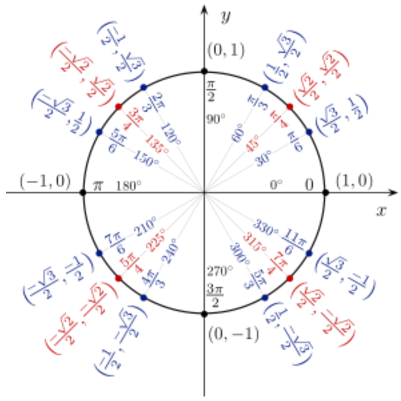
# List of trigonometric identities

In mathematics, **trigonometric identities** are equalities that involve trigonometric functions and are true for every value of the occurring variables where both sides of the equality are defined. Geometrically, these are identities involving certain functions of one or more angles. They are distinct from triangle identities, which are identities potentially involving angles but also involving side lengths or other lengths of a triangle.

These identities are useful whenever expressions involving trigonometric functions need to be simplified. An important application is the integration of non-trigonometric functions: a common technique involves first using the substitution rule with a trigonometric function, and then simplifying the resulting integral with a trigonometric identity



Cosines and sines around the unit circle

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# Notation

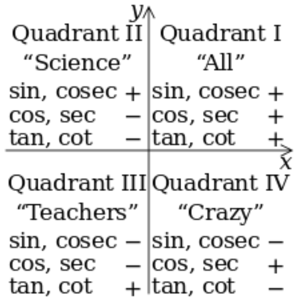
## Angles

This article uses Greek letters such as alpha ($\alpha$), beta ($\beta$), gamma ($\gamma$), and theta ($\theta$) to represent angles. Several different units of angle measure are widely used, including degree, radian, and gradian (gons):

1 full circle (turn) = 360 degree = 2π radian = 400 gon.

If not specifically annotated by (°) for degree or ($^g$) for gradian, all values for angles in this article are assumed to be given in radian.

The following table shows for some common angles their conversions and the values of the basic trigonometric functions:



Signs of trigonometric functions in each quadrant. The mnemonic "**All** **S**cience **T**eachers (are) **C**razy" lists the basic functions ('**All**', **s**in, **t**an, **c**os) which are positive from quadrants I to IV.[1] This is a variation on the mnemonic "All Students Take Calculus".

Conversions of common angles

| Turn | Degree | Radian | Gradian | sine | cosine | tangent |
|---|---|---|---|---|---|---|
| $0$ | $0°$ | $0$ | $0^g$ | $0$ | $1$ | $0$ |
| $\frac{1}{12}$ | $30°$ | $\frac{\pi}{6}$ | $33\frac{1}{3}^g$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ |
| $\frac{1}{8}$ | $45°$ | $\frac{\pi}{4}$ | $50^g$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $1$ |
| $\frac{1}{6}$ | $60°$ | $\frac{\pi}{3}$ | $66\frac{2}{3}^g$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| $\frac{1}{4}$ | $90°$ | $\frac{\pi}{2}$ | $100^g$ | $1$ | $0$ | $\infty$ |
| $\frac{1}{3}$ | $120°$ | $\frac{2\pi}{3}$ | $133\frac{1}{3}^g$ | $\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | $-\sqrt{3}$ |
| $\frac{3}{8}$ | $135°$ | $\frac{3\pi}{4}$ | $150^g$ | $\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-1$ |
| $\frac{5}{12}$ | $150°$ | $\frac{5\pi}{6}$ | $166\frac{2}{3}^g$ | $\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{3}}{3}$ |
| $\frac{1}{2}$ | $180°$ | $\pi$ | $200^g$ | $0$ | $-1$ | $0$ |
| $\frac{7}{12}$ | $210°$ | $\frac{7\pi}{6}$ | $233\frac{1}{3}^g$ | $-\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ |
| $\frac{5}{8}$ | $225°$ | $\frac{5\pi}{4}$ | $250^g$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $1$ |
| $\frac{2}{3}$ | $240°$ | $\frac{4\pi}{3}$ | $266\frac{2}{3}^g$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | $\sqrt{3}$ |
| $\frac{3}{4}$ | $270°$ | $\frac{3\pi}{2}$ | $300^g$ | $-1$ | $0$ | $\infty$ |
| $\frac{5}{6}$ | $300°$ | $\frac{5\pi}{3}$ | $333\frac{1}{3}^g$ | $-\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $-\sqrt{3}$ |
| $\frac{7}{8}$ | $315°$ | $\frac{7\pi}{4}$ | $350^g$ | $-\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $-1$ |
| $\frac{11}{12}$ | $330°$ | $\frac{11\pi}{6}$ | $366\frac{2}{3}^g$ | $-\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{3}}{3}$ |
| $1$ | $360°$ | $2\pi$ | $400^g$ | $0$ | $1$ | $0$ |

Results for other angles can be found at Trigonometric constants expressed in real radicals. Per Niven's theorem, $0,\ 30,\ 90,\ 150,\ 180,\ 210,\ 270,\ 330(,\ 360)$ are the only rational numbers that, taken in degrees, result in a rational sine-value for the corresponding angle within the first turn, which may account for their popularity in examples.[2][3] The analogous condition for the unit radian requires that the argument divided by π is rational, and yields the solutions 0, π/6, π/2, 5π/6, π, 7π/6, 3π/2, 11π/6(, 2π).

## Trigonometric functions

The functions sine, cosine and tangent of an angle are sometimes referred to as the *primary* or *basic* trigonometric functions. Their usual abbreviations are $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$, respectively, where $\theta$ denotes the angle. The parentheses around the argument of the functions are often omitted, e.g. $\sin\,\theta$ and $\cos\,\theta$, if an interpretation is unambiguously possible.

The sine of an angle is defined, in the context of a right triangle, as the ratio of the length of the side that is opposite to the angle divided by the length of the longest side of the triangle (the hypotenuse).

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}.$$

The cosine of an angle in this context is the ratio of the length of the side that is adjacent to the angle divided by the length of the hypotenuse.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}.$$

The tangent of an angle in this context is the ratio of the length of the side that is opposite to the angle divided by the length of the side that is adjacent to the angle. This is the same as the ratio of the sine to the cosine of this angle, as can be seen by substituting the definitions of $\sin$ and $\cos$ from above:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{opposite}}{\text{adjacent}}.$$

The remaining trigonometric functions secant (sec), cosecant (csc), and cotangent (cot) are defined as the reciprocal functions of cosine, sine, and tangent, respectively. Rarely, these are called the secondary trigonometric functions:

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}.$$

These definitions are sometimes referred to as ratio identities.

# Inverse functions

The inverse trigonometric functions are partial inverse functions for the trigonometric functions. For example, the inverse function for the sine, known as the **inverse sine** ($\sin^{-1}$) or **arcsine** ($\arcsin$ or $\operatorname{asin}$), satisfies

$$\sin(\arcsin x) = x \quad \text{for} \quad |x| \leq 1$$

and

$$\arcsin(\sin x) = x \quad \text{for} \quad |x| \leq \frac{\pi}{2}.$$

This article uses the notation below for inverse trigonometric functions:

| **Function** | sin | cos | tan | sec | csc | cot |
|---|---|---|---|---|---|---|
| **Inverse** | arcsin | arccos | arctan | arcsec | arccsc | arccot |

# Pythagorean identities

In trigonometry, the basic relationship between the sine and the cosine is given by the Pythagorean identity:

$$\sin^2\theta + \cos^2\theta = 1,$$

where $\sin^2\theta$ means $(\sin(\theta))^2$ and $\cos^2\theta$ means $(\cos(\theta))^2$.

This can be viewed as a version of the Pythagorean theorem, and follows from the equation $x^2 + y^2 = 1$ for the unit circle. This equation can be solved for either the sine or the cosine:

$$\begin{aligned} \sin\theta &= \pm\sqrt{1-\cos^2\theta}, \\ \cos\theta &= \pm\sqrt{1-\sin^2\theta}. \end{aligned}$$

where the sign depends on the quadrant of $\theta$.

