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The (maximum) independent set problem (MISP) on Riemannian manifolds of higher dimensions

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I. Basic definitions

- Given an undirected and unweighted graph $G = (V, E)$, by a finite set of vertices V and a set of edges $E = \{e_k := (v_i, v_j) \in V^2\}$ on a Riemannian manifold M of $\mathbb{R}^{dim}$ with $dim := \max\{\deg(v_i)\}, \forall v_i \in V$ of $G(V)$.

- For two vertices x and y, we write $x \sim y$ if they are connected by an edge. We say y is a neighbor of x and write $N(x) := \{y \in V : y \sim x\}$ for the set of all neighbors of x.

- The number of vertices connecting to x, is called the degree of x and denoted by $\deg(x)$. Since ~~G is~~ $G(V)$ is finite, by $|V| < \infty$, also $\deg(x)$ is finite. The maximum possible degree of a vertex in a finite graph is $\deg_{max}(x) := |V| - 1$.

- We call a graph $G(V)$ connected if for all two vertices x and y in V, there exists a finite number of vertices $\{v_i\}_{i=1}^{m}$ in V, satisfying that $v_1 := x$, $v_m := y$ and v_i is connected to v_{i+1}, $\forall\ i = 1, 2, \dots, m-1$.

Remark: Without loss of generality, as long not other mentioned, we assume that all in the following considered graphs are connected graphs.
(We introduce not-connected cases, later)

- An <u>Eulerian cycle</u> in an undirected graph is a cycle that uses each edge exactly once. A graph which owns such a cycle is called <u>Eulerian.</u>

If $G(V)$ is a <u>connected</u> graph, then the following statements are equivalent:

1. G is Eulerian.
2. Every vertex $v \in V$ of G have an even degree $\deg(v)$
3. The set E of G is the set union of all edges of pairwise disjunct cycles.

This was proofed by Hierholzer. [Hierholzer].

$\Rightarrow$ A graph G which has <u>only</u> <u>vertices</u> with <u>even</u> <u>degrees</u> is <u>Eulerian.</u>

<u>**Remark:**</u> Without loss of generality, as long not other mentioned, we assume that all in the following considered graphs are <u>Eulerian graphs</u> ($\Rightarrow$ all vertices have even degree)

(we introduce not-Eulerian cases, later)

- We define a <u>vertex path</u>, connecting two vertices $x, y \in V$ by $S := [c_0, c_1, \ldots, c_{n-2}, c_{n-1}]$

or short $= c_0 c_1 c_2 \ldots c_{n-2} c_{n-1}$

with $c_0 := x$ and $c_n := y$ and $c_i \sim c_{i+1}$ for each $i = 0, 1, \ldots, n-2$

- We denote an <u>Eulerian cycle</u> of G[redacted] by

$S_E := [b_0, b_1, \ldots, b_{n-2}, b_{n-1}]$

or short $= b_0 b_1 \ldots b_{n-2} b_{n-1}$

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with length $\ell(S_E) = |E| + 2$, $b_0 = b_{n-1}$, and each ~~ext~~ edge $b_i \sim b_{i+1}$ appears exactly ones.

- Given an Eulerian graph G, its belonging Eulerian cycle is not unique. It can exists more than one Eulerian cycles S_E of G.

- ~~[illegible]~~ Assume, we know one Eulerian cycle $S_{E,1}$ of G. We can derive further Eulerian cycles $S_{E,k\geq 2}$ of G by the set of systematically, cyclically permutations π_k of vertices of $S_{E,1}$ with

$$\{\pi_k : [b_0, b_1, \ldots, b_{n-2}, b_{n-1}]$$
$$\to [b_1, b_2, \ldots, b_{n-3}, b_{n-2}, b_{n-1}, b_0]$$
$$\to [b_2, b_3, \ldots, b_{n-2}, b_{n-1}, b_0, b_1]$$
$$\ldots$$
$$\to [b_{n-1}, b_0, b_1, \ldots, b_{n-3}, b_{n-2}]\}$$

All this Eulerian cycles are in fact the same Eulerian cycle.

- Given an Eulerian cycle S_E of an Eulerian graph G. Each vertex $v_i \in V$ of G appears exactly $\frac{\deg(v_i)}{2}$ times within the Eulerian cycle S_E, except the vertex $v_k = b_0 = b_{n-1}$, which appears exactly $\frac{\deg(v_k)}{2} + 1$ times within the Eulerian cycle S_E.

