7 - Rozkład Poissona

Saturday, 3 February 2024 21:27

Dyskretny rozkład Poissona z parametrem λ

$$X \sim Poisson(\lambda)$$

$$\mathbb{P}[X=k] = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \quad k \geq 0$$

Momenty

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!} = \lambda \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = \lambda$$

$$\mathbb{E}[X^2] = \sum_{k=0}^{\infty} k^2 \cdot e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{(k-1)!} = \lambda \sum_{k=0}^{\infty} (k+1) \frac{\lambda^k}{k!} = \lambda(\lambda+1)$$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda$$

Funkcja tworząca momenty

$$M_X(t) = \mathbb{E}[e^{Xt}] = \sum_{k=0}^{\infty} e^{tk} \cdot e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{\lambda e^t - \lambda} = e^{\lambda(e^t-1)}$$

Suma zmiennych o rozkładzie Poissona ma rozkład Poissona

$$X \sim Poisson(\lambda), \; Y \sim Poisson(\mu)$$

$$\mathbb{P}[X+Y=k] = \sum_{j=0}^{k} \mathbb{P}[X=j \cap Y=k-j] = \sum_{j=0}^{k} e^{-\lambda} \frac{\lambda^j}{j!} \cdot e^{-\mu} \frac{\mu^{k-j}}{(k-j)!}$$

$$= \frac{1}{k!} e^{-(\lambda+\mu)} \sum_{j=0}^{k} \binom{k}{j} \lambda^j \mu^{k-j} = e^{-(\lambda+\mu)} \frac{(\lambda+\mu)^k}{k!}$$

Ograniczenia Chernoffa

$$X \sim Poisson(\lambda)$$

1. Dla $x > \lambda$

$$\mathbb{P}[X \geq x] \leq e^{-\lambda} \frac{(e\lambda)^x}{x^x}$$

2. Dla $x < \lambda$

$$\mathbb{P}[X \leq x] \leq e^{-\lambda} \frac{(e\lambda)^x}{x^x}$$

3. Dla $\delta > 0$

$$\mathbb{P}[X \geq (1+\delta)\lambda] \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\lambda$$

4. Dla $0 < \delta < 1$

$$\mathbb{P}[X \leq (1-\delta)\lambda] \leq \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^\lambda$$

Dowód

1. $\mathbb{P}[X \geq x] \leq \dfrac{e^{\lambda(e^t-1)}}{e^{tx}} = e^{-\lambda + \lambda e^t - tx}$

$t = \ln \frac{x}{\lambda}$

$$\leq e^{-\lambda} \cdot e^{\lambda \cdot \frac{x}{\lambda} - x \ln \frac{x}{\lambda}} = e^{-\lambda} \cdot e^x \cdot \left(\frac{\lambda}{x}\right)^x = e^{-\lambda} \frac{(\lambda e)^x}{x^x}$$

2. $\mathbb{P}[X \leq x] = \mathbb{P}[e^{xt} \geq e^{Xt}] \leq e^{\lambda(e^t-1) - xt}$

$t = \ln \frac{x}{\lambda} \quad \ldots \leq e^{-\lambda + \lambda \cdot \frac{x}{\lambda} - x \ln \frac{x}{\lambda}} = e^{-\lambda} \cdot e^x \cdot \left(\frac{\lambda}{x}\right)^x = e^{-\lambda} \dfrac{(\lambda e)^x}{x^x}$

3. $\mathbb{P}[X \geq (1+\delta)\lambda] \leq e^{\lambda(e^t-1) - (1+\delta)\lambda t}$

$t = \ln 1+\delta \quad \ldots \leq e^{\lambda\delta} \dfrac{1}{(1+\delta)^{\lambda(1+\delta)}} = \left(\dfrac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\lambda$

4. $\mathbb{P}[X \leq (1-\delta)\lambda] \leq e^{\lambda(e^t-1) - (1-\delta)\lambda t}$

$t = \ln 1-\delta \quad \ldots \leq e^{-\delta\lambda} \cdot \dfrac{1}{(1-\delta)^{\lambda(1-\delta)}} = \left(\dfrac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^\lambda$