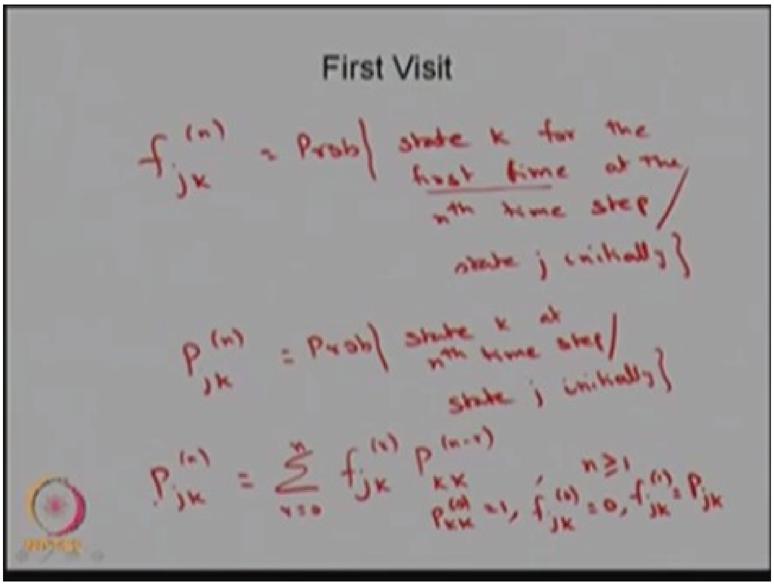

Now I am going to give the next concept called the first visit. We did not come to the classification of a state before that we are developing a few concepts using these concepts we are going to classify the states. The next concept is called a first visit what is the meaning of a first visit? I am going to define the probability mass function as the $f_{jk}^{(n)}$ that means that what is the probability that the system reaches the state k for the first time that is important.

For the first time at the nth time step given that the system starts the state j initially. This is a conditional probability mass function of a system moving from the state j to k. And system reaching the state k at nth time step for the first time that is important. So this is the first time the system reaches the state k at the nth step exactly at the nth step and this conditional probability mass function that I am going to write as the $f_{jk}^{(n)}$.

This is different from the $P_{jk}^{(n)}$. This is also conditional probability whereas this probability is defined what is the probability that the system reaches the state k at the nth time step. Given that it was in the state j initially. This is also conditional probability the only difference is the first time that means that is a possibility the system here the $P_{jk}^{(n)}$ means that is a possibility the system would have come to the state k before nth step also.

So that probability is included whereas as the $f_{jk}^{(n)}$ means that this is the only the nth step it reaches the state k. Therefore, the way I have given the first time conditional this probability and this is not necessarily the first time this is also conditional probability I can relate the f_{jk}

with the P_{jk} both are in the n-step transition probability, but one is for the first time the other one is not necessarily.

I can relate both in the form of $P_{jk}^{(n)}$ that is the n step that is same as $\sum_{r=0}^{n} f_{jk}^{(r)} P_{kk}^{(n-r)}$ or n is greater than or equal to 1. This means if the system is moving from the state j to k in the n step not necessarily the first time that can be written as the union of mutually exclusive events for different r in which the system moves from the state j to k in r steps for the first time and the remaining n-r steps there is a possibility the system would have move the state k to k not necessarily the first time.

And the possible r can be 0 to small n and this n can vary from 1 to infinity. Obviously, we can make out I can give the $P_{kk}^{(0)}$ step that is going to be one and similarly you can make out a $f_{jk}^{(0)}$ also 0 and $f_{jk}^{(1)}$ that is nothing, but the P_{jk}. The first time the system is moving from the state j to k in one step that is same as the one step transition probability.

The first time and one step transition probability is same whereas for n is greater than or equal to 1 then it is going to be combination of the first time with the not necessarily the first time n minus r step transition probability that all possible events that will give all the together final probability. So here we have used the total probability rule as well as the Chapman Kolmogorov equation for the Time Homogeneous Discrete Time Markov Chain to land up giving the relation between the P_{jk} $with f_{jk}$.

Introduction to Probability Theory and Stochastic Processes

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Lecture – 69

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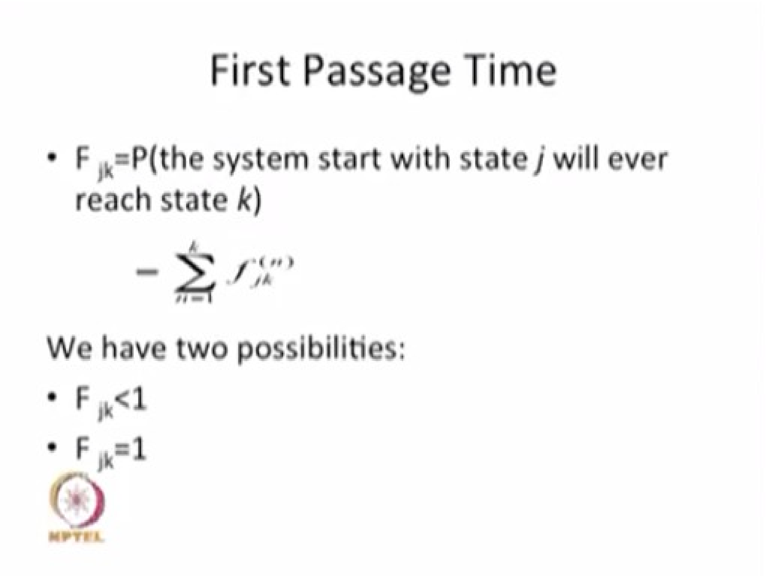

Now I am going to give the next concept called a First Passage Time Distribution. So that is written in the F_{jk} that is nothing but what is the probability that the system start with the state j will ever reach state k. So this probability I am writing as a F_{jk}therefore this same as there is a possibility it would have gone to the state k in n steps first time and all the possible steps for the first time that union will give F_{jk}.

What is the conditional probability that the system is starting from the state j and ever entering into the state k that is all the possible of first time to reaching the state n and all possible n that will give the probability F ever visiting the state k starting with the state j. Now we have two issues or two cases. One what is F_{jk}which is less than 1, what is the situation corresponding to this probability is going to be less than 1.

The other case of interest is when $F_{jk}$is equal to 1 that means with the probability 1 you will be ever visiting the state k by starting from the state j with the probability 1 or whether this probability is going to be less than 1. If it is less than 1 then it is not the correct one that means with the 1 minus of this probability, there is a possibility you would not ever visit the state k if you start from the state j the first case.

The second case that says with the probability 1 you will always reach the state k whatever be the number of steps starting from the state j. So our interest is both less than 1 as well as equal to 1. So the F_{jk}equal to 1 that will give the probability distribution and that distribution is called a First Passage Time Distribution. So this case is our interest and this will give the First Passage Time Distribution.

Because whenever the system is starting from the state j whatever be the number of steps if you are reaching the state k with the probability 1 that means you have the whole mass is 1. And this is going to be the distribution of the First Passage Time.

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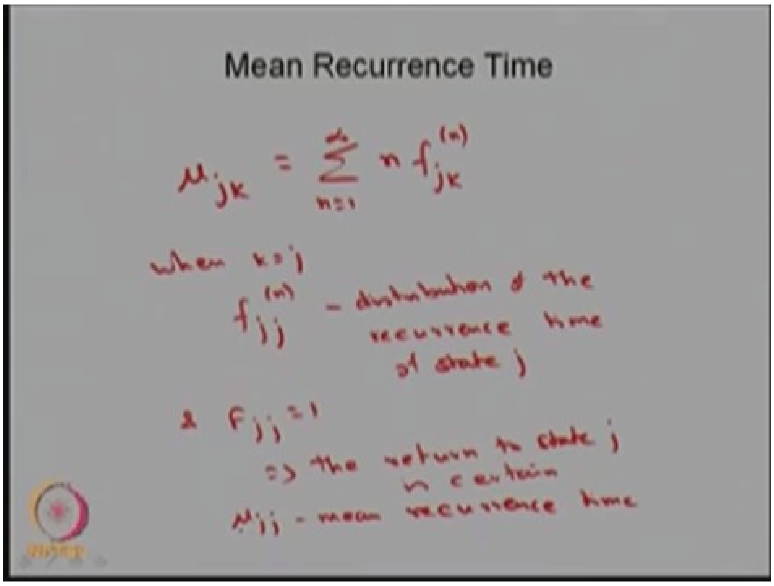

So using this I am going to give the next concept called Mean First Passage Time or Mean Recurrence Time. Mean First Passage Time is same as the Mean Recurrence Time that is defined as μ_{jk}that is nothing but what is the average First Passage Time or average recurrence time whenever the system start from the state j to the state k that is how many steps you have taken.

And what is the probability that starting from state j to k in the n steps and for possible values of n that summation is going to give the Mean First Passage Time or Mean Recurrence Time. When our interest will be when k equal to j return to the same state. So that means f_{jk} of n that will give the Distribution of the Recurrence Time of the state j. And if F_{jj}equal to 1 this corresponding $f_{jj}^{(n)}$is going to be the distribution.

So correspondingly F_{jj} is going to be 1 this implies the return to the state j whenever the system starts from the state j that is certain because that probability is 1 whenever $F_{jj}$is 1 that means the probability 1 if you start from the state j you will definitely come to the state j. Therefore, that is corresponding to $F_{jj}$is equal to 1. And the μ_{jj}that will give what is the Mean Recurrence Time.

The μ_{jj} will give Mean Recurrence Time for the state j. So we are considering the second case in which $F_{jj}$is equal to 1 so that is nothing but the return to the state j whenever the system start from the state j is certain and the small $f_{jj}^{(n)}$will give the distribution of the recurrence time. And our interest is also for the Mean Recurrence Time that can be calculated by using μ_{jj}.

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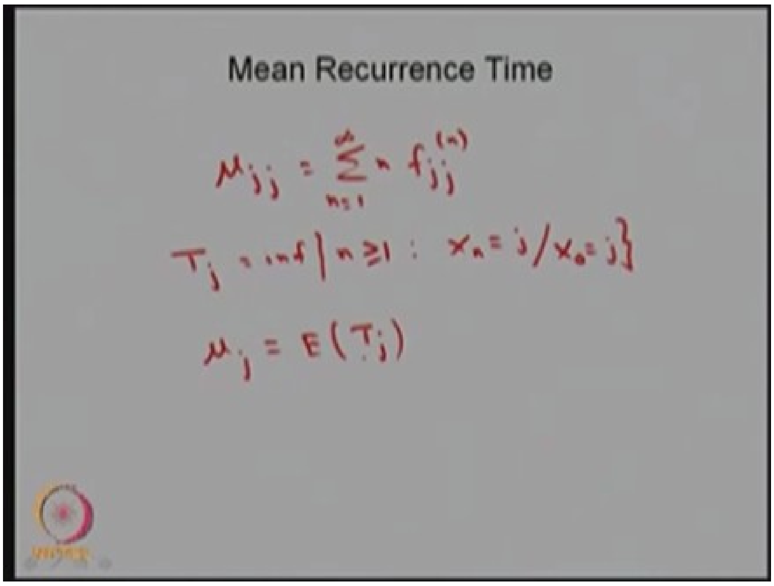

So earlier we have given $\mu_{jj} = \sum_{n=1}^{\infty} n f_{jj}^{(n)}$ by knowing $f_{jj}^{(n)}$you can find out the Mean Recurrence Time for the state j. The same thing can be obtained by using another concept by introducing the random variable that is $T_j = inf\ \{n \geq 1 : X_n = j / X_0 = j\}$.

This is a random variable denoting the first return time to the state j. The first return time, time here it is the step nth step and you find out the first time you return to the state j starting from the state j reaching the state j. So whatever be the first number that integer and that is going to be T_j and this is going to be a random variable. So using this random variable also you can give the definition of Mean Recurrence Time.

Now I can define the Mean Recurrence Time μ_j you do not want the two suffix jj one suffix is enough. So μ_j is nothing but what is the expected or expectation of random variable T_j. So the T_j will give the step that denotes the first return time therefore the expected first passage time that you can write it as the μ_j. So this $\mu_j$and μ_{jj}or both are one of the same.

And here you are finding the distribution and using the distribution you are getting and here you are finding the time and finding the average time using the expectation of T_j. So in both ways one can define the Mean Recurrence Time.

Introduction to Probability Theory and Stochastic Processes
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Lecture - 70
Classification of States

Now we are going for the actual classification of a state using the concept of accessible communicate closed set then communicating class then we have defined a first visit then we have defined a Mean Passage Time or Mean Recurrence Time or Mean First Passage Time. So using these concepts we are going to going to classify the states.

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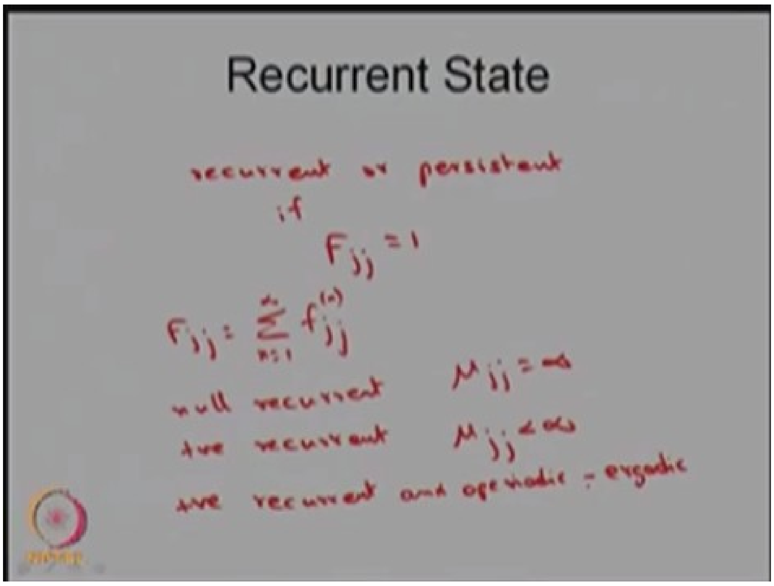

The first definition is a Recurrent State. A state j is said to be recurrent or the other word called a persistent if the F_{jj} equal to 1. If you recall what is F_{jj}. F_{jj} is the probability of ever enter to the state j given that it was in the state j. So the F_{jj} I have given in the summation form of a small $f_{jj}^{(n)}$using the first visit. So, if you recall the F_{jj} is nothing but what are all the possible ways the system can reach the state j as a first visit.

You add all the combination, all the probabilities that is going to be the F_{jj}. So if F_{jj} that means the probability of returning to the same state j if that probability is certain that means if the probability is 1 then that state is going to be the recurrent state. We can classify the recurrent state into two form. One is called the null recurrent. And the other one is called a positive recurrent based on the mean passage time value.

So based on the F_{jj} that is the probability we classify the state is going to be a recurrent state now based on the first passage time distribution the mean first passage time we are going to classify that recurrent state is going to be a null recurrent or positive recurrent accordingly μ_{jj} if it is a finite value then we say that recurrent state is going to be positive recurrent state.

If $\mu_{jj}$is going to be an infinite value that means on average the first passage time is going to be infinite, then that corresponding recurrent state is going to be call as a null recurrent state. So whenever any state is going to be call it as a recurrent state if the probability of ever entering into the state j starting from the state j it is certain or the probability is 1 then that is the recurrent state.

And the recurrent state is going to be called it as a null recurrent if the mean first passage time or mean recurrence time or mean return time is infinity. If that is going to be a finite quantity, then the recurrent state is going to be call it as a positive recurrent state. If any state is going to be a positive recurrent as well as aperiodic then that state is going to be call it as a Ergodic state.

Any state is going to be call it as a Ergodic whenever that state is a positive recurrent as well aperiodic. The aperiodic means the periodicity of that recurrent state is 1 that means that the greatest common divisor of all positive steps in each of the system coming to the same state that value is 1. If the period is 1 and as well as the positive recurrent it should be recurrent as well as positive recurrent that means the mean recurrence time is going to be a finite quantity, then it is going to be call it as a Ergodic state.

In a Markov Chain if all the states are going to be a Ergodic one that means all the states are going to be as a positive recurrent as well as a periodic then we call that Markov Chain itself Ergodic Markov Chain. That means there is a possibility the Markov Chain maybe a irreducible that means you will end up with one class in which all the states are going to form a one closed communicating class.

Suppose if one state is going to be a positive recurrent and aperiodic then all other states are also going to be of the same type and the same period therefore all the states are going to be Ergodic states then that Markov Chain is going to be call it as a Ergodic Markov Chain.

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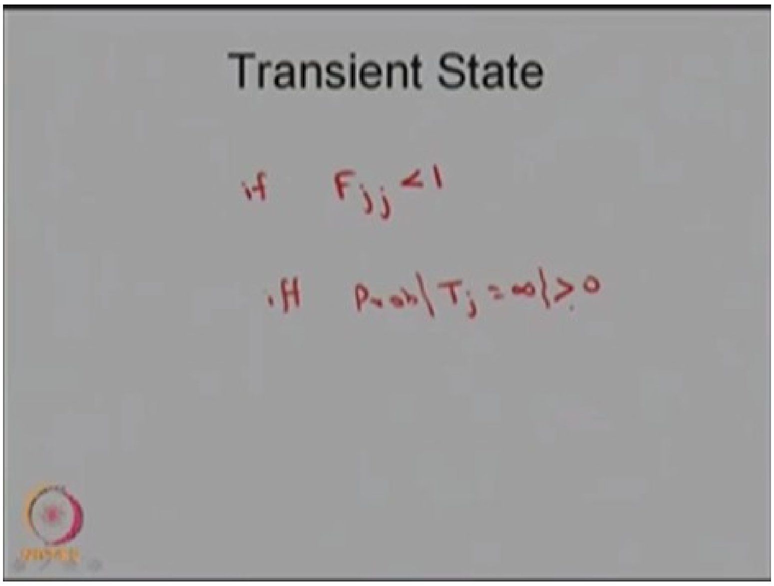

Now I am going to classify the state as a transient state whenever the F_{jj} value is less than 1. If we recall, we have considered only two cases whether the F_{jj} is less than 1 or F_{jj} is equal to 1. Equal to 1 land up recurrent state and $F_{jj}$is less than 1 that gives a transient state. That means the probability of returning to the state j starting from the state j is not certain that means one minus of this probability with that much probability.

The system may not return to the same state j if the system starts from the state j that means with some positive probability because 1 minus these values this value is less than 1 therefore 1 minus of F_{jj} is going to be greater than 0. So with some positive probability the system may not return to the same state if it starts from the state j then that corresponding state is going to be call it as a transient state.

By seeing the one step transition probability matrix or by seeing the state transition diagram of a Discrete Time Markov Chain you can easily come to the conclusion the state is going to be a recurrent state or transient state. Whenever it is going to be a finite number of states this is easy to come to the conclusion if it is a infinite number of states then we need some work to be needed to come to the conclusion whether it is a positive recurrent or null recurrent.

So but easily you can make out the given state is going to be a transient state that you can make out from the state transition diagram or one step transition probability matrix. The conclusion of the state is going to be the transient state that can be given by a random variable T_j also. So the state j is a transient if and only the probability of the $T_j$is equal to infinity and that probability 0. Sorry I had a mistake.

If this probability is strictly greater than 0 the probability of the system returns to the first passage the first passage return time that is infinity if that probability is greater than 0. That means there is a certainly however the system returns to the state j with the infinite amount of time going to take if that event is going to be with the positive probability then that state is going to be the transient state.

So there are two ways you can conclude the given state is going to be the transient state if either $F_{jj}$is less than 1 or the probability of T_j is equal to infinity which is greater than 0.

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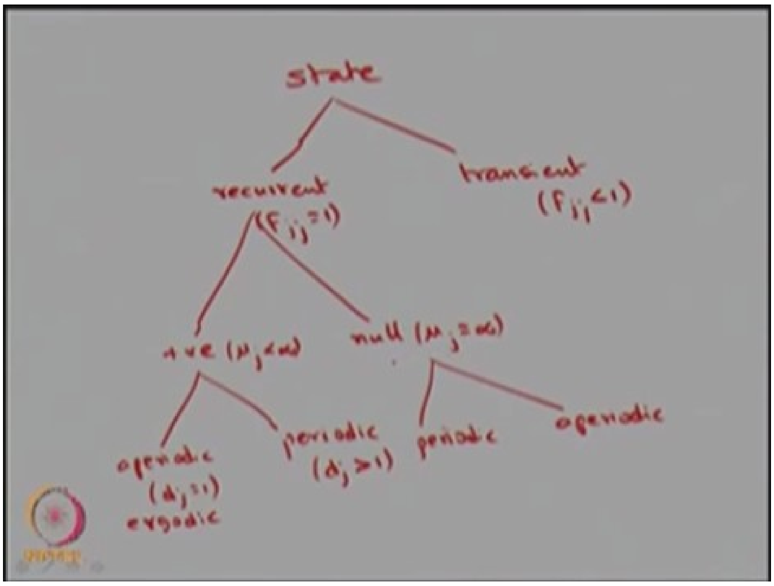

So based on this I can come to the conclusion any state could be recurrent or transient that means this is corresponding to $F_{jj}$is less than 1 and this is corresponding to F_{jj} is equal to 1. I can classify the recurrent sate into two form or either it could be a positive recurrent or null recurrent. Positive recurrent corresponding to the $\mu_j \vee \mu_{jj}$both are one and the same that is going to be a finite value or null recurrent is corresponding to μ_j is equal to infinity.

That means based on the mean recurrence time you can conclude whether it is a positive recurrent or null recurrent. Again, I can classify the positive recurrent into two. One is aperiodic or the other one is periodic. Periodic means that corresponding positive recurrent state that period is greater than 1. Aperiodic means that the d_j is 1. So the periodic positive recurrent state that is going to be call it as an Ergodic state.

Similarly, I can classify the null recurrent state into two. One is periodic and the other one is aperiodic. The absorbing state is a special case of positive recurrent state where the transition probability from a state to itself is 1. So this is a way you can classify the state is a recurrent state or transient state positive recurrent state, null recurrent state again each one could be an Aperiodic or periodic state.

So in this lecture, we started with the few concepts of accessible then communicate then closed set then we have discussed a communicating class. Then, we have discussed what is the meaning of first visit then we have given the first passage time then we have given the mean first passage time distribution or mean recurrence time distribution. So based on those concepts we have classify the state as the recurrent state or transient state.

So this is related to the probability whereas the conclusion of the positive recurrent or null recurrent is related to the average time. So here only it involves the probability that whether in a certain probability the system will come to the same state if the probability 1 whereas here there is uncertainty the system may not come to the state j. If the system starts from the state j.

If there is an uncertainty of returning that means with some positive probability the system would not be back then that state is going to be call it as a transient state. So this you can easily visualize in the state transition diagram of any Discrete Time Markov Chain you can see it whether from by seeing the state transition diagram you can come to the conclusion whether the state is going to be the transient or recurrent.

But through this diagrams you cannot come to the conclusion whether it is going to be a positive recurrent or null recurrent unless otherwise you evaluate this quantity $\mu_j$is going to be $nf_{jj}^{(n)}$

So you find out that summation. So based on the summation values is going to be a finite one or infinite one accordingly that means whether the main recurrence time or mean return time or mean first passage time is going to be a finite quantity or infinite quantity accordingly you can conclude whether that recurrent state is going to be a positive recurrent or null recurrent so here you need a computation.

Whereas by seeing the state transition diagram sometime you can come to the conclusion whether it is a transient state or recurrent state. Now the issue of periodicity. The periodicity is important to conclude whether the limiting distribution exist or not whether that is going to be unique. So you need to find out the Aperiodic or periodic. So if the period is going to be one then that state is going to be call it as aperiodic if the period is greater than 1 then it is a period with that integer.

When it is going to be null recurrent then also you can come to the conclusion whether it is a periodic or Aperiodic. Whenever you have a Markov Chain with the finite number of states then it is easy to find out whether it is going to be a positive recurrent or transient. So a quite good exercise is needed whenever the Markov Chain have a infinite number of states then you need some work to be done for come to the conclusion it is a null recurrent and so on.

In today's lecture with this classification I stop here and all the simple examples and limiting distribution that I will explain in the fourth lecture. Thanks.

Stochastic Processes - 1
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Lecture - 36
Introduction and example of Classification of states

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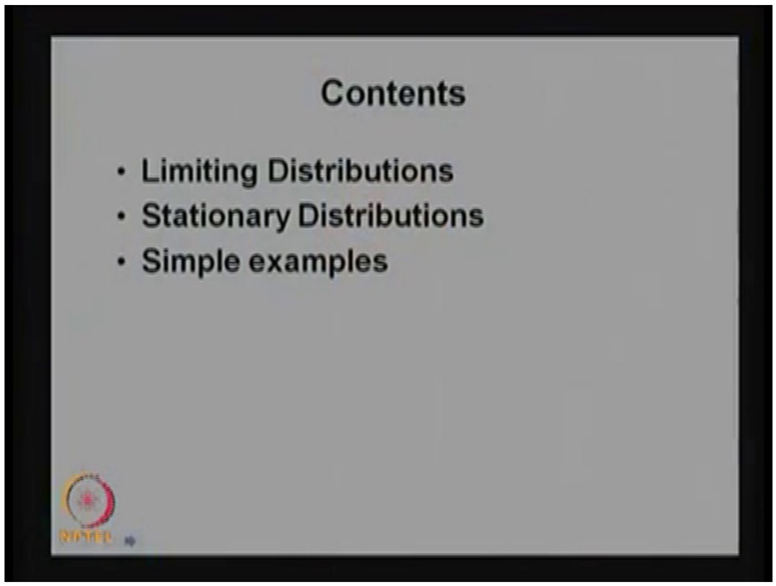

I am planning to explain few examples of classification of the states, then I am going to give the definition of limiting distributions then followed by stationary distributions, then the same examples I am going to explain how to get the stationary distribution if it exists. So, if you recall our earlier lecture, that is our lecture 3, we have given the lot of concepts through those concepts you can classify the states.

The state has a transition state or a recurrent state then the recurrent state can be classified into the positive recurrent state and then null recurrent state and you can find out the periodicity of the states and if the period is going to be 1, then we say that state is going to be the aperiodic state. And if any state is going to be a positive recurrent and aperiodic then we say that state is the ergodic state.

If one step transition probability, if Pi a is equal to 1, then that state is going to be called as a absorbing state. And, we have discussed irreducible Markov chain that means the whole state space is not able to partition into more than one closed communicating classes, then that is going to be closed. That is going to be called as a irreducible Markov chain, otherwise it is a reducible Markov chain.

Now I am going to give simple example through that we are going to explain the classification of the states.

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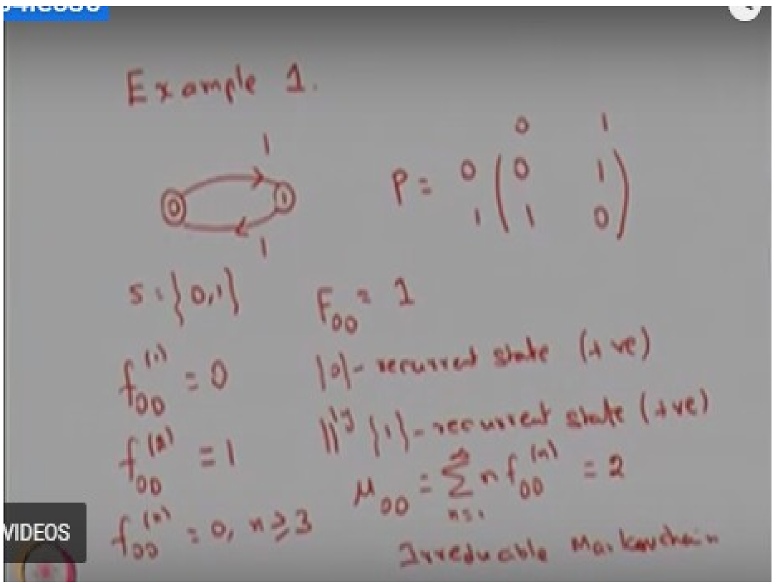

The first example the simplest one. In this first simple example we have only two states, so the states space contains only two elements 0 and 1. The transition, the one step transition probability, from the system is moving from state 0 to 1. that probability is 1. And the system is moving from the state 1 to 0 that probability is also 1. So, the one step transition probability matrix can be obtained from the state transition diagram, both are one and the same.

So, this is the one step transition probability matrix and this the state transition diagram, both are one and the same. So, 0 to 0 that probability 0, 0 to 1 that probability is 1, 1 to 0 that probability is 1, and 1 to 1 is 0. Now we can find out whether these states are going to be a recurrent state or transient state. If you recall to find out the recurrent state or transient state, you have to find out what is the F_{ii}.

So, we start with the state 0, so if you try to find out $f_{00}^{(1)}$, what is the probability that, if the system starts from the state 0 and reaching the state 0 in exactly first step, for the first time. Then that probability is not possible. That is equal to 0. If you try to find out $f_{00}^{(2)}$, first visit to the state 0. Given that started in the state 0 exactly in the second step it reaches the state 0, that is a possible because by seeing the state transition diagram you can make out that the first time.

The system is moving from 0 to 1 and 1 to 0 it is possible coming back to the same state, taking exactly two steps, for the first time. Therefore, $f_{00}^{(2)}$ that probability is 1 and by seeing the state transition diagram, you can visualize, since it comes to the same state, exactly second step. Therefore, all the further steps, for the first time that is not possible. Therefore, all the $f_{00}^{(n)}$, that is going to be 0, for n is greater than or equal to 3.

For n is greater than or equal to 3, the $f_{00}^{(n)}$, is equal to 0. Now if you try to find out what is capital F_{00}that is a probability of ever visiting to state 0, starting from the state 0, that is going to be the summation of $f_{00}^{(n)}$, for all n vary from 1 to infinity if you sum it up, then that is going to be 1. Since $F_{00}$is equal to 1, you can conclude the state 0 is the recurrent state. You can conclude this state 0 is the recurrent state.

Similarly, if you do the same exercise for the state 1, by starting with $f_{11}^{(1)}$what is a probability, $f_{11}^{(2)}$ what is a probability and $f_{11}^{(n)}$and find out the summation, so you land up$F_{11}$is also going to be 1. We can conclude similarly the state 1 that is also recurrent state. Here, after finding the recurrent state, now we can find whether this a positive or null recurrent state.

For that u have to find out what is the mean recurrent state or mean passage time. So, find out what is μ_{00}, that is summation n, $f_{11}^{(1)}$, n varies from 1 to infinity. Here the i is nothing but 00, n $f_{00}^{(n)}$, because this takes the value of $f_{00}^{(2)}$. Therefore, you will get two times 1 and all other quantities are 0, therefore this is going to be 2. And this is going to be a finite quantity.

Therefore, you can conclude state 0 is the positive recurrent state. The same exercise you can do it for μ_{11}, that is also you may land up getting 2. Therefore, you can come to the conclusion state 1, that is also positive recurrent state. So, in this finite discrete time Markov chain you have 2 states. And both are positive recurrent state. And both are communicating states, therefore you have a class that has the two states and the state space is also 0 and 1.

And the closed communicating class is also 0 and 1. Therefore you are not able to partition the state space, into more than one communicating class and so on therefore, we land up this Markov chain is going to be, this Markov chain is going to be the irreducible Markov chain. This Markov chain is irreducible Markov chain, because the state space has only 2 elements. Both the elements are, both the states are communicating each other.

And we land up only one closed communicating class, therefore this is going to be irreducible Markov chain. We can find out what is the periodicity of these states also.

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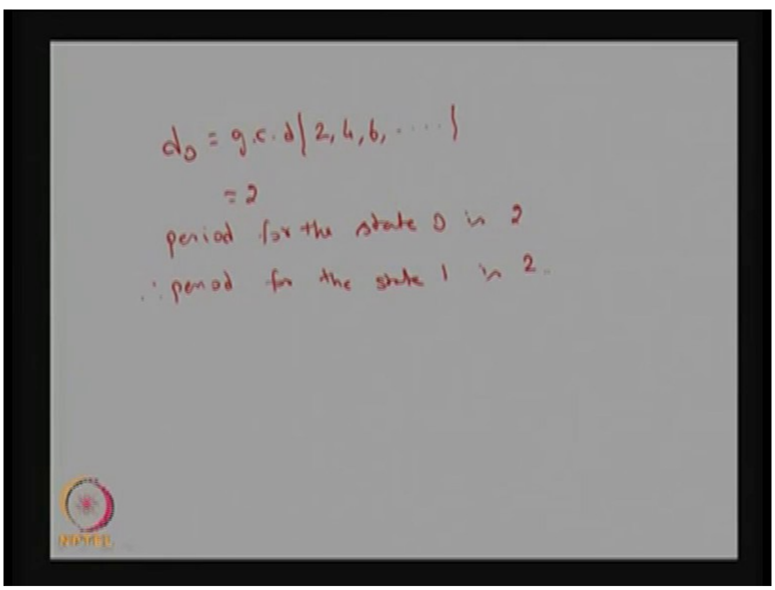

You can find out the periodicity for the state 0, by evaluating the 0, that is nothing but, what is the greatest common divisor of all possible steps, in which the system is coming back to the same state. So, if we find out the system can come to the same state.

If you see the state transition diagram if the system starts from the state 0 coming back to the same state either by 2 steps or 4 steps or 6 steps and so on. You should remember that when you are trying, when you are finding the periodicity, you are finding the number of steps, coming back to the same state, not necessarily the first visit. Whereas the f00 of n, to conclude it is a recurrent state you are finding using the first time reaching that state.

In the exactly and it is tough, so there is a difference. So, the gcd of all the possible steps, in which the system is coming back to the same state. So, it can come back to the same state 0 in 2 steps or 4 steps or 6 steps and so on. so, the gcd is going to be 2, that means the period for the state 2, sorry the state 1, the state 0, period for the state 0 is 2. Similarly, you can find out what is the period for the state 1 also, if you do the same exercise.

But seeing this diagram, you can make out the state 1 also going to have the gcd{2,4,6,8,…}. Therefore, the period for the state 1 also going to be 2. Otherwise also we can conclude, both are communicating states since the period for the state 0 is 2 and since the state 1 communicating with states 0, that means it is accessible in both ways. Therefore, the state 1 is also having the same period.

In conclusion you can make out, if you have a 1 class, with more than 1 states, then all the states are going to have the same period. Therefore, the state 1 is also have the period for the state one, that is also.

That means this example, you have only two states and this is an irreducible Markov chain and both the states are positive recurrent with the period 2. So that is the way using the classification of the states will come to the conclusion of this particular example. Later we are going to find out the limiting distribution and stationary distribution and so on, but for that we need the classification. Here also we can visualize, where the system will be for a longer run, if the system starts from the state 0 or 1.

You can visualize, because it is only two state, by seeing the state transition diagram you can make out, suppose the system start initially in the state 0, at every even number of steps, it will be come back to the stage zero, in a longer run based on the number is going to be even or odd, accordingly the system will be in any one of the states. Similarly, in a longer run, you can make out, if the system starts from the state 1 initially.

All the even number of steps it will be come back to the same state 1, and all the odd number of steps it will be in the state 0, in the longer run also it is going to be happen the same way for even n and odd n, accordingly the system will be in any one of the states. In a longer run also, the system will be any one of these two states only, because it is a irreducible Markov chain. Because these two states are communicating each other.

Therefore, in a longer run, the probability that the system will be in any one of these states, will be some value and only the system will be in any one of these two states only. Later I am going to give the definition of the limiting distribution, through that I am going to explain the same example again.

Introduction to Probability Theory and Stochastic Processes
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Lecture – 72

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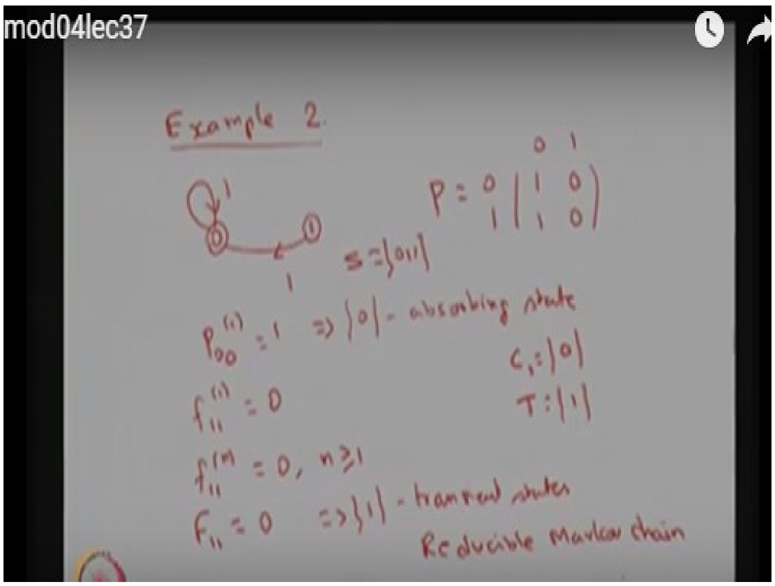

Example 2, here I am going to discuss a reducible Markov Chain. Here also we have an only state. The Probability of system is moving from state 0 to 0 in the next step, the probability is 1 and the system is coming from the state 1 to 0 in 1 step that probability is 1. So, this is the state transition diagram of a Time Homogenous Discrete Time Markov Chain. So, I am going to write what is the 1 Step Transition Probability Matrix for this state transition diagram.

Or for this Discrete Time Markov Chain. So, 0 to 0 one step that probability is 1, 0 to 1 is 0, 1 to 0 is 1, 1 to 1 is 0. You can verify whether this is going to be Stochastic Matrix because each element are lies between 0 to 1 and the row sum is 1. Therefore, this is a Stochastic Matrix so both are equivalent in the state transition diagram and 1 step probability matrix is 1 and the same.

Now we will try to find out what is the Classification of the states. Go for the state 0. The $P_{00}^{(1)}$ that is 1, that is one step a transition of system is moving from state 0 to 0 that is going to 1. This implies the state 0 is an absorbing State. Now we will try to find what is a Classification of a state 1. So, if you find out $f_{11}^{(1)}$ what is the Probability that the system will come to the state 1. Given that it was in the state 1 and the first time we see to the state 1.

Exactly the first step so that is going to be not possible because if the probability is 1. It moved to the state 0 therefore this is going to be 0. And if you find out $f_{11}^{(1)}$, all the subsequent steps also that is also going to be 0 because if the system start from the state 1 in the next step itself it goes to the state 0 with the Probability 1 and it is not coming back. Therefore, now you tried to find out what is the F_{11} that is nothing but the summation of all the f_{ii}'s and that is going to be 0.

If you recall the way you classify the state is going to be recurrent or transient, we said F_{ii} is going to be 1 or F_{ii} is going to be less than 1 so that less than 1 includes F_{ii} is equal to 0. So basically, our interest is to classify whether there is Proper Distribution, the system is coming back to the same state with the probability 1 that is F_{ii} is equal to 1 and all other things we say that a Transient State it includes F_{ii} equal to 0.

So here if the probability 0 the system is not coming back to the state 1, if the system starts from the state 1. This is always a Conditional Probability and this Conditional Probability F_{11} is equal to 0 implies the State 1 is going to be a Transient State. So, whenever any state i, F_{ii} is equal to 1 that concludes the state is going to be recurrent state and whenever the F_{ii} is lies between including 0, excluding 1 that is less than 1, then that state is going to be called it as a Transient State.

Since we have only 2 states that the state space is 0 and 1 and you land up having a 1 absorbing State and 1 Transient State. Therefore, the state space is partition into 1 closed communicating class which has only 1 element and the Transient state is 1. Therefore, I can say the state space S is a partition into closed communicating class C1 which consist of only one element and the collection of all the Transient States that is only one element.

So, this a notation for T collecting all the Transient States in the States space in the DTMC and C1 is the first closed communicating class and which has only 1 element. If any closed communicating class have only 1 element then it is going to be call it as absorbing State. Therefore 0 is an absorbing State and 1 is a Transient State. Since you have a C1 Union T become state Space S. Therefore, this Markov Chain is not an Irreducible Markov Chain.

Therefore, this is called a reducible Markov Chain.

Introduction to Probability Theory and Stochastic Processes
Prof. S. Dharmaraja
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Lecture – 73

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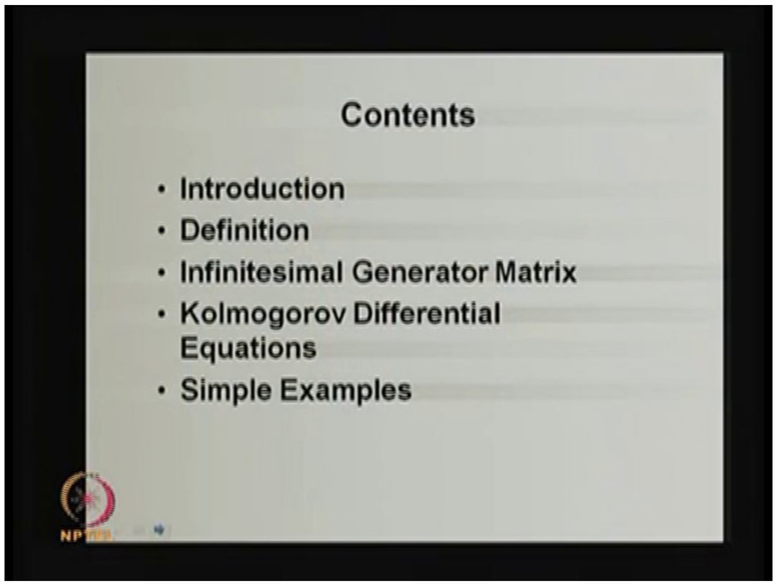

This is Continuous time Markov chain, I am planning for six to eight lectures in this module and I am going to start the lecture 1 with a definition of continuous time Markov chain then the derivation of Chapman Kolmogorov differential equations and I am going to give some simple examples for the continuous time Markov chain and also I am trying to give the stationary and the limiting distributions of continuous time Markov chain in this lecture.

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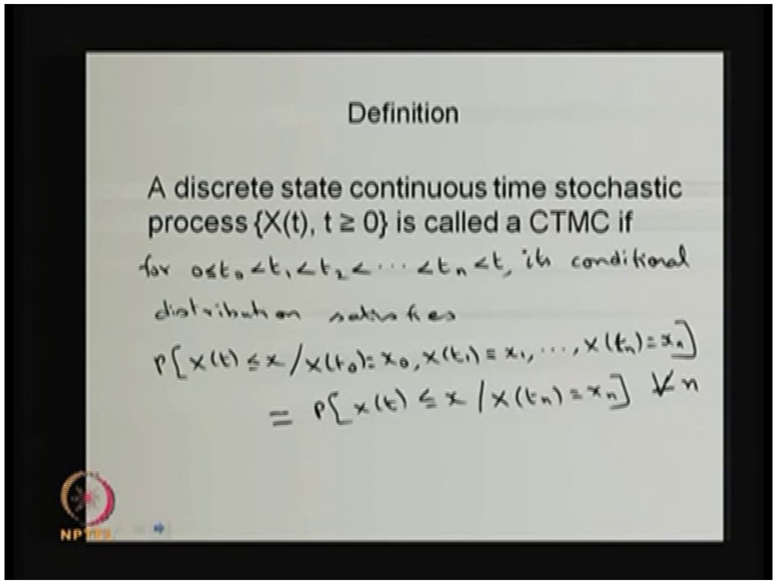

Let me start with the definition, definition of continuous time Markov chain a discrete state continuous time that means the state space is discrete that means the possible values of a the random variable going to take the value for possible values of parameter space that is going to be finite or countably infinite therefore the state space is going to be call it as a discrete.

Continuous time means the parameter space or the possible values of the t that collection is a uncountably infinite, therefore it is called a continuous time that means a parameter space is continuous, so a discrete state continuous time stochastic process $\{X(t), t \geq 0\}$, need not be t greater than or equal to 0 also, but here I am making the very simplest one.

So the X(t) for fixed t is a random variable for every t that collection that is going to be a stochastic process and the state space is discrete and parameter space is continuous and that stochastic process is going to be call it as a continuous time Markov chain if its satisfies the following condition, if you take n time points arbitrary time points n + 1 time points that is t_0 to t_n, you can say it the t0 can be 0 also.

And with this inequality $t_0 < t_1 < t_2 < \ldots < t_n < t$, and you take the any arbitrary t, if this inequality, for fixed t that X(t) is going to be a random variable therefore now we are going to find out the conditional distribution for this n + 1 random variable with the random variable X(t) that means at t_0 you have X(t_0) that's a random variable at t_1, X(t_1) is a random variable.

Similarly, at t_n, X(t_n) is a random variable you have n + 1 random variable with these n random variable given that means it takes already some values with $x_0, x_1, \ldots, x_n$so on respectively, and you are finding the conditional CDF for the random variable X(t), so that means you have n + 2 random variables taken at the arbitrary time points t_0 to t_n as well as small t.

And you are finding the conditional CDF of the random variable X(t) given that already the other n + 1 random variables taken at those arbitrary time points you taken the value $x_0, x_1, \ldots, x_n$ it is taken already these values that conditional distribution conditional CDF if that is same as again it is a conditional CDF of X(t) given the last random variable X(t_n) is equal to x_n.

