Rozdział 6

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Large bipartite subgraph

Every graph with e edges contains bipartite subgraph with at least $\frac{1}{2}e$ edges.

Proof: expected value.

Diagonal Ramsey lower bound (Erdős, 1947)

For any integer k and n if $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, then any edge coloring of K_n contains monochromatic K_k.

Hence $R(k,k) > n - \binom{n}{k}2^{1-\binom{k}{2}} \geq \left(\frac{1}{e\sqrt{2}} + o(1)\right)k 2^{\frac{k}{2}}$

Proof: By Union-Bound such coloring exists.

There exists a coloring with at most $\binom{n}{k}2^{1-\binom{k}{2}}$ monochromatic K_k, hence there is a graph with at least $n - \binom{n}{k}2^{1-\binom{k}{2}}$ vertices without monochromatic K_k.

Dependency graph

Set of events $A_1, \dots, A_n$ in probability space.

Graph on $[n]$ is dependency graph if A_i is independent of events indexed by $[n] \setminus n(i)$.

Lovász Local Lemma (symmetric)

Let A_i be a family of events s.t $\mathbb{P}[A_i] \leq p$ and each of them is dependent on at most d others.

If $p(d+1)e \leq 1$ then there exist an event that none of these events occur.

R(k,k) lower bound (Spencer, 1977)

If $\left(\binom{k}{2}\binom{n}{k-2}+1\right)2^{1-\binom{k}{2}}e \leq 1$ then there exists monochromatic K_k-free graph on n vertices.

In particular: $R(k,k) > \left(\frac{\sqrt{2}}{e} + o(1)\right)k 2^{\frac{k}{2}}$

Proof: LLL - events are dependent if they share an edge.

Lovász Local Lema (Erdős-Lovász, 1975)

Let $B_1, \dots, B_n$ be family of bad events.

If there exist real numbers $x_1, \dots, x_n$ s.t.

$$\mathbb{P}[B_i] \leq x_i \prod_{j \in N(i)}(1-x_j) \quad \forall i \in [n]$$

Then the event avoiding all bad events has probability at least $\prod(1-x_i)$

Proof: We will show by induction that for each $S \subset [n]$:

$$\mathbb{P}\left[B_i \,\middle|\, \bigcap_{j \in S} \overline{B_j}\right] \leq x_i \text{ if } i \notin S$$

We proceed by induction on $|S|$. $|S|=0$ is trivial.

$S_1 := S \cap N(i)$, $S_2 := S \setminus S_1$

$$\mathbb{P}r\left[B_i \,\middle|\, \bigcap_{j \in S} \overline{B_j}\right] = \frac{\mathbb{P}r\left[B_i \cap \bigcap_{j \in S_1} \overline{B_j} \cap \bigcap_{j \in S_2} \overline{B_j}\right]}{\mathbb{P}r\left[\bigcap_{j \in S_1} \overline{B_j} \cap \bigcap_{j \in S_2} \overline{B_j}\right]} = \frac{\mathbb{P}r\left[B_i \cap \bigcap_{j \in S_1} \overline{B_j} \,\middle|\, \bigcap_{j \in S_2} \overline{B_j}\right]}{\mathbb{P}r\left[\bigcap_{j \in S_1} \overline{B_j} \,\middle|\, \bigcap_{j \in S_2} \overline{B_j}\right]}$$

$$\mathbb{P}r\left[B_i \cap \bigcap_{j \in S_2} \overline{B_j} \,\middle|\, \bigcap_{j \in S_2} \overline{B_j}\right] \leq \mathbb{P}r\left[B_i \,\middle|\, \bigcap_{j \in S_2} \overline{B_j}\right] = \mathbb{P}r[B_i] \leq x_i \prod_{j \in N(i)}(1-x_j)$$

$$\mathbb{P}r\left[\bigcap_{j \in S_1} \overline{B_j} \,\middle|\, \bigcap_{j \in S_2} \overline{B_j}\right] = \mathbb{P}r\left[\overline{B_{j_1}} \,\middle|\, \bigcap_{j \in S_2} \overline{B_j}\right] \cdot \mathbb{P}r\left[\overline{B_{j_2}} \,\middle|\, B_{j_1} \cap \bigcap_{j \in S_2} \overline{B_j}\right] \cdots$$

$$\geq (1-x_1)(1-x_2)\cdots(1-x_{|S_1|}) \geq \prod_{j \in N(i)}(1-x_j)$$

Hence $\mathbb{P}r\left[\prod \overline{B_i}\right] = \mathbb{P}r[\overline{B_1}] \cdot \mathbb{P}r[\overline{B_2} \mid \overline{B_2}] \cdots \leq \prod(1-x_i)$.