Generalized quantum XOR-gate for quantum teleportation and state purification in arbitrary dimensional Hilbert spaces

Gernot Alber[1], Aldo Delgado[1], Nicolas Gisin[2], Igor Jex[1,3]
[1] *Abteilung für Quantenphysik, Universität Ulm, D–89069 Ulm, Germany*
[2] *Group of Applied Physics, University of Geneva, 1211 Geneva 4, Switzerland*
[3] *Department of Physics, FJFI ČVUT, Břehová 7, 115 19 Praha 1 - Staré Město, Czech Republic*
(June 1, 2018)

A generalization of the quantum XOR-gate is presented which operates in arbitrary dimensional Hilbert spaces. Together with one-particle Fourier transforms this gate is capable of performing a variety of tasks which are important for quantum information processing in arbitrary dimensional Hilbert spaces. Among these tasks are the preparation of Bell states, quantum teleportation and quantum state purification. A physical realization of this generalized XOR-gate is proposed which is based on non-linear optical elements.

In quantum information processing the quantum XOR-gate [1] plays a fundamental role. In this 2-qubit gate, the first qubit controls the target qubit: if the control is in state $|0>$, the target is left unchanged, but if the control qubit is in state $|1>$ the target's basis states are flipped. Together with one-qubit operations it forms a universal set of quantum gates allowing the implementation of arbitrary unitary operations acting on qubits [2]. It has been demonstrated that it can be used for many practical tasks of quantum information processing with qubits, such as quantum state swapping [3], entangling quantum states [4], performing Bell measurements [5], dense coding [6] and teleportation [7]. Furthermore, in combination with selective measurements it can be used for implementing non-linear quantum transformations of quantum states which may be used for optimal state identification and for state purification [8,9].

For many practical tasks of quantum information processing it is desirable to extend the basic notion of such a quantum XOR-operation to higher dimensional Hilbert spaces. Indeed, most of the physical systems that have been proposed to hold qubits, such as multilevel atoms or ions [10] and multipath-interferometers [11], could equally well encode larger alphabeths. However, there is a considerable degree of freedom involved in such a generalization. Not all such generalizations of the basic quantum XOR-gate of qubits acquire a similar fundamental significance in connection with the universality of quantum operations in higher dimensional Hilbert spaces. A proper generalization is thus a useful tool as it allows one to unify various quantum operations which are of current interest for quantum information processing, such as entangling quantum states, performing Bell measurements, teleportation and purifying quantum states.

In this letter a generalized XOR-gate is proposed which acts on two arbitrary dimensional quantum systems and which inherits all the significant properties of the basic XOR-gate for qubits. In particular we demonstrate that this generalized quantum XOR-gate may be used to entangle two quantum systems with one another, to teleport an unknown quantum state, and to implement non-linear quantum transformations for state purification. A possible physical realization of this quantum gate is proposed which is based on non-linear optical elements.

Let us start by summarizing characteristic properties of the XOR-gate as they are known for qubit systems. For qubits the action of the quantum XOR-gate onto a chosen set of basis states $\{|i\rangle\}$ with $i \in \{0,1\}$ of the Hilbert space of each qubit is defined by

$$XOR_{12}\,|i\rangle_1\,|j\rangle_2 = |i\rangle_1\,|i \oplus j\rangle_2 \ . \tag{1}$$

This transformation has the following characteristic properties: (i) it is unitary and thus reversible, (ii) it is hermitian and (iii) $i \oplus j = 0$ if and only if $i = j$. The first (second) index denotes the state of the control (target) qubit and $\oplus$ denotes addition modulo(2).

