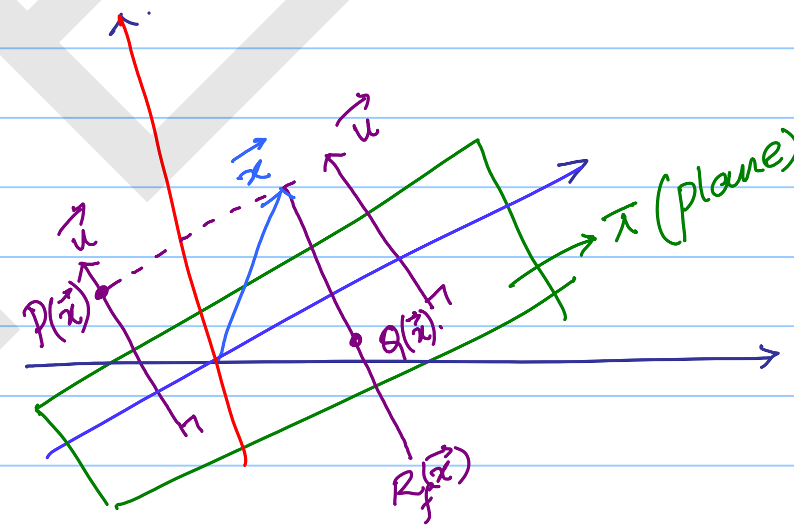
Reflection about a plane passing through the origin and orthogonal to a given vector $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$



Let $\vec{x}$ be any vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

We want to find the reflection of $\vec{x}$ about the plane π.

$$\vec{x} = \text{Proj}_{\vec{u}} \vec{x} + \text{Proj}_{\pi} \vec{x}$$

$$= P(\vec{x}) + Q(\vec{x})$$

$P(\vec{x})$: Proj. of $\vec{x}$ along $\vec{u}$

$Q(\vec{x})$: Proj of $\vec{x}$ onto the plane thro origin and perp. to $\vec{u}$

Ex: $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ $\quad \vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Plane π: $x_2 x_3$ plane.

$$\vec{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ v_3 \end{pmatrix}$$

$$P(\vec{v}) + Q(\vec{v})$$

Let $R_\pi(\vec{x})$ denote the reflection of $\vec{x}$ about the plane π.

For any $\vec{x}$, the midpt of the line segment joining $\vec{x}$ and $R_\pi(\vec{x})$ is the projection of $\vec{x}$ onto the plane π: $Q(\vec{x})$

$$Q(\vec{x}) = \frac{1}{2}\left(\vec{x} + R_\pi(\vec{x})\right)$$

$$\boxed{R_\pi(\vec{x}) = 2Q(\vec{x}) - \vec{x}}$$

where $Q(\vec{x}) = \vec{x} - P(\vec{x})$

Reflection of $\vec{x}$

a) about a line passing thro' the origin and in the direction of $\vec{u}$

$$R_f(\vec{x}) = 2P(\vec{x}) - \vec{x}$$

$P(\vec{x})$: Projection of $\vec{x}$ in the direction of $\vec{u}$.

b) about a plane passing thro' the origin & orthogonal to the vector $\vec{u}$ is given by

$$R_\pi(\vec{x}) = 2Q(\vec{x}) - \vec{x}$$

$Q(\vec{x})$: Proj of $\vec{x}$ onto the plane π.

Let $\pi = x_1 x_2$ plane. $\vec{u} : \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Reflection of $\vec{v}$ about π

$$R_\pi(\vec{v}) = 2Q(\vec{v}) - \vec{v}$$

$Q(\vec{v}) = \vec{v} - P(\vec{v})$ $\quad$ $P(\vec{v})$: Proj of $\vec{v}$ onto $\vec{u}$

$$\Rightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$R_\pi(\vec{v}) = 2Q(\vec{v}) - \vec{v}$$

$$= 2\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ \underline{\underline{-3}} \end{pmatrix}$$

Shears:

Recall : In 2D, shear transf takes a square to a parallelogram.

In 3D, shear takes a cube to a parallelopiped

$$\begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_3 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_3 \\ x_3 \end{bmatrix}$$

Ex: Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\vec{x}' = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$$

Can we get a Shear matrix that transforms $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ to $\begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$?

$$\begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{x_2}{x_1} & 1 & 0 \\ -\frac{x_3}{x_1} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}$$

Useful in Gaussian elimination

Does shear preserve volume?

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + a x_3 \\ x_2 + b x_3 \\ x_3 \end{bmatrix}$$

Vol. is preserved.

Shear does not change the base or height $\rightarrow$ Preserves the volume.

How about rotation transf in 3D?