- We call an Eulerian cycle representation a disruptive bit string representation, if we write for an Eulerian cycle S_E,

$$S_E^{N-1} := [b'_{N-1}, b'_{N-2}, \ldots, b'_1, b'_0]$$
or short $$= b'_{N-1} b'_{N-2} \ldots b'_1 b'_0$$

with $b'_i \in \{0,1\}$, length $\ell(S_E^{N-1}) = |E| + 2 = N$, $i \sim i+1$, which satisfies the following properties:

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1. Direct consecutive bits $b_i b_{i-1}$ are allowed to have the bit patterns 00, 01, and 10. The bit pattern 11 be prohibit.

2. A <u>coupling</u> c of bits $b \in S_E^{N-1'}$ be given by $c_{\alpha := i,\dots,j} := \{(b_i, \dots, b_j) \mid b_i = \dots = b_j, \forall i,j : i \neq j \wedge j \neq \cancel{i+1}\}$ and $|c_{\alpha := i,\dots,j}| \geq 2$. Furthermore, for all couplings c_α and c_β, we have $c_\alpha \cap c_\beta = \emptyset$. We will denote all bits b which belong to the same coupling c by $\overset{c}{b}$. The bit string $S_E^{N-1'}$ consists of at least one coupling of bits, which is $c_{N-1,0} := (b_{N-1}, b_0)$.

<u>Problem statement (Maximization of the number of 1s in $S_E^{N-1'}$)</u>

Given a disruptive bit string $S_E^{N-1'}$ by its total length $N \in \mathbb{N}$ and a fixed set of given couplings $c \in \mathfrak{C}$. We ask for the values of $S_E^{N-1'}$, so that the number of 1s gets maximized in $S_E^{N-1'}$ if we count 1s of bits belonging to the same coupling only once within each coupling.

<u>Remark:</u> The given Problem statement describes the question of the <u>Maximum Independent Set Problem</u> (short: MISP)

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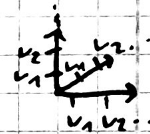
II. Decomposition

- Given an undirected and unweighted graph $G(V)$ on a Riemannian manifold M of $\mathbb{R}^{dim}$ with $dim := \max\{deg(v_i)\}$, $\forall v_i \in V$ of $G(V)$.

- Consider a vertices coordination vector $\vec{v} = (v_1, v_2, \dots, v_n)^T$, $\underset{n:=|V|}{}$ $\forall v_i \in V$. The unity norm between two direct consecutive vertices of V according their ascending index order, be 1. $(d(v_i, v_{i+1}) = 1)$

- The edges of $G(V)$ on $\mathbb{R}^{dim}$ can be represented by an $\mathbb{R}^{dim}$ Tensor structure, called <u>adjacence Tensor</u> T.
  T is given by $\underbrace{\vec{v} \otimes \vec{v} \otimes \vec{v} \otimes \dots \otimes \vec{v}}_{dim-times}$.
  (<u>Ex.</u> 3D )
  ($t_{v_1 v_1 v_2} = v_1 \circ v_1 \circ v_2$)

- If it exists a mapping $\phi: \mathbb{R}^{dim} \to \mathbb{R}^2$ of $G(V)$, then we call $G(V)$ a flat surface graph in terms of ϕ, and T of the mapped flat surface graph equals the well known classical adjacence matrix representation of $G(V)$. ⊛

⊛ <u>Attention:</u> Flat surface <u>doesn't</u> mean planar. It only means its structure can be represented by a well know 2D-adjacence matrix!

- The <u>inner concatenation</u> $\circ$ of T entries is ~~given by~~ denoted by $\underset{dim-factors}{t_{ijk\dots}} := \underbrace{v_i \circ v_j \circ v_k \circ \dots}_{dim-factors}$ or short $v_i v_j v_k \dots$ and called <u>edge conservation</u>. It is defined as follows:

  1. If there exists a mapping $\phi: \mathbb{R}^{dim} \to \mathbb{R}^2$ of $G(V)$, so that $G(V)$ can be considered as a flat surface graph in terms of ϕ, then we can denote each entry of $T^V_{dim=2}$ ($T^V_{dim=2}$ being the $dim=2$ Tensor of a flat surface graph $G(V)$ in terms of ϕ) by $t_{ij} := v_i v_j$, $i, j \in \{1, 2, \dots, n\}$, which describes the

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edge characteristics between the two vertices v_i and v_j

by

$$t_{ij} := edge(i,j) := \begin{cases} 1 & \text{if there exists an edge between } v_i \text{ and } v_j \\ 0 & \text{else} \end{cases}$$

