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Curvature Tensors of a Finsler Space

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Abstract

In the present theoretical analysis, It is defined as the vector space, whose metric is defined by a function, which satisfies the conditions of being a Finsler space. Just as a Riemannian space is locally Euclidean a Finsler space may be regarded as locally Minkowskian space.

Keywords:-*Tensor, Finisler Space, Minkowskian space. connections and covariant*

1. Introduction

A system of n quantities X^i whose transformation law under (1.1) is equivalent to that of the $\dot{x}^{i'}$ is called a contra variant vector attached to the point $P(x^i)$ of X_n. Such contra variant vector constitutes the elements of our new vector space. Hence the totality of all contra variant vectors attached to $P(x^i)$ of X_n is the tangent space denoted by $T_n(P)$ or $T_n(x^i)$.

Further, since the transformation (3.2b) is homogeneous, we may regard the tangent space as 'Centred' affine spaces, the centre or origin corresponding to the values $\dot{x}^1 = 0,\ \dot{x}^2 = 0, ..., \dot{x}^n = 0.$

Indicatrix

We consider that the function $F(x^i, \dot{x}^i)$ is defined for all the line elements $(x^i, \dot{x}^i)$ over the region R of X_n. The equation $F(x^i, \dot{x}^i) = 1$(x^i fixed, $\dot{x}^i$ variable) represent a $(n-1)$ dimensional locus in P i.e. a hyper surface. This hyper surface plays the role of the unit sphere in the geometry of the vector space $T_n(P)$ and is called indicatrix.

Minkowskian Space

It is defined as the vector space, whose metric is defined by a function, which satisfies the conditions *A* to *C* of being a Finsler space. Just as a Riemannian space is locally Euclidean a Finsler space may be regarded as locally Minkowskian space.

Metric Tensor

With the help of equations (3.2a), we can easily see that the set of quantities $g_{ij}(x^i, \dot{x}^i)$ defined by the equation (2.6) form the components of a covariant tensor of rank 2. Also it is clear that $g_{ij}(x^i, \dot{x}^i)$ are positively homogeneous of degree zero in $\dot{x}^i$ and are symmetric in its indices *i* and *j*.

Because of the homogeneity condition (A) for the function $F(x^i, \dot{x}^i),$ we have

$$F^2(x, \dot{x}) = g_{ij}(x, \dot{x})\dot{x}^i \dot{x}^j. \qquad (1.1)$$

The inverse of g_{ij} denoted by g^{ij} is defined as

$$g_{ij}(x,\dot{x})\, g^{ik}(x\,\dot{x}) = \delta_j^k = \begin{cases} 1 & \text{if } \; k = i \\ 0 & \text{if } \; k \neq j \end{cases} \tag{1.2}$$

where δ_j^k is the well known kroncker delta.

Definition

The tensor with covariant components g_{ij} and contra variant components g^{ij} is called the metric tensor or the first fundamental tensor of the Finsler space F_n.

The tensor $C_{ijk}(x,\dot{x})$ defined by

$$C_{ijk}(x,\dot{x}) \overset{def}{=} \frac{1}{2}\frac{\partial g_{ij}(x,\dot{x})}{\partial \dot{x}^k} \overset{def}{=} \frac{1}{4}\frac{\partial^3 F^3(x,\dot{x})}{\partial \dot{x}^i \partial \dot{x}^j \partial \dot{x}^k} \tag{1.3}$$

is positively homogeneous of degree-1 in $\dot{x}^i$ and is symmetric in all three of its indices. This tensor is known as Cartan's C-tensor and satisfy the following conditions

$$C_{ijk}(x,\dot{x})\dot{x}^i = C_{ijk}(x,\dot{x})\dot{x}^j = C_{ijk}(x,\dot{x})\dot{x}^k = 0 \tag{1.4}$$

and

$$\frac{\partial C_{ijk}(x,\dot{x})}{\partial x^h}\dot{x}^i = \frac{\partial C_{ijk}(x,\dot{x})}{\partial x^h}\dot{x}^j = \frac{\partial C_{ijk}(x,\dot{x})}{\partial x^h}\dot{x}^k = 0. \tag{1.5}$$

2. Dual Tangent Space

Corresponding to each contra variant vector $\dot{x}^i$ of $T_n(P)$, we have may associate a covariant vector $\dot{x}_i$ defined by the relation

$$\dot{x}_i = g_{ij}(x,\dot{x})\dot{x}^j \tag{2.1}$$

where it may be noted that the directional argument in the g_{ij} coincides with the vector $\dot{x}^i$ under consideration.