Dividing this identity by either $\sin^2\theta$ or $\cos^2\theta$ yields the other two Pythagorean identities:

$$1 + \tan^2\theta = \sec^2\theta \quad \text{and} \quad 1 + \cot^2\theta = \csc^2\theta.$$

Using these identities together with the ratio identities, it is possible to express any trigonometric function in terms of any other (up to a plus or minus sign):

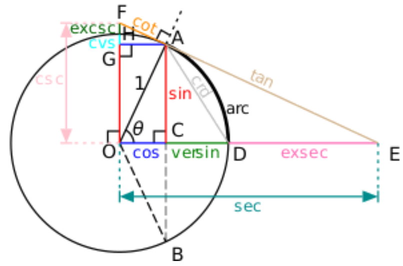
Each trigonometric function in terms of the other five.[4]

| **in terms of** | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ | $\csc\theta$ | $\sec\theta$ | $\cot\theta$ |
|---|---|---|---|---|---|---|
| $\sin\theta =$ | $\sin\theta$ | $\pm\sqrt{1-\cos^2\theta}$ | $\pm\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$ | $\frac{1}{\csc\theta}$ | $\pm\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$ | $\pm\frac{1}{\sqrt{1+\cot^2\theta}}$ |
| $\cos\theta =$ | $\pm\sqrt{1-\sin^2\theta}$ | $\cos\theta$ | $\pm\frac{1}{\sqrt{1+\tan^2\theta}}$ | $\pm\frac{\sqrt{\csc^2\theta-1}}{\csc\theta}$ | $\frac{1}{\sec\theta}$ | $\pm\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$ |
| $\tan\theta =$ | $\pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$ | $\pm\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$ | $\tan\theta$ | $\pm\frac{1}{\sqrt{\csc^2\theta-1}}$ | $\pm\sqrt{\sec^2\theta-1}$ | $\frac{1}{\cot\theta}$ |
| $\csc\theta =$ | $\frac{1}{\sin\theta}$ | $\pm\frac{1}{\sqrt{1-\cos^2\theta}}$ | $\pm\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$ | $\csc\theta$ | $\pm\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$ | $\pm\sqrt{1+\cot^2\theta}$ |
| $\sec\theta =$ | $\pm\frac{1}{\sqrt{1-\sin^2\theta}}$ | $\frac{1}{\cos\theta}$ | $\pm\sqrt{1+\tan^2\theta}$ | $\pm\frac{\csc\theta}{\sqrt{\csc^2\theta-1}}$ | $\sec\theta$ | $\pm\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$ |
| $\cot\theta =$ | $\pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$ | $\pm\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$ | $\frac{1}{\tan\theta}$ | $\pm\sqrt{\csc^2\theta-1}$ | $\pm\frac{1}{\sqrt{\sec^2\theta-1}}$ | $\cot\theta$ |

# Historical shorthands

The versine, coversine, haversine, and exsecant were used in navigation. For example, the haversine formula was used to calculate the distance between two points on a sphere. They are rarely used today.

| Name | Abbreviation | Value[5][6] |
|---|---|---|
| versed sine, versine | $\operatorname{versin}\theta$<br>$\operatorname{vers}\theta$<br>$\operatorname{ver}\theta$ | $1-\cos\theta$ |
| versed cosine, vercosine | $\operatorname{vercosin}\theta$<br>$\operatorname{vercos}\theta$<br>$\operatorname{vcs}\theta$ | $1+\cos\theta$ |
| coversed sine, coversine | $\operatorname{coversin}\theta$<br>$\operatorname{covers}\theta$<br>$\operatorname{cvs}\theta$ | $1-\sin\theta$ |
| coversed cosine, covercosine | $\operatorname{covercosin}\theta$<br>$\operatorname{covercos}\theta$<br>$\operatorname{cvc}\theta$ | $1+\sin\theta$ |
| half versed sine, haversine | $\operatorname{haversin}\theta$<br>$\operatorname{hav}\theta$<br>$\operatorname{sem}\theta$ | $\frac{1-\cos\theta}{2}$ |
| half versed cosine, havercosine | $\operatorname{havercosin}\theta$<br>$\operatorname{havercos}\theta$<br>$\operatorname{hvc}\theta$ | $\frac{1+\cos\theta}{2}$ |
| half coversed sine, hacoversine cohaversine | $\operatorname{hacoversin}\theta$<br>$\operatorname{hacovers}\theta$<br>$\operatorname{hcv}\theta$ | $\frac{1-\sin\theta}{2}$ |
| half coversed cosine, hacovercosine cohavercosine | $\operatorname{hacovercosin}\theta$<br>$\operatorname{hacovercos}\theta$<br>$\operatorname{hcc}\theta$ | $\frac{1+\sin\theta}{2}$ |
| exterior secant, exsecant | $\operatorname{exsec}\theta$<br>$\operatorname{exs}\theta$ | $\sec\theta-1$ |
| exterior cosecant, excosecant | $\operatorname{excosec}\theta$<br>$\operatorname{excsc}\theta$<br>$\operatorname{exc}\theta$ | $\csc\theta-1$ |
| chord | $\operatorname{crd}\theta$ | $2\sin\frac{\theta}{2}$ |



All of the trigonometric functions of an angle $\theta$ can be constructed geometrically in terms of a unit circle centered at $O$. Many of these terms are no longer in common use.

# Reflections, shifts, and periodicity

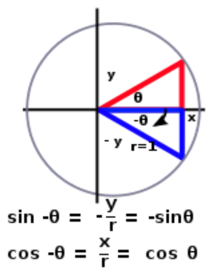
By examining the unit circle, the following properties of the trigonometric functions can be established.

## Reflections

When a direction, represented by an angle $\theta$ enclosed with the $x$-direction, is reflected in a line with direction $\alpha$, then the angle $\theta'$ of this reflected direction has the value

$$\theta' = 2\alpha - \theta.$$

This way, reflections in the directions $0$ and $\pi$ radian ($0°$ and $180°$) generate equally looking results (see picture). The values of the trigonometric functions of these angles $\theta,\ \theta'$ for specific angles $\alpha$ satisfy simple identities: either they are equal, or have opposite signs, or employ the complementary trigonometric function.



Reflecting θ in α=0 (α=π)

| $\theta$ reflected in $\alpha = 0$[7] odd/even identities | $\theta$ reflected in $\alpha = \frac{\pi}{4}$ co-function identities[8] | $\theta$ reflected in $\alpha = \frac{\pi}{2}$ | $\theta$ reflected in $\alpha = \pi$ compare to $\alpha = 0$ |
|---|---|---|---|
| $\sin(-\theta) = -\sin\theta$ | $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$ | $\sin(\pi-\theta) = +\sin\theta$ | $\sin(2\pi-\theta) = -\sin(\theta) = \sin(-\theta)$ |
| $\cos(-\theta) = +\cos\theta$ | $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$ | $\cos(\pi-\theta) = -\cos\theta$ | $\cos(2\pi-\theta) = +\cos(\theta) = \cos(-\theta)$ |
| $\tan(-\theta) = -\tan\theta$ | $\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$ | $\tan(\pi-\theta) = -\tan\theta$ | $\tan(2\pi-\theta) = -\tan(\theta) = \tan(-\theta)$ |
| $\csc(-\theta) = -\csc\theta$ | $\csc\left(\frac{\pi}{2}-\theta\right) = \sec\theta$ | $\csc(\pi-\theta) = +\csc\theta$ | $\csc(2\pi-\theta) = -\csc(\theta) = \csc(-\theta)$ |
| $\sec(-\theta) = +\sec\theta$ | $\sec\left(\frac{\pi}{2}-\theta\right) = \csc\theta$ | $\sec(\pi-\theta) = -\sec\theta$ | $\sec(2\pi-\theta) = +\sec(\theta) = \sec(-\theta)$ |
| $\cot(-\theta) = -\cot\theta$ | $\cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$ | $\cot(\pi-\theta) = -\cot\theta$ | $\cot(2\pi-\theta) = -\cot(\theta) = \cot(-\theta)$ |

## Shifts and periodicity

By shifting round the arguments of trigonometric functions by certain angles, it is sometimes possible that changing the sign or applying complementary trigonometric functions express particular results more simply. Some examples of shifts are shown below in the table.

- A **full turn**, or $360°$, or $2\pi$ radian does not change anything along the unit circle and makes up the smallest interval for which the trigonometric functions $\sin$, $\cos$, $\sec$, and $\csc$ repeat their values, and is thus their period. Shifting arguments of any periodic function by any integer multiple of a full period preserves the function value of the unshifted argument.
- A **half turn**, or $180°$, or $\pi$ radian is the period of $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and $\cot(x) = \frac{\cos(x)}{\sin(x)}$, as can be seen from these definitions and the period of the defining trigonometric functions. So shifting the arguments of $\tan(x)$ and $\cot(x)$ by any multiple of $\pi$, does not change their function values.

  For the functions $\sin$, $\cos$, $\sec$, and $\csc$ with period $2\pi$ half a turn is half of their period. For this shift they change the sign of their values, as can be seen from the unit circle again. This new value repeats after any additional shift of $2\pi$, so all together they change the sign for a shift by any odd multiple of $\pi$, i.e., by $(2k+1)\cdot\pi$, with $k$ an arbitrary integer. Any even multiple of $\pi$ is of course just a full period, and a backward shift by half a period is the same as a backward shift by one full period plus one shift forward by half a period.

- A **quarter turn**, or $90°$, or $\frac{\pi}{2}$ radian is a half period shift for $\tan(x)$ and $\cot(x)$ with period $\pi$ ($180°$), and yields the function value of applying the complementary function to the unshifted argument. By the argument above this also holds for a shift by any odd multiple $(2k+1)\cdot\frac{\pi}{2}$ of the half period.