2. Be $G(V)$ a graph on a $\mathbb{R}^{dim}$ Riemannian manifold M.

The edge conservation of T^V_{dim} for each entry is given by $t_{i_1 i_2 \dots i_{dim}}$, $i_1, i_2, \dots, i_{dim} \in \{1, 2, \dots, n\}$,

with $t_{i_1 i_2 \dots i_{dim}} := v_{i_1} v_{i_2} \cdots v_{i_{dim}}$, which describes the edge characteristics between the vertices $\{v_{i_1}, v_{i_2}, \dots, v_{i_{dim}}\}$

by

$$t_{i_1 i_2 \dots i_{dim}} := edge(i_1, i_2, \dots, i_{dim}) := \begin{cases} (*) & 1 \\ & 0 \end{cases}$$

(*) Consider a graph $G(V)$ on $\mathbb{R}^{dim > 2}$ on a Riemannian manifold M. We define an edge of an higher dimensional case > 2 recursively in the following way:

- We assume that for dimension $\underline{dim}$ $edge(i_1, i_2, \dots, i_{dim})$ defines an edge characteristic between the vertices $v_{i_1}, v_{i_2}, \dots, v_{i_{dim}}$
- Then the edge characteristic for dimension $\underline{dim+1}$, $edge(i_1, i_2, \dots, i_{dim}, i_{dim+1})$ defines an edge characteristic between the vertices $v_{i_1}, v_{i_2}, \dots, v_{i_{dim}}, v_{i_{dim+1}}$ by

  $edge(i_1, i_2, \dots, i_{dim}, i_{dim+1})$ induced an edge characteristic on the concatenation of the edge characteristic $edge(i_1, i_2, \dots, i_{dim})$ with $v_{i_{dim+1}}$: $edge(i_1, i_2, \dots, i_{dim}, i_{dim+1}) :=$

  $$edge(edge(i_1, i_2, \dots, i_{dim}), i_{dim+1})$$

→ We consider subgraphs, at first.

- Be A a subset of V, $A \subseteq V$, such that $G(A)$ be a <u>connected</u> <u>subgraph</u> of $G(V)$ with size $|A| \geq 1$.

- We denote the graph which is given by the remaining vertices $V \setminus A$ by $G(V-A)$.

- The subset D of vertices $D \subseteq V$, called <u>influencers of A</u>, denoted by D_A (or only D if it is clear which subset we are talking about because of the context), be the set of all vertices d_i with:

$$\{\forall d_i \in V: \exists (edge(d_i, x_e) \wedge edge(d_i, a_u)), \text{with } x_e \in (V \setminus A), \text{ and } a_u \in A$$
$$\text{or } \exists (edge(d_i, a_u) \wedge edge(d_i, a_{u'})), \text{with } a_u, a_{u'} \in A\}$$

→ Now, we want to analyse a decomposition of $G(V)$ with ~~simili~~ similarities to the MIS algorithm proposed by Tarjan in [TarjanDec] (section: '3.4. Maximum independent sets').

- We consider a set $\mathcal{A}$ of connected subgraphs A_i of $G(V)$ by $\mathcal{A} := \{A_i\}$ with pairwise $A_i \cap A_j = \emptyset$ and $\mathcal{A} \neq \emptyset$. Furthermore, there exists no edge (a_i, a_j), $a_i \in A_i$ and $a_j \in A_j$.
- We denote the MIS of ~~$\mathcal{A}$ by~~ $G(\mathcal{A})$ by I'.

- We introduce a <u>split function</u> sp which splits a graph $G(V)$ into $G(\mathcal{A})$ and $G'' := G(V-\mathcal{A})$.

- During splitting, sp induces over the <u>impact</u> $imp(V-\mathcal{A})$ a <u>dimension recalibration</u> on $G^*(V-\mathcal{A})$ by $sp \rightarrow imp(V-\mathcal{A})$, $\mathbb{R}^{dim} \longrightarrow \mathbb{R}^{dim'}$, with $dim' := \max\{deg(v_i)\}$ $\forall v_i \in (V \setminus \mathcal{A})$, and an <u>edge influence</u> inf on the graph $G(V-\mathcal{A})$ in the following way:

1. inf introduces a new weighted virtual edge
$(e^w := (d_i, d_j))$ $e^w_{dim'} = (e^w_{dim'-1}, d_j)$, $d_j \in D_{\mathcal{A}}$ and
d_u of $e^w_{dim'-1}$ [redacted] with $d_u \in D_{\mathcal{A}}$, $D_{\mathcal{A}}$ be our set of
influencers of $\mathcal{A}$, ~~[illegible]~~, on the
graph $G(V-\mathcal{A})$, if there exists a MIS size reducing
influence on I' of $G(\mathcal{A})$, denoted by $\inf(I', e^w)$,
because of the establishing of the edge e^w
(if there not already existed that edge before)

2. If there already existed that edge $e^w := (e^w_{dim'-1}, d_j)$
before, we don't change anything and call it a
neutral, id, influence.