So this n + 1 time points are arbitrary time points so if it satisfies for all n for every n that means the conditional distribution of n + 1 random variable is same as the conditional distribution of the last random variable if this property is satisfied by the discrete state continuous time stochastic process for arbitrary time points than that stochastic process is called a continuous time Markov chain.

This is very important concept this is called the Markov property that means the t is sort of the future, so what is the probability that the random variable will be in some state at the future time point t given that you know the present state that is where the system is in time point t_n that is small x_n and I know the past information starting from X(t_0) till X(t_{n-1}).

I know the information that means what is the probability that a future the random variable X(t) will be in some state given that it was in the states x_0 at time point t_0, it was in the state x_1 at the time point t_1 so on, latest at the time point tn the system was in the state x_n that is same as what is the probability that the future the random variable will be in some state at time point t given that it is now in the state x_n at the time point t_n.

That means a future given present as well as the past information is same as future given only the present which and independent of the past information that is called the memoryless property or Markov property. So since these properties satisfied by the stochastic process which has the state space is a discrete and the parameter space is continuous than that stochastic process is called continuous time Markov chain, so this is the definition.

Now we are going to give some more properties over the continuous time Markov chain and some simple examples as well as the I am going to explain the limiting distribution and the stationary distribution for continuous time Markov chain in this lecture.

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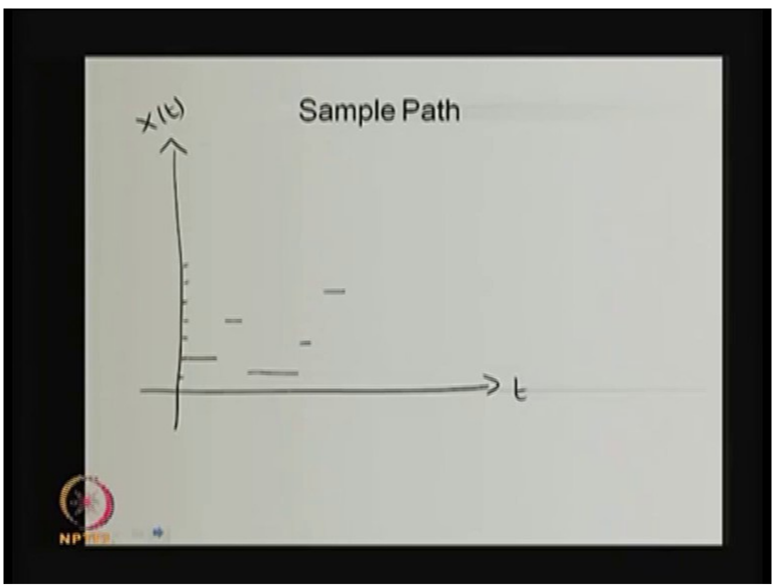

Let me show the sample path over the time t that is x axis, the y axis is X(t), so the system was in some state at time point 0, it was in the same state for some time then it moved into the some other state then it was there in that state for some time then it moved into some other state and so on, if you see the sample path the following observation the system can stay in some state for some amount of time after that it will move to the some state.

So there is no equal interval of system going to be in some state also, it can be some positive amount of time the system can be in the some discrete states, so here the observations are the state space is discrete whereas the parameter space is continuous and the time spent in each state that is going to be a some positive amount of time before moving into any other states so this is the observation in the sample path which I have drawn.

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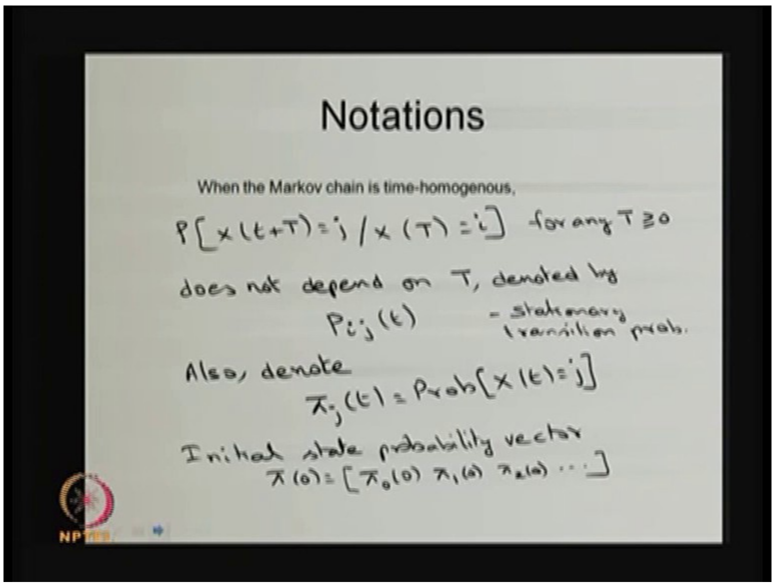

Now I am going for few notations to study or to study the behaviour of a continuous time Markov chain, whenever the Markov chain that means here it is a continuous time Markov chain it is a time homogeneous then the P[X(t+T) = j / X(t) = i], that does not depend on T.

Here we assume that the state changes from i to j, at a future time point t + T, this transition probability says the system was in the state i at the time point t, let me draw the simple diagram.

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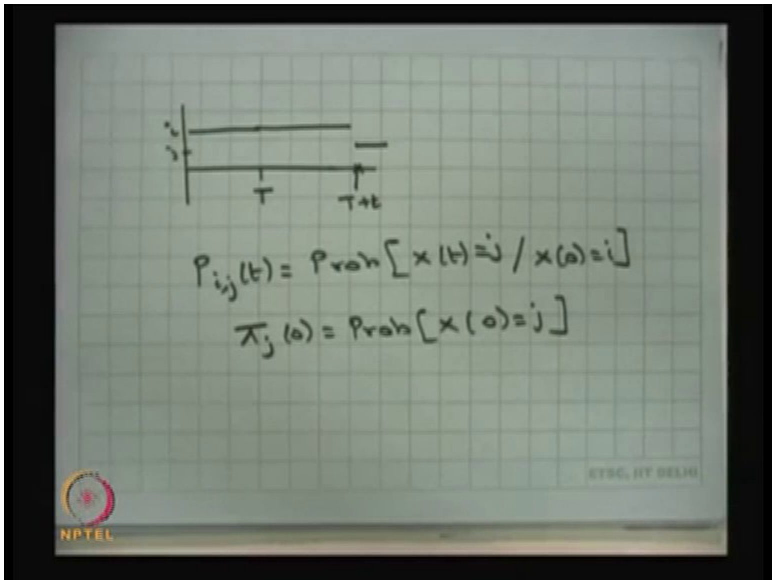

The system was in the state i at the T, then what is the probability that the system will be in the state j. what is the probability that the system will be in the state j at the time point T + t, it is independent of a T, whenever the Markov chain is going to be a time homogeneous for any T greater than or equal to 0, that means actual time does not matter only the length matters, the length of the transition time, that means the small t is matters not the T.

Whenever it is a time homogeneous that is that we can denote it as a $P_{ij}(t)$, because it depends on only the interval not the actual time, therefore it is a function of t, $P_{ij}(t)$, that means that is a transition probability the system so the same thing can be written as the $P_{ij}(t)$, this a notation what is the transition probability that the system was - what is the probability that the system will be in the state j given that it was in the state i at time 0.

Since it is a valid for any interval of T to T + t, it is independent of T therefore I can represent in this transition probability as probability that the system in the state j at time t given that it was in the state i at time 0, this denoted by $P_{ij}(t)$, so this notation you should remember it's a transition probability with the suffix two letters i, j of t, this also call it as a stationary transition probability.

Stationery means it is a time invariant only the length of the interval matters similarly, I am denoting the next notation $\pi_j(t)$, the $P_{ij}(t)$, is a conditional probability, whereas the $\pi_j(t)$ that is a unconditional one, what is the probability that the system will be in the state j at time t, there is a possibility system would have been coming to the state j before time t for at time 0 itself or it would have come before just before t.

Whatever it is this probability will give the interpretation what is the probability that the system will be interested j at time t only it gives the information at that time t, this is unconditional probability. I need another notation for a initial state probability vector also, that is $\pi(0)$, $\pi(0)$, is a vector which consists of entities what is the probability that the system was in the state 0 at time 0.

Therefore, this I can write it as$\pi_j(0)$, that is nothing but what is the probability that the system was in state j at time 0 so this is a meaning of $\pi_j(0)$, what is the probability that the system will be in the state sorry, the system was in the state j at time 0 that is $\pi_j(0)$, with these entities you are framing the vector that is $\pi(0)$,

So in this we are giving you a three notations, one is the transition probability $P_{ij}(t)$, that is a conditional probability, the other one is unconditional probability that is $\pi_j(t)$, and initial state probability vector$\pi(0)$.

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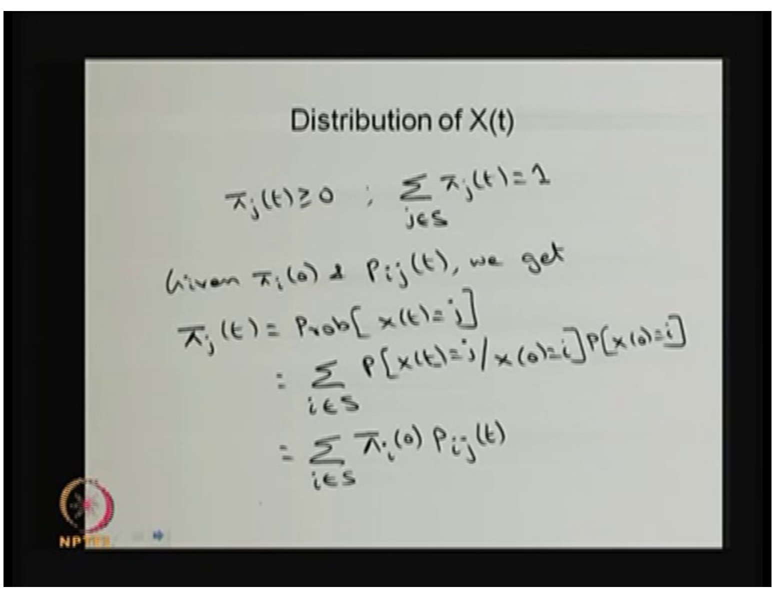

Using these I am trying to find out what is the distribution of X(t) for any time t, for any time t, X(t) will make a stochastic process here it is a continuous time Markov chain, the default one is a time homogeneous continuous time Markov chain and our interest is to find out what is the distribution of the random variable X(t), it has the probability mass function that is $\pi_j(t)$, and if you make a summation over S, where S is the state space that summation is going to be 1.

If I know the initial state probability vector with the entities $\pi_i(0)$,, as well as if I know the transition probability of system moving from the state i to j from 0 to t, I can able to find out what is the probability mass function system being in the state j at time t, that is $\pi_j(t)$ = P{X(t) =j}.

That is same as, I can make a summation I can make a conditional what is the probability that the system will be in the state j at time t given that it was in the state i multiplied by what is the probability that a system was in the state i at time 0, for all possible values of i, where S is nothing but the state space. I know that pi sorry, I know that the P{sX(0) = i}= $\pi_i(0)$.

And this transition probability since the Markov chain is a time homogeneous, so 0 to t that is nothing but 0 to 0 is the time point and t is any time point and i is the state in which the system was in the state in the at time 0, so $P_{ij}(t)$, if I multiply $\pi_i(0) P_{ij}(t)$for all possible values of i, I will get the probability that the system will be in the state j at time t, that means if you want to find out the distribution of X(t) for any time t.

I need initial state probability vector as well as the transition probability of system moving from one state to other states, this is given usually the initial state probability vector is given, so what do we want to find out his $P_{ij}(t)$, so how to find the $P_{ij}(t)$, that derivation I am going to do it in the another two three slides.

Introduction to Probability Theory and Stochastic Processes
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Lecture - 74

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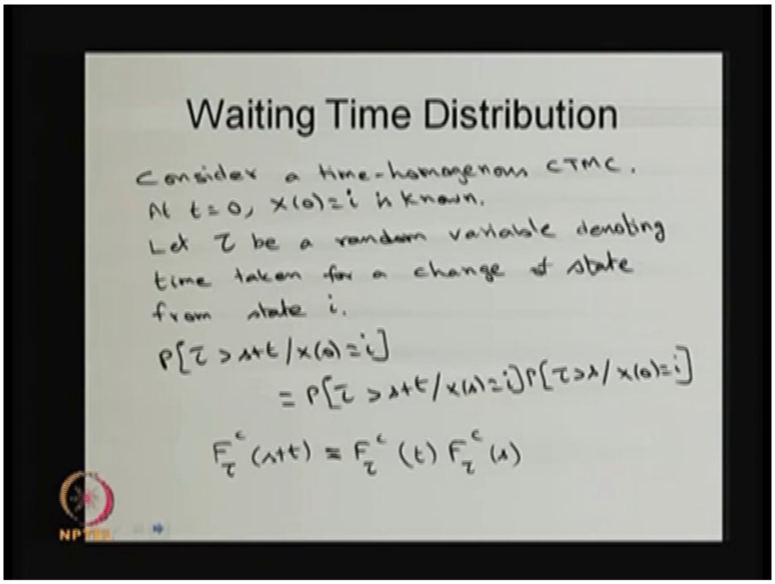

So before going to the P_{ij}, you see the sample path of a time homogeneous continuous time Markov chain.

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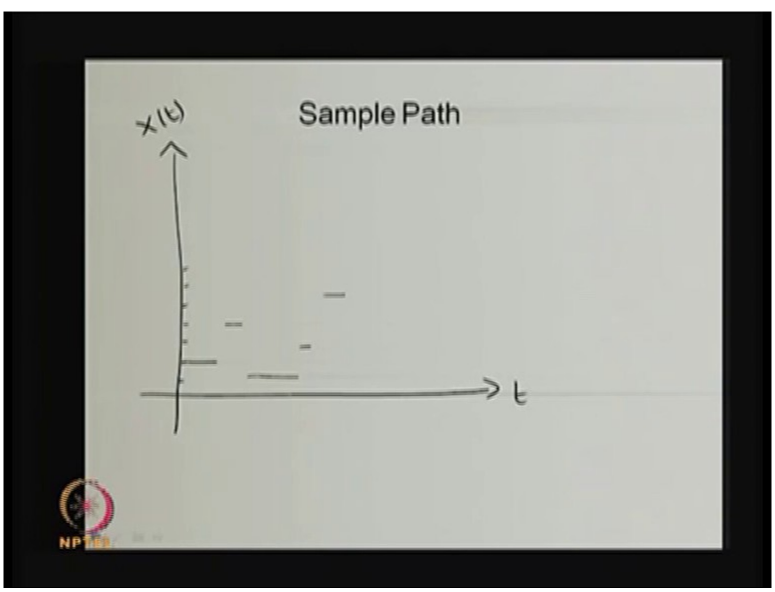

As I said the system is staying for some positive amount of time in any state before moving into any other states, our interest is what is the distribution or what is the waiting time distribution of system being in any state before moving into any other states that is our interest to find out. So how we are going to find out that I am going to explain that is called the Waiting Time Distribution.

That means what is the distribution of a time spending in any state for a time homogeneous continuous time Markov chain before moving into any other states, I assume that at time 0 system was in the state i, that means X(0) = i, that is known or the P{X(0) = i} = 1. Let me make out the random variable τ that is a random variable denoting the time taken for a change of state from the state i,

Change of state means it does not matter which state it goes my interest is to find out what is the waiting time distribution for the state i, the time spent in the state i, for that let me make a simple graph. So this is t and this is X(t).

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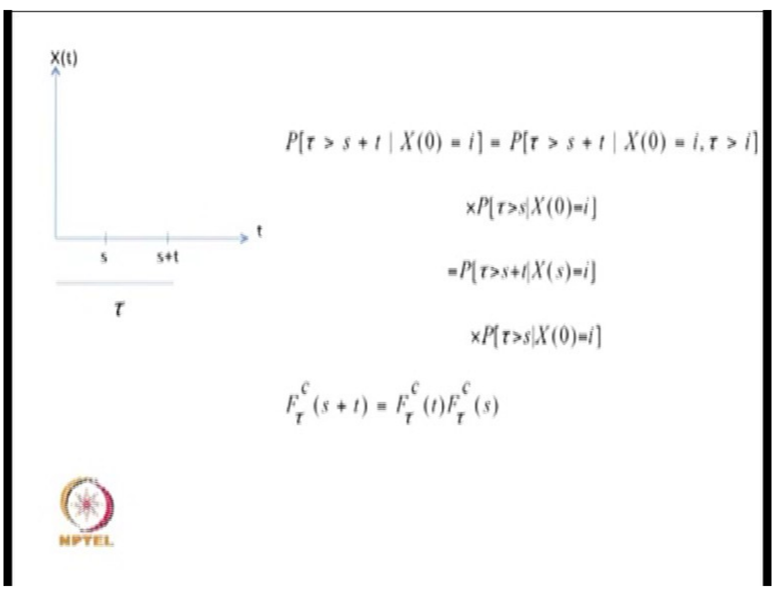

Suppose you assume that the system was in the state i at the time point 0 after some time it moved into some other state, okay at the time point s it was in the state i, at the time point t also it moved into the some other state, so the τ here is nothing but the time spent in the state i from here to here so that is a random variable. So what I am going to do I am going to find out what is a compliment CDF for the random variable τ.

That is a - what is the $P\{\tau > s+t / X(0) = i\}$, that is same as the probability of the $\tau > s + t$ given that I can introduce one more condition $\tau > s$, than I can multiply by using the total theorem of $P\{\tau > s / X(0) = i\}$. That is same as, the first one I can rewrite as a $P\{\tau > s+t / X(s) = i\}$.

Because $X(0) = i$, as well as $\tau > s$, where τ is the time spent in the state i, therefore I can make out $X(s) = i$, by combining these two concept multiplied by the $P\{\tau > s / X(0) = i\}$, that is same expression here. Now the $P\{\tau > s+t / X(s) = i\}$, that I can rewrite because the - this Markov chain is a time homogeneous Markov chain.

So, the s to s + t, that is same as the complement CDF of the random variable τ for the time t, because it is s to s + t, since its Markov chain is a time homogeneous only the length is matters that is the interval of length t. Therefore, this is nothing but the compliments CDF for the random variable τ the time point t multiplied by, this is nothing but 0 to s, so this is the compliment CDF of the random variable τ the time point s.

Whereas the left hand side is the compliment CDF for the random variable τ for the point s + t. So what we got the result is the compliment CDF of the unknown random variable τ at the time point s + t that is same as the product of complement CDF at the time point s and t, so this is valid for all s and t greater than 0, so we have to find out what is the random variable or what is the distribution going to satisfies this compliment CDF at the time point s + t, same as the product of compliments CDF at the time point s and t.

If any distribution satisfies this compliments CDF property then we can find out the distribution further and the variable τ, so in this derivation we have use the time homogeneous property as well as the total probability rule as well as we have use the Markov property therefore it lands up the compliment CDF satisfying this equation.

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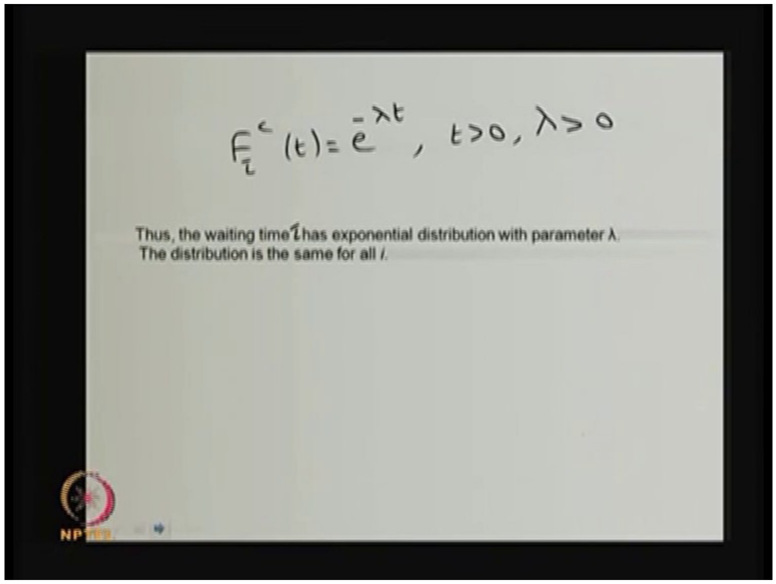

Now we have to find out what is the distribution going to satisfies this property so if you substitute any function with e power any parameter λ with the exponential of $e^{-\lambda t}$ the previous equation is going to be satisfied as long as the function is of the form $e^{-\lambda t}$, for λ is greater than 0 and t is greater than 0, since the compliment CDF is $e^{-\lambda t}$.

Therefore, the CDF of the unknown random variable τ, that is 1 - $e^{-\lambda t}$ for t greater than 0, for some λ and you know that if the CDF of the random variable is 1 - $e^{-\lambda t}$ for t greater than 0 and λ > 0, then that random variable is a exponentially distributed random variable.

So we can conclude the amount of time or the time taken by the system staying in the state i, and that time is exponentially distributed that is a continuous random variable whose distribution is exponential distribution with the parameter λ, even we can specify λ_i, that means it is going to be a function of it depends on the i, that means if the random variable is going to spent in some state.

And that is always exponential distribution with some parameter λ and that parameter λ may depend on the state i, that means if I go back to the sample path I can say that the - that the time the system spending in this particular state that is exponential distributed with some parameters then it moved into some other state. The time spending in this state that is also exponentially distributed with some other it could be some other parameter.

It depends on that particular state then it moved into some other state and the time spending in this state that is also exponentially distributed and later we can conclude all these the time spending in each state because of it is a Markov property satisfied. The time spent in this state - the time spent in this state all are exponentially distributed which is independent of the other.

So all are going to be mutually independent random variables then only the Markov property is going to be satisfied that means whenever the system is moving from one state to another state we will have a exponentially distributed time spending in each state and they are form a mutually independent.
And since the exponential distribution has the memoryless property, the system spending in this state, if you just observe at any time t and what is the probability that the system will be for some more time in the same state given that it was spending already this much time in this state, then that is also exponentially distribution because of memoryless property of exponential distribution and which is independent of how much time spent in the same state already. Therefore, the Markov property is going to be satisfied throughout the time whether the system spending in this state or the other state and so on.

So the Markov property is going to be satisfied for all the time points and that time spending in each state is exponentially distributed and all the random variables spending in each state all are going to be mutually independent random variables. Now, we found out what is the time spending in each state and that is exponential distribution with some parameter λ_i and the distribution is same for all i, whereas the value of the parameter λ maybe depends on the i.

Introduction to Probability Theory and Stochastic Processes
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Lecture – 75

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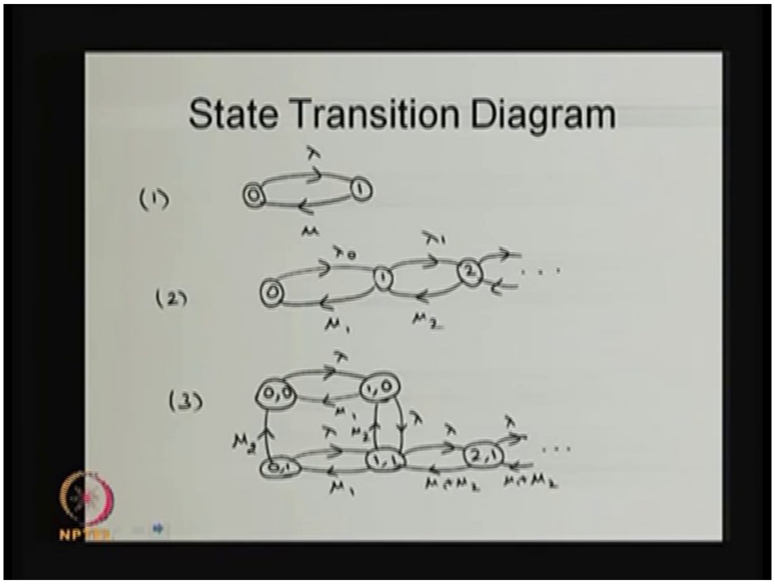

Now I am going to give few state transition diagrams for the time homogeneous continuous time Markov chain, you see the first example it has only two states 0 and 1 so the state space S is 0, 1 and the time spending the state 0, before moving into the state 1, that is exponentially distributed with the parameter λ, once the system come to the state 1 the time spent in the state 1 before moving into the state 0 that is exponentially distributed with the parameter μ.

λ is strictly greater than 0 and μ is also strictly greater than 0, that mean you know the exponential distribution has been 1 divided by the parameter, therefore the average time spent in the state 0, before moving into the state 1 that is 1/ λ, the average time spending in the state 1 before moving in to the state 0 that is 1/μ, since it is a two state so over the time the system will be in the state 0 or 1.

And you can classify the states also, the way we have discussed in the discrete time Markov chain, since both the states are communicating, both the states are accessible from each other - each other direction therefore both the states are communicating each other, since the state space is 0 and 1 and both the states are communicating each other therefore this is a irreducible Markov chain, for a irreducible Markov chain all the states are of the same type.

For a finite Markov chain we have at least 1 positive recurrent state therefore both the states are going to be a positive recurrent state, but here there is no periodicity for the continuous time Markov chain, therefore we can conclude the first example both the states are a positive recurrent and the Markov chain is irreducible Markov chain.

So the continuous amount of time system spending in state 0 and 1 that is exponentially distributed with the parameters which I discussed earlier. Now I am moving into the second example in the second example we have a state space is countably infinite and the system spending in this state 0 before moving into the state 1 that is exponentially distributed with a parameter λ_0.

Whereas the state 1, the system can spend exponential amount of time, amount of time spending in the state 1 before moving in to the state 2 that is exponentially distributed with parameter λ_1, and similarly, the system spending in the state 1 before moving into the state 0, that is exponentially distributed with the parameter μ_1.

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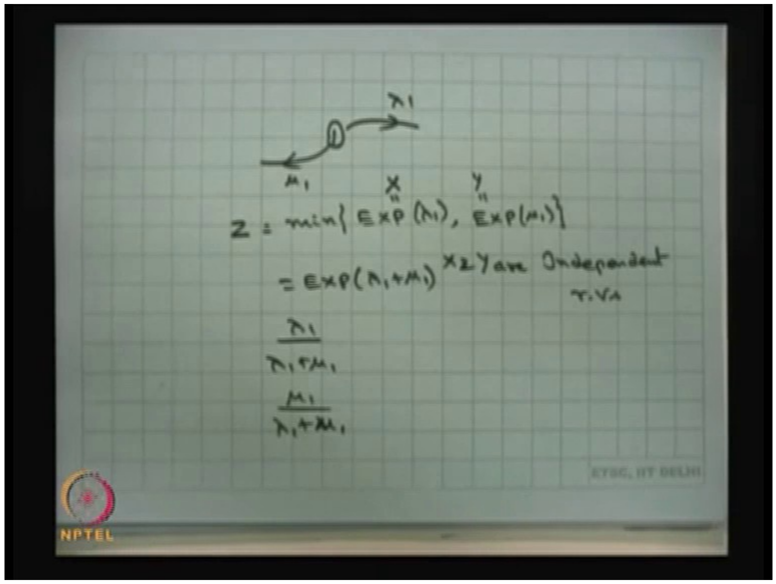

Therefore, this is μ_1 and this is λ_1, therefore the time spending in the state 1 before moving into any other state that is going to be minimum of the exponentially distributed with the parameter λ_1, one random variable you can call it as X and the you can call it as a another random variable that is exponentially distributed with the parameter μ_1.

Therefore the amount of time spending in the state 1 before moving into any other state that just now we have concluded that waiting time distribution is exponentially distributed that will come from here also, so here these two random variables are independent, X and Y are independent random variables both the random variables are independent.

Therefore the time spending in the state 1 before moving into any other state that is going to be minimum of the random variable with exponential distributed parameter λ_1 and the random variable which follows exponential distribution with the parameter μ_1, you know that the minimum of two exponential as long as both the random variables are independent random variable, then this is also going to be exponential distribution with the parameters - with the parameter $\lambda_1+\mu_1$, as long as both random variables are independent and both are exponential you can do it as a homework minimum of two exponential are going to be exponentially with the parameter $\lambda_1 + \mu_1$.

Therefore, the time spending in the state 1 that is exponential distribution with the parameter $\lambda_1 + \mu_1$. Also one can discuss what is the probability that the system moving into the state 2 before moving into the state 1 that is a $\frac{\lambda_1}{\lambda_1 + \mu_1}$, similarly what is the probability that the system moving into the state 0, before moving into the state 2, that is $\frac{\mu_1}{\lambda_1 + \mu_1}$, that also you one can find out, so what is the conclusion here is the time spending in the state 1, that is exponential distribution with a parameter $\lambda_1 + \mu_1$.

Similarly, the time spending in the state 2 that is suppose if it is λ_2 then $\lambda_2 + \mu_2$, so this is the one type of a continuous time Markov chain. The third example this is also continuous time Markov chain. It is sort of two dimensional Markov chain with the labelling with a 0, 0; 1, 0; 2, 0 and so on, so all the labelling which is parameter for the exponential distribution.

So the change from the discrete time Markov chain state transition diagram and the state transition diagram of a continuous time Markov chain, here there is no self-loop and the labels are the parameters for exponential distribution, whereas the discrete time Markov chain it is a one step transition probability going from one state to other states. Here the labels the arrow gives the - the time spending in the state exponential distribution with the parameter $\lambda_0$and moving into the state 1 and so on.

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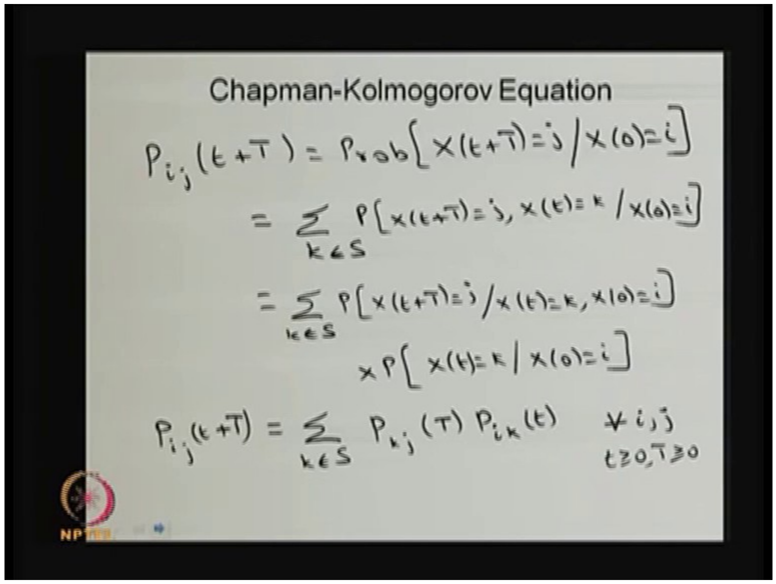

Now I am going to find out how - now I am going to find out the $P_{ij}(t)$, for that I am going to do the derivation starting with the Chapman Kolmogorov equation. let me start with what is the transition probability of system in moving from i to j, during the time 0 to t + T that is nothing but what is a $P\{X(t+T)=j/X(0)=i\}$.

That is same as I can in between make a some other state I can make a one more state k at time point t for all possible values of k also I will get the same result, that is same as I can make a summation over k, k belonging to S, S is state space that is same as what is the $P\{X(t+T)=j/X(0)=i\}$.
As well as it was in the state k at small t also multiplied by what is the transition probability of system moving from 0 to t, from the state i to k, that is same as, the first conditional probability you see this is same as the Markov property in which we have discussed in the definition of a continuous time Markov chain there I have discussed the CDF Cumulative Distribution Function here it is the probability mass function where this is a conditional probability mass function.

What is the conditional probability mass function of the system will be in the state j at time point small t +T given that it was in the state i at the time point 0 as well as it was in the state k at the time point t, and you know that $0 < t < t + T$, because the way we made it as all these values are greater than 0, therefore by using the Markov property of a continuous time Markov chain.

So this is same as what is the probability that the system was in the state k at time t and move into the state j at the time point t + T, again you use the time homogeneous property, first you use the Markov property, therefore this is a transition probability of t to t + T moving from the state k to j, then use the time homogeneous property therefore only the length matters therefore t to t +T that is same as 0 to T.

Therefore the system is moving from the state k to j, from 0 to T, that is $P_{kj}(t)$, the second one, it's a transition probability system is moving from state i to k during the interval 0 to T, therefore this is i to k of t, so this is valid for all i, j with the t greater than or equal to 0 and T is also greater than or equal to 0, therefore the left hand side is the transition probability of system is moving from the state i to j, from 0 to t + T.

That is same as summation over I can rewrite in a different way i to k in the interval 0 to small t, k to j instead of small t to small t + T, because of the time homogeneous and I am just making 0 to T, therefore this is valid for all values of k summation, this equation is called the Chapman Kolmogorov equation for a time homogeneous continuous time Markov chain.

Because here for this transition probability we have used Markov property as well as the time homogeneous property also, therefore this is a Chapman Kolmogorov equation of a transition probability of system moving from i to j, in small t + T can be broken into product of these for all possible values of t, so like this you can break it many more ways with the summation for all for different state of k.

Using this, we are going to find out the transition probability of $P_{ij}(t)$, you remember to find out the distribution of X(t) you need an initial state probability vector as well as the transition probability $P_{ij}(t)$, the initial state probability vector is always given you have to find out the $P_{ij}(t)$, once you know the $P_{ij}(t)$, you can find out the distribution of X(t) for any time t.

Introduction to Probability Theory and Stochastic Processes

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Lecture – 76

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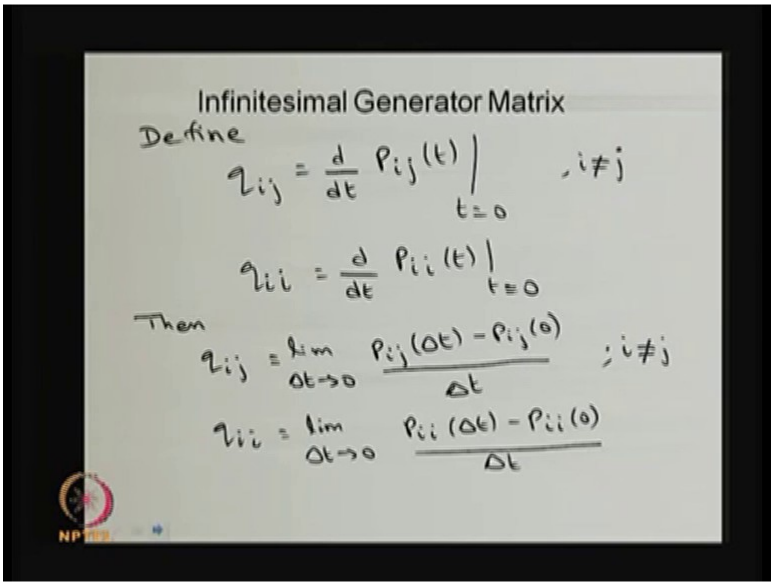

I am going to define the quantity called q_{ij} and later this is going to form a matrix that is going to be called it as a Infinitesimal Generator Matrix. So let me start with the definition q_{ij} that is nothing, but take a derivative of $P_{ij}(t)$that is a function of t. You can find out the derivative. It is differential function only, so you take a derivative then substitute a t equal to 0 for all i is not equal to j.

Then you define q_{ii}that also in the same way separately because the $q_{ii}$the diagonal element is going to be different from all other elements therefore I am defining separately. You know how to find out the derivative, derivative of $P_{ij}(t)$ with respect to t that is nothing, but the limited Δt times to 0 the difference divided by the Δt. Since a $P_{ij}(t)$ is a transition probability of system moving from i to j.

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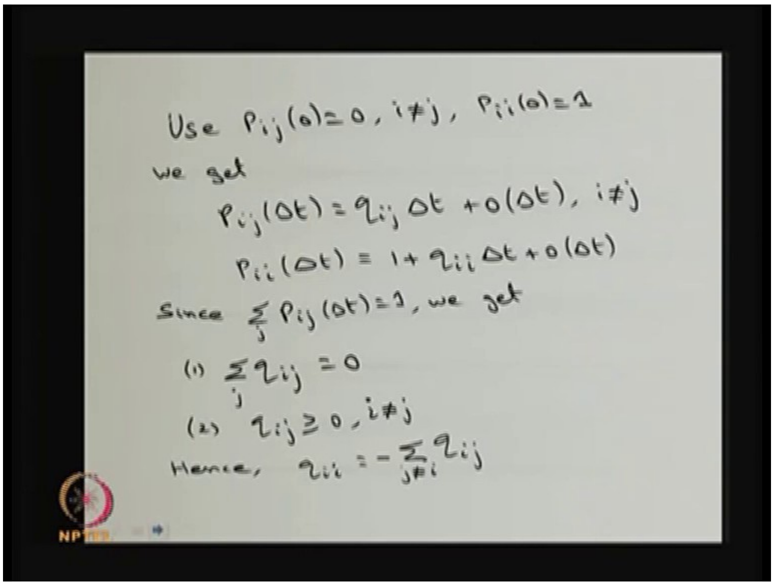

You can use $P_{ij}(0)=0$ for i is not equal to j for j is equal to i that is $P_{ii}(0)=1$,that means what is a transition probability of system moving from the state i to i in the interval 0 that is same as 1 that probability is 1. So use this in the previous limit in this $P_{ij}(0)=0$ and $P_{ii}(0)=1$ you substitute then the limit Δt tends to 0. Therefore, the P_{ij} (Δt) this will go to this side.

So q_{ij}Δt therefore this is going to be P_{ij} (Δt) is nothing, but the q_{ij}Δt + o(Δt) that means as Δt tends to 0 this whole quantity will tends to 0. Similarly, you substitute $P_{ii}(0)=1$ here therefore $P_{ii}(0)$Δt that is same as this will come to this side. So 1 + q_{ii}Δt + o(Δt). So this o(Δt) that also tends to 0 as a Δt tends to be 0.

You know that the summation of P_{ij} even at the time point of Δt the small negligible time point of Δ t at that time also over the i that is equal to 1. Therefore, if you sum it up you can conclude the left hand side is a probability, right hand for i is not equal to j you have q_{ij} whereas that is second expression you have 1+ q_{ii} therefore using the property of $\sum P_{ij}=1$ you will get a $\sum_j q_{ij}=0$.

When you add both the equations for all j you will get a $\sum_j q_{ij}=0$ as well as all $q_{ij}$the quantities are going to be greater than or equal to 0 from the first one because the left hand side is a probability and this is multiplied by the Δt. A Δt is always greater than 0. Therefore, the q_{ij} is going to be greater than 0 for all i not equal to j whereas if i add over all the j that is going to be 0.

Therefore, you will get the q_{ii}that is nothing, but you make the $\sum_{j \neq i} q_{ij}$ then you make a minus sign so that is going to be the q_{ii}. That means the diagonal element is nothing, but make the row sum except the diagonal term and put the minus sign that is going to be diagonal term. Therefore, when you make a row sum that is going to be 0. The details of the proof can be found in the reference books.

So the quantity q_{ij}that has the property the row sum is going to be 0 and other than the diagonal elements are greater than or equal to 0. Therefore, the diagonal element is going to be summation of all the other terms with the minus sign.

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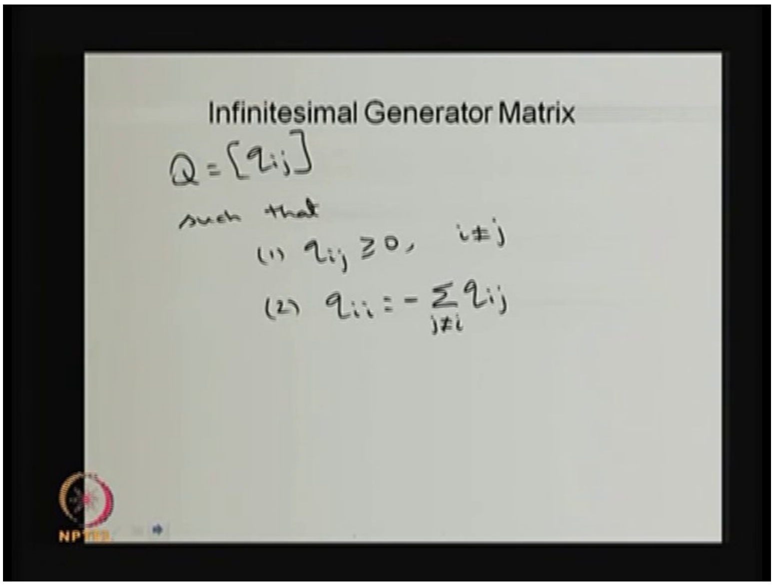

So using this we can make a matrix that is going to be Q matrix. The entity is a q_{ij}such that satisfies the property $q_{ij}$is always greater than or equal to 0 for i is not equal to j whereas the diagonal element is minus of summation.

Therefore, it has the property the row sum is going to be 0. So the difference between this matrix and the one step transition probability matrix in the DTMC that is a probability matrix so the entries are probability values from 0 to 1 and the summation row sum is 1 whereas here because $q_{ij}$are obtained by differentiating the P_{ij}. These are all the rates and these rates are always greater than or equal to 0 other than the diagonal elements and the diagonal elements are a minus with the summation of all other row elements.

So this matrix is called Infinitesimal Generator Matrix. Some books they use the word rate matrix also and whereas here the rates are placed in other than the diagonal elements and sum of the rates could be 0. That means the probability of system moving from that particular state to that particular state is not possible that probability is 0 or there is a in a small interval of time there is the transition is not possible.

So whenever the rates are greater than 0 that means there is a positive probability that the system can have a transition of system moving from i to j.

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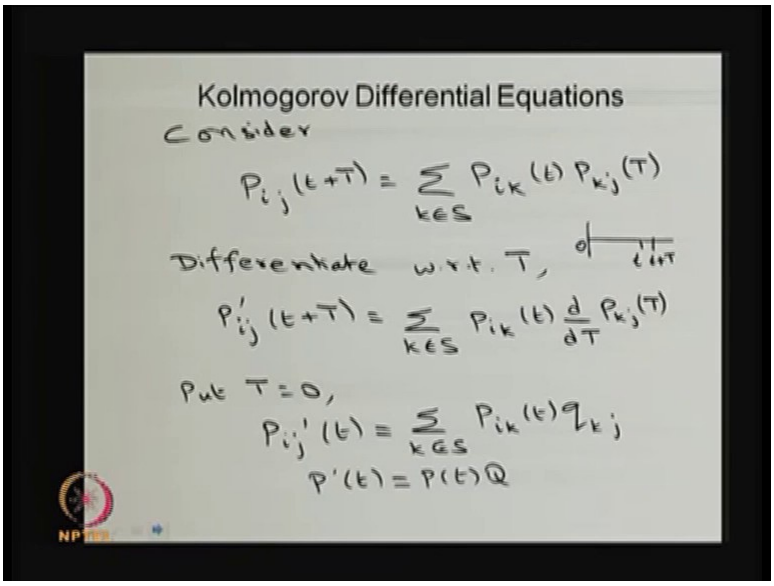

So we have defined the Q matrix. Now using the Q matrix we are going to find out the $P_{ij}(t)$. So let me start with the Chapman–Kolmogorov equation. Now I am going to differentiate with respect to T that means I make the interval 0 to t + T as a 0 to t then I make a t to t+T differentiate with respect to T. Therefore, the left hand side is going to be I have written with a dash so the derivative come inside the $P_{kj}(t)$.

Then I am substituting T = 0. So basically I am making a system to move from state 0 to t then there is a smaller interval of time from t to t +T that is the interpretation of this. Then substituting T = 0 I will get the left hand side is going to be $P_{ij}'(t)$ that is same as the summation over this. Whereas this is nothing, but the way we have defined the Infinitesimal Generator Matrix entities.

So this is nothing, but the q_{kj} that the rate in which the system is moving from the state k to j. In a matrix form I can make it as a $P_{ij}(t)$is going to form a matrix. So the $P'(t)$ that is same as a P(t)Q. So this is the matrix and P(t) is also matrix and this is the $P'(t)$ means each entities are differentiated with respect to time t. So this is in the matrix form and this equation is called a forward Kolmogorov differential equation because the derivation goes from 0 to t.

Then t to t +T were considering as a very small interval of time. Therefore, this equation is called forward Kolmogorov differential equation.

