

Title: Ad hoc method for independent sets Slug: adhocindependentsets Date: 2020-05-20 19:30 Category: Math Tags: graph theory, independent sets, algorithm, complexity Author: Carolin Zöbelein Summary: Today, we want to determine independent sets of an arbitrary undirected graph. Which can be done by a trivial ad hoc method, described as follows...

Today, we want to determine independent sets of an arbitrary undirected graph. Which can be done by a trivial ad hoc method, described as follows.

## Algorithm

We have given an arbitrary undirected graph  $G = (V, E)$  with  $\deg(u_i) \geq 1$  for all  $u_i \in V$ . The set  $V_0 \subseteq V$  of vertices with  $\deg(u_i) = 0$  will not be considered. Since, we are interested in the independent sets of  $G$ , the set  $V_0$  is a trivial case which, and its subsets, can be simply added to any final solution of independent sets. Our following considerations will work for every arbitrary kind of graph, so we don't have to make any other further assumption regarding possible special kinds of graphs respectively sub-graphs, at the moment.

We start our independent set generation with the assumption that all edges  $(u_i, u_j) \in E$  are removed from our graph  $G$ . For this trivial case, we would have the one largest independent set easily given by  $S_0 := \{u_1, u_2, u_3, u_4, \dots, u_{|V|-1}, u_{|V|}\}$ .

Now, we take an arbitrary edge  $e_1 := (u_1, u_2)$  from the set of the belonging edges  $E$  of  $G$ . From the definition of an independent set follows that  $u_1$  and  $u_2$  can not be element of the same independent set of the same time. Since  $u_1 \in S_0 \wedge u_2 \in S_0$  is given, we have to split  $S_0$  into the two new independent sets  $S_0 = \{u_1, u_3, u_4, \dots, u_{|V|-1}, u_{|V|}\}$  and  $S_1 = \{u_2, u_3, u_4, \dots, u_{|V|-1}, u_{|V|}\}$ .

In the next step, we take edge  $e_2$ . Here, we have now to differ between three cases. The two vertices of  $e_2$  both lie completely in  $S_0$  or both lie in  $S_1$  or in  $S_0$  and  $S_1$ , at the same time. Which means our possible outcome for  $e_2 := (u_1, u_3)$  would be  $S_0 \setminus \{u_1\}$ ,  $S_0 \setminus \{u_3\}$  and  $S_1$ , or for  $e_2 := (u_2, u_3)$  we get  $S_0$ ,  $S_1 \setminus \{u_2\}$  and  $S_1 \setminus \{u_3\}$ , or finally for  $e_2 := (u_3, u_4)$  we get  $S_0 \setminus \{u_3\}$ ,  $S_0 \setminus \{u_4\}$ ,  $S_1 \setminus \{u_3\}$  and  $S_1 \setminus \{u_4\}$ .

So, we see that we can generate all of our independent sets by a simple, naive way of taking an edge of  $E$  and looking if this edge is an element of an already generated independent set  $S_i$ . If yes then split this set into two new ones, by copy  $S_i$  to a second set  $S_j$  and removing the first vertex of  $e$  from  $S_i$  and removing the second vertex of  $e$  from  $S_j$ , else nothing has to be done.

By doing this for all edges  $e$  of  $E$ , we finally get all independent sets of  $G$ .

## Complexity

The complexity of this algorithm is easy to determine. Assume we have given the worst case scenario of a complete graph  $G$  (a graph in which every pair of distinct vertices is connected by a unique edge) with  $|V|$  vertices and edges whose number is given by  $|E| = \binom{|V|}{2} = \frac{1}{2}|V|^2 - \frac{1}{2}|V|$ . To determine all independent sets, we have to do our algorithm for  $|E|$  steps, in which we will have to check for this edge in every already existing independent set and finally split it into two. Hence, in the  $|E|$ 'th step, we have to check  $2^{|E|}$  independent sets for the current edge. For the complexity we get with this  $\mathcal{O}\left(2^{|V|^2}\right)$ .

## Outlook

All of this seems not to look very exciting, but we will see that we can derive nice things from it, in future blog posts.