Recap:

Gaussian Elimⁿ

Row operations

RREF

Rank of A.

Row operⁿˢ preserve the solⁿ. space

Focus now:

Det, A^{-1}, Cramer's rule.

* $A^{n\times n}$

determinant: fn

$\text{Det}(A) \longrightarrow \mathbb{R} \to$ real no

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

$$A = \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

$$\det(A^{3\times3}) = a_{11} \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$\det(A^{2\times2})$ = Area of ||gm determined by $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$, $\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$

$\det(A^{3\times3})$: Vol. of parallelopiped.

(A_1) (A_2) & (A_3)

Note: $\det(A^{n\times n})$, A, diagonal

$$A = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \ d_n \end{bmatrix}$$

$$\det(A) = d_1 \cdot d_2 \cdots d_n$$
$$= \prod_{i=1}^{n} d_i$$

(2) U: Upper Δ^{lar} matrix.

$$\det(U) = \prod_{i=1}^{n} u_{ii}$$

(3) $A^{n\times n}$, $\det(cA) = c^n(\det(A))$
for c real scalar.

(4) $\det(A) = 0 \Rightarrow A$ is singular.

(5). If $\det(A) \neq 0$ $\text{rank}(A) = n$.

$A^{n\times n}$ matrix.

For a Sq. matrix $A^{n\times n}$: Equivalent Statements.

(i) A is Singular

(ii) $\text{Rank}(A) < n$

(iii) $\det(A) = 0$

(iv) A is not row eqt to $I_{n\times n}$.

(v) $Ax = 0$ has non trivial Soln.

(vi) $Ax = b$ does not have unique Soln

For non singular A, equivalent Statements:

(i) A is non singular

(ii) $\text{Rank}(A) = n$

(iii) $\text{Det}(A) \neq 0$

(iv) A is row equivalent to $I_{n \times n}$.

(v) $Ax = 0$ has ONLY Trivial Soln.

(vi) $Ax = b$ has unique soln $\underline{x}$ for every given $\underline{b}$.

* $\text{Det}(I_{n \times n}) = 1$.

* For matrix with 2 equal rows $\det(A) = 0$

* $\text{Det}(A^T) = \text{Det}(A)$.

* $\det(AB) = \det(A)\det(B)$

*

* Row Swapping

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

Row Swap $R_1 \leftrightarrow R_2$

$$A_{RS} : \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$$

$$\det(A_{RS}) = a_{21}a_{12} - a_{11}a_{22} = -\det(A).$$

$\Rightarrow$ Row swap changes sign of det.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ $\det(A) = 7-6 = 1.$

$B = \begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix}$ $\det(B) = 16-12 = 4.$

$A+B = \begin{bmatrix} 3 & 5 \\ 7 & 15 \end{bmatrix}$

$\det(A+B) = 45-35 = 10.$

$\det(A) + \det(B) = 5$

$\Rightarrow \det(A+B) \neq \det(A) + \det(B)$

$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\det(C) = 1$

$D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\det(D) = -1$

$C+D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\det(C+D) = 1-1 = 0$

$\det(C) + \det(D) = 1-1 = 0.$

$\Rightarrow$ $\text{Det}(A+B) \neq \det(A) + \det(B)$
$\underline{\text{always}}$

Matrix Inverse:

$a \in \mathbb{R} \quad a \neq 0$

$a^{-1} \in \mathbb{R}; \quad a \cdot a^{-1} = a^{-1} \cdot a = 1$

a^{-1}: Multiplicative Inverse of a.
Unique.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Can we find B such that

$$AB \stackrel{?}{=} I \; ?$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} \\ a_{21} b_{11} + a_{22} b_{21} & a_{21} b_{12} + a_{22} b_{22} \end{bmatrix}$$