- We denote the new adjacence Tensor of $G(V-\mathcal{A})$ by
$T^{V-\mathcal{A}}_{dim'}$ if we introduced weighted virtual edges e^w which
give us the amount of negative changing of I' with
$T^{V-\mathcal{A}}_{dim'}$ entries of e^w's with valus ≤ -1, $\forall e^w$ in $T^{V-\mathcal{A}}_{dim'}$

- We identify edges [redacted] in $T^{V-\mathcal{A}}_{dim'}$ of [redacted] $V \setminus \mathcal{A}$
with edge(d_i, y_u) (*1) [redacted] or edge(y_u, y_e) (*2), $d_i \in D_{\mathcal{A}}$
and $y_u, y_e \in V \setminus \{\mathcal{A} \cup D_{\mathcal{A}}\}$, with $d_i \sim y_u$ respectively
$y_u \sim y_e$ as prohibit pattern '11' bit string connections
and hence mark this edges in the weighted adjacence Tensor
$T^{V-\mathcal{A}}_{dim'}$ with an [redacted] entry value of $-\infty$!

(*1) and (*2): This is for the [redacted] flat surface graph case. In generally
we consider of course edge$(\text{edge}_{dim'-1}, y_e)$
with d_i or $y_u \in \text{edge}_{dim'-1}$.

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- We denote the so called <u>standadized new adjacence Tensor</u> of $G(V-\mathcal{A})$ by $T^{(V-\mathcal{A})'}_{dim'}$ in which we set all of the entry values of the weighted adjacence Tensor $T^{V-\mathcal{A}}_{dim'}$ entries which are $\neq 0$ to 1.

- Looking at the weighted adjacence Tensor $T^{V-\mathcal{A}}_{dim'}$, we want to give a short, sketchy discussion regarding (maximum) independent sets.

1. Tensor $T^{V-\mathcal{A}}_{dim'}$ is symetrically according it its dimensional diagonals. ($\Rightarrow$ for example $v_1 v_2 v_3 = v_1 v_3 v_2 = v_2 v_3 v_1 \ldots$)

2. We call diagonals of the form $\underbrace{v_{i_1} v_{i_1} v_{i_1} \cdots v_{i_1}}_{dim'-\text{times}}$ 1-diagonal forms.

3. We don't have multiple edges. Means e.g. $v_1 v_2 v_2 = v_1 v_2$ with one edge between v_1 and v_2

4. If we sum the values of $T^{V-\mathcal{A}}_{dim'}$ entries, because of symmetry of $T^{V-\mathcal{A}}_{dim'}$, we only sum entries of the valid symmetry area. Additionaly, we have to take into account that during determing $T^{V-\mathcal{A}}_{dim'}$, entries of diagonals are generated by multiple counting of changings. Hence, the sum of diagonal entries is given by the norming to -1 over the number of diagonal entries over which we sum $\left((\text{sum } t_{diagonal}) \frac{1}{|\{t_{diagonal}\}|} \right)$

5. Possibility discussion for finding maximum independent sets in $T^{(V-\mathcal{A})'}_{dim'}$ for the example of a flat surface graph $G(V-\mathcal{A})$ on $\mathbb{R}^2$.

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Case 1

$$
\begin{array}{ccccc|c}
v_1 & v_2 & v_3 & v_4 & v_5 & \\
\hline
0 & & -1 & & & v_1 \\
 & & & ⊛ & & v_2 \\
 & & 0 & & & v_3 \\
 & & & 0 & 0 & v_4 \\
 & & & & 0 & v_5
\end{array}
$$

⊛ entries which are not given are < -1.

→ $\{v_4, v_5\}$ IS of size 2 on $G(V-\mathcal{A})$
$\Rightarrow |I| = |I'| + |IS| = |I'| + 2$

→ $\{v_1, v_3\}$ set of size 2 on $G(V-\mathcal{A})$
but reducing the IS I' of $G(\mathcal{A})$ by -1.

