14 - rozkład jednostajny

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$X \sim U[a,b]$

Dystrybuanta

$$F(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b \leq x \end{cases}$$

Gęstość

$$f(x) = \begin{cases} 0 & x \leq a \\ \frac{1}{b-a} & a \leq x \leq b \\ 0 & b \leq x \end{cases}$$

Wartość oczekiwana i momenty

$$\mathbb{E}[X] = \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{1}{2} \frac{b^2 - a^2}{b-a} = \frac{1}{2}(a+b)$$

$$\mathbb{E}[X^k] = \int_a^b x^k \frac{1}{b-a}\, dx = \frac{1}{k+1} \frac{b^{k+1} - a^{k+1}}{b-a} = \frac{1}{k+1} \sum_{j=0}^{k} a^j b^{k-j}$$

Funkcja tworząca momenty

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_a^b e^{tx} \frac{1}{b-a}\, dx = \frac{1}{b-a} \cdot \frac{e^{tx}}{t} \Big|_a^b$$

$$M_X(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}$$

Wariancja

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{1}{3}(a^2 + ab + b^2) - \frac{1}{4}(a+b)^2 = \frac{(b-a)^2}{12}$$

Rozkład poniżej progu

$$\mathbb{P}[X \leq c \mid X \leq d] = \frac{\mathbb{P}[X \leq c \cap X \leq d]}{\mathbb{P}[X \leq d]} = \frac{\mathbb{P}[X \leq c]}{\mathbb{P}[X \leq d]} = \frac{c-a}{d-a}$$

$$\mathbb{P}[X \leq c \mid X \leq d] \sim U[a,d]$$

K-ta statystyka

$X_1, X_2, \ldots, X_n \sim U[0,b]$ $\quad Y_1, Y_2, \ldots, Y_n$ - posortowane X_i

$$\mathbb{P}[Y_1 \geq y] = \mathbb{P}[\min(X_1, X_2, \ldots, X_n) \geq y] = \left(1 - \frac{y}{b}\right)^n$$

$$\mathbb{E}[Y_1] = \int_{x=0}^{b} \mathbb{P}[Y_1 \geq x]\, dx = \int_{x=0}^{b} \left(1 - \frac{x}{b}\right)^n dx = \int_{t=0}^{1} (1-t) \cdot b\, dt = \frac{b}{n+1}$$

Dalej indukcja: $\mathbb{E}[Y_{k+1}] = \mathbb{E}[Y_k] + \dfrac{b - \frac{k \cdot b}{n+1}}{n-k+1} = (k+1)\dfrac{b}{n+1}$

$$\mathbb{E}[Y_k] = \frac{k \cdot b}{n+1}$$