Title: Cyclotomic polynomials for primes Slug: cyclotomic-polynomials-for-primes Date: 2018-07-18 05:33 Category: Math Tags: cyclotomic polynomials, primes, number theory Author: Samdney

Currently, I'm spending my time with cyclotomic polynomials. So, I don't want to miss the possibility also to mention a small note about the connection of this polynomials to the set of prime numbers.

At first, we have a cyclotomic polynomial in the following way

$$\phi_n\left(x\right) = \prod_{\substack{1\leq g\leq n\\ ggT(g,n)=1}} \left(x - e^{\frac{2\pi i g}{n}}\right) \tag{1}$$

with $n, g \in \mathbb{N}$, respectively

$$\begin{aligned} x^n - 1 &= \prod_{1\leq g\leq n} \left(x - e^{\frac{2\pi i g}{n}}\right) \\ &= \prod_{d|n} \prod_{\substack{1\leq g\leq n\\ ggT(g,n)=d}} \left(x - e^{\frac{2\pi i g}{n}}\right) \\ &= \prod_{d|n} \phi_{n/d}\left(x\right) \\ &= \prod_{d|n} \phi_d\left(x\right) \end{aligned} \tag{2}$$

. In the case of pime numbers $p \in \mathbb{P}$, we also have

$$\phi_p\left(x\right) = 1 + x + x^2 + \cdots + x^{p-1} = \sum_{k=0}^{p-1} x^k \tag{3}$$

. From this follows the first rewriting

$$\begin{aligned} \phi_p\left(x\right) &= \sum_{k=0}^{p-1} x^k \\ &= \sum_{k=0}^{p-1} e^{k\ln(x)} \\ &= e^{0\ln(x)} + \sum_{k=1}^{p-1} e^{k\ln(x)} \\ &= 1 + \frac{e^{\ln(x)}\left(e^{(p-1)\ln(x)} - 1\right)}{e^{\ln(x)} - 1} \end{aligned} \tag{4}$$

, since

$$\sum_{k=1}^{n} e^{kA} = \frac{e^{A}\left(e^{An}-1\right)}{e^{A}-1}$$

. Now we will solve this equation for p.

$$\begin{aligned}
\phi_p\left(x\right) &= 1+\frac{e^{\ln(x)}\left(e^{(p-1)\ln(x)}-1\right)}{e^{\ln(x)}-1} \\
\left(\phi_p\left(x\right)-1\right)\left(e^{\ln(x)}-1\right) &= e^{\ln(x)}\left(e^{(p-1)\ln(x)}-1\right) \\
\frac{\left(\phi_p\left(x\right)-1\right)\left(e^{e^{\ln(x)}}-1\right)}{e^{\ln(x)}}+1 &= e^{(p-1)\ln(x)} \\
\left(p-1\right)\ln\left(x\right) &= \ln\left(\frac{\left(\phi_p\left(x\right)-1\right)\left(e^{\ln(x)}-1\right)}{e^{\ln(x)}}+1\right) \\
\left(p-1\right)\ln\left(x\right) &= \ln\left(\frac{\left(\phi_p\left(x\right)-1\right)\left(x-1\right)+x}{x}\right) \\
\ln\left(x\right)\left(\left(p-1\right)+1\right) &= \ln\left(\left(\phi_p\left(x\right)-1\right)\left(x-1\right)+x\right) \\
p &= \ln\left(\frac{\left(\phi_p\left(x\right)-1\right)\left(x-1\right)+x}{x}\right) \qquad (5)
\end{aligned}$$

Additionally, we have our second, very easy to see, connection for prime numbers

$$\begin{aligned}
\phi_p\left(x\right) &= x^{p}-1 \\
&= \sum_{k'=0}^{p} x^{k'} - \sum_{k''=0}^{p-1} x^{k''} - 1 \\
&= \phi_{p+1}\left(x\right)-\phi_p\left(x\right)-1 \qquad (6)
\end{aligned}$$

. I have no idea for which this can be useful, but I think it is nice to know :P.