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Maximal ideal

In mathematics, more specifically in ring theory, a **maximal ideal** is an ideal that is maximal (with respect to set inclusion) amongst all *proper* ideals.[1][2] In other words, *I* is a maximal ideal of a ring *R* if there are no other ideals contained between *I* and *R*.

Maximal ideals are important because the quotient rings of maximal ideals are simple rings, and in the special case of unital commutative rings they are also fields.

In noncommutative ring theory a **maximal right ideal** is defined analogously as being a maximal element in the poset of proper right ideals, and similarly, a **maximal left ideal** is defined to be a maximal element of the poset of proper left ideals. Since a one sided maximal ideal *A* is not necessarily two-sided, the quotient *R*/*A* is not necessarily a ring, but it is a simple module over *R*. If *R* has a unique maximal right ideal, then *R* is known as a local ring, and the maximal right ideal is also the unique maximal left and unique maximal two-sided ideal of the ring, and is in fact the Jacobson radical J(*R*).

It is possible for a ring to have a unique maximal ideal and yet lack unique maximal one sided ideals: for example, in the ring of 2 by 2 square matrices over a field, the zero ideal is a maximal ideal, but there are many maximal right ideals.

Contents

Definition

There are other equivalent ways of expressing the definition of maximal one-sided and maximal two-sided ideals. Given a ring *R* and a proper ideal *I* of *R* (that is $I \neq R$), *I* is a maximal ideal of *R* if any of the following equivalent conditions hold:

- There exists no other proper ideal *J* of *R* so that $I \subsetneq J$.
- For any ideal *J* with $I \subseteq J$, either $J = I$ or $J = R$.
- The quotient ring *R*/*I* is a simple ring.

There is an analogous list for one-sided ideals, for which only the right-hand versions will be given. For a right ideal *A* of a ring *R*, the following conditions are equivalent to *A* being a maximal right ideal of *R*:

- There exists no other proper right ideal *B* of *R* so that $A \subsetneq B$.
- For any right ideal *B* with $A \subseteq B$, either $B = A$ or $B = R$.
- The quotient module *R*/*A* is a simple right *R* module.

Maximal right/left/two-sided ideals are the dual notion to that of minimal ideals.

Examples

- If **F** is a field, then the only maximal ideal is {0}.
- The maximal ideals of the polynomial ring $\mathbb{C}[x]$ are principal ideals generated by $x - c$ for some c\i.
- In the ring **Z** of integers, the maximal ideals are the principal ideals generated by a prime number

- More generally, all nonzero prime ideals are maximal in a principal ideal domain.
- The maximal ideals of the polynomial ring $K[x_1,...,x_n]$ over an algebraically closed field K are the ideals of the form $(x_1 - a_1,...,x_n - a_n)$. This result is known as the weak nullstellensatz.

Properties

- An important ideal of the ring called the Jacobson radical can be defined using maximal right (or maximal left) ideals.
- If R is a unital commutative ring with an ideal m, then $k = R/m$ is a field if and only if m is a maximal ideal. In that case, R/m is known as the residue field. This fact can fail in non-unital rings. For example, is a maximal ideal in , but 2{\r is not a field.
- If L is a maximal left ideal, then R/L is a simple left R module. Conversely in rings with unity, any simple left R module arises this way. Incidentally this shows that a collection of representatives of simple left R modules is actually a set since it can be put into correspondence with part of the set of maximal left ideals of R.
- **Krull's theorem** (1929): Every nonzero unital ring has a maximal ideal. The result is also true if "ideal" is replaced with "right ideal" or "left ideal". More generally, it is true that every nonzero finitely generated module has a maximal submodule. Suppose I is an ideal which is not R (respectively, A is a right ideal which is not R). Then R/I is a ring with unity, (respectively, R/A is a finitely generated module), and so the above theorems can be applied to the quotient to conclude that there is a maximal ideal (respectively maximal right ideal) of R containing I (respectively, A).
- Krull's theorem can fail for rings without unity. A radical ring, i.e. a ring in which the Jacobson radical is the entire ring, has no simple modules and hence has no maximal right or left ideals. See regular ideals for possible ways to circumvent this problem.
- In a commutative ring with unity, every maximal ideal is a prime ideal. The converse is not always true: for example, in any nonfield integral domain the zero ideal is a prime ideal which is not maximal. Commutative rings in which prime ideals are maximal are known as zero-dimensional rings, where the dimension used is the Krull dimension.

Generalization

For an R module A, a **maximal submodule** M of A is a submodule $M \neq A$ for which for any other submodule N, if $M \subseteq N \subseteq A$ then $N=M$ or $N=A$. Equivalently, M is a maximal submodule if and only if the quotient module A/M is a simple module. Clearly the maximal right ideals of a ring R are exactly the maximal submodules of the module R_R.

Unlike rings with unity however, *a module does not necessarily have maximal submodules*. However, as noted above, finitely generated nonzero modules have maximal submodules, and also projective modules have maximal submodules.

As with rings, one can define the radical of a module using maximal submodules.

Furthermore, maximal ideals can be generalized by defining a **maximal sub-bimodule** M of a bimodule B to be a proper sub-bimodule of M which is contained by no other proper sub-bimodule of M. So, the maximal ideals of R are exactly the maximal sub-bimodules of the bimodule ${}_RR_R$.

References

1. Dummit, David S.; Foote, Richard M. (2004). *Abstract Algebra* (3rd ed.). John Wiley & Sons. ISBN 0-471-43334-9.
2. Lang, Serge (2002). *Algebra*. Graduate Texts in Mathematics. Springer. ISBN 0-387-95385-X.

- Anderson, Frank W.; Fuller, Kent R. (1992), *Rings and categories of modules*, Graduate Texts in Mathematics, **13** (2 ed.), New York: Springer-Verlag, pp. x+376, doi:10.1007/978-1-4612-4418-9, ISBN 0-387-97845-3, MR 1245487
- Lam, T. Y. (2001), *A first course in noncommutative rings*, Graduate Texts in Mathematics, **131** (2 ed.), New York: Springer-Verlag, pp. xx+385, doi:10.1007/978-1-4419-8616-0, ISBN 0-387-95183-0, MR 1838439