Recall :

If $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ are

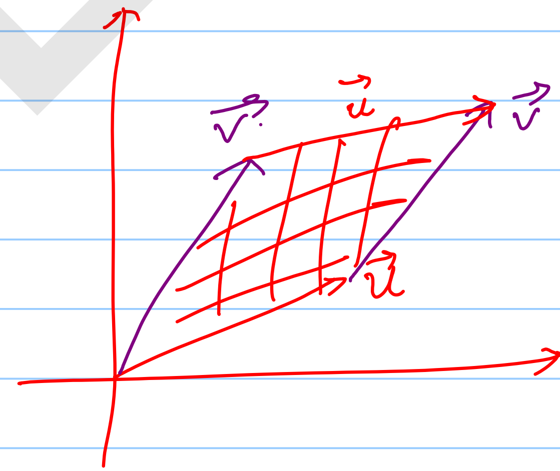
2 vectors s.t the matrix

A is $A = \begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix} =$

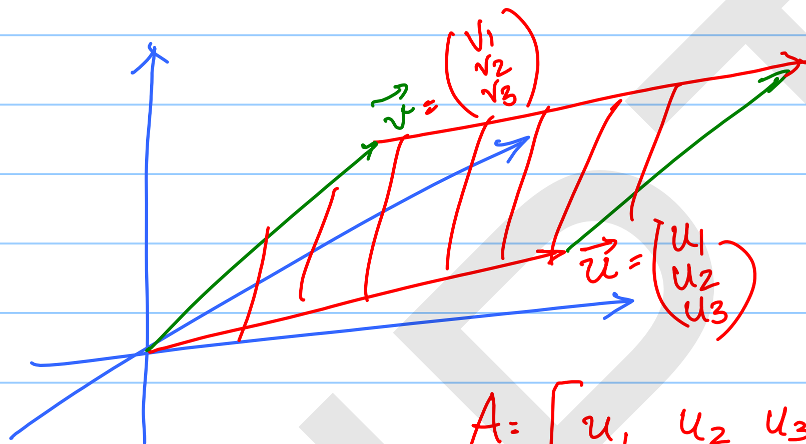
$\Rightarrow A = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$

$Det(A) = u_1 v_2 - u_2 v_1$

= Area of the $\|$gm formed by these two vectors



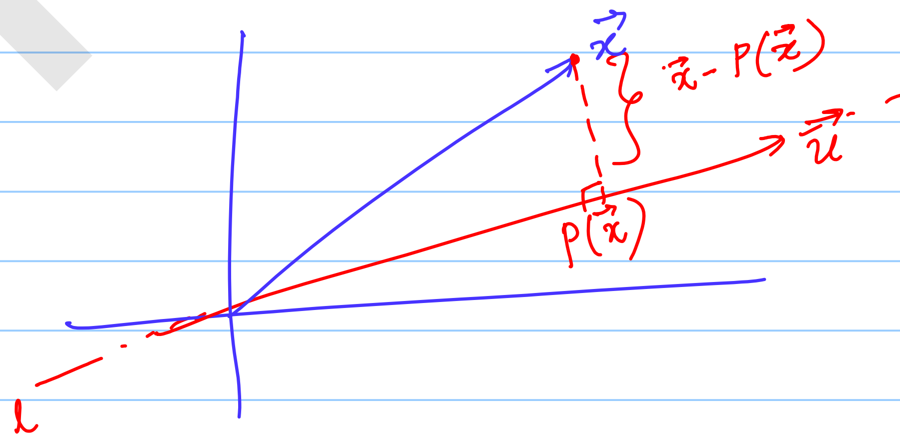
Let $\vec{u} = \begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ & $\vec{v} = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$