$\Rightarrow |I| = |I'| - 1 + |set|$
$= |I'| - 1 + 2 = |I'| + 1$

→ $\{v_1\}$ or $\{v_5\}$ IS of size 1 on $G(V-\mathcal{A})$
$\Rightarrow |I| = |I'| + |IS| = |I'| + 1$

$\Rightarrow$ $\{v_4, v_5\}$ leads us to the MIS of $G(V)$

$\Rightarrow$ In this case, we can determine the MIS of $G(V)$ by solving the MIS problem of the standardized adjacence Tensor $T^{(V-\mathcal{A})'}_{dim'}$

Case 2

$$
\begin{array}{ccccc|c}
v_1 & v_2 & v_3 & v_4 & v_5 & \\
\hline
0 & & -1 & & & v_1 \\
 & & & ⊛ & & v_2 \\
 & & -1 & & & v_3 \\
 & & & 0 & 0 & v_4 \\
 & & & & 0 & v_5
\end{array}
$$

→ $\{v_4, v_5\}$ IS of size 2 on $G(V-\mathcal{A})$
$\Rightarrow |I| = |I'| + |IS| = |I'| + 2$

→ $\{v_1\}$ IS of size 1 on $G(V-\mathcal{A})$
$\Rightarrow |I| = |I'| + |IS| = |I'| + 1$

→ $\{v_1, v_3\}$ set of size 2 on $G(V-\mathcal{A})$
but reducing the IS I' of $G(\mathcal{A})$ by $-1-1=-2$.

$\Rightarrow |I| = |I'| - 2 + |set|$
$= |I'| - 2 + 2 = |I'|$

$\Rightarrow$ $\{v_4, v_5\}$ leads us to the MIS of $G(V)$.

$\Rightarrow$ In this case, we can determine the MIS of $G(V)$ by solving the MIS problem of the standardized adjacence Tensor $T^{(V-\mathcal{A})'}_{dim'}$

Remarks: (for flat surface graphs)

- For each subgraph A of $\mathcal{A}$, we can get maximal one with $v_i v_j = 0$, $v_i \neq v_j$, entry.

Case 3

$$\begin{array}{ccccc|c} v_1 & v_2 & v_3 & v_4 & v_5 & \\ \hline -1 & & 0 & & & v_1 \\ & & & ⊛ & & v_2 \\ & & -1 & & & v_3 \\ & & & 0 & 0 & v_4 \\ & & & & 0 & v_5 \end{array}$$

→ $\{v_3\}$ or $\{v_1\}$ set of size 1 on $G(V-A)$

but reducing the IS I' of $G(A)$ by -1

$\Rightarrow |I| = |I'| - 1 + |set|$
$= |I'| - 1 + 1 = |I'|$

→ $\{v_1, v_3\}$ set of size 2 on $G(V-A)$ but reducing the IS I' of $G(A)$ by -1 if v_1 and v_3 (like in our example) are elements of the same D_{A_i}, and -2 if v_1 and v_3 belong to different $D_{A_i} \neq D_{A_j}$.

⇒ For our example, we get
$|I| = |I'| - 1 + |set|$
$= |I'| - 1 + 2 = |I'| + 1$
(else: $|I| = |I'| - 2 + 2 = |I'|$)

→ $\{v_4, v_5\}$ IS of size 2 on $G(V-A)$
$\Rightarrow |I| = |I'| + |IS| = |I'| + 2$

⇒ $\{v_4, v_5\}$ leads us the MIS of $G(V)$

⇒ In this case, we can determine the MIS of $G(V)$ by solving the MIS problem of the standardized adjacence Tensor $T^{(V-A)'}_{dim'}$

Case 4

$$\begin{array}{ccccc|c} v_1 & v_2 & v_3 & v_4 & v_5 & \\ \hline -1 & & 0 & & & v_1 \\ & & & ⊛ & & v_2 \\ & & -1 & & & v_3 \\ & & & 0 & -1 & v_4 \\ & & & & 0 & v_5 \end{array}$$

→ $\{v_3\}$ or $\{v_1\}$ set of size 1 on $G(V-A)$ but reducing the IS I' of $G(A)$ by -1.

