Least Squared Solutions

In 2D:

$$2x_1 + x_2 = 3$$
$$2x_1 + x_2 = 4$$

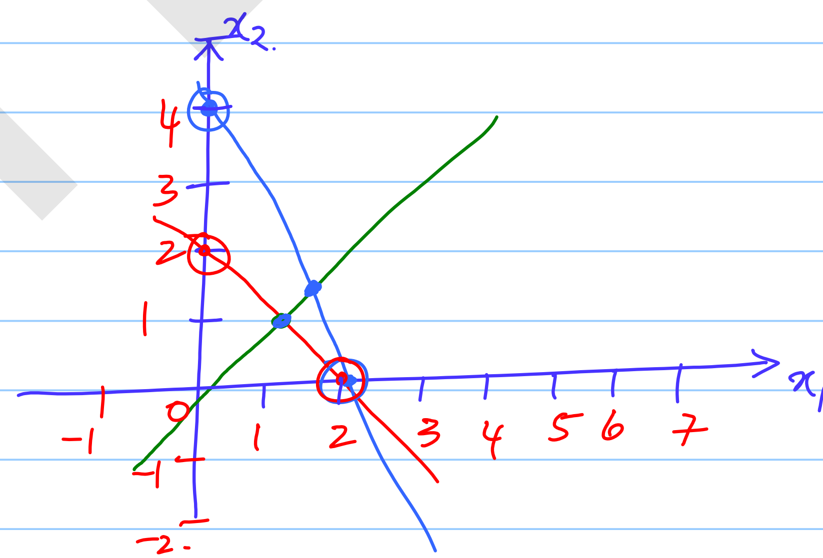
No Solution exists.

Ex: $x_1 + x_2 = 2 \quad (\ell_1)$

$x_1 - x_2 = 0 \quad (\ell_2)$

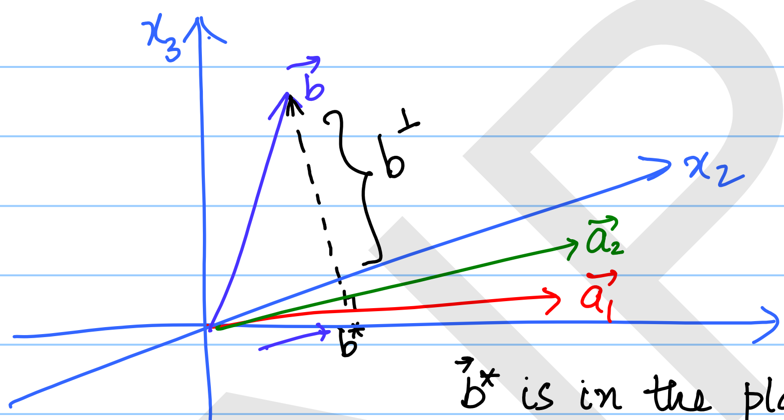
$2x_1 + x_2 = 4. \quad (\ell_3)$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix} \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$$



In the vector perspective / Col. picture

$$x_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$$



$\vec{b}^*$ is in the plane π_{a_1,a_2}

Once we get b^* the proj. of $\vec{b}$ on to the plane π_{a_1,a_2}, we can find Scalars x_1^* and x_2^* s.t we can express b^* as l.c. of $\vec{a}_1$ & $\vec{a}_2$.

We know that

$$\vec{b} = \vec{b}^* + \vec{b}^\perp$$

$\vec{b}^*$ is closest to $\vec{b}$ in the plane π_{a_1,a_2}, $b^\perp$ is orthog. to π_{a_1,a_2}.

$\vec{a}_1^T \vec{b}^\perp = 0$; $\vec{a}_2^T \vec{b}^\perp = 0$

$\vec{a}_1$ & $\vec{a}_2$ are the cols of A.

