Algebraic closure

In mathematics, particularly abstract algebra, an **algebraic closure** of a field *K* is an algebraic extension of *K* that is algebraically closed. It is one of many closures in mathematics.

Using Zorn's lemma[1][2][3] or the weaker ultrafilter lemma,[4][5] it can be shown that every field has an algebraic closure, and that the algebraic closure of a field *K* is unique up to an isomorphism that fixes every member of *K*. Because of this essential uniqueness, we often speak of *the* algebraic closure of *K*, rather than *an* algebraic closure of *K*.

The algebraic closure of a field *K* can be thought of as the largest algebraic extension of *K*. To see this, note that if *L* is any algebraic extension of *K*, then the algebraic closure of *L* is also an algebraic closure of *K*, and so *L* is contained within the algebraic closure of *K*. The algebraic closure of *K* is also the smallest algebraically closed field containing *K*, because if *M* is any algebraically closed field containing *K*, then the elements of *M* that are algebraic over *K* form an algebraic closure of *K*.

The algebraic closure of a field *K* has the same cardinality as *K* if *K* is infinite, and is countably infinite if *K* is finite.[3]

Contents

Examples

- The fundamental theorem of algebra states that the algebraic closure of the field of real numbers is the field of complex numbers.
- The algebraic closure of the field of rational numbers is the field of algebraic numbers.
- There are many countable algebraically closed fields within the complex numbers, and strictly containing the field of algebraic numbers; these are the algebraic closures of transcendental extensions of the rational numbers, e.g. the algebraic closure of **Q**(π).
- For a finite field of prime power order *q*, the algebraic closure is a countably infinite field that contains a copy of the field of order q^n for each positive integer *n* (and is in fact the union of these copies).[6]

Existence of an algebraic closure and splitting fields

Let $S = \{f_\lambda | \lambda \in \Lambda\}$ be the set of all monic irreducible polynomials in *K*[*x*]. For each $f_\lambda \in S$, introduce new variables $u_{\lambda,1}, \ldots, u_{\lambda,d}$ where $d = \operatorname{degree}(f_\lambda)$. Let *R* be the polynomial ring over *K* generated by $u_{\lambda,i}$ for all $\lambda \in \Lambda$ and all $i \leq \operatorname{degree}(f_\lambda)$. Write

$$f_\lambda - \prod_{i=1}^{d}(x - u_{\lambda,i}) = \sum_{j=0}^{d-1} r_{\lambda,j} \cdot x^j \in R[x]$$

with $r_{\lambda,j} \in R$. Let *I* be the ideal in *R* generated by the $r_{\lambda,j}$. Since *I* is strictly smaller than *R*, Zorn's lemma implies that there exists a maximal ideal *M* in *R* that contains *I*. The field K_1=*R*/*M* has the property that every polynomial f_λ with coefficients in *K* splits as the product of $x - (u_{\lambda,i} + M)$, and hence has all roots in K_1. In the same way, an extension K_2 of K_1 can be constructed, etc. The union of all these extensions is the algebraic closure of K, because any polynomial with coefficients in this new field has its coefficients in some K_n with sufficiently large n, and then its roots are in K_{n+1}, and hence in the union itself.

It can be shown along the same lines that for any subset S of $K[x]$, there exists a splitting field of S over K.

Separable closure

An algebraic closure K^{alg} of K contains a unique separable extension K^{sep} of K containing all (algebraic) separable extensions of K within K^{alg}. This subextension is called a **separable closure** of K. Since a separable extension of a separable extension is again separable, there are no finite separable extensions of K^{sep}, of degree > 1. Saying this another way, K is contained in a *separably-closed* algebraic extension field. It is unique (up to isomorphism).[7]

The separable closure is the full algebraic closure if and only if K is a perfect field. For example, if K is a field of characteristic p and if X is transcendental over K, $K(X)(\sqrt[p]{X}) \supset K(X)$ is a non-separable algebraic field extension.

In general, the absolute Galois group of K is the Galois group of K^{sep} over K.[8]

See also

- Algebraically closed field
- Algebraic extension
- Puiseux expansion

References

1. McCarthy (1991) p.21
2. M. F. Atiyah and I. G. Macdonald (1969). *Introduction to commutative algebra*. Addison-Wesley publishing Company. pp. 11–12.
3. Kaplansky (1972) pp.74-76
4. Banaschewski, Bernhard (1992), "Algebraic closure without choice.", *Z. Math. Logik Grundlagen Math.*, **38** (4): 383–385, Zbl 0739.03027 (https://zbmath.org/?format=complete&q=an:0739.03027)
5. Mathoverflow discussion (https://mathoverflow.net/questions/46566/is-the-statement-that-every-field-has-an-algebraic-closure-known-to-be-equivalent)
6. Brawley, Joel V.; Schnibben, George E. (1989), "2.2 The Algebraic Closure of a Finite Field", *Infinite Algebraic Extensions of Finite Fields* (https://books.google.com/books?id=0HNfpAsMXhUC&pg=PA22), Contemporary Mathematics, **95**, American Mathematical Society, pp. 22–23, ISBN 978-0-8218-5428-0, Zbl 0674.12009 (https://zbmath.org/?format=complete&q=an:0674.12009)
7. McCarthy (1991) p.22
8. Fried, Michael D.; Jarden, Moshe (2008). *Field arithmetic*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. **11** (3rd ed.). Springer-Verlag. p. 12. ISBN 978-3-540-77269-9. Zbl 1145.12001 (https://zbmath.org/?format=complete&q=an:1145.12001)

- Kaplansky, Irving (1972). *Fields and rings*. Chicago lectures in mathematics (Second ed.). University of Chicago Press. ISBN 0-226-42451-0. Zbl 1001.16500.
- McCarthy, Paul J. (1991). *Algebraic extensions of fields* (Corrected reprint of the 2nd ed.). New York: Dover Publications. Zbl 0768.12001.