$\Rightarrow |I| = |I'| - 1 + |set| = |I'| - 1 + 1 = |I'|$

→ $\{v_1, v_3\}$ set of size 2 on $G(V-A)$ but reducing the IS of I' of $G(A)$ by -1
$\Rightarrow |I| = |I'| - 1 + 2 = \underline{|I'| + 1}$

→ $\{v_4\}$ or $\{v_5\}$ IS of size 1 on $G(V-A)$
$\Rightarrow |I| = |I'| + |IS| = \underline{|I'| + 1}$

→ $\{v_4, v_5\}$ set of size 2 on $G(V-A)$ but reducing the IS I' of $G(A)$ by -1.
$\Rightarrow |I| = |I'| - 1 + |set|$
$= |I'| - 1 + 2 = \underline{|I'| + 1}$

⇒ We see, that in this case, we get the MIS of $G(V)$ with size $|I'| + 1$

! for: $\{v_1, v_3\}$, $\{v_4\}$, $\{v_5\}$, $\{v_4, v_5\}$

⇒ Discussion, see next page

Discussion of case 4:

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1. Assume we take the ~~standardized Tensor~~ standardized adjacence Tensor $T^{(V-A)'}_{dim}$. So, we have:

$$\begin{array}{ccccc|c} v_1 & v_2 & v_3 & v_4 & v_5 & \\ \hline 1 & & 0 & & & v_1 \\ & & & & & v_2 \\ & & 1 & & & v_3 \\ & & & 0 & 1 & v_4 \\ & & & & 0 & v_5 \end{array}$$

We get as MIS:

$\{v_4\}$ and $\{v_5\}$

⇒ We can derive the set $\{v_4, v_5\}$ by having an additional look at our original ~~Tensor~~ weighted Tensor $T^{(V-A)}_{dim}$, by taking v_4 and v_5 and seeing that $v_4v_5 = -1$, so just an reducing by -1 by taking an additional vertex to v_4 (respectively taking an additional vertex to v_5) ⇒ So, we get the same total size solution of $|I|$, $G(V)$, by $\{v_4, v_5\}$

→ $\{v_1, v_3\}$ would be not find by an MIS solution of $G(V-A)$. We only get it, if we ~~see~~ notize that $v_1v_3 = 0$ and that all diagonal entries of the participating vertices are -1 in $T^{(V-A)}_{dim}$. So, we would also get this set of vertices.

→ If we put all together, we get totally: $\{v_1, v_3\} \cup \{v_4\}$ and $\{v_1, v_3\} \cup \{v_5\}$ and $\{v_1, v_3\} \cup \{v_4, v_5\}$ all with the same total MIS size of $G(V-A)$ by $|I| = |I'| + 1 + 1 = |I'| + 2$.

2. Assume, we take the standardized adjacence Tensor $T^{(V-A)'}_{dim}$, but put all diagonal entries to 0.

$$\begin{array}{ccccc|c} v_1 & v_2 & v_3 & v_4 & v_5 & \\ \hline 0 & & 0 & & & v_1 \\ & & & & & v_2 \\ & & 0 & & & v_3 \\ & & & 0 & 1 & v_4 \\ & & & & 0 & v_5 \end{array}$$

We get the MIS: $\{v_1, v_3\}$ with $|I| = |I'| + 2$ which is obvousily wrong

⇒ So it is ~~necesse~~ necessary still to check the results on the original Tensor $T^{(V-A)}_{dim}$

⇒ Furthermore, with an MIS algorithm on this Tensor here, we would not find $\{v_4\}$, $\{v_5\}$ and $\{v_4, v_5\}$!

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- We assume, that we are able to identify the MIS in $G(\mathcal{V})$, which we denote by I'.

- We look again at the weighted adjacence Tensor $T^{V-\mathcal{V}}_{dim'}$.

<u>1. Case:</u> The MIS of $G(V-\mathcal{V})$ is given by a set of vertice entries in $T^{V-\mathcal{V}}_{dim'}$ in ~~which are all~~ entries? ~~with~~ have value 0. In which all entries of the vertices of this set have value 0.

$\Rightarrow$ We denote this MIS by I'' and recognize that the already known MIS of $G(\mathcal{V})$ by <u>I' doesn't get reduced</u> in its size

$\Rightarrow$ The total MIS of $G(V)$ is given by:

$$I = I' \cup I'' \quad \text{and} \quad I'' \text{ in } T^{V-\mathcal{V}}_{dim'}$$

consists of only 0 entries, so we have no participating edge in I''. (similar to the case: $I'' \cap C = \emptyset$ in [TarjanDec])

<u>2. Case:</u> The MIS of $G(V-\mathcal{V})$ is given by a set of vertices in $T^{V-\mathcal{V}}_{dim'}$, so that at least one vertex of this set of vertices does have an entry value in $T^{V-\mathcal{V}}_{dim'}$ of $\neq 0$.