$\Rightarrow \quad A^T b^\perp = 0$

we know that

$b^\perp = \vec{b} - \vec{b}^*$

$A^T(\vec{b} - \vec{b}^*) = 0$

Since $\vec{b}^*$ is in the plane π_{a_1,a_2}

$\vec{b}^* = Ax^* \qquad x^* = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$$A^T(\vec{b} - A\vec{x}^*) = 0$$

$$\Rightarrow \quad A^T\vec{b} - A^TA\vec{x}^* = 0$$

$$\Rightarrow \quad A^TA\vec{x}^* = A^Tb. \longrightarrow (1)$$

We had $A\vec{x} = \vec{b}$, we now solve

$$A^TA\vec{x}^* = A^Tb.$$

A^TA is a square matrix & symmetric too!

The solution x^* to (1), when it has one, is the one that minimizes the error

$$\|A\vec{x}^* - \vec{b}\|^2.$$

$\vec{x}^*$: Least sq. soln

In our example

$$A^Tb = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

$$x^* = (A^T A)^{-1} A^T b$$

$$\Rightarrow \left(\frac{1}{\det(A^T A)} \left(\mathrm{Adj}\,(A^T A) \right) \right) \left(A^T b \right)$$

$$= \frac{1}{14} \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 18/14 \\ 16/14 \end{bmatrix} //$$

Inner Product & Gram Schmidt Orthonormalizn, QR decomp.

Given 2 vectors $\vec{u}$ & $\vec{v}$, we define the inner product of $\vec{u}, \vec{v}$ as

$$\langle \vec{u}, \vec{v} \rangle = \underline{\underline{v^T u}} = \underline{\underline{u^T v}} = \langle \vec{v}, \vec{u} \rangle$$

(i) For any scalar k,

$$\langle k\vec{u}, \vec{v} \rangle = \langle \vec{u}, k\vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$$

(ii) For any 3 vectors $\vec{u}, \vec{v}$ & $\vec{w}$, we have

$$\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, w \rangle.$$

(iii) For any vector $\vec{v}$ $\langle \vec{v}, \vec{v} \rangle \geq 0$

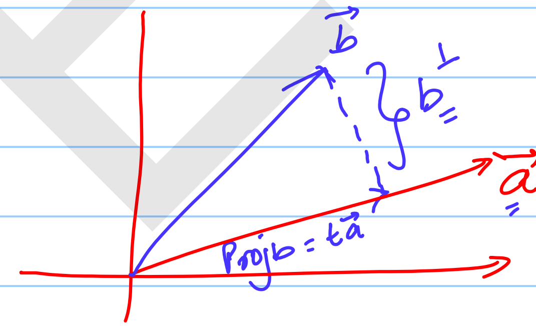
$\langle \vec{v}, \vec{v} \rangle = 0$ if & only if $\vec{v}$ is the zero vector.

Given 3 vectors $\vec{u}, \vec{v}$ & $\vec{w}$,

linearly indep, Can we find

3 vectors Say, $\vec{\varphi}_1, \vec{\varphi}_2$ & $\vec{\varphi}_3$

Orthogonal to each other



$$\vec{b}^{\perp} = \vec{b} - \mathrm{Proj}_{\vec{a}}\, \vec{b}$$

$$\vec{b}^{\perp} \cdot \vec{a} = 0$$

$$\vec{\varphi}_1 = \vec{u}$$

$$\vec{\varphi}_2 = \vec{v} - \left(\mathrm{Proj}_{\vec{\varphi}_1} \vec{v}\right)$$

$$= \vec{v} - \frac{\vec{v} \cdot \vec{\varphi}_1}{\vec{\varphi}_1 \cdot \vec{\varphi}_1} \vec{\varphi}_1$$

$$\vec{\varphi}_3 = \vec{w} - \frac{\vec{w} \cdot \vec{\varphi}_1}{\vec{\varphi}_1 \cdot \vec{\varphi}_1} \vec{\varphi}_1 - \frac{\vec{w} \cdot \vec{\varphi}_2}{\vec{\varphi}_2 \cdot \vec{\varphi}_2} \vec{\varphi}_2$$

