Spring 2017 18.600 Final Exam Solutions

1. (10 points) Let X be an exponential random variable with parameter $\lambda = 1$.

   (a) Compute $E[3X^{13}]$. **ANSWER:** Recall that $\int_0^\infty e^{-x} x^n = n!$ (which can be taken as one of the definitions of $n!$). This implies that $E[X^n] = n!$ and hence $E[3X^{13}] = 3 \cdot 13!$

   (b) Compute the conditional probability $P[X > 10 | X > 5]$ **ANSWER:** An exponential random variable of rate λ is greater than a with probability $e^{-\lambda a}$. Thus

   $$P[X > 5, X > 10]/P[X > 5] = P[X > 10]/P[X > 5] = e^{-10}/e^{-5} = e^{-5}.$$

   Alternatively, this follows from "memoryless" property of exponentials. (If X is the time a bus arrives, then *given* no bus for first five units, conditional probability of five more units without bus is same as original probability of five units without bus.)

   (c) Let $Y = X^2$. Compute the cumulative distribution function F_Y. **ANSWER:**

   $$F_Y(a) = P(Y \le a) = P(X^2 \le a) = P(X \le \sqrt{a}) = F_X(\sqrt{a}) = 1 - e^{-\sqrt{a}}.$$

2. (10 points) Sally is a pleasant texting companion. Each of her texts consists of a string of five emoji, each chosen independently from the same probability distribution. Precisely, if $X = (X_1, X_2, X_3, X_4, X_5)$ represents one of her texts, then each X_i is independently equal to

   Person Shugging with probability 1/4,
   Face with Tears of Joy with probability 1/4,
   Face Blowing a Kiss with probability 1/8,
   Face with Rolling Eyes with probability 1/8,
   Love Heart with probability 1/16,
   Thinking Face with probability 1/16,
   See-No-Evil Monkey with probability 1/16,
   Hula-hooping Statue of Liberty with probability 1/16.

   (a) Compute the entropy of one of Sally's emoji. That is, compute $H(X_1)$. **ANSWER:** $\frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \frac{1}{8} \cdot 3 + \frac{1}{16} \cdot 4 + \frac{1}{16} \cdot 4 + \frac{1}{16} \cdot 4 + \frac{1}{16} \cdot 4 = 11/4$.

   (b) Compute the entropy of an entire text, i.e., compute $H(X) = H(X_1, X_2, X_3, X_4, X_5)$. **ANSWER:** Since the X_i are i.i.d we have $H(X) = 5H(X_1) = 55/4$.

   (c) Suppose you try to figure out the value of X_1 by asking a series of yes or no questions to someone who knows the value. Assume you use the strategy that minimizes the expected number of questions you need. How many questions do you expect to ask? **ANSWER:** We know that $H(X) = 11/4$ is the minimum number possible. Since we can divide the space exactly in half (in terms of probability) with each question, this minimum is achievable, and the answer is 11/4.

3. Let X_1, X_2, X_3 be i.i.d. random variables, each with probability density function $\frac{1}{\pi(1+x^2)}$.

(a) Assume that a and b are fixed positive constants and write $Y = aX_1 + b$. Compute the probability density function f_Y. **ANSWER:** Recall that in general $f_{aZ}(x) = a^{-1}f_Z(x/a)$ and $f_{Z+b}(x) = f_Z(x - b)$. Applying that here we have

$$f_Y(x) = a^{-1}f_{X_1}((x - b)/a) = a^{-1}\frac{1}{\pi(1 + (x - b)^2/a^2)}.$$

(b) Compute the probability that $X_1 \in [-1, 1]$. Give an explicit number. (Recall spinning flashlight story.) **ANSWER:** The flashlight at $(0, 1)$ goes over a $\pi/2$ range of angles as it shifts from pointing to $(0, 1)$ to $(0, -1)$, as compared to the total π range of angles. So the answer is $(\pi/2)/\pi = 1/2$.

(c) Compute the probability density function for $Z = (X_1 + X_2 + X_3)/3$. **ANSWER:** The average of Cauchy random variables is again Cauchy so $f_Z(x) = \frac{1}{\pi(1+x^2)}$.