For the four other trigonometric functions a quarter turn also represents a quarter period. A shift by an arbitrary multiple of a quarter period, that is not covered by a multiple of half periods, can be decomposed in an integer multiple of periods, plus or minus one quarter period. The terms expressing these multiples are $(4k \pm 1)\cdot\frac{\pi}{2}$. The forward/backward shifts by one quarter period are reflected in the table below. Again, these shifts yield function values, employing the respective complementary function applied to the unshifted argument.
Shifting the arguments of $\tan(x)$ and $\cot(x)$ by their quarter period ($\frac{\pi}{4}$) does not yield such simple results.

| Shift by one quarter period | Shift by one half period[9] | Shift by full periods[10] | Period |
|---|---|---|---|
| $\sin(\theta \pm \frac{\pi}{2}) = \pm\cos\theta$ | $\sin(\theta + \pi) = -\sin\theta$ | $\sin(\theta + k\cdot 2\pi) = +\sin\theta$ | $2\pi$ |
| $\cos(\theta \pm \frac{\pi}{2}) = \mp\sin\theta$ | $\cos(\theta + \pi) = -\cos\theta$ | $\cos(\theta + k\cdot 2\pi) = +\cos\theta$ | $2\pi$ |
| $\tan(\theta \pm \frac{\pi}{4}) = \frac{\tan\theta \pm 1}{1 \mp \tan\theta}$ | $\tan(\theta + \frac{\pi}{2}) = -\cot\theta$ | $\tan(\theta + k\cdot\pi) = +\tan\theta$ | $\pi$ |
| $\csc(\theta \pm \frac{\pi}{2}) = \pm\sec\theta$ | $\csc(\theta + \pi) = -\csc\theta$ | $\csc(\theta + k\cdot 2\pi) = +\csc\theta$ | $2\pi$ |
| $\sec(\theta \pm \frac{\pi}{2}) = \mp\csc\theta$ | $\sec(\theta + \pi) = -\sec\theta$ | $\sec(\theta + k\cdot 2\pi) = +\sec\theta$ | $2\pi$ |
| $\cot(\theta \pm \frac{\pi}{4}) = \frac{\cot\theta \pm 1}{1 \mp \cot\theta}$ | $\cot(\theta + \frac{\pi}{2}) = -\tan\theta$ | $\cot(\theta + k\cdot\pi) = +\cot\theta$ | $\pi$ |

# Angle sum and difference identities

These are also known as the *addition and subtraction theorems* or *formulae*. The identities can be derived by combining right triangles such as in the adjacent diagram, or by considering the invariance of the length of a chord on a unit circle given a particular central angle. Furthermore, it is even possible to derive the identities using Euler's identity although this would be a more obscure approach given that complex numbers are used.

For acute angles $\alpha$ and $\beta$, whose sum is non-obtuse, a concise diagram (shown) illustrates the angle sum formulae for sine and cosine: The bold segment labeled "1" has unit length and serves as the hypotenuse of a right triangle with angle $\beta$; the opposite and adjacent legs for this angle have respective lengths $\sin\beta$ and $\cos\beta$. The $\cos\beta$ leg is itself the hypotenuse of a right triangle with angle $\alpha$; that triangle's legs, therefore, have lengths given by $\sin\alpha$ and $\cos\alpha$, multiplied by $\cos\beta$. The $\sin\beta$ leg, as hypotenuse of another right triangle with angle $\alpha$, likewise leads to segments of length $\cos\alpha\sin\beta$ and $\sin\alpha\sin\beta$. Now, we observe that the "1" segment is also the hypotenuse of a right triangle with angle $\alpha + \beta$; the leg opposite this angle necessarily has length $\sin(\alpha + \beta)$, while the leg adjacent has length $\cos(\alpha + \beta)$. Consequently, as the opposing sides of the diagram's outer rectangle are equal, we deduce

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

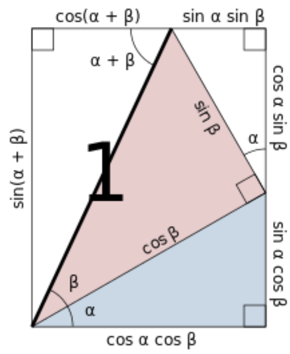


Illustration of angle addition formulae for the sine and cosine. Emphasized segment is of unit length.

Relocating one of the named angles yields a variant of the diagram that demonstrates the angle difference formulae for sine and cosine.[11] (The diagram admits further variants to accommodate angles and sums greater than a right angle.) Dividing all elements of the diagram by $\cos\alpha\cos\beta$ provides yet another variant (shown) illustrating the angle sum formula for tangent.

| | |
|---|---|
| **Sine** | $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ [12][13] |
| **Cosine** | $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ [13][14] |
| **Tangent** | $\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$ [13][15] |
| **Cotangent** | $\cot(\alpha \pm \beta) = \frac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}$ [13][16] |
| **Arcsine** | $\arcsin x \pm \arcsin y = \arcsin\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$ [17] |
| **Arccosine** | $\arccos x \pm \arccos y = \arccos\left(xy \mp \sqrt{\left(1-x^2\right)\left(1-y^2\right)}\right)$ [18] |
| **Arctangent** | $\arctan x \pm \arctan y = \arctan\left(\frac{x \pm y}{1 \mp xy}\right)$ [19] |
| **atan2** | $\operatorname{atan2}(y_1, x_1) \pm \operatorname{atan2}(y_2, x_2) = \operatorname{atan2}(y_1 x_2 \pm y_2 x_1, x_1 x_2 \mp y_1 y_2)$ |
| **Arccotangent** | $\operatorname{arccot} x \pm \operatorname{arccot} y = \operatorname{arccot}\left(\frac{xy \mp 1}{y \pm x}\right)$ |

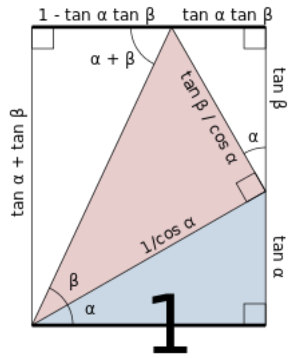


Illustration of the angle addition formula for the tangent. Emphasized segments are of unit length.

## Matrix form

The sum and difference formulae for sine and cosine can be written in matrix form as:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$
$$= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}.$$

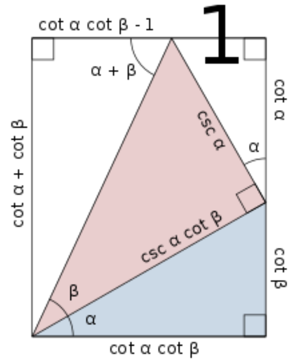


Illustration of the angle addition formula for the cotangent. Top right segment is of unit length.

The matrix inverse for a rotation is the rotation with the negative of the angle

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}^{-1} = \begin{pmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix},$$

which is also the matrix transpose.

These formulae show that these matrices form a representation of the rotation group in the plane (technically, the special orthogonal group $\mathrm{SO}(2)$), since the composition law is fulfilled and inverses exist. Furthermore, matrix multiplication of the rotation matrix for an angle $\alpha$ with a column vector will rotate the column vector counterclockwise by the angle $\alpha$.

## Sines and cosines of sums of infinitely many angles

When the series $\sum_{i=1}^{\infty} \theta_i$ converges absolutely then

$$\sin\left(\sum_{i=1}^{\infty} \theta_i\right) = \sum_{\text{odd } k \geq 1} (-1)^{\frac{k-1}{2}} \sum_{\substack{A \subseteq \{1,2,3,\dots\} \\ |A|=k}} \left(\prod_{i \in A} \sin\theta_i \prod_{i \notin A} \cos\theta_i\right)$$

$$\cos\left(\sum_{i=1}^{\infty} \theta_i\right) = \sum_{\text{even } k \geq 0} (-1)^{\frac{k}{2}} \sum_{\substack{A \subseteq \{1,2,3,\dots\} \\ |A|=k}} \left(\prod_{i \in A} \sin\theta_i \prod_{i \notin A} \cos\theta_i\right).$$

Because the series $\sum_{i=1}^{\infty} \theta_i$ converges absolutely, it is necessarily the case that $\lim_{i\to\infty} \theta_i = 0$, $\lim_{i\to\infty} \sin\theta_i = 0$, and $\lim_{i\to\infty} \cos\theta_i = 1$. In particular, in these two identities an asymmetry appears that is not seen in the case of sums of finitely many angles: in each product, there are only finitely many sine factors but there are cofinitely many cosine factors. Terms with infinitely many sine factors would necessarily be equal to zero.

When only finitely many of the angles $\theta_i$ are nonzero then only finitely many of the terms on the right side are nonzero because all but finitely many sine factors vanish. Furthermore, in each term all but finitely many of the cosine factors are unity.

## Tangents and cotangents of sums

Let $e_k$ (for $k$ = 0, 1, 2, 3, ...) be the $k$th-degree elementary symmetric polynomial in the variables

$$x_i = \tan\theta_i$$

for $i$ = 0, 1, 2, 3, ..., i.e.,

$$\begin{aligned}
e_0 &= 1 \\
e_1 &= \sum_i x_i &&= \sum_i \tan\theta_i \\
e_2 &= \sum_{i<j} x_i x_j &&= \sum_{i<j} \tan\theta_i \tan\theta_j \\
e_3 &= \sum_{i<j<k} x_i x_j x_k &&= \sum_{i<j<k} \tan\theta_i \tan\theta_j \tan\theta_k \\
&\ \ \vdots && \ \ \vdots
\end{aligned}$$

Then

$$\begin{aligned}
\tan\left(\sum_i \theta_i\right) &= \frac{\sin\left(\sum_i \theta_i\right) / \prod_i \cos\theta_i}{\cos\left(\sum_i \theta_i\right) / \prod_i \cos\theta_i} \\
&= \frac{\sum_{\text{odd } k \geq 1} (-1)^{\frac{k-1}{2}} \sum_{\substack{A \subseteq \{1,2,3,\dots\} \\ |A|=k}} \prod_{i \in A} \tan\theta_i}{\sum_{\text{even } k \geq 0} (-1)^{\frac{k}{2}} \sum_{\substack{A \subseteq \{1,2,3,\dots\} \\ |A|=k}} \prod_{i \in A} \tan\theta_i} = \frac{e_1 - e_3 + e_5 - \cdots}{e_0 - e_2 + e_4 - \cdots} \\
\cot\left(\sum_i \theta_i\right) &= \frac{e_0 - e_2 + e_4 - \cdots}{e_1 - e_3 + e_5 - \cdots}
\end{aligned}$$

using the sine and cosine sum formulae above.

