Rozdział 12

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Regularity Lemma

Edge density

$$d(A,B) = \frac{e(A,B)}{|A|\cdot|B|}$$

ε-regularity

$d = d(A,B)$. A and B are ε-regular iff:

$A' \subset A$, $B' \subset B$ s.t. $|A'| \geq \varepsilon|A|$, $|B'| \geq \varepsilon|B| \Rightarrow |d - d(A',B')| < \varepsilon$

Szemerédi Regularity Lemma 1978

For $\varepsilon > 0$ and integer m, there exist M s.t every large enough graph on n vertices can be partitioned into $k+1$ sets of vertices s.t.:

$m \leq k \leq M$

$|V_0| \leq \varepsilon n$

$|V_1| = |V_2| = \dots = |V_k|$

all but εk^2 pairs of sets V_i, V_j are ε regular.

Proof: Energy increment argument. We start by any partition and if there are any issues we split it further.

Degree lemma

If (X,Y) is ε-regular with density d, then for $B \subset Y$ s.t $|B| \geq \varepsilon|Y|$:

$|\{x \in X : |N(x) \cap B| \leq (d-\varepsilon)|B|\}| \leq \varepsilon|X|$

Generalized degree lemma

$|\{\bar{x} \in X^k : N(\bar{x}) \cap B| < (d-\varepsilon)^k |B|\}| \leq \varepsilon k |X|^k$

Triangle counting lemma

Let (A,B), (B,C) and (A,C) be ε-regular pairs with $|A| = |B| = |C| = n$ and all pairwise densities $\geq 2\varepsilon$.

Then $\#\Delta$ is at least $(1-2\varepsilon)(d(A,B)-\varepsilon)(d(B,C)-\varepsilon)(d(A,C)-\varepsilon)$

Counting lemma

Let H be k vertex graph. Let $V_1, \dots, V_k$ be disjoint sets of n vertices. s.t. all pairs are ε-regular. Let d_{ij} be density between V_i, V_j. if $d_{ij} > C\varepsilon$ (C depends only on H), then graph $V_1 \cup \dots \cup V_k$ contains at least $\prod(d_{ij} - C\varepsilon)\, n^k$ copies of H.

Degree form

For every $\varepsilon > 0$, $d \in [0,1]$ there exists integer M s.t. any n-vertex graph G can be partitioned into $k+1$ parts and subgraph $G' \subset G$ s.t

$V_0 < \varepsilon n$

$|V_1| = |V_2| = \dots = |V_k| \leq \lceil \varepsilon n \rceil$

$k \leq M$

there are no edges inside V_i in G'.

$d_{G'}(v) > d_G(v) - (d+\varepsilon)n$

all pairs (V_i, V_j) in G' are either ε-regular with density 0 or $\geq d$.

Graph removal lemma

For every $c > 0$ and graph H on k vertices there exists $a > 0$ s.t. every n-vertex graph can be made H-free by removing at most cn^2 edges or contains $a n^k$ copies of H.

Roth, 1953

For every $\varepsilon > 0$ there exist n_0 s.t if $A \subseteq [n \geq n_0]$ then A contains 3-element arithmetic progression.

Embedding lemma

For any $d \in [0,1)$, $\Delta \geq 1$ there exists ε_0 s.t for any $\varepsilon \leq \varepsilon_0$:

let H be a graph with $\Delta(H) \leq \Delta$, $s \in \mathbb{N}$ and $R = R(\varepsilon, d)$ be reduced graph of G with parts of size $\geq \frac{2s}{d^\Delta}$.

Then if H is subgraph of $R[s]$ then it's subgraph of G.