(d) Compute the probability density function for $W = (X_1 + 2X_2)/3$. [Hint: use (c).] **ANSWER:** The average of X_2 and X_3 is Cauchy, so $X_2 + X_3$ has the same density function as $2X_2$. Hence the answer is the same as in (c), i.e., $f_W(x) = \frac{1}{\pi(1+x^2)}$

4. Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables each of which is equal to 5 with probability $1/2$ and -5 with probability $1/2$. Write $Y_0 = 85$ and $Y_n = Y_0 + \sum_{i=1}^n X_i$ for $n > 0$.

(a) Find the probability that the sequence $Y_0, Y_1, Y_2, \ldots$ reaches 50 before it reaches 100. **ANSWER:** Write p_{50} for probability of reaching 50 first and p_{100} for probability of hitting 100 first. Then by the optional stopping theorem $50p_{50} + 100p_{100} = 85$ and $p_{50} + p_{100} = 1$. Solving gives $p_{50} = 3/10$.

(b) Let T be the smallest $n \geq 0$ for which Y_n is an integer multiple of 500. Compute $E[Y_T]$. **ANSWER:** Eventually Y_n will reach 0 or 500. By optional stopping theorem $E[Y_T] = Y_0 = 85$.

(c) Compute $E[3Y_{10} - 4Y_8|Y_5]$ in terms of Y_5. In other words, recall that the conditional expectation $E[3Y_{10} - 4Y_8|Y_5]$ can be understood as a random variable and express this random variable as a simple function of Y_5. **ANSWER:** $E[Y_{10}|Y_5 = Y_5]$ and $E[Y_8|Y_5] = Y_5$. By additivity of conditional expectation $E[3Y_{10} - 4Y_8|Y_5] = -Y_5$.

5. (10 points) Let X be a normal random variable with mean zero and variance one.

(a) Compute the moment generating function for X using the "complete the square" trick. (Show your work.) **ANSWER:**

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{tx} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x^2 - 2tx + t^2)/2 + t^2/2} dx = e^{t^2/2}.$$

(b) Fix positive constants a, b, and c and compute $E[aX^2 + bX + c]$ in terms of a, b, and c.
ANSWER: Additivity of expectation gives

$$E[aX^2 + bX + c] = E[aX^2] + E[bX] + E[c] = a + c.$$

6. Ten people are taking a class together.

(a) Let K be the number of ways to divide the ten people into five groups, two people per group, so that each person in the class has a partner. Find K. **ANSWER:** $10!/2^5$ ways to get ordered set of five partnerships. Dividing by 5! we get $K = 10!/(2^5 5!)$ to get unordered division into five partnerships.. Alternatively, line people up, choose parner for first person (9 choices), partner for next un-partnered person (7 choices) etc. to get $K = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1$.

(b) Assume that one of these K ways is chosen uniformly at random on one day, and another is chosen uniformly at random (independently) on a subsequent day. Let N be the number of people who have the same partner on both days. What is $E[N]$?
ANSWER: Each person has 1/9 chance of getting same partner on second day. Additivity of expectation gives $E[N] = 10/9$.

(c) Compute $P(N = 6)$. You can use K in your answer if you did not compute an explicit value for K. **ANSWER:** Fix partnerships for first day. Then there are K ways to choose partnerships for second day. How many of these ways result in $N = 6$? Well, there are $\binom{5}{3}$ ways to decide which partnerships will stay unchanged on second day. Pick one of remaining four people, and there are two ways to assign that person a partner *different* from the previous day. So the number of ways to get $N = 6$ is $\binom{5}{3} \cdot 2$ and $P(N = 6) = \frac{\binom{5}{3} \cdot 2}{K}$.

7. (10 points) Suppose that the pair (X, Y) is uniformly distributed on the semi-circle $\{(x, y) : x^2 + y^2 \leq 1, x \geq 0\}$.

(a) Compute the joint probability density $f_{X,Y}(x, y)$. **ANSWER:** $f_{X,Y}(x, y) = 2/\pi$ if (x, y) in semi-circle, zero otherwise.

(b) Compute the conditional expectation $E[X|Y]$ as a function of Y. **ANSWER:** Given a value for Y in $[-1, 1]$, the value X is conditionally uniform on $[0, \sqrt{1 - Y^2}]$ so $E[X|Y] = \sqrt{1 - Y^2}/2$ for $Y \in [-1, 1]$.

