Vector Spaces.

* Consider 2 vectors $\vec{u}$ & $\vec{v}$, both 3 component vectors

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \& \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Consider $\underline{\underline{\vec{w}}} = \vec{u} + \vec{v}$

$$= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$$

For some scalar, α, real

$$\alpha \cdot \vec{u} = \alpha \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \alpha \cdot u_1 \\ \alpha \cdot u_2 \\ \alpha \cdot u_3 \end{pmatrix}$$

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$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \qquad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$$

$$A + B = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix}$$

k, Scalar, real

$$k \cdot A = k \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} k \cdot a_1 & k \cdot a_2 \\ k \cdot a_3 & k \cdot a_4 \end{pmatrix}$$

Algebraic properties like add^n & Scalar multiplic^n on a set of real $m \times n$ matrices & that of n-component vectors → Are similar.

Generaliz^n of objects that possess similar properties as above.

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Vector Space: Algebraic Structure with operations having properties similar to that of operations on n-component vectors.

Vector Space over $\mathbb{R}$. $\mathbb{R}$: Real field.

Defn: A vector space over $\mathbb{R}$, is a nonempty set V of vectors, along with an operation called vector addition, a rule for adding 2 vectors from V and another operation called the scalar multiplication, a rule for multiplying a real no × a vector from V, with the following properties:

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For every vector u, v, w in $\underline{V}$

and real numbers $\underline{\alpha}, \beta$,

(i) $u + v \in V$ → Closure under addition.

(ii) $\alpha u \in V$ → Closure under Scalar Multiplic^{cn}

(iii) $u + v = v + u$ Commutativity

(iv) $u + (v + w) = (u + v) + w$ associativity

(v) There exists the element $\underline{0} \in V$ s.t

$u + \underline{0} = u = \underline{0} + u$ (Identity element for addn)

(vi) There exists $-u \in V$ s.t $\left.\begin{matrix} \\ u + (-u) = (-u) + u = \underline{0} \end{matrix}\right\}$ Additive Inverse.

(vii) $\alpha(u + v) = \alpha u + \alpha v$. Distributive

(viii) $(\alpha + \beta) u = \alpha u + \beta u$ Multiplic^{n} over Add.

(ix) $(\alpha\beta) u = \alpha(\beta u)$ ⇒ Associativity

(x) $1 \cdot u = u$ → Identity element

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Every element of $\mathcal{V}$: vector.

Some examples of vector spaces:

Ex: 1. Let $\mathcal{V} = \mathbb{R}^2 \rightarrow$ Set of all 2 component vectors.

Ex: 2: Let $\mathcal{V} = \left\{ \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}, \; x_1 \in \mathbb{R} \right\}$.

$$= \left\{ x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Let $\vec{u} = \begin{pmatrix} u_1 \\ u_1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix}$

Let α be a real scalar.

(i) $\vec{u} + \vec{v} = \begin{pmatrix} u_1 \\ u_1 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_1 \end{pmatrix}$

$$= \begin{pmatrix} u_1 + v_1 \\ u_1 + v_1 \end{pmatrix} \in \mathcal{V}.$$

$\mathcal{V}$ is closed under vector addition.

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(ii) For Scalar α,

$$\alpha \vec{u} = \alpha \cdot \begin{pmatrix} u_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} \alpha u_1 \\ \alpha u_1 \end{pmatrix}$$

$\alpha \vec{u} \in V$.

(iii) $\underline{0}$ vector: $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in V$.

$\therefore V = \left\{ \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}, x_1 \in \mathbb{R} \right\}$ is a vector space!

Ex: 3: $M^{2\times 2}$: Set of all real Symmetric 2×2 matrices.

$$A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}.$$

$$A + B = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ a_2 + b_2 & a_3 + b_3 \end{pmatrix} \in M^{2\times 2}$$

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For real scalar α,

$$\alpha A = \alpha \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a_1 & \alpha a_2 \\ \alpha a_2 & \alpha a_3 \end{pmatrix} \in M^{2\times 2}.$$

$M^{2\times 2}$ closed under scalar multiplicⁿ

$$\underline{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in M^{2\times 2}.$$

$\therefore$ $M^{2\times 2}$ is a vector space over $\mathbb{R}$.