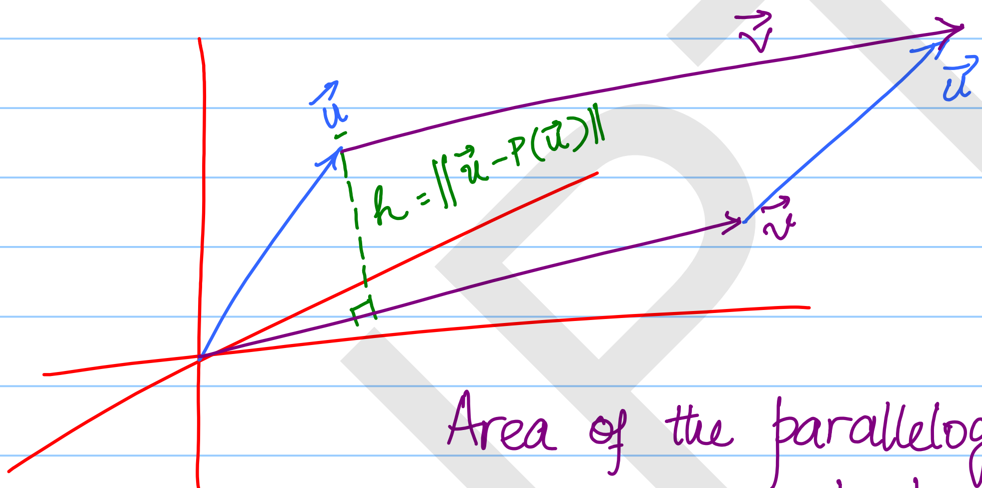

$$A = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

Recall:

The distance from a point $\vec{x}$ to the line along $\vec{u}$ $(\vec{u} \neq \vec{0})$



$$\|\vec{x} - P(\vec{x})\| = \frac{\sqrt{(\vec{x}\cdot\vec{x})(\vec{u}\cdot\vec{u}) - (\vec{x}\cdot\vec{u})^2}}{\|\vec{u}\|}$$




Area of the parallelogram

$= b \times h$

Area $= b \times h$

$b : \|\vec{v}\|.$

$$h = \|\vec{u} - P(\vec{u})\| = \frac{\sqrt{(\vec{u}\cdot\vec{u})(\vec{v}\cdot\vec{v}) - (\vec{u}\cdot\vec{v})^2}}{\|\vec{v}\|}$$

$\therefore$ Area $= b \times h$

$$= \cancel{\|\vec{v}\|} \cdot \frac{\sqrt{(\vec{u}\cdot\vec{u})(\vec{v}\cdot\vec{v}) - (\vec{u}\cdot\vec{v})^2}}{\cancel{\|\vec{v}\|}}$$

Area of the parallelogram

$$= \sqrt{(\vec{u}\cdot\vec{u})(\vec{v}\cdot\vec{v}) - (\vec{u}\cdot\vec{v})^2}$$

$$= \sqrt{\|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}\cdot\vec{v})^2} \longrightarrow (1)$$

Recall $\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$

$$\text{Area} = \sqrt{\|\vec{u}\|^2\|v\|^2 - (\|\vec{u}\|\,\|\vec{v}\|\cos\theta)^2}$$

$$\text{Area} = \|\vec{u}\|\,\|\vec{v}\|\sin\theta$$

Consider:

$$u_1 x_1 + u_2 x_2 + u_3 x_3 = 0 \longrightarrow$$
$$v_1 x_1 + v_2 x_2 + v_3 x_3 = 0 \longrightarrow$$

find the solution the above sys. of eqns.

$$A\vec{x} = \vec{0}$$

$$= \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$$
$$\Rightarrow u^T x = 0 \Rightarrow \vec{u} \cdot \vec{x} = 0$$
$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$
$$v_1 x_1 + v_2 x_2 + v_3 x_3 = 0 \Rightarrow v^T x = 0 \Rightarrow \vec{v} \cdot \vec{x} = 0$$

$\vec{u} \cdot \vec{x} = 0$

$\vec{v} \cdot \vec{x} = 0$

$\Rightarrow$ Find the vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Orthogonal to both $\vec{u}$ & $\vec{v}$