The number of terms on the right side depends on the number of terms on the left side.

For example:

$$\begin{aligned}
\tan(\theta_1 + \theta_2) &= \frac{e_1}{e_0 - e_2} = \frac{x_1 + x_2}{1 - x_1 x_2} = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}, \\
\tan(\theta_1 + \theta_2 + \theta_3) &= \frac{e_1 - e_3}{e_0 - e_2} = \frac{(x_1 + x_2 + x_3) - (x_1 x_2 x_3)}{1 - (x_1 x_2 + x_1 x_3 + x_2 x_3)}, \\
\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) &= \frac{e_1 - e_3}{e_0 - e_2 + e_4} \\
&= \frac{(x_1 + x_2 + x_3 + x_4) - (x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4)}{1 - (x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4) + (x_1 x_2 x_3 x_4)},
\end{aligned}$$

and so on. The case of only finitely many terms can be proved by mathematical induction.[20]

### Secants and cosecants of sums

$$\sec\left(\sum_i \theta_i\right) = \frac{\prod_i \sec\theta_i}{e_0 - e_2 + e_4 - \cdots}$$

$$\csc\left(\sum_i \theta_i\right) = \frac{\prod_i \sec\theta_i}{e_1 - e_3 + e_5 - \cdots}$$

where $e_k$ is the $k$th-degree elementary symmetric polynomial in the $n$ variables $x_i = \tan\theta_i$, $i = 1, \ldots, n$, and the number of terms in the denominator and the number of factors in the product in the numerator depend on the number of terms in the sum on the left.[21] The case of only finitely many terms can be proved by mathematical induction on the number of such terms.

For example,

$$\sec(\alpha+\beta+\gamma) = \frac{\sec\alpha\sec\beta\sec\gamma}{1 - \tan\alpha\tan\beta - \tan\alpha\tan\gamma - \tan\beta\tan\gamma}$$

$$\csc(\alpha+\beta+\gamma) = \frac{\sec\alpha\sec\beta\sec\gamma}{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}.$$

# Multiple-angle formulae

| | |
|---|---|
| $T_n$ is the $n$th Chebyshev polynomial | $\cos(n\theta) = T_n(\cos\theta)$ [22] |
| $S_n$ is the $n$th spread polynomial | $\sin^2(n\theta) = S_n(\sin^2\theta)$ |
| de Moivre's formula, $i$ is the imaginary unit | $\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$ [23] |

### Double-angle, triple-angle, and half-angle formulae

#### Double-angle formulae

$$\sin(2\theta) = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\cot(2\theta) = \frac{\cot^2\theta - 1}{2\cot\theta}$$

$$\sec(2\theta) = \frac{\sec^2\theta}{2-\sec^2\theta}$$

$$\csc(2\theta) = \frac{\sec\theta\csc\theta}{2}$$

#### Triple-angle formulae

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta = 4\sin\theta\sin\left(\frac{\pi}{3}-\theta\right)\sin\left(\frac{\pi}{3}+\theta\right)$$

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta = 4\cos\theta\cos\left(\frac{\pi}{3}-\theta\right)\cos\left(\frac{\pi}{3}+\theta\right)$$

$$\tan(3\theta) = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} = \tan\theta\tan\left(\frac{\pi}{3}-\theta\right)\tan\left(\frac{\pi}{3}+\theta\right)$$

$$\cot(3\theta) = \frac{3\cot\theta - \cot^3\theta}{1-3\cot^2\theta}$$

$$\sec(3\theta) = \frac{\sec^3\theta}{4-3\sec^2\theta}$$

$$\csc(3\theta) = \frac{\csc^3\theta}{3\csc^2\theta - 4}$$

#### Half-angle formulae

$$\sin\frac{\theta}{2} = \operatorname{sgn}\left(2\pi - \theta + 4\pi\left\lfloor\frac{\theta}{4\pi}\right\rfloor\right)\sqrt{\frac{1-\cos\theta}{2}}$$

$$\text{where } \operatorname{sgn} x = \pm 1 \text{ according to whether } x \text{ is positive or negative.}$$

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

$$\cos\frac{\theta}{2} = \operatorname{sgn}\left(\pi + \theta + 4\pi\left\lfloor\frac{\pi-\theta}{4\pi}\right\rfloor\right)\sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$

$$\begin{aligned}\tan\frac{\theta}{2} &= \csc\theta - \cot\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta}\\ &= \frac{1-\cos\theta}{\sin\theta} = \frac{-1\pm\sqrt{1+\tan^2\theta}}{\tan\theta} = \frac{\tan\theta}{1+\sec\theta}\end{aligned}$$

$$\cot\frac{\theta}{2} = \csc\theta + \cot\theta = \pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta}$$

[24][25]

Also

$$\tan\frac{\eta+\theta}{2} = \frac{\sin\eta+\sin\theta}{\cos\eta+\cos\theta}$$

$$\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right) = \sec\theta + \tan\theta$$

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$$

#### Table

These can be shown by using either the sum and difference identities or the multiple-angle formulae.

| | Sine | Cosine | Tangent | Cotangent |
|---|---|---|---|---|
| **Double-angle formulae**[26][27] | $\begin{aligned}\sin(2\theta) &= 2\sin\theta\cos\theta\\ &= \frac{2\tan\theta}{1+\tan^2\theta}\end{aligned}$ | $\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta\\ &= 2\cos^2\theta - 1\\ &= 1 - 2\sin^2\theta\\ &= \frac{1-\tan^2\theta}{1+\tan^2\theta}\end{aligned}$ | $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$ | $\cot(2\theta) = \frac{\cot^2\theta -}{2\cot\theta}$ |
| **Triple-angle formulae**[22][28] | $\begin{aligned}\sin(3\theta) &= -\sin^3\theta + 3\cos^2\theta\sin\theta\\ &= -4\sin^3\theta + 3\sin\theta\end{aligned}$ | $\begin{aligned}\cos(3\theta) &= \cos^3\theta - 3\sin^2\theta\cos\theta\\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$ | $\tan(3\theta) = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$ | $\cot(3\theta) = \frac{3\cot\theta -}{1-3c}$ |
| **Half-angle formulae**[24][25] | $\sin\frac{\theta}{2} = \operatorname{sgn}\left(2\pi - \theta + 4\pi\left\lfloor\frac{\theta}{4\pi}\right\rfloor\right)\sqrt{\frac{1-\cos\theta}{2}}$ <br> $\left(\text{or } \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}\right)$ | $\cos\frac{\theta}{2} = \operatorname{sgn}\left(\pi + \theta + 4\pi\left\lfloor\frac{\pi-\theta}{4\pi}\right\rfloor\right)\sqrt{\frac{1+\cos\theta}{2}}$ <br> $\left(\text{or } \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}\right)$ | $\begin{aligned}\tan\frac{\theta}{2} &= \csc\theta - \cot\theta\\ &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\\ &= \frac{\sin\theta}{1+\cos\theta}\\ &= \frac{1-\cos\theta}{\sin\theta}\end{aligned}$ <br> $\tan\frac{\eta+\theta}{2} = \frac{\sin\eta+\sin\theta}{\cos\eta+\cos\theta}$ <br> $\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right) = \sec\theta+\tan\theta$ <br> $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$ <br> $\tan\frac{\theta}{2} = \frac{\tan\theta}{1+\sqrt{1+\tan^2\theta}}$ <br> $\text{for } \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ | $\begin{aligned}\cot\frac{\theta}{2} &= \csc\theta + \cot\\ &= \pm\sqrt{\frac{1+c}{1-c}}\\ &= \frac{\sin\theta}{1-\cos\theta}\\ &= \frac{1+\cos\theta}{\sin\theta}\end{aligned}$ |

The fact that the triple-angle formula for sine and cosine only involves powers of a single function allows one to relate the geometric problem of a compass and straightedge construction of angle trisection to the algebraic problem of solving a cubic equation, which allows one to prove that trisection is in general impossible using the given tools, by field theory.

A formula for computing the trigonometric identities for the one-third angle exists, but it requires finding the zeroes of the cubic equation $4x^3 - 3x + d = 0$, where $x$ is the value of the cosine function at the one-third angle and $d$ is the known value of the cosine function at the full angle. However, the discriminant of this equation is positive, so this equation has three real roots (of which only one is the solution for the cosine of the one-third angle). None of these solutions is reducible to a real algebraic expression, as they use intermediate complex numbers under the cube roots.

