18.440 Practice Final Exam: 100 points
Carefully and clearly show your work on each problem (without
writing anything that is technically not true) and put a box
around each of your final computations. This practice exam
deals only with the portion of the course after the second
midterm, while the actual final exam will cover the entire course.

I. (20 points) Let X be a random variable with finite mean μ and variance
σ^2. The central limit theorem states that if X_i are independent instances
of X, then the quantities

$$T_n := \frac{\sum_{i=1}^n X_i - n\mu}{\sigma\sqrt{n}}$$

converge in law to a standard normal random variable. That is

$$\lim_{n\to\infty} P\{T_n < a\} = \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

Prove, using the following steps, that the moment generating functions of
the T_n (assuming they exist and are well defined) converge to the moment
generating function of a standard normal random variable.

1. Explain why it suffices to consider mean zero, variance one random
   variables X. You may then assume below that $\mu = 0$ and $\sigma^2 = 1$.

2. Let $M_X(t) = \mathbb{E}e^{tX}$. Show that $M_{T_n}(t) = M_X(t/\sqrt{n})^n$. Show also
   that $M_X''(0) = 1$ and $M_X'(0) = 0$.

3. Let $L_X(t) = \log M_X(t)$ and show that $L_{T_n}(t) = nL_X(t/\sqrt{n})$.

4. Show that $L_X'(0) = 0$ and $L_X''(0) = \sigma^2 = 1$.

5. Show that if N is a standard normal random variable then
   $L_N(t) = t^2/2$.

6. Use Taylor approximation to show that, for each fixed t,
   $\lim_{n\to\infty} L_{T_n}(t) = L_N(t)$. [Recall that Taylor approximation states
   that if R is twice differentiable at zero then
   $R(t) = R(0) + R'(0)t + R''(0)t^2/2 + o(t^2)$, for small t.]

Solution: Check the central limit theorem derivation in the textbook and
slides.

II. (15 points) State the strong and weak laws of large numbers and explain why the strong law of large numbers implies the weak law of large numbers.

Solution: Check the law of large number explanations in the textbook and slides.

III. (10 points) Let X and Y be the outcomes of independent die rolls, and let $Z = X + Y$. (Assume that these are 3-sided dice, taking the values 1, 2, and 3 with equal probability.) Compute the following:

1. The entropies $H(X)$, $H(Y)$, $H(Z)$, and $H(X, Y)$.

2. Show that $H(X, Y) = H(X, Z)$.

3. Verify by explicitly calculating both sides that
   $H(X, Z) = H(Z) + H_Z(X)$.

Solution:

1. $H(X) = H(Y) = -\log \frac{1}{3} = \log 3$ and
   $H(X, Y) = H(X) + H(Y) = 2 \log 3$.

2. Like the pair (X, Y), the pair (X, Z) takes 9 values, all with equal probability. So $H(X, Z) = -\log \frac{1}{9} = 2 \log 3$.

3. The variable Z takes 5 values: $2, 3, 4, 5$ and 6 with probabilities $\frac{1}{9}$, $\frac{2}{9}$, $\frac{3}{9}$, $\frac{2}{9}$, and $\frac{1}{9}$. Now,

$$
\begin{aligned}
H_Z(X) &= \sum_{j=2}^{6} P\{Z = j\} H_{Z=j}(X) \\
&= \frac{1}{9} \log 1 + \frac{2}{9} \log 2 + \frac{3}{9} \log 3 + \frac{2}{9} \log 2 + \frac{1}{9} \log 1 \\
&= \frac{4}{9} \log 2 + \frac{1}{3} \log 3.
\end{aligned}
$$

And

$$H(Z) = \frac{1}{9}(-\log\frac{1}{9}) + \frac{2}{9}(-\log\frac{2}{9}) + \frac{3}{9}(-\log\frac{3}{9}) + \frac{2}{9}(-\log\frac{2}{9}) + \frac{1}{9}(-\log\frac{1}{9})$$
$$= \frac{4}{9}\log 3 + \frac{4}{9}(2\log 3 - \log 2) + \frac{1}{3}\log 3$$
$$= \frac{5}{3}\log 3 - \frac{4}{9}\log 2.$$

So indeed $H(Z) + H_Z(X) = 2\log 3 = H(X, Z)$.

IV. (10 points) Elaine's trusty old car has three states: broken (in Elaine's possession), working (in Elaine's possession), and in the shop. Denote these states B, W, and S.

1. Each morning the car starts out B, it has a .5 chance of staying B and a .5 chance of switching to S by the next morning.

2. Each morning the car starts out S, it has a .75 chance of staying S and a .25 chance of switching to W by the next morning.

