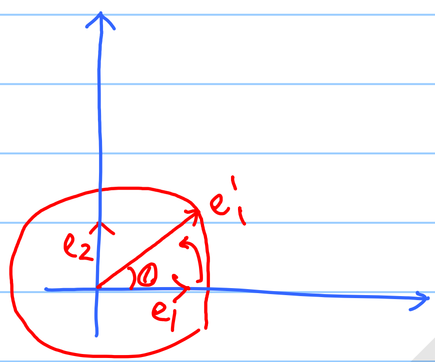
Rotation



Rotation by an angle θ counterclockwise direction.

$$e_1' = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

What is the matrix that effects rotation of a vector by an angle θ in the counterclockwise direction?

$$e_2' = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \overset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \overset{\vec{x}\,'}{\begin{bmatrix} x_1 \cos\theta - x_2 \sin\theta \\ x_1 \sin\theta + x_2 \cos\theta \end{bmatrix}}$$

$$\Downarrow$$

Length of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \sqrt{x_1^2 + x_2^2} = \|x\|$

$$\|\vec{x}\,'\|^2 = (x_1 \cos\theta - x_2 \sin\theta)^2 + (x_1 \sin\theta + x_2 \cos\theta)^2$$

$$= x_1^2 \cos^2\theta + x_2^2 \sin^2\theta - \cancel{2 x_1 x_2 \cos\theta \sin\theta} + x_1^2 \sin^2\theta + x_2^2 \cos^2\theta + \cancel{2 x_1 x_2 \sin\theta \cos\theta}$$

$$= x_1^2 (\cos^2\theta + \sin^2\theta) + x_2^2 (\cos^2\theta + \sin^2\theta)$$

$$= x_1^2 + x_2^2 \Rightarrow \|\vec{x}_1'\| = \sqrt{x_1^2 + x_2^2}$$

Rotation by an angle θ in 2D preserves the length

$\vec{x} \longrightarrow \vec{x'}$ is R_θ (Counterclockwise)

$\vec{x'} \longrightarrow \vec{x}$ $\Rightarrow$ Rotate by an angle $-\theta$

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$R_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R_\theta^T$$

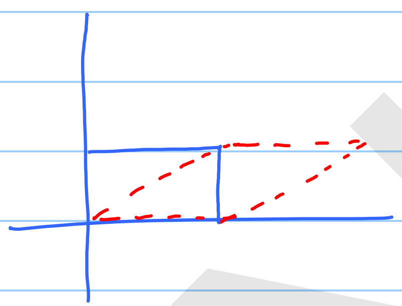
Rotation does not change the area.

Shears

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longrightarrow \vec{y} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$

or

$$\vec{y'} = \begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix}$$



$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \quad a_{11} x_1 + a_{12} x_2 = x_1 + x_2$$

$$a_{21} x_1 + a_{22} x_2 = 0 x_1 + x_2 .$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \longrightarrow \vec{v} = \begin{bmatrix} u_1 + k u_2 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 + k u_2 \\ u_2 \end{bmatrix}$$

Recall elementary row operation of the type $R_i \leftarrow R_i + \alpha R_j$

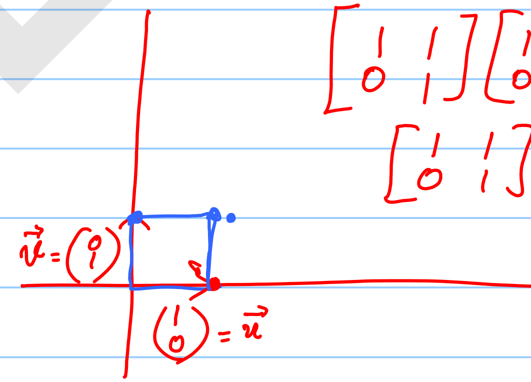
$$\begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 + \alpha x_1 \end{bmatrix}$$

α : Scalar.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$

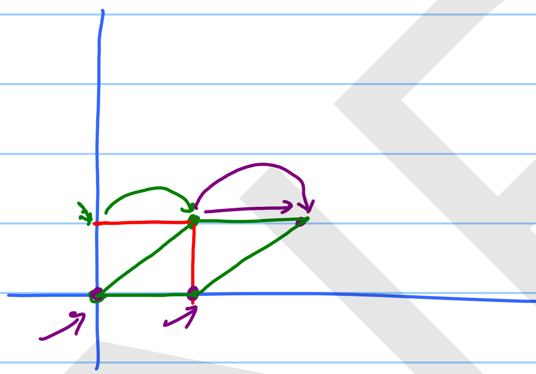
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Unit square : $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$



$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow$ Matrix that converts regular font to italics.

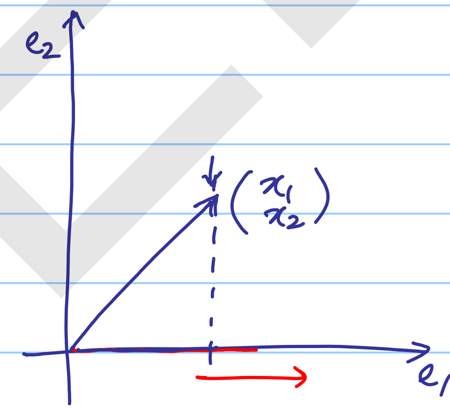
$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

$$\boxed{R_i \leftarrow R_i + kR_j}.$$

Projections:

Orthogonal projections.

→ Very important concept in linear Algebra.



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$$

Along e_1 axis the second component of any vector is 0

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} a_{11} x_1 + a_{12} x_2 = \underline{1} x_1 + \underline{0} x_2 \\ a_{21} x_1 + a_{22} x_2 = \underline{0} x_1 + \underline{0} x_2 \end{array}$$

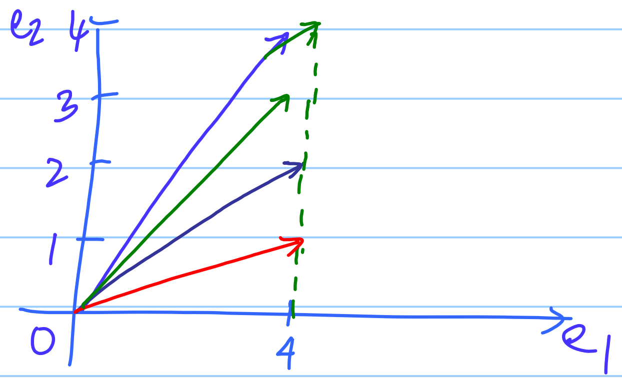
$\Rightarrow$ $P \to$ Matrix projects every vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ onto e_1 axis

Proj. onto e_1 axis

$$P_{e_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Projecting onto e_2 - axis

$$P_{e_2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



Projection onto e_1-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Projection Matrices – Not Invertible

"Transform" $\Leftrightarrow$ " Reversible."