Notes: The (maximum) independent set problem (MISP) on Riemannian manifolds of higher dimensions

Carolin Zöbelein*

October 14, 2020

Abstract

We consider the determination of (maximum) independent sets of arbitrary graphs with the help of higher dimensional tensor based representations and subgraph decomposition by weighted based introduction of virtual edges. Finally, we do a comparison of our algorithm with a similiar one given by Tarjan in *Decomposition by clique separators*. In contrast to Tarjan's algorithm, we propose an algorithm which offers the possibility to choose the atom subgraphs freely instead of handling mandatory atom subgraphs given by a decomposition.

Keywords: Maximum Independent Sets, Independent Sets, Graphs, Graph Theory, Riemannian Manifolds, Higher Dimensions, Tensors

Contents

Preamble

The following content is a sketch for discussion purposes only, without warranty for mathematical completeness.

1 Basic definitions

Given an undirected and unweighted graph $G = (V, E)$, by a finite set of vertices V and a set of edges $E := \{e_k := (v_i, v_j) \in V^2\}$ on a Riemannian manifold M of $\mathbb{R}^{dim}$ with $dim := max\{deg\,(v_i)\}, \forall\, v_i \in V$ of $G\,(V)$.

For two vertices x and y, we write $x \sim y$ if they are connected by an edge. We say y is a neighbor of x and write $N\,(x) := \{y \in V \,:\, y \sim x\}$ for the set of all neighbors of x.

*The author believes in the importance of the independence of research and is funded by the public community. If you also believe in this values, you can find ways for supporting the author's work here: `https://research.carolin-zoebelein.de/funding.html`, Email: `contact@carolin-zoebelein.de`, PGP: D4A7 35E8 D47F 801F 2CF6 2BA7 927A FD3C DE47 E13B, `https://research.carolin-zoebelein.de`, CV availabe at `https://research.carolin-zoebelein.de/files/cv_longversion.pdf`, 😎

The number of vertices connecting to x, is called the degree of x and denoted by $deg(x)$. Since $G(V)$ is finite, by $|V| < \infty$, also $deg(x)$ is finite. The maximum possible degree of a vertex in a finite graph is $deg_{max}(x) := |V| - 1$.

We call a graph $G(V)$ connected if for all two vertices x and y in V, there exists a finite number of vertices $\{v_i\}_{i=1}^{m}$ in V, satisfying that $v_1 := x$, $v_m := y$ and v_i is connected to v_{i+1}, $\forall\, i = 1, 2, \ldots, m-1$.

Remark 1.1. Without loss of generality, as long not other mentioned, we assume that all in the following considered graphs are connected graphs (We introduce not-connected cases, later).

An Eulerian cycle in an undirected graph is a cycle that uses each edge exactly once. A graph which owns such a cycle is called Eulerian. If $G(V)$ is a *connected* graph, then the following statements are equivalent:

1. G is Eulerian
2. Every vertex $v \in V$ of G have an even degree $deg(v)$
3. The set E of G is the set union of all edges of pairwise disjunct cycles

This was proofed by Hierholzer [?].

$\Rightarrow$ A graph G which has only vertices with even degrees is Eulerian.

Remark 1.2. Without loss of generality, as long not other mentioned, we assume that all in the following considered graphs are Eulerian graphs ($\Rightarrow$ All vertices have even degree), (We introduce not-Eulerian cases, later).

We define a vertex path, connecting two vertices $x, y \in V$ by $S := [c_0, c_1, \cdots, c_{n-2}, c_{n-1}]$ or short $= c_0 c_1 c_2 \cdots c_{n-2} c_{n-1}$ with $c_0 := x$ and $c_n := y$ and $c_i \sim c_{i+1}$ for each $i = 0, 1, \cdots, n-2$.

We denote an Eulerian cycle of G by $S_E := [b_0, b_1, \cdots, b_{n-2}, b_{n-1}]$ or short $= b_0 b_1 \cdots b_{n-2} b_{n-1}$ with length $l(S_E) = |E| + 2$, $b_0 = b_{n-1}$, and each edge $b_i \sim b_{i+1}$ appears *exactly ones.*

Given an Eulerian graph G, its belonging Eulerian cycle is not unique. It can exist more than one Eulerian cycles S_E of G.

