CSCI-4961 - RPI, 05/20 - 08/16/2024

HW $|1\rangle$: One Qubit

Due date: Wed. 11:59pm, May 29

Contributor: ______________________________ *Lecturer: Yanglet Liu*

Topics: Arithmetic and Algebra; Complex numbers $\mathbb{C}$; Superposition; One-qubit gates.

$$\boxed{|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle}$$

Overview

- This homework is due by 11:59pm on Wednesday, May 29
- You may work on this problem set in a group of up to three students; students are encouraged to re-organize teams for different tasks
- Besides the textbook, you may use ChatGPT or any online materials, but **please state clearly the info sources**.
- Please start this homework early and ask questions during Yue's OHs; also ask questions on the Discussion Forum, but be careful not to give any answers away
- Please be concise in your written answers; even if your solution is correct, if it is not well-presented and clear, you may still lose points
- You can type or hand-write (or both) your solutions to the required graded problems below; all work must be organized in one PDF that lists all teammate names
- You are strongly encouraged to use LaTeX, in particular for mathematical symbols; see the corresponding hw1.tex file as a starting point and example

Videos on Qubit and Superposition

Video [1] (5 min, TED-Ed): Schrödinger's cat: A thought experiment in quantum mechanics
Video [2] (14 min): Single qubit and its logic gates
Video [3] (10 min): Mapping the qubit state onto the Bloch Sphere
Video [4] (18 min): A Beginner's Guide To Quantum Computing

[1] https://www.youtube.com/watch?v=UjaAxUO6-Uw
[2] https://www.youtube.com/watch?v=rD_fH7O-D5Y&t=77s
[3] https://www.youtube.com/watch?v=lqWSziZJsLs
[4] https://www.youtube.com/watch?v=JRIPV0dPAd4

Notations

- $x, y, a, b \in \mathbb{R}$; $\alpha, \beta \in \mathbb{C}$
- Imaginary unit $\boxed{i = \sqrt{-1}}$
- Complex number $z \in \mathbb{C}$: $z = x + iy$ or $z = a + ib$
- Conjugate $z^* = a - ib$, or $\bar{z}$
- Modulus (a.k.a., length or norm) $|z| = \sqrt{z^* z} = \sqrt{z z^*} = \sqrt{x^2 + y^2}$
- *Dirac notation*: $|0\rangle$, $|1\rangle$ (called *computational basis*).
- Quantum bit (qubit) $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.

Practice Problems

The problems below are practice problems that will not be reviewed or graded. We encourage you to work on these problems as you study and learn the course material.

Bits (page 2 - 11)

- **Exercise** 1.1, 1.2, 1.4
- **Exercise** 1.5, 1.6
- **Exercise** 1.10

Qubits (page 83 - 86)

- **Exercise** 2.6, 2.7
- **Exercise** 2.8, 2.9

Graded Problems

The problems below are required and will be graded.

Q1. Complex Numbers on Complex Plane (Section 2.2.3, pp. 80 - 83)

1). Given the imaginary unit $\boxed{i = \sqrt{-1}}$, solve the quadratic equation $x^2 + 5x + 10 = 0$ for x.

Hint: The solution to $ax^2 + bx + c = 0$ is given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

2). $\boxed{\text{Euler's formula } e^{i\theta} = \cos\theta + i\sin\theta \text{ and Euler's identity } e^{i\pi} + 1 = 0}$.

On the complex plane with a circle with radius $= 1$, please mark the five points for $\theta = 0, 1, \frac{\pi}{2}, \pi, 2\pi$, respectively.

Q2. Spherical Coordinates and Cartesian Coordinates (Section 2.4)

$$\boxed{|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle\,, \quad 0 \le \theta \le \pi, \quad 0 \le \phi < 2\pi}$$

1). $\boxed{\text{Global phases are physically irrelevant}}$

(Exercise 2.13 on pp. 92) Is there a measurement that can distinguish the following pairs of states? If yes, give such a measurement. If no, explain your reasoning.

(a). $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $e^{i\pi/8}\ |+\rangle = \frac{e^{i\pi/8}}{\sqrt{2}}(|0\rangle + |1\rangle)$.

(b). $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

(c). $|0\rangle$ and $e^{i\pi/4}\,|0\rangle$.

2). On the Block sphere, find the position for the following states (qubits). Please give the answers in (θ, ϕ) coordinates

(a). $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle\ = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle),$
$|i\rangle\ = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle), \quad |-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle).$

(b). $\frac{3+i\sqrt{3}}{4}\,|0\rangle - \frac{1}{2}\,|1\rangle$

3). Given $\theta = \pi/3$ and $\phi = 5\pi/6$, please find the corresponding Cartesian coordinates (x, y, z). Please also give corresponding Cartesian coordinates for $|+\rangle$, $|-\rangle$, $|i\rangle$, and $|-i\rangle$, respectively.

Q3. One-Qubit Gates (Section 2.6)

$\boxed{\text{Quantum gates are linear maps that keep the total probability equal to } 1.}$

Hint: Please use the definition of gates and the linearity of functions below. NOT linear algebra in Chapter 3.

$\boxed{\text{Linearity of function: } f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)}$.

1). (Exercise 2.27 on pp. 104) Prove that

(a) $XZXZ(\alpha|0\rangle + \beta|1\rangle) = -(\alpha|0\rangle + \beta|1\rangle)$.

(b) $ZXZX(\alpha|0\rangle + \beta|1\rangle) = -(\alpha|0\rangle + \beta|1\rangle)$

2). (Exercise 2.30 on pp. 107) Work out the math to show that

(a) $H|-\rangle = |1\rangle$.

(b) $H|-i\rangle = e^{-i\pi/4}|i\rangle \equiv |i\rangle$