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Principal ideal domain

In abstract algebra, a **principal ideal domain**, or **PID**, is an integral domain in which every ideal is principal, i.e., can be generated by a single element. More generally, a principal ideal ring is a nonzero commutative ring whose ideals are principal, although some authors (e.g., Bourbaki) refer to PIDs as principal rings. The distinction is that a principal ideal ring may have zero divisors whereas a principal ideal domain cannot.

Principal ideal domains are thus mathematical objects that behave somewhat like the integers, with respect to divisibility: any element of a PID has a unique decomposition into prime elements (so an analogue of the fundamental theorem of arithmetic holds); any two elements of a PID have a greatest common divisor (although it may not be possible to find it using the Euclidean algorithm). If *x* and *y* are elements of a PID without common divisors, then every element of the PID can be written in the form *ax* + *by*.

Principal ideal domains are noetherian, they are integrally closed, they are unique factorization domains and Dedekind domains. All Euclidean domains and all fields are principal ideal domains.

Principal ideal domains appear in the following chain of class inclusions:

> **commutative rings** ⊃ **integral domains** ⊃ **integrally closed domains** ⊃ **GCD domains** ⊃ **unique factorization domains** ⊃ **principal ideal domains** ⊃ **Euclidean domains** ⊃ **fields** ⊃ **finite fields**

Contents

Examples

Examples include:

- *K*: any field,
- **Z**: the ring of integers,[1]
- *K*[*x*]: rings of polynomials in one variable with coeficients in a field. (The converse is also true; that is, if *A*[*x*] is a PID, then *A* is a field.) Furthermore, a ring of formal power series in one variable over a field is a PID since every ideal is of the form (x^k),
- **Z**[*i*]: the ring of Gaussian integers,[2],
- **Z**[ω] (where ω is a primitive cube root of 1): the Eisenstein integers,
- Any discrete valuation ring, for instance the ring of *p*-adic integers $\mathbb{Z}_p$.

Examples of integral domains that are not PIDs:

- **Z**[*x*]: the ring of all polynomials with integer coefficients --- it is not principal because the ideal generated by 2 and *X* is an example of an ideal that cannot be generated by a single polynomial.
- *K*[*x*,*y*]: rings of polynomials in two variables. The ideal (*x*,*y*) is not principal.

Modules

The key result is the structure theorem: If *R* is a principal ideal domain, and *M* is a finitely generated *R*-module, then M is a direct sum of cyclic modules, i.e., modules with one generator. The cyclic modules are isomorphic to R/xR for some $x \in R$[3] (notice that x may be equal to 0, in which case R/xR is R).

If *M* is a free module over a principal ideal domain *R*, then every submodule of *M* is again free. This does not hold for modules over arbitrary rings, as the example $(2, X) \subseteq \mathbb{Z}[X]$ of modules over $\mathbb{Z}[X]$ shows.

Properties

In a principal ideal domain, any two elements *a,b* have a greatest common divisor, which may be obtained as a generator of the ideal *(a,b)*.

All Euclidean domains are principal ideal domains, but the converse is not true. An example of a principal ideal domain that is not a Euclidean domain is the ring $\mathbb{Z}\left[(1+\sqrt{-19})/2\right]$.[4][5] In this domain no q and r exist, with 0≤|r|<4, so that $(1+\sqrt{-19}) = (4)q + r$, despite $1+\sqrt{-19}$ and 4 having a greatest common divisor of 2.

Every principal ideal domain is a unique factorization domain (UFD).[6][7][8][9] The converse does not hold since for any UFD *K*, *K*[*X*,*Y*] (the rings of polynomials in 2 variables) is a UFD but is not a PID. (To prove this look at the ideal generated by $\langle X, Y \rangle$. It is not the whole ring since it contains no polynomials of degree 0, but it cannot be generated by any one single element.)

1. Every principal ideal domain is Noetherian.
2. In all unital rings, maximal ideals are prime. In principal ideal domains a near converse holds: every nonzero prime ideal is maximal.
3. All principal ideal domains are integrally closed.

The previous three statements give the definition of a Dedekind domain, and hence every principal ideal domain is a Dedekind domain.

Let *A* be an integral domain. Then the following are equivalent.

1. *A* is a PID.
2. Every prime ideal of *A* is principal.[10]
3. *A* is a Dedekind domain that is a UFD.
4. Every finitely generated ideal of *A* is principal (i.e., *A* is a Bézout domain) and *A* satisfies the ascending chain condition on principal ideals.
5. *A* admits a Dedekind–Hasse norm.[11]

A field norm is a Dedekind-Hasse norm; thus, (5) shows that a Euclidean domain is a PID. (4) compares to:

- An integral domain is a UFD if and only if it is a GCD domain (i.e., a domain where every two elements have a greatest common divisor) satisfying the ascending chain condition on principal ideals.

An integral domain is a Bézout domain if and only if any two elements in it have a gcd *that is a linear combination of the two.* A Bézout domain is thus a GCD domain, and (4) gives yet another proof that a PID is a UFD.

See also

- Bézout's identity

Notes

1. See Fraleigh & Katz (1967), p. 73, Corollary of Theorem 1.7, and notes at p. 369, after the corollary of Theorem 7.2
2. See Fraleigh & Katz (1967), p. 385, Theorem 7.8 and p. 377, Theorem 7.4.

3. See also Ribenboim (2001), p. 113 (https://books.google.com/books?id=u5443xdaNZcC&pg=PA113), proof of lemma 2.
4. Wilson, Jack C. "A Principal Ring that is Not a Euclidean Ring." Math. Mag **46** (Jan 1973) 34-38 [1] (https://www.jstor.org/stable/2688577)
5. George Bergman, *A principal ideal domain that is not Euclidean - developed as a series of exercises* PostScript file (http://math.berkeley.edu/~gbergman/grad.hndts/nonEucPID.ps)
6. Proof: every prime ideal is generated by one element, which is necessarily prime. Now refer to the fact that an integral domain is a UFD if and only if its prime ideals contain prime elements.
7. Jacobson (2009), p. 148, Theorem 2.23.
8. Fraleigh & Katz (1967), p. 368, Theorem 7.2
9. Hazewinkel, Gubareni & Kirichenko (2004), p.166 (https://books.google.com/books?id=AibpdVNkFDYC&pg=PA166), Theorem 7.2.1.
10. T. Y. Lam and Manuel L. Reyes, A Prime Ideal Principle in Commutative Algebra (http://math.berkeley.edu/~mreyes/oka1.pdf) Archived (https://web.archive.org/web/20100726160025/http://math.berkeley.edu/~mreyes/oka1.pdf) 2010-07-26 at the Wayback Machine.
11. Hazewinkel, Gubareni & Kirichenko (2004), p.170 (https://books.google.com/books?id=AibpdVNkFDYC&pg=PA170), Proposition 7.3.3.

References

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External links

- Principal ring on MathWorld