Rozdział 4

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N_s - number of K_s in graph

Moon-Moser theorem

For every n-vertex graph and $s \geq 2$:

$$N_{s+1} \geq \frac{N_s}{s^2-1}\left(\frac{s^2 N_s}{N_{s-1}} - n\right)$$

Proof: We count number of triples (S', x, y) s.t. S' is a copy of K_{s-1} and x and y are adjacent to S'.

By Cauchy-Schwarz: $\sum d(S)^2 \cdot N_{s-1} \geq \left(\sum d(S)\right)^2 \geq s^2 N_s^2$

$$\Rightarrow \sum d(S)^2 \geq \frac{s^2 N_s^2}{N_{s-1}}.$$

Then we double count giving lower bound and obtain desired bound.

Corollary

Let G be a graph with at least $\left(1-\frac{1}{x}\right)\frac{n^2}{2}$ edges.

For every $s \geq 1$: $N_{s+1} \geq \left(1-\frac{s}{x}\right)\frac{n}{s+1} N_s$

Proof: induction on s.

$$N_{s+1} \geq \frac{N_s}{s^2-1}\left(\frac{N_s \cdot s^2}{N_{s-1}} - n\right) \geq \frac{N_s}{s^2-1}\left(\frac{\left(1-\frac{s-1}{x}\right)\frac{n}{s} \cdot N_{s-1} \cdot s^2}{N_{s-1}} - n\right)$$

$$= \frac{N_s}{(s+1)(s-1)} \cdot ns\left(1-\frac{s-1}{x}\right) = N_s \cdot \frac{n}{s+1} \cdot \frac{s}{s-1}\left(1-\frac{s-1}{x}\right) \geq \left(1-\frac{s}{x}\right)\frac{n}{s+1} \cdot N_s$$

Corollary

For any $\varepsilon > 0$ and any $s \geq 1$ there exist $c > 0$ s.t. any n-vertex graph with at least $\left(1-\frac{1}{s}+\varepsilon\right)\frac{n^2}{2}$ edges:

$$N_{s+1} \geq c \cdot n^{s+1}$$

Proof: By previous corollary with $\frac{1}{x} := \frac{1}{s} - \varepsilon \Rightarrow s - x = -\varepsilon x s$

$$N_{s+1} \geq \prod_{i=1}^{s}\left(\left(1-\frac{i}{x}\right)\frac{n}{i+1}\right) \cdot N_1 = \frac{n^{s+1}}{x^s (s+1)!} \prod_{i=1}^{s}(x-i)$$

$$\geq n^{s+1} \cdot \frac{(x-s)^s}{x^s (s+1)!} = n^{s+1} \cdot \frac{(\varepsilon x s)^s}{x^s (s+1)!} = \frac{(\varepsilon s)^s}{(s+1)!} \cdot n^{s+1}$$

Erdős-Simonovits (1983)

For any $\varepsilon > 0$ and graph F there exist $c > 0$ s.t. every n-vertex graph with at least $ex(n, F) + \varepsilon n^2$ edges contains at least $c n^{V(F)}$ copies of F.

Proof: Auxiliary lemma that dense graph has many dense subgraphs. Every such subgraph contains a copy of F.