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Index of a subgroup

In mathematics, specifically group theory, the **index** of a subgroup H in a group G is the "relative size" of H in G: equivalently, the number of "copies" (cosets) of H that fill up G. For example, if H has index 2 in G, then intuitively half of the elements of G lie in H. The index of H in G is usually denoted $|G : H|$ or $[G : H]$ or $(G:H)$.

Formally, the index of H in G is defined as the number of cosets of H in G. (It is always the case that the number of left cosets of H in G is equal to the number of right cosets.) For example, let $\mathbf{Z}$ be the group of integers under addition, and let $2\mathbf{Z}$ be the subgroup of $\mathbf{Z}$ consisting of the even integers. Then $2\mathbf{Z}$ has two cosets in $\mathbf{Z}$ (namely the even integers and the odd integers), so the index of $2\mathbf{Z}$ in $\mathbf{Z}$ is two. To generalize,

$$|\mathbf{Z} : n\mathbf{Z}| = n$$

for any positive integer n.

If N is a normal subgroup of G, then the index of N in G is also equal to the order of the quotient group G / N, since this is defined in terms of a group structure on the set of cosets of N in G.

If G is infinite, the index of a subgroup H will in general be a non-zero cardinal number. It may be finite - that is, a positive integer - as the example above shows.

If G and H are finite groups, then the index of H in G is equal to the quotient of the orders of the two groups:

$$|G : H| = \frac{|G|}{|H|}.$$

This is Lagrange's theorem, and in this case the quotient is necessarily a positive integer.

Contents

Properties

- If H is a subgroup of G and K is a subgroup of H, then

$$|G : K| = |G : H|\,|H : K|.$$

- If H and K are subgroups of G, then

$$|G:H\cap K|\leq|G:H|\,|G:K|,$$

with equality if $HK = G$. (If $|G : H \cap K|$ is finite, then equality holds if and only if $HK = G$.)

- Equivalently, if H and K are subgroups of G, then

$$|H:H\cap K|\leq|G:K|,$$

with equality if $HK = G$. (If $|H : H \cap K|$ is finite, then equality holds if and only if $HK = G$.)

- If G and H are groups and φ: $G \to H$ is a homomorphism, then the index of the kernel of φ in G is equal to the order of the image:

$$|G:\ker\ \varphi|=|\mathrm{im}\ \varphi|.$$

- Let G be a group acting on a set X, and let $x \in X$. Then the cardinality of the orbit of x under G is equal to the index of the stabilizer of x:

$$|Gx|=|G:G_x|.$$

This is known as the orbit-stabilizer theorem.

- As a special case of the orbit-stabilizer theorem, the number of conjugates gxg^{-1} of an element $x \in G$ is equal to the index of the centralizer of x in G.
- Similarly, the number of conjugates gHg^{-1} of a subgroup H in G is equal to the index of the normalizer of H in G.
- If H is a subgroup of G, the index of the normal core of H satisfies the following inequality:

$$|G:\mathrm{Core}(H)|\leq|G:H|!$$

where ! denotes the factorial function; this is discussed further below.

  - As a corollary, if the index of H in G is 2, or for a finite group the lowest prime p that divides the order of G, then H is normal, as the index of its core must also be p, and thus H equals its core, i.e., is normal.
  - Note that a subgroup of lowest prime index may not exist, such as in any simple group of non-prime order, or more generally any perfect group.

Examples

- The alternating group A_n has index 2 in the symmetric group S_n, and thus is normal.
- The special orthogonal group $SO(n)$ has index 2 in the orthogonal group $O(2)$, and thus is normal.
- The free abelian group $\mathbf{Z} \oplus \mathbf{Z}$ has three subgroups of index 2, namely

$$\{(x,y)\mid x\text{ is even}\},\quad\{(x,y)\mid y\text{ is even}\},\quad\text{and}\quad\{(x,y)\mid x+y\text{ is even}\}.$$

- More generally, if p is prime then $\mathbf{Z}^n$ has $(p^n - 1) / (p - 1)$ subgroups of index p, corresponding to the $p^n - 1$ nontrivial homomorphisms $\mathbf{Z}^n \to \mathbf{Z}/p\mathbf{Z}$.
- Similarly, the free group F_n has $p^n - 1$ subgroups of index p.
- The infinite dihedral group has a cyclic subgroup of index 2, which is necessarily normal.

Infinite index

If H has an infinite number of cosets in G, then the index of H in G is said to be infinite. In this case, the index $|G : H|$ is actually a cardinal number. For example, the index of H in G may be countable or uncountable, depending on whether H has a countable number of cosets in G. Note that the index of H is at most the order of G, which is realized for the trivial subgroup, or in fact any subgroup H of infinite cardinality less than that of G.

Finite index

An infinite group G may have subgroups H of finite index (for example, the even integers inside the group of integers). Such a subgroup always contains a normal subgroup N (of G), also of finite index. In fact, if H has index n, then the index of N can be taken as some factor of $n!$; indeed, N can be taken to be the kernel of the natural homomorphism from G to the permutation group of the left (or right) cosets of H.

A special case, $n = 2$, gives the general result that a subgroup of index 2 is a normal subgroup, because the normal group (N above) must have index 2 and therefore be identical to the original subgroup. More generally, a subgroup of index p where p is the smallest prime factor of the order of G (if G is finite) is necessarily normal, as the index of N divides $p!$ and thus must equal p, having no other prime factors.

