Some properties of dot product.

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\vec{u} \cdot \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

(i) $$= u_1^2 + u_2^2 = \|\vec{u}\|^2$$

If $\|\vec{u}\|^2 = 0 \Rightarrow u_1^2 + u_2^2 = 0$

$$\vec{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(ii) $\vec{u}$ & $\vec{v}$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\vec{w} = \vec{u} + \vec{v}$$

$$\vec{w} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\|\vec{w}\|^2 = w_1^2 + w_2^2$$

$$= (u_1 + v_1)^2 + (u_2 + v_2)^2$$

$$= u_1^2 + 2u_1v_1 + v_1^2 + u_2^2 + 2u_2v_2 + v_2^2$$

$$= \underbrace{u_1^2 + u_2^2}_{\|\vec{u}\|^2} + \underbrace{v_1^2 + v_2^2}_{\|\vec{v}\|^2} + \underbrace{2(u_1v_1 + u_2v_2)}_{2\,\vec{u}\cdot\vec{v}}$$

$$\|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\,\vec{u}\cdot\vec{v}$$

$$\vec w = \vec u + \vec v$$

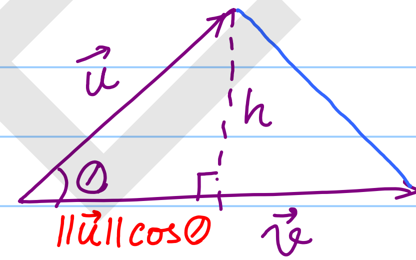
$$\|\vec w\|^2 = \vec w \cdot \vec w$$

$$= (\vec u + \vec v)\cdot(\vec u + \vec v) \leftarrow$$

$$\|\vec u\|^2 + \|\vec v\|^2 + 2\,\vec u\cdot\vec v \leftarrow$$

$$= \vec u\cdot\vec u + \vec v\cdot\vec v + \vec u\cdot\vec v + \vec v\cdot\vec u$$

$$\boxed{(\vec u+\vec v)\cdot(\vec u+\vec v) = \|\vec u\|^2 + \|\vec v\|^2 + 2\,\vec u\cdot\vec v}$$



From trig. ideas we know that the height

$$h = \|\vec u\|\cos\theta$$

$$\Rightarrow h^2 = \|\vec u\|^2\cos^2\theta$$

$$= \|\vec u\|^2(1-\sin^2\theta) \longrightarrow (1)$$

Similarly

$$h^2 = \|\vec{v}-\vec{u}\|^2 - (\|\vec{v}\| - \|\vec{u}\|\cos\theta)^2 \quad - (2)$$

$$\|u\|^2(1-\sin^2\theta) = \|\vec{v}-\vec{u}\|^2 - (\|\vec{v}\| - \|\vec{u}\|\cos\theta)^2$$

$$\Rightarrow \|\vec{v}-\vec{u}\|^2 = \|\vec{v}\|^2 + \|\vec{u}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

$\rightarrow$ Law of cosines

(3)

$$\|\vec{v}-\vec{u}\|^2 = (\vec{v}-\vec{u})\cdot(\vec{v}-\vec{u})$$
$$= \|\vec{v}\|^2 + \|\vec{u}\|^2 - 2\,\vec{u}\cdot\vec{v} \quad \rightarrow (4)$$

Compare (3) & (4) we get

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$\Rightarrow \boxed{\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}}$$

If $\theta = 90^\circ \Rightarrow \vec{u}\cdot\vec{v} = 0$

Comments:

If $\theta = 90^\circ$, $\cos\theta = 0 \Rightarrow \vec{u}\cdot\vec{v} = 0$.

If θ is acute, then $\cos\theta > 0 \Rightarrow \vec{u}\cdot\vec{v} > 0$

If θ is obtuse, then $\cos\theta < 0 \Rightarrow \vec{u}\cdot\vec{v} < 0$

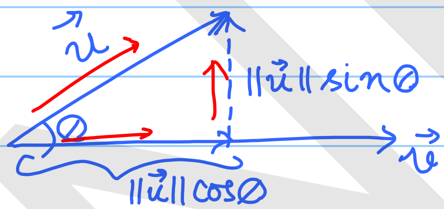
If $\theta = 0$ $\quad \vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos 0$

$$= \|\vec{u}\|\,\|\vec{v}\|$$

Orthogonal Projections:

Projection of a vector $\vec{u}$ in the direction of $\vec{v}$

→ Creates a footprint of length $= \|\vec{u}\|\cos\theta$



Orthogonal projection of $\vec{u}$ onto $\vec{v}$

$$\Rightarrow \text{proj}_{\vec{v}}\, u = \|\vec{u}\|\cos\theta \boxed{\frac{\vec{v}}{\|\vec{v}\|}}$$

$$= \cancel{\|\vec{u}\|}\, \frac{\vec{u}\cdot\vec{v}}{\cancel{\|\vec{u}\|}\,\|\vec{v}\|}\, \frac{\vec{v}}{\|\vec{v}\|}$$

$$\text{Proj}_{\vec{v}}\, u = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\,\vec{v}$$

The vector $\vec{u}$ can be decomposed into 2 orthogonal components

$$\vec{u} = \vec{u}_{\parallel} + \vec{u}_{\perp}$$

$\vec{u}_{\parallel}$ = $\cos\theta$ component

$\vec{u}_{\perp}$: $\sin\theta$ component

$$\vec{u}_{\perp} = \vec{u} - \text{proj}_{\vec{v}}\,\vec{u} = \vec{u} - \vec{u}_{\parallel}$$

Summary:

→ Ideas of vectors & points

→ Points are fixed w.r.to a coordinate system

→ Vectors → Not fixed

→ Coordinate dep & indep operations

→ Length of vectors

→ Combination of vectors

→ L.I & LD vectors.

→ Dot product

→ $\cos\theta$