Recap:

$Ax = b$ Solving $Ax = b \Rightarrow$

(i) Unique

(ii) Infinitely many $\rightarrow$ Consistent

(iii) No Soln — Inconsistent.

Gaussian Elimin^n:

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$
$$\vdots \qquad\qquad \vdots$$
$$a_{m1} + \cdots + a_{mn}x_n = b_m$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Homogeneous: $A\underline{x} = \underline{0}$

Soln: $(x_1, x_2, \dots x_n) = (t_1 \dots t_n)$.

Ex: $2x_1 + x_2 = 3$
$x_1 + 2x_2 = 3$.

$x_1 = 1$; $x_2 = 1$

$2(1) + 1 = 3$
$1 + 2(1) = 3$

* <u>Soln:</u>
No
Unique
Infinitely many

Gaussian Elimination.

Systematic procedure

$$Ax = b$$

$A^{m\times n}$, $x^{n\times 1}$, $b^{m\times 1}$

Augmented Matrix = $[A \mid b]$

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ & & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

<u>Ex : 1</u>

$$x_1 + 2x_2 + 3x_3 = 3$$
$$2x_1 - x_2 + x_3 = 1$$
$$x_1 - x_2 + x_3 = 0$$

$[A \mid b]$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

a) Algebraic operⁿs - Do not alter Solutions

b) Simpler Systems.

* Elementary row operⁿs

a) $R_i \leftrightarrow R_j$

b) $R_i \leftarrow \alpha R_i \quad \alpha \neq 0$

c) $R_i = R_i + \alpha R_j \quad \alpha \neq 0$

Ex:

$$2x_1 + 3x_2 + 3 = 6$$
$$2x_1 - x_2 + 2x_3 = 3$$
$$2x_1 - x_2 - x_3 = 0$$

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 2 & -1 & 2 \\ 2 & -1 & -1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

$$[A \mid b] = \left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 2 & -1 & 2 & 3 \\ 2 & -1 & -1 & 0 \end{array}\right]$$

$$R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 2 & -1 & -1 & 0 \\ 2 & -1 & 2 & 3 \end{array}\right]$$

$$R_3 \leftarrow R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 2 & -1 & -1 & 0 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

$$R_2 \leftarrow R_2 - R_1$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & -4 & -2 & -6 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

$R_3 \leftarrow (R_3)\frac{1}{3}$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & -4 & -2 & -6 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \quad 2x_1 + 3x_2 + x_3 = 6$$
$$-4x_2 - 2x_3 = -6$$
$$x_3 = 1$$

$x_3 = 1;$
$x_2 = 1$
$x_1 = 1.$

Soln: $x_1 = 1; \; x_2 = 1; \; x_3 = 1.$

$2x_1 + 3x_2 + x_3 = 6 = 2(1) + 3(1) + 1$

We can verify the other 2 eqns hold

$$2x_1 + 3x_2 + x_3 = 6$$
$$-4x_2 - 2x_3 = -6$$
$$x_3 = 1$$

&

$Ax = b$ for $A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 2 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$

have same solutions.

Soln set same $\Rightarrow$ Equivalent Systems

Ex: 2:
$$x_1 + x_2 + x_3 = 3$$
$$-x_1 - x_2 + x_3 = 1.$$

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ -1 & -1 & 1 & 1 \end{array}\right]$$

$$R_2 \leftarrow R_2 + R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$R_2 \leftarrow R_2/2$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\Rightarrow \quad x_1 + x_2 + x_3 = 3$$
$$\underline{\underline{x_3 = 2.}}$$

$\Rightarrow x_1 + x_2 = 1. \Rightarrow$ line $(1, 0)$ & $(0, 1)$

$x_1 = 1 - x_2.$

$x_2 = t$, arbitrary $\quad x_1 = 1 - t$

Soln: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1-t \\ t \\ 2 \end{pmatrix}$

$\Rightarrow$ Infinitely many solutions.

Reduced row Echelon form RREF.

- $A^{m \times n}$ in RREF

(i) Row with first nonzero number must be 1. $\longrightarrow$ Leading 1.

(ii) Rows of entire zeros $\rightarrow$ Bottom of the matrix

(iii) 2 Successive rows with leading 1's.

$$\begin{bmatrix} 1 & * & * & * \\ * & * & 1 & * \end{bmatrix}$$

(iv) Col. with Leading 1 must have 0 everywhere else in the column.

Procedure: Gauss Jordan Tech.

Ex:

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Note: $R_1, R_2, R_3 \to$ Leading 1's.

No zero rows.

A is in RREF.

Ex: $A = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & 0 & 5 \\ 0 & 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

A is in RREF.

Row echelon form:

Prop (i) to (iii) holds.

$\rightarrow$ Gaussian Elimin^n.

$A = \left[\begin{array}{ccc|c} 1 & 3 & 1 & 6 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$ Row Echelon form

$B = \left[\begin{array}{cccc|c} 0 & 1 & 3 & 2 & 4 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ R.E.F

Steps to Solve: $Ax = b$

1) Augment A with b.
2) Row operations to get RREF
3) Soln $\to$ Inspection
   (or) Parametric form.

Note:

$A^{m\times n}$.

(1) $Ax = 0$

# of Leading 1s $< n$

$\Rightarrow Ax = 0 \Rightarrow$ Infinitely many Soln

Non trivial.

(2) # of Leading 1s $= n$.

$\Rightarrow Ax = 0$ Unique / Trivial Soln.

(3) $A^{m \times n}$ has a unique RREF for $A \neq 0$ matrix.

(4) Given RREF (A), we can find if $Ax = b$ has soln or not

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 0 & 5 \\ 0 & 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & \boxed{0 & 7} \end{array}\right].$$

Inconsistency. $\rightarrow$ No Soln.

$$2x_1 + 3x_2 + x_3 = 6$$
$$2x_1 - x_2 + 2x_3 = 3$$
$$2x_1 - x_2 - x_3 = 0$$

$$[A|b] = \left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 2 & -1 & 2 & 3 \\ 2 & -1 & -1 & 0 \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & -4 & 1 & -3 \\ 2 & -1 & -1 & 0 \end{array}\right]$$

$$R_3 \leftarrow R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & -4 & 1 & -3 \\ 0 & -4 & -2 & -6 \end{array}\right]$$

$$R_3 \leftarrow R_3 - R_2.$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & -4 & 1 & -3 \\ 0 & 0 & -3 & -3 \end{array}\right]$$

$$R_3 \leftarrow (R_3)/-3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & -4 & 1 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$R_2 \leftarrow R_2 - R_3$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & -4 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_2 \leftarrow R_2/(-4)$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_1 \leftarrow R_1 - R_3$

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_1 \leftarrow R_1 - 3R_2$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_1 \leftarrow R_1/2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 1 \\ x_2 = 1 \\ x_3 = 1 \end{array}$$

Ex 2: $x_1 + x_2 + x_3 = 3$

$-x_1 - x_2 + x_3 = 1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ -1 & -1 & 1 & 1 \end{array}\right]$$

$R_2 \leftarrow R_2 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$R_2 \leftarrow R_2/2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$R_1 \leftarrow R_1 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$x_3 = 2$; $x_1 + x_2 = 1 \Rightarrow x_1 = 1 - x_2$.

PIVOT or Leading Variables.

x_1 & x_3.

x_2 : FREE Variable.

RANK of A:

No. of pivot Variables in RREF of A.

Ex 1: $\text{Rank}(A) = 3$

Ex 2: $\text{Rank}(A) = 2$.

$\#$ Pivots $+$ $\#$ free $=$ $\#$ of cols$(A) = n$.