## Sine, cosine, and tangent of multiple angles

For specific multiples, these follow from the angle addition formulae, while the general formula was given by 16th-century French mathematician François Viète.

$$\sin(n\theta) = \sum_{k \text{ odd}} (-1)^{\frac{k-1}{2}} \binom{n}{k} \cos^{n-k}\theta \sin^k\theta,$$
$$\cos(n\theta) = \sum_{k \text{ even}} (-1)^{\frac{k}{2}} \binom{n}{k} \cos^{n-k}\theta \sin^k\theta\,,$$

for nonnegative values of $k$ up through $n$.

In each of these two equations, the first parenthesized term is a binomial coefficient, and the final trigonometric function equals one or minus one or zero so that half the entries in each of the sums are removed. The ratio of these formulae gives

$$\tan(n\theta) = \frac{\sum_{k \text{ odd}} (-1)^{\frac{k-1}{2}} \binom{n}{k} \tan^k\theta}{\sum_{k \text{ even}} (-1)^{\frac{k}{2}} \binom{n}{k} \tan^k\theta}\,.$$

### Chebyshev method

The Chebyshev method is a recursive algorithm for finding the $n$th multiple angle formula knowing the $(n-1)$th and $(n-2)$th values.[29]

$\cos(nx)$ can be computed from $\cos((n-1)x)$, $\cos((n-2)x)$, and $\cos(x)$ with

$$\cos(nx) = 2 \cdot \cos x \cdot \cos((n-1)x) - \cos((n-2)x).$$

This can be proved by adding together the formulae

$$\cos((n-1)x + x) = \cos((n-1)x)\cos x - \sin((n-1)x)\sin x$$
$$\cos((n-1)x - x) = \cos((n-1)x)\cos x + \sin((n-1)x)\sin x.$$

Similarly, $\sin(nx)$ can be computed from $\sin((n-1)x)$, $\sin((n-2)x)$, and $\cos(x)$ with

$$\sin(nx) = 2 \cdot \cos x \cdot \sin((n-1)x) - \sin((n-2)x).$$

This can be proved by adding formulae for $\sin((n-1)x + x)$ and $\sin((n-1)x - x)$.

Serving a purpose similar to that of the Chebyshev method, for the tangent we can write:

$$\tan(nx) = \frac{\tan((n-1)x) + \tan x}{1 - \tan((n-1)x)\tan x}\,.$$

### Tangent of an average

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = -\frac{\cos\alpha - \cos\beta}{\sin\alpha - \sin\beta}$$

Setting either $\alpha$ or $\beta$ to 0 gives the usual tangent half-angle formulae.

### Viète's infinite product

$$\cos\frac{\theta}{2} \cdot \cos\frac{\theta}{4} \cdot \cos\frac{\theta}{8} \cdots = \prod_{n=1}^{\infty} \cos\frac{\theta}{2^n} = \frac{\sin\theta}{\theta} = \operatorname{sinc}\theta.$$

(Refer to sinc function.)

## Power-reduction formulae

Obtained by solving the second and third versions of the cosine double-angle formula.

| Sine | Cosine | Other |
|---|---|---|
| $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$ | $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$ | $\sin^2\theta\cos^2\theta = \frac{1-\cos(4\theta)}{8}$ |
| $\sin^3\theta = \frac{3\sin\theta - \sin(3\theta)}{4}$ | $\cos^3\theta = \frac{3\cos\theta + \cos(3\theta)}{4}$ | $\sin^3\theta\cos^3\theta = \frac{3\sin(2\theta) - \sin(6\theta)}{32}$ |
| $\sin^4\theta = \frac{3 - 4\cos(2\theta) + \cos(4\theta)}{8}$ | $\cos^4\theta = \frac{3 + 4\cos(2\theta) + \cos(4\theta)}{8}$ | $\sin^4\theta\cos^4\theta = \frac{3 - 4\cos(4\theta) + \cos(8\theta)}{128}$ |
| $\sin^5\theta = \frac{10\sin\theta - 5\sin(3\theta) + \sin(5\theta)}{16}$ | $\cos^5\theta = \frac{10\cos\theta + 5\cos(3\theta) + \cos(5\theta)}{16}$ | $\sin^5\theta\cos^5\theta = \frac{10\sin(2\theta) - 5\sin(6\theta) + \sin(10\theta)}{512}$ |

and in general terms of powers of $\sin\theta$ or $\cos\theta$ the following is true, and can be deduced using De Moivre's formula, Euler's formula and the binomial theorem.

| | Cosine | Sine |
|---|---|---|
| if $n$ is odd | $\cos^n\theta = \frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}}\binom{n}{k}\cos\left((n-2k)\theta\right)$ | $\sin^n\theta = \frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}}(-1)^{\left(\frac{n-1}{2}-k\right)}\binom{n}{k}\sin\left((n-2k)\theta\right)$ |
| if $n$ is even | $\cos^n\theta = \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{2}{2^n}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos\left((n-2k)\theta\right)$ | $\sin^n\theta = \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{2}{2^n}\sum_{k=0}^{\frac{n}{2}-1}(-1)^{\left(\frac{n}{2}-k\right)}\binom{n}{k}\cos\left((n-2k)\theta\right)$ |

## Product-to-sum and sum-to-product identities

The product-to-sum identities or prosthaphaeresis formulae can be proven by expanding their right-hand sides using the angle addition theorems. See amplitude modulation for an application of the product-to-sum formulae, and beat (acoustics) and phase detector for applications of the sum-to-product formulae.

| Product-to-sum[30] |
|---|
| $2\cos\theta\cos\varphi = \cos(\theta-\varphi)+\cos(\theta+\varphi)$ |
| $2\sin\theta\sin\varphi = \cos(\theta-\varphi)-\cos(\theta+\varphi)$ |
| $2\sin\theta\cos\varphi = \sin(\theta+\varphi)+\sin(\theta-\varphi)$ |
| $2\cos\theta\sin\varphi = \sin(\theta+\varphi)-\sin(\theta-\varphi)$ |
| $\tan\theta\tan\varphi = \frac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{\cos(\theta-\varphi)+\cos(\theta+\varphi)}$ |
| $\prod_{k=1}^{n}\cos\theta_k = \frac{1}{2^n}\sum_{e\in S}\cos(e_1\theta_1+\cdots+e_n\theta_n)$ where $S=\{1,-1\}^n$ |

| Sum-to-product[31] |
|---|
| $\sin\theta\pm\sin\varphi = 2\sin\left(\frac{\theta\pm\varphi}{2}\right)\cos\left(\frac{\theta\mp\varphi}{2}\right)$ |
| $\cos\theta+\cos\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$ |
| $\cos\theta-\cos\varphi = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$ |

### Other related identities

- $\sec^2 x + \csc^2 x = \sec^2 x\csc^2 x.$ [32]
- If $x + y + z = \pi$ (half circle), then

  $$\sin(2x)+\sin(2y)+\sin(2z) = 4\sin x\sin y\sin z.$$

- **Triple tangent identity:** If $x + y + z = \pi$ (half circle), then

  $$\tan x+\tan y+\tan z = \tan x\tan y\tan z.$$

  In particular, the formula holds when $x$, $y$, and $z$ are the three angles of any triangle.
  (If any of $x$, $y$, $z$ is a right angle, one should take both sides to be $\infty$. This is neither $+\infty$ nor $-\infty$; for present purposes it makes sense to add just one point at infinity to the real line, that is approached by tan $\theta$ as tan $\theta$ either increases through positive values or decreases through negative values. This is a one-point compactification of the real line.)

- **Triple cotangent identity:** If $x + y + z = \frac{\pi}{2}$ (right angle or quarter circle), then

  $$\cot x+\cot y+\cot z = \cot x\cot y\cot z.$$

### Hermite's cotangent identity

Charles Hermite demonstrated the following identity.[33] Suppose $a_1, \ldots, a_n$ are complex numbers, no two of which differ by an integer multiple of π. Let

$$A_{n,k} = \prod_{\substack{1\le j\le n \\ j\neq k}}\cot(a_k-a_j)$$

(in particular, $A_{1,1}$, being an empty product, is 1). Then

$$\cot(z-a_1)\cdots\cot(z-a_n) = \cos\frac{n\pi}{2}+\sum_{k=1}^{n}A_{n,k}\cot(z-a_k).$$

The simplest non-trivial example is the case $n = 2$:

$$\cot(z-a_1)\cot(z-a_2) = -1+\cot(a_1-a_2)\cot(z-a_1)+\cot(a_2-a_1)\cot(z-a_2).$$

### Ptolemy's theorem

Ptolemy's theorem can be expressed in the language of modern trigonometry as:

If $w + x + y + z = \pi$, then:

{\displaystyle {\begin{aligned}\sin(w+x)\sin(x+y)&=\sin(x+y)\sin(y+z)&{\text{(trivial)}}\\&=\sin(y+z)\sin(z+w)&{\text{(trivial)}}\\&=\sin(z+w)\sin(w+x)&{\text{(trivial)}}\\&=\sin w\sin y+\sin x\sin z.&{\text{(significant)}}\end{aligned}}}

(The first three equalities are trivial rearrangements; the fourth is the substance of this identity.)