Assume, we know one Eulerian cycle $S_{E,1}$ of G. We can derive further Eulerian cycles $S_{E,k>1}$ of G by the set of systematically, cyclically permutations π_k of vertices of $S_{E,1}$ with

$$\begin{aligned}\{\pi_k \,:\, & [b_0, b_1, b_2 \cdots, b_{n-3}, b_{n-2}, b_{n-1}] \\ & \to [b_1, b_2, b_3 \cdots, b_{n-2}, b_{n-1}, b_0] \\ & \to [b_2, b_3, b_4 \cdots, b_{n-1}, b_0, b_1] \\ & \cdots \\ & \to [b_{n-1}, b_0, b_1, \cdots, b_{n-4}, b_{n-3}, b_{n-2}]\}\end{aligned}$$

All this Eulerian cycles are in fact the same Eulerian cycle.

Given an Eulerian cycle S_E of an Eulerian graph G. Each vertex $v_i \in V$ of G appears exactly $deg(v_i)/2$ times within the Eulerian cycle S_E, except the vertex $v_k := b_0 = b_{n-1}$, which appears exactly $deg(v_k)/2 + 1$ times within the Eulerian cycle S_E.

We call an Eulerian cycle representation a disruptive bit string representation, if we write for an Eulerian cycle S_E, $S_E^{N-1'} := [b'_{N-1}, b'_{N-2}, \cdots, b'_1, b'_0]$ or short $= b'_{N-1} b'_{N-2} \cdots b'_1 b'_0$ with $b'_i \in \{0, 1\}$, length $l\left(S_E^{N-1'}\right) = |E| + 2 = N$, $i \sim i+1$, which satisfies the following properties:

1. Direct consecutive bits $b'_i b'_{i-1}$ are allowed to have the bit patterns 00, 01 and 10. The bit pattern 11 be prohibit.

2. A coupling c of bits $b' \in S_E^{N-1'}$ be given by $c_{\alpha:=i,\ldots,j} := \{(b_i, \ldots, b_j) \mid b_i = \cdots = b_j, \forall\, i, j \,:\, i \neq j\}$ and $|c_{\alpha:=i,\ldots,j}| \geq 2$. Furthermore, for all couplings c_α and c_β, we have $c_\alpha \cap c_\beta = \emptyset$. We will denote all bits b' which belong to the same coupling c by $b^{c'}$. The bit string $S_E^{N-1'}$ consists of at least one coupling of bits, which is $c_{N-1,0} := \left(b'_{N-1}, b'_0\right)$.

Problem Statement 1 (Maximization of the number of 1s in $S_E^{N-1'}$)**.** *Given a disruptive bit string $S_E^{N-1'}$ by its total length $N \in \mathbb{N}$ and a fixed set of gien couplings $c \in \mathcal{C}$. We ask for the values of $S_E^{N-1'}$, so that the number of* 1*s gets maximized in $S_E^{N-1'}$ if we count* 1*s of bits belonging to the same coupling only once within each coupling.*

Remark 1.3. The given problem statement describes the question of the **M**aximum **I**ndependent **S**et **P**roblem (short: MISP).

2 Decomposition

Given an undirected and unweighted graph $G(V)$ on a Riemannian manifold M of $\mathbb{R}^{dim}$ with $dim := max\{deg(v_i)\}, \forall\, v_i \in V$ of $G(V)$.

Consider a vertices coordination of vector $\vec{v} = (v_1, v_2, \ldots, v_n)^T$, $\forall\, v_i \in V$, $n := |V|$. The unity norm between two direct consecutive vertices of V according their ascending index order, be 1 ($d(v_i, v_{i+1}) = 1$).

The edges of $G(V)$ on $\mathbb{R}^{dim}$ can be represented by an $\mathbb{R}^{dim}$ tensor structure, called adjacence tensor T. T is given by $\underbrace{\vec{v} \otimes \vec{v} \otimes \vec{v} \otimes \cdots \otimes \vec{v}}_{\text{dim}-\text{times}}$.

If it exists a mapping $\phi : \mathbb{R}^{dim} \to \mathbb{R}^2$ of $G(V)$, then we call $G(V)$ a flat surface graph in terms of ϕ and T of the mapped flat surface graph equals the well known classical adjacence matrix representation of $G(V)$[1].