Let us now consider the problem of generalizing the quantum XOR-gate to higher dimensional Hilbert spaces. The desired generalized quantum XOR-gate (GXOR-gate) should act on two D -dimensional quantum systems. In analogy with qubits we will call these two systems qudits. The basis states $|i\rangle$ of each qudit are labeled by elements in the ring Z_D which we denote by the numbers $i = 0, ..., D-1$ with the usual rules for addition and multiplication *modulo(D)*. In principle, the GXOR-gate could be defined in a straightforward way by using Eq.(1) and by performing $i \oplus j$ *modulo(D)*, i. e.

$$GXOR_{12}\,|i\rangle_1\,|j\rangle_2 = |i\rangle_1\,|i \oplus j\rangle_2 \,. \tag{2}$$

However, with this GXOR-gate one cannot purify quantum states with the help of non-linear quantum transformations as $-i \neq i$ in Z_D for $D > 2$. Moreover, the GXOR-gate defined in (2) is unitary but not hermitian for $D > 2$. Therefore it is no longer its own inverse. Thus, the inverse GXOR-gate has to be obtained from the GXOR-gate of Eq.(2) by iteration, i.e. $GXOR_{12}^{-1} = (GXOR_{12})^{D-1} = GXOR_{12}^{\dagger} \neq GXOR_{12}$. All these inconvenient properties of this preliminary definition (2) can be removed by the alternative definition

$$GXOR_{12}\,|i\rangle_1\,|j\rangle_2 = |i\rangle_1\,|i \ominus j\rangle_2 \,. \tag{3}$$

In Eq.(3) $i \ominus j$ denotes the difference $i - j$ modulo (D). In the special case of qubits the definition of Eq.(3) reduces to Eq.(1) as $i \ominus j \equiv i \oplus j$ *modulo*(2). Furthermore, this definition preserves all the properties of Eq.(1) also for arbitrary values of D, namely it is unitary, hermitian and $i \ominus j = 0 \; modulo(D)$ if and only if $i = j$.

The GXOR-gate of Eq.(3) admits a natural extension to control and target systems with continuous spectra. In this case the basis states $|i\rangle$ are replaced by the basis states $\{|x\rangle\}$ with the continuous variable $x \in \mathbf{R}$. These new basis states are assumed to satisfy the orthogonality condition $\langle x|\, y\rangle = \delta\,(x-y)$. Furthermore, as the dimension D tends to infinity the modulo operation entering Eq.(3) can be omitted. Thus, for continuous variables the action of the GXOR-gate becomes

$$GXOR_{12}\,|x\rangle_1\,|y\rangle_2 = |x\rangle_1\,|x-y\rangle_2\,. \tag{4}$$

Let us note that this definition for the case of continuous variables is different from the generalized XOR-gate proposed in Ref. [12]. This latter gate is not hermitian whereas the GXOR-gate of Eq.(4) is both unitary and hermitian. The GXOR-gate of Eq. (4) can be represented in terms of a translation and a space inversion, namely

$$GXOR_{12}\,|x\rangle_1\,|y\rangle_2 = \hat{\Pi}_2\, e^{(-i\hat{P}_y^{(2)}\hat{x}^{(1)})}\,|x\rangle_1\,|y\rangle_2\,. \tag{5}$$

Thereby $\hat{P}_y^{(2)}$ denotes the canonical momentum operator which is conjugate to the position operator $\hat{y}^{(2)}$ acting on quantum system 2 and $\hat{\Pi}_2$ is the corresponding operator of space inversion.

With the help of the GXOR-gate of Eq.(3) a variety of quantum operations can be implemented which are of central interest for quantum information processing. As a first application let us consider the preparation of a basis of entangled states from separable ones. If $|l\rangle|m\rangle$ with $l, m, = 0, ..., D-1$ denotes an orthonormal basis of separable states an associated basis of entangled two-particle states is given by

$$|\psi_{lm}\rangle = GXOR_{12}[(F|l\rangle)_1|m\rangle_2]. \tag{6}$$

Thereby F denotes the discrete Fourier transformation, i.e. $F|l\rangle = (1/\sqrt{D})\sum_{k=0}^{D-1} exp(i2\pi lk/D)|k\rangle$. For qubits this unitary quantum transformation leads to the well known basis of four Bell states. In the simplest higher dimensional case of $D = 3$, for example, the first few states of this entangled generalized Bell basis are given by

$$\begin{aligned}
|\psi_{00}\rangle &= \frac{1}{\sqrt{3}}[|00\rangle + |11\rangle + |22\rangle],\\
|\psi_{10}\rangle &= \frac{1}{\sqrt{3}}[|00\rangle + e^{i2\pi/3}|11\rangle + e^{-i2\pi/3}|22\rangle],\\
|\psi_{20}\rangle &= \frac{1}{\sqrt{3}}[|00\rangle + e^{-i2\pi/3}|11\rangle + e^{i2\pi/3}|22\rangle],\\
|\psi_{01}\rangle &= \frac{1}{\sqrt{3}}[|02\rangle + |10\rangle + |21\rangle], ...
\end{aligned} \tag{7}$$