### Finite products of trigonometric functions

For coprime integers $n$, $m$

$$\prod _{k=1}^{n}\left(2a+2\cos \left({\frac {2\pi km}{n}}+x\right)\right)=2\left(T_{n}(a)+{(-1)}^{n+m}\cos(nx)\right)$$

where $T_n$ is the Chebyshev polynomial

The following relationship holds for the sine function

{\displaystyle \prod _{k=1}^{n-1}\sin \left({\frac {k\pi }

## Linear combinations

For some purposes it is important to know that any linear combination of sine waves of the same period or frequency but different phase shifts is also a sine wave with the same period or frequency, but a different phase shift. This is useful in sinusoid data fitting, because the measured or observed data are linearly related to the $a$ and $b$ unknowns of the in-phase and quadrature components basis below, resulting in a simpler Jacobian, compared to that of $c$ and $\varphi$.

### Sine and cosine

The linear combination, or harmonic addition, of sine and cosine waves is equivalent to a single sine wave with a phase shift and scaled amplitude,[34][35][36]

{\displaystyle a\sin x+b\cos

where the original amplitudes $a$ and $b$ sum in quadrature to yield the combined amplitude $c$,

{\displaystyle

and, using the atan2 function, the initial value of the phase angle $x + \varphi$ is obtained by

\varphi

### Arbitrary phase shift

More generally, for an arbitrary phase shift, we have

{\displaystyle a\sin x+b\sin(x+\theta

where

{\displaystyle c={\sqrt

and

\varphi =\operatorname

### More than two sinusoids

The general case reads[37]

$$\sum _{i}a_{i}\sin(x+\theta _{i})=a\sin(x+\theta ),$$

where

a^{2}=\sum _{i,j}a_{i}a_{j}\cos(\thet

and

\tan \theta ={\frac {\sum _{i}a_{i}\sin \theta _{i}}{\sum

See also Phasor addition.

## Lagrange's trigonometric identities

These identities, named after Joseph Louis Lagrange, are:[38][39]

{\displaystyle {\begin{aligned}\sum _{n=1}^{N}\sin(n\theta )&={\frac {1}{2}}\cot {\frac {\theta }{2}}-{\frac {\cos \left(\left(N+{\frac {1}{2}}\right)\theta \right)}{2\sin \left({\frac {\theta }{2}}\right)}}\\\sum _{n=1}^{N}\cos(n\theta )&=-{\frac {1}{2}}+{\frac {\sin

A related function is the following function of $x$, called the Dirichlet kernel.

$${\displaystyle 1+2\cos x+2\cos(2x)+2\cos(3x)+\cdots +2\cos(nx)={\frac {\sin \left(\left(n+{\frac {1}{2}}\right)x\right)}{\sin \left({\frac {x}{2}}\right)}}.}$$

see proof.

# Other sums of trigonometric functions

Sum of sines and cosines with arguments in arithmetic progression:[40] if $\alpha \neq 0$, then

$${\displaystyle {\begin{aligned}&\sin \varphi +\sin(\varphi +\alpha )+\sin(\varphi +2\alpha )+\cdots \\[8pt]&{}\qquad \qquad \cdots +\sin(\varphi +n\alpha )={\frac {\sin {\frac {(n+1)\alpha }{2}}\cdot \sin \left(\varphi +{\frac {n\alpha }{2}}\right)}{\sin {\frac {\alpha }{2}}}}\quad {\text{and}}\\[10pt]&\cos \varphi +\cos(\varphi +\alpha )+\cos(\varphi +2\alpha )+\cdots \\[8pt]&{}\qquad \qquad \cdots +\cos(\varphi +n\alpha )={\frac {\sin {\frac {(n+1)\alpha }{2}}\cdot \cos \left(\varphi +{\frac {n\alpha }{2}}\right)}{\sin {\frac {\alpha }{2}}}}.\end{aligned}}}$$

For any $a$ and $b$:

{\displaystyle a\cos x+b\sin x={\sqrt

where $\operatorname{atan2}(y, x)$ is the generalization of $\arctan(\frac{y}{x})$ that covers the entire circular range.

\sec x\pm \tan x=\tan \left({\frac {\pi }{4}}\pm

The above identity is sometimes convenient to know when thinking about the Gudermannian function, which relates the circular and hyperbolic trigonometric functions without resorting to complex numbers.

If $x$, $y$, and $z$ are the three angles of any triangle, i.e. if $x + y + z = \pi$, then

{\displaystyle \cot x\cot y+\cot y\cot

# Certain linear fractional transformations

If $f(x)$ is given by the linear fractional transformation

f(x)={\frac {(\cos \alpha )x-\sin \alpha }{(\sin \alpha )x+\cos \alpha }}

and similarly

g(x)={\frac {(\cos \beta )x-\sin \beta }{(\sin \beta )x+\cos \beta }}

then

{\displaystyle f{\big (}g(x){\big )}=g{\big (}f(x){\big )}={\frac {{\big (}\cos(\alpha +\beta ){\big )}x-\sin(\alpha +\beta )}{{\big (}\sin(\alpha +\beta ){\big

More tersely stated, if for all $\alpha$ we let $f_\alpha$ be what we called $f$ above, then

$$f_\alpha \circ f_\beta = f_{\alpha+\beta}.$$

If $x$ is the slope of a line, then $f(x)$ is the slope of its rotation through an angle of $-\alpha$.

# Inverse trigonometric functions

$$\arcsin x + \arccos x = \frac{\pi}{2}$$

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$$

$$\arctan x + \arctan \frac{1}{x} = \begin{cases} \frac{\pi}{2}, & \text{if } x > 0 \\ -\frac{\pi}{2}, & \text{if } x < 0 \end{cases}$$

$$\arctan \frac{1}{x} = \arctan \frac{1}{x+y} + \arctan \frac{y}{x^2 + xy + 1}$$ [41]

## Compositions of trig and inverse trig functions

$$\begin{aligned} \sin(\arccos x) &= \sqrt{1-x^2} & \tan(\arcsin x) &= \frac{x}{\sqrt{1-x^2}} \\ \sin(\arctan x) &= \frac{x}{\sqrt{1+x^2}} & \tan(\arccos x) &= \frac{\sqrt{1-x^2}}{x} \\ \cos(\arctan x) &= \frac{1}{\sqrt{1+x^2}} & \cot(\arcsin x) &= \frac{\sqrt{1-x^2}}{x} \\ \cos(\arcsin x) &= \sqrt{1-x^2} & \cot(\arccos x) &= \frac{x}{\sqrt{1-x^2}} \end{aligned}$$

## Relation to the complex exponential function

With the unit imaginary number $i$ satisfying $i^2 = -1$,

$$e^{ix} = \cos x + i \sin x$$ [42] (Euler's formula),

$$e^{-ix} = \cos(-x) + i \sin(-x) = \cos x - i \sin x$$

$$e^{i\pi} = -1$$ (Euler's identity),

$$e^{2\pi i} = 1$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$ [43]

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$ [44]

$$\tan x = \frac{\sin x}{\cos x} = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}.$$

These formulae are useful for proving many other trigonometric identities. For example, that $e^{i(\theta+\varphi)} = e^{i\theta}\, e^{i\varphi}$ means that

$$\cos(\theta+\varphi) + i \sin(\theta+\varphi) = (\cos\theta + i \sin\theta)(\cos\varphi + i \sin\varphi) = (\cos\theta \cos\varphi - \sin\theta \sin\varphi) + i (\cos\theta \sin\varphi + \sin\theta \cos\varphi).$$

That the real part of the left hand side equals the real part of the right hand side is an angle addition formula for cosine. The equality of the imaginary parts gives an angle addition formula for sine.

## Infinite product formulae

For applications to special functions, the following infinite product formulae for trigonometric functions are useful:[45][46]

$$\begin{aligned} \sin x &= x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2}\right) & \cos x &= \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 \left(n - \frac{1}{2}\right)^2}\right) \\ \sinh x &= x \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{\pi^2 n^2}\right) & \cosh x &= \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{\pi^2 \left(n - \frac{1}{2}\right)^2}\right) \\ \frac{\sin x}{x} &= \prod_{n=1}^{\infty} \cos \frac{x}{2^n} & |\sin x| &= \frac{1}{2} \prod_{n=0}^{\infty} \sqrt[2^{n+1}]{\left|\tan(2^n x)\right|} \end{aligned}$$

## Identities without variables

In terms of the arctangent function we have[41]

$$\arctan \frac{1}{2} = \arctan \frac{1}{3} + \arctan \frac{1}{7}.$$

The curious identity known as Morrie's law,

$$\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ = \frac{1}{8},$$

is a special case of an identity that contains one variable:

$$\prod_{j=0}^{k-1} \cos(2^j x) = \frac{\sin(2^k x)}{2^k \sin x}.$$

The same cosine identity in radians is

$$\cos \frac{\pi}{9} \cos \frac{2\pi}{9} \cos \frac{4\pi}{9} = \frac{1}{8}.$$

Similarly,

$$\sin 20^\circ \cdot \sin 40^\circ \cdot \sin 80^\circ = \frac{\sqrt{3}}{8}$$

is a special case of an identity with the case x = 20:

$$\sin x \cdot \sin(60^\circ - x) \cdot \sin(60^\circ + x) = \frac{\sin 3x}{4}.$$