$\Rightarrow$ We also denote MIS by I'' and recognize that the already known MIS of $G(\mathcal{V})$ by I' <u>get reduced</u> in its size by the entries $\neq 0$ in $T^{V-\mathcal{V}}_{dim'}$, which belonging vertices are building a clique on $e^{m}_{i} := (e^{m}_{dim-1}, d_j), dim'$, edges.(*) We denote the reduced MIS of $G(\mathcal{V})$ by $I_{(\{v\})}$, $\{v\}$ being the set of vertices with entries $\neq 0$.

$\Rightarrow$ The total MIS of $G(V)$ is given by:

$$I = I_{(\{v\})} \cup I'' \quad \text{and} \quad I'' \text{ in } T^{V-\mathcal{V}}_{dim'}$$

(*) Not sure. I think the case of having only one vertex with an edge on themselve is also possible.

consists of at least one $\neq 0$ entry, so we have at least one participating edge in I''.

(similar to the case: $v \in I'' \cap C$ in [TarjanDeco])

Remark to this case:

~~entries~~ MIS vertex entries $\neq 0$ of I'' in $T^{V-A}_{\dots}$ are given by the virtual weighted edges which we introduced and are in fact 'connections' between $d_i \in D_A$ vertices.

- Until now, we assumed that $G(V)$ ~~and~~ and $G(V-A)$ be always connected graphs.

- Given an ~~(undirected) and~~ not connected graph G.
  Then there exists $G_1, G_2, G_3, \dots$ part graphs such that $G(V) = \bigcup_i G_i$ with $V = \bigcup_i V_i$ and $E := \bigcup_i E_i$ for all $G_i(V_i, E_i)$.
  The MIS of $G(V)$ is given by the union of the MIS_i of $G_i(V_i, E_i)$, for all G_i with $MIS_{V(totally)} := \bigcup_i MIS_i$.

- Until now, we assumed that $G(V)$ and $G(V-A)$ be always Eulerian graphs.

- Why we use Eulerian graphs(?)
  The efficiency of the proposed MIS approach is mostly influenced by the choice of vertices for the subgraphs A_i of A. (their vertex edge structure and degrees).

  The easiest ~~way~~ way of choosing the most favorite subgraph with certain characteristics is given by going through a known Eulerian cycle sequence, from which we can grab A_i sets with $\mathcal{O}(\text{length}(S_E))$ time complexity.

- In the beginning we talked about Eulerian graphs and Hierholzer's algorithm for graphs $G(V)$ which can be represented by an classical adjacence matrix, our $T^{V}_{dim=2}$ Tensor on $\mathbb{R}^2$.

- Without loss of generality, we can assume that there exists for each grapha on $\mathbb{R}^{dim>2}$ manifold regarding its belonging T^{V}_{dim} representation, there exists ~~an~~ recursive mapping for $e_{dim} = (e_{dim-1}, v_{dim})$ with $e_{dim-1} \mapsto e_{dim}$.

  Hence, there exists a valid representation mapping of an Eulerian cycle of a $T_{dim=2}$ representation by an edge concatenation to an Eulerian cycle of a $T^{V}_{dim>2}$ representation and vice versa.

  $\Rightarrow$ It exists an Hierholzer's algorithm equivalent algorithm for graphs with ~~an~~ a $T^{V}_{dim>2}$ representation.

- Given a connected graph $G(V)$. Assume G is <u>not</u> Eulerian, which means it exists vertices $v_i' \in V$ with $deg(v') = odd$.

  From basic graph theory, we know that for every graph G, we have $2|E| = \sum_{v_i \in V} deg(v_i)$.

  $\Rightarrow$ We know that we always have an even number of vertices with odd ~~got~~ degree.

