Consider $A\vec{x} = \vec{b}$

$A \in \mathbb{R}^{m \times n}$ $\quad x \in \mathbb{R}^n$ $\quad b \in \mathbb{R}^m$.

$$A = \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & & A_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

$$x \in \mathbb{R}^n = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad b \in \mathbb{R}^m = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$A\vec{x} = \vec{b}$

$= x_1(\vec{A_1}) + x_2(\vec{A_2}) + \cdots + x_n(\vec{A_n}) = \vec{b}$

To solve $A\vec{x} = \vec{b} \Rightarrow$ find scalars

$x_1, x_2 \ldots x_n$ s.t

$\vec{b}$ can be expressed as l.c. of

cols. of A

$\Rightarrow \vec{b} = x_1\vec{A_1} + x_2\vec{A_2} + \cdots + x_n\vec{A_n}$

2 Cases arise:

(1) We can find scalars

$x_1, x_2 \dots x_n$ s.t

$\vec{b} = x_1\vec{A}_1 + x_2\vec{A}_2 + \dots + x_n\vec{A_n}$

(2) There exists no scalars $x_1, x_2 \dots$

x_n s.t

$\vec{b} = x_1\vec{A}_1 + \dots + x_n\vec{A_n}$

Case 1:

There exists scalars $x_1, \dots, x_n$

s.t $\vec{b} = x_1 \vec{A_1} + \dots + x_n \vec{A_n}$

$\Rightarrow$ $\vec{b}$ can be expressed as

Cols. of A.

Set of all possible $\vec{b}$ vectors that can be expressed as

$$\vec{b} = \sum_{i=1}^{n} x_i A_i$$

Let this set be R_A.

$b \in \mathbb{R}^m$

Is R_A nonempty? YES.

$0 \in \mathbb{R}^m$ (zero vector) is an element of R_A.

Let $\vec{u}, \vec{v} \in \mathbb{R}^m$ be elements of R_A.

(1) $\leftarrow \vec{u} = \alpha_1 \vec{A}_1 + \alpha_2 \vec{A}_2 + \dots + \alpha_n \vec{A}_n$
for some real scalars $\alpha_1, \dots \alpha_n$.

$\vec{v} = \beta_1 \vec{A}_1 + \beta_2 \vec{A}_2 + \dots + \beta_n \vec{A}_n \rightarrow$ (2)

for some real scalars $\beta_1, \beta_2 \dots \beta_n$.

(1) + (2)

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_m+v_m \end{pmatrix}$$

$$\vec{u}+\vec{v} = \alpha_1 \vec{A_1} + \alpha_2 \vec{A_2} + \cdots + \alpha_n \vec{A_n} + \beta_1 \vec{A_1} + \beta_2 \vec{A_2} + \cdots + \beta_n \vec{A_n}$$

$$\vec{w} = \vec{u}+\vec{v} = \begin{pmatrix} u_1+v_1 \\ \vdots \\ u_m+v_m \end{pmatrix} = \underbrace{(\alpha_1+\beta_1)}_{\gamma_1}\vec{A_1} + \underbrace{(\alpha_2+\beta_2)}_{\gamma_2}\vec{A_2} + \cdots + \underbrace{(\alpha_n+\beta_n)}_{\gamma_n}\vec{A_n}$$

$$\vec{w} = \gamma_1 \vec{A_1} + \gamma_2 \vec{A_2} + \cdots + \gamma_n A_n$$

$$\Rightarrow \vec{w} = \vec{u}+\vec{v} \in R_A$$

R_A is closed under vector addition.

$$\vec{u} = \alpha_1 \vec{A_1} + \alpha_2 \vec{A_2} + \cdots + \alpha_n \vec{A_n}$$

Let c be a real scalar

$$c\vec{u} = \begin{pmatrix} c_1 u_1 \\ \vdots \\ c_m u_m \end{pmatrix} = c\left[\alpha_1 \vec{A_1} + \alpha_2 \vec{A_2} + \cdots + \alpha_n \vec{A_n}\right]$$

$$= \underbrace{(c\alpha_1)}_{k_1}\vec{A_1} + (c\alpha_2)\vec{A_2} + \cdots + (c\alpha_n)\vec{A_n}$$

$$= k_1 \vec{A_1} + k_2 \vec{A_2} + \cdots + k_n \vec{A_n}$$

$c\vec{u}$ = LC. of cols. of A.

R_A is non empty, closed under vector addition & scalar multiplicn

Hence R_A is a subspace of $\mathbb{R}^m$

R_A : Range of A

Column space of A

Col. sp(A).

$\mathbb{R}^n$

(1) $N_A \rightarrow$ Subspace of $\mathbb{R}^n$

$N_A : \left\{ x \in \mathbb{R}^n : A\vec{x} = \vec{0}_m \right\}$.

$\mathbb{R}^m$.

R_A or ColSp(A)

$R_A : \left\{ b \in \mathbb{R}^m : b = \sum_{i=1}^{n} x_i A_i \quad \vec{b} = A\vec{x} \right\}$