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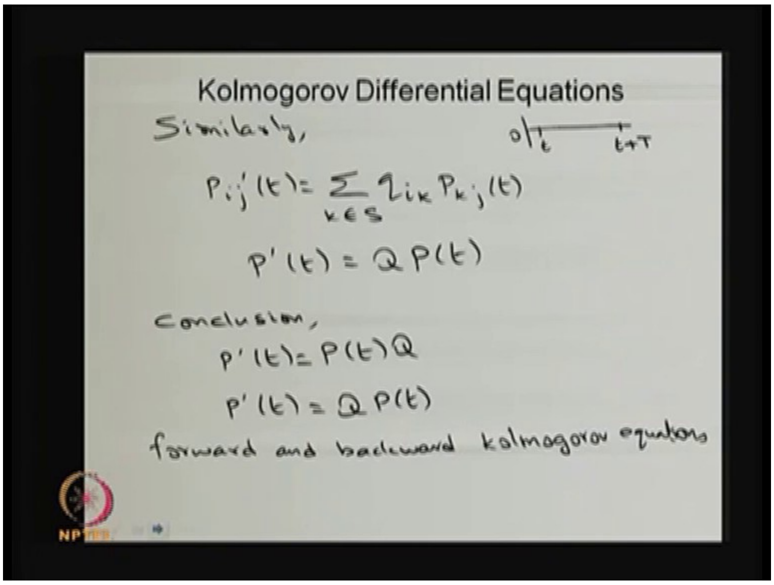

In the same way, if you do 0 to t that has a that has a small interval of time and a t to t + T then I will get the $P'(t)$ =QP(t) that is called the backward Kolmogorov differential equation. Whether you frame a forward equation or a backward Kolmogorov equation if you solve that equation you will get the $P_{ij}(t)$. If you solve $P'(t)$ =P(t)Q that is a forward equation.

The $P'(t)$ = QP(t) that is a backward equation. If you solve the equation with the initial condition because it is a differential equation so you need a initial condition what is a probability, what is a transition probability of system moving from i to j at time 0. If you know the initial condition by supplying that solving this equation you will get the $P_{ij}(t)$. Once you know the $P_{ij}(t)$ then you can get the distribution of X(t).

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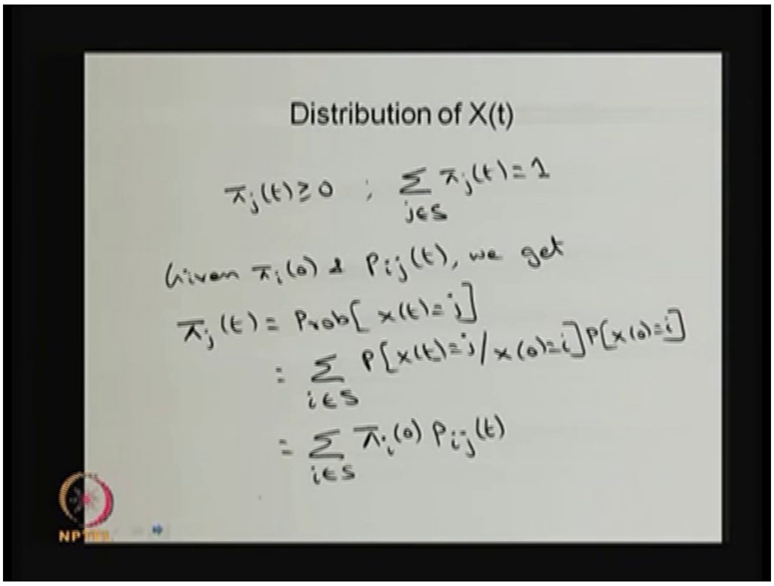

So once you know the $P_{ij}(t)$ the given is $\pi_i(0)$ and by solving that forward or backward Kolmogorov differential equation you will get the $\pi_j(t)$ using these two you can get the $\pi_j(t)$. So for a given $\pi_i(0)$ and $P_{ij}(t)$ that means the transition probability and the initial state probably vector one can find out the distribution of X(t). So in this lecture I have started with the Markov process then I have discussed the definition of a Continuous Time Markov Chain and also I have given what is the distribution of time spending in any state before moving into any other state.

And also I explain the Infinitesimal Generator Matrix using that how to find out the transition probability of $P_{ij}(t)$ from the Chapman-Kolmogorov equation and we got a forward as well as the backward Kolmogorov differential equations by solving a forward or backward Kolmogorov differential equation one can get the $P_{ij}(t)$ that is the transition probability. Using this equation, you can get the π_j¿) that is nothing, but the distribution of X(t).

With this, let me stop this lecture and in the next lecture I will go for simple example of a

Continuous Time Markov Chain as well as the stationary limiting distribution and steady state distribution in the next lecture.

Introduction to Probability Theory and Stochastic Processes

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Lecture – 77

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In the lecture 2, I am planning to discuss the limiting distribution, stationary distribution and a steady state distribution followed by that I am planning to give a description about the birth, death processes and also some simple examples for the limiting distribution, stationary, steady state distributions and birth, death processes.

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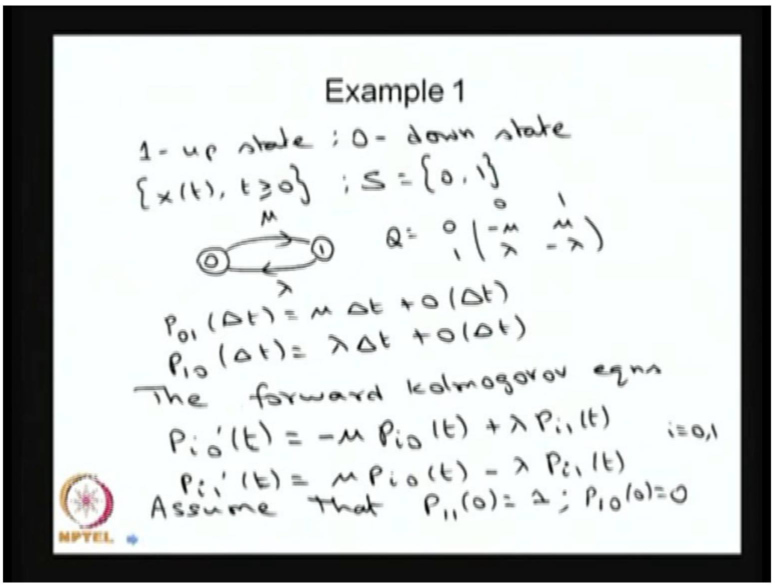

Before I go to the limiting distribution, let me give the example for the Continuous Time Markov Chain to get the time dependent solution. This example is the very simplest example that is a two state Continuous Time Markov Chain the default one is a Time Homogenous. The state space is 1 and 0. One you can consider as upstate or operational sate and 0 is a downstate or non operational state.

So this can be visualized for any model in which the whole dynamics can be described with the two state and Markov property is satisfied. The system going from the state 1 to 0 or the time spent in the state 1 before moving into the state 0 that is exponentially distributed with a parameter λ. Once it is failed that means the systems is in the downstate. The time spent in the repair time that is exponentially distributed.

The parameter μ so once the repair is over the system is in operational state therefore it is in the upstate. So 0 is related to the downstate and 1 is related to the upstate and μ is nothing, but the mean $1/\mu$ is the Mean time for the repair and $1/ \lambda$ is the mean time of a failure and the failure time is exponentially distributed with the parameter λ and the repair time is exponentially distributed with a parameter μ.

This is a state transition diagram for the two states CTMC. The corresponding Q matrix the Infinitesimal Generator Matrix that it consists of a two cross two matrix. The system going from the state is 0 to 1 that rate is μ. The system going from the state 1 to 0 that rate is λ and the diagonal values are minus of summation of upper values that row sum. So 0 to 0 is $-\mu$ and 1 to 1 is $- \lambda$. Therefore, the rates are other than the other than diagonal elements.

And the diagonal elements are minus of sum of the row values other than that diagonal element. So this is nothing, but in a very small interval of time Δt the system is moving from the state 0 to 1 that probability the probability of system moving from the state 0 to 1 that is nothing, but the downstate to the upstate in a very small interval of time Δt why you are finding the probability of Δt.

Since the model is a Time Homogeneous only the interval matter not the actual time or you can visualize this as the sometime t to $t + \Delta t$ also. So this is the interval of Δ small negligible interval Δt the system is moving from the state 0 to 1 that probability is nothing but the rate means the rate. The rate is nothing but the repair rate so the rate $\mu\Delta t + o(\Delta t)$ it's a small o.

Order of Δt means as Δt tends to 0 the order of Δt will be 0. Similarly, you can visualize the probability of system moving from the state 1 to 0 in the interval Δt in a small interval Δt that is same as the failure rate λΔt that is the small interval of time +o(Δt). So this order of Δt also tends to 0 as Δt tends to 0. So using this I can make the forward Kolmogorov equation.

I can go for writing a forward Kolmogorov equation or backward Kolmogorov equation, but forward Kolmogorov equation is easy to make out so if the system is in the state i at time 0 what is the net rate the system will be the state 1 at the time t that net rate is nothing, but what are all the inflow that probability rate minus what are all the outflows that is the way you can visualize the right hand side.

So all the positive terms are related to the incoming rates and all the negative terms related to the outgoing rates. So since it is a two state model if the system is in the state 0 at time T there is a possibility it has not moved anywhere from the state 0 or it would have come from the state 1. Therefore, the incoming will be state 1. Therefore, the system will be in the state 1 at a time t and given that the starting from the state i that probability multiplied by the rate sort of inflow minus because we are writing the equation for the state 0.

Therefore, it is not moved from the state 0 that is a with the rate μ it can move to the state 0 to 1. Therefore, -μ times it does not move from the state 0 therefore - μ times the probability of being in the state 0 at time t given that it was in the state i at time 0 that probability multiplied by minus μ that is an outflow and the λ times $P_{i1}(t)$ that is a inflow.

Therefore, in the left hand side, it is a derivative of the function t it is a probability function. So $P_{i0}'(t)$ that is nothing but the net rate being in the system at time t in a state 0 given that it was in the state i at time 0 that net rate is same as the inflow minus outflow with the corresponding rates. Similarly, you can write the equation for the state 1 that means you start from the state 1 either you would have come from the state 0 to the 1 or you did not move from the state 1.

Therefore, - λ $P_{i1}(t)$ + μ $P_{i0}(t)$ that is the net rate corresponding to the state 1. So now we are able to write the forward Kolmogorov equation. So this is a interpretation of the forward

Kolmogorov equation. You can write easily by making a matrix $P_{ij}'(t)$ that is equal to P(t)Q where Q is the Infinitesimal Generator Matrix then also you will get the same things so I am just giving the interpretation.

Now my interest is to find out the time dependent or transient solution for this two state CTMC for that this is a different differential equation we need initial condition to solve these equations. So I make the assumption at time 0 the system is in the state 1. Therefore, the transition probability of system the $P_{11}(0)$that is equal to 1 since I made the assumption the system was in the state 1 at time 0 therefore being in the state 0 that is going to be 0. So I need this both the initial condition there to solve the equation.

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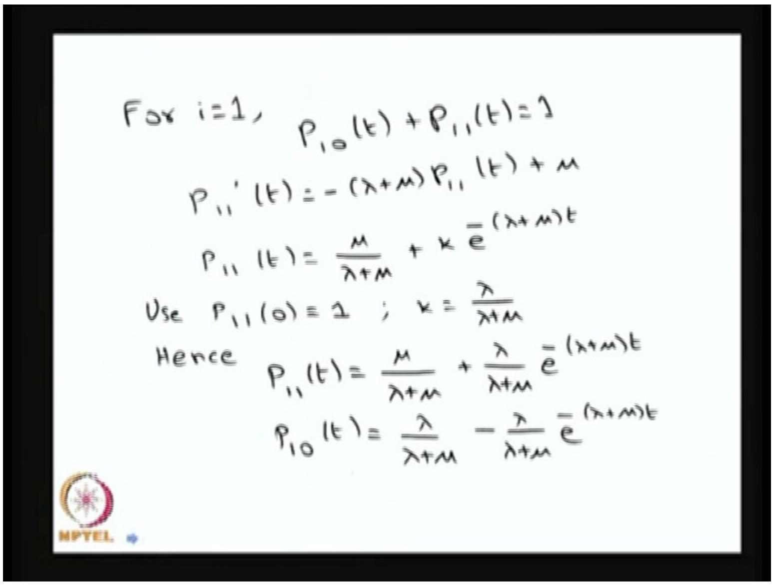

So let me start since I made the initial condition state is 1 therefore i is equal to 1 so I will have the first equation that is I always have the summation of the probability at time t. This is a transition probability are going to be 1 the summation and also I have a two different differential equations. So what I can do I can take the second equation in this then instead of $P_{10}(t)$, I can use the summation of probabilities is equal to 1.

Therefore, instead of $P_{10}(t)$,I can use the $P_{10}(t)$is nothing, but $1-P_{11}(t)$, I can substitute in the second equation therefore I will get $P_{11}'(t)=-(\lambda+\mu)P_{11}(t)+¿$ μ, substituting $P_{10}(t)$ is equal to $1-P_{11}(t)$ in the second equation the previous slide. Now I have to solve this a differential equation the unknown is $P_{11}(t)$ conditional probability.

And I have to use an initial condition $P_{11}(t)=1$using that I will get $P_{11}(t)=\frac{\mu}{\lambda+\mu}+k\,e^{-(\lambda+\mu)t}$that constant I can find out using this initial condition therefore $k=\frac{\lambda}{\lambda+\mu}$. So the $P_{11}(t)$ is equal to substituting $k=\frac{\lambda}{\lambda+\mu}$ in this equation I will get the $P_{11}(t)$.

Once I know the $P_{11}(t)$ use the first equation so I will get $P_{10}(t)$ is equal to 1- $P_{11}(t)$ therefore $P_{10}(t)$ that is equal to this expression. You can cross check this now if you add both the equations you will get a 1 and if you put t equal to 0 you will get the initial condition also correctly and if you put a t tends to infinity that we are going to discuss in the limiting distribution if you put a t tends to infinity in this equation.

You will get a $\frac{\mu}{\lambda+\mu}$, $\frac{\lambda}{\lambda+\mu}$. So this is for the t tends to infinity. Therefore, if you make a matrix the limit n tends to infinity of the limit.

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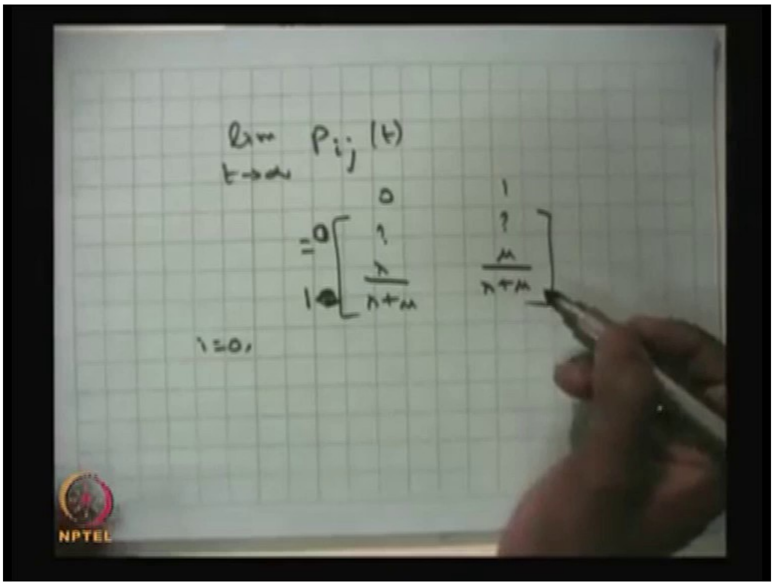

If you find out the limiting distribution of a limit t tends to infinity of P(t) so you will get the matrix and this matrix as t tends to infinity for this example is set two cross two matrix. And that consists of a for different values you will have a now you are doing for the second row

therefore that is equal to $\frac{\lambda}{\lambda+\mu}$ and this is equal to $\frac{\mu}{\lambda+\mu}$.

So if the system starts from the state 1 at a t tends to infinity the system will be in the state 0 with a probably $\frac{\lambda}{\lambda+\mu}$ and the system will be in the state 1 with a probability $\frac{\mu}{\lambda+\mu}$. Similarly, if you go for i equal to 0 you will get the same derivation and you can fill up what is the element here.

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$$\begin{bmatrix} \frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \\ \frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \end{bmatrix}$$

So this is the limiting distribution of probability matrix and if you see the rows are going to be identical. So you will have a same identical row in this row also so that means you will get the limiting distribution. I will discuss the limiting distribution, after giving one more example I will explain in detail. So this is the transition probability system starting from the state 1 and being in the state 1 or 0 at a time t.

Introduction to Probability Theory and Stochastic Processes
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Lecture – 78

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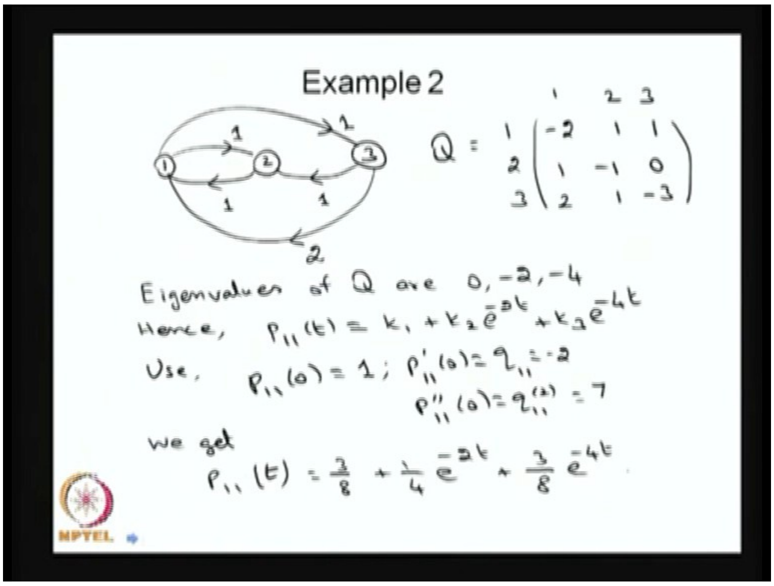

I am going to give one more example this has 3 states and this is a state transition diagram and the values are nothing, but the rates in which the system is moving from one state to other states. So that is the difference between the state transition diagram of a DTMC and the CTMC. So this is a rate in which the system is moving from one state to another state and some arcs are not there that means there is a no way the system is moving from the state 2 to 3 in a small interval of time.

Whereas all the other possibilities I have given. So the corresponding Q matrix it is a three cross three matrix and you can make out all the row sum are going to be 0 and the diagonal elements are minus of some of other values the same rows and other than the diagonal elements the values are greater than or equal to 0. My interest is to find out the time dependent solution for this example also.

I can make a forward Kolmogorov equation $P^{'}(t)=P(t)Q$ it is a 3 cross 3 matrix therefore I will have a 3 equations and I have one equation I can have summation of probabilities equal to 1 and I can start with the initial condition the system being in the state 1 at time 0 the probability is 1, I can start with that and I can solve those three equations with the initial condition and I can get the solution that is a one way.

Since it is a finite state CTMC there are many ways to get the time dependent solution basically you have to solve the system of different differential equations with the initial conditions. Here I am using the Eigen value method that means find the Eigen values for the Q matrix. Therefore, use the Eigen value and Eigen vector concept and you get the $P_{11}(t)$ with the unknown k_1, k_2, k_3 and to find the unknown to k_1, k_2, k_3 use the initial condition.

Here I am using the initial condition as well as the Q matrix values the q_{11} that means the element corresponding to the 1, 1 that is nothing but the $P_{11}'(0)$. Similarly, if I go for $Q^2$and $q_{11}(2)$. The element in the 1, 1 in the Q^2 that is nothing but $P_{11}''(0)$. Therefore now I can use these three initial conditions to get the unknown values k_1, k_2, k_3. So once I know the k_1, k_2, k_3, I can substitute that for the $P_{11}¿$) is equal to this much.

Similarly, I can go for finding the $P_{12}(t) \wedge P_{13}(t)$. I do not want $P_{13}¿$) in the same way because once I know the $P_{11}(t)$ and $P_{12}(t)$. So $P_{13}(t)$ is nothing but 1 minus of that those two probabilities because there is summation of probabilities is equal to 1. So this is the other way of getting the time dependent solution the transition probability of system being in the state j given that it was in the state i at time 0.

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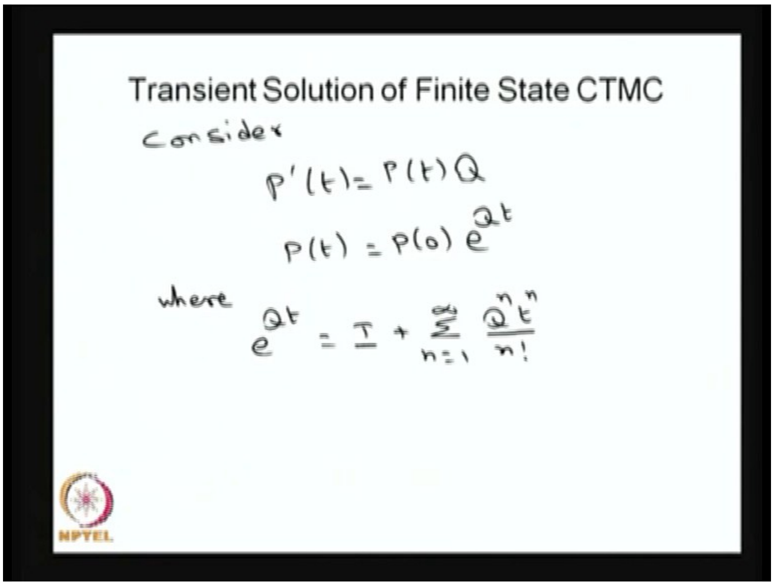

Suppose the CTMC has the finite state space then I can use the exponential matrix also to get the time dependent solution that what I have given this way.

So start with the forward equation. Therefore, the solution is going to be $P(t)=P(0)e^{Qt}$. P(t) is a matrix P(0) is the matrix e^{Qt} that is also again going to be a matrix exponential matrix therefore I am writing it $e^{Qt}$is nothing but Q is the matrix and t is the real value. So if greater than or equal to 0 therefore $e^{Qt}$is going to be I matrix.

I matrix is nothing but the identical matrix of order whatever the state number plus the $\sum_{n=1}^{\infty} \frac{Q^n t^n}{n!}$. So the whole thing is going to be exponential matrix and using that you can get the P(t). I am not going detailed for how to compute this $e^{Qt}$and so on, but whenever you have CTMC the finite space through this method also one can get the time dependent solution.

So with this I have completed the examples for the CTMC to find out the time dependent or transient probabilities.

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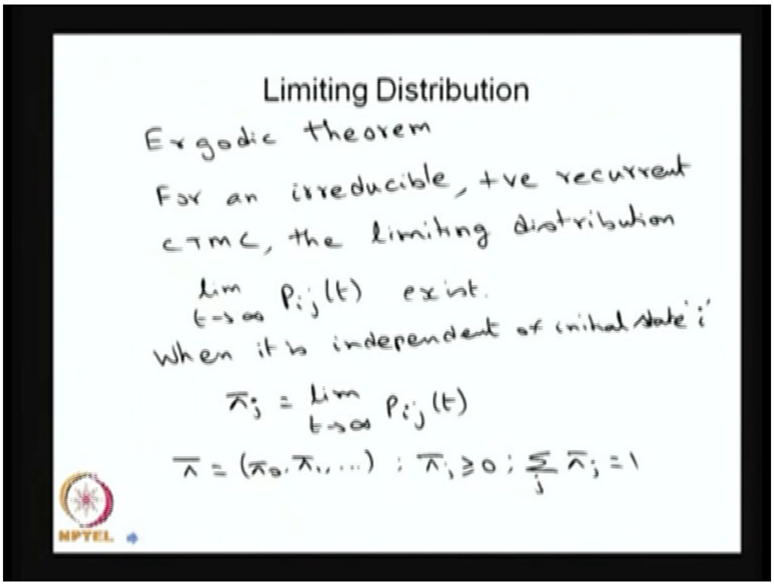

Now, I am moving into the limiting distribution the way we discuss the limiting distribution for the DTMC the same concept can be used for the CTMC also. The change is instead of one step transition probability matrix here we have to use the Infinitesimal Generator Matrix in a different way. So I am first giving the Ergodic Theorem, whenever the CTMC is irreducible that means all the states are communicating with all other states.

Since all the states are communicating with all other states. So if one is of the particular type it is a positive recurrent then all the other states are going to be a positive recurrent. If one is going to be a null recurrent then all the other states also going to be a null recurrent. So here I am making the assumption the CTMC is irreducible as well as all the states are positive recurrent then the limiting distribution always exist.

Suppose, it is independent of the initial state, it need not be a independent of initial states suppose the same thing is independent of initial state then I can write that limiting probability is $P_{ij}(t)$. Since it is independent of i. I can write it as the π_j. Then I can form a vector and since it is a limiting distribution it is a probability distribution. Therefore, the probabilities are these probabilities are always greater than or equal to 0.

And the summation of probability is going to be 1. It would not be less than 1 that is the Ergodic theorem says. Whenever you have a irreducible CTMC with all the states are positive recurrent then as t tends to infinity the system has the distribution limiting distribution. If it is independent of initial states, then you can label with the π_j as the probabilities.

And this probability distribution satisfies it is a probability mass function therefore it satisfies the probability mass function conditions. That means whenever you have a dynamical system in which it is irreducible model and all the states are positive recurrent that means the mean recurrence time is going to be finite value. Then that system is called it as a Ergodic system or the Ergodic concept can be used therefore as t tends to infinity you can get the limiting distribution.

If it is independent of initial state means, whatever be the state, you are going to do it for the discrete end stimulation for the dynamical system that is for a Ergodic system then the initial condition state does not matter to get the limiting distribution. Later we are going to give some few examples how to find out the limiting distribution.

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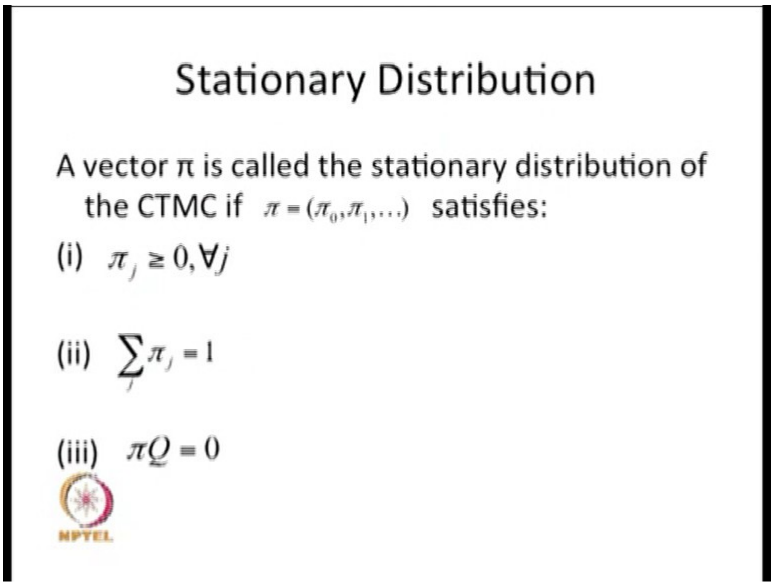

I am explaining the stationary distribution also. The stationary distribution the way I have discuss the DTMC the CTMC also same. So I have a vector if the vector satisfies these three conditions; probabilities therefore greater than or equal to zero, summation is equal to 1 and you should be able to solve the solution and get the π. It is a homogenous situation. So you need a second condition to have the non zero probabilities.

So if you solve π Q = 0 along with the $\sum \pi_j = 1$ and if this π_j exist then the CTMC has the stationary distribution. The similar way I have discussed the stationary distribution for the DTMC model also instead of πQ = 0 we had a π P = π. So if any vector satisfies that π P = π and $\sum \pi_i = 1$ and all the $\pi_i$are greater than or equal to 0.

Then that is going to be a stationary distribution for DTMC. The same way if π Q = 0 and $\sum \pi_j = 1$, $\pi_j$are greater than or equal to 0. If this is satisfied by any vector, then that is going to be the stationary distribution for a Time Homogenous CTMC. Every time we are discussing the default CTMC that is the time homogenous CTMC.

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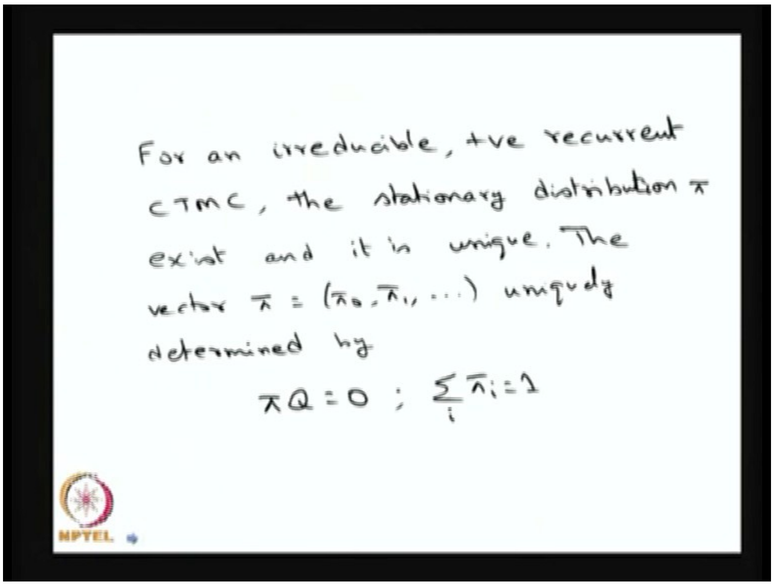

The main result for the stationary distribution whenever you have an irreducible positive recurrent CTMC the stationary distribution exists and that is going to be unique. Whenever the CTMC is a positive recurrent as well as irreducible there is no need of periodicity in the CTMC whereas the same stationary distribution for the DTMC we have included one more condition that is aperiodic, but for the CTMC there is no periodicity for the state.

Therefore, as long as the system is irreducible and the positive recurrent then the stationary distribution exist and it is unique and by solving these equations you can get the unique stationary distribution.

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Lecture – 79

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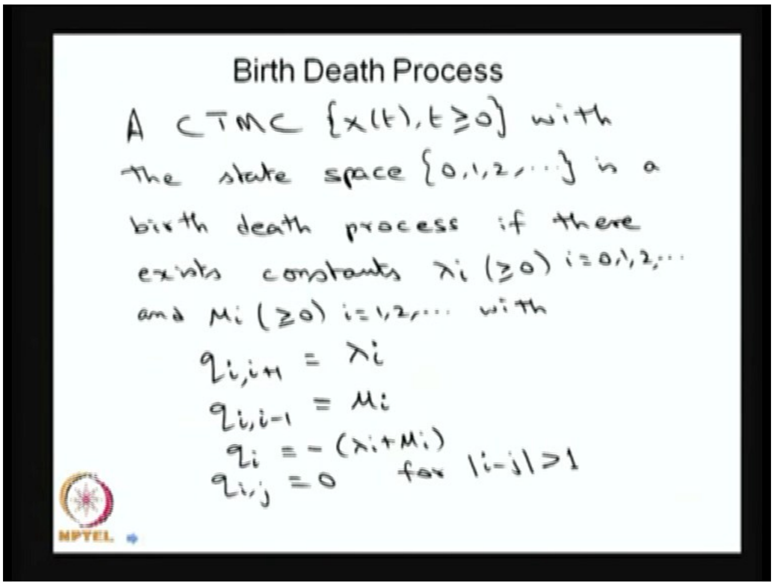

Now I am moving into the special case of Continuous Time Markov Chain that is a birth death process. This is a very important time homogenous Continuous Time Markov Chain because many of the scenario can be mapped with the birth death process either with the finite state or infinite state. Let me first give the definition of birth death process.

I started with Continuous Time Markov Chain it is a time homogenous Continuous Time Markov Chain with the state space countably infinite it can be finite also that CTMC is going to be call it as a birth death process. If there exist a constant λ_i and μ_i such that and this are all are nothing, but the Infinitesimal Generator Matrix elements and this is i to i +1 that rate is always λ_i.

And rate in which the system is moving from the state i to i -1 that rate is μ_i and the diagonal elements are $-(\lambda_i+\mu_i)$. Whereas all the other rates, the system is moving from the state i to j other than i to i +1, i to i - 1 and i to i and all other rates are always 0, absolute of i minus j is greater than 1.

That means you will have the Infinitesimal Generator Matrix in which you will only have a tridiagonal matrix and all other elements are going to be 0. I can write down the condition so that it land up the rates are going to be only λ_i and μ_i so on, not all other rates are going to be 0.

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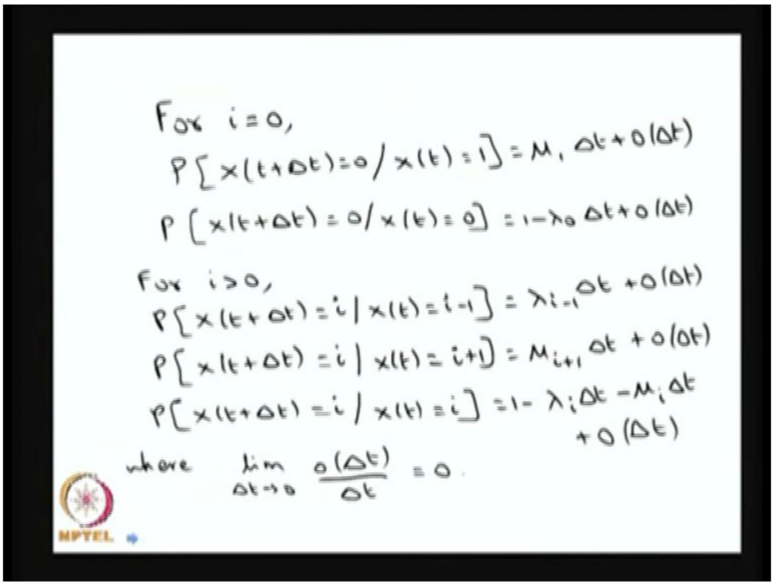

So if I start with i is equal to 0 the system is moving from the state 1 to 0 in the interval of delta t because it is a time homogenous model. So this is nothing but this probability the system is moving from the state 1 to 0 in the interval of Δt that is nothing, but the rate is $\mu_1 \Delta t + o(\Delta t)$

Similarly, the system is moving from the state 0 to 0 from the time t to t + Δt or during the interval Δt that is nothing, but 1 $-\lambda_0 \Delta t + o(\Delta t)$. This $\mu_i$and $\lambda_0$and so on, these values are always going to be greater than or equal to 0 strictly greater than 0 also. For i is greater than 0 the system is moving from the state i to i that is 1 $-\lambda_i \Delta t - \mu_i \Delta t + o(\Delta t)$.
Whereas the system is moving from i + 1 to i one step backward that is $\mu_{i+1} \Delta t$. The system is moving from the state i - 1 to i or i is greater than 0 that is a forward one step more that is $\lambda_{i-1} \Delta t + o(\Delta t)$. This is the order of Δt it maybe a function of Δt need not be the same as the Δt tends to 0 this quantities are going to be 0 or of Δt divided by Δt is going to be 0.

Therefore, this is the way the system is moving from the one state to either one step forward or either one step backward or move anywhere. So these are only three possibilities with these probabilities. Therefore, we land up the Q matrix is going to be the system is moving from the state i to i +1 forward one more that rate is λ_i.

The system is moving from the i to i -1 one step backward that is μ_i or the system being in the same state that rate is $-(\lambda_i+\mu_i)$. Therefore, there is no other move from the system from one state to all other states either one step forward or one step backward.

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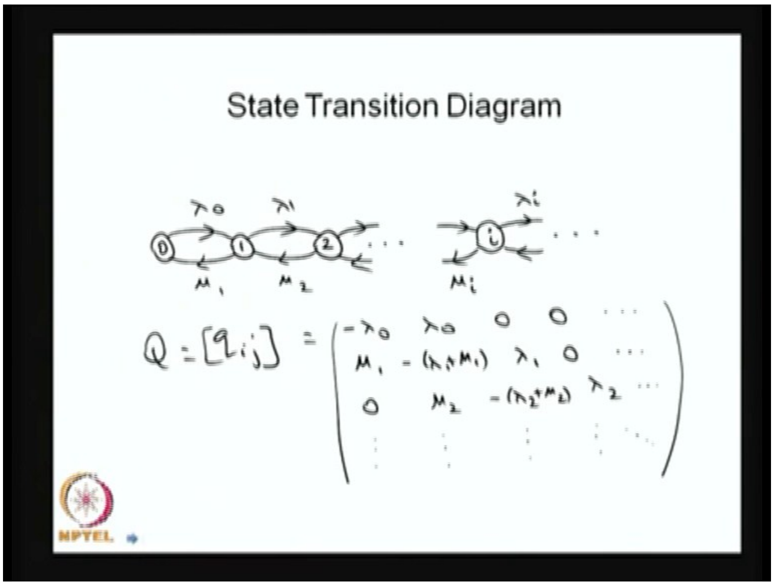

So this can be visualized in the state transition diagram. Since I started with the state space 0 to infinity there is a possibility you can have a label from some negative integers to the positive integers so we can always transform into something therefore default scenario or the simplest one I discussed from 0 to infinity. Therefore, we can visualize whatever will be the label that can be transformed in a one to one fashion.

So this is the rate in which the system is moving from the state 0 to 1 that rate is λ_0. The system is moving from the state 1 to 2 that rate is λ_1 or the system is moving from the state 1 to 0 that rate is μ_1. Therefore the time spent in the state 1 before moving into any other states that is a minimum of the time spending in the state 1 before moving into the state 2 or the system time spending in the state 1 before moving into the state 0.

So both are exponentially distributed with the parameters λ_1 and the μ_1 and the minimum of that time is the spending time or the waiting time in the state 1 that is going to be exponential distribution with the parameter $\lambda_1 + \mu_1$ because both are independent. The time spending in the state 1 before moving into the state 2 and similarly the time spending in the state 1 before moving into state 0.

And both the random variables are independent that is the assumption therefore it is going to be exponentially distributed, the time spending in the state 1 that is exponentially distributed with the parameter $\lambda_1 + \mu_1$. Like that you can discuss for all other states so whenever you have a birth death process the system either move one step forward or one step backward then it is a called a birth death process.

Therefore, here this $\lambda_i' s$is called the system is moving from one state to forward one step therefore this $\lambda_i' s$ is called the birth rates. The system is moving from one state to previous one state and the corresponding rate $\mu_i, \mu_1, \mu_2, \mu_3$and so on and these rates are going to be call it as death rates.
So λ_i are nothing, but the birth rates that means the rate in which the system is moving from the state i to i +1 that depends on i. Therefore, that rate is λ_i. The system is moving from the state i to i - 1 that is related to the death by 1 that is a function of i. Therefore, that death rate is μ_i. So λ_i are the birth rate and the μ_i are the death rates.

Suppose example the system moving from the state 2 to 1 the death rate will be μ_2. So you can fill up the Q matrix if you see the Q matrix it is a tridiagonal matrix.

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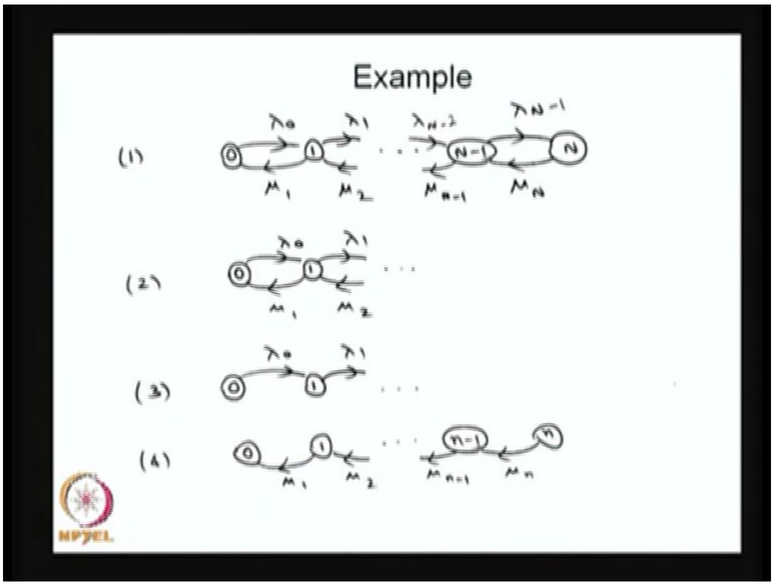

So here I am giving few examples of a birth death process. The first example consist of the first example is a finite state model. The birth rates are $\lambda_0, \lambda_1, till\, \lambda_{N-1}$. The death rates are μ_1, μ_2 , μ_N. It is a finite state birth death process.

The second example is the infinite state of birth death process. The third example the all the death rates are 0 that is also possible. The fourth example all the birth rates are 0 that is also possible, but one can discuss the state classification also. The first one it is a finite state model all the states are communicating with all other states therefore it is a irreducible positive recurrent birth death process.

The second one is the infinite state all the states are communicating with all other states it is a Irreducible, but one cannot conclude without knowing the values about the λ_0 and λ_i and μ_i one cannot conclude it is a positive recurrent or null recurrent. The mean recurrence time that is going to be a finite one then you can conclude it is positive recurrent otherwise it is null recurrent.

So as such you cannot discuss now the positive recurrent or null recurrent but you can conclude it is a recurrent state. The third example the system keeps moving forward therefore all the states are transient states. It is not irreducible it is a reducible model all the states are transient states that mean as t tends to infinity the system will be in some infinite state. So one cannot define the infinite state therefore the limiting distribution would not exist in this situation.

The fourth example it is a finite model, but all the states are not communicating with all other states therefore it is a not a irreducible it is a reducible model. Whenever the system starts from some state other than 0 over the time the system is keep moving backward and once it reaches the state 0, it will be forever. Therefore, state 0 is absorbing barrier.

Introduction to Probability Theory and Stochastic Processes

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Lecture – 80

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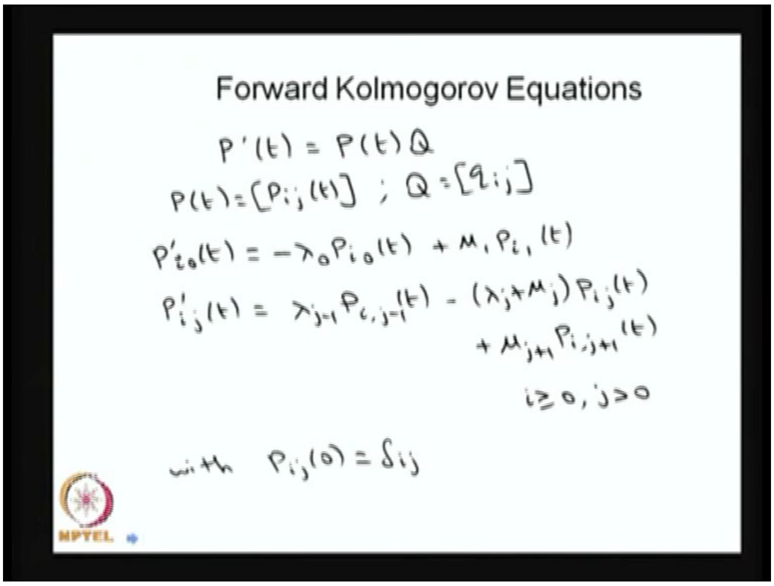

We are discussing the forward Kolmogorov equation for a special case of Continuous-time Markov Chain that is the birth death process. For a birth death process, the Q matrix is a tridiagonal matrix. Therefore, you will have the equations from the forward Kolmogorov equation, you will have a only two terms in the right hand side for the first equation and you will have only three terms, the diagonal element and two off-diagonal elements.

Therefore, the first equation one can discuss first, the P_{i0}' that is nothing, but the system is not moved from the state zero, moving from the state zero that rate is λ_0, therefore not moving $-\lambda_0$ times the probability and the system can come from the state one with the rate μ_1 . Therefore, $\mu_1 P_{i1}(t)$.

For all other equations either the system comes from the previous state with the rate λ_{j-1} or it comes from the forward one state with the rate μ_{j+1}or not moving anywhere. So these are all the all possibilities therefore with these three possibilities you have a three terms in the right hand side and that is the net rate for any strategy. So if you solve this equation with this initial condition, δ_{ij}, you will have the solution of a P_{ij}.

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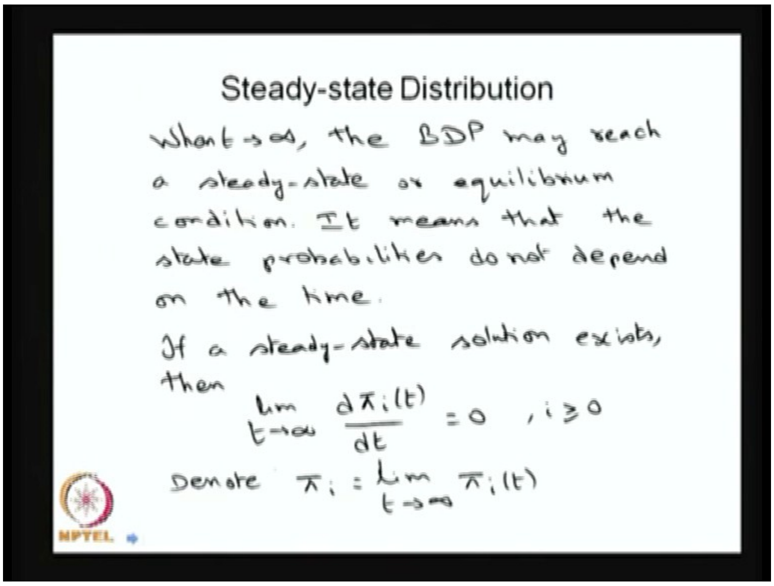

Here I am discussing the steady state distribution, the way I have discussed the limiting distribution that is a limit t tends to infinity, $P_{ij}(t)$exist, then it is called the limiting distribution and the stationary distribution is nothing but for the DTMC is π = πP, $\sum_i \pi_i = 1$. For the CTMC πQ = 0, and $\sum_i \pi_i = 1$. That is going to be the steady state distribution, stationary distribution.

