Flag Algebra

We consider linear combinations of graphs with real coefficients.

For some graphon W, $f_W(G) = d(G,W)$ and we extend it linearly.

$$d(H,W) = \sum_{G\in G_n} d(H,G)\, d(G,W) \quad \text{(probabilities)}$$

Hence $f_W\left(H - \sum_{G\in G_n} d(H,G)\, G\right) = 0$

Let A_0 be linear space generated by such elements for all H, n.

Let A be algebra $\mathbb{R}G/A_0$.

Multiplication

We want multiplication to: $f_W(H_1)\cdot f_W(H_2) = f_W(H_1\cdot H_2)$

$$H_1\cdot H_2 = \sum_{G\in G_{|H_1|+|H_2|}} d(H_1,H_2,G)\, G$$

where $d(H_1,H_2,G)$ is probability that random subset of $|H_1|+|H_2|$ vertices of G randomly split into parts of size $|H_1|$ and $|H_2|$ induces H_1 and H_2.

Rooted graphs

G^R - set of graphs with $|R|$ distinguishable vertices inducing R.

$f_W^{x_1,\dots,x_{|R|}}(H)$ - density of H in W conditioned, that first $|R|$ variables induce R.

f_W^R - probability distribution of $f_W^{x_1,\dots,x_{|R|}}$ under condition that $x_1,\dots,x_{|R|}$ induce R.

Rooted multiplication

$H_1, H_2 \in G^R$

probability $|H_1|+|H_2|-|R|$ vertices split accordingly induces H_1 and H_2

$$H_1\cdot H_2 = \sum_{G\in G^R_{|H_1|+|H_2|-|R|}} d(H_1,H_2,G)\, G$$

Flag Algebra

A_0^R: $H - \sum_{G\in G_n} d(H,G)\, G$

$A = \mathbb{R}G^R/A_0^R$

$[\![\ ]\!]: A^R \to A$

$a\in A^R$, W s.t. $d(R^\circ, W) > 0$

$f_W([\![a]\!]) = f_W([\![R]\!])\,\mathbb{E}[f_W^R(a)]$ } we want

$H\in G^R$

H_0 - H without root, $q(H)$ - prob. that choosing the vertices of R at random in H_0 creates H.

$[\![H]\!] = q(H)\cdot H_0$

Hence $$f_W([\![R]\!])\,\mathbb{E}[f_W^R(H)] = \frac{(|H|-|R|)!}{|Aut(H)|}\int_{[0,1]^{|H|}} \prod_{ij\in e(H)} W(x_i,x_j)\prod_{ij\notin e(H)}\left(1-W(x_i,x_j)\right)$$

Cauchy-Schwarz for flags

$$[\![a\cdot b]\!]^2 \le [\![a^2]\!]\cdot[\![b^2]\!]$$