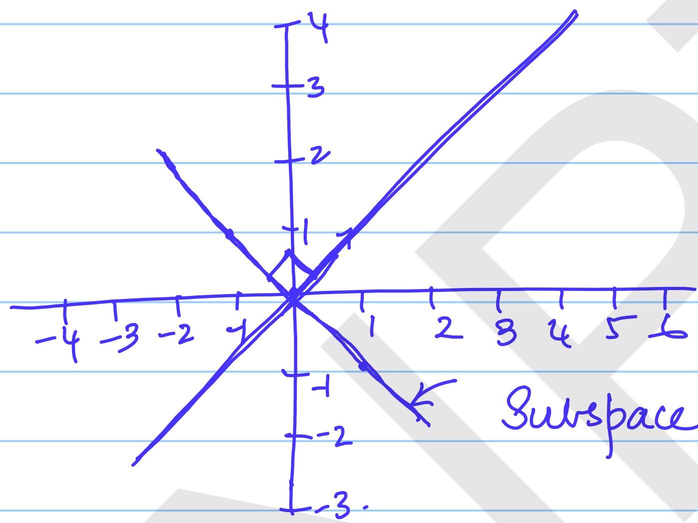
Ex 1: $x_1 + x_2 = 0$

Solve for x_1 & x_2.



Subspace of $\mathbb{R}^2$

$1x_1 + 1x_2 = 0$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$.

$= x_1 + x_2 = 0$.

Set of vectors.

Ex: 2. $1x_1 + 1x_2 + 2x_3 = 0$.

Set of vectors orthogonal to the vector $(1, 1, 2)$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_1 + x_2 + 2x_3 = 0$$

$$\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_3 = t, \quad x_2 = s.$

$x_1 = -s - 2t.$

Solution : $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -s-2t \\ s \\ t \end{pmatrix}$

$$\begin{pmatrix} -s-2t \\ s \\ t \end{pmatrix} = s\underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{x_1x_2 \text{ plane}} + t\underbrace{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}_{x_1x_3 \text{ plane}}$$

Recall: If $\vec{u}$ & $\vec{v}$ are two vectors then, for a real Scalar c, $c\vec{u}\cdot\vec{v} = \vec{u}\cdot c\vec{v} = c(\vec{u}\cdot\vec{v})$.

If $\vec{u}\cdot\vec{v} = 0$

$c\vec{u}\cdot\vec{v} = 0$ for real scalar c.

The set of vectors orthogonal to $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is the plane passing thro' the origin & defined by $\left\{ s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Ex 3:

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$$
$$b_1 x_1 + b_2 x_2 + b_3 x_3 = 0.$$

We want that vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Orthogonal to both $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ & $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Cross product of $\underbrace{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}$

will be orthogonal to both

$\underbrace{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}$

Ex: 4: $2x_1 + x_2 = 0$
$x_1 + 2x_2 = 0$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ Orthogonal to both $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$A\vec{x} = \vec{0}$

The subspace is entire $\mathbb{R}^2$

We want set of vectors Orthogonal to every vector in $\mathbb{R}^2$

$\Rightarrow$ This set has only zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ is a subspace of $\mathbb{R}^2$.

Definition:

Given a vector Space V of dimension 'd' and a Subspace W of V, we define the Orthogonal Complement Subspace of W as the set of vectors in V which are Orthogonal to W.

The Orthogonal Complement Subspace is denoted by $W^{\perp}$

Let V be a d-dim vector space and W be a k-dim subspace of V.

$$W^{\perp} = \{x : x \cdot w = 0 \text{ for every } w \in W\}$$

Examples:

(1) $V = \mathbb{R}^2$

$$W = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, x_1 \in \mathbb{R} \right\}$$

$$W^{\perp} = \left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix}, x_2 \in \mathbb{R} \right\}$$

Ex: 2: $V = \mathbb{R}^2$

$W = \mathbb{R}^2 \rightarrow \dim(W) = 2$

$W^\perp = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \rightarrow$ Trivial Subspace of $\mathbb{R}^2$.

$\dim(W^\perp) = 0$

$\dim(W) + \dim(W^\perp) = \dim(V)$.

Ex: 3: $V = \mathbb{R}^3$

$W = \left\{ \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}, x_1 \in \mathbb{R} \right\}$. $\dim(W) = 1$

$W^\perp : \left\{ \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} x_2, x_3 \in \mathbb{R} \right\}$.

$= \left\{ x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, x_2, x_3 \in \mathbb{R} \right\}$

$\dim(W^\perp) = 2$ $\quad \dim(W^\perp) + \dim(W) = 2 + 1 = 3$