(c) Express $E[X^3 \cos(Y)]$ as a double integral. You do not have to compute the integral.
ANSWER:
$$\int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \frac{2}{\pi} x^3 \cos(y) dx dy.$$

8. (10 points) Let X, Y, Z be i.i.d. exponential random variables, each with parameter $\lambda = 1$. Compute the probability density functions for the following random variables.

(a) $2X + 1$ **ANSWER:** $f_{2X+1}(x) = \frac{1}{2}e^{-(x-1)/2}$ for $x \in [1, \infty)$.

(b) $X + Y + Z$. **ANSWER:** This is a Γ distribution: $f_{X+Y+Z}(x) = x^2 e^{-x}/2$ on $[0, \infty)$.

(c) $\min\{X, Y, Z\}$. **ANSWER:** Exponential with $\lambda = 3$, so density is $3e^{-3x}$ on $[0, \infty)$.

9. (10 points) A certain athletically challenged monkey has difficulty climbing a ladder. The ladder has seven rungs. At each given second, the monkey can be on any of the seven rungs of the ladder — or on the "zeroth" rung, meaning on the ground). If $0 \le m \le 6$ and the monkey is on the mth rung of the ladder at a given second, then at the next second the monkey will be at rung $m + 1$ with probability $1/2$ and back at rung 0 with probability $1/2$. If the monkey is at level 7, then with probability 1 the monkey will go back to 0 at the next step.

(a) Give the 8 by 8 matrix describing the monkey's transition probabilities. (You don't have to write all 64 entries. Just write the entries that are non-zero.) **ANSWER:**

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) If the monkey is at state zero at a given time, which is the probability that the monkey is on the 7th rung of the ladder seven seconds later. (That is, what is the probability the monkey goes straight from the bottom to the top without falling once?) **ANSWER:** $2^{-7} = 1/128$.

(c) Compute $(\pi_0, \pi_1, \ldots, \pi_7)$ where π_i is the fraction of the time the monkey spends in state i, over the long term. (Hint: first see if you can express each π_j as an integer multiple of π_{j+1} for $j < 7$. Then see if you can express each π_j as a multiple of π_7.) **ANSWER:**

$$\begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 & \pi_6 & \pi_7 \end{pmatrix} \cdot \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 & \pi_6 & \pi_7 \end{pmatrix}.$$

We find that each $\pi_j = 2\pi_{j+1}$ for $j \in \{0, 1, 2, 3, 4, 5, 6\}$. Thus the row vector has the form $\begin{pmatrix} 128\pi_7 & 64\pi_7 & 32\pi_7 & 16\pi_7 & 8\pi_7 & 4\pi_7 & 2\pi_7 & \pi_7 \end{pmatrix}$. and since values add up to 1 we get $\pi_7 = 1/255$ and the answer is

$$\begin{pmatrix} 128/255 & 64/255 & 32/255 & 16/255 & 8/255 & 4/255 & 2/255 & 1/255 \end{pmatrix}.$$

10. (10 points) Let $X_1, X_2, \ldots, X_{300}$ be independent random real numbers, each chosen uniformly on the interval $[0, 100]$. Let $S = \sum_{i=1}^{300} X_i$.

(a) Compute $E[S]$ and $\text{Var}(S)$. **ANSWER:** $E[S] = 100E[X_1] = 15000$. Recall that variance of uniform random variable on $[0, 1]$ is $1/12$, so $\text{Var}[X_1] = 10000/12$ and $\text{Var}[S] = 3000000/12 = 250000$.

(b) Give a Poisson approximation for the probability that $3 \leq X_j < 4$ for exactly 3 of the values $j \in \{1, 2, \ldots, 300\}$. **ANSWER:** The number of values in that interval is roughly Poisson with parameter $\lambda = 3$, so probability we get $k = 3$ is

$$e^{-\lambda}\lambda^k/k! = e^{-3}3^3/3! = \frac{9}{2e^3}.$$

(c) Give an interval $[a, b]$ such that $E[S] = (a+b)/2$ and $P(S \in [a, b]) \approx .95$. (You may use the fact that $\int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \approx .95$.) **ANSWER:** Standard deviation of S is $\sqrt{250000} = 500$, and CLT implies chance of S being within two standard deviations of $E[S]$, i.e., in $[14000, 16000]$ is about .95.

18.600 Probability and Random Variables
Fall 2019