3. Each morning the car starts out W, it has a .75 chance of staying W, and a .25 chance of switching to B by the next morning.

Over the long term, on what percentage of days does the car start out in state W?

Solution: Ordering the states B, W, S, we may write the Markov chain matrix as

$$M = \begin{pmatrix} .5 & 0 & .5 \\ .25 & .75 & 0 \\ 0 & .25 & .75 \end{pmatrix}.$$

We find the stationarity probability vector $\pi = (\pi_B, \pi_W, \pi_S) = (.2, .4, .4)$ by solving $\pi M = \pi$ (with components of π summing to 1). So $\pi_W = .4$.

V. (20 points)

1. Let $X_1, X_2, \ldots$ be independent random variables with expectation one. In which of the cases below is the sequence Y_i necessarily a martingale?

   (a) $Y_n = X_n - 1$ **NO**

   (b) $Y_n = X_n^2 - 1$ **NO**

   (c) $Y_n = 7$ **YES**

   (d) $Y_n = n - \sum_{i=1}^n X_i$ **YES**

   (e) $Y_n = n - \sum_{i=1}^n i^2 X_i$ **NO**

   (f) $Y_n = \prod_{i=1}^n X_i$ **YES**

   (g) $Y_n = \sum_{i=1}^n X_n^2 - 2n$ **NO**

   (h) $Y_n = \prod_{i=1}^n X_i^2$ **NO**

2. Let Y_n be a martingale. Which of the following is necessarily a stopping time for Y_n?

   (a) The smallest n for which $Y_n > 17$. **YES (assuming it's almost surely finite)**

   (b) The largest n for which $Y_n = 17$. **NO**

   (c) The smallest value n for which $Y_n = Y_{n+1}$. **NO**

3. Give an example of a martingale M_n and a stopping time T such that with probability one, $M_0 = 1$ but $\mathbb{E}M_T = 0$. Explain why your example does not violate the optional stopping theorem.
   SOLUTION: Write $Y_n = 1 + \sum_{i=1}^n X_i$ where each X_i is independently -1 with probability .5 and 1 with probability .5 and let T be the smallest n for which that $Y_n = 0$. This does not violate the optional stopping theorem because there is no bound on how large Y_n can become before time T.

VI. (15 points) This problem asks you to complete a few calculations that arise in the derivation of the Black-Scholes formula. Let X be a mean zero normal random variable with variance σ^2. Compute the following (you may use the function $\phi(a) := \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answers):

1. $\mathbb{E}e^X$.

2. $\mathbb{E}e^X 1_{X>A}$ for a fixed constant A.

3. $\mathbb{E}1_{X>A}$.

4. $\mathbb{E}G(e^X)$ where $G(a) := \begin{cases} 0 & a < K \\ a - K & a \geq K \end{cases}$.

5. How do the answers above change if X is a normal random variable with mean $-\sigma^2/2$ and variance σ^2?

6. BONUS: Can you explain how the answers above imply the Black-Scholes formula for the price of a European call option on a stock whose risk neutral probability density is log-normal?

Solution: Check the homework solutions on the Black Scholes derivation (as well as the slides on Black Scholes and the review problem in the last lecture).

VII. (10 points) Assume zero dividends/interest and that the strike price for a European call option on a stock at a fixed maturity date T and strike price K is given by $C(K)$. Suppose that $C(K) = e^{-K}$ for all $K \geq 0$, and answer the following:

1. What must the present value of the stock be?

2. What is the risk neutral probability that the stock price will lie in the interval $[5, 10]$ at maturity?

3. What is the present value of a contract that pays X^2 at maturity if the stock price at maturity is X?

Solution: Let X be the value of the stock at time T. Let f_X be the risk neutral probability density function and F_X the corresponding cumulative distribution function. Derive (or recall from lecture) the important facts

1. $C'(x) = F_X(x) - 1$

2. $C''(x) = f_X(x)$.

Thus,

1. $F_X(x) = 1 - e^{-x}$ on $[0, \infty)$

2. $f_X(x) = e^{-x}$ on $[0, \infty)$

Assuming no arbitrage, the present value of the stock is $E[X] = 1$, the risk neutral probability it will belong to $[5, 10]$ is $F_X(10) - F_X(5) = e^{-5} - e^{-10}$ and the present value of the contract paying X^2 at maturity is the expectation $E[X^2] = 2$. We remark that the form of f_X we find here (the exponential density function) is not typical for the risk neutral probability of an ordinary stock (more common is for f_X to be approximately log-normal, as in Black-Scholes theory, but with fatter tails).

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