The *innner concatenation* $\circ$ of T entries is denoted by $t_{\underbrace{ijk\cdots}_{\text{dim}-\text{factors}}} := \underbrace{v_i \circ v_j \circ v_k \circ \ldots}_{\text{dim}-\text{factors}}$ or short $v_i v_j v_k \cdots$ and called edge conservation. It is defined as follows:

1. If there exists a mapping $\phi : \mathbb{R}^{dim} \to \mathbb{R}^2$ of $G(V)$, so that $G(V)$ can be considered as a flat surface graph in terms of ϕ, then we can denote each entry of $T^V_{dim=2}$ ($T^V_{dim=2}$ being the $dim = 2$ tensor of a flat surface graph $G(V)$ in terms of ϕ) by $t_{ij} := v_i v_j$, $i, j \in \{1, 2, \ldots, n\}$, which describes the edge characteristics between the two vertices v_i and v_j by

$$t_{ij} := edge(i,j) = \begin{cases} 1 & \text{if there exists an edge between } v_i \text{ and } v_j \\ 0 & \text{otherwise} \end{cases}$$

2. Be $G(V)$ a graph on a $\mathbb{R}^{dim}$ Riemannian manifold M. The edge conservation of T^V_{dim} for each entry is given by $t_{i_1, i_2, \ldots, i_{dim}}$, $i_1, i_2, \ldots, i_{dim} \in \{1, 2, \ldots, n\}$, with $t_{i_1, i_2, \ldots, i_{dim}} := v_{i_1} v_{i_2} \cdots v_{i_{dim}}$, which describes the edge characteristics between the vertices $\{v_{i_1}, v_{i_2}, \ldots, v_{i_{dim}}\}$ by

[1] Attention: Flat surface **doesn't** mean planar. It only means its structure can be represented by a well known 2D-adjacence matrix!

$$t_{i_1 i_2 \ldots i_{dim}} := edge\,(i_1, i_2, \ldots, i_{dim}) = \begin{cases} 1 & (\star) \text{ see explanation beneath} \\ 0 & (\star) \text{ see explanation beneath} \end{cases}$$

($\star$) Consider a graph $G\,(V)$ on $\mathbb{R}^{dim>2}$ on a Riemannian manifold M. We define an edge of an higher dimensional case > 2 recursively in the following way:

(a) We assume that for dimension <u>dim</u> $edge\,(i_1, i_2, \ldots, i_{dim})$ defines an edge characteristic between the vertices $v_{i_1}, v_{i_2}, \cdots v_{i_{dim}}$.

(b) Then the edge characteristic for dimension <u>$dim+1$</u>, $edge\,(i_1, i_2, \ldots, i_{dim}, i_{dim+1})$ defines an edge characteristic between the vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_{dim}}, v_{i_{dim+1}}$ by $edge\,(i_1, i_2, \ldots, i_{dim}, i_{dim+1})$ induced an edge characteristic on the concatenation of the edge characteristic $edge\,(i_1, i_2, \ldots, i_{dim})$ with $v_{i_{dim+1}}$, $edge\,(i_1, i_2, \ldots, i_{dim}, i_{dim+1}) := edge\,(edge\,(i_1, i_2, \ldots, i_{dim})\,, i_{dim+1})$.

$\rightarrow$ We consider subgraphs, at first.

Be A a subset of V, $A \subseteq V$, such that $G\,(A)$ be a <u>connected subgraph</u> of $G\,(V)$ with size $|A| \geq 1$.

We denote the graph which is given by the remaining vertices $V \setminus A$ by $G\,(V-A)$.

The subset D of vertices $D \subseteq V$, called <u>influencers of A</u>, denoted by D_A (or only D if it is clear which susbset we are talking about because of the context), be the set of all vertices d_i with:

$$\begin{aligned} \{\forall d_i \in V \ : & \exists\,(edge\,(d_i, x_l) \wedge edge\,(d_i, a_k))\,, \text{with } x_l \in (V \setminus A) \text{ and } a_k \in A, \\ \text{or} \quad & \exists\,(edge\,(d_i, a_k) \wedge edge\,(d_i, a_{k'}))\,, \text{with } a_k, a_{k'} \in A\} \end{aligned}$$

Now, we want to analyse a decomposition of $G\,(V)$ with similarities to the MIS algorithm proposed by Tarjan in [?] (section: '3.4. Maximum independent sets').

