origin and the point $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

If HSE has non-trivial soln, then it has infinitely many solns.

$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is a soln. $k\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is also a soln to $2x_1 + x_2 = 0$, $4x_1 + 2x_2 = 0$

k: real.

$2x_1 + x_2 = 3$ $\quad x_1 = 1.5 \quad 1 \quad 0 \quad \ldots$
$4x_1 + 2x_2 = 6$ $\quad x_2 = 0 \quad 1 \quad 3 \quad \ldots$

Infinitely many solns.

Complete Soln to

$$2x_1 + x_2 = 3$$
$$4x_1 + 2x_2 = 6$$

is given by

$$\boxed{x = x_p + k\, x_H}$$

x_p : particular Soln.
x_H : Solution to Homogeneous sys. of eqn.

Linear Transformations of vectors in 2D.

In geometry we focus on

a) describing objects that can be generated

b) transforming these objects

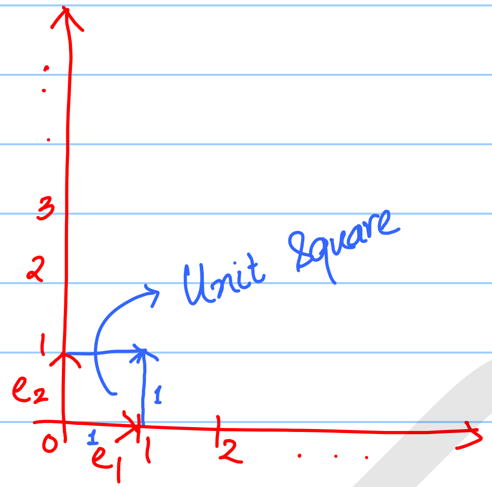
Ex: Consider $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

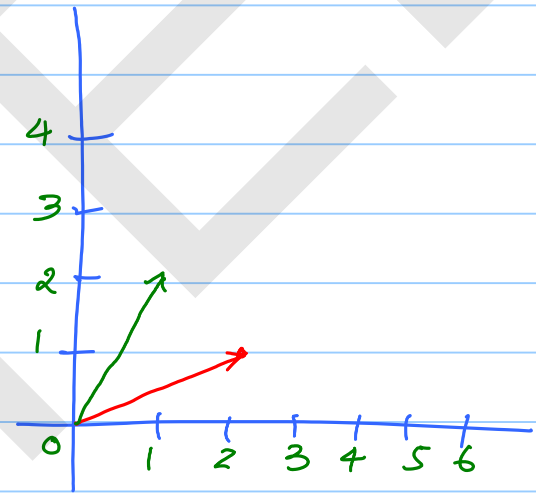
$\Rightarrow v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= v_1 e_1 + v_2 e_2.$

$\vec{v} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} = 5 e_1 + 4 e_2$



$\vec{v} = v_1 \vec{e}_1 + v_2 \vec{e}_2 \rightarrow$ Linear Comb^n of $\vec{e}_1$ and $\vec{e}_2$



$\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$\vec{w} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \underline{\underline{k_1}}\, \vec{u} + \underline{\underline{k_2}}\, \vec{v}$ $\quad k_1 = 1$ & $k_2 = 1$

We can capture the linear transformation of a vector in a matrix form.

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = k_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad 2k_1 + k_2 = 3$$
$$k_1 + 2k_2 = 3.$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$\Downarrow$

Maps $k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$

For ex: $\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$

* Some linear transformations of vectors in 2D.

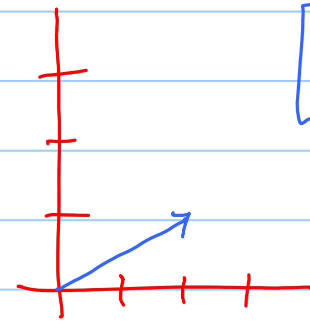
* Scaling a vector.

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \longrightarrow \begin{bmatrix} kv_1 \\ kv_2 \end{bmatrix} \quad k : \text{Scalar}$$

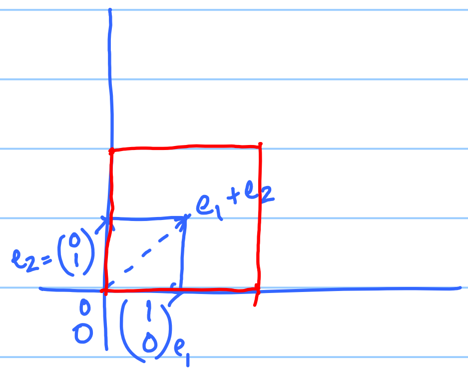
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} kv_1 \\ kv_2 \end{bmatrix}$$

$$\Rightarrow \quad a_{11}v_1 + a_{12}v_2 = kv_1 + 0v_2$$
$$a_{21}v_1 + a_{22}v_2 = 0v_1 + kv_2.$$

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} kv_1 \\ kv_2 \end{bmatrix}$$



$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$



$\Rightarrow$ Area of unit Square $= 1$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

Scaling transformation affects the area of the object that gets scaled.

Reflections in 2D

$$\begin{bmatrix} ? \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} \checkmark$$

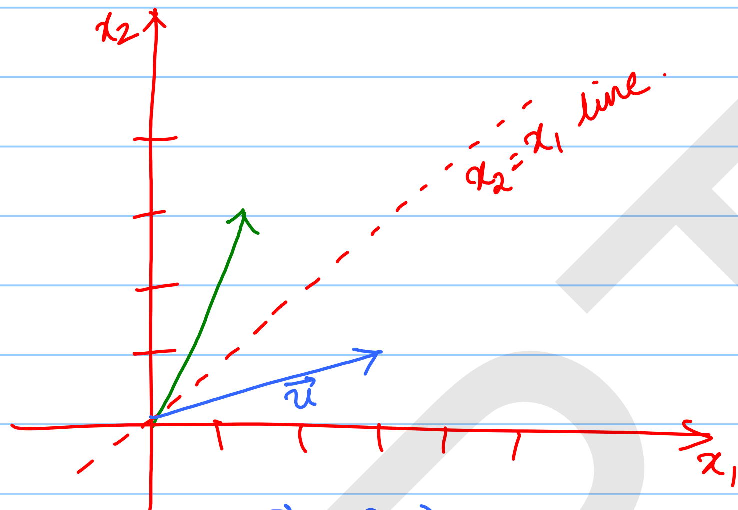
$$\Rightarrow a_{11} v_1 + a_{12} v_2 = 0 v_1 + v_2$$
$$a_{21} v_1 + a_{22} v_2 = 1 v_1 + 0 v_2$$

Reflection transf: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ✓
R.

Elementary row operation of row swapping is carried out using the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

In order to undo the effect of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, apply the same transform^n again.

Inverse of reflection matrix R is R itself



$\vec{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

1) Are there any more linear transformations?
2) Can we reverse the effect of a l.t?