* Check if the set of vectors $S = \left\{ \begin{pmatrix} x_1 \\ 2 \end{pmatrix}, x_1 \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^2$. → NO.

* If $A^{2\times2}$ is the set of all invertible 2×2 real matrices, then check if $A^{2\times2}$ is a subspace of $M^{2\times2}$, set of all real matrices. NO.

* Is the set S of all 3 component vectors in $\mathbb{R}^3$, S as given below a subspace of $\mathbb{R}^3$

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \right\}.$$

NO

$\mathbb{R}^n \to \vec{x} \in \mathbb{R}^n = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

Ordered n-tuple of real numbers.

Let $\vec{u}_1 = \begin{pmatrix} u_{11} \\ u_{21} \\ \vdots \\ u_{n1} \end{pmatrix} \dots \vec{u}_k = \begin{pmatrix} u_{1k} \\ u_{2k} \\ \vdots \\ u_{nk} \end{pmatrix}$

Let $\vec{v}$ be a vector, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

$\vec{v}$ is a linear combination of $\vec{u}_1, \dots \vec{u}_k$ if and only if there exists real scalars $\alpha_1, \alpha_2 \dots \alpha_k$ s.t

$$\vec{v} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \dots + \alpha_k \vec{u}_k$$

Applied Linear Algebra
- Stephen Boyd & others

Suppose $\vec{u}_1, \vec{u}_2 \dots \vec{u}_k$ correspond to different audio tracks over the same period of time

$\vec{u}_1 \to$ Voice $\quad \vec{u}_2 \to$ Violin ...

$$\vec{u} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \dots + \alpha_k \vec{u}_k$$

Represents mixture of audio

Signals/tracks where

$|\alpha_i| \Rightarrow$ Relative loudness.

Ex: Check if

$\vec{u} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$ is a l.c. of

$\vec{u_1} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ & $\vec{u_2} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Soln: To find Scalars $\alpha_1, \alpha_2 \in \mathbb{R}$

s.t $\vec{u} = \alpha_1 \vec{u_1} + \alpha_2 \vec{u_2}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}$ $\vec{x} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$

$A\vec{x} = \vec{b}$

RREF of $[A|b]$

$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 1 & 4 \end{array}\right]$ $R_2 \leftarrow R_2 - R_1$;
$R_3 \leftarrow R_3 - 2R_1$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & -1 & 0 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - \frac{R_2}{2}} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & 0 & 4 \end{array}\right]$$

Inconsistent

$\Rightarrow$ No soln.

$\Rightarrow$ No α_1, α_2 exists satisfying the 3 eqns.

$\therefore$ we say that the vector $\vec{b} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$ cannot be expressed as a l.c. of $\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ &

$\vec{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Linear Independence.

Given $\vec{u}_1, \vec{u}_2 \dots \vec{u}_n \in R^n$

what is the l.c. of these vectors that gives the $\vec{0}$?

$\Rightarrow \quad \vec{0} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \dots + \alpha_n \vec{u}_n$

What are the values of the Scalars?

Let $S = \{\vec{u}_1, \vec{u}_2 \dots \vec{u}_n\}$ be a nonempty Subset of R^n

Set S is l.i set if and only if there exists real scalars $\alpha_1 \dots \alpha_n$ s.t the equation

$$\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \dots + \alpha_n \vec{u}_n = \vec{0}$$

implies $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$
(Scalars are 0).

$\Rightarrow$ The only l.c that gives the zero vector is the TRIVIAL L.C with scalars $= 0$.

If there exists atleast one $\alpha_i \neq 0$, then S is a linearly dep. set.

NOTE: Any subset of a vector space V, that contains the $\vec{0}$ is a l. dep. set.

(2) Suppose $\vec{u}$ is a l.c. of the vectors $\vec{u}_1, \vec{u}_2 \dots \vec{u}_n$ in $\mathbb{R}^n$

$$\vec{u} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \dots + \alpha_n \vec{u}_n$$

When $\vec{u}_1, \vec{u}_2 \dots \vec{u}_n$ is a linearly indep set then $\vec{u}$ can be expressed as a unique l.c. of $\vec{u}_1 \dots \vec{u}_n$

Proof:

Suppose there exists 2 l.c. for $\vec{u}$ in terms of the l.i vectors $\vec{u}_1 \dots \vec{u}_n$

$$\vec{u} = \alpha_1 \vec{u}_1 + \dots + \alpha_n \vec{u}_n \rightarrow (1)$$

$$\vec{u} = \beta_1 \vec{u}_1 + \dots + \beta_n \vec{u}_n \rightarrow (2)$$

Subtracting (2) from (1)

$$\vec{0} = (\alpha_1 - \beta_1)\vec{u}_1 + \dots + (\alpha_n - \beta_n)\vec{u}_n$$

$$= \gamma_1 \vec{u}_1 + \dots + \gamma_n \vec{u}_n$$

$l.d$: linearly dep.

This contradicts the fact that

$\vec{u}_1, \vec{u}_2 \dots, \vec{u}_n$ are $l.i$.

$\therefore$ α_i's $= 0$ for all i's $1, \dots n$.

$\therefore$ $\alpha_1 - \beta_1 = 0$, $\alpha_2 - \beta_2 = 0$ $\dots$ $\alpha_n - \beta_n = 0$

$\Rightarrow$ $\alpha_i = \beta_i$ for $i = 1 \dots n$.

$\Rightarrow$ If $\vec{u}_1, \vec{u}_2 \dots \vec{u}_n$ are $l.i$ vectors in $\mathbb{R}^n$, then for every $\vec{u} \in \mathbb{R}^n$, there is a unique $l.c$ of $\vec{u}_1, \vec{u}_2 \dots \vec{u}_n$

* A set of 2 vectors in $\mathbb{R}^n$ is $\underline{l.d}$ set if one of them is a multiple of the other.

* A set that contains only one non zero vector is a $l.i.$ set.

How do we check if given set of vectors is l.i or not?

Define a matrix A whose cols are the given vectors.

Solve the homogeneous sys. of eqns $A\vec{x} = \vec{0}$

if we get only trivial soln, then the vectors are l.i.

If we get non trivial soln, the vectors are l.d.

$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}$ $A\vec{x} = \vec{0}$

$x_1 = 0, x_2 = 0.$