Now I am discussing the steady state distribution, that is nothing but when t tends to infinity the birth death process may reach steady state or equilibrium condition. That means the state probability's does not depend on time. That is a meaning of a steady state distribution. As t tends to infinity, whenever we say the birth death process reaches a steady state or at equilibrium, that state probability does not depend on time.

That means if a steady state solution exist, since the state probability does not depend on time t, the derivative of the time dependent state probability at time t, that derivative at t tends to infinity becomes zero. If the steady state solution exists. Since the state probability's does not depend on time t as t tends to infinity, I can write as a $\pi_i = \lim_{t \to \infty} \pi_i(t)$.

So this is different from the way we discussed earlier that conditional probability $P_{ij}(t)$. But using $P_{ij}(t)$, one can find out what is $\pi_j(t), \pi_i(t)$.

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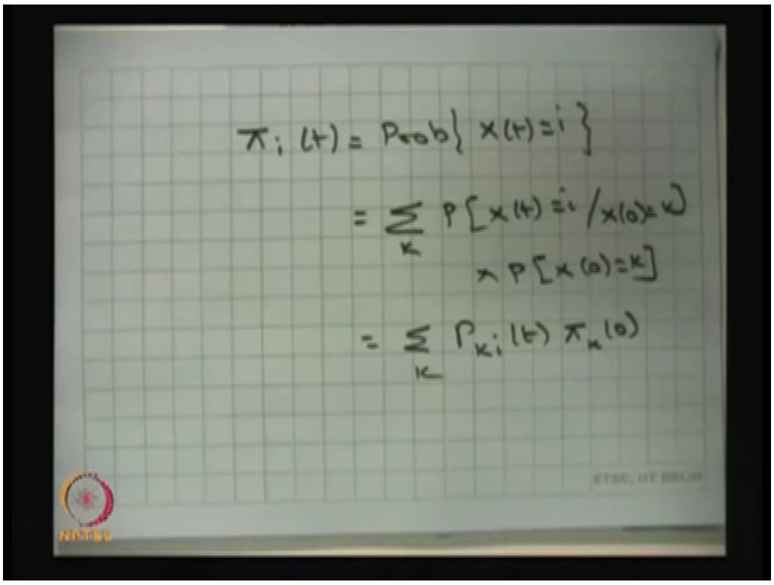

That is nothing but the π(t) that I have given in the first lecture for the CTMC. The $\pi_i(t)$ that is nothing but what is the probability that the system will be in the state i at times. That is same as what is the probability that the system will be in the state i given that it was in the state some k at times zero multiplied by what is the probability that it was in the state k at times.

That is nothing but summation of k and this is nothing but the transition probability and this is nothing but the initial probability vector element. So using $P_{ki}(t)$ or $P_{ji}(t)$ that is a conditional probability, one can get the unconditional probability. This is nothing but the distribution of X(t).

So this is the probability mass function, probability mass at state i. So now what I am defining, whenever the steady state distribution exists, that means it is independent of time t. Therefore, $\lim_{t \to \infty} \pi_i(t) = \pi_i$ and whenever the steady state solution exists, I can use $\lim_{t \to \infty} \pi_i'(t) = 0$.

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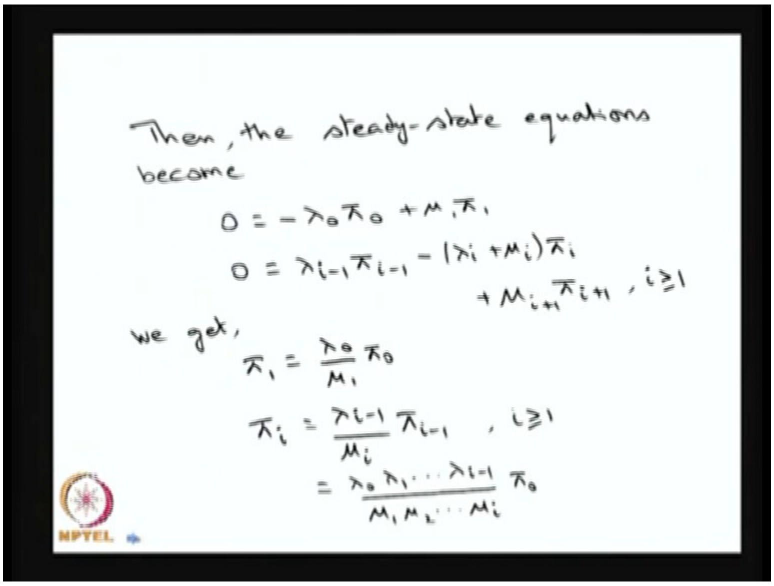

Therefore, I am going to use these two to get the steady state probabilities for the birth death process. Since, $\lim_{t \to \infty} \pi_i'(t) = 0$, therefore, all the left hand side in the forward Kolmogorov equation that is going to be zero, the right hand side you will have a , $\lim_{t \to \infty} \pi_i(t)$, that can be written as the $\pi_0 \wedge \pi_1$.

So the way we write the conditional probability for, $P_{ij}$with the Kolmogorov forward equation, you can write the similar equation for the unconditional probability π_i also. So now I am putting the left hand side zeros because of this condition, limit t tends to infinity, the derivatives equal to zero and the right hand side I am using as a t tends to infinity, this probability is nothing but the π_i.

Therefore, it is going to be $-\lambda_0 \pi_0 + \mu_1 \pi_1$ and all other equation as three terms. In this homogeneous equation and you need a one normalising condition. So from this homogeneous equation, I can get regressively π_i in terms of π_0. So from the first equation, I can get a π_1 in terms of π_0 and the second equation.

I can get a π_2 in terms of first π_1 then I can get a π_1 in terms of π_0. Therefore, regressively, I can get $\pi_i$in terms of π_0 for all i greater than or equal to 1.

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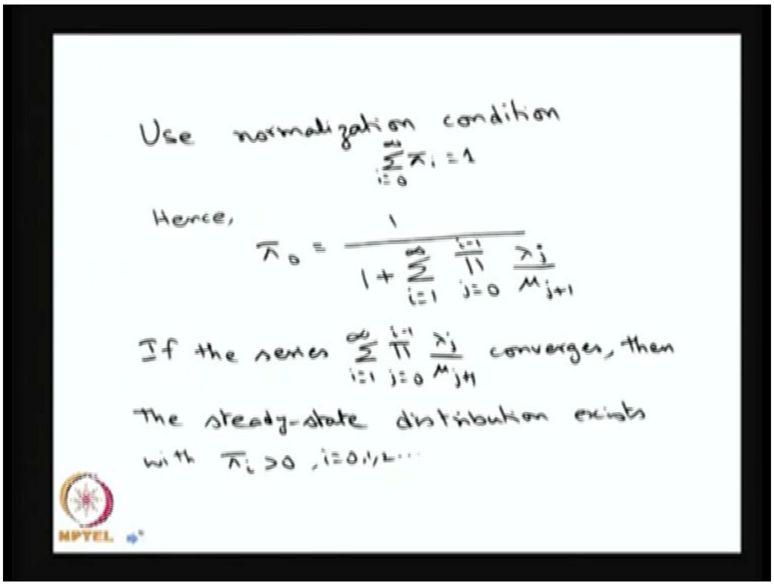

Now I can use a normalising condition, $\sum_i \pi_i = 1$, therefore, I will get a $\pi_0 = \frac{1}{1 + \sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}}}$.

Since we need steady state probabilities and all the π_i are in terms of π_0. As long as the denominator converges, you will have a π_0 is greater than zero. So ones the π_0 is greater than zero, then you will get all the π_i with the $\sum_i \pi_i = 1$.

So whenever these series converges, then I will have a steady state distribution with the positive probability and a summation of probability is going to be 1. So this is the condition for a steady state distribution for a birth death process because we started with a birth death process forward Kolmogorov equation using these two conditions we have simplified into this form and use a normalising condition and get the π_0.

As long as the summation is or the series converges, then we will have the steady state. If the series diverges, that means by substituting the values for the λ_i's and μ_i's and if the series denominator series diverges, then the π_0 is going to be zero in turn all the π_i's are equal to zero therefore the steady state distribution will not exist if the denominator series diverges.

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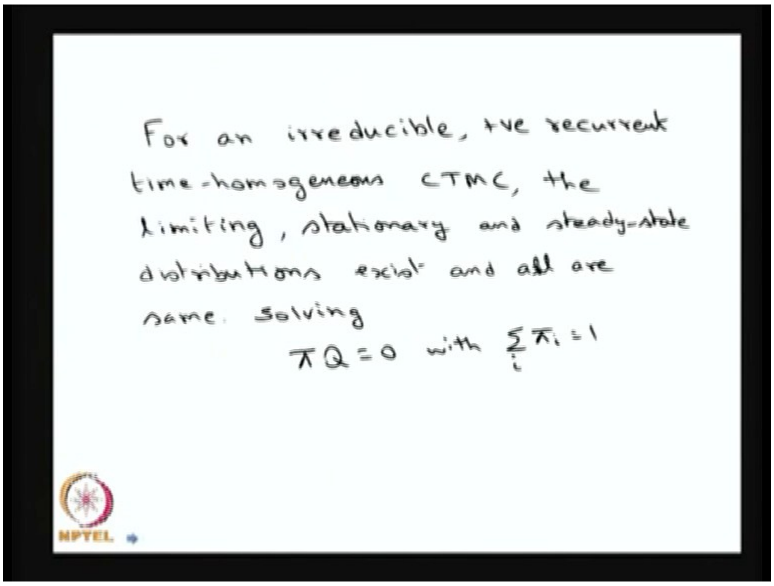

I am going to give a one simple result, for a irreducible positive recurrent time homogeneous CTMC, we know that a limiting distribution exist, a stationary distribution exist. Now I am including the steady state distribution also exist, I have given for a steady state distribution for the birth rate process, not for the CTMC but here I am giving the result for the CTMC. All the three distribution exist and all are going to be same.

Whenever the CTMC is a time homogeneous irreducible positive recurrent, all these three distributions are same and one can evaluate, one can solve this two equation by $\pi Q = 0$ and with the $\sum_i \pi_i = 1$, you can get the limiting distribution, stationary distribution or steady state or equilibrium distribution.

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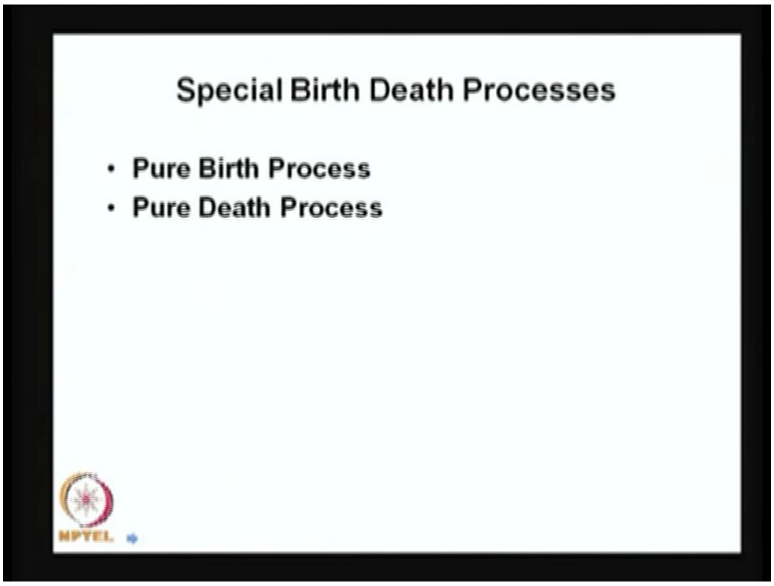

As a special case of birth death process, I am going to discuss these two processes in this lecture.

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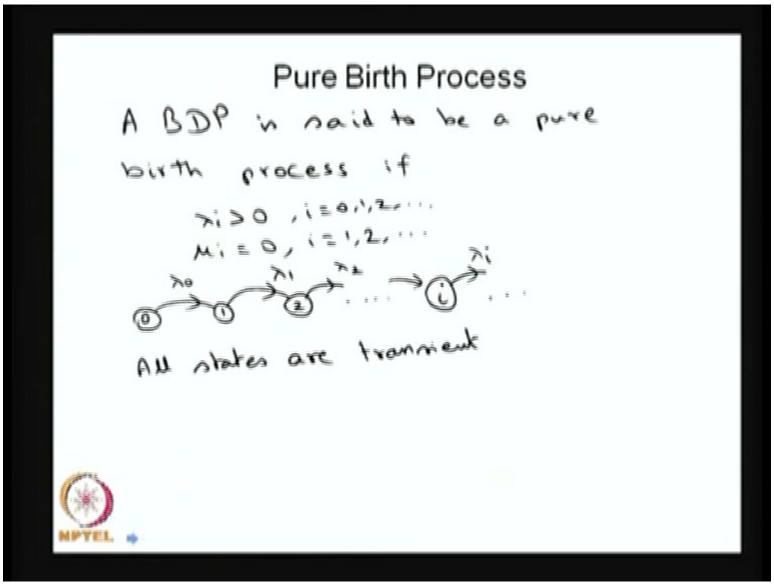

Whenever, we say the birth death process is a pure birth process, that means all the death rates are going to be zero, we started with a birth death process with the only λ_i's ≥ 0 and the μ_i's = 0, then it is going to be call it as a pure birth process. There is a one special case of pure birth process with the λ_i's are going to be constant, that is λ, that is a Poisson process.

I am going to discuss in the next lecture and in these pure birth process, these λ_i's are the function of i. Here all the states are transient states.

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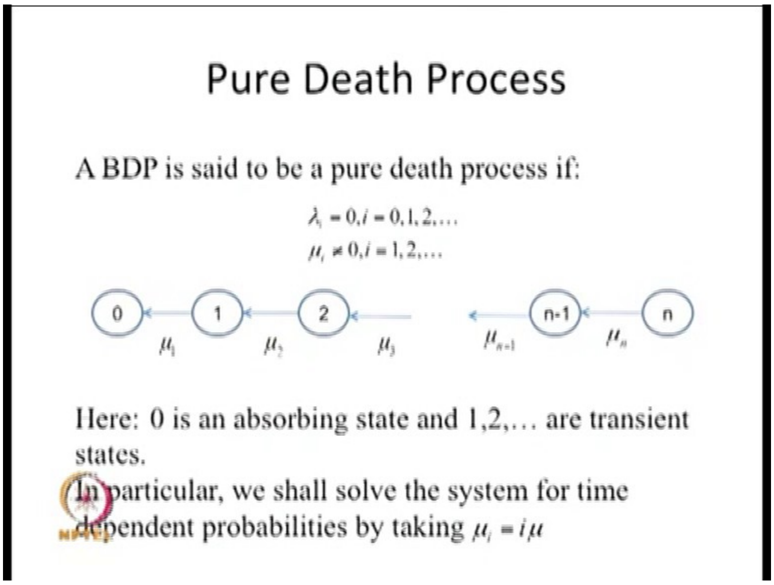

Here I am discussing the pure death process. A birth death process is said to be a pure death process if the birth rates are zero and the death rates are non zero. In particular, we shall obtain the time dependent probabilities of a pure death process in which the death rates μ_i's are equal to iμ.

As I given the example, as a fourth example in the birth death process, this state zero is a absorbing barrier. Therefore, the state zero is absorbing state and all other states are going to be transient state. And here the limiting distribution exists and one can also find time dependent probabilities for this model.

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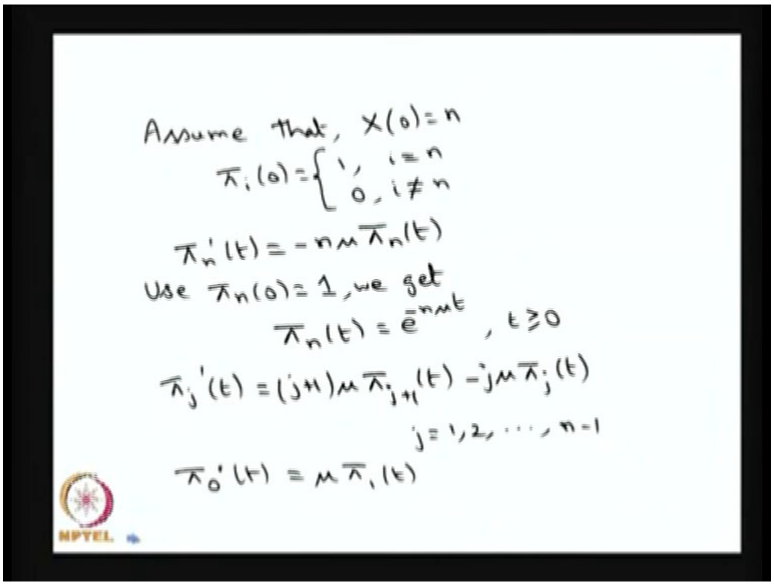

Suppose you start with the assumption, the system at time zero, in the system is in the state n, at times zero the system in the state n at times zero. With that assumption, I can frame the equation that is the $\pi_n'(t) = -n\mu\,\pi_n(t)$.

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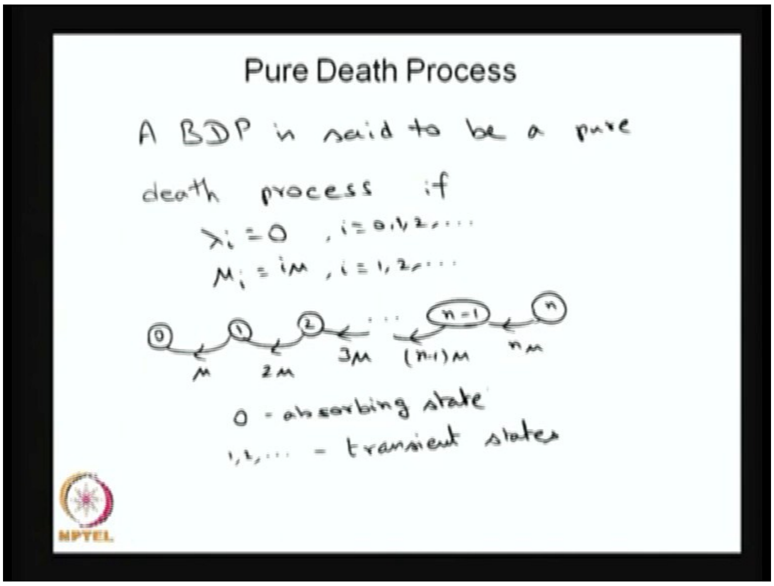

That means the rate in which the system is in the state n that is nothing but not moving to the state n - 1 with the rate -nμ. Therefore, the equation for the state n that is a $\pi_n'(t)$ that is equal to not moving from the state n therefore minus that outgoing rate that is nμ being the state is n therefore $\pi_n(t)$. I can use the initial condition $\pi_n(0)=$¿1, so I will get $\pi_n(t)$.

For the second equation, I have to go for what is the equation for the state n minus 1. So the $\pi'_{n-1}(t)$, that is nothing but either the system come from the state n or not moving from the state n minus 1. Therefore, system coming from the state n that is a nμ times the system being the state n - (n – 1)μ$\pi_{n-1}(t)$. So we will have a two terms in the right hand side coming from the one forward state or not moving from the same state.

So you will have a two terms for j is equal to 1 to n - 1. For the last state, that is the state zero, the system comes from the state 1. Since the state zero is absorbing states, there is no second term. So it is going to be μ$\pi_n(t)$. So you know $\pi_n(t)$, use the $\pi_n(t)$in the equation for n -1 and get the π_{n-1}, like that you find out till π_1. Use the π_1 to get the $\pi_0(t)$. Use the recursive way.

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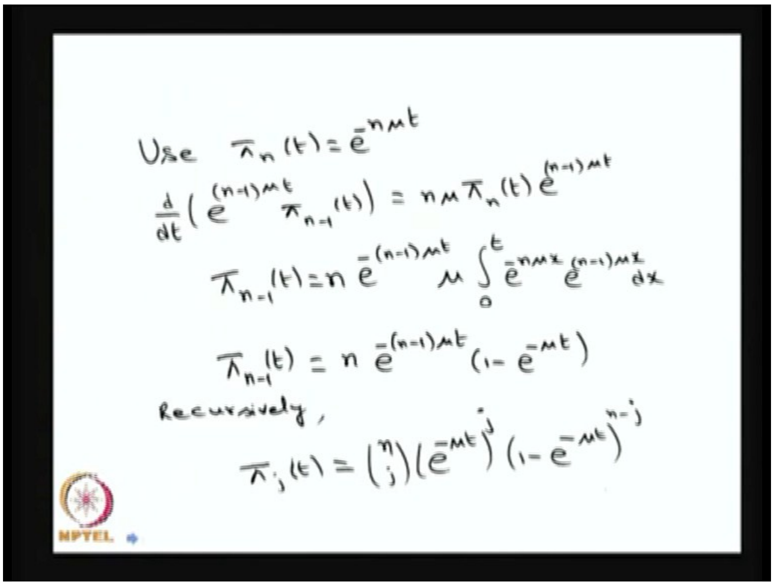

So using the recursive way, you will get the $\pi_j(t)={}^nC_j\left(e^{-\mu t}\right)^j\left(1-e^{-\mu t}\right)^{n-j}$, this is survival probability of system being in the state and $\left(1-e^{-\mu t}\right)^{n-j}$. Suppose the system being in the state j, that means from the state n this many combinations would have come and the survival probability is $e^{-\mu t}$and that power.

So this is nothing but the probability $p^j(1-p)^{n-j}$. Therefore, this π_j follows the binominal distribution with the survival probability $e^{-\mu t}$being in the state j.

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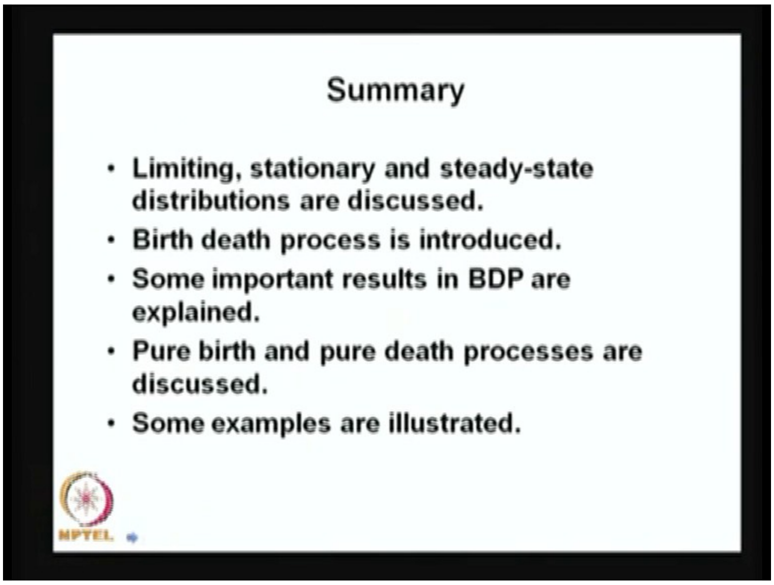

So for the pure death process, I have explained the time dependent probabilities of being in the state j, that is a unconditioned probability. So with this the summary of this lecture is, I have discussed the limiting, stationary and a steady state distribution, I have introduced a birth death process. Some important results also discussed and at the end, I have discussed the pure birth and pure death process also. In the next lecture, I am going to explain the important pure birth process that is the Poisson process.

Thanks.

Introduction to Probability Theory and Stochastic Processes
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Lecture – 81

In this lecture, we are going to discuss Poisson process and its application. So let me start with the Poisson process definition, then I give some properties in the Poisson process and I also present some examples. Poisson process is a very important stochastic process. Whenever something happens in some random way occurrence of some event and if it satisfies a few properties, then we can model using a Poisson process.

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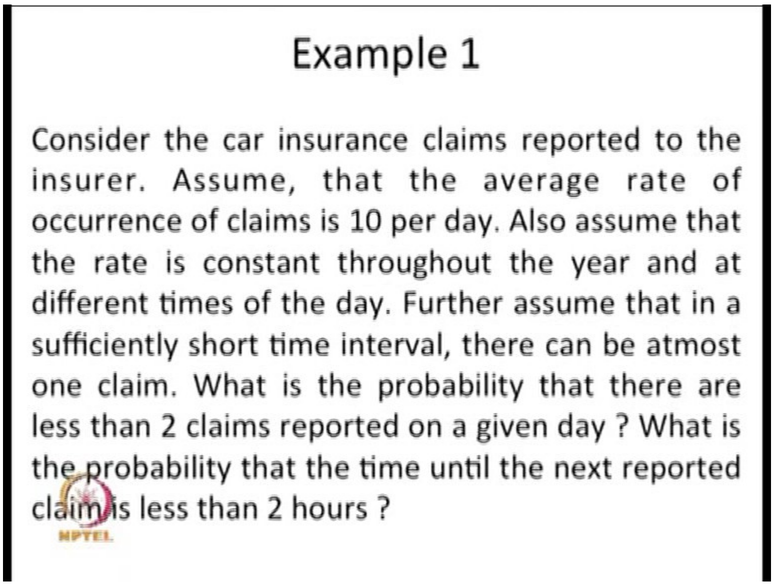

And Poisson process has some important properties whereas the other stochastic processes will not be satisfied with those properties.

Therefore, the Poisson process is a very important stochastic process for the many modelling in applications like telecommunication or wireless networks or any computer systems or anything, any dynamical system in which the arrival comes in some pattern and satisfies few properties.

So before moving into the actual definition of Poisson process, I am going to give one simple example and through this example, I am going to relate the Poisson process definition then later I am going to solve the same example also. This is the example number 2, example 1 I have something else.

Consider a car insurance claim reported to insurer. It need not be car insurance, you can think of any motor car, motor insurance or any particular type of vehicle or whatever it is. Assume that the average rate of occurrence of claims 10 per day. It is an average rate per day. Therefore, it is a rate, per day the average rate is 10. Also assume that this rate is a constant throughout the year and at different times of a day.

So even though these quantities are average quantities, there is a possibility some day, there is no claim reported at all or the there are someday more than some 30, 40 claims reported. And all the possibilities are there but we make the assumption, the average rate is a constant throughout the year at the different times of a day also. Further assume that, in a sufficiently short time interval there can be utmost one claim.

Suppose you think of a very small interval of 1 minute or 5 minutes or whatever, a very small quantity comparing to the because here I have given the average rate is 10 per day. Therefore, whatever the time you think of a very negligible, in that the probability of a or it is sufficiently small interval of time, there is a possibility of only maximum one claim can be reported.

The question is what is the probability that there are less than two claims reported on a given day, what is a probability that less than two claims reported means what is a probability that in a given day either no claim or one claim. Also we are asking the second question, what is the probability that the time until the next reported claim is less than 2 hours. Suppose some time, one claim is reported, what is the probability that the next time is going to be reported before 2 hours.

We started with this problem, the car insurance claims reported, therefore the claims are nothing but some event and these events are occurring over the time. Suppose you make the assumption of a sufficiently smaller interval of time utmost one claim can happen and average rate of occurrence of claim is a constant throughout the time.

So with this assumption, one we can think of a sort of arrival process, pure birth process, satisfying some condition and that may lead into Poisson process. So this same example, we are going to consider it again also. Now I am going for definition of a Poisson process. How one can derive the Poisson process.

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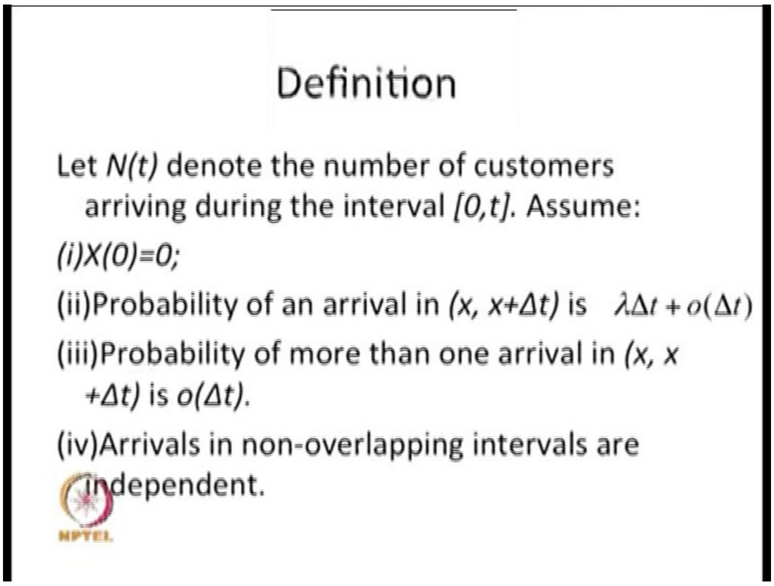

Poisson process is a stochastic process with some conditions. So how one can derive the Poisson process. For that let me start with the random variable N(t), that denotes the number of customers arriving during the interval 0 to time t. That means how many arrivals takes place in the interval 0 to t. That means for fixed t, N(t) is a random variable. Over the time this N(t) collection, that is a stochastic process.

I am making some four assumptions. With these assumptions, I am going to conclude the N(t) is going to be a stochastic process. The first assumption, not X(0), but N(0) = 0. A time zero, the number of customers is zero, N(0) = 0. It is a wrong N(0). Second one, the probability of arrival in a interval x to x+ Δt that is a $\lambda\Delta$t where $\lambda \geq 0$.

That means probability that only one arrival is going to take place in an interval of Δt that probability is a λ Δt. For us very, very small interval Δt. It is independent of x that means it is increments are stationary. That property I am going to introduce in this assumption. The probability of more than one arrival in the interval x to x + Δt is negligible.

That means utmost maximum one arrival can occur in a very small interval of time, that is the assumption, that I am specifying in third one. The fourth assumption arrivals in non-overlapping intervals are independent. That means, if the arrival occurs in some interval and another some non-overlapping interval, then those arrivals are going to be form a independent. That means there is no dependency over the non-overlapping intervals arrivals going to occur or not.

So with these four assumptions N(0) = 0 and probability of one arrival is $\lambda\ \Delta t$ in a small interval, more than one arrival occurrence in a interval Δt, where Δt is very small, that is the probability is negligible and non-overlapping intervals arrival are independent.

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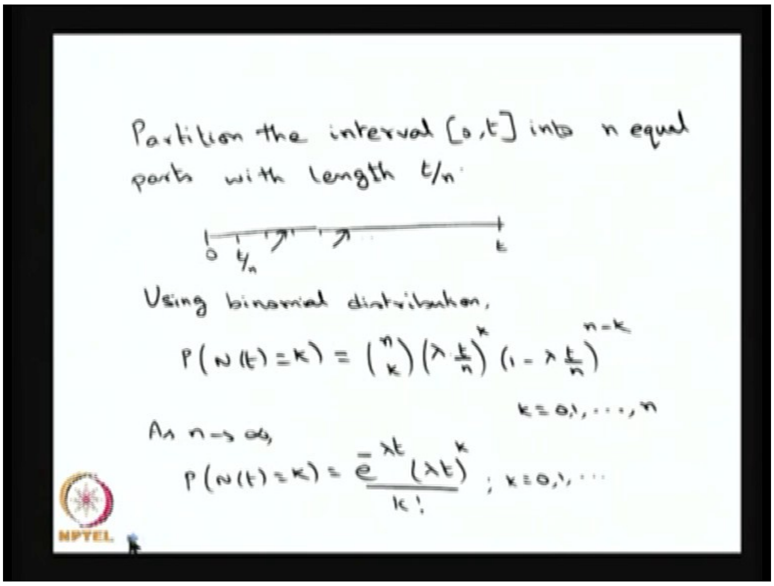

So with this derivation, I am going to find out the distribution of N(t). To find the distribution of N(t), first I am doing partitioning the interval 0 to t into n equal parts with the length t/n. The way I use the, the way I partition the interval 0 to t into n pieces, such that t/n is going to be very small interval, so that means I have to partition that interval 0 to t in such a way that the t/n is going to be as small as.

Therefore, I can use those assumption of a probability of occurring one arrival in that interval of length t/n, that probability is $\frac{\lambda t}{n}$ and the probability of not occurring a event in that interval t by n is 1 - $\frac{\lambda t}{n}$. So I can use those concepts for that I have to partition the interval 0 to t into n parts with sufficient larger n, therefore t/n is going to be smaller.

Now, since I partition these intervals into n pieces, n parts, I can think of at each part, I can think of a binomial or Bernoulli distribution at each pieces, therefore, all the non-overlapping intervals occurrence are independent therefore I can think of it is accumulation of a n independent Bernoulli trials. Since it is a n independent Bernoulli trials for each intervals t/n of length t/n.

Therefore, the total number of events occur in the interval 0 to t by partitioning into n equal parts. This is a sort of what is a probability that k event occurs in the interval 0 to in the time duration 0 to t as a n partition. So out of n equal parts, what is the probability that k events occur in the interval 0 to t. That is nothing but since it is each interval is going to form a Bernoulli distribution with a probability p = $\frac{\lambda t}{n}$.

Therefore, the total number is going to be binomial distribution with the parameters n and p where p = $\frac{\lambda t}{n}$. Therefore, this is the probability mass function of a k event occurs out of n equal parts. Therefore, $^{n}C_{k}\left(\frac{\lambda t}{n}\right)^{k}\left(1-\frac{\lambda t}{n}\right)^{n-k}$. Now the running index for k goes from 0 to n. That means there is a possibility no event takes place in the interval 0 to t or maximum of n event takes place in all n intervals.

So this is for sufficiently large n such that the t/n is smaller. We take n tends to infinity to understand the limiting behaviour of the scenario as the partition becomes final. Now I can go for n tends to infinity, what will happen, as n tends to infinity, if you do the simplification here as n tends to infinity, that simplification I am not doing in this presentation as a limit n tends to infinity, the whole thing will land up $e^{-\lambda t}\frac{(\lambda t)^{k}}{k!}$.

Now the k running index is a 0, 1, 2 and so on. This you can use the concept the binomial distribution as n tends to infinity and p tends to 0, your n into p becomes the λ. So that will give the Poisson distribution. The limiting case of a binomial distribution is a Poisson distribution.

So using that logic, these binomial distribution mass as n intends to infinity. This becomes a Poisson distribution mass function. So this is nothing but the right hand side is the probability mass function for a Poisson distribution with a parameter λt. And this is a random variable N(t) for a fixed t. Therefore, for fixed t, N(t) is a Poisson distributed random variable with a parameter λt where $\lambda > 0$.

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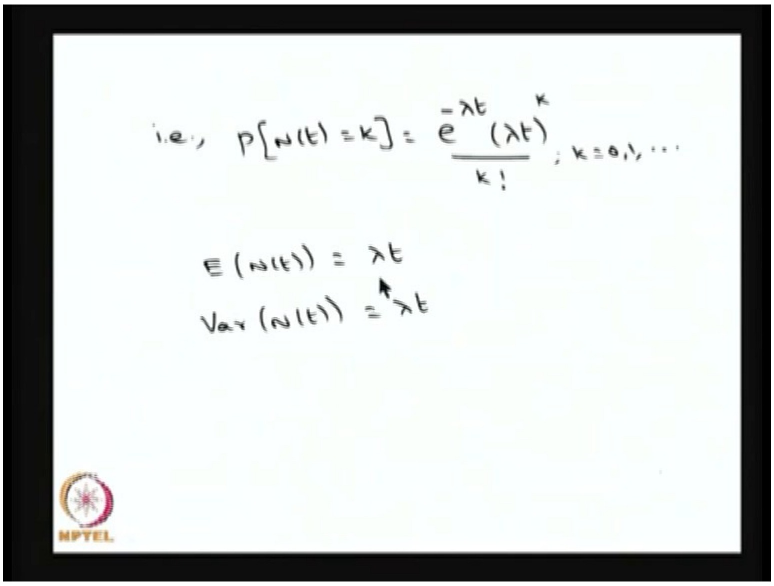

Therefore, we can conclude the stochastic process related to the N(t) or fixed t N(t) is a Poisson distribution. Therefore, the stochastic process N(t) over the $t \geq 0$ that is nothing but a Poisson process. So from the Poisson distribution, we are getting Poisson process because each random variable is a Poisson distributed with a parameter λt.

Therefore, that collection of random variables is a Poisson process with the parameter λt. Since it is a Poisson distributed random variable for fixed t, you can get the mean and variance and all other moments also by using the probability mass function of N(t).

Introduction to Probability Theory and Stochastic Processes

Prof. S. Dharmaraja
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Lecture - 82

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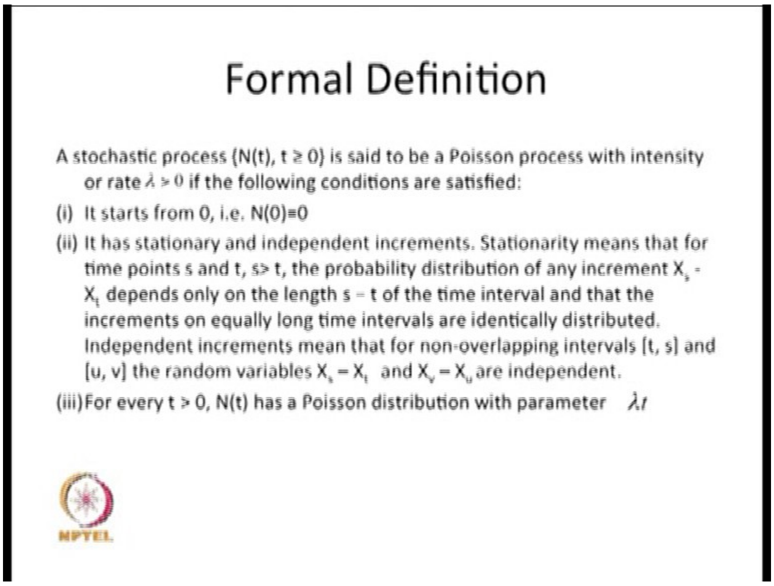

Formally we define Poisson process as follows. A stochastic process $N(t), t \geq 0$ is said to be a Poisson process with the intensity or rate $\lambda > 0$ in the following conditions are satisfied. First condition, it starts from 0, that is $N(0) = 0$. Second condition, the increments are stationary and independent, stationarity means that for time points s and t, s greater than t.

The probability distribution of any increment N(s)-N(t) depends only on the length s - t of the time interval and that the increments on equally long time intervals are identically distributed. Independent increments means that for any non-overlapping intervals, t, s and u, v, the random variables N(s)-N(t) and N(v)-N(u) are independent. For t greater than 0, N(t) has a Poisson distributed random variable with a parameter λt.

And the difference of the random variables defined over non-overlapping intervals are independent. λt is a cumulative rate t time t. The Xi's are independent and identically distributed random variables with some distribution function G independent of the Poisson process $N(t), t \geq 0$. It is Markov in nature because the two queues act independently and are themselves M/M/1 queuing system, which satisfied the Markov property.

Assuming that each queue behaves as the M/M/1 queue, the details of the proof can be found in the reference books because q_{ij}'s are obtained by differentiating the P_{ij}'s. For every t greater than 0, N(t) has a Poisson distribution with the parameter λt. Like that you can go for many more increments also.

For illustration, I have made it with the two increments, that means the occurrence of an arrival during this non-overlapping intervals are independent and stationary mean it is a time invariant only the length matters and not the actual time. Third one, for every t, N(t) has a Poisson distribution with the parameter λt. So the Poisson logic is coming into the fourth condition only.

The first condition is started 0, increments are stationary and increments are independent. The third condition for fixed t, N(t) is a Poisson distribution random variable with the parameter λt. Therefore, this stochastic process is called a Poisson process. Now we can relate the way we have done the derivation. We have taken care these three assumptions starting at time 0,0.

Increments are stationary that we have taken and increments are independent that is non-overlapping intervals are independent. Then we have derived, we are getting the distribution of the random variable N(t) is a Poisson distributed random variable. Therefore, this is a Poisson process.

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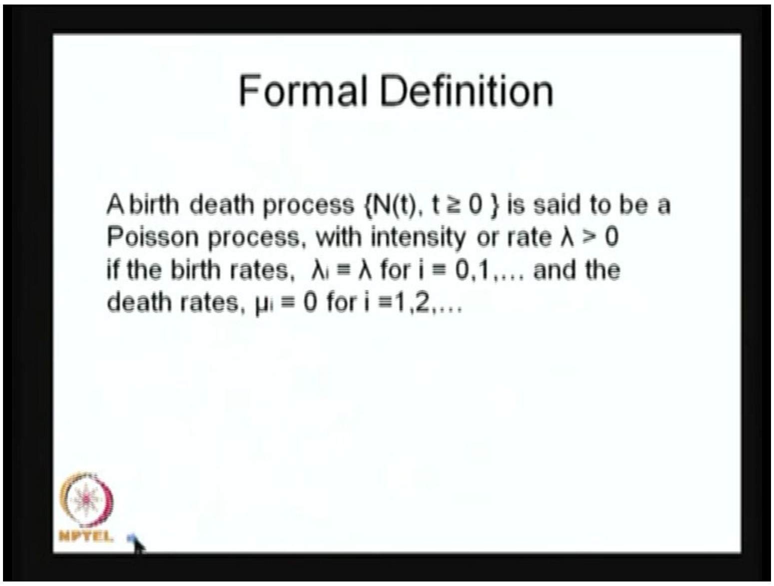

The another way of defining the Poisson process, we can start with the birth death process. You know that birth death process is a special case of a Continuous-time Markov Chain. It is a special case of a Markov process also. So you can think of a stochastic process, then the special cases are Markov process. Then the special cases are Continuous-time Markov Chain. Then you have a special case, that is a birth death process.

So you can define the Poisson process from the birth death process also. A birth death process N(t) is said to be a Poisson process with intensity or rate λ if a birth rates are constant for all i and the death rates are 0. You start from the birth death process with all the birth rates are same, that means it is a special case of pure birth process in which birth rates are constant for all the states. And the death rates are 0.

Then also you will get the Poisson process.

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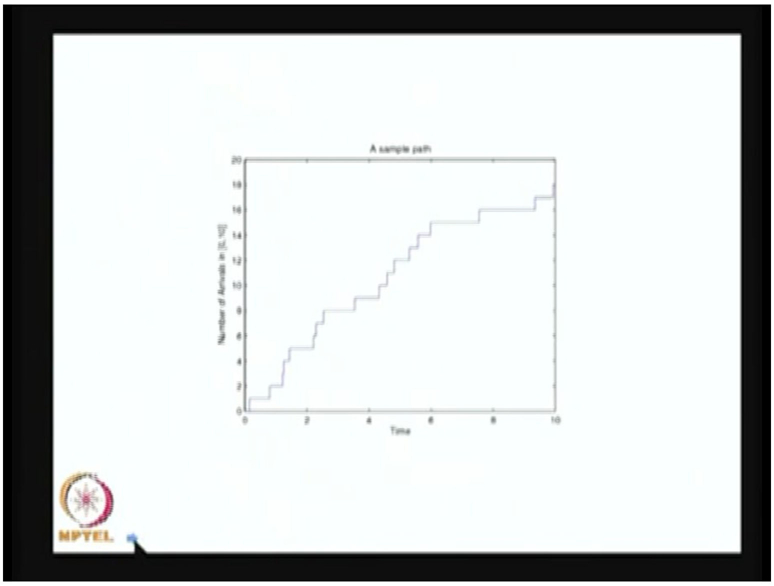

Here I am giving a sample path for the Poisson process, so this is a created using the MATLAB write the simple code of a Poisson process. Then, you develop the sample path, that means at time 0, the system at 0, at some time one arrival takes place. Therefore, the system land up 1, therefore, the y axis is nothing but the N(t). So at this time one arrival takes place therefore the number of customers in the system, number of arrivals till this time, that is 1.

So it is a right continuous function. The value at that point and the right limit is same as both are same which is different from the left limit of the arrival epoch, arrival time of epoch. So the system was in the state 1, till the next arrival takes place. So suppose the arrival takes place here, then the N(t) = 2 at this time point in which the arrival epoch and the right limit and so on.

So this is the way, therefore, the system at any time it will be the same value or it will be incremented by only one unit. The Poisson process sample path will be with the one unit step by increment at any time, there is no way the two steps the system can move forward at even in a very small interval of time, the system will move into the only one step, that you can visualise here.

Therefore, you can go back to the assumptions which we have started the derivation N(0) = 0 in a very small interval of time utmost one event can takes place and the difference of the random variables defined over non-overlapping intervals are independent. And increments are also stationary. So those things you cannot able to visualise in the sample path. So this is just one sample path over the time and N(t).