We consider a set $\mathcal{A}$ of connected subgraphs $G\,(A_i)$ of $G\,(V)$ by $\mathcal{A} := \{A_i\}$ with pairwise $A_i \cap A_j = \emptyset$ and $\mathcal{A} \neq \emptyset$. Furthermore, there exists no edege (a_i, a_j), $a_i \in A_i$ and $a_j \in A_j$.

We denote the MIS of $G\,(\mathcal{A})$ by I'.

We introduce a <u>split function</u> sp which splits a graph $G\,(V)$ into $G\,(\mathcal{A})$ and $G'' := G\,(V-\mathcal{A})$.

During splitting, sp induces over the <u>impact</u> $imp\,(V-\mathcal{A})$ a <u>dimension recalibration</u> on $G\,(V-\mathcal{A})$ by $sp \rightarrow imp\,(V-\mathcal{A})$, $\mathbb{R}^{dim} \rightarrow \mathbb{R}^{dim'}$, with $dim' := max\{deg\,(v_i)\}$, $\forall\, v_i \in (V \setminus \mathcal{A})$, and an <u>edge influence</u> inf on the graph $G\,(V-\mathcal{A})$ in the following way:

1. inf introduces a <u>new weighted virtual edge</u> $(e^w := (d_i, d_j))$ $e^w_{dim'} = \left(e^w_{dim'-1}, d_j\right)$, $d_j \in D_{\mathcal{A}}$ <u>and</u> d_k of $e^w_{dim'-1}$ with $d_k \in D_{\mathcal{A}}$, $D_{\mathcal{A}}$ be our set of influencers of $\mathcal{A}$, on the graph $G\,(V-\mathcal{A})$, if there exists a MIS <u>size reducing influence</u> on I' of $G\,(\mathcal{A})$, denoted by $inf\,(I', e^w)$, because of the establishing of the edge e^w (if there not already existed that edge before).
2. If there already existed that edge $e^w := \left(e^w_{dim'-1}, d_j\right)$ before, we don't change anything and call it a <u>neutral</u>, id, influence.

We denote the <u>new adjacence tensor</u> of $G\,(V-\mathcal{A})$ by $T^{V-\mathcal{A}}_{dim'}$ if we introduced <u>weighted</u> virtual edges e^w which give us the amount of negative changing of I' with $T^{V-\mathcal{A}}_{dim'}$ entries of e^w's with values ≤ -1, $\forall\, e^w$ in $T^{V-\mathcal{A}}_{dim'}$.

We identify edges in $T_{dim'}^{V-\mathcal{A}}$ of $V \setminus \mathcal{A}$ with $edge\,(d_i, y_k)$ or $edge\,(y_k, y_l)^2$, $d_i \in D_{\mathcal{A}}$ and $y_k, y_l \in V \setminus \{\mathcal{A} \cup D_{\mathcal{A}}\}$, with $d_i \sim y_k$ respectively $y_k \sim y_l$ as prohibit pattern '11' bit string connections and hence mark this edges in the weighted adjacence tensor $T_{dim'}^{V-\mathcal{A}}$ with an entry value of '$-\infty$'.

We denote the so called standadized new adjacence tensor of $G\,(V - \mathcal{A})$ by $T_{dim'}^{(V-\mathcal{A})'}$ in which we set all of the entry values of the weighted adjacence tensor $T_{dim'}^{V-\mathcal{A}}$, which are $\neq 0$, to 1.

Looking at the weighted adjacence tensor $T_{dim'}^{V-\mathcal{A}}$, we want to give a short, sketchy, discussion regarding (maximum) independent sets.