The totality of all covariant vectors $\dot{x}_i$ given by the equation (2.1) associated to a point P, is called the dual tangent space of X_n at P and is denoted by $T_n'(P)$.

Magnitude of A Vector, Angle and Orthogonality

Let X^i be a vector and $\dot{x}^i$ be an arbitrary fixed direction. Then the scalar $|X|$ is given by

$$|X|^2 = g_{ij}(x,\dot{x})\, X^i X^j \tag{3.1}$$

is called the square of the magnitude of the vector X^i for the pre assigned direction $\dot{x}^i$. The Minkowskian magnitude of a vector X^i is defined as

$$|X|^2 = g_{ij}(x^i,\, X^i)\, X^i X^j. \tag{3.2}$$

The Minkowskian cosine corresponding to two arbitrary directions λ^i and μ^i is defined by the ratio

$$\cos(\lambda,\mu)=\frac{g_{ij}(x^k,\lambda^k)\lambda^i\mu^j}{F(x^k,\lambda^k)F(x^k,\mu^k)}. \tag{3.3}$$

From the ratio (3.3), the Minkowskian cosine is not symmetrical in its arguments λ^i and μ^i. However, for an arbitrary fixed direction $\dot{x}^i$, the cosine for two directions λ^i,μ^i can be written as

$$\cos(\lambda,\mu)=\frac{g_{ij}(x^k,\dot{x}^k)\lambda^i\mu^j}{\left[g_{ij}(x^k,\dot{x}^k)\lambda^i\lambda^j\right]^{1/2}\left[g_{ij}(x^k,y^k)\mu^i\mu^j\right]^{1/2}}. \tag{3.4}$$

This expression is symmetric in λ and μ but it depends on the original choice of the direction $\dot{x}^i$. Therefore, it is basically different from (3.3).

Definition (3.1):
The vector μ^i is said to be orthogonal with respect to the vector λ^i iff

$$g_{ij}(x^k,\lambda^k)\lambda^i\mu^j=0. \tag{3.5}$$

Thus, we see that orthogonality is not a symmetrical relationship between the two vectors λ^i and μ^i.

Definition (3.2):
The vector λ^i and μ^i are called orthogonal (for a pre assigned direction $\dot{x}^i$) iff

$$g_{ij}(x^k,\dot{x}^k)\lambda^i\mu^j=0. \tag{3.6}$$

This definition of orthogonality is symmetrical in λ^i and μ^i.

Connection and Covariant Differentiation in F_n:

The connection theories of Finsler space F_n have been studied by many authors. These theories may broadly be divided into two types. In one F_n is constructed of the line elements and is used by most of the researchers [17] [5] [2] [13] and the other is derived term Minkowkian tangent spaces [11] [3].

(A) Finsler Connection

The Finsler connection $F\Gamma$ of a Finsler space F_n is a triad $(F^i_{jk},N^i_k,C^i_{jk})$ of a V-connection F^i_{jk}, a non linear connection N^i_k and a vertical connection C^i_{jk} [17]. In general, the vertical connection C^i_{jk} is different from Cartan's *C*-tensor obtained from C_{ijk} given by the equation (1.3). However, there are certain Finsler connections to be discussed, in which two quantities (vertical connections and Cartan's *C*-tensor) are identical.