As the GXOR-gate is hermitian it can also be used to disentangle this basis of generalized Bell states again by inverting Eq.(6). This basic disentanglement property is of practical significance as it enables one to reduce Bell measurements to measurements of separable states. Examples where these latter types of measurements are of central interest are dense coding [6] and quantum teleportation schemes [7].

The basis of entangled Bell states resulting from Eq.(6) can be used for teleporting an arbitrary D-dimensional quantum state from A (Alice) to B (Bob). For this purpose let us assume that A and B share an entangled pair of particles prepared in state $|\psi_{lm}\rangle$ as defined by Eq.(6). If A wants to teleport an unknown quantum state $|\chi\rangle = \sum_{n=0}^{D-1}\alpha_n|n\rangle$ to B she has to perform a Bell measurement which yields one of the entangled basis states of Eq.(6) as an output state (compare with Fig. (1)). Conditioned on the measurement result of Alice, Bob has to perform an appropriate unitary transformation onto his particle which prepares this latter particle in state $|\chi\rangle$. This arbitrary dimensional teleportation scheme rests on the identity

$$\begin{aligned}
|\chi\rangle|\psi_{jk}\rangle_{23} &= \sum_{l,m=0}^{D-1}|\psi_{lm}\rangle_{12}\frac{e^{-i2\pi jm/D}}{D}U_{lm}|\chi\rangle,\\
U_{lm}|n\rangle &= e^{-i2\pi n(l-j)/D}|n-k-m\rangle.
\end{aligned} \tag{8}$$

This basic relation for teleportation for an arbitrary dimensional state $|\chi\rangle$ can be derived in a straightforward way from Eqs. (3) and (6). The classical communication requires $2\log_2(D)$ bits, which is the minimum necessary in all quantum teleportation schemes.

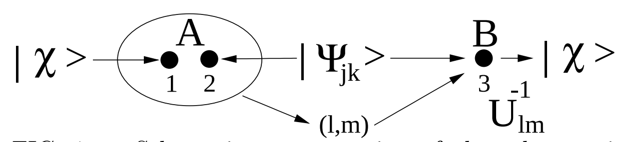

FIG. 1. Schematic representation of the teleportation scheme involving Bell measurements onto the generalized Bell states of Eq.(6).

Together with filtering measurements acting on a target quantum system t the GXOR-gate of Eq. (3) can be used to implement non-linear transformations of quantum states of a control system c. This can be demonstrated most easily by considering the case of two qudits which are prepared in the quantum states σ^t and σ^c initially. Let us perform the quantum operation

$$T(\sigma^c, \sigma^t) = \frac{A\,(\sigma^c \otimes \sigma^t)\,A^\dagger}{Tr[A\,(\sigma^c \otimes \sigma^t)\,A^\dagger]} \tag{9}$$

on these two qudits with

$$A = (\mathbf{1_c} \otimes P_-)\;GXOR_{ct}. \tag{10}$$

Thereby $\mathbf{1_c}$ denotes the identity operator acting in the Hilbert space of the control system and $P_- = |0\rangle_{tt}\langle 0|$ is the projector onto state $|0\rangle_t$ of the target qudit. With the decomposition

$$\sigma^c = \sum_{ij}^{D-1} \sigma^c_{ij} \left|i\right\rangle_{cc} \left\langle j\right| ,$$

$$\sigma^t = \sum_{ij}^{D-1} \sigma^t_{ij} \left|i\right\rangle_{tt} \left\langle j\right| \tag{11}$$