1. Tensor $T_{dim'}^{V-\mathcal{A}}$ is symetrically according its dimensional diagonals. ($\Rightarrow$ for example $v_1v_2v_3 = v_1v_3v_2 = v_2v_3v_1 = \cdots$)
2. We call diagnonals of the form $\underbrace{v_{i_1}v_{i_1}v_{i_1}\cdots v_{i_1}}_{dim'-times}$ 1-diagonal forms
3. We don't have multiple edges. Means e.g. $v_1v_2v_2 = v_1v_2$ with one edge between v_1 and v_2
4. If we sum the values of $T_{dim'}^{V-\mathcal{A}}$ entries, because of symmetry of $T_{dim'}^{V-\mathcal{A}}$, we only sum entries of the valid symmetry area. Additionally, we have to take into account that during determing $T_{dim'}^{V-\mathcal{A}}$, entries of diagonals are generated by multiple counting of changings. Hence, the sum of diagonal entries is gien by the norming to -1 over the number of diagonal entries over which we sum $((\text{sum } t_{diagonal})\, 1/|\{t_{diagonal}\}|)$
5. Possibility discussion for finding maximum independent sets in $T_{dim'}^{(V-\mathcal{A})'}$ for the example of a flat surface graph $G\,(V - \mathcal{A})$ on $\mathbb{R}^2$.

   Case 1 (see figure 1, ($\star$) entries which are not given are < -1)

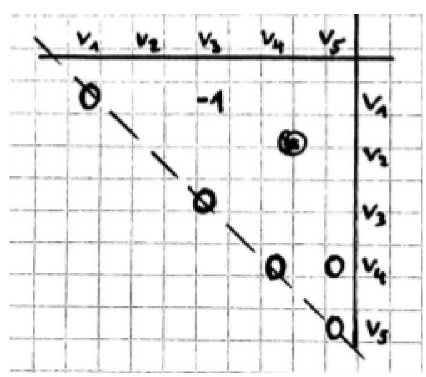

Figure 1: Case 1

$\rightarrow \{v_4, v_5\}$ IS of size 2 on $G\,(V - \mathcal{A})$
$\Rightarrow |I| = |I'| + |IS| = |I'| + 2$

$\rightarrow \{v_1, v_3\}$ set of size 2 on $G\,(V - \mathcal{A})$ but reducing the IS I' of $G\,(\mathcal{A})$ by -1.
$\Rightarrow |I| = |I'| - 1 + |set| = |I'| - 1 + 2 = |I'| + 1$

$\rightarrow \{v_1\}$ or $\{v_2\}$ IS of size 1 on $G\,(V - \mathcal{A})$
$\Rightarrow |I| = |I'| + |IS| = |I'| + 1$

[2] This is for the flat surface graph case. In generally, we consider of course $edge\,(edge_{dim'-1}, y_l)$ with d_i or $y_k \in edge_{dim'-1}$.

$\Rightarrow \{v_4, v_5\}$ leads us to the MIS of $G(V)$

$\Rightarrow$ In this case, we can determine the MIS of $G(V)$ by solving the MIS problem of the standardized adjacence tensor $T_{dim'}^{(V-\mathcal{A})'}$.

Remark 2.1. For each subgraph A_i of $\mathcal{A}$ (for flat surface graphs) we can get maximal one $v_i v_j = 0$, with $v_i \neq v_j$, entry.

<u>Case 2</u> (see figure 2)

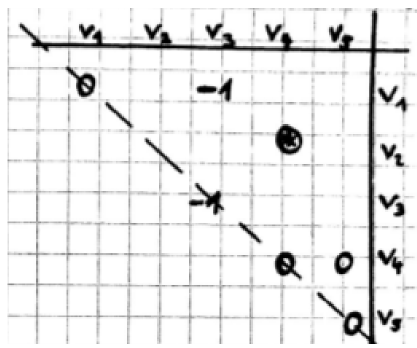

Figure 2: Case 2

$\rightarrow \{v_4, v_5\}$ IS of size 2 on $G(V - \mathcal{A})$
$\Rightarrow |I| = |I'| + |IS| = |I'| + 2$

$\rightarrow \{v_1\}$ IS of size 1 on $G(V - \mathcal{A})$
$\Rightarrow |I| = |I'| + |IS| = |I'| + 1$

$\rightarrow \{v_1, v_3\}$ set of size 2 on $G(V - \mathcal{A})$ but reducing the IS I' of $G(\mathcal{A})$ by $-1 - 1 = -2$.
$\Rightarrow |I| = |I'| - 2 + |set| = |I'| - 2 + 2 = |I'|$

$\Rightarrow \{v_4, v_5\}$ leads us to the MIS of $G(V)$.

