Ex: 1. $2x_1 + x_2 = b_1$

$x_1 + 2x_2 = b_2$.

Col. Sp(A) where $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$\vec{b} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ \& } \begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\text{Basis for } \mathbb{R}^2} \rightarrow$ l.i. vectors in $\mathbb{R}^2$

$$\vec{b} = x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\left\{ \vec{b} = x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ is the entire $\mathbb{R}^2$.

$\therefore$ Col. Sp(A) : Entire $\mathbb{R}^2$.

Ex: Col. Sp(A), : $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

$\vec{b} = x_1 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= 2x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= (2x_1 + x_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Col. Space (A) = line through the origin & passing thro' $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Col. Sp(A) → 1D subspace of $\mathbb{R}^2$.

Null Sp(A) → 1D subspace of $\mathbb{R}^2$.

↳ line thro' the origin & $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Ex: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$\vec{b} = x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \underbrace{(x_1 + x_3)}_{k} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\text{Col Sp}(A) = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right)$$

$\Rightarrow$ Plane passing thro' the origin & defined by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\rightarrow$ 2D Subspace of $\mathbb{R}^3$.

Consider A^T of a matrix $A \in \mathbb{R}^{m \times n}$

$A^T \in \mathbb{R}^{n \times m}$.

$$(A^T)\, y^{m \times 1} = c^{n \times 1}.$$

N_{A^T} (Null space of A^T):

Set of solutions, $y \in \mathbb{R}^m$, s.t

$$A^T y = 0^{n \times 1}$$

N_{A^T} is a subspace of $\mathbb{R}^m$.

Given $A \in \mathbb{R}^{m \times n}$.

Set of all $c \in \mathbb{R}^n$ which can be expressed as l.c. of cols of A^T.

$$\underline{\underline{c}}^{n\times 1} = \alpha_1 A_1^T + \alpha_2 A_2^T + \cdots + \alpha_m A_m^T.$$

is called Column space of A^T
Range of A^T
Row space of A.

$\underline{\underline{\mathbb{R}^n}}$	$\underline{\underline{\mathbb{R}^m}}$
a) Null space of A N_A.	a) Null space of A^T
b) Col. Space of A^T R_{A^T}. or Colsp(A^T) Row Space of A	b) Col. Sp(A) Range of A.

* Null Space of A:

Def for nullity | Dimension of null sp(A) is called nullity of A.

n_A, ν_A.

If the null space of A is

(a) only the trivial subspace of $\mathbb{R}^n$ i.e., $\{\vec{0}\}$, $\nu_A = 0$

$\Rightarrow N_A$ is a 0D subsp of $\mathbb{R}^n$.

(b) a line passing through the origin, then the nullity is 1

$\nu_A = 1$.

c) a plane passing thro' the origin, then

$\nu_A = 2$.

No. of free variables in the RREF of A = Nullity of A.

= No. of zero rows in RREF of A.

Dimension of the Col. Sp(A) is called the rank of the matrix A. Denoted as r_A or ρ_A.

No. of linearly indep. cols of A is called the rank of A.

= No. of pivot variables in the RREF of the matrix A.

= No. of non-zero rows in the RREF of A.

Let $R(A)$ be the RREF of A.

We know that the number of pivot variables in $R(A)$ plus the number of free variables in $R(A)$ = No. of col. in A.

Rank-Nullity Theorem:

$$\text{Rank}(A) + \text{Nullity}(A) = \text{No. of Cols of } A.$$

$$\Downarrow$$

$$\underbrace{\#\text{ of pivot var} + \#\text{ of free var.}}_{\text{in RREF}(A)} = \#\text{ of Cols. of } A.$$