CSCI-4961 - RPI, 05/20 - 08/16/2024

Lecture $|3\rangle$: One Qubit (3/3)

May 21, Tue. 6:30pm –8:35pm

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Topics: Qubit and One-Qubit Gates.

Quantum gates are linear maps that keep the total probability equal to 1.

$$\boxed{\boldsymbol{U}|\psi\rangle = \alpha\boldsymbol{U}|0\rangle + \beta\boldsymbol{U}|1\rangle}$$

where $\boldsymbol{U}$ is a unitary matrix.

Linearity of function: $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$.

Pop Quizzes and Exercises

1). Verify the distribution law of Boolean algebra (pp. 39), $A + (BC) = (A + B)(A + C)$

2). (Exercise 2.22, on pp. 99) Consider a map $\boldsymbol{U}$ that transforms the Z-basis states as follows:

$$\begin{aligned} \boldsymbol{U}|0\rangle &= |0\rangle + |1\rangle \\ \boldsymbol{U}|1\rangle &= |0\rangle - |1\rangle \end{aligned} \tag{6}$$

Say $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is a normalized quantum state, i.e., $|\alpha|^2 + |\beta|^2 = 1$.
Calculate $\boldsymbol{U}|\psi\rangle$; Is $\boldsymbol{U}$ a valid quantum gate?

3). (Exercise 2.6, pp.85) If we measure a qubit $\frac{1+i\sqrt{3}}{3}|0\rangle + \frac{2-i}{3}|1\rangle$ or $\frac{1+i\sqrt{3}}{3}|0\rangle + \frac{1+2i}{3}|1\rangle$, what are the probabilities of getting $|0\rangle$ and $|1\rangle$, respectively?
After measurement, if we get $|0\rangle$, what will happen if we measure the same qubit a second time?

4). (Exercise 2.8 and 2.9, pp. 86) A qubit $\frac{e^{i\pi/8}}{\sqrt{5}}|0\rangle + \beta|1\rangle$, what is a possible value of β? Given a qubit $A(2e^{i\pi/6}|0\rangle - 3|1\rangle)$, two questions: 1). find A; and 2). If we measure it, what are the probabilities of getting $|0\rangle$ and $|1\rangle$, respectively?

5). **Exercise 2.6, 2.31** (pp. 104. 108)
1). Calculate $Z^{217}X^{101}Y^{50}(\alpha|0\rangle + \beta|1\rangle)$;
2). Calculate $Y^{51}H^{99}T^{36}Z^{25}|0\rangle$.

6). **Exercise 2.28** (pp. 105)

7). **Example** (pp. 106-108) Using Definition of H gate, show 1). $H|+\rangle$, and $H|i\rangle$; 2). $HSTH|0\rangle$

Outline

- Linear maps
- One-Qubit gates
- Summary of Chapter 2

Notations

- Quantum bit (qubit) $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.
- Quantum gate: $\boldsymbol{U}$
- $\equiv$ denotes "equivalent to"

7 Linear Maps

Linear Maps (Section 2.6.1, on pp. 99)

Example: consider a quantum gate $\boldsymbol{U}$:

$$\begin{aligned} \boldsymbol{U}|0\rangle &= \frac{\sqrt{2}-i}{2}|0\rangle - \frac{1}{2}|1\rangle, \\ \boldsymbol{U}|1\rangle &= \frac{1}{2}|0\rangle + \frac{\sqrt{2}+i}{2}|1\rangle \end{aligned} \tag{7}$$

A quantum gate is *linear*, recall the linearity of function such that

$\boxed{\text{Linearity of function: } f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)}$.

We can distribute $\boldsymbol{U}$ across a superposition state (qubit) $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ as follows (pp. 99):

$$\begin{aligned} \boldsymbol{U}|\psi\rangle &= \alpha\boldsymbol{U}|0\rangle + \beta\boldsymbol{U}|1\rangle \\ &= \left(\alpha\frac{\sqrt{2}-i}{2} + \beta\frac{1}{2}\right)|0\rangle + \left(-\alpha\frac{1}{2} + \beta\frac{\sqrt{2}+i}{2}\right)|1\rangle \end{aligned} \tag{8}$$

Let us check whether it is *valid* quantum gate.

- assuming $|\alpha|^2 + |\beta|^2 = 1$,
- summing the norm-square of each amplitude to see if it is still 1.

(pp. 99)

$$\left|\alpha\frac{\sqrt{2}-i}{2}+\beta\frac{1}{2}\right|^2+\left|-\alpha\frac{1}{2}+\beta\frac{\sqrt{2}+i}{2}\right|^2=1 \tag{9}$$

Probability-preserving linear maps.

$\boxed{\text{Quantum gates are linear maps that keep the total probability equal to } 1.}$

8 One-Qubit Gates

Section 2.6.3

Examples of Lab1: X, Y, Z, S, T, H

$\boxed{\text{One-qubit gates are rotations on the Block sphere.}}$

Exercise 2.28, pp. 105

We leave Section 2.3.3, 2.3.4, 2.6.4 to later lectures.

9 Summary of Chapter 2

9.1 Superposition: Qubit

$$\boxed{|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle}$$

where amplitudes $\alpha, \beta \in \mathbb{C}$ satisfy that $|\alpha|^2 + |\beta|^2 = 1$.

