linear transformation.

$A^{2\times2}$ matrix

Suppose A is the matrix representing a linear transformⁿ then

$$A(\alpha\vec{u}+\beta\vec{v}) = \alpha A\vec{u} + \beta A\vec{v}$$

for some scalars α & β and vectors $\vec{u}$ & $\vec{v}$

$\vec{u}, \vec{v}$ the l.c. of $\vec{u}, \vec{v}$ as

$$\vec{w} = \alpha\vec{u} + \beta\vec{v}.$$

$$A\vec{w} = A(\alpha\vec{u}+\beta\vec{v})$$

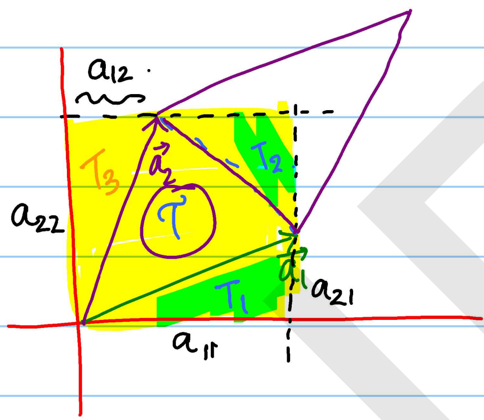
$$= \alpha A\vec{u} + \beta A\vec{v}$$

for some scaling factor k,

$$A(k\vec{u}) = k(A\vec{u})$$

Consider a matrix $A^{2\times 2}$ as below

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$



Area of triangle T

$$T = (a_{11} \cdot a_{22}) - T_1 - T_2 - T_3.$$

Area of ΔT_1 : $\frac{1}{2} a_{11} a_{21}$

Area of ΔT_3 : $\frac{1}{2} a_{12} \cdot a_{22}$

Area of ΔT_2 : $\frac{1}{2}(a_{11} - a_{12})(a_{22} - a_{21})$

$$T = a_{11}a_{22} - \cancel{\frac{a_{11}a_{21}}{2}} - \cancel{\frac{a_{12}a_{22}}{2}} - \frac{a_{11}a_{22}}{2} + \cancel{\frac{a_{12}a_{22}}{2}} + \cancel{\frac{a_{11}a_{21}}{2}} - \frac{a_{12}a_{21}}{2}$$

$$T = a_{11}a_{22} - \frac{a_{11}a_{22}}{2} - \frac{a_{21}a_{12}}{2}$$

$$= \frac{a_{11}a_{22}}{2} - \frac{a_{21}a_{12}}{2}.$$

Area of the parallelogram $= 2T = \cancel{2}\left(\frac{a_{11}a_{22} - a_{21}a_{12}}{\cancel{2}}\right)$

$$\boxed{\text{Area of } \|gm = a_{11}a_{22} - a_{21}a_{12}.} \rightarrow \text{Determinant of } A$$

where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

① If $\det(A) = 1 \rightarrow$ A does not change the area

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\det(R_\theta) = \cos^2\theta - (-\sin^2\theta) = 1.$$

$\det(R_\theta) = 1 \longrightarrow$ linear Transf A Does not change the area.

② If $0 \leq \det(A) \leq 1$, the linear Transf A Shrinks the area.

③ If $\det(A) = 0 \rightarrow$ Rank deficient

④ If $\det(A) > 1 \rightarrow$ Area expanded

⑤ If $\det(A) < 0 \Rightarrow$ Orientation of the object is changed.

Recall: CRAMER'S RULE.

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$\Rightarrow \quad x_1 = \frac{\det\begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}}{\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}$$

$$x_2 = \frac{\det\begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}}{\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}$$

If $\det(A) \neq 0$, then A is invertible!

For a linear transf $A^{2\times 2}$ to be invertible,

$$\det(A) \neq 0.$$

When will $\det(A^{2\times 2}) = 0$?

$$\det(A^{2\times 2}) = a_{11}a_{22} - a_{12}a_{21}$$

If $\det(A^{2\times 2}) = 0 \Rightarrow a_{11}a_{22} - a_{12}a_{21} = 0$

$\Rightarrow a_{11}a_{22} = a_{12}a_{21}$

$\Rightarrow \dfrac{a_{11}}{a_{21}} = \dfrac{a_{12}}{a_{22}}$.

$\Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ det $= 0$ if the cols of $A^{2\times 2}$ are multiples of each other

For ex: $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

$\det(A) = 4 - 4 = 0$

System of linear eqns

2 equations & 2 Unknowns.

$$a_{11} x_1 + a_{12} x_2 = b_1$$
$$a_{21} x_1 + a_{22} x_2 = b_2$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

When does $A^{2\times 2} x^{2\times 1} = b^{2\times 1}$ have

(i) Unique Soln?

(ii) Infinitely many Solns?

(iii) No Solutions?