Eqs. (9) and (10) may be rewritten in the form

$$T(\sigma^c,\sigma^t) = \frac{\sum_{ijkl}^{D-1} \sigma^c_{ij}\sigma^t_{kl} \left|i\right\rangle_{cc} \left\langle j\right| \otimes P_- \left|i \ominus k\right\rangle_{tt} \left\langle j \ominus l\right| P_-}{\sum_{ikl}^{D-1} \sigma^c_{ii}\sigma^t_{kl} \left\langle 0|i \ominus k\right\rangle_{tt} \left\langle i \ominus l|0\right\rangle}. \tag{12}$$

Assuming that both control and target qudit are prepared in the same state initially, i.e. $\sigma^c \equiv \sigma^t$, and using the basic property $i \ominus j = 0 \; modulo(D)$ if and only if $i = j$ of the GXOR-gate of Eq.(3) it turns out that Eq.(9) is equivalent to the relations

$$T(\sigma^c, \sigma^t \equiv \sigma^c) = \sigma^c_{output} \otimes P_-,$$

$$\sigma^c_{output} = \frac{\sum_{ij}^{D-1} \left(\sigma^c_{ij}\right)^2 \left|i\right\rangle_{cc} \left\langle j\right|}{\sum_{i}^{D-1} \left(\sigma^c_{ii}\right)^2}. \tag{13}$$

As a result of the quantum operation (9) the combined system formed by the control and the target qudit forms a factorizable state with the target qudit being in state $|0\rangle\langle 0|$. According to Eq.(13) the density matrix elements of σ^c with respect to the computational basis $|i\rangle$ $(i = 0, ..., D-1)$ have been squared. This final state is prepared with probability $p_c = \sum_i^{D-1} (\sigma^c_{ii})^2$. From Eq. (13) it is easy to verify that the quantum operation (9) has the following basic properties: (i) it maps density matrices onto density matrices, (ii) it is not injective and non-linear, (iii) there are states invariant under the transformation, and (iv) it maps pure states onto pure states. It is also possible to extended the quantum operation of Eq. (9) to cases in which there is more than one control system and in which both the control and the target systems are composite quantum systems each of which consists of M qudits. In this case σ^c describes a general M-qudit state of the form

$$\sigma^c = \sum_{\mathbf{ij}} \sigma^c_{\mathbf{ij}} \left|\mathbf{i}\right\rangle_{cc} \left\langle \mathbf{j}\right| , \tag{14}$$

with $\mathbf{i} = (i_1, ..., i_M)$ and $\mathbf{j} = (j_1, ..., j_M)$. In Eq.(9) the operator A has to be replaced by

$$A = (\mathbf{1_c} \otimes P_-)\Pi_{j=1}^M \Pi_{i=1}^N GXOR^{(j)}_{ct_i} \tag{15}$$

with the projection operators $P_- = \Pi_{i=1}^N \otimes P_{t_i}$ and $P_{t_i} = \left|\mathbf{0}\right\rangle_{t_i t_i} \left\langle \mathbf{0}\right|$ onto state $\left|\mathbf{0}\right\rangle_{t_i}$ of the M-qudit target system t_i. Thereby the GXOR-gate $GXOR^{(j)}_{ct_i}$ operates on the j-th qudit of the control and of the i-th target system. The resulting final state of the control system is given by

$$\sigma^c_{output} = \frac{\sum_{\mathbf{ij}} (\sigma^c_{\mathbf{ij}})^{1+N} \left|\mathbf{i}\right\rangle_{cc} \left\langle \mathbf{j}\right|}{\sum_{\mathbf{i}} (\sigma^c_{\mathbf{ii}})^{1+N}}. \tag{16}$$

and is prepared with probability $p_c = \sum_{\mathbf{i}} (\sigma^c_{\mathbf{ii}})^{1+N}$.