For the case $x$ = 15,

$$\sin 15^\circ \cdot \sin 45^\circ \cdot \sin 75^\circ = \frac{\sqrt{2}}{8},$$

$$\sin 15^\circ \cdot \sin 75^\circ = \frac{1}{4}.$$

For the case $x$ = 10,

$$\sin 10^\circ \cdot \sin 50^\circ \cdot \sin 70^\circ = \frac{1}{8}.$$

The same cosine identity is

$$\cos x \cdot \cos(60^\circ - x) \cdot \cos(60^\circ + x) = \frac{\cos 3x}{4}.$$

Similary,

$$\cos 10^\circ \cdot \cos 50^\circ \cdot \cos 70^\circ = \frac{\sqrt{3}}{8},$$

$$\cos 15^\circ \cdot \cos 45^\circ \cdot \cos 75^\circ = \frac{\sqrt{2}}{8},$$
$$\cos 15^\circ \cdot \cos 75^\circ = \frac{1}{4}.$$

Similarly,

$$\tan 50^\circ \cdot \tan 60^\circ \cdot \tan 70^\circ = \tan 80^\circ,$$

$$\tan 40^\circ \cdot \tan 30^\circ \cdot \tan 20^\circ = \tan 10^\circ.$$

The following is perhaps not as readily generalized to an identity containing variables (but see explanation below):

$$\cos 24^\circ + \cos 48^\circ + \cos 96^\circ + \cos 168^\circ = \frac{1}{2}.$$

Degree measure ceases to be more felicitous than radian measure when we consider this identity with 21 in the denominators:

$$\cos\frac{2\pi}{21} + \cos\left(2\cdot\frac{2\pi}{21}\right) + \cos\left(4\cdot\frac{2\pi}{21}\right) \\ + \cos\left(5\cdot\frac{2\pi}{21}\right) + \cos\left(8\cdot\frac{2\pi}{21}\right) + \cos\left(10\cdot\frac{2\pi}{21}\right) = \frac{1}{2}.$$

The factors 1, 2, 4, 5, 8, 10 may start to make the pattern clear: they are those integers less than $\frac{21}{2}$ that are relatively prime to (or have no prime factors in common with) 21. The last several examples are corollaries of a basic fact about the irreducible cyclotomic polynomials the cosines are the real parts of the zeroes of those polynomials; the sum of the zeroes is the Möbius function evaluated at (in the very last case above) 21; only half of the zeroes are present above. The two identities preceding this last one arise in the same fashion with 21 replaced by 10 and 15, respectively.

Other cosine identities include:[47]

$$2\cos\frac{\pi}{3} = 1,$$
$$2\cos\frac{\pi}{5} \times 2\cos\frac{2\pi}{5} = 1,$$
$$2\cos\frac{\pi}{7} \times 2\cos\frac{2\pi}{7} \times 2\cos\frac{3\pi}{7} = 1,$$

and so forth for all odd numbers, and hence

$$\cos\frac{\pi}{3} + \cos\frac{\pi}{5} \times \cos\frac{2\pi}{5} + \cos\frac{\pi}{7} \times \cos\frac{2\pi}{7} \times \cos\frac{3\pi}{7} + \cdots = 1.$$

Many of those curious identities stem from more general facts like the following:[48]

$$\prod_{k=1}^{n-1} \sin\frac{k\pi}{n} = \frac{n}{2^{n-1}}$$

and

$$\prod_{k=1}^{n-1} \cos\frac{k\pi}{n} = \frac{\sin\frac{\pi n}{2}}{2^{n-1}}$$

Combining these gives us

$$\prod_{k=1}^{n-1} \tan \frac{k\pi}{n} = \frac{n}{\sin \frac{\pi n}{2}}$$

If $n$ is an odd number ($n = 2m + 1$) we can make use of the symmetries to get

$$\prod_{k=1}^{m} \tan \frac{k\pi}{2m+1} = \sqrt{2m+1}$$

The transfer function of the Butterworth low pass filter can be expressed in terms of polynomial and poles. By setting the frequency as the cutoff frequency, the following identity can be proved:

$$\prod_{k=1}^{n} \sin \frac{(2k-1)\pi}{4n} = \prod_{k=1}^{n} \cos \frac{(2k-1)\pi}{4n} = \frac{\sqrt{2}}{2^n}$$

## Computing π

An efficient way to compute π is based on the following identity without variables, due to Machin:

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$$

or, alternatively, by using an identity of Leonhard Euler:

$$\frac{\pi}{4} = 5 \arctan \frac{1}{7} + 2 \arctan \frac{3}{79}$$

or by using Pythagorean triples:

$$\pi = \arccos \frac{4}{5} + \arccos \frac{5}{13} + \arccos \frac{16}{65} = \arcsin \frac{3}{5} + \arcsin \frac{12}{13} + \arcsin \frac{63}{65}.$$

Others include

$$\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3};$$ [49][41]

$$\pi = \arctan 1 + \arctan 2 + \arctan 3.$$ [49]

$$\frac{\pi}{4} = 2 \arctan \frac{1}{3} + \arctan \frac{1}{7}.$$ [41]

Generally, for numbers $t_1, \ldots, t_{n-1} \in (-1, 1)$ for which $\theta_n = \sum_{k=1}^{n-1} \arctan t_k \in (\pi/4, 3\pi/4)$, let $t_n = \tan(\pi/2 - \theta_n) = \cot \theta_n$. This last expression can be computed directly using the formula for the cotangent of a sum of angles whose tangents are $t_1, \ldots, t_{n-1}$ and its value will be in $(-1, 1)$. In particular, the computed $t_n$ will be rational whenever all the $t_1, \ldots, t_{n-1}$ values are rational. With these values,

$$\begin{aligned}
\frac{\pi}{2} &= \sum_{k=1}^{n} \arctan(t_k) \\
\pi &= \sum_{k=1}^{n} \operatorname{sign}(t_k) \arccos\left(\frac{1-t_k^2}{1+t_k^2}\right) \\
\pi &= \sum_{k=1}^{n} \arcsin\left(\frac{2t_k}{1+t_k^2}\right) \\
\pi &= \sum_{k=1}^{n} \arctan\left(\frac{2t_k}{1-t_k^2}\right),
\end{aligned}$$

where in all but the first expression, we have used tangent half-angle formulae. The first two formulae work even if one or more of the $t_k$ values is not within $(-1, 1)$. Note that when $t = p/q$ is rational then the $(2t, 1 - t^2, 1 + t^2)$ values in the above formulae are proportional to the Pythagorean triple $(2pq, q^2 - p^2, q^2 + p^2)$.

For example for $n = 3$ terms,

$$\frac{\pi}{2} = \arctan\left(\frac{a}{b}\right) + \arctan\left(\frac{c}{d}\right) + \arctan\left(\frac{bd-ac}{ad+bc}\right)$$

for any $a, b, c, d > 0$.

## A useful mnemonic for certain values of sines and cosines

For certain simple angles, the sines and cosines take the form $\frac{\sqrt{n}}{2}$ for $0 \le n \le 4$, which makes them easy to remember

$$\begin{array}{ccccccccc}
\sin(0) & = & \sin(0^\circ) & = & \dfrac{\sqrt{0}}{2} & = & \cos(90^\circ) & = & \cos\left(\dfrac{\pi}{2}\right) \\
\sin\left(\dfrac{\pi}{6}\right) & = & \sin(30^\circ) & = & \dfrac{\sqrt{1}}{2} & = & \cos(60^\circ) & = & \cos\left(\dfrac{\pi}{3}\right) \\
\sin\left(\dfrac{\pi}{4}\right) & = & \sin(45^\circ) & = & \dfrac{\sqrt{2}}{2} & = & \cos(45^\circ) & = & \cos\left(\dfrac{\pi}{4}\right) \\
\sin\left(\dfrac{\pi}{3}\right) & = & \sin(60^\circ) & = & \dfrac{\sqrt{3}}{2} & = & \cos(30^\circ) & = & \cos\left(\dfrac{\pi}{6}\right) \\
\sin\left(\dfrac{\pi}{2}\right) & = & \sin(90^\circ) & = & \dfrac{\sqrt{4}}{2} & = & \cos(0^\circ) & = & \cos(0) \\
 & & & & \uparrow & & & & \\
 & & & & \text{These} & & & & \\
 & & & & \text{radicands} & & & & \\
 & & & & \text{are} & & & & \\
 & & & & 0,\,1,\,2,\,3,\,4. & & & &
\end{array}$$

### Miscellany

With the golden ratio $\varphi$:

$$\cos\frac{\pi}{5} = \cos 36^\circ = \frac{\sqrt{5}+1}{4} = \frac{\varphi}{2}$$

$$\sin\frac{\pi}{10} = \sin 18^\circ = \frac{\sqrt{5}-1}{4} = \frac{\varphi^{-1}}{2} = \frac{1}{2\varphi}$$

Also see trigonometric constants expressed in real radicals.

