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Noetherian ring

In mathematics, more specifically in the area of abstract algebra known as ring theory, a **Noetherian ring** is a ring that satisfies the ascending chain condition on left and right ideals; that is, given any chain of left (or right) ideals:

$$I_1 \subseteq \cdots \subseteq I_{k-1} \subseteq I_k \subseteq I_{k+1} \subseteq \cdots$$

there exists an n such that:

$$I_n = I_{n+1} = \cdots .$$

Noetherian rings are named after Emmy Noether.

The notion of a Noetherian ring is of fundamental importance in both commutative and noncommutative ring theory, due to the role it plays in simplifying the ideal structure of a ring. For instance, the ring of integers and the polynomial ring over a field are both Noetherian rings, and consequently, such theorems as the Lasker–Noether theorem, the Krull intersection theorem, and the Hilbert's basis theorem hold for them. Furthermore, if a ring is Noetherian, then it satisfies the descending chain condition on *prime ideals*. This property suggests a deep theory of dimension for Noetherian rings beginning with the notion of the Krull dimension.

Contents

Characterizations

For noncommutative rings, it is necessary to distinguish between three very similar concepts:

- A ring is **left-Noetherian** if it satisfies the ascending chain condition on left ideals.
- A ring is **right-Noetherian** if it satisfies the ascending chain condition on right ideals.
- A ring is **Noetherian** if it is both left- and right-Noetherian.

For commutative rings, all three concepts coincide, but in general they are different. There are rings that are left-Noetherian and not right-Noetherian, and vice versa.

There are other, equivalent, definitions for a ring R to be left-Noetherian:

- Every left ideal I in R is finitely generated, i.e. there exist elements $a_1, \ldots, a_n$ in I such that $I = Ra_1 + \ldots + Ra_n$.[1]
- Every non-empty set of left ideals of R, partially ordered by inclusion, has a maximal element with respect to set inclusion.[1]

Similar results hold for right-Noetherian rings.

For a commutative ring to be Noetherian it suffices that every prime ideal of the ring is finitely generated.[2]

Properties

- Any commutative principal ideal ring is Noetherian, since every ideal of such a ring is generated by a single element. In particular, every principal ideal domain and every Euclidean domain is Noetherian.
- $\mathbf{Z}$ is a Noetherian ring, a fact which is exploited in the usual proof that every non-unit integer is divisible by at least one prime, although it's usually stated as "every non-empty set of integers has a *minimal* element with respect to divisibility".
- If R is a Noetherian ring, then $R[X]$ is Noetherian by the Hilbert basis theorem. By induction, $R[X_1, ..., X_n]$ is a Noetherian ring. Also, $R[[X]]$, the power series ring is a Noetherian ring.
- If R is a Noetherian ring and I is a two-sided ideal, then the factor ring R/I is also Noetherian. Stated differently, the image of any surjective ring homomorphism of a Noetherian ring is Noetherian.
- Every finitely-generated commutative algebra over a commutative Noetherian ring is Noetherian. (This follows from the two previous properties.)
- A ring R is left-Noetherian if and only if every finitely generated left R-module is a Noetherian module.
- Every localization of a commutative Noetherian ring is Noetherian.
- A consequence of the Akizuki-Hopkins-Levitzki Theorem is that every left Artinian ring is left Noetherian. Another consequence is that a left Artinian ring is right Noetherian if and only if right Artinian. The analogous statements with "right" and "left" interchanged are also true.
- A left Noetherian ring is left coherent and a left Noetherian domain is a left Ore domain.
- A ring is (left/right) Noetherian if and only if every direct sum of injective (left/right) modules is injective. Every injective module can be decomposed as direct sum of indecomposable injective modules.
- In a commutative Noetherian ring, there are only finitely many minimal prime ideals.
- In a commutative Noetherian domain R, every element can be factorized into irreducible elements. Thus, if, in addition, irreducible elements are prime elements, then R is a unique factorization domain.

Examples

- Any field, including fields of rational numbers, real numbers, and complex numbers, is Noetherian. (A field only has two ideals — itself and (0).)
- Any principal ideal domain, such as the integers, is Noetherian since every ideal is generated by a single element.
- A Dedekind domain (e.g., rings of integers) is Noetherian since every ideal is generated by at most two elements. The "Noetherian" follows from the Krull–Akizuki theorem. The bounds on the number of the generators is a corollary of the Forster–Swan theorem (or basic ring theory).
- The coordinate ring of an affine variety is a Noetherian ring, as a consequence of the Hilbert basis theorem.
- The enveloping algebra U of a finite-dimensional Lie algebra $\mathfrak{g}$ is a both left and right noetherian ring; this follows from the fact that the associated graded ring of U is a quotient of $\operatorname{Sym}(\mathfrak{g})$, which is a polynomial ring over a field; thus, noetherian.[3] For the same reason, the Weyl algebra, and more general rings of differential operators, are Noetherian.[4]
- The ring of polynomials in finitely-many variables over the integers or a field.