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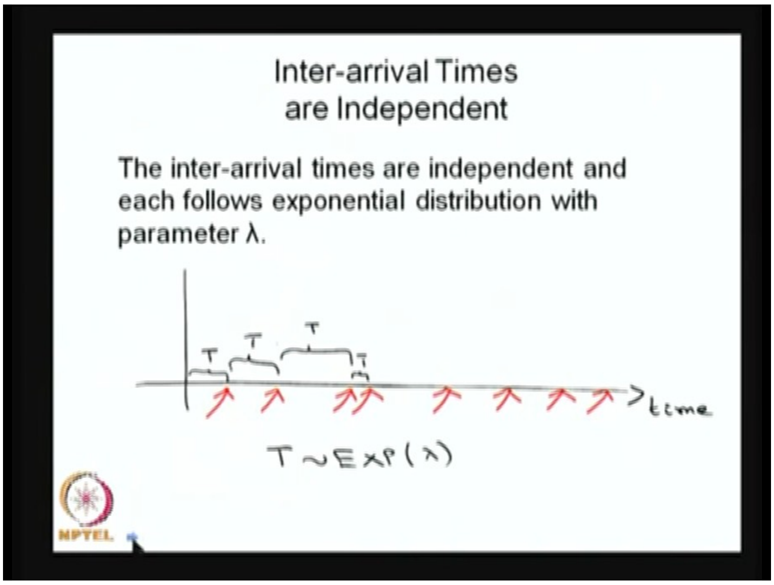

The second one inter-arrival times are independent, as well as we can conclude the inter-arrival times are exponentially distributed also. The inter-arrival times are independent and each follow exponential distribution with the parameter λ. What is the meaning of inter-arrival times, a time 0, the system is in the state 0. First arrival occurs at this point, second arrival occurs this time point and third, fourth and so on.

The inter-arrival times means what is the time taken for the first arrival, then what is the interval of time taken for the first arrival to the second arrival and second to the third and so on. So that is the inter-arrival time. So whenever you have a Poisson process that means the arrival of event occur over the time that follows a Poisson process.

Then this inter-arrival time, suppose I make it as a random variable T and those random variables going to follow exponential distribution with the same parameter λ and all the inter-arrival times also independent. That means these are all identically distributed random variable, I can go for different random variable label also X1, X2, X3, X4 and so on. So all those random variables are i.i.d random variables and each follows exponential distribution with the parameter λ.

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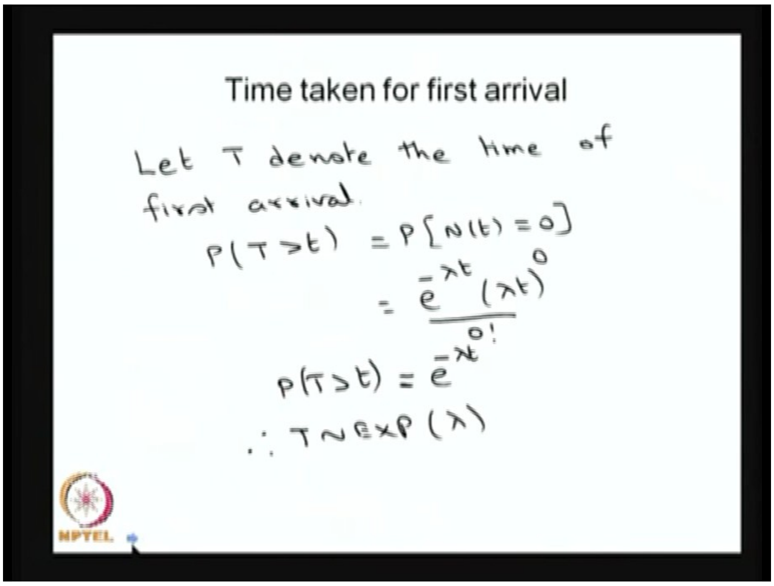

So this can be proved easily. Let me start giving the proof for the first arrival time. That means the first one from 0 to the first arrival, like that you can go for the other arrivals also using the other properties or you can use the multidimensional random variable distribution concept and use the function of a random variable and you can get the distribution also.

But here I am finding the distribution for the first arrival. So let T denote the time of first arrival. My interest is to find out what is a distribution of T. I know that this is going to be a continuous random variable because it is a time, so anytime the first arrival can occur. So, since it is a continuous random variable, I can find out the CDF of the random variable or complement CDF.

So here I am finding first the complement CDF using that I am going to find out the distribution. Let me start with the probability that the first arrival is going to takes place after time t. What is the meaning of that, the first arrival is going to occur after time t. That means till time t there is no arrival. So both the events are equivalent events. The $P\{T > t\} = P\{N(t) = 0\}$.

That means no event takes place till time t because the N(t) denotes the number of arrival of customers during the interval 0 to small t, both are closed, 0 to t. Therefore, N(t) = 0 that

means till time t nobody turned up, that is equivalent of the first arrival is going to takes place after t. I do not know the distribution of T but I know what is the P{N(t) = 0}.
That is why, I am writing this relation. So once I substitute the probability mass at 0 for the random variable N(t), just now we have proved that N(t) for fixed t is a Poisson distribution random variable with a parameter λt. Therefore, I know what is the probability mass at 0. So substitute the probability mass function with the 0. I will get $e^{-\lambda t}$ that is a complement CDF of the random variable T.

Once I know the complement CDF, I can find out of the CDF, from the CDF I can compare the CDF of some standard continuous random variable. I can conclude this is nothing but exponential distribution with the parameter λ because this is a complement CDF at time t. Therefore, it is a λt. So I conclude the distribution of a T is exponential distribution with the parameter λ.

That means the first time of arrival, this random variable that is a continuous random variable and the continuous random variable follows exponential distribution with the parameter λ. Since, I know the increments are independent, increments are stationary and so on, I can use the similar logic for inter-arrival time of this time also, then that is also going to follow exponential distribution. Since the increments are independent.

So this is the first time and this is second time. Therefore, the inter-arrival times also going to be independent. That means, whenever you have a Poisson process, that means the arrival occurs over the time in a very small interval, maximum one arrival takes place and the probability of one arrival in that small interval is λ Δt, from that you will get the λ. So you can conclude that is a Poisson distribution, Poisson process.

So ones the arrival follows a Poisson process, the inter-arrival times are exponential and independent. So from the Poisson process, one can get the inter-arrival times are exponential distribution and independent. The converse also true, that means if some arrival follows with the inter-arrival times exponential distribution and all the inter-arrival times are independent.

Then you can conclude the arrival process is going to form a Poisson process. That means arrival process and Poisson process implies the inter arrival times are exponentially distributed and are independent. Similarly, inter-arrival times are independent as well as exponentially distributed with the parameter λ, then the arrival process is a Poisson process with a parameter λ.

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Lecture - 83

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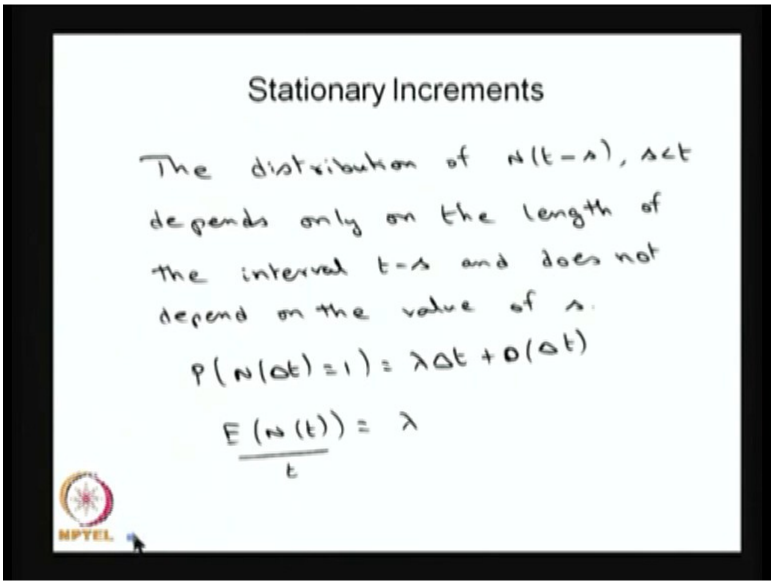

Now I am going for the stationary increment. The distribution of N(t-s) depends only on the length of the interval t - s and does not depends on the value of s. That means, during the interval Δt, the one arrival is going to $\lambda \Delta t$, order of Δt that will tends to zero as Δt tends to zero.

That means, the stationary increment means if you find out the rate that means you find out the average per unit of time, then that is going to be constant. So this is the assumption we have taken it in the car insurance problem, the average rate per unit day, that is going to be constant and that is an assumption we have taken at going to be a constant throughout the year and also the different times of a day.

So here also we will get whenever we have a Poisson process, then the average rate is going to be a constant because of the stationary increment.

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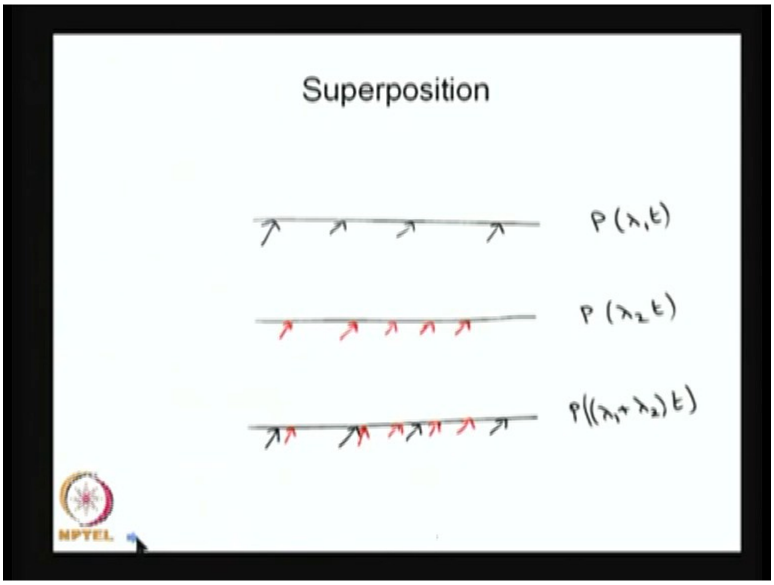

The next property, suppose you have a Poisson process of a one arrival and you have a Poisson process of the other arrival, that means one type of arrival is a Poisson process with the parameter λ_1 and another type of arrival who that is also Poisson process with the parameter λ_2. As long as both are independent, the arrivals are independent, then the together superposition.

That is going to be again Poisson with the parameter $\lambda_1+\lambda_2$. You can add the parameter. That means for fixed t that is going to be a Poisson distributed random variable with the parameter $(\lambda_1+\lambda_2)$t. Whenever you have a two independent or more than one independent Poisson process arrival, then the merging or the superposition will be again Poisson process as long as they are mutually independent with the parameter is nothing but the sum of those parameters.

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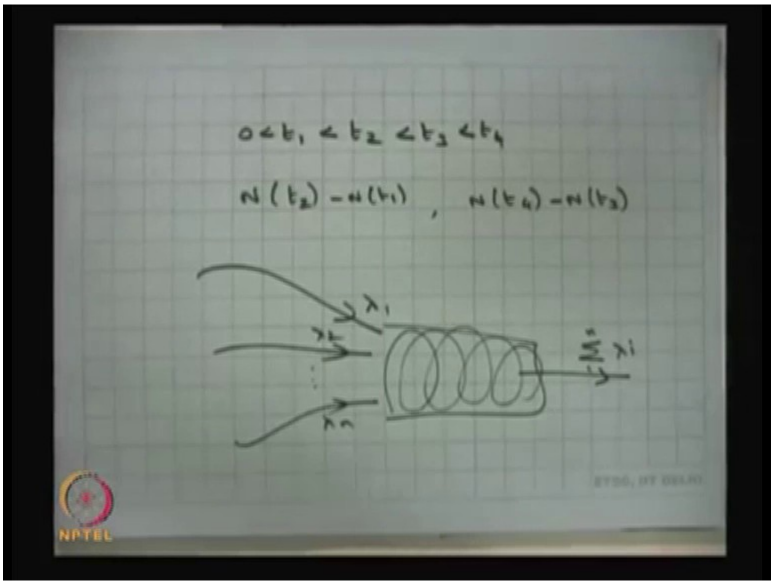

That is, you can combine many Poisson process as streams into one stream and that is going to be a Poisson stream with the parameter sum of parameters λ_1 ¿ λ_n. So this is possible, this is used in many telecommunication application, that means, suppose, you have a Poisson arrival of a packets from the different streams and all the streams are mutually dependent, the arrival are independent.

Then the total number of packets arriving into the particular switch or router, whatever it is. Then the multiplexed one, that is going to be always Poisson process, that arrival follows a Poisson process with the parameters are sum of, parameter is nothing but the sum of these parameters as long as they are Poisson as well as independent.

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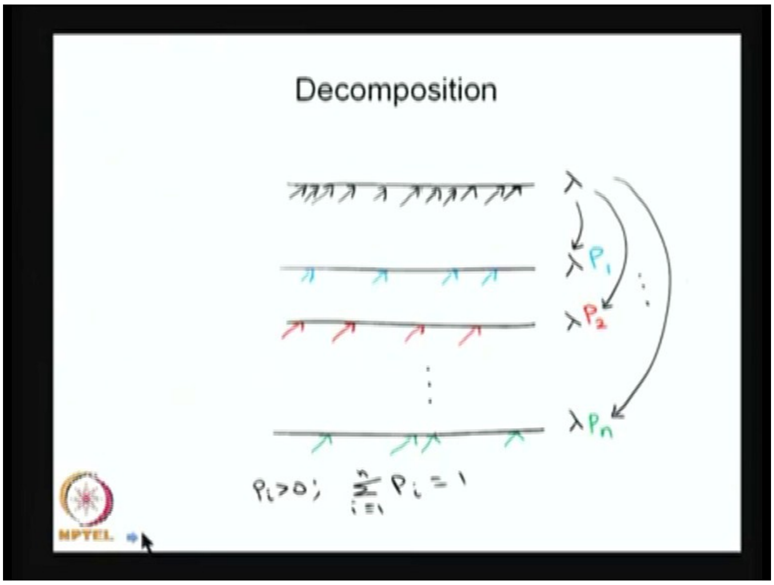

The next property, decomposition, suppose if you have a one Poisson stream, you can decompose into many Poisson streams with the sum proportion. So that proportions are the p_1, p_2 and p_n's. So one Poisson stream can be split into n Poisson streams with the parameter λ p_1, λ $p_2$where each p_i's > 0. $\sum_i p_i = 1$.

That means these are all the probabilities. With these probabilities you can split one Poisson stream into many Poisson streams. So here I have made a n Poisson streams, that means the same arrival is with some probability p_1, it lands up here with some probability p_1, this put up here, with some probability p_n is put up here. So the split of one Poisson stream into n Poisson streams is allowed.

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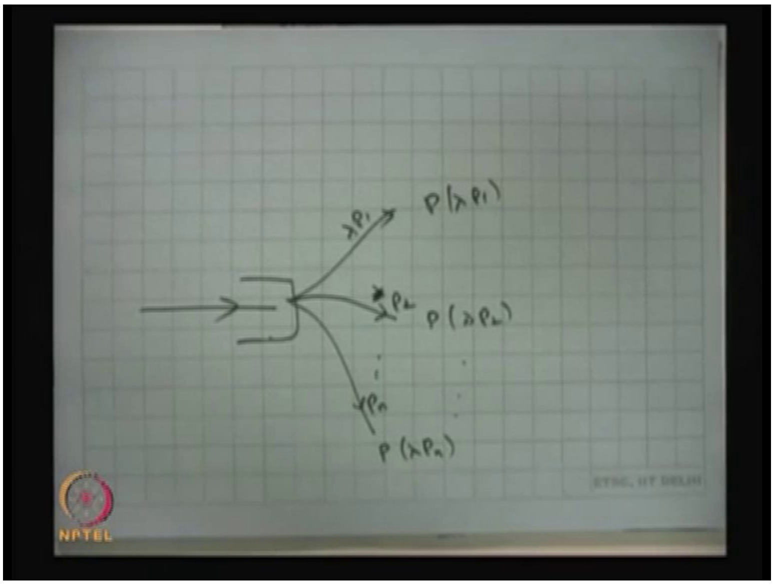

That means the same example, if you have a one router and from the router if the arrival is splitted into many streams with a probability p_1, it goes to the first stream with the probability p_2 it goes to the second stream and with the probability p_n it goes to the last stream. Then each one is going to be a Poisson process with the parameter λ p_1 and λ p_2 and so on, λ p_n.

So the split is possible as well as the super imposition is also possible from the Poisson process. So this also has a many more applications in the telecommunication networks. One type of packet arrival can be splitted into n proportions p_1, p_2, p_n's and each one is going to be a Poisson process.

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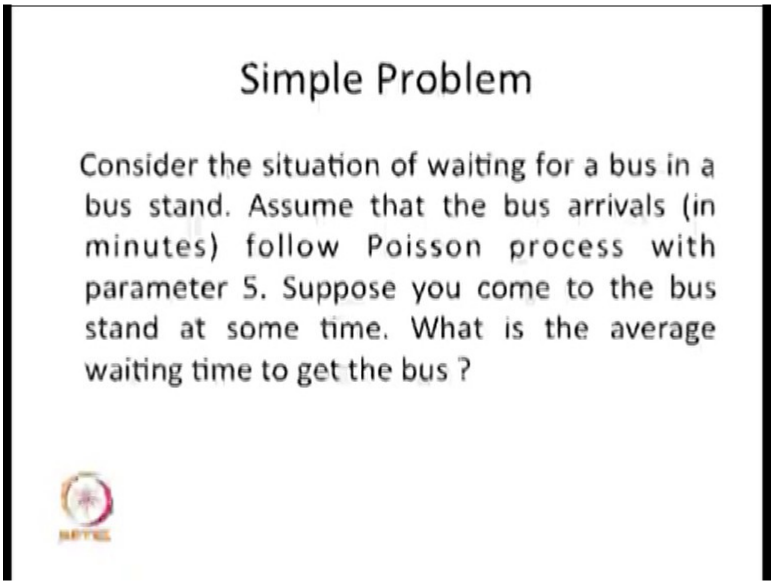

Now I am going to give the first example to illustrate the Poisson process. Consider the situation of a waiting for a bus in a bus stand. Assume that the bus arrivals in minutes follow a Poisson process with the parameter 5. With the rate, the parameter here that is nothing but the intensity or rate. Suppose you come to the bus stand at some time, what is the average waiting time to get the bus.

When you land up bus stand, there is a possibility the bus would have come before some time, the time in which the next bus is about to come, you are going to take that bus and till that time, you are going to wait in the bus stand, that is the waiting time. So the waiting time is a random variable. So that is a continuous random variable. The question is what is the average waiting time.

One can find out the distribution of the waiting time also. Here the question is what is the average waiting time. So then, what I can do, I can use the Poisson concept here.

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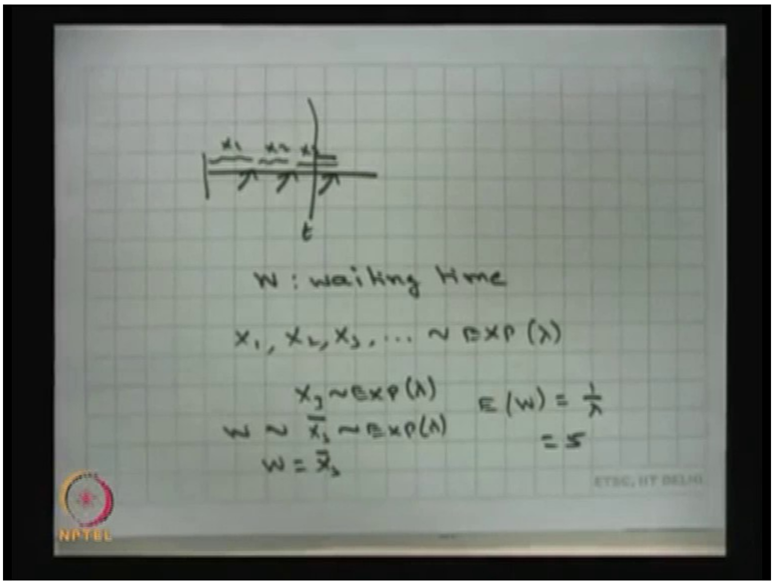

The arrival follows the arrival of a bus follows a Poisson process. Suppose at some time you come to the bus stand and suppose the bus is going to come at this time, your waiting time is this much. So suppose, you make W is going to be your waiting time, W is going to be your waiting time, the question is what is an average waiting time. Just now I have explained the Poisson process as the property, the inter-arrival times are exponential distribution.

The inter-arrival times are exponential distribution and all the times are, all the inter-arrival times are independent also. Therefore, this X_1 and this is X_2 and this is X_3, so X_1, X_2, X_3like that so many, all the inter-arrival times, that is going to follow exponential distribution with the parameter λ. Since the waiting time is going to be the remaining time of arrival of the third bus.

So the W, the waiting time is same as a the remaining or residual time of the third bus to come into the bus stand. So X_3 is exponential distribution with the parameter λ. The residual lifetime of X_3, suppose I make it as a notation $\acute{X}_3$, the residual lifetime, residual time of arrival, not life time, residual arrival time of the third bus coming to the bus stand, that is also going to be exponential distribution.

This is because of the memoryless property, the residual time is also, whenever, some time is exponentially distributed, some random variable time is exponentially distributed, then the residual time is also going to be exponentially distributed using the memoryless property.

Therefore, residual arrival time of bus to come to the bus stand, that is also exponential distribution with the parameter same λ.

So this is same as the W. The waiting time W is same as a residual time. Therefore, W is always going to be exponentially distributed with the parameter λ. That means the waiting time for the bus to come to the bus stand to catch. So the W is exponentially distributed therefore the question is what is a average waiting time. So average waiting time is nothing but one divided by the parameter.

So here it says the Poisson process with the intensity 5, that rate is λ, that is the mean inter-arrival times between the buses is 5 minutes, that means the mean inter-arrival times between the buses is 5 minutes is nothing but it is exponentially distributed with the parameter that is average 5 minutes therefore that is the same thing. Therefore, that is equal to 5 minutes.

Because the way I have given the clue, the mean interval between the buses is 5 minutes, that means the average of X_i's that is equal to 5 minutes. So that is as same as your waiting time because it is exponential distribution therefore the residual is also exponential distribution, therefore you can use the same value, therefore the average is going to be 5 minutes. So using Poisson process one can find out the different results related to the number of arrivals.

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Lecture - 84

Next I am going to give some more process related to the Poisson process.

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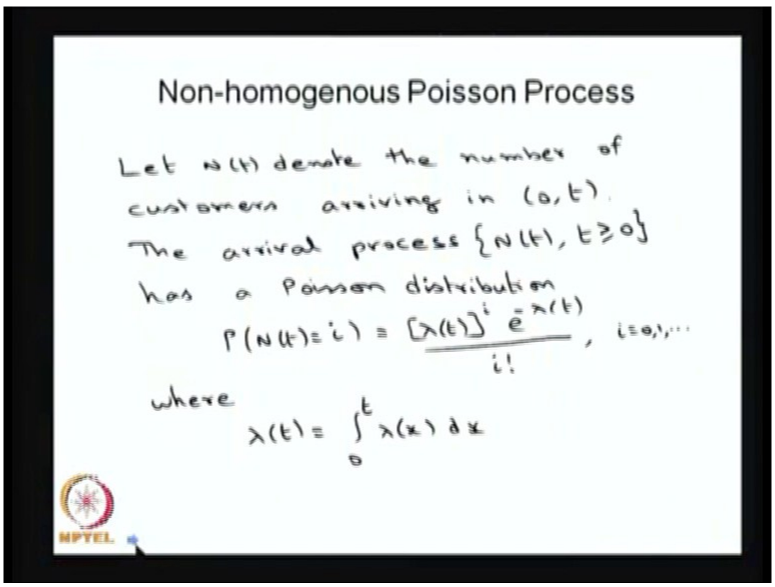

The first one is the non-homogeneous Poisson process, let N(t) you know the number of customers arriving in the interval 0 to t, the arrival process has a Poisson distribution but here the change instead of the mean arrival rate is a constant mean arrival rate is a constant λ but here it is a function of t, λ(t) is the cumulative rate t time t that is a change from the Poisson process then this stochastic process is called non homogeneous Poisson process.

Instead of mean arrival rate is a constant, here the λ(t) is a function of t therefore this stochastic process is called non homogeneous Poisson process.

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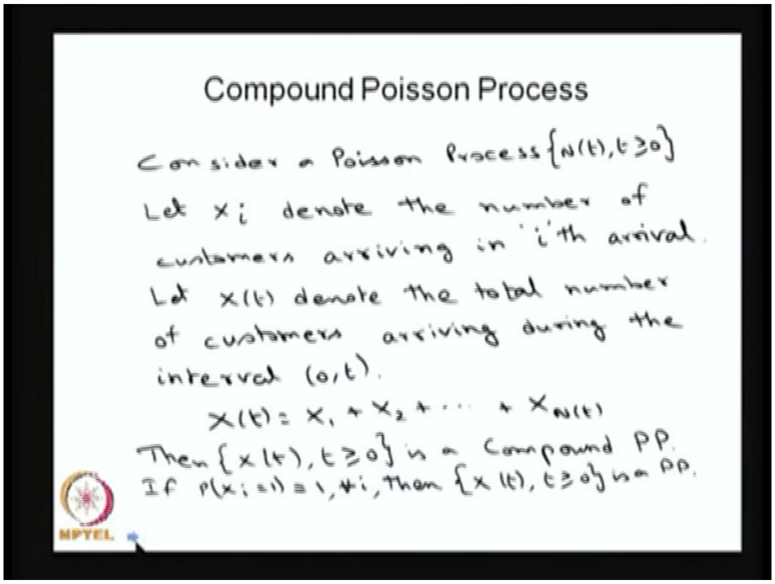

Second one Compound Poisson process using Poisson process one can develop a little complicated stochastic process related to the arrival that is called compound Poisson process, consider a Poisson process N(t) then you define a random variable X_i denote the number of customers arriving at the ith time point of arrival, X_1 denotes how many arrivals takes place at the time of first arrival, first arrival time point X_2 will be what is 2nd time of arrival, how many arrival takes place.

Therefore, I am making a new random variable X(t) that involves t that denotes the total number of customers arriving during the interval 0 to t that means it is going to be how many arrival takes place in the first time point X_1, how many arrival takes place at the second time of arrival that is X_2 and so on + $X_{N(t)}$, here N(t) is a random variable and how many arrival takes place that is X_i's altogether that is going to be the total number of arrivals.

X_i's are independent and identically distributed random variables with some distribution function G independent of the Poisson process N(t), so this is nothing but a random sum because these are all the random variable and how many random variables you are going to add that depends on the value of N(t) over the t.

Therefore, this is a random sum of X_i's with N(t), obviously these two are independent X_i's are independent of N(t), and since it is a number of our customers arrival during the in the ith time point, therefore X_i's are discrete random variable - X_i's are discrete random variable and N(t) is also Poisson process, therefore X(t) is going to be a discrete state continuous time stochastic process.

And we are using a Poisson process to get these stochastic processes therefore it is called a compound Poisson process. one can deduce Poisson process from a compound Poisson process by substituting each X_i takes a value only one unit that means the number of customers arriving at the ith time point is going to be only one that means if I make a probability of X_i takes the value only one that probability is one for all i.

Then I will have only one value possible till N(t), then this is going to be a Poisson process, suppose the probability of X_i is supposed the X_i's are going to be a discrete random variable with the possible values are 0, 1, 2 and so on, then the X(t) is going to be a Poisson process

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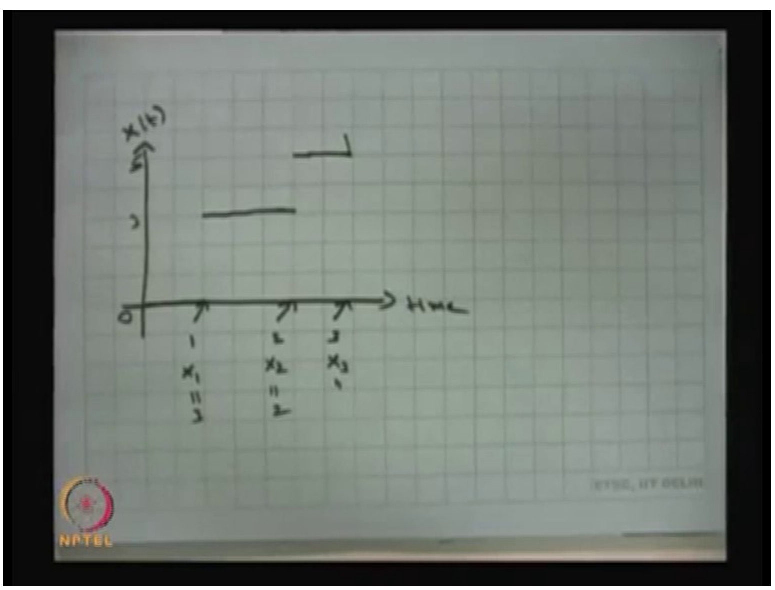

I can make a simple example path for the compound Poisson process this is over the time and this is over the X(t), suppose these are all the time points in which arrival time point so this is the first arrival time point and this is the second arrival time point and this is the third arrival time point. It can be anywhere in the continuous time therefore this is called discrete state continuous time stochastic process.

So here I am relating with the random variable X_1, this X_2 random variable this is X_3 so till the first arrival till the first time of arrival the number of customers in the system is 0, at the first time of arrival the X_1 suppose you think you make the assumption X_1 takes a value 3 therefore this will be incremented by 3 till the second arrival, at the time of second arrival suppose you assume that this takes value 2 with some probability.

Probability of X_2 takes the value 2 is greater than 0 so you have assumed the value 2, it can take in any other value also so it is incremented by 2 till it takes a third arrival the value is so this is 0 this is 3 then 3 + 2, 5 at this time whatever be the number of arrival accordingly this can take some value.

So the difference between the compound Poisson process and the Poisson process in the Poisson process the increment will be only one-unit increment over the time whenever the time it which the arrival occur - arrival time approves, whereas here wherever the time of arrival time approves the number of customers entered that need not be 1, it can be more than or equal to 1 so that is the way the jump goes, therefore this is called a compound Poisson process.

So we have seen two variations of Poisson process one is a non-homogeneous Poisson process the other one is a compound Poisson process.

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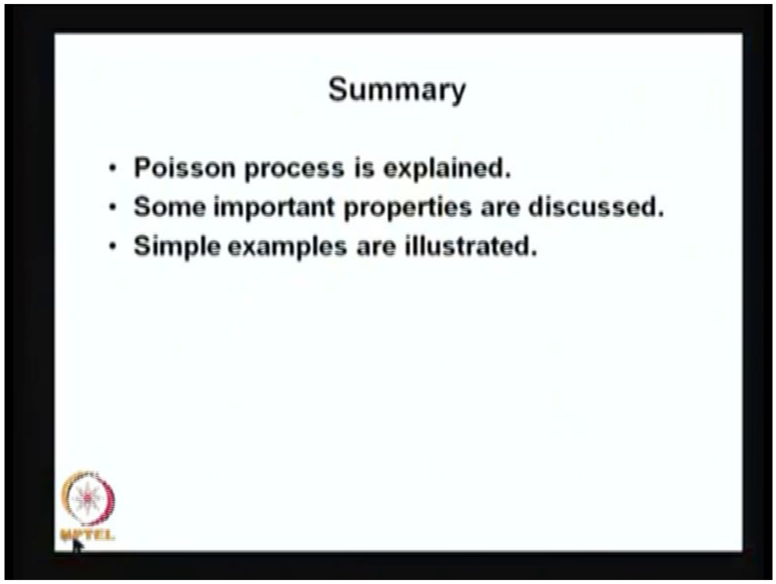

So before I go to the, I complete let me give the solution for the first example - second example which I started. That is discussing the car insurance problem.

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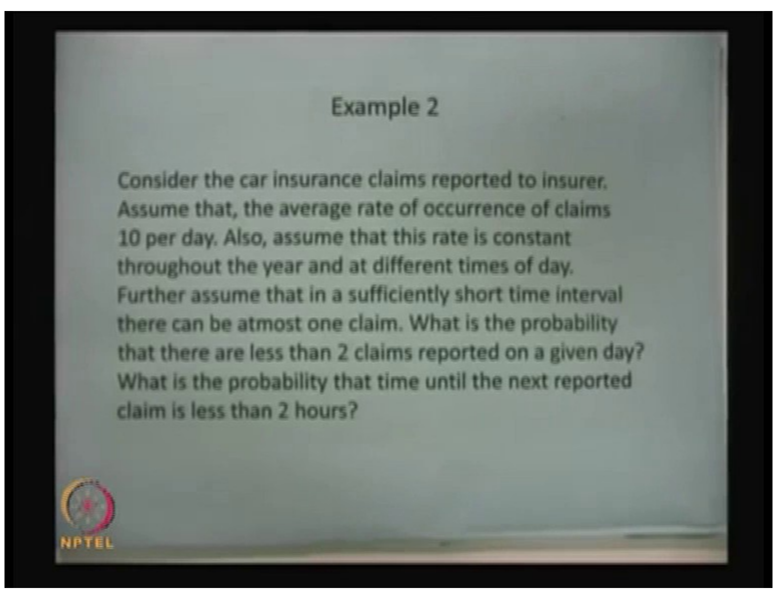

We have discussed two problems, the first problem is related to the bus stand bus arrival issues and this is the car insurance problem, so in this problem we have not assume the Poisson process, but the problem is related to the Poisson process one can assume it is a form of Poisson process, because you see the assumptions the average rate of occurrence of claims is a 10 per day.

Also the rate is a constant and in a very small interval of time at most one claim can happen the questions are what is the probability that there are less than 2 claims reported on a given day, since the increments are stationary so any day you can think of with the only the interval what is the probability that the time until the next reported claim is less than 2 hours, so this is related to use the exponential distribution because the inter arrival time are exponential distribution.

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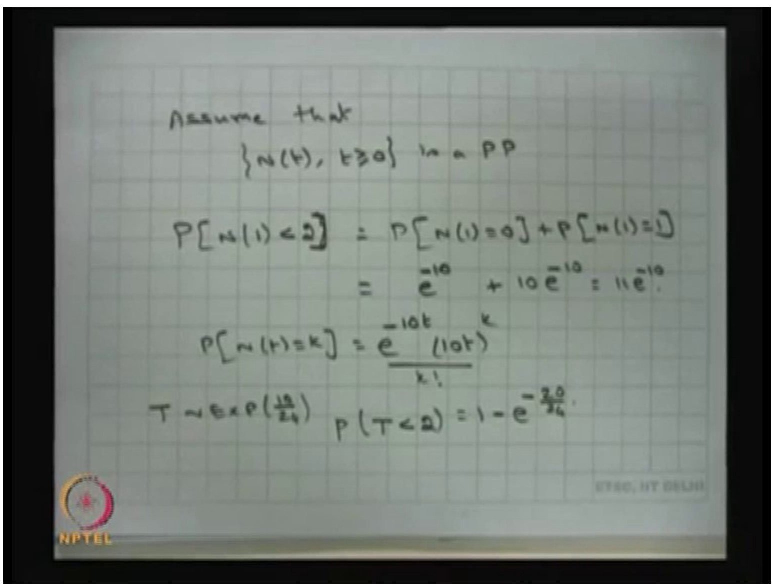

So for the first question you can assume that - you can assume that the N(t) is nothing but a number of a - number of insurance car insurance claims reported to the insurer that has Poisson process you can assume that N(t) is a Poisson process based on the assumptions given in the problem, once you assume that this is a Poisson process the question is what is the probability that there are less than 2 claims reported on a given day.

So the given day 2 days you can shift in to 0 to 2 days itself, because of the increments are stationary so the question is nothing but what is the P{N(1) < 2} in a given day a day, so what is the P{N(1) < 2}, that is nothing but what is the P{N(1) = 0} and P{N(1) =1}

Therefore, the probability is added so you substitute since it is N(t) is a Poisson process the P{ N(t) = k} = $e^{-\lambda t}\frac{(\lambda t)^k}{k!}$, here the λ is 10 per day, $e^{-10t}\frac{(10t)^k}{k!}$, so this is the probability mass function for the random variable N(t) for fixed t. Therefore, P{N(1) = 0}, that is nothing but e^{-10t}here the t is 1, 1 day + P{N(1) = 1}.

You substitute here therefore you will get $10e^{-10t}$, so the answer is 11 e^{-10t}, numerically you can get what is a value, so the probability that there are two claims reported on a given day is a $11e^{-10t}$.

The second question what is the probability that time until the next reported claim is less than 2 hours, so this is equivalent of the next reporting claim is less than 2 hours that means the residual time of the next claim that is going to happen the one claim is going to happen less than 2 hour that means you can use the inter arrival time that is exponential distribution with the parameter λ here the λ is a 10 or 10/2 hours.

Therefore, you should convert the values so it is a 10/24, claim can happen at any day throughout 24 hours, therefore 10 per day therefore it is 10 / 24 per hour, so that T is a exponentially distributed with a parameter 10/ 24 now the question is what is the probability that the time until the next report claim is less than 2 hours that means what is the P{T< 2}.

That is nothing but - that is nothing but since it is exponential distribution and you know the CDF of the random variable T so the P{T < 2} is nothing but a 1- $e^{\frac{-2*10}{24}}$ so 20 by 24, so the answer is 1- $e^{\frac{-20}{24}}$that is the probability that the next report claim is going to be occur before 2 hours, so with these I have completed two examples also. So in this lecture we have discussed a Poisson process and we have illustrated two examples for the Poisson process also, some important properties also discussed in this.

Thanks.

Introduction to Probability Theory and Stochastic Processes
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Module – 12
Simple Markovian Queueing Models
Lecture - 85

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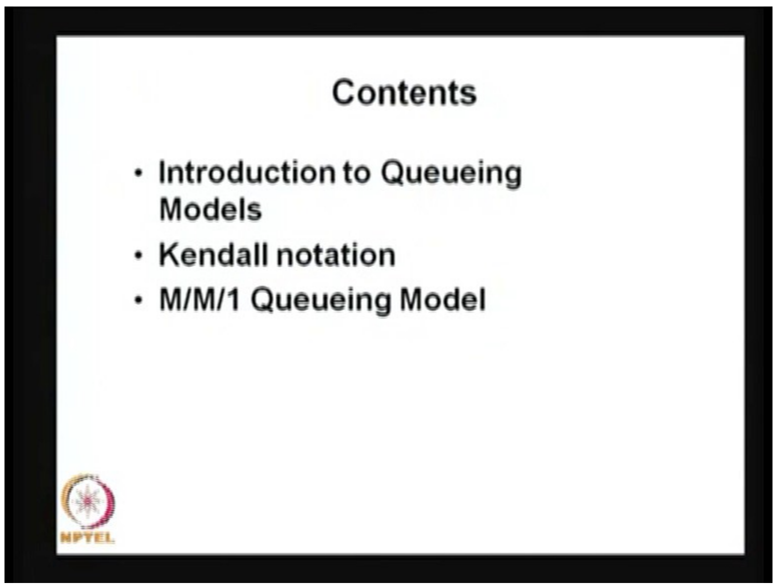

In this talk I am going to discuss the queuing models, so for that I am going to give the introduction to the queuing models then I am going to discuss the Kendall notation then followed by that the simplest queuing model M/M/1 queue model will be discussed, and this is going to be the applications of continuous time Markov chain in queuing models, so in this lecture I am going to discuss only the simplest queuing model M/M/1 queues.

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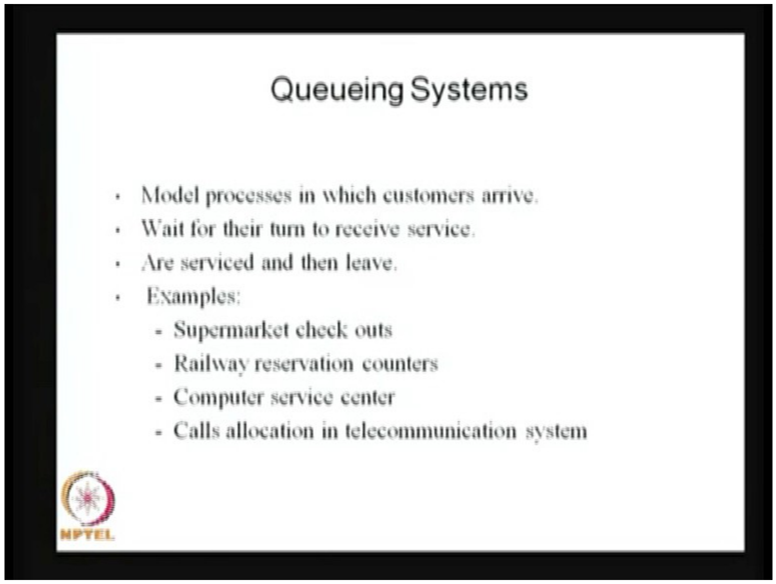

So how one can define the queuing system, you can see many examples in which whenever you go to the supermarket to get some items or you see the railway station counters or you can see the computer service center many PC's are there and printers and so on, so how the queueing system is created and also you can see the examples in the calls allocation in telecommunication systems in all those examples you can see something is getting served and leave the system.

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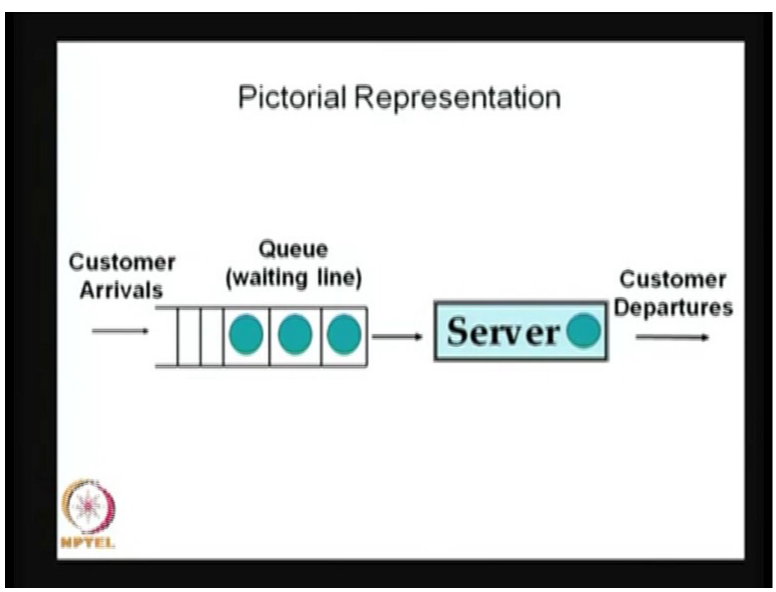

We can give the queuing system, we can represent the queuing system in a pictorial form some customers are coming into the system and waiting for their service, once the service is over then they departure from the system, so this is the way one can visualize the queuing system in a pictorial form.

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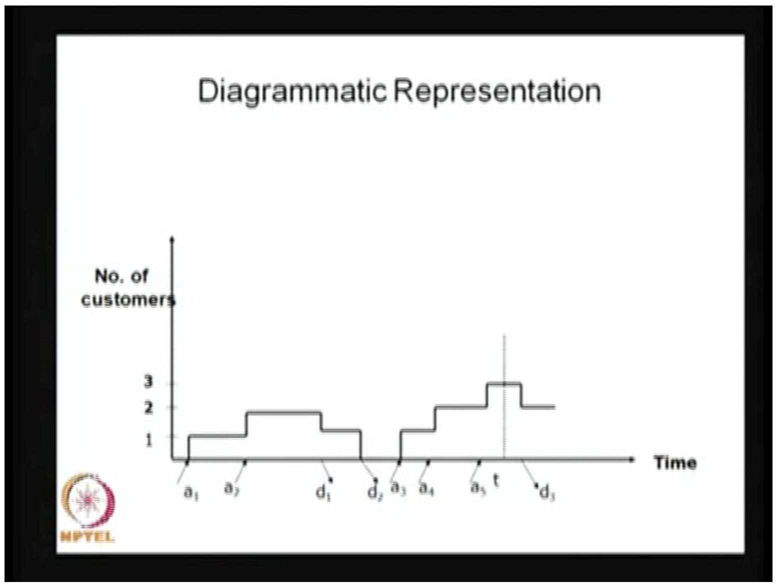

This is a diagrammatic representation and x axis is the time and y axis is the number of customers in the system, suppose at time a_1, the first customer enters into the system then the number of customers in the system is incremented by 1, the customer who entered the system is to getting the service during his service time the next customer enters the system that with the time point a_2.

Therefore, now that number of customers in the system is 2 going on at this time point the first customer service is over so he departure from the system that is d_1, that time point in which the first customer service is over, now the number of customers in the system is 1 the time point d_2 the second customer service also gets over, now the number of customers in the system is 0.