### An identity of Euclid

Euclid showed in Book XIII, Proposition 10 of his *Elements* that the area of the square on the side of a regular pentagon inscribed in a circle is equal to the sum of the areas of the squares on the sides of the regular hexagon and the regular decagon inscribed in the same circle. In the language of modern trigonometry, this says:

$$\sin^2 18^\circ + \sin^2 30^\circ = \sin^2 36^\circ.$$

Ptolemy used this proposition to compute some angles in his table of chords.

## Composition of trigonometric functions

This identity involves a trigonometric function of a trigonometric function:[50]

$$\cos(t\sin x) = J_0(t) + 2\sum_{k=1}^{\infty} J_{2k}(t)\cos(2kx)$$

$$\sin(t\sin x) = 2\sum_{k=0}^{\infty} J_{2k+1}(t)\sin\big((2k+1)x\big)$$

$$\cos(t\cos x) = J_0(t) + 2\sum_{k=1}^{\infty}(-1)^k J_{2k}(t)\cos(2kx)$$

$$\sin(t\cos x) = 2\sum_{k=0}^{\infty}(-1)^k J_{2k+1}(t)\cos\big((2k+1)x\big)$$

where $J_i$ are Bessel functions.

## Calculus

In calculus the relations stated below require angles to be measured in radians; the relations would become more complicated if angles were measured in another unit such as degrees. If the trigonometric functions are defined in terms of geometry, along with the definitions of arc length and area, their derivatives can be found by verifying two limits. The first is:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1,$$

verified using the unit circle and squeeze theorem. The second limit is:

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0,$$

verified using the identity $\tan\frac{x}{2} = \frac{1-\cos x}{\sin x}$. Having established these two limits, one can use the limit definition of the derivative and the addition theorems to show that $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$. If the sine and cosine functions are defined by their Taylor series, then the derivatives can be found by differentiating the power series term-by-term.

$$\frac{d}{dx}\sin x = \cos x$$

The rest of the trigonometric functions can be differentiated using the above identities and the rules of differentiation:[51][52][53]

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\cot x = -\csc^2 x, \qquad \frac{d}{dx}\operatorname{arccot} x = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\sec x = \tan x \sec x, \qquad \frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\csc x = -\csc x \cot x, \qquad \frac{d}{dx}\operatorname{arccsc} x = \frac{-1}{|x|\sqrt{x^2-1}}$$

The integral identities can be found in List of integrals of trigonometric functions. Some generic forms are listed below.

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

### Implications

The fact that the differentiation of trigonometric functions (sine and cosine) results in linear combinations of the same two functions is of fundamental importance to many fields of mathematics, including differential equations and Fourier transforms.

### Some differential equations satisfied by the sine function

Let $i = \sqrt{-1}$ be the imaginary unit and let ∘ denote composition of differential operators. Then for every **odd** positive integer $n$,

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{d}{dx} - \sin x\right) \circ \left(\frac{d}{dx} - \sin x + i\right) \circ \cdots$$
$$\cdots \circ \left(\frac{d}{dx} - \sin x + (k-1)i\right) (\sin x)^{n-k} = 0.$$

(When $k = 0$, then the number of differential operators being composed is 0, so the corresponding term in the sum above is just $(\sin x)^n$.) This identity was discovered as a by-product of research in medical imaging.[54]

# Exponential definitions

| Function | Inverse function[55] |
|---|---|
| $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ | $\arcsin x = -i\ln\left(ix + \sqrt{1-x^2}\right)$ |
| $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ | $\arccos x = -i\ln\left(x + \sqrt{x^2-1}\right)$ |
| $\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i\left(e^{i\theta} + e^{-i\theta}\right)}$ | $\arctan x = \frac{i}{2}\ln\left(\frac{i+x}{i-x}\right)$ |
| $\csc\theta = \frac{2i}{e^{i\theta} - e^{-i\theta}}$ | $\operatorname{arccsc} x = -i\ln\left(\frac{i}{x} + \sqrt{1-\frac{1}{x^2}}\right)$ |
| $\sec\theta = \frac{2}{e^{i\theta} + e^{-i\theta}}$ | $\operatorname{arcsec} x = -i\ln\left(\frac{1}{x} + i\sqrt{1-\frac{1}{x^2}}\right)$ |
| $\cot\theta = \frac{i\left(e^{i\theta} + e^{-i\theta}\right)}{e^{i\theta} - e^{-i\theta}}$ | $\operatorname{arccot} x = \frac{i}{2}\ln\left(\frac{x-i}{x+i}\right)$ |
| | |
| $\operatorname{cis}\theta = e^{i\theta}$ | $\operatorname{arccis} x = \frac{\ln x}{i} = -i\ln x = \arg x$ |

# Further formulae for the case $\alpha + \beta + \gamma = 180°$

The following formulae apply to arbitrary plane triangles and follow from $\alpha + \beta + \gamma = 180°$, as long as the functions occurring in the formulae are well-defined (the latter applies only to the formulae in which tangents and cotangents occur).

$$\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\cdot\tan\beta\cdot\tan\gamma$$

$$\cot\beta\cdot\cot\gamma+\cot\gamma\cdot\cot\alpha+\cot\alpha\cdot\cot\beta=1$$

$$\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}=\cot\frac{\alpha}{2}\cdot\cot\frac{\beta}{2}\cdot\cot\frac{\gamma}{2}$$

$$\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=1$$

$$\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$-\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$$

$$\cos\alpha+\cos\beta+\cos\gamma=4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}+1$$

$$-\cos\alpha+\cos\beta+\cos\gamma=4\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}-1$$

$$\sin(2\alpha)+\sin(2\beta)+\sin(2\gamma)=4\sin\alpha\sin\beta\sin\gamma$$

$$-\sin(2\alpha)+\sin(2\beta)+\sin(2\gamma)=4\sin\alpha\cos\beta\cos\gamma$$

$$\cos(2\alpha)+\cos(2\beta)+\cos(2\gamma)=-4\cos\alpha\cos\beta\cos\gamma-1$$

$$-\cos(2\alpha)+\cos(2\beta)+\cos(2\gamma)=-4\cos\alpha\sin\beta\sin\gamma+1$$

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2\cos\alpha\cos\beta\cos\gamma+2$$

$$-\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2\cos\alpha\sin\beta\sin\gamma$$

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=-2\cos\alpha\cos\beta\cos\gamma+1$$

$$-\cos^2\alpha+\cos^2\beta+\cos^2\gamma=-2\cos\alpha\sin\beta\sin\gamma+1$$

$$-\sin^2(2\alpha)+\sin^2(2\beta)+\sin^2(2\gamma)=-2\cos(2\alpha)\sin(2\beta)\sin(2\gamma)$$

$$-\cos^2(2\alpha)+\cos^2(2\beta)+\cos^2(2\gamma)=2\cos(2\alpha)\,\sin(2\beta)\,\sin(2\gamma)+1$$

$$\sin^2\left(\frac{\alpha}{2}\right)+\sin^2\left(\frac{\beta}{2}\right)+\sin^2\left(\frac{\gamma}{2}\right)+2\sin\left(\frac{\alpha}{2}\right)\,\sin\left(\frac{\beta}{2}\right)\,\sin\left(\frac{\gamma}{2}\right)=1$$

# Miscellaneous

## Dirichlet kernel

The **Dirichlet kernel** $D_n(x)$ is the function occurring on both sides of the next identity:

$$1+2\cos x+2\cos(2x)+2\cos(3x)+\cdots+2\cos(nx)=\frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}.$$

The convolution of any integrable function of period 2π with the Dirichlet kernel coincides with the function's $n$th-degree Fourier approximation. The same holds for any measure or generalized function.

## Tangent half-angle substitution

If we set

$$t=\tan\frac{x}{2},$$

then[56]

$$\sin x=\frac{2t}{1+t^2};\qquad \cos x=\frac{1-t^2}{1+t^2};\qquad e^{ix}=\frac{1+it}{1-it}$$

where $e^{ix}=\cos x+i\sin x$, sometimes abbreviated to $\operatorname{cis} x$.

When this substitution of $t$ for $\tan\frac{x}{2}$ is used in calculus, it follows that $\sin x$ is replaced by $\frac{2t}{1+t^2}$, $\cos x$ is replaced by $\frac{1-t^2}{1+t^2}$ and the differential $dx$ is replaced by $\frac{2\,dt}{1+t^2}$. Thereby one converts rational functions of $\sin x$ and $\cos x$ to rational functions of $t$ in order to find their antiderivatives.

# See also

- Derivatives of trigonometric functions
- Exact trigonometric constants (values of sine and cosine expressed in surds)
- Exsecant
- Half-side formula
- Hyperbolic function

- Laws for solution of triangles:
  - Law of cosines
    - Spherical law of cosines
  - Law of sines
  - Law of tangents
  - Law of cotangents
  - Mollweide's formula
- List of integrals of trigonometric functions
- Proofs of trigonometric identities
- Prosthaphaeresis
- Pythagorean theorem
- Tangent half-angle formula
- Trigonometry
- Trigonometric constants expressed in real radicals
- Uses of trigonometry
- Versine and haversine
- Mnemonics in trigonometry

# Notes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# External links

- Construction proof for sine and cosine of the sum of two angles
- Values of sin and cos, expressed in surds, for integer multiples of 3° and of $5\frac{5}{8}$°, and for the same angles csc and sec and tan