If a Finsler connection is given, the *h*- and *v*-covariant derivatives of any tensor field T^i_j are defined as

$$T^i_{j|k}=\partial_k T^i_j+T^m_j F^i_{mk}-T^i_m F^m_{jk} \tag{4.1}$$

and

$$T^i_{j|k}=\dot{\partial}_k T^i_j+T^m_j C^i_{mk}-T^i_m C^m_{jk} \tag{4.2}$$

respectively, where

$$d_k=\partial_k-N^m_k\dot{\partial}_m, \tag{4.3}$$

$$\partial_k={\partial}/{\partial x^k},\ \dot{\partial}_k={\partial}/{\partial \dot{x}^k},$$

$\left(_{|k}\right)$ and $\left(|_k\right)$ denotes the *h* and *v*-covariant derivatives respectively.

For any Finsler connection $(F^i_{jk}, N^i_k, C^i_{jk})$ we have five tensors which are expressed as follows:

The $(h)h$-torsion tensor:

$$T^i_{jk} = F^i_{jk} - F^i_{kj}, \tag{4.4}$$

The $(v)V$-torsion tensor:

$$S^i_{jk} = C^i_{jk} - C^i_{kj}, \tag{4.5}$$

The $(h)hv$-torsion tensor: C^i_{jk} = as the connection C^i_{jk}, (4.6)

The $(v)h$-torsion tensor: $R^i_{jk} = d_k N^i_j - d_j N^i_k$, (4.7)

The $(v)hv$-torsion tensor: $P^i_{jk} = \dot{\partial}_k N^i_j - F^i_{kj}$. (4.8)

The deflection tensor field D^i_j of a Finsler connection is given by

$$D^i_j = \dot{x}^k N^i_j - F^i_{kj}. \tag{4.9}$$

When a Finsler metric is given, various Finsler connections may be defined from the metric. The well known examples are the Rund connection, the Cartan connection and the Berwald connection which are given below.

(B) Rund Connection

As in Riemannian geometry, the Christoffel's symbols of first and second kinds have been defined as

$$\gamma_{hij}(x,\dot{x}) = \frac{1}{2}(\partial_j g_{hi} + \partial_h g_{ij} - \partial_i g_{jh}) \tag{4.10}$$

and

$$\gamma^h_{ij}(x,\dot{x}) = g^{hk}(x,\dot{x})\gamma_{ikj}(x,\dot{x}). \tag{4.11}$$

From the definition it is clear that $\gamma_{ikj}(x,\dot{x})$ is symmetric in its extreme indices and $\gamma^h_{ij}(x,\dot{x})$ is symmetric in its lower indices and satisfy the relation

$$\partial_k g_{ij}(x,\dot{x}) = \gamma_{ijk}(x,\dot{x}) + \gamma_{jik}(x,\dot{x}). \tag{4.12}$$

The symbols $\Gamma^h_{ij}(x,\dot{x})$ are defined as

$$\Gamma^h_{ij}(x,\dot{x}) = \gamma^h_{ij}(x,\dot{x}) - C^h_{im}(x,\dot{x})\gamma^m_{kj}(x,\dot{x})\dot{x}^k \tag{4.13}$$

where

$$C^h_{ij}(x,\dot{x}) = g^{hk}(x,\dot{x})C_{ikj}(x,\dot{x}) \tag{4.14}$$

and Cartan's C-tensor C_{ikj} is defined by (1.3).

For a vector X^i the components $\dfrac{\delta X^i}{\delta t}$ defined by

$$\frac{\delta X^i}{\delta t} = \frac{dX^i}{dt} + \Gamma^i_{jk}(x,\dot{x})X^j \frac{dx^k}{dt} \tag{4.15}$$

form the contra variant components of a vector. The process of differentiation given by (4.15) is called 'δ-differentiation'.