  $\Rightarrow$ We can map G to an Eulertan graph by connecting always two of the odd degree vertices $\{u, v\}$ by one new introduced helper vertex $h \in H$ (be H the set of helper vertices) and hence we get two new edges (u, h) and (h, v)

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Consequences of adding helper vertices regarding MIS's:

1. **Case:** The helper vertex h connects two vertices u and v which belong to the same independent set S, which means $u \in S \wedge v \in S$ and hence $(u,v) \notin E$.
   $\Rightarrow$ Since u and v are still not directly connected with each other, they will still belong to the same independent set. Since, each of them have a connection to h, h will become a ~~member~~ member of an other, second, independent set. So, we will get our original independent set S unchanged and any other second independent set with $S' \cup \{h\}$

2. **Case:** The helper vertex h connects two vertices u and v which belong to two different independent sets S_i, S_j with $S_i \nsubseteq S_j$, which means $u \in S_i \wedge v \in S_j$ and hence $(u,v) \in E$
   $\Rightarrow$ Since u and v are already directly connected, also with a connection to h, they will still both belong to two different sets. Since, each of them have a connection to h, h will become a member of an other, third, independent set. So, we will get our original two independent sets S_i and S_j unchanged and any other third **independent** set with $S_k \cup \{h\}$.

3. It's clear, that an helper vertex is not allowed to be ~~an~~ element of a MIS solution, else ~~it~~ this solution wouldn't be, ~~in generally~~, ~~not~~ anymore automatically the MIS solution of the real, original, given graph $G(V)$.

⇒ After generating the Eulerian cycle sequence S_E, we have to take care that helper vertices are not be ~~taken~~ considered for finding the MIS solutions of the ~~subgraph~~ graphs $G(\mathcal{A})$ and $G(V-\mathcal{A})$.

⇒ In the disruptive bit string representation $S_E^{\omega-1}$, the belonging bits of helper vertices have to be set to 0 mandatory

⇒ To be sure, that helper vertices are not considered for the IS solutions of $G(V-\mathcal{A})$, all entries of $T^{V-\mathcal{A}}_{dim'}$ with $v_{i_1}v_{i_2}\cdots v_{i_{dim'}}$ with at least one v_{i_j} being a helper vertex are set to '$-\infty$'. (Marked as not allowed vertices to choose).

⇒ The identification of known helper vertices within the Eulerian cycle sequence can be done in $\mathcal{O}(\text{length}(S_E))$ time complexity.

- Finally at this point, we want still give a sketchy discussion regarding the choice of $\mathcal{A}$ respectively a single subgraph A_i.

- Remember the general property of $\mathcal{A}$ with A_i ~~being~~ building connected subgraphs $G(A_i)$ of $G(A)$ and $A_i \cap A_j = \emptyset$, $\mathcal{A} \neq \emptyset$. Furthermore, there exits no edge (a_i, a_j), $a_i \in A_i$ and $a_j \in A_j$ between subsets.

- Let's look at a single subset A_i, its belonging subgraph $G(A_i)$ and the influencers of A_i, D_{A_i}.

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- We ~~denote~~ define $n := |D_{A_i}|$.
- If the edge influence inf introduces a new weighted virtual edge of the form $e^w_{dim'} := (e^w_{dim'-1}, d_{i_1})$, $e^w_{dim'-2} := (e^w_{dim'-2}, d_{i_2})$, ... $e^w_{dim'-(dim'-1)} := (d_0, d_{i_{dim'-1}})$, $d_0 = d_{i_1} = d_{i_2} = \dots = d_{i_{dim'-1}} \in D_{A_i}$, we call the edge $e^w_{dim'}$ an (1, dim')-edge (this vertex is only connected to themselve).
- If $d_0 = d_{i_2} = d_{i_3} = \dots = d_{i_{dim'-1}}$ and $d_0 \neq d_{i_1} \in D_{A_i}$, we call the edge $e^w_{dim'}$ an (2, dim'-1, 1)-edge, and so on...

- The tuple $(x, a, b, c, \dots)$ means, we have x different vertices. The first of them appears a times, the second of them appears b times and so on...

- Assume now, we have an subgraph $G(A_i)$ which only introduces edges of the (1, dim') form on $T^{V-A_i}_{dim'}$. This means we only get new edges on themselves for $d_i \in D_{A_i}$. All the entries, appart from the Tensor diagonal entries, stay the same means

  $T^{V-A_i}_{dim'} = T^V_{dim}$ for non-diagonal entries.

  So, we have „mainly" the same Tensor like before, just reduced by the vertices of set A_i.
- If we consider for example (2, dim'-1, 1) edges instead, we now can have edges between two $d_i, d_u \in D_{A_i}$ vertizes, which didn't have an edge before.

⇒ Hence, the kind of choice we make regarding the decision which subgraphs A_i we choose, has a strong influence on the characteristics of $T^{V-A}_{dim'}$ Tensors and hence also on our algorithm approach for finding (M)IS's.

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