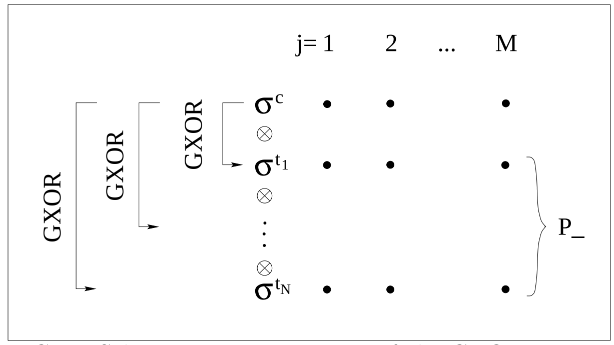

FIG. 2. Schematic representation of the GXOR-gates and projections involved in the non-linear quantum transformation of Eq. (16). The qudits are represented by dots. The dots of the first line represent the M qudits of the control system. The dots of the following lines represent the $M \times N$ qudits of the N target systems $t_1, t_2, ..., t_N$. The GXOR-gate $GXOR^{(j)}_{ct_i}$ acts on the $j-th$ qudit of the control and target system t_i with $j \in \{1, 2, ..., M\}$ and $i \in \{1, 2, ..., N\}$. The operator P_- projects the state of the whole systems onto state $\left|\mathbf{0}\right\rangle \left\langle \mathbf{0}\right|$ with $\left|\mathbf{0}\right\rangle = \left|0\right\rangle_1 \left|0\right\rangle_2 ... \left|0\right\rangle_{MN}$.

In general, also the non-linear quantum transformation of Eq. (16) has invariant states. This suggests to use this non-linear quantum transformation for the purification of quantum states of a two-qudit system. For the special case of a control system consisting of two-qubits such a purification scheme has already been proposed previously [9]. In order to discuss an analogous purification scheme in arbitrary dimensional Hilbert spaces we start from the observation that for $M = 2$ the entangled basis state $|\psi_{00}\rangle$ of Eq.(6) is a fixed point of the non-linear two-particle quantum map of Eq.(16). Thus this map may be used to purify quantum states towards the entangled state $|\psi_{00}\rangle$. In order to exemplify the convergence properties of this purification process let us assume that initially we start from a Werner state of the form

$$\sigma^c = \lambda|\psi_{00}\rangle\langle\psi_{00}| + (1-\lambda)\mathbf{1}/D^2. \tag{17}$$

This state may result from a physical situation where two spatially separated parties, say A(lice) and B(ob), want to share the entangled basis state $|\psi_{00}\rangle$ but with a probability of $(1-\lambda)$ the transmission of this entangled pair leads to unwanted noise represented by the chaotic state $\mathbf{1}/D^2$. This quantum state σ^c is non-separable if and only if $\lambda > \lambda_D = (1+D)^{-1}$ [13] so that a purification scheme based on Eq.(16) can succeed only for these values of λ. In order to maximize the range of convergence of a purification scheme based on Eq.(16) let us introduce an additional unitary twirling transformation [14] of the form $U_A \otimes U_B^*$ which is performed by parties A and B locally after each iteration of the non-linear quantum map (16). Thus, at each step of the purification process the mapping

$$\sigma^c \to U_A \otimes U_B^* \frac{\sum_{\mathbf{ij}} (\sigma^c_{\mathbf{ij}})^2 \, |\mathbf{i}\rangle_{cc} \langle \mathbf{j}|}{\sum_{\mathbf{i}} (\sigma^c_{\mathbf{ii}})^2} U_A^\dagger \otimes U_B^{*\dagger} \tag{18}$$

is performed. Thereby the local unitary transformation redistributes all states. The only state which is left invariant by this redistribution procedure is the entangled state $|\psi_{00}\rangle$. In principle, the local unitary transformation U can be chosen arbitrarily. However, numerical simulations indicate that the region of convergence of the purification process can be improved considerably by choosing two different types of local unitary twirling transformations which are used alternatively. As an example, let us choose for U alternatively a discrete Fourier transform involving all D states $|0\rangle, ...|D-1\rangle$ and a discrete Fourier transform involving the $D-1$ states $|0\rangle, ...|D-2\rangle$ only. Numerical simulations which we have performed for dimensions $2 \leq D \leq 20$ demonstrate clearly that the purification procedure involving these two local unitary transformations is capable of purifying all non-separable Werner states of the form of Eq.(17). Thus, it is expected that this maximal range of convergence of this purification scheme also applies to all higher dimensional Hilbert spaces.