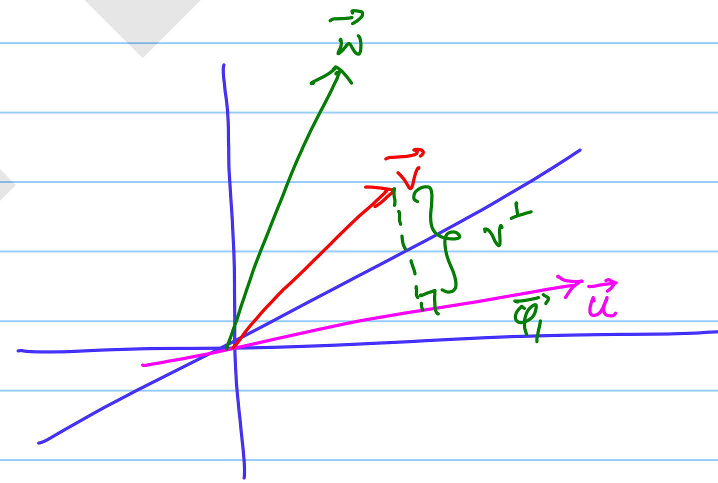
$$\vec{e}_1 = \frac{\vec{\varphi}_1}{\|\vec{\varphi}_1\|}$$

$$\vec{e}_2 = \frac{\vec{\varphi}_2}{\|\vec{\varphi}_2\|}$$

$$\vec{e}_3 = \frac{\vec{\varphi}_3}{\|\vec{\varphi}_3\|}$$

$$\vec{\varphi}_1 \cdot \vec{\varphi}_2 = 0 \quad ; \quad \vec{\varphi}_1 \cdot \vec{\varphi}_3 = 0$$

$$\vec{\varphi}_2 \cdot \vec{\varphi}_3 = 0$$



Suppose $A = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$

$$\vec{u} = \vec{\varphi}_1 = \|\vec{\varphi}_1\| \vec{e}_1$$

$$\vec{v} = \vec{\varphi}_2 + \mathrm{Proj}_{\vec{\varphi}_1} \vec{v}$$

$$= \|\vec{\varphi}_2\| \vec{e}_2 + \frac{\vec{v} \cdot \vec{\varphi}_1}{\vec{\varphi}_1 \cdot \vec{\varphi}_1} \|\varphi_1\| \vec{e}_1$$

$$\vec{w} = \vec{\varphi}_3 + \mathrm{Proj}_{\vec{\varphi}_1} \vec{w} + \mathrm{Proj}_{\vec{\varphi}_2} \vec{w}$$

$$\underline{\underline{\vec{w}}} = \underbrace{\|\vec{\varphi}_3\| \vec{e}_3 + \frac{w \cdot \varphi_1}{\vec{\varphi}_1 \cdot \vec{\varphi}_1} \|\vec{\varphi}_1\| \vec{e}_1 + \frac{\vec{w} \cdot \vec{\varphi}_2}{\vec{\varphi}_2 \cdot \vec{\varphi}_2} \|\vec{\varphi}_2\| \vec{e}_2}$$

$$A = \begin{bmatrix} \|\vec{\varphi}_1\| \vec{e}_1 & \|\vec{\varphi}_2\| \vec{e}_2 + \frac{\vec{v} \cdot \vec{\varphi}_1}{\varphi_1 \cdot \varphi_1} \|\varphi_1\| e_1 & \end{bmatrix}$$

$$= \underset{Q}{\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}} \underset{R}{\begin{bmatrix} \|\varphi_1\| & \frac{v \cdot \vec{\varphi}_1}{\varphi_1 \cdot \varphi_1} \cdot \|\varphi_1\| & \frac{w \cdot \varphi_1}{\varphi_1 \cdot \varphi_1} \|\varphi_1\| \\ 0 & \|\varphi_2\| & \frac{w \cdot \varphi_2}{\varphi_2 \cdot \varphi_2} \|\varphi_2\| \\ 0 & 0 & \|\varphi_3\| \end{bmatrix}}$$

$$\boxed{A = QR}$$

Prod. of Orthogonal Matrix
and an upper triangular
matrix.

Orthogonal Matrix : Q

(i) Cols. of Q are all unit vectors

(ii) Col. of Q are orthogonal to each other.

(iii) $Q^{-1} = Q^T$.