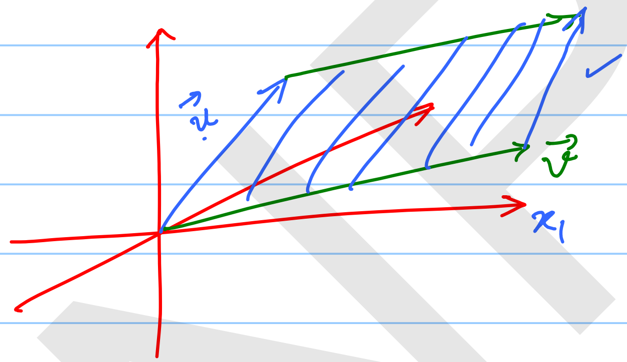
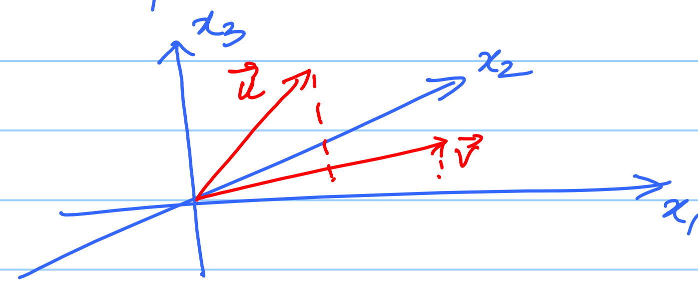
Recall: If $\vec{u}$ & $\vec{v}$ are vectors in $\mathbb{R}^3$ and linearly independent, then the area of the parallelogram with sides along $\vec{u}$ & $\vec{v}$ =

$$\text{Area of } ||gm = \|\vec{u} \times \vec{v}\|$$



Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

Suppose we project $\vec{u}$ & $\vec{v}$ onto the $x_1 x_2$-plane



Projection of $\vec{u}$ & $\vec{v}$ onto x_1x_2 plane results in

$$\begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \ \& \ \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}$$

Area of the parallelogram π_{12} determined by these two vectors is length $\begin{bmatrix} 0 \\ 0 \\ u_1v_2 - u_2v_1 \end{bmatrix} = |u_1v_2 - u_2v_1|$

Similarly, we can project $\vec{u}$ & $\vec{v}$ onto x_2x_3 plane

$$\begin{pmatrix} 0 \\ u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \\ v_3 \end{pmatrix}$$

Area of parallelogram π_{23} is the length of the cross product vector $\begin{bmatrix} u_3v_2 - u_2v_3 \\ 0 \\ 0 \end{bmatrix} \Rightarrow |u_3v_2 - u_2v_3|$

Projection of $\vec{u}, \vec{v}$ onto the x_1x_3 plane.

$$\begin{pmatrix} u_1 \\ 0 \\ u_3 \end{pmatrix} \& \begin{pmatrix} v_1 \\ 0 \\ v_3 \end{pmatrix}$$

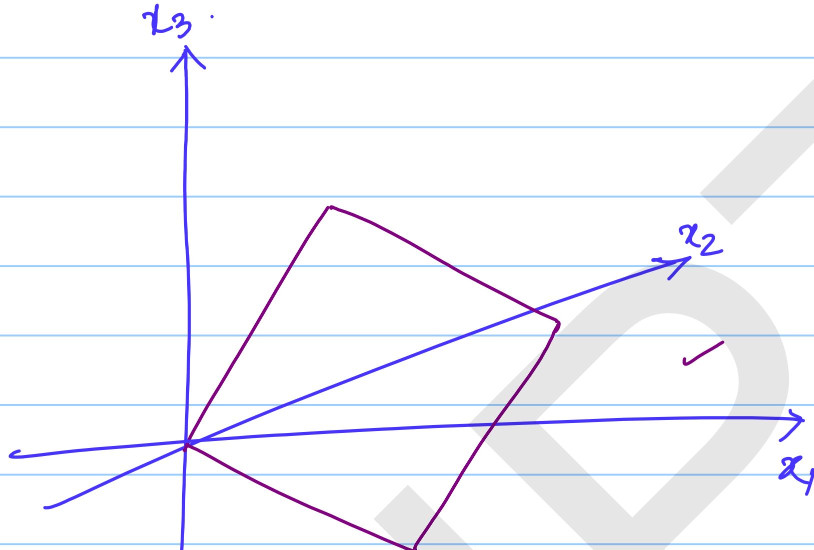
Area of the parallelogram, $\bar{\lambda}_{13}$ determined by these 2 vectors is $|u_3v_1 - u_1v_3|$.