Let us finally discuss a possible physical realization of the GXOR-gate defined by Eq.(3) which is based on non-linear optical elements. For this purpose we assume that the two quantum systems which are going to be entangled are two modes of the radiation field. The basis states $|i\rangle_1$ $(i = 0, ..., D-1)$ of the first quantum system are formed by n-photon states of mode one with $0 \leq n \leq D-1$. The basis states of the second quantum system $|k\rangle_2$ $(k = 0, ..., D-1)$ are formed by Fourier transformed n-photon states of this latter mode, i.e. $|k\rangle_2 = 1/\sqrt{D} \sum_{n=0}^{D-1} \exp(i2\pi kn/D)|n\rangle_2$. Let us further assume that the dynamics of these two modes of the electromagnetic field are governed by the Kerr-effect [15]. Thus, in the interaction picture their Hamiltonian is given by $H = \hbar\chi a_1^\dagger a_1 a_2^\dagger a_2$ with the creation and annihilation operators $a_{1,2}^\dagger$ and $a_{1,2}$ of modes 1 and 2, respectively. For the sake of simplicity the nonlinear susceptibility χ is assumed to be real-valued and positive. Preparing intitially both quantum systems in state $|i\rangle_1|k\rangle_2$ after an interaction time of magnitude $t = 2\pi/(D\chi)$ this two-mode system ends up in state $|\psi\rangle_{12} = |i\rangle_1|k-i\rangle_2$. Applying to this latter state a time reversal transformation which may be implemented by the process of phase conjugation [15] we finally arrive at the desired state $|i\rangle_1|i-k\rangle_2$. Thus this combination of a Kerr-interaction with a time reversal transformation is capable of realizing the GXOR-gate of Eq. (3).

In summary, a generalized quantum XOR-gate has been proposed which acts on two quantum systems in arbitrary dimensional Hilbert spaces. This quantum gate is unitary and hermitian and preserves characteristic properties of the basic quantum XOR-gate acting on qubits. It has been demonstrated that together with one-particle Fourier-transformations this GXOR-gate is capable of performing various important elementary tasks of quantum information processing in arbitrary dimensional Hilbert spaces, such as the preparation of entangled basis states (the so-called Bell states), quantum teleportation and quantum state purification. Physically this proposed quantum gate can be implemented optically, for example, with the help of the Kerr-effect and with the help of phase conjugation. These applications demonstrate the usefulness of the presented GXOR-gate as a basic and unifying concept for several problems of quantum information processing in arbitrary dimensional Hilbert spaces.

This work is supported by the DFG (SPP 'Quanteninformationsverarbeitung'), by the ESF programme on 'Quantum Information Theory and Quantum Computation' and by the European IST-1999-11053 EQUIP project. A.D. acknowledges support by the DAAD. I.J is supported by the A. von Humboldt foundation and by the Ministry of Education of the Czech Republic. We are grateful to Steve Barnett for stimulating discussions.

[1] C. Monroe et al., Phys. Rev. Lett. **75**, 4714 (1995).
[2] A. Barenco et al., Phys. Rev. Lett. **74**, 4083 (1994); D. DiVincenzo, Phys. Rev. A **51** 1015 (1995); A. Barenco et al., Phys. Rev. A **52**, 3457 (1995).
[3] R. Feynman, Opt. News **11**, 11 (1985).
[4] A. Barenco et al., SIAM Journal of Computing **26**, 1541 (1997).
[5] S. L. Braunstein, A. Mann, M. Revzen, Phys. Rev. Lett. **68**, 3259 (1992).
[6] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[7] C. H. Bennett et al., Phys. Rev. Lett. **70**, 1895 (1993).
[8] C. H. Bennett et al., Phys. Rev. Lett. **76**, 722 (1996).
[9] H, Bechmann-Pasquinucci, B. Huttner, N. Gisin, Phys. Lett. A **242**, 198 (1998).
[10] Ch. Roos et al., Phys. Rev. Lett. **83**, 4713 (1999).
[11] J. Brendel, N. Gisin, W. Tittel, and H. Zbinden. Phys. Rev. Lett. **82**, 2594 (1999).
[12] S. L. Braunstein, Phys. Rev. Lett. **80**, 4084 (1998).
[13] A. O. Pittenger and M. H. Ruben, quant-ph/0001110.
[14] M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4206 (1999).
[15] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics*, (Cambridge, Cambridge, 1995).