Orthogonal Compl. Subspace.

Let $V = \mathbb{R}^n$, Let $\underline{W}$ be a Subspace of $\mathbb{R}^n$ of dimension 'k'.

$W^\perp = \{\vec{v} \in \mathbb{R}^n \mid \vec{v}\cdot\vec{w} = 0$ for every $\vec{w} \in W\}$

(1) Let $V = \mathbb{R}^n$, W be a Subsp of $\mathbb{R}^n$.
Then $W^\perp$ is also a subsp of $\mathbb{R}^n$

$$W \cap W^\perp = \{\vec{0}\}$$

Proof: Let $\vec{v} \in W \cap W^\perp$

$\Rightarrow \vec{v} \in W$ & $\vec{v} \in W^\perp$

$\Rightarrow \vec{v}\cdot\vec{v} = 0$

$\vec{v}\cdot\vec{v} = 0 \Rightarrow \vec{v} = \vec{0}$.

$\Rightarrow W \cap W^\perp = \{\vec{0}\}$.

* $V = R^n$, W subspace of R^n.
↳ k-dim.

Let $\mathcal{B}$ be an orthonormal basis for V.

$\therefore \mathcal{B} = \{u_1, u_2 \ldots u_k, u_{k+1}, \ldots u_n\}$

$$u_i \cdot u_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Let $\{u_1, \ldots u_k\}$ be the orthonormal basis for W

Then $\{u_{k+1}, \ldots, u_n\}$ is a basis for $W^\perp$

Let $\underline{S}$ be the span of $\{u_{k+1} \ldots u_n\}$

Note that $\{u_{k+1} \ldots u_n\}$ is l.i & is a basis for $W^\perp$.

To establish span S & $W^\perp$ to be the same, we show

$S \subseteq W^\perp$ & $W^\perp \subseteq S$

$\Rightarrow S = W^\perp$

To prove $S \subseteq W^{\perp}$

Any vector $\vec{x}$ of the form

$$\vec{x} = d_{k+1}\vec{v}_{k+1} + \cdots + d_n \vec{v}_n$$

for some scalars $d_{k+1} \ldots d_n$.

is orthogonal to every $\vec{w} \in W$

Let $\vec{w} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ for some scalars $c_1 \ldots c_k$.

$$\Rightarrow \vec{x}\cdot\vec{w} = 0 \Rightarrow (d_{k+1}\vec{v}_{k+1} + \cdots + d_n\vec{v}_n)\cdot(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) = 0$$

Also $\{\vec{v}_{k+1}, \ldots \vec{v}_n\}$ is orthonormal & orthogonal to every vector $\{\vec{v}_1 \ldots \vec{v}_k\}$

Hence $\vec{x} \in W^{\perp} \Rightarrow S \subseteq W^{\perp}$.

To show that $W^{\perp} \subseteq S$. we must show than any vector in $W^{\perp}$ is also in the Span S.

Let $\vec{x} \in W^{\perp}$. Since $\{v_1, \ldots, v_n\}$ is is Orthonormal basis for $\mathbb{R}^n$

$$\vec{x} = \cancel{(\vec{x} \cdot \vec{v}_1)}^{0} \vec{v}_1 + \cdots + \cancel{(x \cdot \vec{v}_k)}^{0} \vec{v}_k + (\vec{x} \cdot \vec{v}_{k+1}) \vec{v}_{k+1} + \cdots + (\vec{x} \cdot \vec{v}_n) \vec{v}_n$$

$\Rightarrow \vec{x} = (\vec{x} \cdot \vec{v}_{k+1}) \vec{v}_{k+1} + \cdots + (\vec{x} \cdot \vec{v}_n) \vec{v}_n$

$\Rightarrow \vec{x} \in$ Span S of $\{\vec{v}_{k+1} \ldots \vec{v}_n\}$

$\Rightarrow W^{\perp} \subseteq S$

$\Rightarrow \boxed{W^{\perp} = S.}$

(2) $\dim(W) + \dim(W^{\perp}) = \dim(V)$

(3) Let $W \subseteq \mathbb{R}^n$ & $\{\vec{u}_1 \dots \vec{u}_k\}$ is an orthonormal basis for W.

For any vector $\vec{v} \in \mathbb{R}^n$, we define the orthogonal proj. of $\vec{v}$ onto W as

$$\text{proj}_W \vec{v} = (\vec{v}\cdot\vec{u}_1)\vec{u}_1 + (\vec{v}\cdot\vec{u}_2)u_2 + \dots (\vec{v}\cdot\vec{u}_k)\vec{u}_k$$

(4) Let W be a subsp of $\mathbb{R}^n$. Every vector $\vec{v} \in \mathbb{R}^n$ can be expressed uniquely as

(*) $\underline{\underline{\vec{v} = \vec{v}_W + \vec{v}_{W^{\perp}}}}$ where

$\vec{v}_W \in W$ & $\vec{v}_{W^{\perp}} \in W^{\perp}$.

