Eigenvalues & eigenvectors

$$A^{2\times 2}\, x^{2\times 1} = b^{2\times 1}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Is it possible to have a vector $\vec{u} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ s.t

$$A\vec{u} = \lambda \vec{u} \,?$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

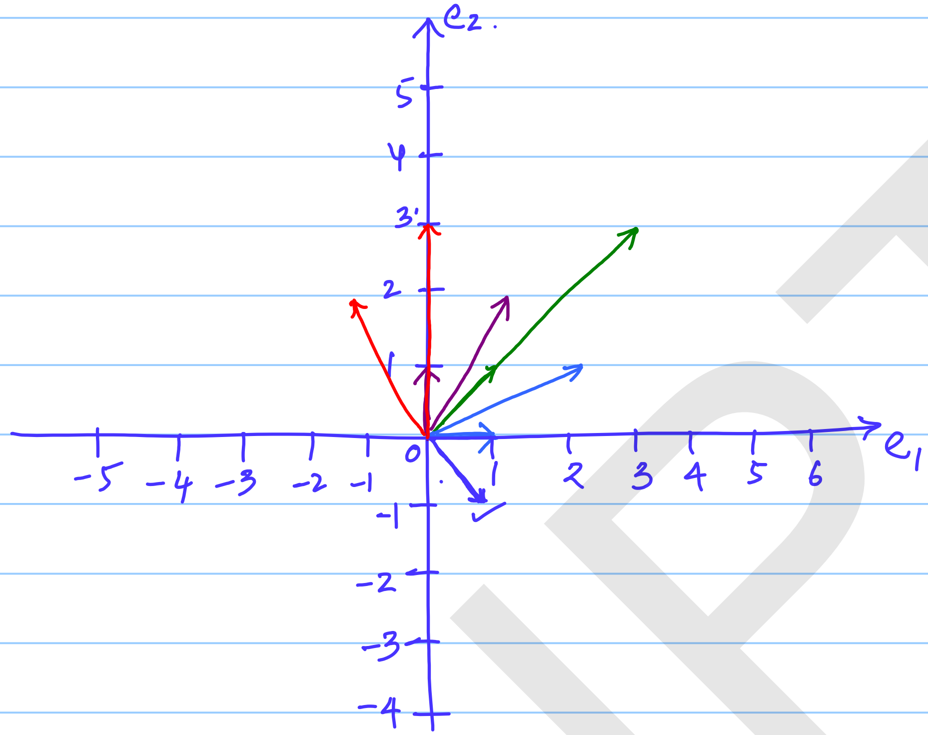
$$\vec{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \vec{b} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \vec{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$



$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \underline{\underline{3}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \underline{\underline{1}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A vector $\vec{u}$ is called an eigenvector to the matrix A if

$A\vec{u} = \lambda\vec{u}$ for some scalar λ; λ : eigenvalue.

$\vec{u}$ is the eigenvector associated with the eigenval. λ.

$A\vec{u} = \lambda\vec{u}$

$A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \lambda\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ λ: Scalar.

How do we find λ, and how do we find $\vec{u}$?

$A\vec{u} = \lambda\vec{u}$ $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$A\vec{u} = \lambda I\vec{u}$ $I : \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow (A - \lambda I)\vec{u} = \vec{0} \longrightarrow (1)$

$$\begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$: Soln to the Homogeneous system $(A-\lambda I)\vec{u} \stackrel{!}{=} \vec{0}$

$\Rightarrow$ Non trivial Soln $\vec{u}$ to $(A-\lambda I)\vec{u} = \vec{0}$

Recall: If $\underline{\text{det of } A^{2\times 2} = 0}$, then the $\underline{\text{HSE}}$ $A\vec{x} = \vec{0}$ has nontrivial soln.

$\Rightarrow (A - \lambda I)\vec{u} = \vec{0}$ $\quad \vec{u} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore$ $\boxed{\det(A - \lambda I) = 0}$

Characteristic eqn of A.

$$\det\Big(\underbrace{\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}}^{A - \lambda I}\Big) = 0$$

$$\Rightarrow (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

$$\Rightarrow \lambda^2 - \lambda\underbrace{(a_{11} + a_{22})}_{\text{Sum of diag = Trace}} + \underbrace{a_{11}a_{22} - a_{21}a_{12}}_{\det(A)} = 0.$$

Roots of the char. eqns = eigenval.

Product of eigenval = det(A)

Sum of eigenval (A) $\overset{\text{of } A}{=}$ Trace(A).

λ_1, λ_2 are 2 roots of char. eqn. of $A^{2\times 2}$.

$\lambda_1 \rightarrow$ eigen val.

$\lambda_2 \rightarrow$ eigen val.

HSE

$(A-\lambda_1 I)\vec{u} = \vec{0} \longrightarrow (1)$

$\hookrightarrow$ eigen vector associated with λ_1

$(A-\lambda_2 I)\vec{v} = \vec{0}$

$\hookrightarrow$ eig. vect assoc. with λ_2.

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

char eqn: $\det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = 0$

$\Rightarrow (2-\lambda)^2 - 1 = 0$

$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$

$\lambda_1 = 3, \quad \lambda_2 = 1.$

$\lambda_1 \cdot \lambda_2 = 3 \cdot 1 = 3 = \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\lambda_1 + \lambda_2 = 3+1 = 4 =$ Sum of diag. elements of A.

$\lambda_1 = 3$

$$\begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -u_1 + u_2 = 0$

$u_1 - u_2 = 0$

$\Rightarrow u_1 = u_2.$

$\therefore$ eigenvector: $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_1 \end{pmatrix} = u_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_1 = 3. \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 1 \rightarrow$ eigenvector: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

1. Is it always that for a 2x2 matrix we get real eigenval?

Char eqn: $\lambda^2 - \lambda(\text{Trace}) + \det(A) = 0$.

→ Quadratic eqn.

Both roots can be complex.

⇒ No real eigenvalues ⇒ No real eigenvectors.

⇒ No fixed direction.

R_θ : Rotates every vector by an angle θ.

R_{90°

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{90^\circ} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Char. eq: $\det\begin{pmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{pmatrix} = 0$

$\Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1, \ \lambda = \pm i$

2) What if the roots of char eqn of $A^{2\times2}$ are repeated?

$$\lambda^2 - \lambda(\text{Sum of diag}) + \text{Det}(A) = 0.$$

$$\lambda_1 = \lambda_2 = \lambda.$$

If eigenvalues are repeated for a 2x2 matrix A, we have only one eigenvector
$\rightarrow$ 1 fixed direcⁿ.

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow$ Shear.

Char. eqn $\det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = 0$

$$\Rightarrow (1-\lambda)^2 = 0$$

$$\Rightarrow \underline{\underline{\lambda = 1}} \text{ twice.}$$

$$(A - \lambda I)\vec{u} = \vec{0}$$

$$\begin{pmatrix} 1-1 & 1 \\ 0 & 1-1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$0u_1 + u_2 = 0$

$\Rightarrow u_2 = 0.$

$0u_1 = 0$ $\quad u_1$ can take any value.

$\therefore$ the eigenvector is

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} k \\ 0 \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

k: real.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$\lambda_1 = 3,$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \Rightarrow$

If $A^{2\times2}$ is a real Symmetric matrix with distinct eigenvalues, the corresponding eigenvectors are Orthogonal to each other.