Powers of 2 and k-digits structures

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Wed 01 Jun 2022 in Math, Carolin Zöbelein

In my paper Powers of 2 whose digits are powers of 2 (see also https://research.carolin-zoebelein.de/public.html#bib6), I'm discussing digits of powers of 2, and which conditions are necessary to get for them powers of 2, too.

Given be the set of powers of 2 by $P_y = 2^y$, $y \in \mathbb{N}_0$. It is unknown if, apart from $P_{y=0} = 2^0 = 1$, $P_{y=1} = 2^1 = 2$, $P_{y=2} = 2^2 = 4$, $P_{y=3} = 2^3 = 8$ and $P_{y=7} = 2^7 = 128$, there exist more P_y's whose digits are powers of 2 (A130693 in the On-line Encyclopedia of Integer Sequences (OEIS) http://oeis.org/A130693 [Dre07]) [Wel97], too.

Looking at the set of powers of 2's [Slo], we know that a m-digit power of 2 by P_y, has a periodicity of $\varphi\left(5^k\right) = 4 \cdot 5^{k-1}$ for the last $k \leq m$ digits, starting at 2^k [YY64]. Taking the known periodicity of the last k-digits into account, we want to discuss properties for the last $k' > k$ digits, for fixed last k-digits of P_y.

Notation. If we write 2^y_k, we are talking about the k'th digit (counted from right to left, starting counting by 1) of 2^y, in base 10 representation. For step sizes we write $d^{k+1}_{y,k}$, meaning the step size of the $k+1$-digit, starting by 2^y, with a k-digit periodicity. Furthermore, we will denote the set of all one-digit powers of 2 by $\mathcal{P}_2 := \{1, 2, 4, 8\}$.

For this, at first, we also considered k-digit structures of powers of 2 in generally, and used the following two lemmas as starting point for our proofs in the mentioned paper.

Lemma 2.1 (k-digits structure). Let be $P_y = 2^y$, $y \in \mathbb{N}_0$, and the last $k^\star$-digits periodical with $\varphi\left(5^{k^\star}\right) = 4 \cdot 5^{k^\star-1}$, for all $2^y \geq 2^{k^\star}$, $k^\star \geq 2$. Then for $2^{k+k^\star+\varphi\left(5^k\right)}$, $k \in \left[k^\star, k^\star + \varphi\left(5^{k^\star-1}\right) - 1\right]$, the last k-digits are given by $2_1^{1+k^\star+\varphi\left(5^1\right)} \cdot 2^{k-1}$, with $k - x \approx \left(1 - \log_{10}(2)\right) k - k^\star \log_{10}(2)$ leading zeros for $k \geq 2$, and at least one leading zero for $k \geq 3$.

Proof. We know, that for the last k-digits $2^{k+k^\star+\varphi\left(5^k\right)} \sim 2^{k+k^\star}$, which have $x \approx (k + k^\star) \log_{10}(2)$ digits. Since, we also have the periodicity $\varphi\left(5^k\right)$, we directly get $k - x \approx \left(1 - \log_{10}(2)\right) k - k^\star \log_{10}(2)$ for the number of leading zeros. Looking at $0 \leq k - x$, we receive $k \gtrsim k^\star \frac{\log_{10}(2)}{1-\log_{10}(2)}$, and hence $k \geq 2$ by the constraint $k^\star \geq 2$, and for $1 \geq k - x$, with $k = k^\star$, we recive $k \gtrsim \frac{1}{1-2\log_{10}(2)}$, and hence $k \geq 3$. Finally it is easy to see, that the statement is always satisfied for $k \geq k^\star$, because of $k^\star \gtrsim k^\star \frac{\log_{10}(2)}{1-\log_{10}(2)} \approx 0.4k^\star$ for $k = k^*$.

Lemma 2.2 ($k^\star$-digits fixed structure). Let be $P_y = 2^y$, $y \in \mathbb{N}_0$, and the last $k^\star$-digits periodical with $\varphi\left(5^{k^\star}\right) = 4 \cdot 5^{k^\star-1}$, for all $2^y \geq 2^{k^\star}$, $k^\star \geq 2$. Then for $2^{k+k^\star+\varphi\left(5^k\right)}$, $k \in \left[k^\star, k^\star + \varphi\left(5^{k^\star-1}\right) - 1\right]$, the last $k+1$ to $k+\delta k$-digits are fixed for at least $\delta k = k^\star$ digits.

Proof. Consider $\left(2^{k+k^\star+\varphi\left(5^k\right)} - 2_1^{1+k^\star+\varphi\left(5^1\right)} \cdot 2^{k-1}\right) \cdot 10^{-k} \cdot 2^{\varphi\left(5^{\delta k}\right)} \approx \left(2^{(k+1)+k^\star+\varphi\left(5^{k+1}\right)} - 2_1^{1+k^\star+\varphi\left(5^1\right)} \cdot 2^{(k+1)-1}\right)$
$\cdot 10^{-(k+1)} \left(2^{k+k^\star+\varphi\left(5^k\right)} - 2_1^{1+k^\star+\varphi\left(5^1\right)} \cdot 2^{k-1}\right) \cdot 2^{\varphi\left(5^{\delta k}\right)} \approx \left(2^{k+k^\star+\varphi\left(5^k\right)} \cdot 2^{4\varphi\left(5^k\right)} - 2_1^{1+k^\star+\varphi\left(5^1\right)} \cdot 2^{k-1}\right)$
$\cdot\, 5^{-1}$, for which we can equating the coefficients with approximation. We look at $\varphi\left(5^{\delta k}\right) \approx 4\varphi\left(5^k\right)$, and receive $\delta k \approx \lfloor \log_5\left(4 \cdot 5^k\right) \rfloor \approx \lfloor 1.86k \rfloor \approx k$. Finally, we can conclude $\delta k \gtrsim k^\star$ for $k \in \left[k^\star, k^\star + \varphi\left(5^{k^\star-1}\right) - 1\right]$.

References

[Dre07] Gregory P. Dresden. A130693 - OEIS: Powers of 2 whose digits are powers of 2. http://oeis.org/A130693, 07 2007. (Accessed on 2021/07/18). [Slo] N. J. A. Sloane. Table of n, 2 n for n = 0..1000 - OEIS. http://oeis.org/A000079/b000079.txt. (Accessed on 2021/08/08). [Wel97] David Wells. The Penguin dictionary of curious and interesting numbers. Penguin, 1997. [YY64] AM Yaglom and IM Yaglom. Challenging mathematical problems with elementary solutions. I, 1964.