$$AB = I.$$

$$\Rightarrow \quad a_{11} b_{12} = -a_{12} b_{22}$$

$$\Rightarrow \quad \boxed{b_{12} = -\frac{a_{12}}{a_{11}} b_{22}}$$

$$a_{21} b_{11} = -a_{22} b_{21}$$

$$\boxed{b_{21} = -\frac{a_{21}}{a_{22}} b_{11}}$$

$$b_{11} = \frac{a_{22}}{a_{11} a_{22} - a_{12} a_{21}}$$

$$b_{22} = \frac{a_{11}}{a_{11} a_{22} - a_{12} a_{21}}$$

$$b_{12} = \frac{-a_{12}}{a_{11} a_{22} - a_{12} a_{21}}$$

$$b_{21} = \frac{-a_{21}}{a_{11} a_{22} - a_{12} a_{21}}.$$

$\det(A)$.

$$B = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$\det(A) \neq 0$ B exists

$$B = A^{-1}.$$

$$BA = I.$$

$$AB = I.$$

A^{-1} exists if $\det(A) \neq 0$.

$A^{m\times n}$: Inverse not defined for $m \neq n$.

Properties of Inverses:

(i) A, B invertible of same size

$(AB)^{-1} = B^{-1}A^{-1}$

Let $AB = C$

$C^{-1} = (AB)^{-1}$

$CC^{-1} = I$

$(AB)B^{-1}A^{-1} = I$.

$$\boxed{C^{-1} = B^{-1}A^{-1}}$$

- Product of any no. of invertible matrices is invertible.

$\rightarrow$ and inverse of prod = Prod. of inverse in the reverse order.

* A is invertible $\Rightarrow$ A^{-1} is also invertible

$$\Rightarrow (A^{-1})^{-1} = A.$$

* $c \neq 0$; cA is invertible if A is invertible & $(cA)^{-1} = c^{-1}A^{-1}$

* $A \rightarrow$ invertible $\Rightarrow$ A^T is invertible

$$(A^T)^{-1} = (A^{-1})^T.$$

$A^{n\times n} \neq 0$

$$A^{-1} = \frac{\operatorname{Adj} A}{\det(A)}.$$

$\operatorname{Adj}(A)$ for $A^{2\times 2}$

$$\operatorname{Adj}(A) = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$A^{n\times n}$, A^{-1} computation tedious.

$AA^{-1} = I$.

↳ elements of A^{-1}.

Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

$$B = A^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$AB = I$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \leftarrow R_2 - 3R_1$ & $R_3 \leftarrow R_3 - 2R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{array}\right]$$

$R_3 \leftarrow R_3 - \dfrac{R_2}{5}$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & 0 & -18/5 & -7/5 & -1/5 & 1 \end{array}\right]$$

$R_3 \leftarrow R_3 / (-18/5)$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & 0 & 1 & 7/18 & 1/18 & -5/18 \end{array}\right]$$

$$R_2 \leftarrow R_2 + 7R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & 0 & -5/18 & 25/18 & -35/18 \\ 0 & 0 & 1 & 7/18 & 1/18 & -5/18 \end{array}\right]$$

$$R_1 \leftarrow R_1 - 3R_3.$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -3/18 & -3/18 & 15/18 \\ 0 & -5 & 0 & -5/18 & 25/18 & -35/18 \\ 0 & 0 & 1 & 7/18 & 1/18 & -5/18 \end{array}\right]$$

$$R_1 \leftarrow R_1 + (2/5)\, R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/18 & 7/18 & 1/18 \\ 0 & -5 & 0 & -5/18 & 25/18 & -35/18 \\ 0 & 0 & 1 & 7/18 & 1/18 & -5/18 \end{array}\right]$$

$$R_2 \leftarrow R_2/(-5)$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/18 & 7/18 & 1/18 \\ 0 & 1 & 0 & 1/18 & -5/18 & 7/18 \\ 0 & 0 & 1 & 7/18 & 1/18 & -5/18 \end{array}\right]$$

$$\underbrace{\begin{matrix} -5/18 & 7/18 & 1/18 \\ 1/18 & -5/18 & 7/18 \\ 7/18 & 1/18 & -5/18 \end{matrix}}_{A^{-1}}$$

$A^{-1}A = I$ (Can be verified)