Examples: "plus", "minus", "i", and "minus i"; (θ, ϕ) coordinates, and (x, y, z) coordinates.

$$\begin{aligned}
|+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), & \left(\frac{\pi}{2},\ 0\right), && (1,0,0)\\
|-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), & \left(\frac{\pi}{2},\ \pi\right), && (-1,0,0)\\
|i\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle), & \left(\frac{\pi}{2},\ \frac{\pi}{2}\right), && (0,1,0)\\
|-i\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle), & \left(\frac{\pi}{2},\ \frac{3\pi}{2}\right), && (0,-1,0)
\end{aligned} \tag{10}$$

9.2 Measurement

We measure a qubit $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ and get either $|0\rangle$ and $|1\rangle$

$$\begin{cases} |0\rangle\,, & \text{with prob. } |\alpha|^2 \\ |1\rangle\,, & \text{with prob. } |\beta|^2 \end{cases} \tag{11}$$

$\boxed{\text{The probability is given by the norm-square of the amplitude.}}$

Examples: $\frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/6}|1\rangle)$; $\frac{2}{3}|0\rangle + \frac{1-2i}{3}|1\rangle$.

$\boxed{\text{Measurement collabpses the qubit.}}$

The outcome of flipping a coin is a *random variable* (r.v.) X:

$$X = \begin{cases} \text{“Head”}, & \text{with prob. } 1/2 \\ \text{“Tail”}, & \text{with prob. } 1/2 \end{cases} \tag{12}$$

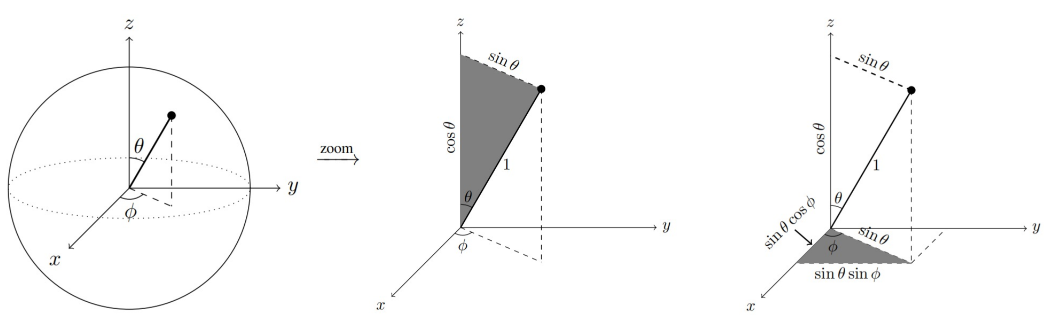
which can be represented as a vector $\boldsymbol{p}$:

$$\boldsymbol{p} = \begin{bmatrix} \frac{1}{2} \\ \\ \frac{1}{2} \end{bmatrix} \tag{13}$$

Similarly, we can represent a qubit $|\psi\rangle$ as a vector

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \\ \beta \end{bmatrix} = \alpha \underbrace{\begin{bmatrix} 1 \\ \\ 0 \end{bmatrix}}_{|0\rangle} + \beta \underbrace{\begin{bmatrix} 0 \\ \\ 1 \end{bmatrix}}_{|1\rangle} \tag{14}$$

9.3 Block Sphere



Normalization (Section 2.3.2): we need to make sure that the total probability is 1, i.e., $|\alpha|^2 + |\beta|^2 = 1$.

Example: $A(\sqrt{2]}\,|0\rangle + i\,|1\rangle)$, we can get $A = \frac{1}{\sqrt{3}}e^{i\theta}$ for any real θ. We can pick $A = \frac{1}{\sqrt{3}}$.
Global Phase (Section 2.4.1): Example

$$\underbrace{e^{i\theta}}_{\text{global phase}} \left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle \right) \tag{15}$$

$\boxed{\text{Global phases are physically irrelevant.}}$

Spherical Coordinates (Section 2.4.2)

$$\boxed{|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle\,, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi}$$

Example (pp. 93): $\frac{3+i\sqrt{3}}{4}|0\rangle - \frac{1}{2}|1\rangle$
(θ, ϕ) coordiantes: $\theta = \frac{\pi}{3}$, $\phi = \frac{5\pi}{6}$.

$\boxed{\text{A qubit can be any point on the Block sphere.}}$

Cartesian coordinates

$$\begin{cases} x = \sin\theta\cos\phi, \\ y = \sin\theta\sin\phi, \\ z = \cos\theta. \end{cases} \tag{16}$$

Example (pp. 96): Given $\theta = \pi/3, \phi = 5\pi/6$, we have

$$\begin{cases} x = \sin(\pi/3)\cos(5\pi/6) = \frac{\sqrt{3}}{2}\frac{-\sqrt{3}}{2} = -\frac{3}{4}, \\ y = \sin(\pi/3)\sin(5\pi/6) = \frac{\sqrt{3}}{2}\frac{1}{2} = -\frac{\sqrt{3}}{4} \\ z = \cos(\pi/3) = \frac{1}{2}. \end{cases} \tag{17}$$

Videos

Video [19] (10 min): Mapping the qubit state onto the Bloch Sphere
Video [20] (10 min): Quantum Computers, Explained With Quantum Physics
Extra time: HW1

[19] https://www.youtube.com/watch?v=lqWSziZJsLs
[20] https://www.youtube.com/watch?v=jHoEjvuPoB8