The third customer enter at the time point a_3, so during this interval the system was empty, so like that the system is keep increasing whenever one customer enter into the system and decreasing by going one whenever the service is completed. So this is the diagrammatic representation of any queuing system.

Here I made the assumption it is a very simplest one only one customer entering into the system and only once customer is getting served and leave the system and so on, so this is the simple way of simple diagrammatic representation of the queuing system.

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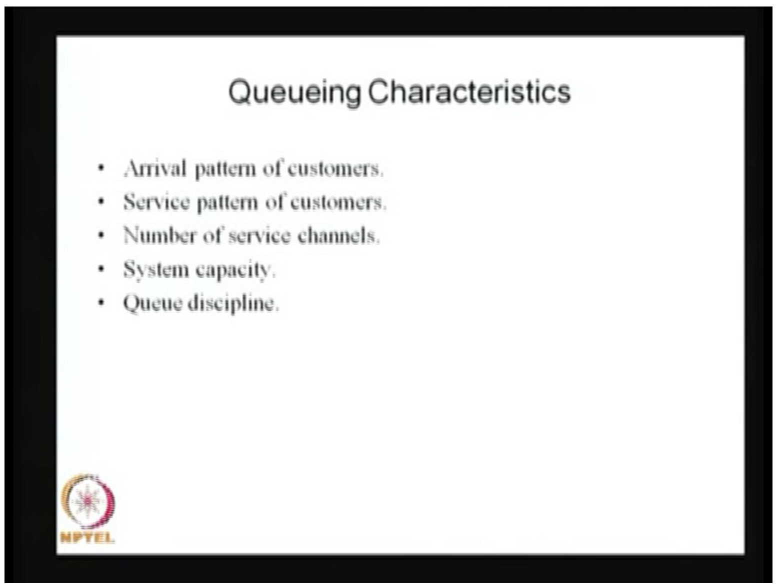

So to define the queuing system you need a few important characteristics using that one can easily frame the queuing system. So for that you need the first information that is arrival pattern of customers, how the customers are entering into the system how frequently whether the customers are coming in a very constant interval of time or in random fashion, if it is constant then we say the inter arrival time is a deterministic.

If the customers are entering into the system with the inter arrival time that is some random variable then we should know what is distribution of inter arrival time, so this information is needed to define the queuing system the arrival pattern that includes whether it is a deterministic or probabilistic, if it is a deterministic then what is the inter arrival time that constant time if it is a probabilistic then what is a distribution and so on.

Similarly, after the customers entering into the system you should know how the service takes place, whether the service time for each customer who enter into the system is it a constant or random, if it is a constant amount of service for each customer then what is a time how much time it takes for each service, if it is a probabilistic then what is the distribution of service time. Then, the third important information are the characteristic is a number of servers in the system.

How many service channels are available to do the service whether you have only one server in the system are more than one or countably infinite numbers, so according to that the queuing system may vary, the third information is number of - number of servers in the system. The fourth information, that is system capacity whether the capacity is a finite one or infinite capacity accordingly the number of customers in the system may go maximum the finite capacity.

Or the infinite number of customers can wait in the system to get the service, therefore the system capacity is also important characteristic. The fourth one queuing discipline, when the customers are entered into the system whether they are getting served or whether they are placed in a first come first order or first come last service or random fashion or priority based and so on.

So the queueing discipline also important to know - to know the how the queuing system is at any time, to know that dynamics of number of customers in the system and you should know how the queuing discipline is taken care. Similarly, the service discipline also, how the service is also takes place during the picking the customers for the service, so these are all the minimum important information to characterize the queuing system.

One is a arrival pattern, second is a service pattern and third is a number of servers, the fourth is a capacity of the system and the service discipline or queuing discipline.

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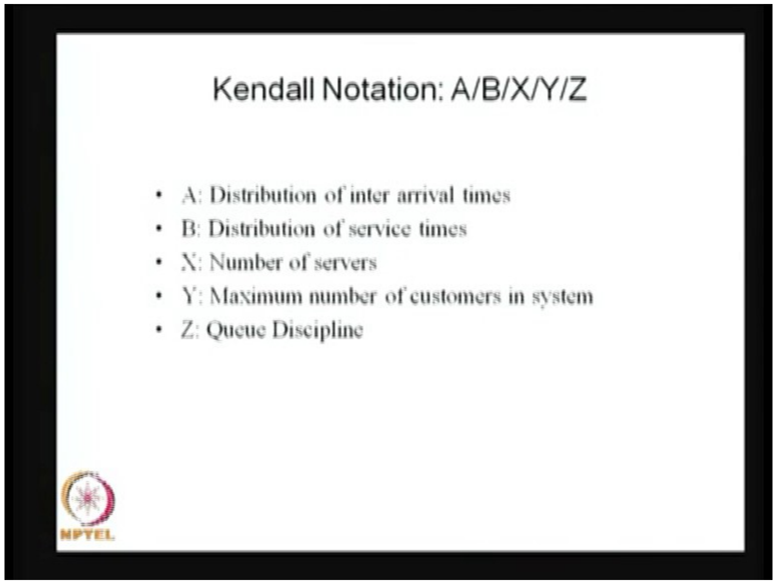

So based on that, the Kendall made a notation and that notation is called a Kendall notation, the Kendall notation consists of A/B/X/Y/Z, so the possible values you are going to assign for A,B,X,Y,Z, accordingly one can define the queuing system and each letter is corresponding to some important characteristics of the queuing system.

A denotes the arrival pattern information here the A denotes the distribution of inter arrival time, the letter corresponding to the A. The second one B, whatever the letters you are going to assign for the second one that denotes the distribution of the service time, the way I have said the characteristic the first one is arrival pattern, second one is service pattern and so on, the same way we have given the Kendall notation.

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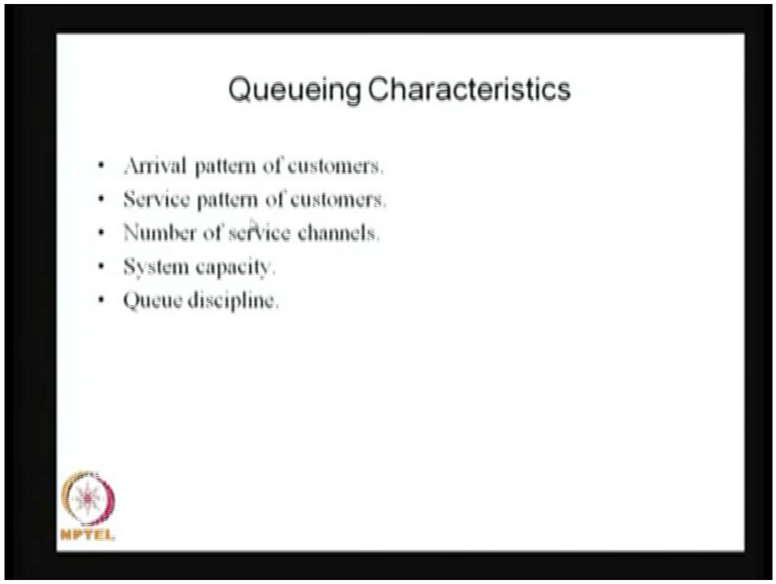

So the capital A is for the - the letter whatever the letter you are going to assign for A, that is for the distribution of inter arrival time and B is for the service time distribution. The third one X, whatever the number you are going to write that is the number of servers in the system. The fourth one what is the capacity of the system.

The fifth one, what is the queueing discipline whether it is a first Come first served, last come first served, priority, random and so on. Now I am going to give what are all the different possible values for these letters.

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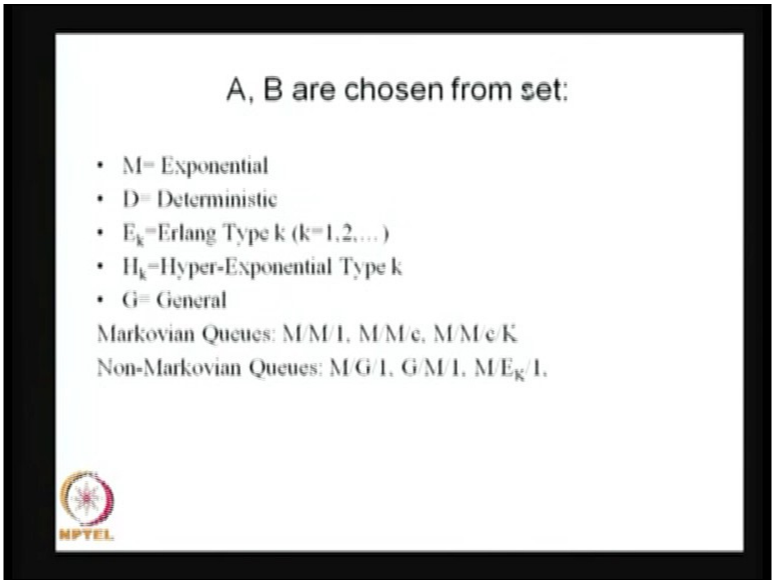

The first two, A is for the distribution of inter arrival time, B is for the distribution of service time, the both can be chosen from these letters, if you write M in the first place that means the inter arrival time is exponentially distributed, even though it is exponentially distributed we use the letter M because of exponential distribution satisfies the memoryless property or Markovian property so to denote that we use the letter M.

So whenever you write M in the place of A or the second place B then that means the inter arrival time is exponentially distributed or service time is exponentially distributed respectively, suppose you write the letter D in the place of A or B that means that distribution is a deterministic that means it is going to take - it is not a probabilistic its takes a constant amount of time whether you placed in the first or second accordingly.

So it is going to be a constant amount of time going to take for the inter arrival time or service time whenever you place it in A or B respectively, similarly if you use the letter E_k that means it is a Erlang distribution of type k or we can say Erlang distribution of a stage k, that can be 1, 2 and so on, that means the inter arrival time is a Erlang distributed with the stage k if you place it in the first letter.

Similarly, $H_k$means hyper exponential distribution of a type k, whenever you have a inter arrival time is other than exponential, deterministic and so on, so usually other than exponential you can use the letter G, G means general distribution, general distribution is also it is a known distribution the only thing is it is other than exponential distribution, so either you can use the letter M, D, E_k, H_k, or G.

So G can be other than M itself in the usual or in general form, it is a known distribution that other than exponential we use the letter called G for general distribution, so these are all the possible values for the A and B, whereas the third one is the number of servers in the system and the fourth one is a capacity of the system and the fifth one is queueing discipline, the default discipline is a first come first served.

Therefore, no need to write the fifth information and the sixth information also there what is the population of the customers who are entering into the system, the default population is infinite that means from infinite source the customers are entering into the system, that is the sixth information, as long as we will not write, as long as system in which the population is infinite as well as the giving discipline is first come first served, then we will not write.

So we write only the first four information, that is a inter arrival time distribution, second one is service time distribution, the third one is number of servers and the fourth information is capacity of the system, so in these examples the inter arrival time and service time both are exponentially distributed by default they are independent also.

And the third letter denotes number of servers in the system, so here only one server in the system, here c means it can be greater than or equal to 1, that is multi server system and fourth letter K means the capacity of the system, suppose we did not write the fourth information here that means it is an infinite capacity system and this is also infinity capacity system.

And since the inter arrival time and service time are exponentially distributed this model is called the Markovian queues, because it satisfies the Markov property whereas non Markovian queues either service time or the inter arrival time can be a non-exponential distribution or non-exponential distribution in default we can use a letter general distribution.

So whenever G comes in the first place or the second place then we use non expo - when we then we say it is a non-exponent - non Markovian queues and if the fourth letter is missing that means it is an infinite capacity system.

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Lecture - 86

There are many applications of queuing system, we are going to discuss the abstract queuing system in the further lecture.

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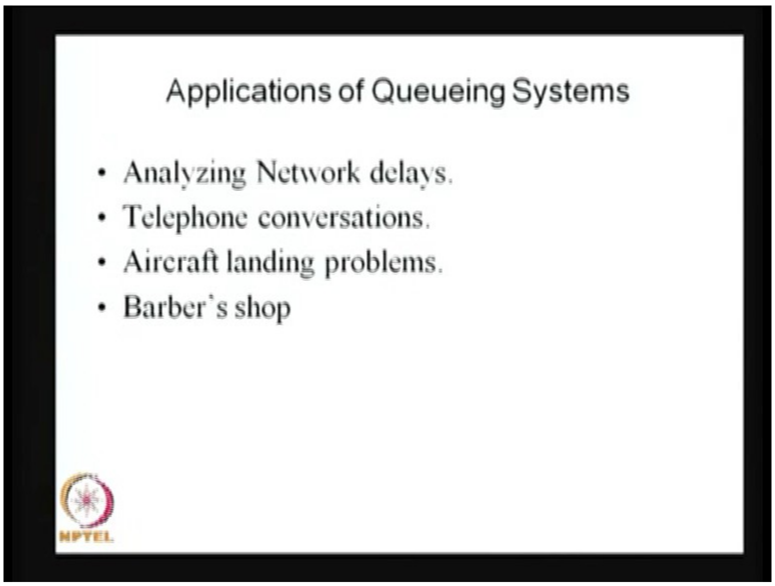

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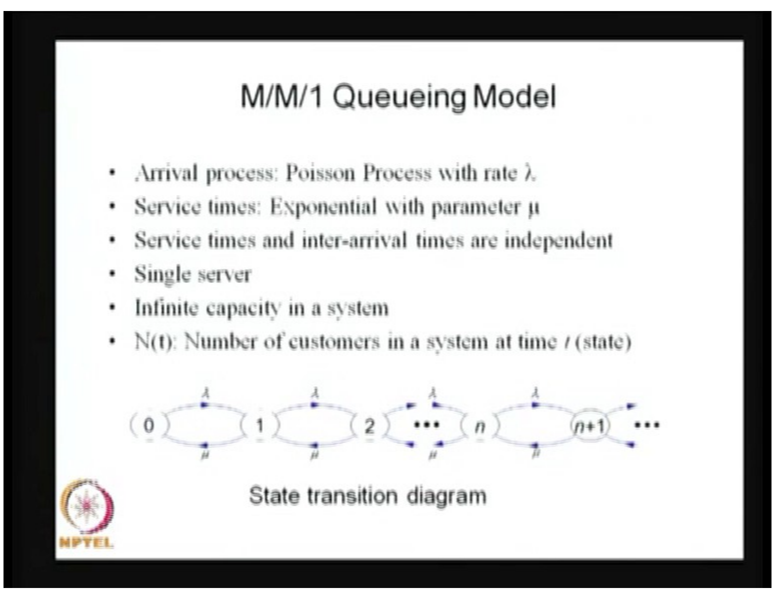

The easiest or the simplest queuing model that is a Markovian queuing model that is M/M/1 queuing model, later we are going to correlate to the birth death process also, in the M/M/1 queuing model the inter arrival time is exponentially distributed as I discussed the Poisson process in the previous lecture whenever you have arrival follows Poisson process then the inter arrival time follows exponential distribution and are independent also.

So here the first information is arrival process follows a Poisson process with intensity or rate λ, that means the inter arrival times are independent and each one is exponentially distributed with a parameter λ. The second information that is service time, service times are exponentially distributed with a parameter μ and the service times are independent for each customer and that's also independent with the arrival process.

That means there is no dependency over the arrival pattern with the service pattern, service times and inter arrival times are independent. Then the third information only one server in the system that’s a queuing system in which only one server. And the fourth information is missing that means it is a default its infinite capacity model - infinite capacity model.

Now our interest is to find out the behavior of a queuing system or the behavior of number of customers in the system at any time t therefore you can define a random variable N(t), that is nothing but the number of customers in the system at time t, therefore this is going to follow form a stochastic process over the t, since the inter arrival time is exponentially distributed and the service times are exponentially distributed.

The memory less property is going to be satisfied throughout all the time, therefore this stochastic process there is a discrete state continuous time stochastic process satisfying the Markov property, therefore this is a Markov process, since inter arrival time is exponentially distributed and the service time is exponentially distributed and both are independent and the service time is also independent for each customers.

Therefore this stochastic process satisfies the memoryless property at all-time points, therefore these discrete state because the possible values of N(t), since it is a number of customers the possible values are 0,1, 2 and so on, countably infinite therefore it is a discrete state and you are observing the system over the time therefore it is a continuous time.

Therefore this stochastic process is the discrete state continuous time stochastic process satisfying the Markov property based on these assumptions, therefore N(t) is a Markov process since the state space is a discrete therefore this is a Markov chain, therefore this is a continuous time Markov chain, therefore N(t) is a CTMC, so one can write the state transition diagram for the - for this CTMC.

That means the possible states are 0, 1, 2 and so on, so this will form a nodes and you try to find out what is the rate in which the system is moving from one state to other state, since it is a M/M/1 queue model queuing model, therefore whenever the system is in the - whenever the system is in the state 0 by the inter arrival time which is exponentially distributed the number of customers in the system will be incremented by 1.

Therefore that rate will be λ or the system moving from the state 0 to 1 it spends exponentially distributed amount of time here before moving into the state 1, once the system come to the state 1 either one more arrival is possible or the customer who is under service then service could have been finished, therefore the service time is exponentially distributed with a parameter μ, therefore the system goes from the state 1 to 0 with a parameter μ.

Similarly from 1 to 2 because of the inter arrival time is exponentially distributed with the parameter λ, therefore this is λ, since the arrival follows a Poisson process in a very small interval of time only one customer is possible with the probability $\lambda \Delta t$ and so on, therefore there is no way the system goes from one state to jump into more than one state that is not possible forward.

So only one step forward is possible because of the arrival process follows the Poisson process and since we have only one server in the system, the system also decremented by only one level below, therefore this is going to form a birth death process, the reason for this CTMC going to be a birth death process because of the arrival process follows a Poisson process so whatever the assumptions we have it for the Poisson process that is going to be satisfied.

And since we have only one server in the system and he does the service for only one customer at a time after finishing that server - after finishing the customer service then it move into the next service immediately and so on, if the customers are available in the queue, therefore the system goes to the one step one state below by only one move only, it won't move from 2 to 0 or 3 to 1 and so on, therefore this CTMC is a birth death process.

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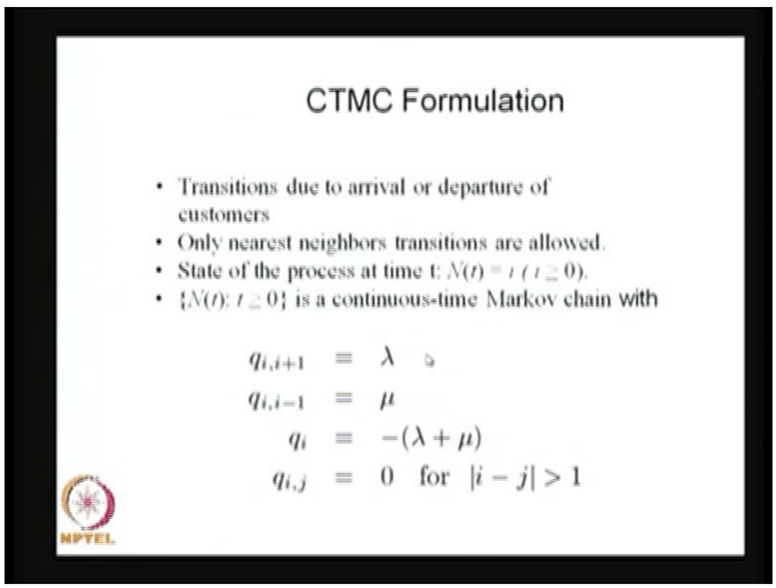

Therefore I am connecting the CTMC with the M/M/1 queue in particular the CTMC the birth death process, because of the transitions due to arrival or departure of a customer and only nearest neighbors transitions are allowed, because of the assumptions which we have made, therefore this is going to a continuous time Markov chain with the rate in which the system moves from the state i to i + 1, that rate is λ.

And the system moves from the state i to i - 1 that rate is μ, and all other rates are going to be 0 other than the diagonal element and this rates also constant not state dependents rates, therefore this is a birth death process with the birth rates λ and the death rates μ.

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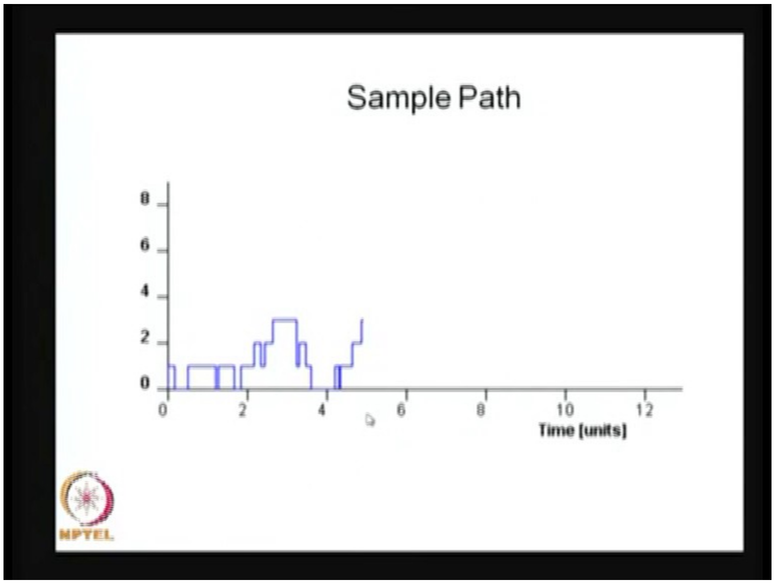

So this is a sample path suppose a time 0, one customer in the system then it services over then the second customer enter into the system, now the number of customers in the system is 1 and so on, so that means this duration is the service time for the first customer and from this point to this point that is a inter arrival time of the second customer enter into the system.

And from this time point to this time point that is the service time for the second customer which is independent of the service time for the first customer and this is the time point the second customer enter and this is the time point in which the third customer enters therefore the inter arrival time is from this point to this point and so on, so this is a dynamics of a number of customers in the system over the time.

Therefore, this stochastic process is a discrete state continuous time stochastic process satisfying the Markov property, therefore this is a continuous time Markov chain. So later I am going to simulate the M/M/1 queuing model using some simulation technique.

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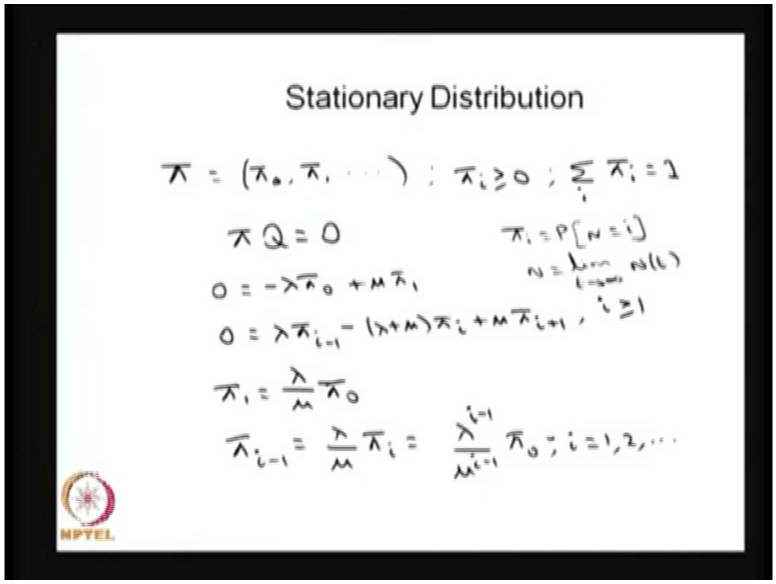

So the conclusion is the underlying stochastic process for the M/M/1 queuing model is a birth death process N(t) is a stochastic process, so this stochastic process is a birth death process.

Therefore now we are going to discuss the stationary distribution time dependent probabilities and so on. So how to find the stationary distribution, solve $\pi Q = 0$, pi is the vector consists of a π_i's, where π_i's are nothing but what is the probability that N customers in the system what is the probability that i customers in the system in a long run, so that long run is defined in this way the N(t) is a stochastic process as t tends to infinity, the number of customer in the system in a long run that is going to be the N.

And π_i is nothing but a probability that i customers in the system in a longer run, so now we are going to solve $\pi Q = 0$ with the normalized equation $\sum_i \pi_i =$ ¿¿ 1, so once you frame the equation you will get a π_1, in terms of π_0 and π_{i-1} in terms of a first π_i, then substitute recursively you will get in terms of π_0.

So since it is a homogeneous equation you will get all π_i's in terms ofπ_0, so use the normalizing equation $\sum_i \pi_i =$¿¿ 1, you will get π_0.

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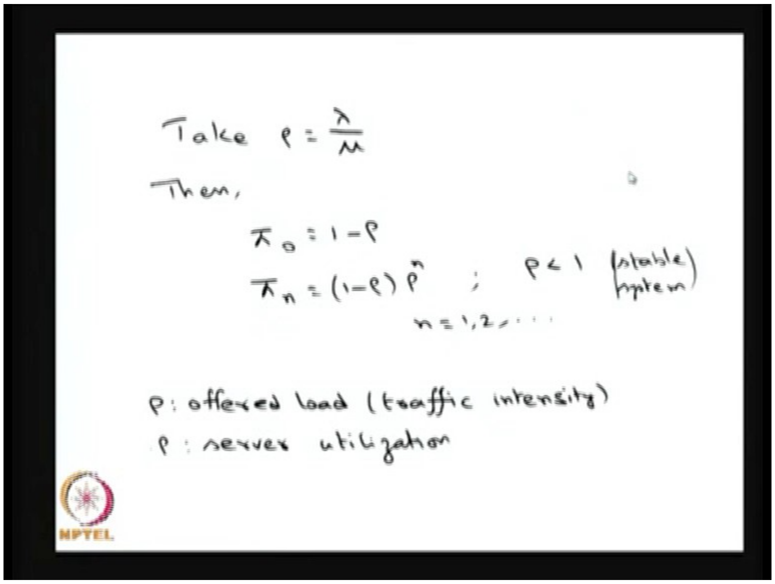

So the $\pi_0 =$¿1-ρ, where $\rho = \frac{\lambda}{\mu}$and since I am relating this stochastic process with a birth death process with a infinite capacity, if you recall the stationary distribution exist as long as the denominator of π_0 that series converges, so that will converge only if $\frac{\lambda}{\mu} <$¿ 1, if $\frac{\lambda}{\mu} \geq 1$ then that denominator diverges accordingly you won't get the stationary distribution.

So to have a stationary distribution you need rho has to be less than 1, that also you can intuitively say whenever system is stable that is corresponding to ρ < 1in that you will have a stationary distribution that means in a longer run this is the proportion of the time the system will be empty and the π_n is nothing but the n customers in the system in a longer run, that is a $(1-\rho)\rho^n$, where ρ < 1.

This ρ can be visualized as the offered load also, because the λ is nothing but the mean arrival rate and the μ is a mean service rate and this ratio will give the offered load and $1 - \pi_0$ that is the probability that the system is non-empty and that is nothing but the server utilization.

Server utilization is nothing but what is the probability that the server is busy, the server will be busy as long as the system is non-empty, so the ρ is the server utilization that can be obtained in the from this formula and in a longer run the server utilization is ρ.

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Lecture – 87

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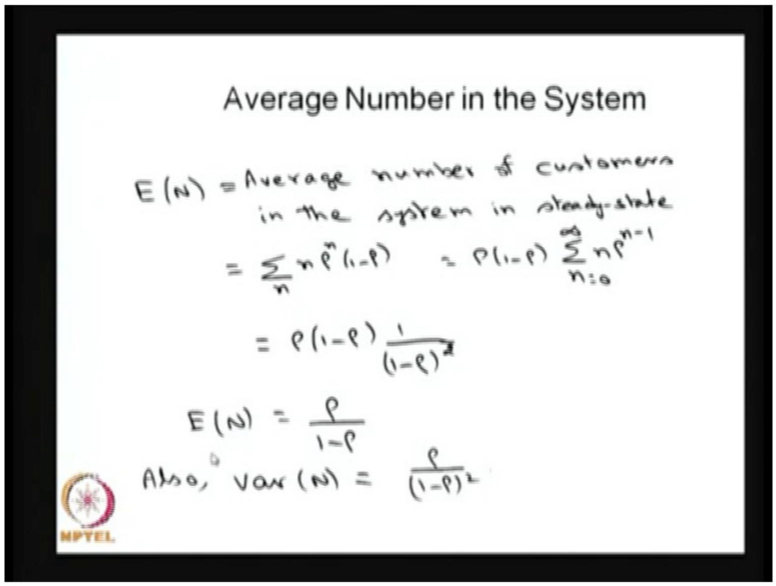

Other than stationary distribution one can find out the average measures also in the system, so suppose you make E(N) that is nothing but the average number of customers in the system in steady state, since you know the probability distribution substitute $\pi_n^{'}s$here therefore $\sum_n n\pi_n$, that he is going to be the average number of customers in the system, if you do little simplification you will get $\frac{\rho}{1-\rho}$, where $\rho < 1$.

So this is an average number of customers in the system and also one can get a variance of the number of customers in the system also for that you have to find out the $E(N^2)$ then using that formula you can get the variance of N also, so here we are getting a mean and variance of number of customers in the system in steady state.

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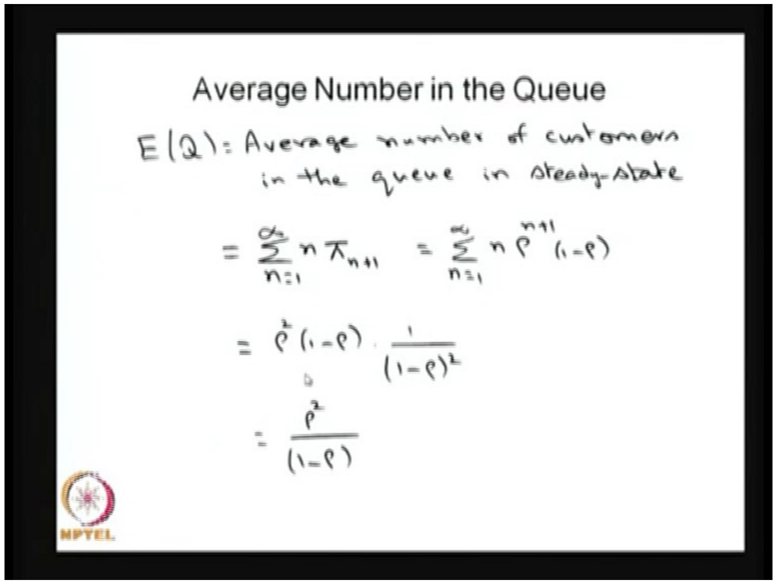

Also one can find average number in the queue, so the letter Q is a random variable, and here we are finding the E[Q], that is average number of customers in the queue that means before getting the service, how many customers in the system we have only one server in the system and whenever the service is going on and all other arriving customers will be queued, that means when n + 1 customers in the system, n people are in the queue.

Therefore, $\sum_{n} n\pi_{n+1}$, do the simplification you will get average number of customers in the queue also substitute the π_{n+1}, from the one I have discussed in the stationary distribution.

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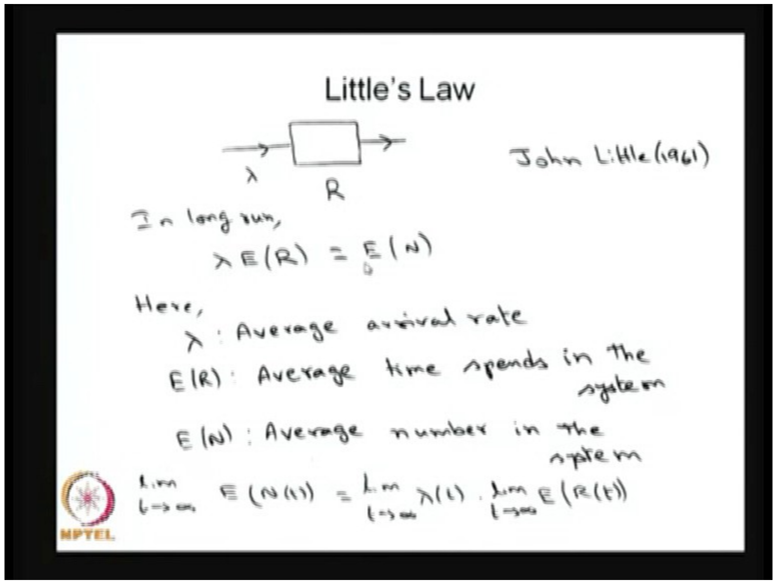

Here I am going to relate the average measures using the Little's law this is proven by John little 1961, this is valid for any system in which arrival comes into the pattern with the arrival rate λ, and R is a time spend in the system and leave the system after the service or whatever things are over, then in a longer than one can say the arrival rate multiplied by the average time spent in the system that is same as average number in the system.

So this relation is valid for whatever be the underlying distribution - whatever be the underlying distribution of the service, underlying distribution of the arrival what it says if you have a system in which the arrival rate is mean arrival rate is λ and the mean time spent in the system is expectation of R than that product will give average number in the system.

Since indirectly it says whenever the system has a long run in a stable system the expectation of a average number of customers during the interval 0 to t, as t tends to infinity that is going to be have a limit expectation of a N and the arrival rate λ (t), that also has the mean arrival rate as t tends to infinity that is also going to be a sum having a limit constant λ.

And similarly the average spent by the customers in the system at any time t and if you make t tends to infinity that expectation quantity also has a limit, therefore you will have a λ E[R] =

E[N], now using this Little's law I am going to find out the measures for the M/M/1 queue model.

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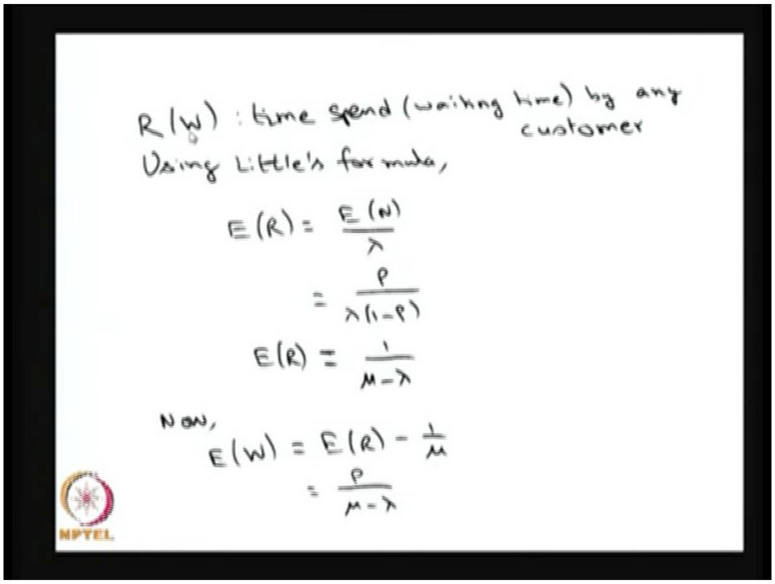

So suppose R denotes the time spent in the system by the customer and W denotes the waiting time by any customer in the system, then I can use the Littles formula, the Little's law in the previous one, so if I know the mean arrival rate and if I know mean number of customers in the system in a longer run using these I can find out the average time spent in the system.

If I know to use the, using Little's law If I know the average number of customers in the system longer run and if I know the arrival rate then I can find out the average time spent in the system in a longer run, similarly I can once I know the average time spent in the system if I subtract the average time of my own service then that is going to be the average time waiting in the queue.

So this is the average time waiting in the queue that is same as average time spent in the system minus my own average service time. In the M/M/1 queue model the service time is exponentially distributed with a parameter μ, therefore the average is $1/\mu$, so the difference will give the average time waiting in the queue by any customer not only the average measures for the M/M/1 queue one can find out the actual distribution for R as well as W also.

Because this is a very simplest Markovian queuing model, whereas for all other models it is little complicated but still one can get it, so this is the easy model in which one can find out the distribution of the time spent in the system as well as the time R as well as the waiting time by a customer in the queue.

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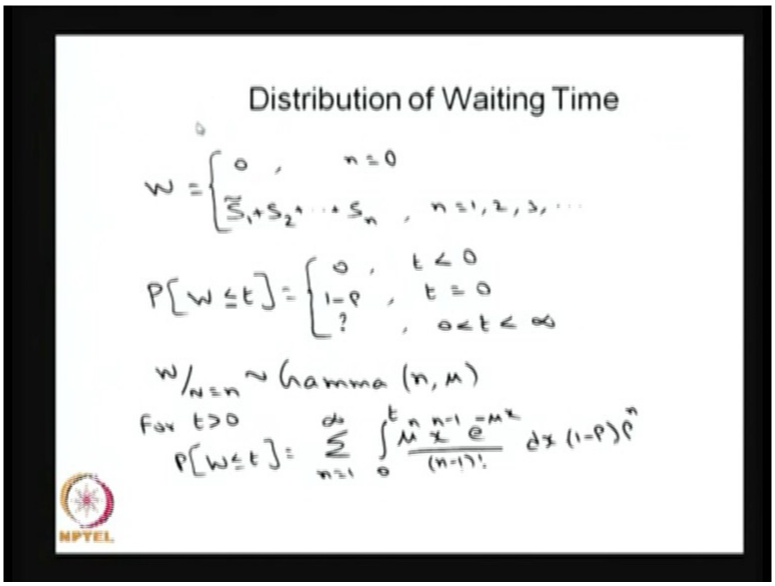

First let us go for finding out the distribution of waiting time, waiting time means if no one in the system when you arrive then your waiting time is 0, you are immediately going to get the service so the service time is your time spent in the system usually the time spent in the system is the time of your service + time of waiting time, so here I am finding the only the distribution of waiting time first so whenever the system is 0, you are waiting time is 0.

whenever no customer in the system in the waiting time is 0, whenever more than or equal to one customer in the system then the waiting time is same as the remaining service time for the customer who is under service plus the customers in the queue before you join in the queue so those peoples service time addition the residual are the remaining service time of the customer who the first customer who is under service.

So this total time is the waiting time whenever the system is non-empty, whenever the system is empty then the waiting time is 0, therefore the W is a random variable either it takes the value 0 or it takes a value greater than 0 based on the time - time of service of previous n people in the ahead of you, therefore W is a mixed random variable which has the probability mass function at 0 as well as a density function between the intervals 0 to infinity.

So let me try for finding out the CDF of this random variable, so the CDF is going to be 0, till 0 at 0 it has the CDF 1 - ρ, because when the waiting time is 0 that is equivalent of no one in the system so in the long run no one in the system that is a π_0.

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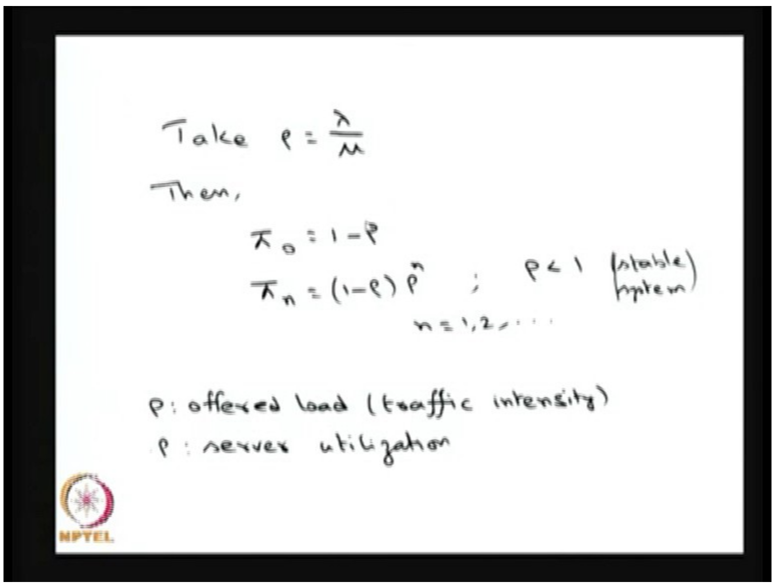

And the π_0 probability is 1- ρ, system is empty in a longer run that is 1 - ρ, therefore the CDF at 0 that is same as 1 - ρ of that is π_0 between the interval 0 to infinity when we have to find out the distribution of W.

Whenever n customers before you join in the system that conditional distribution the distribution of a W given the number of customers in the system is n that distribution is nothing but the service time of the n customers, the first customers remaining service time, the service time of the first customers is exponential distribution, the residual or remaining service time of the first customer that is also exponential distribution because of memoryless property.

So this is exponential distribution, this is second customer service times that is exponential distribution and similarly, for the n th customer also service time is exponentially distributed and the way we made assumption all the service times are independent and each one is exponentially distributed with a parameter μ, therefore this is a sum of n independent exponentially distributed random variables.

Therefore, the sum of n exponentials that is going to be a gamma distribution with the parameters n and μ, there are many ways of finding out the distribution but here I am just explaining through the distribution concept this is sum of n independent exponential distribution therefore you can conclude it is gamma distribution with a parameters n and μ, once you know the conditional distribution our interest is to find out the unconditional one.

That means for t is greater than 0 and CDF at the point t, that is nothing but what is the conditional density probability density and what is the probability of n customers in the system that multiplication with the possible n will give the CDF between the interval 0 to t, so I have a density function of a gamma distribution probability density function with a parameters n and μ.

And this is a probability density function multiplied and integration between 0 to t that will give the CDF and unconditional multiplied by probability of n customers in the system that with the summation that will give the unconditional, therefore the CDF is going to be summation n is equal to 1 to infinity integration 0 to t of the probability density function of a gamma distribution multiplied by n customer in the system.

If you do the simplification you will get the $1-e^{-\mu(1-\rho)t}$.

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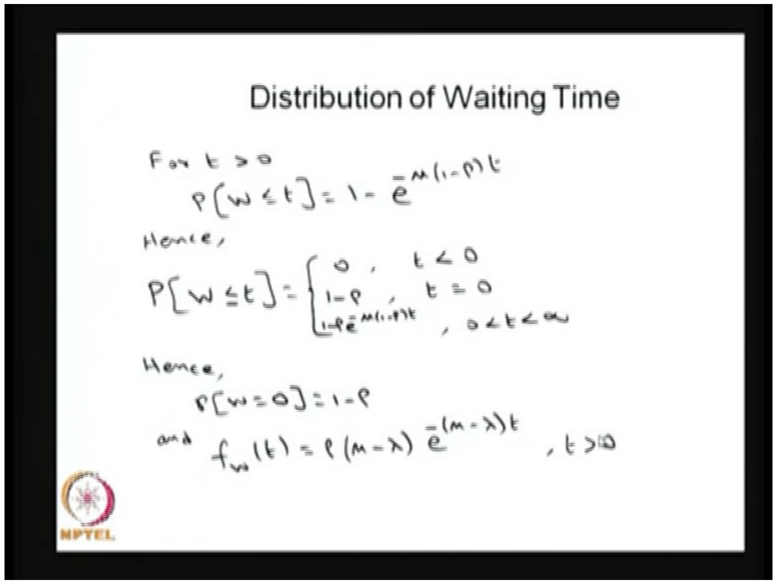

Therefore we can substitute here - here I made a mistake so here it is multiplied by ρ, so $1-\rho e^{-\mu(1-\rho)t}$, that is going to be the, so once you are getting the CDF you can conclude this is a mixed random variable with the probability mass at 0 is 1 - ρ.

And the density function between the interval 0 to infinity that is a $\rho(\mu-\lambda)e^{-(\mu-\lambda)t}$, that is the probability density function for a distribution of waiting time.

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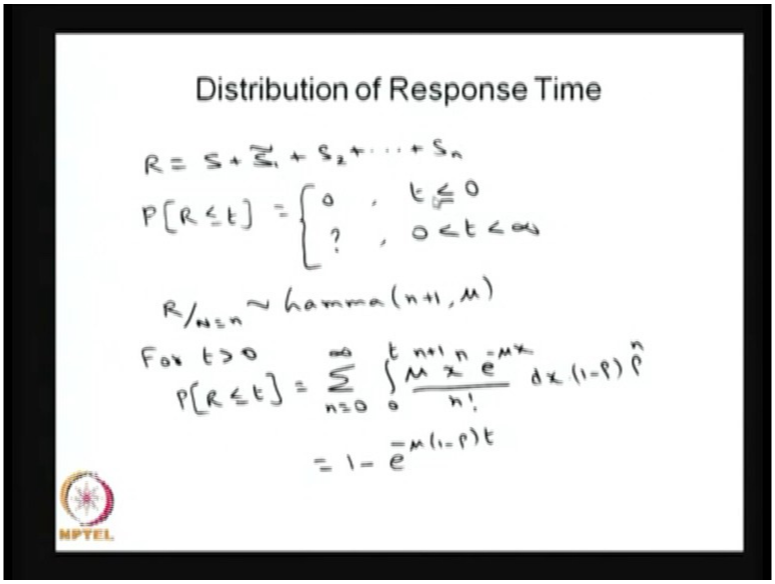

Similarly, one can get the distribution of response time also or the total time spent in the system, the total time spent in the system that is nothing but that's a random variable and there is a dual service time of the first customer who is in the system plus all the remaining n customers in the system in the queue plus your own service time, therefore here this is not a mixed random variable this is a continuous random variable.