Prove that (*) $= \vec{V} = \vec{V}_W + \vec{V}_{W^{\perp}}$ is unique for $\vec{V}_W \in W$ & $\vec{V}_{W^{\perp}} \in W^{\perp}$.

$$\vec{v} = \vec{v}_W + v_{W^\perp}$$

$$\|\vec{v}\|^2 = \left(\vec{v}_W + \vec{v}_{W^\perp}\right)\cdot\left(\vec{v}_W + \vec{v}_{W^\perp}\right)$$

$$= \|\vec{v}_W\|^2 + \|v_{W^\perp}\|^2$$

$\rightarrow$ Pythagoras Theorem.

Orientation of fundamental Subspaces of $A \in \mathbb{R}^{m\times n}$.

$\mathbb{R}^n$	$\mathbb{R}^m$
N_A	N_{A^T}.
R_{A^T}	R_A.

Consider $\mathbb{R}^n$. Let $A \in \mathbb{R}^{m \times n}$.

Let $\vec{x} \in \mathcal{N}_A$. $\Rightarrow A\vec{x} = \vec{0}_m$.

For every vector $\vec{x} \in \mathcal{N}_A$, $A\vec{x} = \vec{0}_m$

$\Rightarrow A\vec{x}$ is orthogonal to every vector $\vec{u} \in \mathbb{R}^m$. for $\vec{x} \in \mathcal{N}_A$.

$\Rightarrow A\vec{x} \cdot \vec{u} = 0$ for $x \in \mathcal{N}_A$.

$= (Ax)^T \cdot u = 0$ $\Rightarrow x^T(A^Tu) = 0$

$= x^T(A^Tu) = 0$

$\Rightarrow \underbrace{x}_{\Downarrow \; \mathcal{N}_A} \cdot \underbrace{(A^Tu)}_{\Downarrow \; \text{element of } R_{A^T}} = 0$ $x \in \mathcal{N}_A, \vec{u} \in \mathbb{R}^m$

$\Rightarrow \boxed{\begin{aligned} \mathcal{N}_A &= R_{A^T}^{\perp} \\ R_{A^T} &= \mathcal{N}_A^{\perp} \end{aligned}}$

$\mathbb{R}^n$:

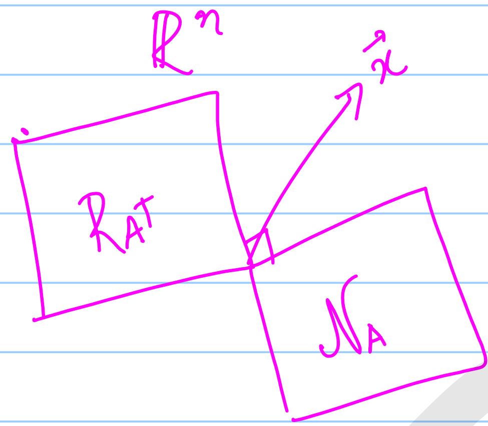
$\mathcal{N}_A$ is Orthog. Comp. of R_{A^T}.

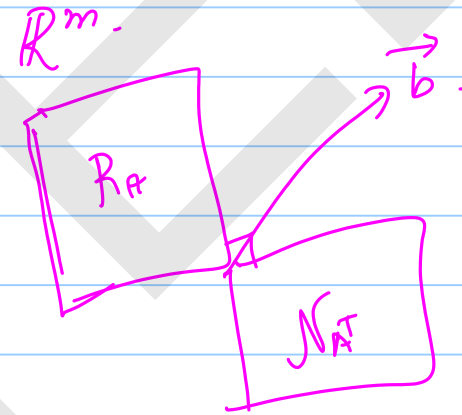
R_{A^T} is Orthog Comp of $\mathcal{N}_A$.

<u>$\mathbb{R}^m$:</u>

$\mathcal{N}_{A^T} = R_A^{\perp}$

$R_A = \mathcal{N}_{A^T}^{\perp}$





$\vec{x} = \vec{x}_{N_A} + \vec{x}_{R_{A^T}}$.

$\vec{b} = \vec{b}_{R_A} + \vec{b}_{N_{A^T}}$.

4 fundamental Subsp. of $A^{m\times n}$

				Dimension
(1)	N_A,	ν_A	$\rightarrow$	Nullity
(2)	R_A,	r_A	$\rightarrow$	Rank
(3)	N_{A^T}	ν_{A^T}		
(4)	R_{A^T}	r_A	=	Rank.