Linear Maps in 3D

Scaling :

Map that enlarges or shrinks a vector

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

Let $\vec{v}' = \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} s v_1 \\ s v_2 \\ s v_3 \end{bmatrix}$

where s is a scalar.

$$\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s v_1 \\ s v_2 \\ s v_3 \end{bmatrix}$$

$$\begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s_1 v_1 \\ s_2 v_2 \\ s_3 v_3 \end{bmatrix}$$

s_1, s_2 & s_3 are scalars

If a specific s_i $(i = 1, 2$ or $3)$ is s.t $0 < s_i < 1$, then we get a shrink in that direction.

For ex:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2v_1 \\ 0.5v_2 \\ -v_3 \end{bmatrix}$$

$\Rightarrow$ Stretch along the e_1 direction
Shrink along the e_2 direction
and a flip along e_3 direction

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Suppose we have the unit cube defined by $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

then the vol. of unit cube $= 1$

With scaling factors s_{11}, s_{22} & s_{33}, we have

$$\begin{bmatrix} s_{11} & 0 & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & s_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_{11} & 0 & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & s_{33} \end{bmatrix}$$

New vol $= s_{11} \cdot s_{22} \cdot s_{33}$

Scaling changes the vol. of an object by a factor = Product of the scaling factors.

Reflections in 3D.

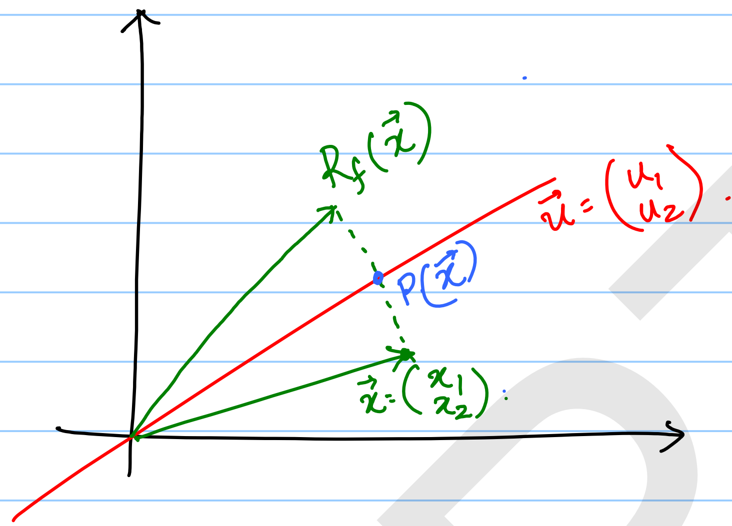
Recall: Reflections in 2D were about lines passing thro' the origin.

In 3D, Reflections: Can be about a line passing thro' the origin or a plane thro' the origin.

Reflections in 2D.

Let R_f denote the map that assigns to every vector $\vec{x}$, the reflection of $\vec{x}$ in the line along the vector $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$



Let $R_f(\vec{x})$ be the point s.t the midpoint of the line segment joining $\vec{x}$ & $R_f(\vec{x})$ is the projec^n of $\vec{x}$ to the line along $\vec{u}$

$$P(\vec{x}) = \frac{1}{2}\left(\vec{x} + R_f(\vec{x})\right)$$

$$\Rightarrow \boxed{R_f(\vec{x}) = 2P(\vec{x}) - \vec{x}}$$

Ex: $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ✓ $\quad \vec{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

$$R_f(\vec{x}) = 2P(\vec{x}) - \vec{x}$$

$$P(\vec{x}) = \frac{\vec{x}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}$$

$$= \frac{u^T x}{u^T u}\,\vec{u}$$

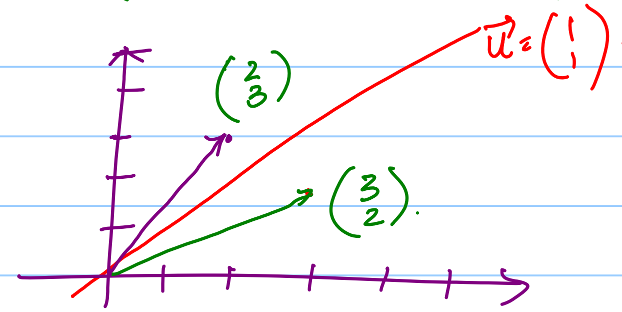
$$= \frac{\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{5}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5/2 \\ 5/2 \end{pmatrix}$$

$$R_f(\vec{x}) = 2P(\vec{x}) - \vec{x}$$

$$= 2\begin{pmatrix} 5/2 \\ 5/2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$



Extending the Same idea, we have, the reflection of $\vec{x}$

$R_f(\vec{x})$, where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and reflection about the line thro' the origin & in the direction of $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$.

$$\boxed{R_f(\vec{x}) = 2P(\vec{x}) - \vec{x}}$$

Ex: $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\quad \vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$P(\vec{x}) = \frac{x^T u}{u^T u}\vec{u} = \frac{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$R_f(\vec{x}) = 2P(\vec{x}) - \vec{x}$$

$$= 2\begin{pmatrix}2\\2\\2\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix}.$$

Question: What is the reflection of a vector about a plane passing thro' the origin & Orthogonal to $\vec{u} = \begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$?