Because your service time is a continuous random variable which is exponentially distributed, therefore the R is going to be sum of your own service + the remaining service of the first person in the system if and so on, till the n th customer who is in the queue, therefore this is the CDF of the random variable R, here also one can argue when n customer in the system before him who enter into the system that is a sum of exponential independent random variable and so on.

Therefore, this is going to be a gamma distribution with the parameters n + 1, µ and for t greater than 0, find out the CDF using the first conditional then unconditional multiplied by $(1 - \rho)\rho^n$ summation over n is equal to 0 to infinity, because there is a possibility no one in the system or 1 customer, 2 customer and so on, therefore the running index is 0 to infinity, do the simplification you will get $1 - e^{-\mu(1-\rho)t}$.

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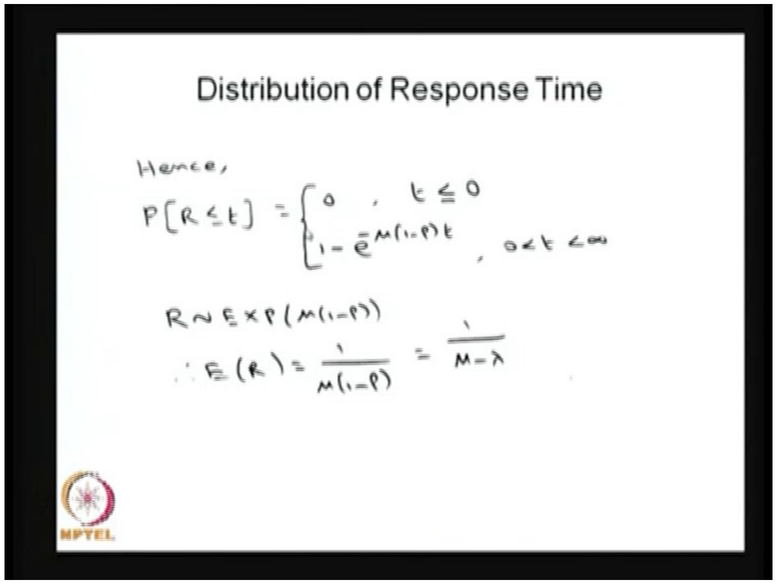

Therefore, you can substitute here and if you see the CDF is same as the CDF of exponential distribution with a parameter that is μ(1- ρ), therefore you can conclude the total time spent in the system is exponentially distributed with a parameters μ(1- ρ), if you find out the average time that is going to be 1 divided by the parameter that is this.

The same thing you got it in the average response time from the Little's formula using once you know the value of λ and expected number in the system using Little's law you got expectation of time spent in the system that is same result, so here we are getting first finding the distribution of time spent in the system or response time then we are finding the average time.

Introduction to Probability Theory and Stochastic Processes
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Lecture – 88

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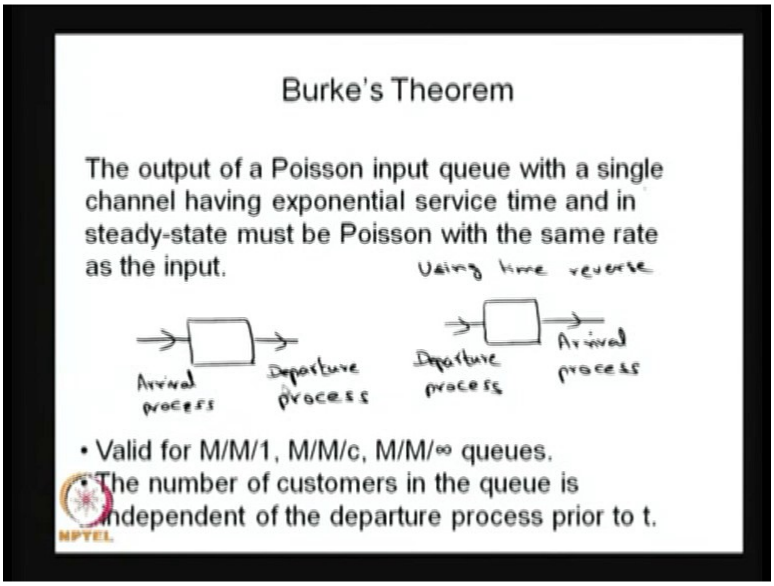

Here I am giving the concept of output process. The arrival follows the Poisson process for the M/M/1 queue and the service is exponentially distributed which is independent of the arrival process and the customers leave the system. Now the question is what is the distribution of the departure process. That means, what is the inter departure time.

After first customer leaves how much time it takes for the second customer leave the system and then the third customer how much time it takes for the inter departure time and therefore what is the distribution of the departure process. That is given by the Burke's Theorem. The output of a Poisson input queue with a single channel having exponential service time and in steady state must be a Poisson with the same rate as the input.

So whenever you have a system in which the arrival follows the Poisson and whenever the system has a single channel and the service time is exponentially distributed, in a longer run the departure process is also going to be a Poisson process and the rate will be the same rate as the arrival process. So this can be proved, but here I am giving the interpretation using the time reverse process, because in a steady state this model is going to satisfy the time reverse model.

Therefore, this stationary distribution exists and if we make this M/M/1 queuing model, the underlying birth death process satisfies the time reversibility equation. Therefore, using the time reverse you can conclude the departure process, you can reverse it and that is going to be independent of the arrival process and this is also going to be again Poisson process.

So using the time reverse concept one can prove the departure process is independent of the arrival process and departure process is also Poisson process with their same rate as the arrival rate. And even though I said it is the single channel having exponential service time and this is valid for M/M/1 queue, the multi-server Markovian queue as well as the infinite server Markovian queue also.

So all those models can be combined with the single channel having exponential service time. Whether it is a single server or multi server or infinite server, these results hold good. And the next result is the number of customers in the queue is independent of the departure process prior to it that it also satisfied.

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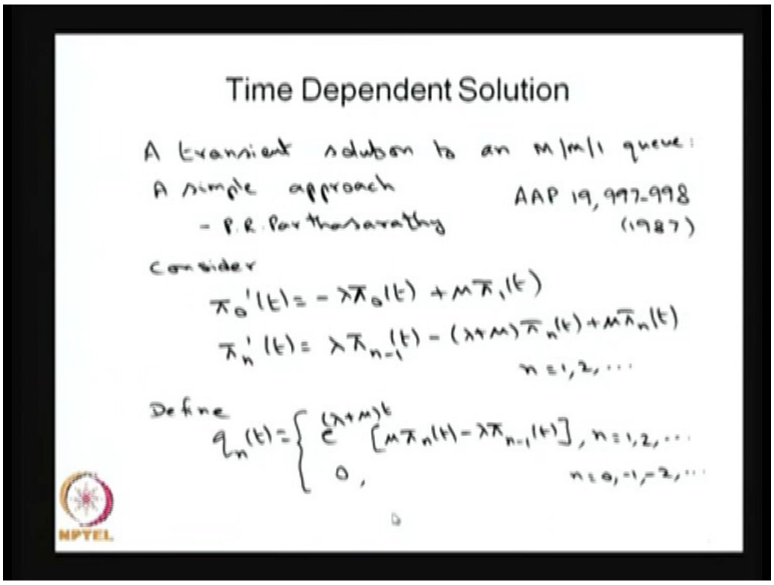

Now we are giving the time dependent solution of a M/M/1 queue. There are many more methods to find out the time dependent solution for a M/M/1 queue. It started with the spectral method and the combinatorial method and also the difference equation method. Like that there are many more methods in the literature to find out the time dependent solution and here I am presenting the time depending the time dependent solution by P.R. Parthasarthy and this work has appeared in the advanced applied probability volume number 19, 1987.

So in this paper he has considered the system of differential equation that is nothing but the forward Kolmogorov equation and making a simple function $q_n(t)$, that is the difference of π_n with the multiplication of $e^{(\lambda+\mu)t}$. So once you use this definition, once you convert this system of difference equation with $q_n(t)$ by making a proper generating function.

That is of the form $\sum_{n=-\infty}^{\infty} q_n(t)s^n$, therefore this is sort of generating function in terms of $q_n(t)$ where $q_n(t)$is for n is equal to one to infinity. This is of difference of $\mu\pi_n - \lambda\pi_{n-1}$, multiplication $e^{(\lambda+\mu)t}$ and for n is equal to zero, minus 1, -2 and so on zero.

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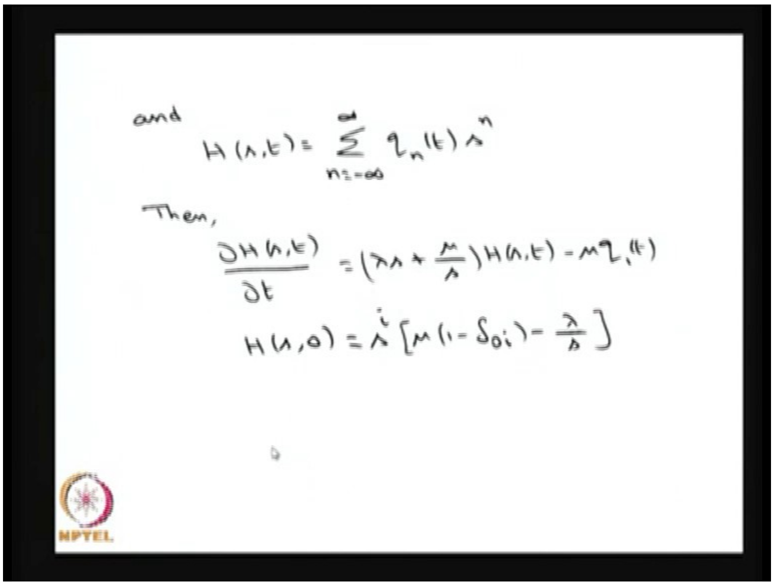

Therefore, you have a generating function. So you can covert the whole difference differential equation in terms of π_n multiplied by the one partial differential equation with the initial condition also changes because if you assume that the i customers in the system at times zero and this is going to be initial condition for function H(s,t) at t equal to zero.

So now the question is you have to solve this equation with this initial condition, this pde using this initial condition.

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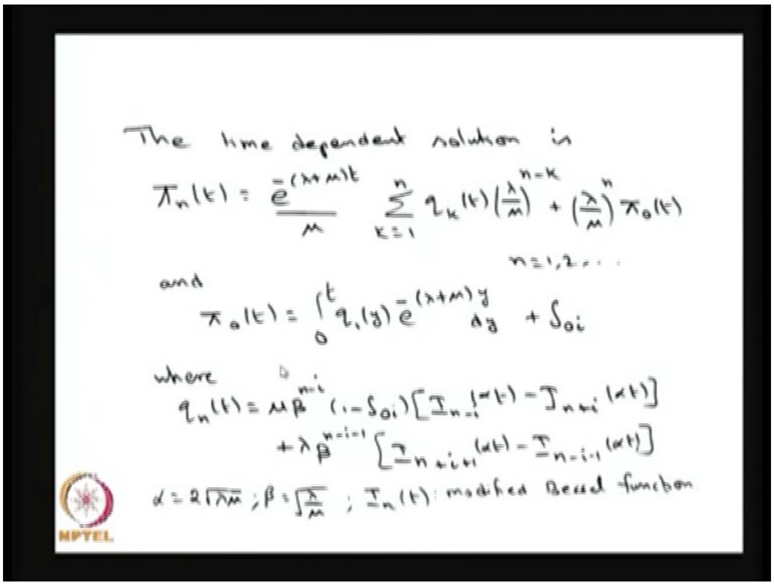

So use some identity of modified Bessel function one can get the solution $\pi_n(t)$ in terms of $\pi_0(t)$, where $\pi_0(t)$ you can get it in terms of q_1 where all the q_n's satisfies this equation that is in terms of the modified Bessel function. So one can see the complete solution in this paper.

But here I am giving the very simple approach of getting the time dependent solution for the M/M/1 queue by changing this system of differential equation multiplied by one pde the initial condition and solve that pde and obtaining the π_n and π_0 in terms of modified Bessel Function.

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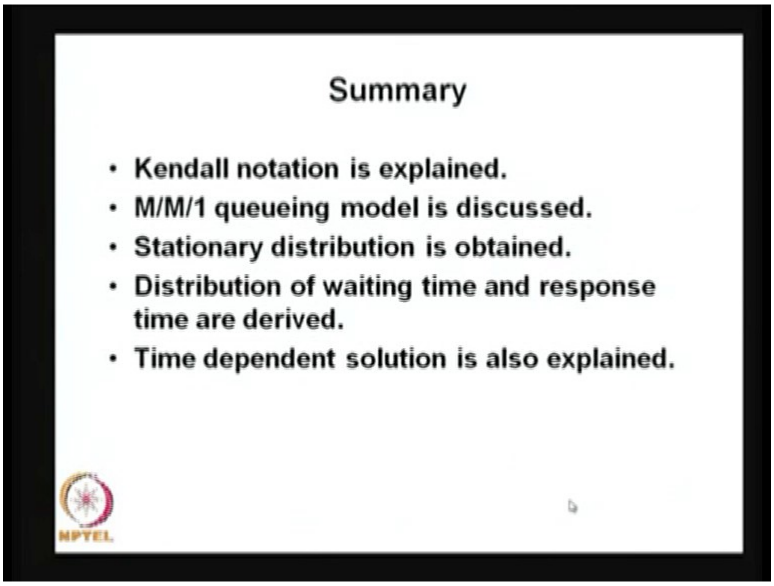

Before I go to the summary let me give the simulation of M/M/1 queue.

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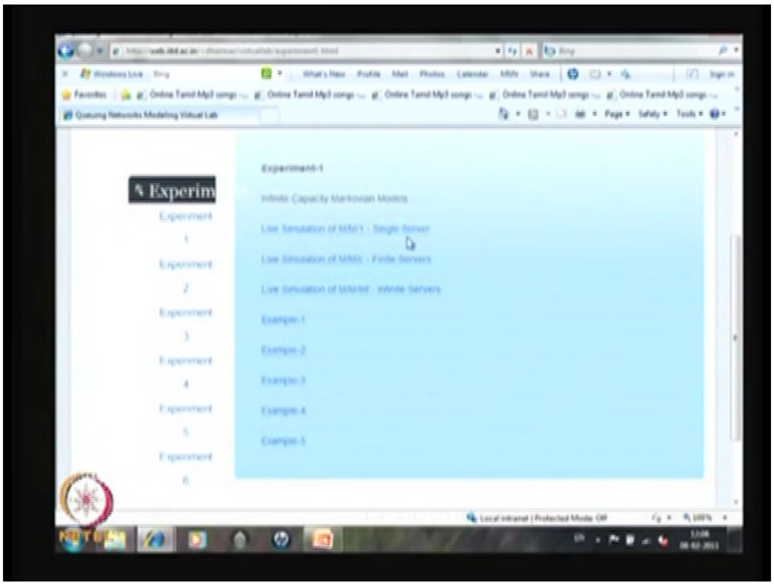

So this is the queuing network modelling lab. So in this queuing network modeling lab, one can simulate the queuing network models.

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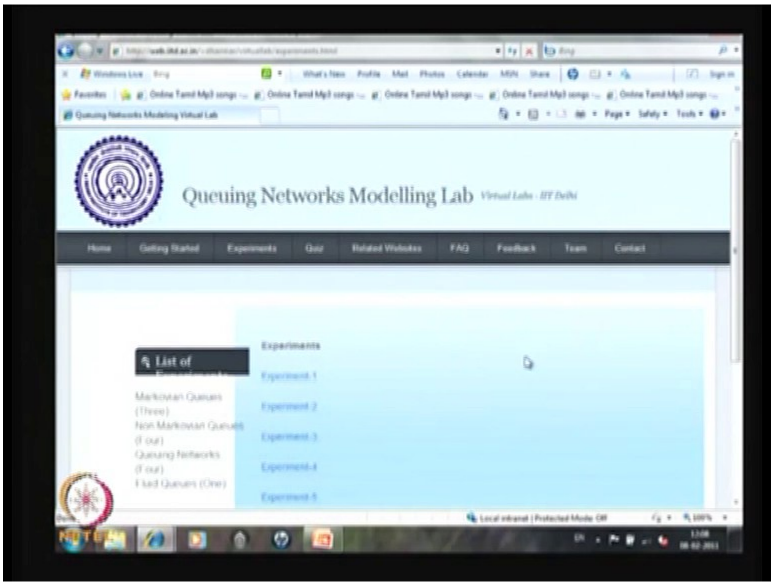

So in this I am going to explain how to simulate the M/M/1 queue and first experiment is nothing but live stimulation of M/M/1 queue, single server as well as you can simulate a multi-server queue model and you can go for the infinite server model also. So here I am simulating the M/M/1 queuing model.

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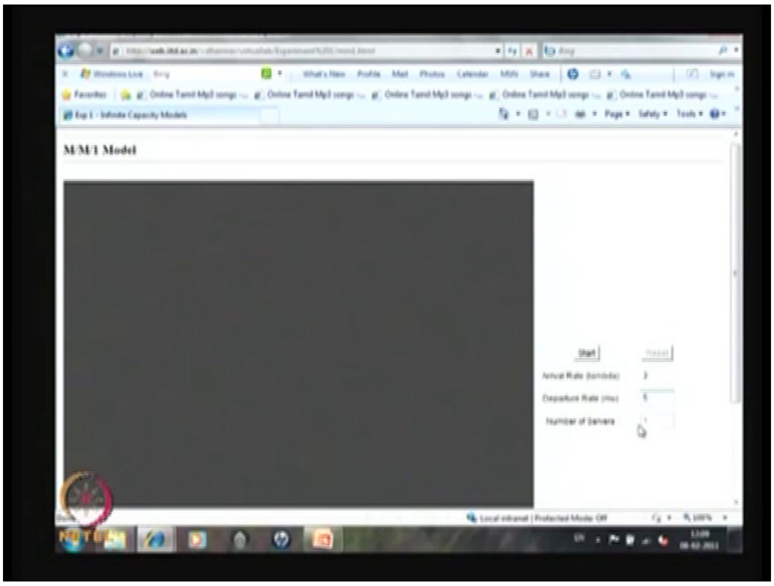

So to simulate the M/M/1 queuing model, you need the information about the inter arrival time that is exponential distribution you need a parameter λ, the value of λ as well as you need a value of μ, that μ is nothing but the service rate. So suppose you supply the arrival rate, suppose the arrival rate is two and the departure rate is 5, the number of servers, since it is M/M/1 queue therefore it is already one is placed, it is the number of servers.

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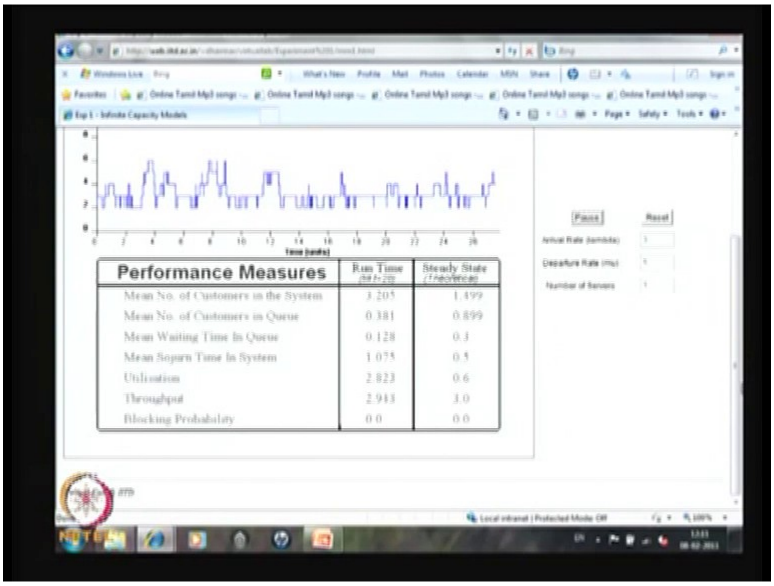

So you can start. So this is a way this system increases. So this is the actual simulation goes with that, it is the time x axis and y is the number of customers in this system and here the information is how many customers entered till this time, that is 15 customers entered and nobody is blocked because it is a M/M/1 queuing system.

Therefore all the customers who are entering it will be queued and how many customers are served during this time and the number of customers in the orbit this is nothing to do with the M/M/1 queue, this is for the retrial queues and now how many customers are in the system at this time and here this table gives the performance measures, the one we have calculated the average number of customers in the system, E[R].

And the average number of customers in the queue E of this is mean number of customers in the system that is E[N], the mean number of customers in the queue E[Q], mean waiting time that is E[W], mean sojourn time in the system, sojourn time, spending time, response time all are the same, the mean sojourn time in the system is nothing but E[R]. So this is nothing but the E[R], this is nothing but E[W], this is nothing but E[Q].

And this is nothing but the E[N] and the utilization is nothing but what is the probability that - so here I am giving the run time, what is the average values till this time and what is the result is going to be in longer run in a steady state and blocking probability is here zero because the system is infinite capacity model, therefore there is no one blocked, therefore the blocking probability is zero. So this is the way we can reset and you can give some other values and you can start again. And you get another simulation also.

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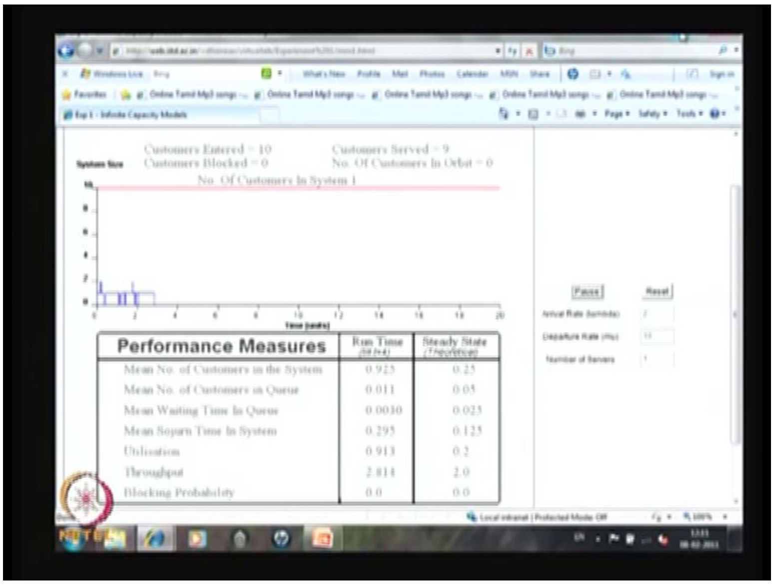

And initially it gives the fixed steady state results in the steady state theoretical result and the run time is nothing but what is the result over the time. With this let me complete the simulation.

So in the summary, we have started with the Kendall notation and M/M/1 queue is discussed. Stationary distribution, waiting time distribution, waiting time distribution is discussed for the M/M/1 queue and also the time dependent solution and I have given the simulation of M/M/1 queue also.

Introduction to Probability Theory and Stochastic Processes
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Lecture – 89

In this lecture we are going to consider the other simple Markovian Queueing models as an application of a continuous time Markov chain.

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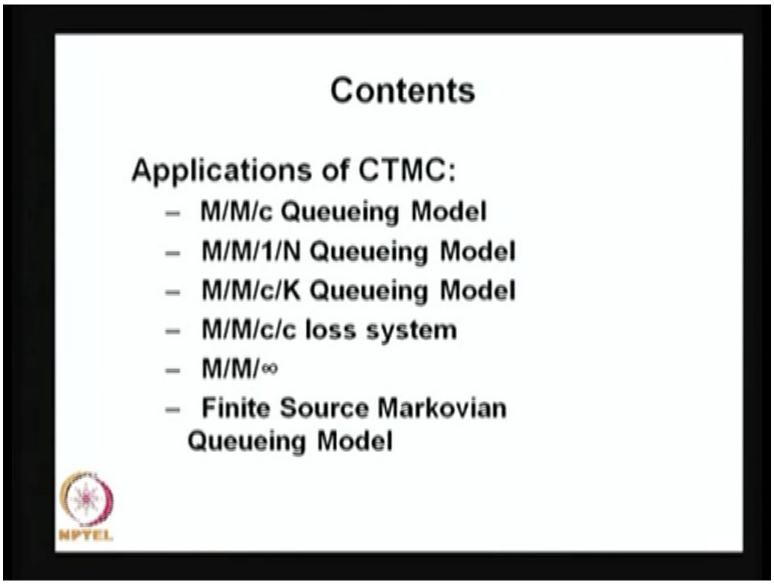

So in this lecture, I am planning to discuss other than M/M/1 queueing model. I am going to discuss a simple Markovian Queueing models starting with the M/M/c infinity queueing model.

Then the finite capacity model Markovian setup M/M/1/N Queueing model. Then I am going to discuss the multi-server finite capacity model; that is M/M/c/K Queueing model. After that I am going to discuss the loss system; that is M/M/c/c model. For an infinite server model that is M/M/∞ also I am going to discuss. At the end, I am going to discuss the Finite Source Markovian Queueing model also. Whereas the other 5 models, the population is infinite source, so the last one is the Finite Source Markovian Queueing model also I am going to discuss as the application of continuous time Markov chain.

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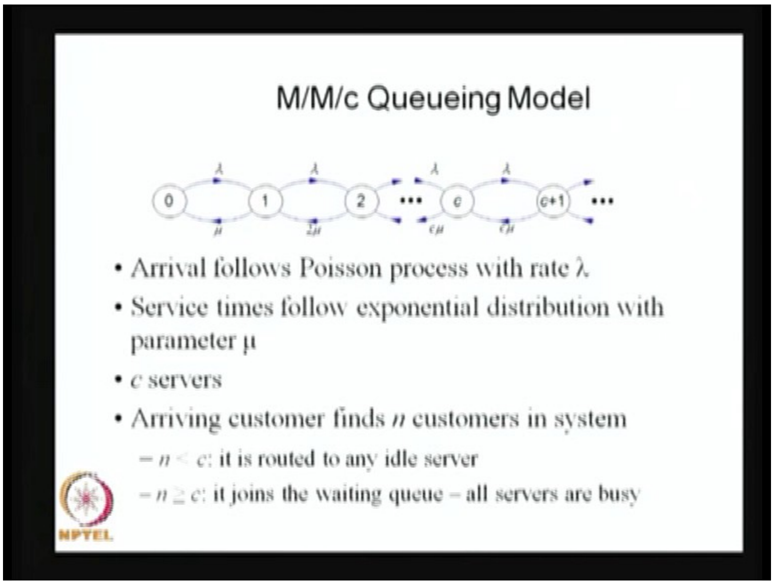

The first model is a multi-server infinite capacity Markovian Queueing model. The letter M denotes the inter arrival time is the exponentially distributed its parameter λ.

The service time by the each server that is exponentially distributed with the parameter μ and all we have a more than one servers; suppose you can consider as a c, where c is a positive integer and all the servers are identical and each server is doing the service which is exponentially distributed with the parameter μ, which is independent of the all other servers and the service time is independent with the inter arrival time also.

With these assumptions, if you make a random variable X(t) is the number of customers in the system at any time t that is a stochastic process. Since the possible values of number of customers in the system at any time t that is going to be 0, 1, 2 and so on. Therefore, it is a discrete state and you are observing the queueing system at any time t therefore it is a continuous time.

So, discrete state continuous time stochastic process and if you observe the system is keep moving into the different states because of a either arrival or the service completion from the any one of the c servers. So suppose, there are no customer in the system and the system moves from the state 0 to 1 by one arrival. So the inter arrival time is exponentially distributed therefore the rate in which the system is moving from the state 0 to 1 is λ.

Like that you can visualise the rates for a system moving from 1 to 2, 2 to 3 and so on. Whereas, whenever the system size is 1, 2 and so till c, since we have a c number of a servers in the system; whoever entering into the system they will start getting the service immediately. Suppose the system goes from the state 1 to 0, that means the customer enter into the system and he gets the service immediately.

And the service time is exponentially distributed with the parameter μ. Therefore, whenever the service is completed the system goes form the state 1 to 0, therefore the rate is μ. Whereas from 2 to 1, there are 2 customers in the system and both are under service at any time if any one of the severs; if any one of the servers complete the service, then the system moves from 2 to 1.

So the service completion will be minimum of the service time of the both the servers. Since each server is doing the service are exponentially distributed with the parameter μ; therefore, the minimum of a two exponential and both are independent also. Therefore, that is also going to be an exponentially distributed with the sum of parameters so it is going to be parameter will be μ+ μ that is 2μ.

So this system moves from the state 2 to 1 will be; the rate will be 2μ. Like that it will be keep going till the state from c to c -1, that means we have c servers. Therefore, whenever the systems size is also less than or equal to c; that means all the customers are under service. Now we will discuss the rate in which the system is moving from the state c + 1 to c.

The system state is a c + 1, that means when the number of customers in the system that is c + 1. We have c servers; therefore, one customer will be waiting for the service, waiting in the queue. Therefore, this system is moving from c + 1 to c that is nothing but one of the server completed the service out of c servers. Therefore, the rate will be the service time; completion service time will be exponentially distribution with the parameter cμ not (c+1)μ.

It is a, we have only c servers therefore the minimum of exponentially distributed with the parameters μ and so on with the c exponentially distributed random variables, therefore that is going to be exponentially distribution with the parameter μ plus μ plus, whereas cμ is therefore it is going to be cμ. Like that the rate will be; the death rate will be cμ after c + 1 onwards.

Whereas from 0 to c, it will be μ, 2μ, 3μ and so on till cμ after that it will be a cμ from the state from c, c + 1 to c, c + 2 to c + 1 and so on and if you see the state transition diagram, you can observe that it is a birth death process. So before that, let me explain; what is the M/M/c infinity means, whenever c customers or c servers or any one of the c servers are available, then the customers will get the service immediately.

If all the c servers are busy, then the customers have to wait till any one of the c servers are going to be completing their service, so that is the way the system works. Therefore, you will have the system size.

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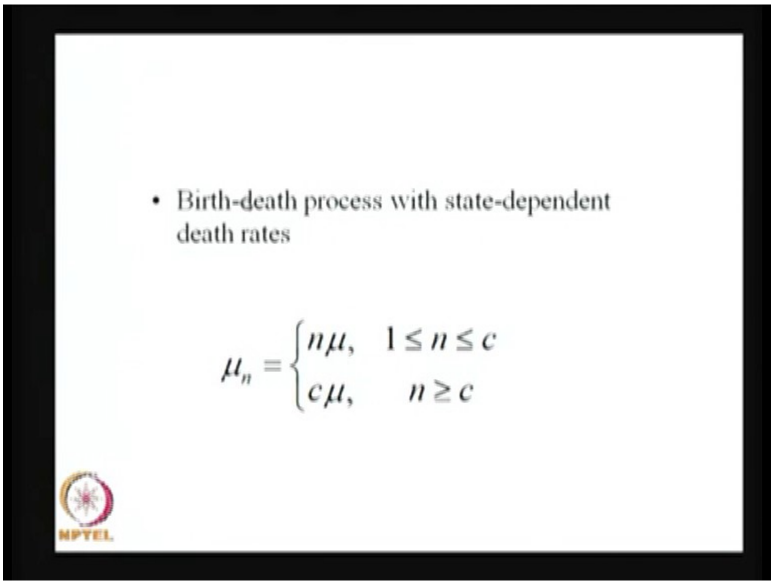

The system size, the underlined stochastic process is going to be a birth death process as a special case of a continuous time Markov chain. Because the transitions are only the neighbour transitions with the forwards rates, that is λ.

And backwards rates are the death rates are going to be μ, 2μ and so on. Therefore, this is a special case of our continuous time Markov chain, the underlined stochastic process for the M/M/c infinity model that is the birth death process. The birth rates are λ whereas the death rates depend on the n the μ_n is the function of n. Therefore, it is called state dependent death rates.

It might not be the function nμ, it can be a function of n, then we can use the word state dependent. So here it is a linear function. So state dependent death rates and the death rates are nμ, whenever n is lies between 1 to c and the μ_n is going to be cμ for n is greater than or equal to c that you can observe it from the state transition diagram also.
The death rates are going to be cμ here also cμ and so on. Therefore, this is the birth death process with the state dependent death rates.

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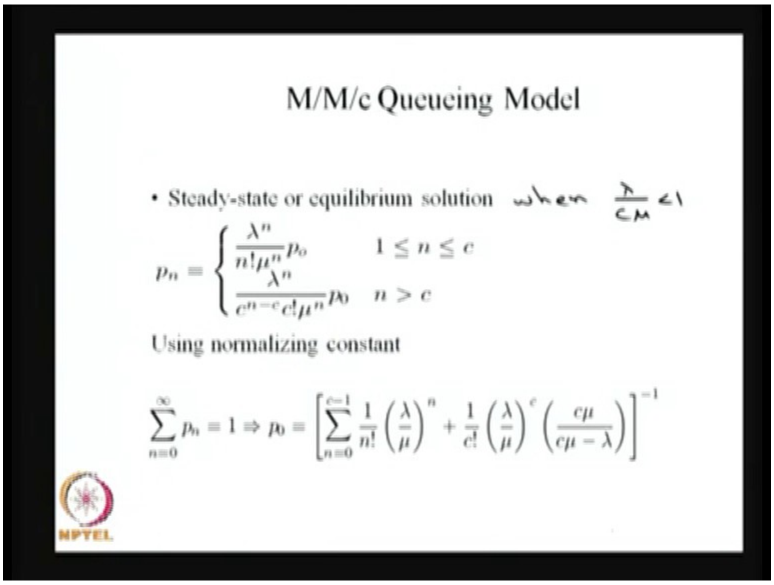

Now our interest is to find out the steady state or equilibrium solution. Since it is an infinite capacity model, if you observed the birth death process with the infinite state space, then you need a condition, so that the steady state probability is exist.

So whenever λ/cμ is less than 1, whenever λ/cμ is less than 1, you can find out the limiting probabilities, so sometimes I use the letter p_n, sometimes I use the word π_n, both are one and the same. So you find out the steady state probability by solving a πQ is equal to; pQ = 0 and the $\sum_i p_i =$¿¿1 and if you recall the birth death processes steady state probabilities the π_0 has the 1 divided by the series.

Whenever the denominator series converges, then you will get the p_n's. So either I use the p_n's or π_n's both are one and the same. So here $\sum_i p_i =$¿¿1 and p and if you make a vector p, pQ = 0, if you solve that equation and the denominator of $p_0$that expression that is going to be converges only if $\lambda/c\mu < 1$. So therefore whenever this condition is there, the queueing system is stable also.

If you put c =1, you will get the M/M/1 queue. So using the normalizing condition you are getting the p_0 and p_0 is 1 divided by this, so this is a series, so this series is going to be converges only if this condition is satisfied. So by solving that equations, you are getting p_n's in terms of p_0 and using normalizing constant you are getting a p_0. Therefore this is the steady state also known as the equilibrium solution for the M/M/c infinity model.

So here we are using the birth death process with the birth rates are λ and the death rates are given in this form and use the same logic of the stationary distribution for the birth death process. Using that, we are getting the steady state or equilibrium solution for the M/M/c model.

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Prof. S. Dharmaraja
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Lecture – 90

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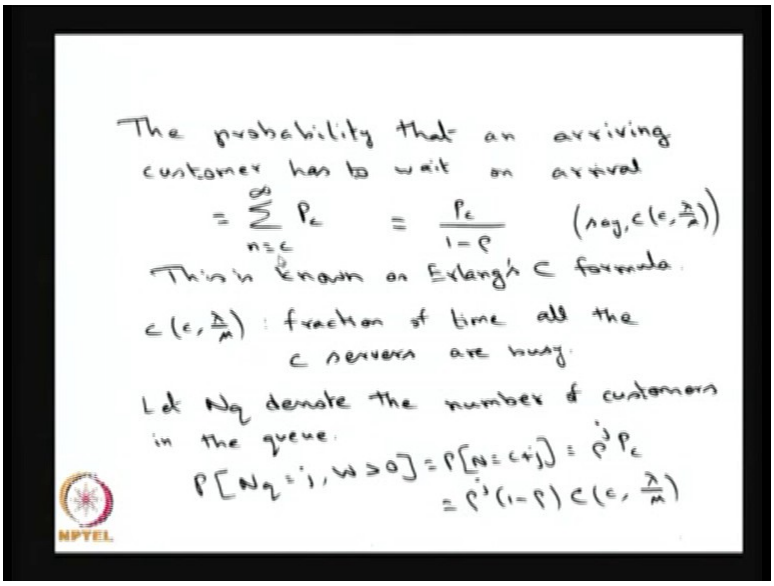

Other than the steady state probability, we can get some more measures. The first one is the probability that the arriving customer has to wait on arrival. What is the probability that the arriving customer has to wait on arrival? So that means the number of customers in the system is greater than or equal to c, then only the customer has to wait. So the probability,

$$\sum_{n=c}^{\infty} p_n.$$

If you had all those probabilities, that is going to be $\frac{p_c}{1-\rho}$. This probability is known as a Erlang C formula for a multi-server infinite capacity model. That I am denoting with the letter (c, $\frac{\lambda}{\mu}$), because you need number of severs in the system and you need λ as well as μ. If I know this quantity, I can find out what is Erlang's C formula. This is very important formula. Using that you can find out what is the optimal C such a way that the probability has to be minimum.

You can find out what is optimal number of severs is needed to have some upper bound probability of arriving customer has to wait. Therefore, this Erlang C formula is very useful in performance analysis of any system. The next quantity is N_q denotes the number of customers in the queue. So either use the letter N_q, earlier I used the letter Q itself. So for that I am finding the joint distribution of what is the probability that the number of customers in the queue is j and the waiting time is going to be greater than zero.

W is used for the waiting time. So the waiting time is going to be greater than zero. That is same as the number of customers in the system that is c + j. What is the probability that j customers in the queue as well as the waiting time is greater than zero that is same as what is the probability that c + j customers in the system. Do the simplification, so you will get this joint probability in terms of Erlang's C formula.

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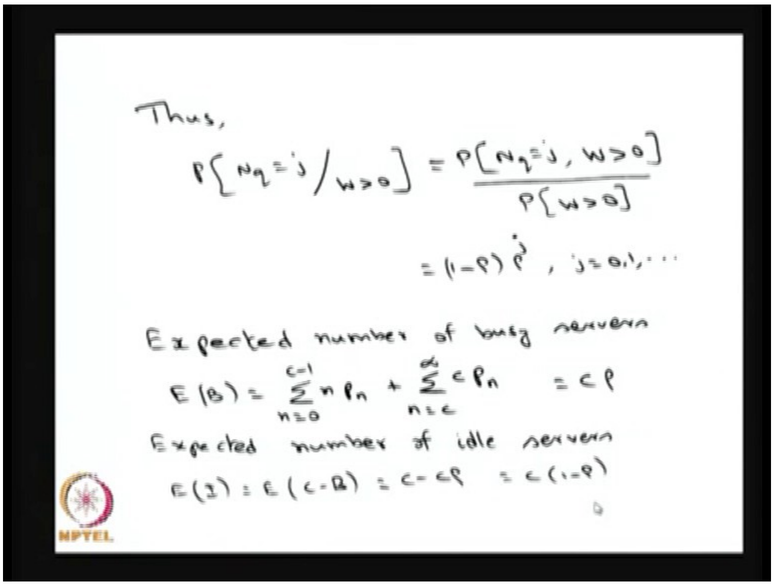

So using that I am finding the conditional probability, what is the conditional probability. What is the conditional probability that j customers in the queue given that the waiting time is greater than zero. If you do little simplification, I will get $(1-\rho)\rho^j$ where $\rho = \lambda/c\mu$. This is nothing but the probability mass function of geometric distribution.

This is the probability mass function of a geometric distribution; therefore, this conditional probability is geometrically distributed with a parameter ρ. From these we can find out the expected number of, the next measure is expected number of busy severs. What is the average number of busy servers?

That is nothing but the $\sum_{n=0}^{c-1}$ np_n, that means whenever the system size is less than c, only those many servers are busy and with the probability. Whenever customer are more than n customers in the system all the c servers are going to be busy, therefore cp_n. You simplify you will get cρ, that is the expected number of busy servers.

Once I know the expected number of busy servers, I can find out what is the expected number of idle servers also, that is a negation, that is expected number of idle servers is nothing but expectation of, it is a random variable. So ideal number is nothing but there are totally c servers in the system, therefore c minus busy servers are B, therefore c- B is same as I.

So the expectation satisfies the linear property, therefore expectation of I is same as expectation of c - B. c is a constant and B is a random variable, therefore it is c - expectation of B, expectation of B just now we got cρ, therefore the expected number of idle server is (c-1)ρ. So other than stationary distribution for the mmc model we are getting what is the probability that arriving customer has to wait.

And we are getting the conditional probability of j customers in the queue given that waiting time is greater than zero as well as these expected quantities we are getting.

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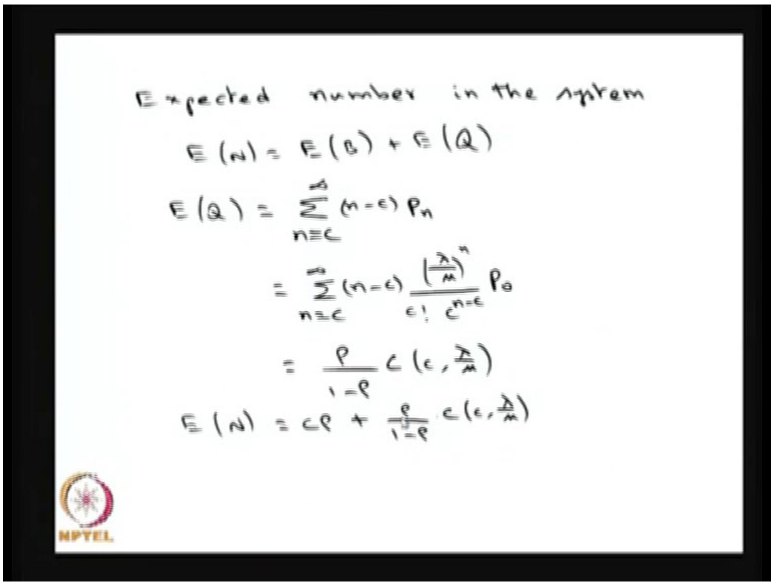

Also we can find out what is the expected number of customers in the system. That is nothing but, expected number is nothing but expected of the busy servers, plus expected number in the queue. Earlier I used the notation N_q and Q are both one the same. So I can compute what is the expectation of Q, it is a little simplification and then I can substitute expectation of Q here.

Therefore, I will get expected number of customers in this system that involves the Erlang C formula. So this Erlang C formula is used to get the expected number of customers in the system and later you can do some optimization over the probability expected number with specified c and $\frac{\lambda}{\mu}$.

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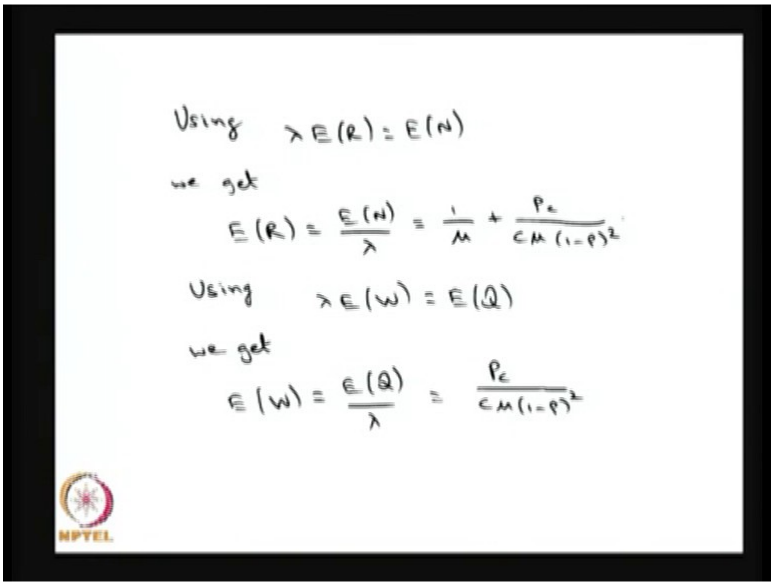

So using little's formula I can find out the expected time spend in the system because I know what is the arrival rate and from the stationary distribution I got expected number in the system in the steady state. Therefore, since I know λ and expectation of N, I can get expectation of R, where R is the response time or sojourn time or total time spend in the system.

So that expectation is going to be, E[N]/λ. Do a little simplification you will get expectation of R. You can apply the little's formula in the queue level also. So this is a system level and you can apply the queue level also. So λ times expectation of waiting time is same as expectation of number of customers in the queue.

So expectation of waiting time or average waiting time is same as E[Q]/ λ. So using, since the M/M/c infinity queue the underlying stochastic process is a birth-death process, therefore we are getting all the measures using the birth-death logic.