In particular, this process gives a well defined parallel displacement. The vector $X^i + dX^i$ of $T_n(x^i + dx^i)$ is said to be obtained from the vector X^i of $T_n(x^i)$ by parallel displacement if $\delta X^i = 0$. Hence, for such a displacement, we have [21]

$$dX^i = -\Gamma^i_{jk}(x,\dot{x})X^j dx^k \tag{4.16}$$

The partial δ- derivative with respect to x^k in the direction $\dot{x}^i$ of the arbitrary tensor $T^i_j(x,\xi)$ is defined by the formula [21]

$$T^i_{j;k} = \partial_k T^i_j + \dot{\partial}_h T^i_j \partial_k \xi^h + T^m_j \Gamma^{*i}_{mk}(x,\dot{x}) - T^i_m \Gamma^{*m}_{jk}(x,\dot{x}), \tag{4.17}$$

where the coefficients $\Gamma^{*m}_{jk}(x,\dot{x})$ is given by

$$\Gamma^{*m}_{jk}(x,\dot{x}) = g^{ih}(x,\dot{x})\Gamma^{*m}_{jk}(x,\dot{x}) \tag{4.18}$$

and

$$\Gamma^{*}_{jhk}(x,\dot{x}) = \gamma_{jhk}(x,\dot{x}) - [C_{khi}(x,\dot{x})\Gamma^i_{jm}(x,\dot{x}) + \\ + C_{hji}(x,\dot{x})\Gamma^i_{km}(x,\dot{x}) - C_{jki}(x,\dot{x})\Gamma^i_{hm}(x,\dot{x})]\dot{x}^m. \tag{4.19}$$

The symbol Γ^{*i}_{jk} is symmetric in its lower indices *j* and *k*, while Γ^i_{jk} is no-symmetric in *j* and *k*. Also, we have

$$\Gamma^{*i}_{jk}\dot{x}^j\dot{x}^k = \Gamma^i_{jk}\dot{x}^j\dot{x}^k = \gamma^i_{jk}\dot{x}^j\dot{x}^k, \tag{4.20}$$

$$\Gamma^i_{jk}\dot{x}^k = \Gamma^{*i}_{jk}\dot{x}^k, \tag{4.21}$$

$$\Gamma^i_{jk}\dot{x}^j = \gamma^i_{jk}\dot{x}^j. \tag{4.22}$$

The partial δ -derivative of the metric tensor $g_{ij}(x,\xi)$ in the direction $\dot{x}^i$ in view of (4.17) is given by

$$g_{ij}(x,\xi);_k = \partial_k g_{ij}(x,\xi) + 2C_{ijh}(x,\xi)\partial_k \xi^h \\ - g_{hj}(x,\xi)\Gamma^{*h}_{ik}(x,\dot{x}) - g_{ih}(x,\xi)\Gamma^{*h}_{jk}(x,\dot{x}) \tag{4.23}$$

If, in particular, $\dot{x}^i = \xi^i$, the above equation reduces to

$$g_{ij}(x,\xi);_k = 2C_{ijk}(x,\xi)\,\xi^h_{;k}. \tag{4.24}$$

We see that the partial δ-derivative of the metric tensor g_{ij} does not vanish in general. Therefore, further developments of theory of Finsler spaces will differ considerably from the established results of Riemannian geometry in which the covariant derivative of the metric tensor vanishes.

Further, it is to be noted that if the vector field ξ^i is stationary, and that is $\xi^i_{;j} = 0$ then the partial δ-differentiation of a tensor field is *h*-covariant derivative with respect to the Rund connection $(\Gamma^{*i}_{jk}, G^i_j, 0)$ where Γ^{*i}_{jk} is V-connection defined by the equation (4.19) and G^i_j is defined by

$$G^i_j(x,\dot{x}) = \dot{\partial}_j G^i,\ 2G^i(x,\dot{x}) = \gamma^i_{jk}\dot{x}^j\dot{x}^k \tag{4.25}$$

and the vertical connection C^i_{jk} vanishes in this triad. Hence the *v*-covariant derivative of a tensor field is identical to the partial derivative with respect to the element of support $\dot{x}^i$ [17] [21].

Conclusion

We see that the partial δ-derivative of the metric tensor g_{ij} does not vanish in general. Therefore, further developments of theory of Finsler spaces will differ considerably from the established results of Riemannian geometry in which the covariant derivative of the metric tensor vanishes.
When a Finsler metric is given, various Finsler connections may be defined from the metric. The well known examples are the Rund connection, the Cartan connection and the Berwald connection which are given below.

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