Rozdział 2

Sunday, 26 January 2025 22:30

$K_{k+1}[t]$ – blowup

Complete multipatite graph with k+1 parts of size t.

Erdős–Stone Theorem (1946)

For any $k,t \geq 1$, $\varepsilon > 0$ there exist n_0:

$$ex(n, K_{k+1}[t]) \leq \left(1 - \frac{1}{k} + \varepsilon\right)\frac{n^2}{2}$$

i.e. for large enough n, n vertex graph G with $e(G) \geq \left(1 - \frac{1}{k} + \varepsilon\right)\frac{n^2}{2}$ contains $K_{k+1}[t]$.

Proof:

From Angle-Form Szemeredi Regularity we obtain graph with edges corresponding to high-density ε-regular pairs. Such graph will contain K_{k+1}, hence for large enough sizes of parts, the original graph contains $K_{k+1}[t]$.

Erdős–Simonovits (1966)

For any graph F s.t. $\chi(F) = k+1$, maximal nr. of edges in n-vertex F-free graph is at most $\left(1 - \frac{1}{k}\right)\frac{n^2}{2} + o(n^2)$

Proof: For any $\varepsilon > 0$ $\left(1 - \frac{1}{k} - \varepsilon\right)\frac{n^2}{2} \leq ex(n, F) \leq \left(1 - \frac{1}{k} + \varepsilon\right)\frac{n^2}{2}$

The upper bound is by Erdős–Stone with $K_{k+1}[v(F)]$.

Lower bound is due $T_k(n)$ being an F-free graph.

Critical edge theorem (Simonovits, 1974)

Let F be graph s.t. $\chi(F) = k+1$. For large enough n, the unique extremal graph for F is $T_k(n)$ iff F has critical edge.

Proof: If $T_k(n)$ is extremal graph for F, then $T_k(n)$ + any edge contains F. $\chi(T_k(n)) = k$, hence this new edge is critical.

The other direction is more involved. We take extremal graph G. $e(G) \geq e(T_k(n))$, hence due to Erdős–Stone it contains large K_k blowup. By technical lemma we show that it must be complete bipartite.

Corollary

$$ex(n, C_{2k+1}) = \left\lfloor \frac{n^2}{2} \right\rfloor$$