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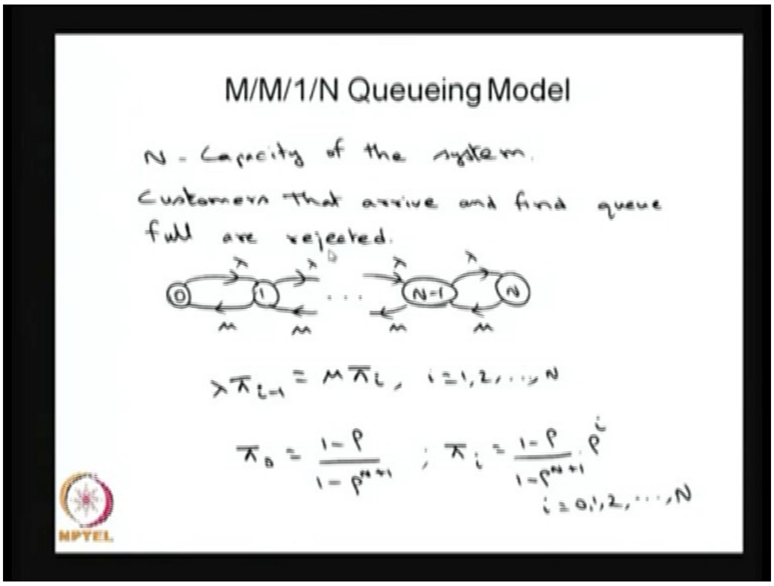

Next I am going for the finite capacity. So the N is the capacity of the system, that means when the customers arrives and find queue full, that customer will be rejected. Therefore, at any time the number of customers in the system if you make it as a random variable, that random variable takes the possible values from zero to N. So the states phase is fine. The number of customers in the system is anytime t that is a random variable and you will have a stochastic process.

And since the entire arrival time is exponentially distributed services, exponentially distributed only one server, finite capacity, therefore the underlying stochastic process is birth-death process, if the birth rate is λ and the death rate is μ. If you see the Q matrix for case 1, infinitesimal generated matrix.

That is a tridiagonal matrix with all the off diagonals or λ 's as well as μ and diagonals are -(λ+ μ) except the first term and last term. Except the first row and last row. Our interest is to get the stationary distribution later I am going to explain the time dependence relation also. So to get the stationary distribution either you write πQ = 0.

And the $\sum \pi_i = 1$and solve that or you write the balance equation the πQ = 0 that will land up a balance equation, so some books writes this as a balance equation. What is the inflow rate and what is the outflow rate, both are going to be same whenever the system reaches equilibrium state. Therefore, the outflow is λ times this and inflow is μ times that, like that you can go for understanding the balance equation for the state and second and so on.

And this also satisfies the, this is also called satisfying the time reversible equation. Therefore, one can use the time reversible property of a birth-death process. So you can find out the π_i's easily using the time reversible equation itself. You do not want to use $\pi Q = 0$ instead of that you can write the time reversible equation since it is satisfied by all the states.

Now you can use the $\sum_{i=0}^{N} \pi_i = 1$, therefore you will get π_0 and here the birth-death process with the finite state space, therefore the π_0 will be one divided by the denominator series, that is the finite series, finite terms in it. Therefore, it always converges immaterial of the value of λ and μ. Therefore, you will get π_0 without any restriction over λ and μ.

So once you get the π_0, you can get π_i in terms of π_0 therefore that is $\frac{1-\rho}{1-\rho^{N+1}} \rho^i$ where $\rho = \lambda/\mu$. So this is the underlying stochastic process as birth-death process with the birth rates λ and death rate is μ. So you can use all the concepts of the birth-death process and you can analyze the system in an easy way. So this is a steady state probability.

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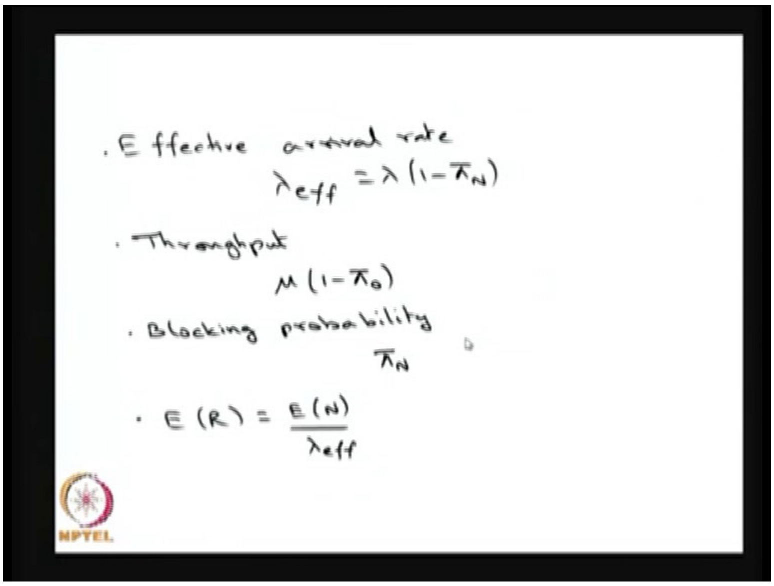

Once you know the steady state probability you can get the other measures also. Here the other important thing is called effective arrival rate. That means the system; the queuing system is finite capacity.

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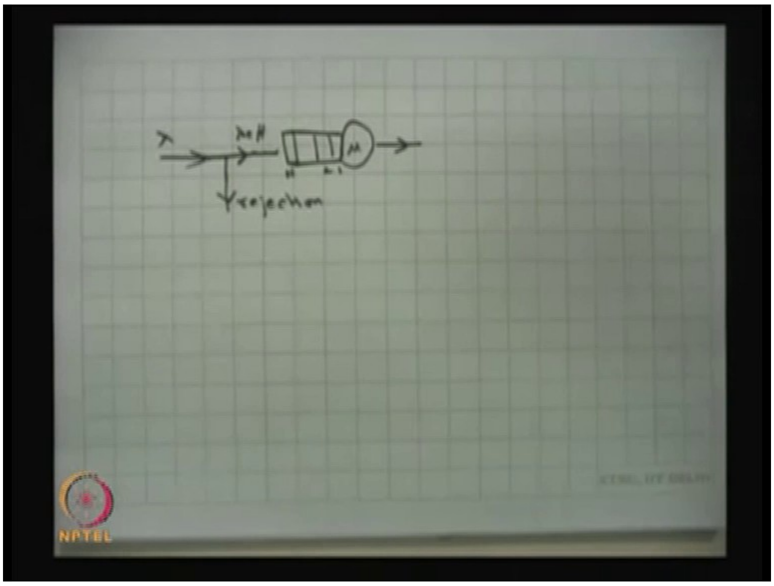

So maximum N customers can wait in this system and the service rate is μ, the arrival rate is λ from the infinite population. So whenever the system size is full, the customer is rejected, there for there is a rejection. After the service is completed the customer leaves the system. So the effective arrival rate is nothing but what is the rate in which the system is, the customers are entering into the system.

So there is a partition here. So the effective arrival rate is λ. That rate will be what is the probability that the system is not full multiplied by the arrival rate λ that is going to be the $\lambda_{effective}$. Whenever the system is not full, that proportion of the time or the probability is 1 - π_N where π_N is the steady state probability, just now we got it.

From here you can get π_N that is probability that the system is full, that is 1- π_N is the probability is that the system is not full and multiplied by the arrival rate that is going to be the $\lambda_{effective}$. And you can also find out the throughput. Throughput is nothing but, what is the rate in which the customers are served per any tough time. The service rate is μ and this is the probability that the system is not empty, $1-\pi_0$.

Therefore, $(1-\pi_0)\mu$ that is the rate in which the customers are served in the M/M/1/N system. Whenever the system is not empty, that probability multiplied by μ that is going to be the throughput. By using the time reversible equation, the $\mu(1-\pi_0)$ you can get in terms of λ equivalent also, but the throughput is the service rate multiplied by what is the probability that the system is not empty.

Since it is a finite capacity system, one can find out the blocking probability also. Blocking probability is nothing but the probability that the customers are blocked. The customers are blocked whenever the system is full. Therefore, the blocking probability is same as the probability that the system is full, that is π_N.

Once we know the steady state probabilities you can find out the average number of customers in the system and using the little's formula you can get, expected time spent in the system by any customer divided by not λ , it is $\lambda_{effective}$ because the effective arrival rate is used in the little's formula not the arrival rate. For a M/M/1 infinity system, the effective arrival rate and the arrival rate are one and the same because there is no blocking, therefore the probability of 1- π_N that is equal to one only.

Therefore, the effective arrival rate and the arrival rate are same for infinite capacity system because there is no blocking. For a finite capacity system, the effective arrival rate has to be computed. Similarly, we have to go for finding the $\lambda_{effective}$for the M/M/c/K model also. So other than stationary distribution or equilibrium probabilities we are getting the other performance measures using the birth death process concepts.

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Lecture – 91

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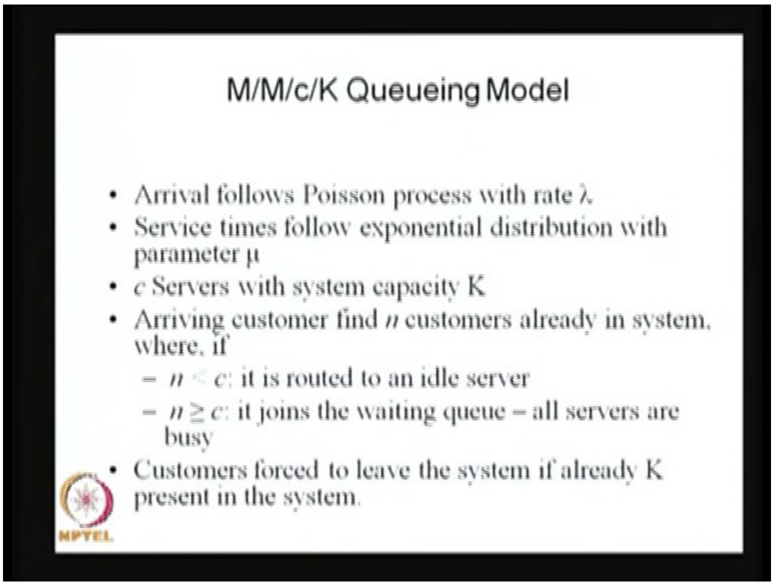

Now I am moving into M/M/c/K model. So here the change is instead of one server in the M/M/1 model you have more than one server c and you have a finite capacity that is K, capacity of the system. So the arrival follows the Poisson process, service is exponential, we have c identical servers. The capacity is K and this is the scenario in which whenever the system size is less than c it will be routed into the ideal server.

And if it is greater than or equal to c, that means all the servers are busy that means the customer has to wait. But if the system size is full, that means c customers are under service and K - c customers are waiting in the queue for the service and then whoever comes it will be rejected force to leave the system.

And therefore you have a waiting as well as blocking because it is a finite capacity there is blocking and since you have, always we choose K such that it is $K \geq c$. If $K = c$, then it is a loss system and then if $K > c$, then K - c customers maximum can wait in the system, in the queue.

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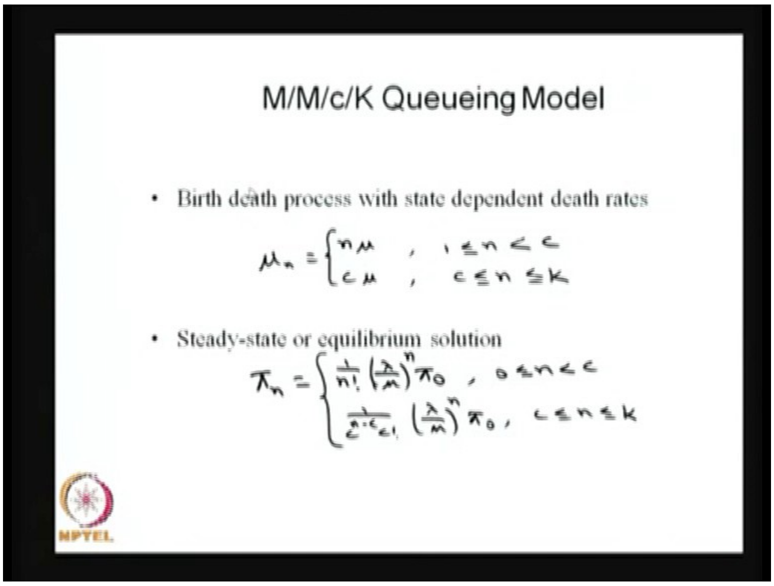

Therefore, the underlying stochastic process, here the stochastic process is again number of customers in the system at any time t. Therefore, this stochastic process is also going to be a continuous time Markov chain because of these assumptions, inter arrivals are exponential distribution service, each service by each server is exponentially distributed and all are independent and so on.

So with these assumptions these stochastic processes is a continuous time Markov chain and at any time only, only one forward or only one backward the system can move. Therefore, it is going to be a birth death process also and the birth rates are λ because it is a infinite source population, so all the λ_n are going to be λ, whereas the death rates are state dependent.

That is going to be nμ, lies between 1 to c, from c to k onwards it is going to be cμ. I have not drawn the state transition diagram for M/M/c/K, but you can visualize the way we have M/M/1/N and M/M/c model, so it is a combination of that, that is going to be the state transition diagram. Since it is a finite capacity model, it is easy to get the steady state and the equilibrium solution.

So first you solve πQ = 0 that means you write π_n in terms of π_0 and use a normalizing constant $\sum_i \pi_i = 1$, using that you will get π_0. So I have not written here. So use the normalizing constant $\sum_i \pi_i = 1$, get the π_0, then substitute π_0 here therefore you will get π_n in terms of π_0 completely.

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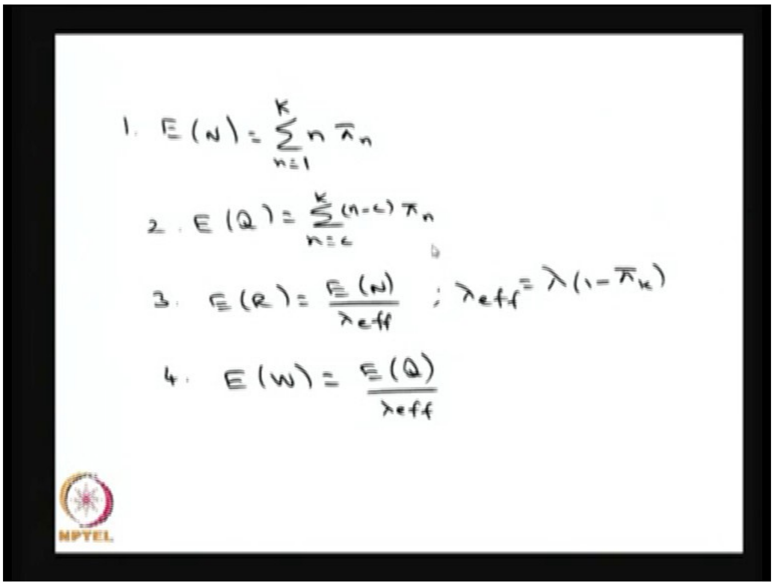

After that you can get all other average measures, the way I have explained M/M/1/N and the M/M/c infinity, the combination of that you can get the average number of customers in the system, average number of customers in the queue. That is $\sum_{n=c}^{K}$ (n -c)π_n, and the average time spend in this system.

Since it is a finite capacity you have to find out the $\lambda_{effective}$, effective arrival rate, that his one minus, its capacity is K, therefore $1-\pi_K$and that is the probably that the system is not full. So the effective arrival rate is λ(1 - π_K), substitute here and get the average time spent in the system and similarly you can find out the average time spent in the queue also using the little's formula.

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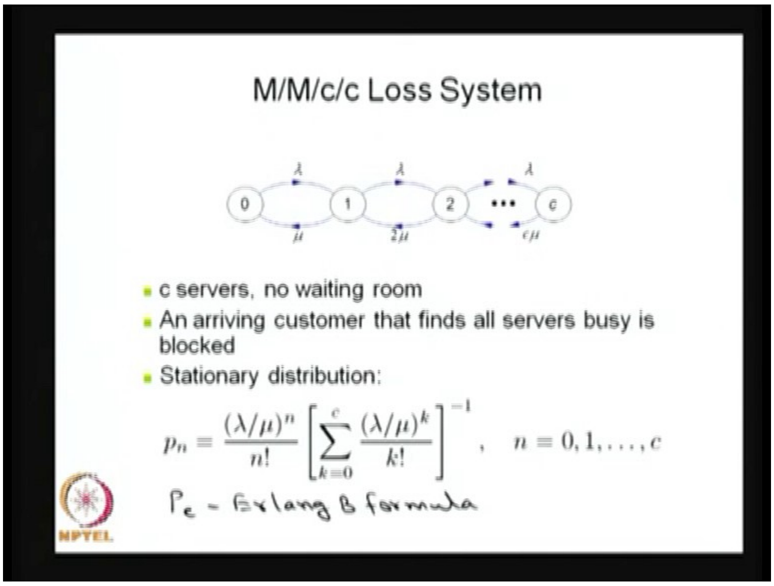

Now I am moving into the fourth simple Markovian queuing model. First I started with the M/M/c infinity, M/M/1/N, then I did M/M/c/K and now I am going for K = c that is loss system. It is not a queuing system because we have c servers and the capacity of the system is also c.

Example is you can think of parking lot which has some c parking lots and the cars coming into the system that is if you make the assumption is inter arrival time is exponentially distributed and the car spending time in each parking lot that is exponentially distributed then the parking lot problem can be visualized as the M/M/c loses too. So here we have c identical servers, no waiting room.

So since it is c capacity and c waiting room you think of self-service with the capacity c, that also you can visualize. So here the inter arrival times are exponentially distributed and service by each server, that is exponentially distributed with the parameter μ therefore the system goes from two to one, one to zero, so on, it is going to be how many customers in the system and completing their service.

Therefore, the time is exponentially distributed with the sum of those parameters accordingly. Therefore, it is going to be 1μ, 2μ till cμ. Since it is a finite capacity and so on, it is a reducible model, positive recurrent. So this listed probability exists, limiting probability also exists and that is same as the equilibrium probabilities also. Therefore, by using pQ =0 and $\sum_i p_i = 1$ you can get the steady state or the equilibrium probabilities, that is p_n.

The p_c, that is nothing but the probability that the system is full and that is same known as the Erlang B formula. So this is also useful to design the system for a given or what is the optimal c such that you can minimize the probability that the system is full, for that you need this formula therefore to do the optimization problem over the c and here we denote p_c, that is Erlang B formula. Whereas Erlang C formula comes from the M/M/c/K model for the last system we will get the Erlang B formula.

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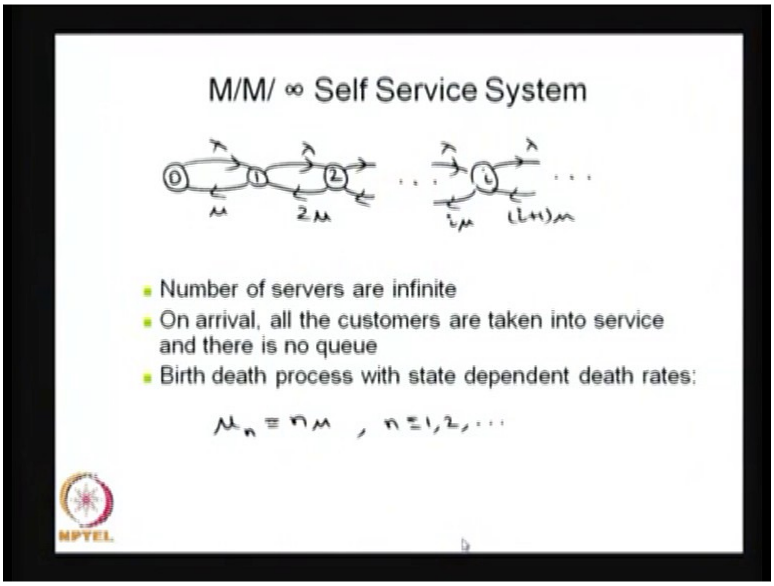

The fifth model, that is M/M/∞, it is not a queuing model, because servers are infinite and limited servers in the system and therefore the customers whoever enters he will get it immediately serviced. The service will be started immediately whereas the service time is exponentially distributed with the parameter μ by each server, all the servers are identical.

The number of servers are infinite here. Therefore, you will have the underlying stochastic process for the system size that is a birth death process with the birth rates are λ because the population is from the infinite source, the death rates are 1μ, 2μ and so on because the number of servers are infinite. So the model which I have discussed in today's lecture, all the five models are the underlying stochastic processes, birth death process.

This is simplest Markovian queuing models.

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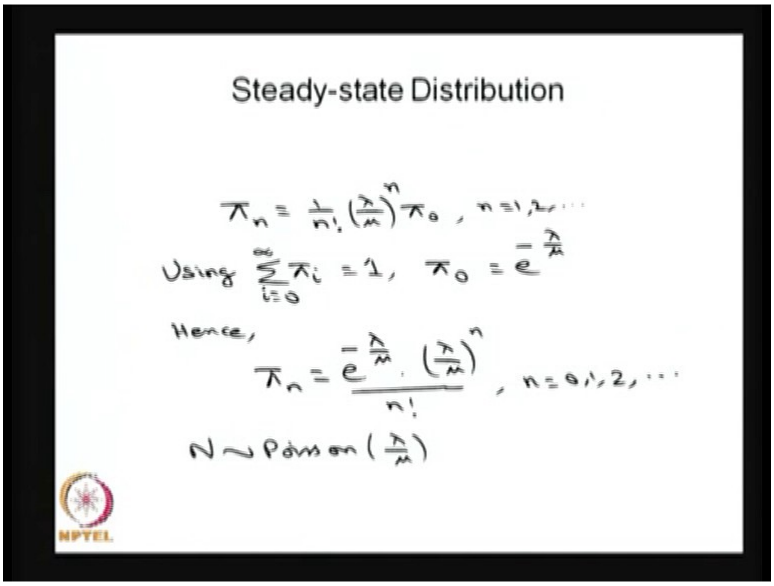

You can get the steady state distribution, use the same theory of birth death process and if you observe these steady state probabilities is of the same Poisson, it is of the form that is probability mass function of a Poisson distribution. Therefore, you can conclude in a steady state, number of customers in the system that is Poisson distributed with the parameter λ/μ.

Because the probability mass function for the π_n is same as the probability mass function of exponentially distributed random variable with the parameter λ/μ.

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Lecture – 92

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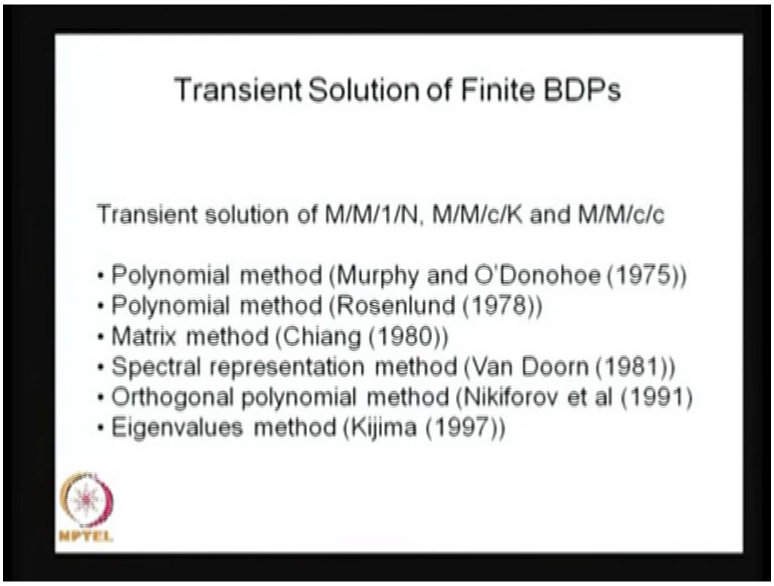

Now I am explaining the transient solution of a finite birth-death process. So using these, one can find out the transient solution of the birth-death process which I have discussed today's class M/M/1/N, M/M/c/K and M/M/c/c also. So the logic is same, that means you have a birth death process with a finite state space. Therefore, the queue matrix is going to be a degree whatever be the number of states in the state space.

And it is going to be a tridiagonal matrix and you know the λ_n's and μ_n's, birth rates as well as the death rates. And the birth rates and death rates are going to be different for these three models. There are many literatures over the transient solution of finite birth-death process started with Murphy and O'Donohoe, he uses the polynomial method. And in 1978 Rosenlund also found the transient solution for the finite BDP using again the different polynomial methods.

And Chiang in 1980, he made a matrix method to this transient solution. Then later Van Doorn gave the solution using spectral representation method. And Nikiforov et al 1991, he also gave the transient solution using orthogonal polynomial. And later Kijima also gave the solution using Eigenvalues method. So these are all the literatures for getting the transient solution of a finite birth-death process.

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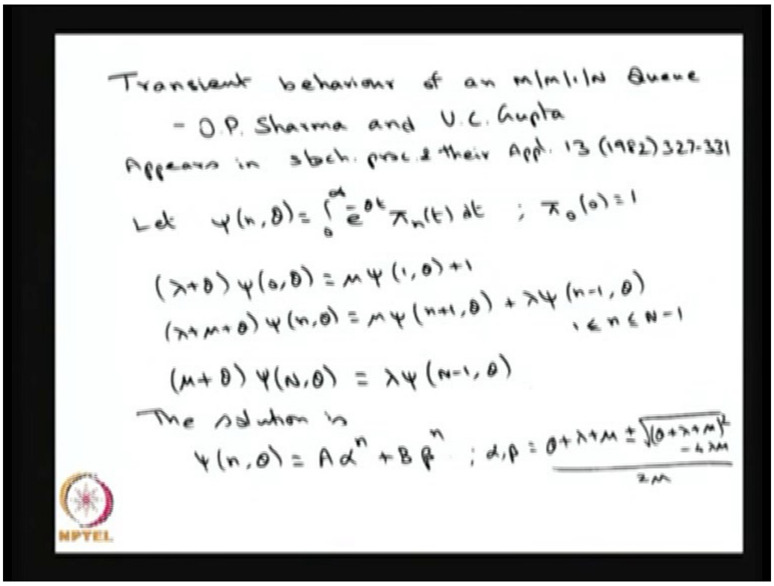

And here I am going to explain how to get the transient behaviour of M/M/1/N queue in a very simplest form, even though there are this many literature and many more literatures for the finite birth-death process. But here I am explaining the overview of how to get the transient behaviour of M/M/1/N queue and this is by O. P. Sharma and U. C. Gupta. It appears in Stochastic processes and their applications, volume 13-1982.

So what this method work you start with the forward Kolmogorov equation.

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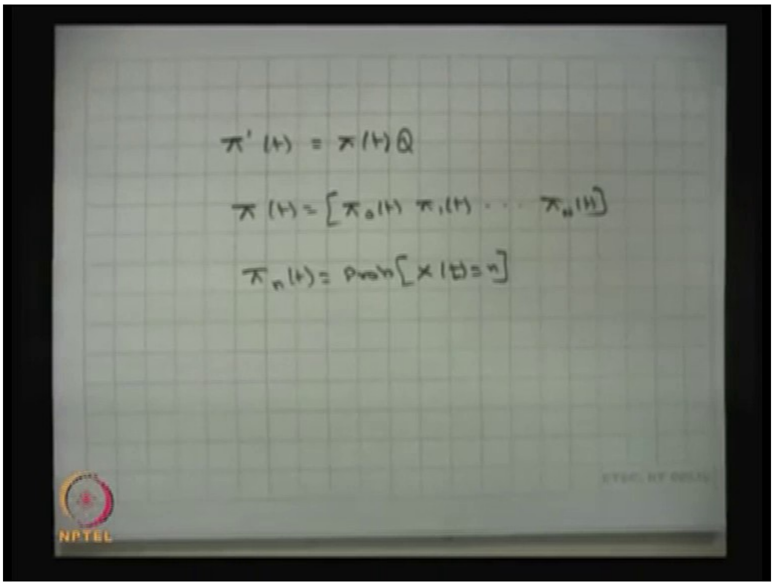

That is π(t), $\pi'(t)$, that is started with $\pi'(t)=\pi(t)$Q matrix, where π is the matrix and π' is the derivatives and Q is the infinitesimal matrix. Take a forward Kolmogorov equation, then use the Laplace transform for each $\pi_n(t)$ you take the, sorry here the $\pi'(t)$ is the vector, it is the distribution of a X(t).

Therefore, this is a vector and this is a vector and Q is a matrix, not the matrix which I said wrongly. So this is a vector and this is a vector and Q is a matrix. So take a Laplace transform for each probability where the $\pi_n(t)$, that is nothing but, so the π(t) is a vector that started with $\pi_0(t)$, $\pi_1(t)$ and so on, $\pi_n(t)$, where $\pi_n(t)$ is nothing but what is the probability?

That the same notation that I started when I discussed the continuous Markov chain, what is the probability that n customers in the system at time t, it is a conditional probability distribution. So $\pi_n(t)$ is the probability that n customers in the system at time t, and $\pi_n(t)$ you get the vector and you make a forward Kolmogorov equation $\pi'(t)=\pi(t)$Q.
And take a Laplace transform, for each $\pi_n(t)$ that exist, because this is a probability and the conditions for the Laplace transform of this function satisfies you can cross check. Therefore, you are taking a Laplace transform, so this is going to be a function of θ. Before taking a Laplace transform, you need an initial condition also. So at time zero, you assume that no customer in the system, at time zero.

Now customer in the system, that means X(0) is equal to zero. Therefore, that probability is going to be one and all other probabilities are going to be zero. That is the initial probability vector. So use this initial probability vector and apply it over the forward Kolmogorov equation taking a Laplace transform, you will get the system of algebraic equation.

Since you are using the $\pi_0(0) = 1$, you will get the first equation with the term one and all other terms are going to be zero. And you know the Laplace transform of derivative of the function. So you substitute, you take a Laplace transform over the forward Kolmogorov equation with this initial condition as well as $\pi_n(0)=0$ for n $\neq 0$. So you will have an algebraic equation that is n + 1 algebraic equations as a function of θ.

You have to solve this algebraic equation system of algebraic equation in terms of θ. Once you are able to solve these and take an inverse Laplace transform and that is going to be the system size at any time t. You can start saying that this is going to be of the solution $A\alpha^n \wedge B\beta^n$, where α and β are given in this form, where α is equal to this plus something and β is equal to minus something, minus square root of this expression.

So you have a α as well as β and now what do you want to find out, if you find out the constant A and B you can get Laplace transform of $\pi_n(t)$. Then you take an inverse Laplace transform and you get the $\pi_n(t)$.

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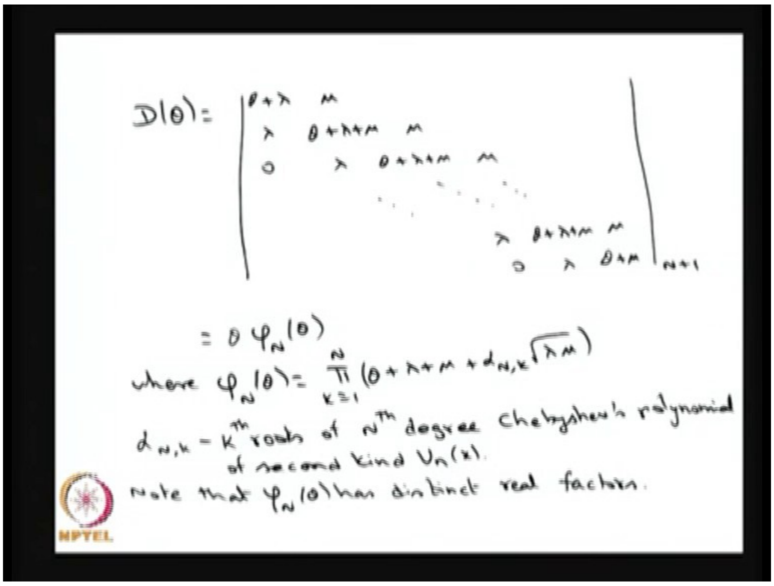

So for that you need the determinant of matrix of this form and here this is nothing but all these values are death rates, these are all the birth rates. And this is corresponding to the M/M/1/N model. And the same logic goes for the transient solution for the M/M/c/K as well as M/M/c/c. So instead of this λ's and μ's you will have a corresponding birth rates and death rates.

But ultimately you will have a N+1 matrix determinant as a function of θ. And since these three models are going to be a irreducible, positive recurrent, the stationary probability and the limiting probabilities exist. Therefore, this determinant going to be always of the form θ times some other function as the degree, as a polynomial of degree N in the function of θ. So this θ is corresponding to the stationary probabilities or the limiting probabilities.

Therefore, always you can get the N +1 order matrix determine that is θ times the polynomial of degree N is a function of θ. For the M/M/1/N model the birth rates are λ and the death rates are μ and you can get this polynomial also in the form of product. The product of θ + λ + $\mu \alpha_{N,k} \sqrt{(\lambda\mu)}$, where $\alpha_{N,k}$ is nothing but the k roots of N-th degree Chebyshev's polynomial of second kind.

There is a relation between the birth-death process with the orthogonal polynomial. For instant, the M/M/1/N model the finite capacity M/M/1/N model, the corresponding orthogonal polynomial for this birth-death process is the Chebyshev's polynomial of the second kind. Similarly, you can say the orthogonal polynomial corresponding to the M/M/c/c model that is Charlier polynomial, like that we can discuss the corresponding orthogonal polynomial for the finite capacity birth-death processes.

So here for the M/M/1/N model this is related to the Chebyshev's polynomial of second kind, that is $U_n(x)$. So once you are able to get the Chebyshev's polynomial roots and that roots is going to play a role in the product form and that is going to be the polynomial. Note that this polynomial has a distinct real factor.

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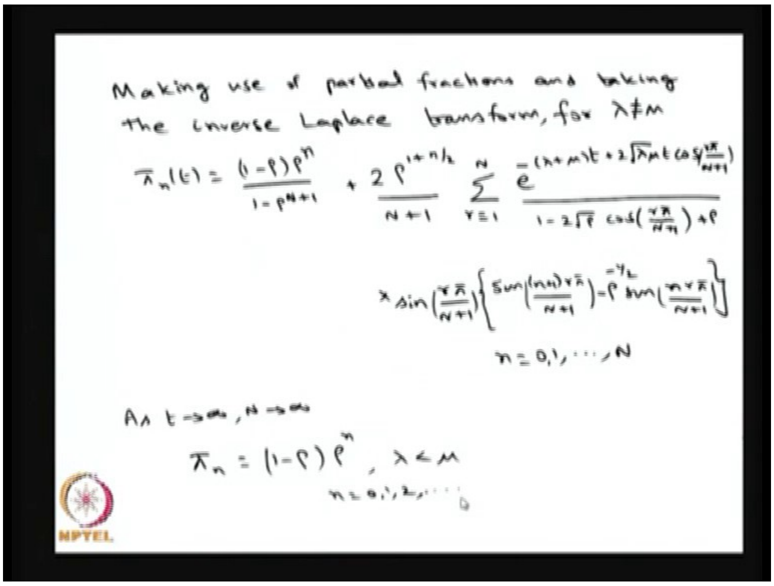

Therefore, you can use the partial fraction, then you can take the inverse Laplace transform to finally you can get the $\pi_n(t)$. I am skipping all the simplification part and main logic is this N+1 th order matrix determinant and that determinants has the factors.

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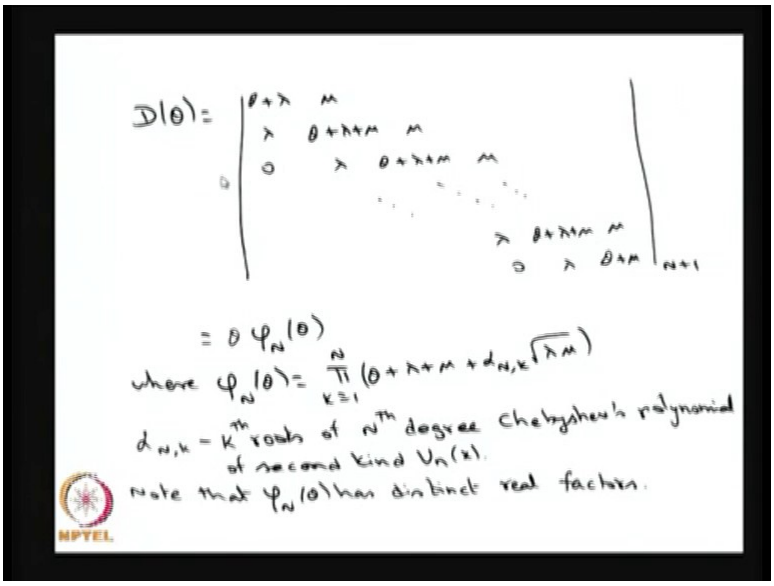

And those factors are related to the Chebyshev's polynomial roots. So once you use all those logics and use the partial fraction.

Then finally you take the inverse Laplace transform, for λ is not equal to μ, you will get steady state or stationary probabilities plus this expression and this is the function of t, $e^{-(\lambda+\mu)t+2\sqrt{\lambda}\mu t\cos\frac{r\pi}{n+1}}$ and denominator this expression multiplied by this.

And here this result is related to the initial condition zero, that means at time zero the system is empty. If the system is not empty, then you will have a one more expression here sin of this minus another term. So that is, you will have a little bigger expression for system size is not empty. And this θ times this, that will give the corresponding partial fraction and so on inverse Laplace it will give the terms which is independent of t and that is related to the steady state probability.

Because if you put t tends to infinity and these quantities are greater than zero. So as t tends to infinity the whole terms will tends to zero. Therefore, as the t tends to infinity, we will have $\pi_n(t)$ is equal to this expression and this is valid for $\rho < 1$. With that condition ρ is less than one, those terms will tend to zero and you will have only this term and that is going to be the steady state or limiting probabilities for M/M/1/N model.

If you make also n tends to infinity along with the t tends to infinity, you will have π_n's that is the steady state probability for the M/M/1 infinity model. So even though I have explained M/M/1/N transient solution in a brief way. But the same logic goes for the M/M/c/c model also, the only difference is this determinant has the λ's and instead of μ's you will have μ_2, μ_3, and so on.

And instead of the Chebyshev's polynomial, you will land up with the Charlier polynomial. But there is a difference between this M/M/1/N model and M/M/c/c model transient solution. Since the Chebyshev's polynomial has a closed form roots, you can find out the factors. So here these are all the factors and you know the factors as well as you can get the closed form expression for the M/M/1/N transient solution where as Charlier polynomial does not have a closed form roots.

Therefore, you will land up with the numerical result for the transient solution for M/M/1/N, M/M/c/c model.

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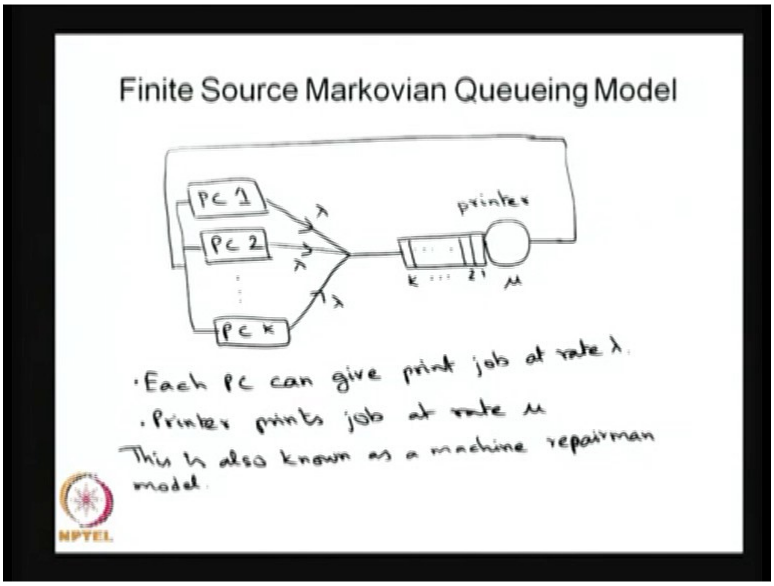

In the case of a continuous time Markov chain, that is a finite source Markovian Queuing models. This model is also known as a machine repairman model and you can think of these PCs are nothing but machines and this is nothing but the repairman. And here the scenario is we have a K PCs and each PC can give a print job and the inter arrival of print jobs that is exponentially distributed by each PC.

Therefore, the print jobs that is follow a arrival process that is a Poisson process with the parameter λ from each PC. And once the print jobs come into the printer, it will wait for the print. And the time taken for each print that is also exponentially distributed with the parameter μ.

And here there is another assumption before the first print is over by the same PC, it cannot give another print command. Therefore, after the print is over by any one particular print job of any PC, then these things will go back to the same thing, then with the inter arrival of print jobs generated that is exponentially distributed, then the print job can come into the printer. So with these assumptions you can think of the Stochastic process.

That means the number of print jobs at any time t in the printer that is going to form a Stochastic process and with the assumption of inter arrival of print jobs, that is exponential and actual printing job that is exponentially distributed and so on. Therefore, this is going to be a birth-death process, with the birth rates or K λ or (K-1)λ and so on, whereas the death rates that is μ, because we have only one repair.

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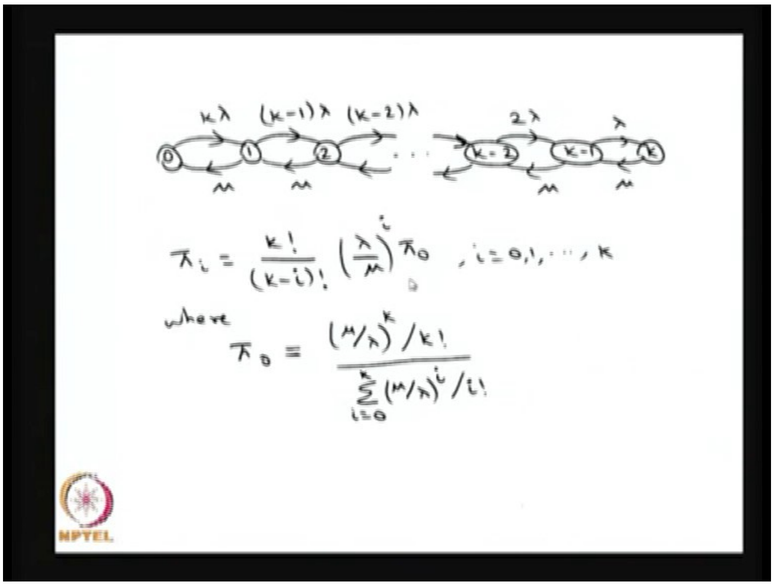

So this is nothing but system size number of jobs in the print job, printer. So therefore that varies from zero to K, because we are making the assumption more than one print job cannot be given by the same PC before the print is over. And from zero to one, the arrival rate will be any one of the K PCs. Therefore, the arrival rate is K λ and already one print job is there in the system printer.

Therefore, out of K -1 PCs one print job can come, therefore the inter arrival time that is exponentially distributed with the parameter (K-1)λ and so on. So this is a way you can visualise the birth rates, whereas the death rates are μ. Once you know the birth rates and death rates you can apply the birth-death process concept to get the steady state probabilities.

So here we are getting the π_i's in terms of π_0, and using the $\sum_i \pi_i = 1$, you are getting $\pi_0$also. And once you know the steady state probability, you can get the all other measures. So the difference is in this model it is a finite source, therefore the birth rates are the function of, it is the state dependent birth rates where as the death rates are μ's only. Simulation of a Queuing model, I will do it in the next lecture.

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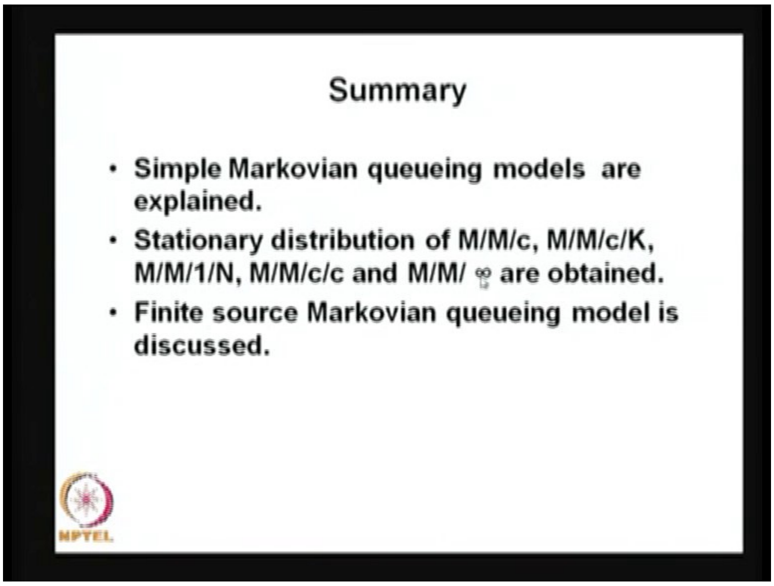

The summary of today's lecture, I have discussed the simple Markovian queueing models other than M/M/1 infinity, that I have discussed in the previous lecture and stationary distribution and all the other performance measures using the birth-death process we have discussed for these queueing models and finally I discussed the finite source Markovian queueing model also.

Thanks.

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