Recall:

Consider the system of equations

$A\vec{x} = \vec{b}$

If $Ax = b$ is consistent, then $Ax = b$

has a) either a unique solution

or b) infinitely many solutions

If $Ax = b$ is inconsistent, then no solution exists.

$\Rightarrow$ look for least squared solutions

Look at $A\vec{x} = \vec{b}$

$A \in \mathbb{R}^{m \times n}$ $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$.

$Ax = b$

$\Rightarrow b = x_1 A_1 + x_2 A_2 + \ldots + x_n A_n$.

where A_i : i^{th} col. of matrix A.

Solving $Ax = b$ is basically finding out the value of scalars $x_1, \dots, x_n$ s.t $\vec{b}$ can be expressed as a l.c. of cols of A

If we can find scalars $x_1, x_2 \dots x_n$, then $b = x_1 A_1 + \dots + x_n A_n$

$\Rightarrow$ $b \in$ Col. Space of A.

If we cannot find scalars $x_1 \dots x_n$, then $b \notin$ col sp(A).

How do we know if a given System of eqns is Consistent or not?

Ex 1: $2x_1 + x_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$
$2x_1 + x_2 =$

Inconsistent Sys. & no soln exists.

$2x_1 + x_2 = b_1$
$2x_1 + x_2 = b_2$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & b_1 \\ 2 & 1 & b_2 \end{array}\right]$$

- RREF
- $R_2 \leftarrow R_2 - R_1$

$$\left[\begin{array}{cc|c} 2 & 1 & b_1 \\ \underline{0 \quad 0} & & \underline{b_2 - b_1} \end{array}\right].$$

For solutions to exist for the given example $\boxed{b_2 - b_1 = 0}$.

$\Rightarrow b_1 - b_2 = 0$

Look at:

$A^T y = 0$.

$$= \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

RREF: $R_2 \leftarrow R_2 - 2R_1$

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow y_2 = t \Rightarrow y_1 = -t$.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} t \\ -t \end{pmatrix} = \left\{ t\begin{pmatrix} 1 \\ -1 \end{pmatrix}, t \in \mathbb{R} \right\}.$$

$1\,b_2 - 1\,b_1 = 0 \therefore b_1 - b_2 = 0$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \underline{b_1 - b_2 = 0}.$$

Sol. HS. Reqd b.

Ex 2:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Q: For what value(s) of b does the above $Ax = b$ have a soln

RREF (A).

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 0 & b_2 \\ 1 & -1 & b_3 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_2$.

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 0 & b_2 \\ 0 & -1 & b_3 - b_2 \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1$

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & -1 & b_2 - b_1 \\ 0 & -1 & b_3 - b_2 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_2$.

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & -1 & b_2 - b_1 \\ 0 & 0 & b_3 - 2b_2 + b_1 \end{array}\right]$$

$\Rightarrow b_1 - 2b_2 + b_3 = 0$ Consistency cond^n

Let us look at $A^T y = 0$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \leftarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$y_3 = t.$

$-y_2 - 2y_3 = 0 \Rightarrow y_2 = -2t.$

$y_1 + y_2 + y_3 = 0.$

$y_1 = -y_2 - y_3 = 2t - t = t.$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = \left\{ t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, t \in \mathbb{R} \right\}$$

$b_1 - 2b_2 + b_3 = 0$

$$\Rightarrow \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0.$$

$=$

CONSISTENCY CONDITION:

$$\vec{b} \cdot \vec{y} = 0, \text{ where}$$

$$\vec{y} \in \mathcal{N}_{A^T}.$$

then

$Ax = b$ will have

Solution!