Two-colorability of random non-uniform hypergraphs

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Hypergraphs

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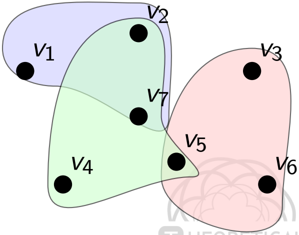
A hypergraph $\mathcal{H} = (V, E)$ is a pair consisting of a set of vertices V and a set of (hyper)edges $E \subseteq \mathcal{P}(V)$.



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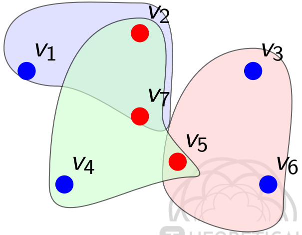
Proper (vertex) 2-coloring

No monochromatic edges.



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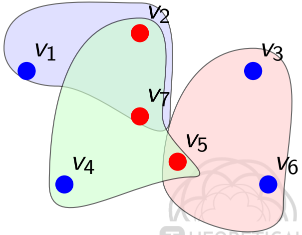
2-coloring is hard

NP-complete for k-graphs, with $k \geq 3$.



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Random k-graphs (generalised Erdős–Rényi model)

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Random non-uniform hypergraphs

For a function $\mathcal{P} : \mathbb{N} \to [0,1]$, let $\widehat{H}(n; \mathcal{P})$ be a distribution of random hypergraphs, where each of $\binom{n}{k}$ possible edges is included independently with probability $\mathcal{P}(k)$.

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- What is the maximum expected number of edges s.t. a random k-graph is 2-colorable w.h.p?

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A random k-graph on n vertices with at least $\frac{\ln 2}{2} \cdot 2^k \cdot n$ expected edges is a.a.s. not-2-colorable.

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With k large enough, a random k-graph with less than $\frac{\ln 2}{2} \cdot \left(2^k - 1 - \frac{1}{\ln 2}\right) \cdot n$ expected hyperedges is a.a.s. 2-colorable.

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The lower-bound was later improved by Coja-Oghlan and Zdeborova.

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Consider a distribution of random non-uniform hypergraphs $\widehat{H}(n;\mathcal{P})$ s.t. the expected number of k-edges is $\lambda_k \cdot 2^k \cdot n$.

Suppose that $\sum_k \lambda_k \geq \frac{\ln 2}{2}$, then $\mathcal{H} \in \widehat{H}(n;\mathcal{P})$ is a.a.s. not-2-colorable.

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Morover, when $\sum_k \frac{\lambda_k}{1-2^{1-k}} < \frac{\ln 2}{2}(1-\varepsilon)$ a random hypergraph is 2-colorable with **positive** probability.

Algorithmic 2-coloring

Algorithm introduced by Achlioptas, Krivelevich, Kim and Tetali (2002).

AKKT algorithm

do $n/2$ times:
    if there exists a 3-tail or 2-tail e:
        pick any two uncolored vertices $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{e}$ and color x *RED*, y *BLUE*
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AKKT succeeds w.h.p on random k-graphs with no more than $c \cdot \frac{2^k}{k} \cdot n$ edges on average, with $c \leq \frac{1}{10}$ for $k \geq 40$.

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AKKT's sufficient condition

If in every phase of the AKKT algorithm the expected number of new 3-tails is bounded by $1 - \varepsilon$ for $\varepsilon > 0$, then a.a.s. the algorithm succeeds in finding a proper 2-coloring.

Distribution of new 3-tails with a single edge size k

- Number of 3-tails at any phase is a Poisson variable.
- At any phase, with the expected number of k-edges equal $\lambda_k \cdot \frac{2^k}{k} \cdot n$, the Poisson's variable parameter is a linear function of λ_k.

