CSCI-4961 - RPI, 05/20 - 08/16/2024

Lecture $|6\rangle$: The Matrix (2/3)

June 4, Tue. 6:30pm –8:35pm

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Topics: Dirac Notation

Quantum gates are linear maps that keep the total probability equal to 1.

$\boxed{\boldsymbol{U}\,|\psi\rangle = \alpha\boldsymbol{U}\,|0\rangle + \beta\boldsymbol{U}\,|1\rangle}$, where $\boldsymbol{U}$ is a unitary matrix.

Pop Quizzes and Exercises

1). (Inner product, Exercise 3.4, pp. 118) Consider

$$|a\rangle = \frac{3+i\sqrt{3}}{4}\,|0\rangle + \frac{1}{2}\,|1\rangle\,, \quad |b\rangle = \frac{1}{4}\,|0\rangle + \frac{\sqrt{15}}{4}\,|1\rangle \tag{32}$$

1). Find $\langle a|b\rangle$; 2). Find $\langle b|a\rangle$; 3). What is the relationship between $\langle a|b\rangle$ and $\langle b|a\rangle$?

2). (Orthonormality, Exercise 3.7, pp. 120) Consider

$$|a\rangle = \frac{3+i\sqrt{3}}{4}\,|0\rangle + \frac{1}{2}\,|1\rangle\,, \quad |b\rangle = \frac{1}{4}\,|0\rangle + x\,|1\rangle \tag{33}$$

1). Find x s.t. $|a\rangle$ and $|b\rangle$ are orthogonal; 2). Find x s.t. $|b\rangle$ is normalized;
3). For what values of x (if any) are $|a\rangle$ and $|b\rangle$ orthonormal?

3). (Orthonormality, Exercise 3.8 on pp. 120)

4). (Exercise 3.16 on pp. 131) Prove that $XY = iZ$ using two different ways:
1). Show $XY|0\rangle = iZ|0\rangle$ and $XY|1\rangle = iZ|1\rangle$.
2). Using matrix representation, show that $XY = iZ$.

5). Exercise 3.17 and 3.18 on pp. 132

6). Exercise 3.20 on pp. 135

News

IBM Quantum Challenge 2024 (Jun 05, 2024 at 9:00AM - Jun 14, 2024 at 4:00 PM):
https://challenges.quantum.ibm.com/2024

Outline

- (Complex) Linear algebra
- Vectors for Qubit
- Matrices for Quantum Gates

Videos

Video [28] (5 min): Smith visiting The Oracle | The Matrix Revolutions
Video [29] (8 min): Best Fight Scenes from the Matrix Trilogy
Video [30] (10 min): Vectors | Ch 1, Essence of linear algebra
Video [31] (5 min): Matrix formulation of quantum mechanics

15 Vectors for Qubits

15.1 Dirac Notation

Paul Dirac, A NEW NOTATION FOR QUANTUM MECHANICS, 1939.

Bra-ket:
Ket: a column vector,
Bra: the conjugate transpose of the column vector.

$$\begin{cases} |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \langle 0| = [1\ 0] \\ |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & \langle 1| = [0\ 1] \end{cases} \tag{34}$$

[28] https://www.youtube.com/watch?v=zBR2LlutVRM
[29] https://www.youtube.com/watch?v=bBdxqt_AFOg
[30] https://www.youtube.com/watch?v=fNk_zzaMoSs&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab
[31] https://www.youtube.com/watch?v=wIwnb1ldYTI&t=144s

$$
\begin{cases} |+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, & \langle +| = \frac{1}{\sqrt{2}}[1\ 0] \\ |-1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}, & \langle -| = \frac{1}{\sqrt{2}}[1\ -1] \end{cases} \tag{35}
$$

$$
\begin{cases} |i\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}, & \langle i| = \frac{1}{\sqrt{2}}[1\ -i] \\ |-i\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}, & \langle -i| = \frac{1}{\sqrt{2}}[1\ i] \end{cases} \tag{36}
$$

A qubit in superposition state

$$
\begin{aligned} |\psi\rangle &= \alpha\,|0\rangle + \beta\,|1\rangle \\ &= \begin{bmatrix}\alpha\\ \\ \beta\end{bmatrix} = \alpha \underbrace{\begin{bmatrix}1\\ \\ 0\end{bmatrix}}_{|0\rangle} + \ \beta \underbrace{\begin{bmatrix}0\\ \\ 1\end{bmatrix}}_{|1\rangle} \end{aligned} \tag{37}
$$

$\boxed{\text{Linearity of function: } f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)}$.

Measuring $|\psi\rangle$, we get

$$
\begin{cases} |0\rangle & \text{w.p. } |\alpha|^2, \\ |1\rangle & \text{w.p. } |\beta|^2, \end{cases} \tag{38}
$$

such that that $|\alpha|^2 + |\beta|^2 = 1$.

Examples: "plus", "minus", "i", and "minus i", ket and bra representations.

$$
\begin{aligned}
|+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), & \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ \\ 1\end{bmatrix}, && \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix} \\
|-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), & \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ \\ -1\end{bmatrix}, && \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix} \\
|i\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle), & \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ \\ i\end{bmatrix}, && \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\end{bmatrix} \\
|-i\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle), & \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ \\ -i\end{bmatrix}, && \frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\end{bmatrix}
\end{aligned} \tag{39}
$$

16 One-Qubit Gates

A quantum gate is *linear*, recall the linearity of function such that

$\boxed{\text{Linearity of function: } f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)}$.

We can distribute $\boldsymbol{U}$ across a superposition state (qubit) $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ as follows (pp. 99):

$$\boldsymbol{U}|\psi\rangle = \alpha\boldsymbol{U}|0\rangle + \beta\boldsymbol{U}|1\rangle \tag{40}$$

Probability-preserving linear maps.
$\boxed{\text{Quantum gates are linear maps that keep the total probability equal to } 1.}$
Section 3.3.2 and Lab1: I, X, Y, Z, S, T, H

Gate	Action on Computational Basis	Matrix Representation
Identity	$I\|0\rangle = \|0\rangle$ $I\|1\rangle = \|1\rangle$	$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
Pauli X	$X\|0\rangle = \|1\rangle$ $X\|1\rangle = \|0\rangle$	$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Pauli Y	$Y\|0\rangle = i\|1\rangle$ $Y\|1\rangle = -i\|0\rangle$	$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
Pauli Z	$Z\|0\rangle = \|0\rangle$ $Z\|1\rangle = -\|1\rangle$	$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Phase S	$S\|0\rangle = \|0\rangle$ $S\|1\rangle = i\|1\rangle$	$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
T	$T\|0\rangle = \|0\rangle$ $T\|1\rangle = e^{i\pi/4}\|1\rangle$	$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$
Hadamard H	$H\|0\rangle = \frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$ $H\|1\rangle = \frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$	$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Figure 10: Common one-qubit gates as matrices

16.1 The Identity Matrix

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{41}$$

$$\boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{42}$$

16.2 X Matrix

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{43}$$