LU Decomposition.

Gauss elimination has 2 parts

(i) Transforming the coefficient matrix to an upper triangular form with forward elimination

(ii) Back Substitution.

$A\vec{x} = \vec{b}$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$\underline{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Applying elementary row transformations to the augmented matrix $[A \mid b]$, we had

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^* & a_{23}^* & b_2^* \\ 0 & 0 & a_{33}^* & b_3^* \end{array}\right]$$

$$U\vec{x} = b^* \qquad b^* = \begin{bmatrix} b_1 \\ b_2^* \\ b_3^* \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^* & a_{23}^* \\ 0 & 0 & a_{33}^* \end{bmatrix}$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \rightarrow \overset{b^*}{\begin{bmatrix} b_1 \\ b_2^* \\ b_3^* \end{bmatrix}}$$

Let E_i denote i^{th} elementary row transform^n.

$$\underbrace{E_{n-1} \cdots E_3 E_2 E_1}\, A = \mathcal{U}$$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \xrightarrow{?} \mathcal{U}$$

$$R_2 \leftarrow R_2 - \frac{b_1}{a_1} R_1 \qquad a_1 \neq 0.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{-b_1}{a_1} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 - \frac{b_1}{a_1} a_2 & b_3 - \frac{b_1}{a_1} a_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - \frac{c_1}{a_1} R_1 \qquad a_1 \neq 0$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-c_1}{a_1} & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 & 0 \\ \frac{-b_1}{a_1} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \right\}$$

$$E_{n-1} \cdots E_2 E_1 A = U$$

U: Upper triangular Matrix

$$A = \underbrace{\boxed{E_1^{-1} E_2^{-1} \cdots E_{n-1}^{-1}}}_{L} U.$$

$A = LU$ $\quad$ L: Lower Δlar Matrix.

$$Ax = b$$

$$L\underbrace{Ux}_{} = b$$

$$Ux = y$$

$$\Rightarrow Ly = b$$

$$\begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$
\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}
$$

we get the values of x_1, x_2, x_3 by back substitution.

LU decomposition.

$$
\begin{array}{ccc|ccc}
 & & & u_{11} & u_{12} & u_{13} \\
 & & & 0 & u_{22} & u_{23} \\
 & & & 0 & 0 & u_{33}. \\
\hline
1 & 0 & 0 & a_{11} & a_{12} & a_{13} \\
l_{21} & 1 & 0 & a_{21} & a_{22} & a_{23} \\
l_{31} & l_{32} & 1 & a_{31} & a_{32} & a_{33}.
\end{array}
$$

Given $a_{i,j}$, we need to find out $l_{i,j}$ & u_{ij}

The elements of A below the diagonal can be written as

$$a_{ij} = l_{i,1} u_{1,j} + \dots + l_{i,j-1} u_{j-1,j} + l_{i,j} u_{jj} \quad j < i$$

elements of A that are on or above the diagonals, we have

$$a_{i,j} = l_{i,1} u_{1,j} + \dots + l_{i,i-1} u_{i-1,j} + l_{ij} u_{ij}$$

$$\text{for } j \geq i$$

$\Rightarrow$

$$l_{i,j} = \frac{1}{u_{jj}} \left(a_{ij} - l_{i,1} u_{1,j} - \dots - l_{i,j-1} u_{j-1,j} \right) \quad j < i$$

$$u_{ij} = a_{ij} - l_{i,1} u_{1,j} - \dots - l_{i,i-1} u_{i-1,j} \quad j \geq i$$