An alternative proof of the result that subgroup of index lowest prime p is normal, and other properties of subgroups of prime index are given in (Lam 2004).

Examples

The above considerations are true for finite groups as well. For instance, the group **O** of chiral octahedral symmetry has 24 elements. It has a dihedral D_4 subgroup (in fact it has three such) of order 8, and thus of index 3 in **O**, which we shall call H. This dihedral group has a 4-member D_2 subgroup, which we may call A. Multiplying on the right any element of a right coset of H by an element of A gives a member of the same coset of H ($Hca = Hc$). A is normal in **O**. There are six cosets of A, corresponding to the six elements of the symmetric group S_3. All elements from any particular coset of A perform the same permutation of the cosets of H.

On the other hand, the group T_h of pyritohedral symmetry also has 24 members and a subgroup of index 3 (this time it is a D_{2h} prismatic symmetry group, see point groups in three dimensions), but in this case the whole subgroup is a normal subgroup. All members of a particular coset carry out the same permutation of these cosets, but in this case they represent only the 3-element alternating group in the 6-member S_3 symmetric group.

Normal subgroups of prime power index

Normal subgroups of prime power index are kernels of surjective maps to p-groups and have interesting structure, as described at Focal subgroup theorem: Subgroups and elaborated at focal subgroup theorem

There are three important normal subgroups of prime power index, each being the smallest normal subgroup in a certain class:

- $\mathbf{E}^p(G)$ is the intersection of all index p normal subgroups; $G/\mathbf{E}^p(G)$ is an elementary abelian group and is the largest elementary abelian p-group onto which G surjects.
- $\mathbf{A}^p(G)$ is the intersection of all normal subgroups K such that G/K is an abelian p-group (i.e., K is an index p^k normal subgroup that contains the derived group $[G,G]$): $G/\mathbf{A}^p(G)$ is the largest abelian p-group (not necessarily elementary) onto which G surjects.
- $\mathbf{O}^p(G)$ is the intersection of all normal subgroups K of G such that G/K is a (possibly non-abelian) p-group (i.e., K is an index p^k normal subgroup): $G/\mathbf{O}^p(G)$ is the largest p-group (not necessarily abelian) onto which G surjects. $\mathbf{O}^p(G)$ is also known as the ***p*-residual subgroup**.

As these are weaker conditions on the groups K, one obtains the containments

$$\mathbf{E}^p(G) \supseteq \mathbf{A}^p(G) \supseteq \mathbf{O}^p(G).$$

These groups have important connections to the Sylow subgroups and the transfer homomorphism, as discussed there.

Geometric structure

An elementary observation is that one cannot have exactly 2 subgroups of index 2, as the complement of their symmetric difference yields a third. This is a simple corollary of the above discussion (namely the projectivization of the vector space structure of the elementary abelian group

$$G/\mathbf{E}^p(G) \cong (\mathbf{Z}/p)^k,$$

and further, G does not act on this geometry, nor does it reflect any of the non-abelian structure (in both cases because the quotient is abelian).

However, it is an elementary result, which can be seen concretely as follows: the set of normal subgroups of a given index p form a projective space, namely the projective space

$$\mathbf{P}(\operatorname{Hom}(G, \mathbf{Z}/p)).$$

In detail, the space of homomorphisms from G to the (cyclic) group of order p, $\operatorname{Hom}(G, \mathbf{Z}/p)$, is a vector space over the finite field $\mathbf{F}_p = \mathbf{Z}/p$. A non-trivial such map has as kernel a normal subgroup of index p, and multiplying the map by an element of $(\mathbf{Z}/p)^\times$ (a non-zero number mod p) does not change the kernel; thus one obtains a map from

$$\mathbf{P}(\operatorname{Hom}(G, \mathbf{Z}/p)) := (\operatorname{Hom}(G, \mathbf{Z}/p)) \setminus \{0\})/(\mathbf{Z}/p)^\times$$

to normal index p subgroups. Conversely, a normal subgroup of index p determines a non-trivial map to $\mathbf{Z}/p$ up to a choice of "which coset maps to $1 \in \mathbf{Z}/p$, which shows that this map is a bijection.

As a consequence, the number of normal subgroups of index p is

$$(p^{k+1} - 1)/(p - 1) = 1 + p + \cdots + p^k$$

for some k; $k = -1$ corresponds to no normal subgroups of index p. Further, given two distinct normal subgroups of index p, one obtains a projective line consisting of $p + 1$ such subgroups.

For $p = 2$, the symmetric difference of two distinct index 2 subgroups (which are necessarily normal) gives the third point on the projective line containing these subgroups, and a group must contain $0, 1, 3, 7, 15, \ldots$ index 2 subgroups – it cannot contain exactly 2 or 4 index 2 subgroups, for instance.

See also

- Virtually
- Codimension

References

- Lam, T. Y. (March 2004), "On Subgroups of Prime Index", *The American Mathematical Monthly*, **111** (3): 256–258, JSTOR 4145135, alternative download

External links

- Normality of subgroups of prime index at PlanetMath.org.
- "Subgroup of least prime index is normal" at Groupprops, The Group Properties Wiki