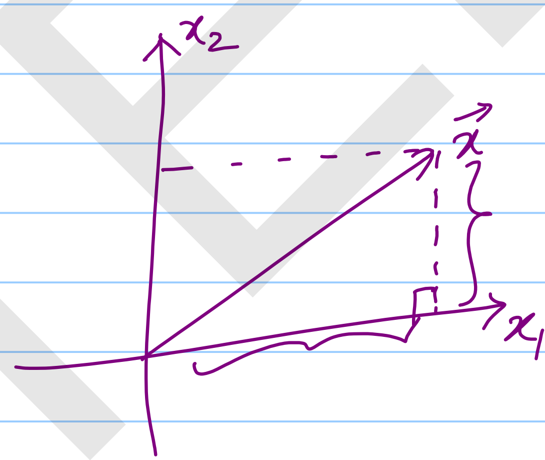
Recall that area of the ||gm determined by $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \& \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$\begin{pmatrix} \text{Area of} \\ \text{||gm } \bar{\lambda} \end{pmatrix}^2 = (u_1v_2 - u_2v_1)^2 + (u_3v_1 - u_1v_3)^2 + (u_2v_3 - u_3v_2)^2$$

$$\begin{pmatrix} \text{Area of} \\ \text{||gm } \bar{\lambda} \end{pmatrix}^2 = \begin{pmatrix} \text{Area of ||gm} \\ \bar{\lambda}_{12} \end{pmatrix}^2 + \begin{pmatrix} \text{Area of} \\ \text{||gm } \bar{\lambda}_{13} \end{pmatrix}^2 + \begin{pmatrix} \text{Area} \\ \text{of ||gm} \\ \bar{\lambda}_{23} \end{pmatrix}^2$$

Generalization of Pythagoras Theorem





$$\|\vec{x}\|^2 = \|\vec{x}_{x_1}\|^2 + \|\vec{x}_{x_2}\|^2$$

Pythagoras Theorem:

Squared length of a vector = Sum of squared length of its proj. onto coordinate axes.

Properties of cross product of 2 vectors:

(i) $\vec{u} \times \vec{u} = \vec{0}$ for any vector $\vec{u}$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\vec{u} \times \vec{u} = \vec{0}$$

(ii) $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$ for every $\vec{u}, \vec{v}$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\vec{v} \times \vec{u} = \begin{bmatrix} v_2 u_3 - v_3 u_2 \\ v_1 u_3 - v_3 u_1 \\ v_1 u_2 - v_2 u_1 \end{bmatrix} \qquad \vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_1 v_3 - u_3 v_1 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

$$\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$$

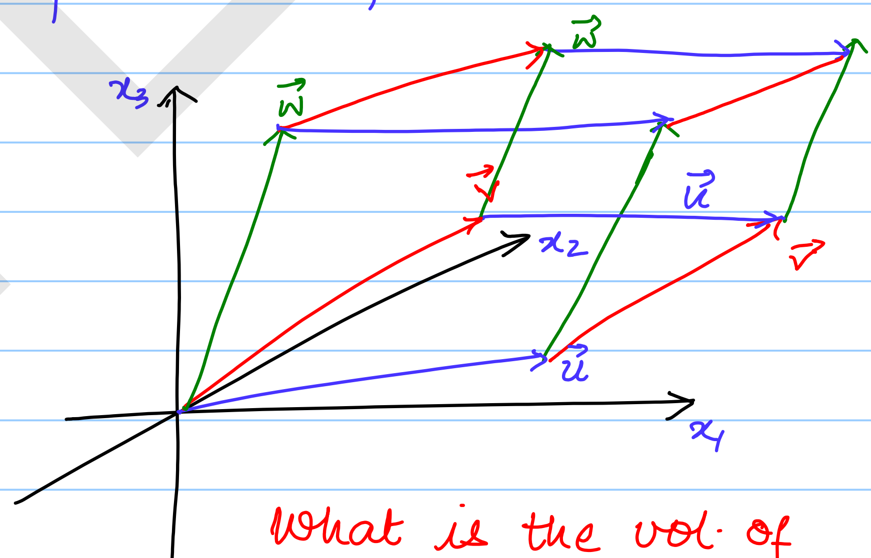
(iii) $(t\vec{u})\times\vec{v} = t(\vec{u}\times\vec{v})$ for t scalar.

(iv) $\vec{u}\times(\vec{v}+\vec{w}) = (\vec{u}\times\vec{v})+(\vec{u}\times\vec{w})$

(v) If $\vec{u}\times\vec{v}=\vec{0}$, then $\vec{u}$ and $\vec{v}$ are linearly dependent

Conversely, if $\vec{u}$ & $\vec{v}$ are linearly dependent, $\vec{u}\times\vec{v}=\vec{0}$

Suppose we have 3 linearly indep vectors $\vec{u}, \vec{v}$ & $\vec{w}$ in $\mathbb{R}^3$



What is the vol. of this parallelopiped?