Rings that are not Noetherian tend to be (in some sense) very large. Here are some examples of non-Noetherian rings:

- The ring of polynomials in infinitely-many variables, X_1, X_2, X_3, etc. The sequence of ideals (X_1), (X_1, X_2), (X_1, X_2, X_3), etc. is ascending, and does not terminate.
- The ring of algebraic integers is not Noetherian. For example, it contains the infinite ascending chain of principal ideals: (2), $(2^{1/2})$, $(2^{1/4})$, $(2^{1/8})$, ...
- The ring of continuous functions from the real numbers to the real numbers is not Noetherian. Let I_n be the ideal of all continuous functions f such that $f(x) = 0$ for all $x \geq n$. The sequence of ideals I_0, I_1, I_2, etc., is an ascending chain that does not terminate.
- The ring of stable homotopy groups of spheres is not Noetherian.[5]

However, a non-Noetherian ring can be a subring of a Noetherian ring. Since any integral domain is a subring of a field, any integral domain that is not Noetherian provides an example. To give a less trivial example,

- The ring of rational functions generated by x and y/x^n over a field k is a subring of the field $k(x,y)$ in only two variables.

Indeed, there are rings that are right Noetherian, but not left Noetherian, so that one must be careful in measuring the "size" of a ring this way. For example, if L is a subgroup of $\mathbf{Q}^2$ isomorphic to $\mathbf{Z}$, let R be the ring of homomorphisms f from $\mathbf{Q}^2$ to itself satisfying $f(L) \subset L$. Choosing a basis, we can describe the same ring R as

$$R = \left\{ \begin{bmatrix} a & \beta \\ 0 & \gamma \end{bmatrix} \middle| \, a \in \mathbb{Z}, \beta \in \mathbb{Q}, \gamma \in \mathbb{Q} \right\}.$$

This ring is right Noetherian, but not left Noetherian; the subset $I \subset R$ consisting of elements with $a=0$ and $\gamma=0$ is a left ideal that is not finitely generated as a left R-module.

If R is a commutative subring of a left Noetherian ring S, and S is finitely generated as a left R-module, then R is Noetherian.[6] (In the special case when S is commutative, this is known as Eakin's theorem.) However this is not true if R is not commutative: the ring R of the previous paragraph is a subring of the left Noetherian ring $S = \mathrm{Hom}(\mathbf{Q}^2, \mathbf{Q}^2)$, and S is finitely generated as a left R-module, but R is not left Noetherian.

A unique factorization domain is not necessarily a noetherian ring. It does satisfy a weaker condition: the ascending chain condition on principal ideals.

A valuation ring is not Noetherian unless it is a principal ideal domain. It gives an example of a ring that arises naturally in algebraic geometry but is not Noetherian.

Primary decomposition

In the ring $\mathbf{Z}$ of integers, an arbitrary ideal is of the form (n) for some integer n (where (n) denotes the set of all integer multiples of n). If n is non-zero, and is neither 1 nor −1, by the fundamental theorem of arithmetic, there exist primes p_i, and positive integers e_i, with $n = \prod_i {p_i}^{e_i}$. In this case, the ideal (n) may be written as the intersection of the ideals $(p_i^{e_i})$; that is, $(n) = \cap_i ({p_i}^{e_i})$. This is referred to as a *primary decomposition* of the ideal (n).

In general, an ideal Q of a ring is said to be *primary* if Q is proper and whenever $xy \in Q$, either $x \in Q$ or $y^n \in Q$ for some positive integer n. In $\mathbf{Z}$, the primary ideals are precisely the ideals of the form (p^e) where p is prime and e is a positive integer. Thus, a primary decomposition of (n) corresponds to representing (n) as the intersection of finitely many primary ideals.

Since the fundamental theorem of arithmetic applied to a non-zero integer n that is neither 1 nor −1 also asserts uniqueness of the representation $n = \prod_i {p_i}^{e_i}$ for p_i prime and e_i positive, a primary decomposition of (n) is essentially *unique*.

For all of the above reasons, the following theorem, referred to as the *Lasker–Noether theorem*, may be seen as a certain generalization of the fundamental theorem of arithmetic:

> **Lasker-Noether Theorem.** Let R be a commutative Noetherian ring and let I be an ideal of R. Then I may be written as the intersection of finitely many primary ideals with distinct radicals; that is:
>
> $$I = \bigcap_{i=1}^{t} Q_i$$
>
> with Q_i primary for all i and $\mathrm{Rad}(Q_i) \neq \mathrm{Rad}(Q_j)$ for $i \neq j$. Furthermore, if:
>
> $$I = \bigcap_{i=1}^{k} P_i$$

> is decomposition of I with $\mathrm{Rad}(P_i) \neq \mathrm{Rad}(P_j)$ for $i \neq j$, and both decompositions of I are *irredundant* (meaning that no proper subset of either $\{Q_1, ..., Q_t\}$ or $\{P_1, ..., P_k\}$ yields an intersection equal to I), $t = k$ and (after possibly renumbering the Q_i) $\mathrm{Rad}(Q_i) = \mathrm{Rad}(P_i)$ for all i.

For any primary decomposition of I, the set of all radicals, that is, the set $\{\mathrm{Rad}(Q_1), ..., \mathrm{Rad}(Q_t)\}$ remains the same by the Lasker–Noether theorem. In fact, it turns out that (for a Noetherian ring) the set is precisely the assassinator of the module R/I; that is, the set of all annihilators of R/I (viewed as a module over R) that are prime.

See also

- Krull–Akizuki theorem
- Noetherian scheme
- Artinian ring
- Artin–Rees lemma
- Krull's principal ideal theorem

References

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4. Hotta, Takeuchi & Tanisaki (2008, §D.1, Proposition 1.4.6)
5. http://math.stackexchange.com/questions/1513353/the-ring-of-stable-homotopy-groups-of-spheres-is-not-noetherian
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