

**18.440 Midterm 1, Spring 2011: 50 minutes, 100 points.**  
**SOLUTIONS**

1. (20 points) Consider an infinite sequence of independent tosses of a coin that comes up heads with probability  $p$ .

- (a) Let  $X$  be such that the first heads appears on the  $X$ th toss. In other words,  $X$  is the number of tosses required to obtain a heads.

Compute (in terms of  $p$ ) the expectation  $E[X]$ . **ANSWER:**  
**geometric random variable with parameter  $p$  has expectation  $1/p$ .**

- (b) Compute (in terms of  $p$ ) the probability that exactly 5 of the first 10 tosses are heads. **ANSWER: binomial probability**  $\binom{10}{5}p^5(1-p)^5$

- (c) Compute (in terms of  $p$ ) the probability that the 5th head appears on the 10th toss. **ANSWER: negative binomial. Need 4 heads in first 9 tosses, 10th toss heads. Probability**  $\binom{9}{4}p^4(1-p)^5p$ .

2. (20 points) Jill sends her resume to 1000 companies she finds on monster.com. Each company responds with probability  $3/1000$  (independently of what all the other companies do). Let  $R$  be the number of companies that respond.

- (a) Compute  $E[R]$ . **ANSWER: binomial random variable with  $n = 1000$  and  $p = 3/1000$ .  $E[R] = np = 3$ .**

- (b) Compute  $\text{Var}[R]$ . **ANSWER: binomial random variable with  $n = 1000$  and  $p = 3/1000$ .  $\text{Var}[R] = np(1-p) = 3(1 - 3/1000)$ .**

- (c) Use a Poisson random variable approximation to estimate the probability  $P\{R = 3\}$ . **ANSWER:  $R$  is approximately Poisson with  $\lambda = 3$ . So  $P\{R = 3\} \approx e^{-\lambda}\lambda^k/k! = e^{-3}3^3/3! = 9e^{-3}/2$ .**

3. (10 points) How many four-tuples  $(x_1, x_2, x_3, x_4)$  of *non-negative* integers satisfy  $x_1 + x_2 + x_3 + x_4 = 10$ ? **ANSWER: represent partition with stars and bars  $**|**||***|*$ . Have  $\binom{13}{3}$  ways to do this.**

4. (10 points) Suppose you buy a lottery ticket that gives you a one in a million chance to win a million dollars. Let  $X$  be the amount you win. Compute the following:

- (a)  $E[X]$ . **ANSWER:  $\frac{1}{10^6}10^6 = 1$ .**

- (b)  $\text{Var}[X]$ . **ANSWER:  $E[X^2] - E[X]^2 = \frac{1}{10^6}(10^6)^2 - 1^2 = 10^6 - 1$ .**

5. (20 points) Suppose that  $X$  is continuous random variable with probability density function  $f_X(x) = \begin{cases} 2x & x \in [0, 1] \\ 0 & x \notin [0, 1] \end{cases}$ . Compute the following:

- (a) The expectation  $E[X]$ . **ANSWER:**

$$\int_{-\infty}^{\infty} f_X(x)xdx = \int_0^1 f_X(x)xdx = \int_0^1 2x^2dx = \frac{2}{3}x^3|_0^1 = 2/3.$$

- (b) The variance  $\text{Var}[X]$ . **ANSWER:**

$$E[X^2] = \int_{-\infty}^{\infty} f_X(x)x^2dx = \int_0^1 f_X(x)x^2dx = \int_0^1 2x^3dx = \frac{2}{4}x^4|_0^1 = 1/2.$$

**So variance is**  $1/2 - (2/3)^2 = 1/2 - 4/9 = 1/18$ .

- (c) The cumulative distribution function  $F_X$ . **ANSWER:**

$$F_X(a) = \int_{-\infty}^a f_X(x)dx = \begin{cases} 0 & a < 0 \\ a^2 & a \in [0, 1] \\ 1 & a > 1 \end{cases}.$$

6. (20 points) A standard deck of 52 cards contains 4 aces. Suppose we choose a random ordering (all  $52!$  permutations being equally likely). Compute the following:

- (a) The probability that *all* of the top 4 cards in the deck are aces.

**ANSWER:** 4! ways to order aces, 48! ways to order remainder. Probability  $4!48!/52!$

- (b) The probability that *none* of the top 4 cards in the deck is an ace. **ANSWER:** choose cards one at a time starting at the top and multiply number of available choices at each stage to get total number. Probability is  $48 \cdot 47 \cdot 46 \cdot 45 \cdot 48!/52!$ .

- (c) The *expected* number of aces among the top 4 cards in the deck.

(There is a simple form for the solution.) **ANSWER:** have probability  $4/52 = 1/13$  that top card is an ace. Similarly, probability  $1/13$  that  $j$ th card is an ace for each  $j \in \{1, 2, 3, 4\}$ . Additivity of expectation gives answer:  $4/13$ .

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18.600 Probability and Random Variables

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