$u_1 x_1 + u_2 x_2 + u_3 x_3 = 0 \longrightarrow (1)$

$v_1 x_1 + v_2 x_2 + v_3 x_3 = 0 \longrightarrow (2)$

$(1) \times v_1 - (2) \times v_2$

$\cancel{u_1 v_1 x_1} + u_2 v_1 x_2 + u_3 v_1 x_3 = 0$

$\cancel{u_1 v_1 x_1} + v_2 u_1 x_2 + v_3 u_1 x_3 = 0$

$\Rightarrow x_2(v_1 u_2 - v_2 u_1) + x_3(v_1 u_3 - v_3 u_1) = 0 .$ (3)

Similarly ① $\times v_2$ − ② $\times u_2$

$$\Rightarrow \quad u_1 v_2 x_1 + \cancel{u_2 v_2 x_2} + u_3 v_2 x_3 = 0$$
$$v_1 u_2 x_1 + \cancel{v_2 u_2 x_2} + v_3 u_2 x_3 = 0$$
$$- \qquad - \qquad -$$

$$\Rightarrow \quad x_1 (u_1 v_2 - u_2 v_1) + x_3 (u_3 v_2 - u_2 v_3) = 0.$$

Choosing

$$x_1 = u_2 v_3 - u_3 v_2$$
$$x_2 = u_3 v_1 - u_1 v_3$$
$$x_3 = u_1 v_2 - u_2 v_1$$

We get $A\vec{x} = \vec{0}$

$$\vec{x} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} = \text{CROSS PRODUCT of } \vec{u} \times \vec{v}$$

If $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$,

we define the cross product of

$\vec{u}$ & $\vec{v}$ as

$$\boxed{\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}}$$

the vector which is perpendicular to both $\vec{u}$ & $\vec{v}$.

Area of the $||^{gm}$ formed by $\vec{u}$ & $\vec{v}$

$$\text{Area} = \sqrt{(\vec{u}\cdot\vec{u})(\vec{v}\cdot\vec{v}) - (\vec{u}\cdot\vec{v})^2}$$

$$= \sqrt{(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)^2}$$

$$= \sqrt{\begin{aligned}&\cancel{u_1^2v_1^2} + u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + \cancel{u_2^2v_2^2} + u_2^2v_3^2 + \\ &u_3^2v_1^2 + u_3^2v_2^2 + \cancel{u_3^2v_3^2} - \cancel{u_1^2v_1^2} - \cancel{u_2^2v_2^2} - \cancel{u_3^2v_3^2} \\ &- 2u_1v_1u_2v_2 - 2u_1v_1u_3v_3 - 2u_2v_2u_3v_3\end{aligned}}$$

$$= \sqrt{\begin{aligned}&u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_3^2 + u_3^2v_1^2 + \\ &u_3^2v_2^2 - 2u_1v_1u_2v_2 - 2u_1v_1u_3v_3 - 2u_2v_2u_3v_3\end{aligned}}$$

$$= \sqrt{(u_1v_2-u_2v_1)^2 + (u_1v_3-u_3v_1)^2 + (u_2v_3-u_3v_2)^2}$$

$$= \sqrt{(u_2v_3-u_3v_2)^2 + (u_3v_1-u_1v_3)^2 + (u_1v_2-u_2v_1)^2}$$

$$\text{Area of } ||^{gm} = \|\vec{u}\times\vec{v}\| = \left\| \begin{bmatrix} u_2v_3-u_3v_2 \\ u_3v_1-u_1v_3 \\ u_1v_2-u_2v_1 \end{bmatrix} \right\|$$

The cross product

$$\vec{u} \times \vec{v} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

has components that look like the determinant of 2x2 matrices. How do we interpret this?

② Suppose $\vec{u}, \vec{v}, \vec{w}$ are 3 l.i. vectors in 3D, what is the geometry formed?