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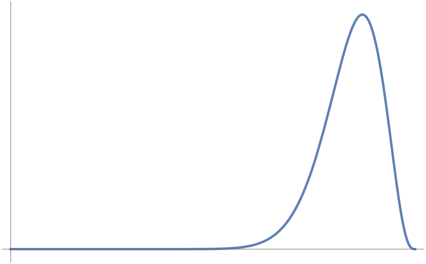

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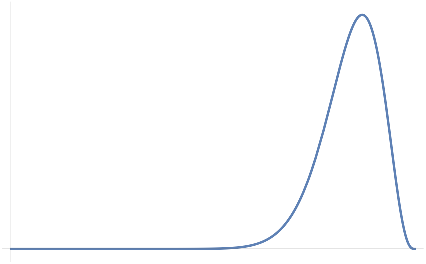

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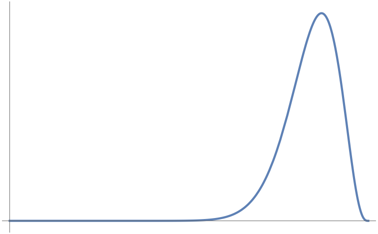
Distribution of new 3-tails - we can do better



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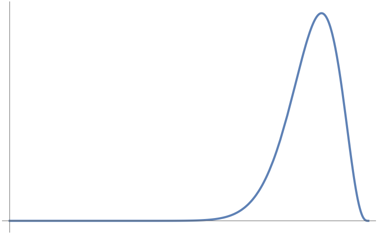

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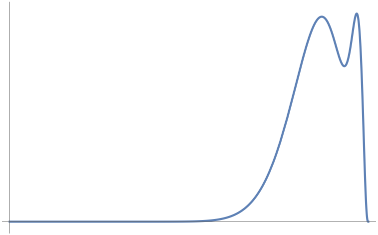

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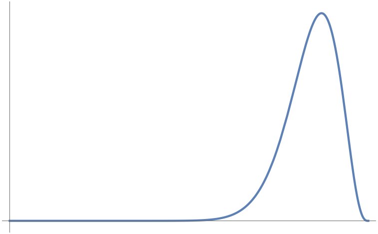

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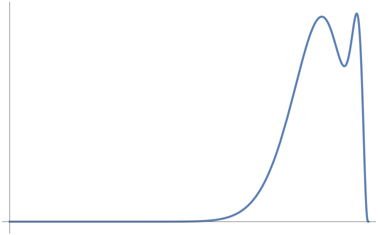

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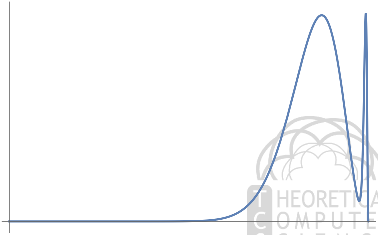

k = 20 and k = 400

Packing more edges

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Unbounded sum of lambdas

For every $c > 0$, there exists a random hypergraph model $\widehat{H}(n; \mathcal{P})$ s.t. the expected number of edges of size k is $\lambda_k \cdot \frac{2^k}{k} \cdot n$, for which

$$\sum_{k \in supp(\mathcal{M})} \lambda_k \geq c$$

and AKKT procedure succeeds a.a.s.

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AKKT aggregating condition

Consider a distribution of random hypergraphs $\widehat{H}(n;\mathcal{P})$ s.t. the expected number of k-edges is $\lambda_k \cdot \frac{2^k}{k} \cdot n$. Let

$$f(x) := \sum_{k \in supp(\mathcal{M})} \frac{\lambda_k}{k} \cdot x^k$$

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Then, for any $\varepsilon > 0$, the AKKT succeeds, when

$$\max_{x \in [0,1]} \left(\frac{16}{3} (1-x)^3 f^{(4)}(x) \right) < 1 - \varepsilon,$$

where $f^{(4)}$ is the fourth derivative of f.

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- The exponent 3 and fourth derivative come from considering 3-tails.

Impact of non-uniformity

- Non-constructive:
  - k-edge weighs 2^{-k}.
  - $\lambda_k \cdot 2^k \cdot n$ k-edges on average.
- Algorithmic:
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Algorithmic* vs. non-constructive upper-bound

	algorithmic*	**non-constructive**
uniform	bounded	$\frac{\ln 2}{2}$
non-uniform	unbounded	$\frac{\ln 2}{2}$

Thank you!