$\Rightarrow$ In this case, we can determine the MIS of $G(V)$ by solving the MIS problem of the standardized adjacence tensor $T_{dim'}^{(V-\mathcal{A})'}$

<u>Case 3</u> (see figure 3)

$\rightarrow \{v_3\}$ or $\{v_1\}$ set of size 1 on $G(V - \mathcal{A})$ but reducing the IS I' of $G(\mathcal{A})$ by -1
$\Rightarrow |I| = |I'| - 1 + |set| = |I'| - 1 + 1 = |I'|$

$\rightarrow \{v_1, v_3\}$ set of size 2 on $G(V - \mathcal{A})$ but reducing the IS I' of $G(\mathcal{A})$ by -1 if v_1 and v_3 (like in our example) are elements of the same D_{A_i}, and -2 if v_1 and v_3 belong to different $D_{A_i} \neq D_{A_j}$.
$\Rightarrow$ For our example, we get $|I| = |I'| - 1 + |set| = |I'| - 1 + 2 = |I'| + 1$
(else: $|I| = |I'| - 2 + 2 = |I'|$)

$\rightarrow \{v_4, v_5\}$ IS of size 2 on $G(V - \mathcal{A})$
$\Rightarrow |I| = |I'| + |IS| = |I'| + 2$

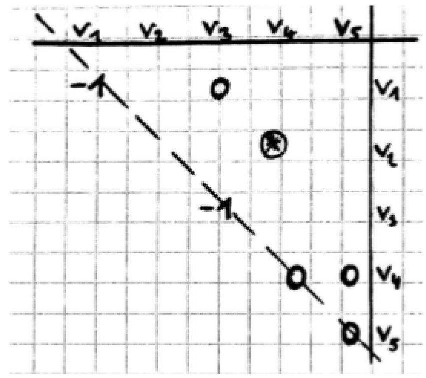

Figure 3: Case 3

$\Rightarrow \{v_4, v_5\}$ leads us to the MIS of $G(V)$.

$\Rightarrow$ In this case, we can determine the MIS of $G(V)$ by solving the MIS problem of the standardized adjacence tensor $T_{dim'}^{(V-\mathcal{A})'}$.

<u>Case 4</u> (see figure 4)

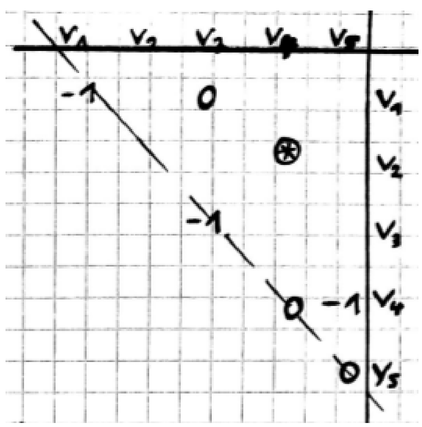

Figure 4: Case 4

$\rightarrow \{v_3\}, \{v_4\}$ set of size 1 on $G(V-\mathcal{A})$ but reducing the IS I' of $G(\mathcal{A})$ by -1.
$\Rightarrow |I| = |I'| - 1 + |set| = |I'| - 1 + 1 = |I'|$

$\rightarrow \{v_1, v_3\}$ set of size 2 on $G(V-\mathcal{A})$ but reducing the IS of I' of $G(\mathcal{A})$ by -1.
$\Rightarrow |I| = |I'| - 1 + 2 = |I'| + 1$

$\rightarrow \{v_4\}$ or $\{v_5\}$ IS of size 1 on $G(V-\mathcal{A})$
$\Rightarrow |I| = |I'| + |IS| = |I'| + 1$

$\rightarrow \{v_4, v_5\}$ set of size 2 on $G(V-\mathcal{A})$ but reducing the IS I' of $G(\mathcal{A})$ by -1.
$\Rightarrow |I| = |I'| - 1 + |set| = |I'| - 1 + 2 = |I'| + 1$

$\Rightarrow$ We see, that in this case, we get the MIS of $G(V)$ with size $|I'|+1$ for: $\{v_1, v_3\}$, $\{v_4\}$, $\{v_5\}$, $\{v_4, v_5\}$

Discussion of case 4:

1. Assume we take the standardized adjacence tensor $T_{dim'}^{(V-\mathcal{A})'}$. So, we have in figure 5:

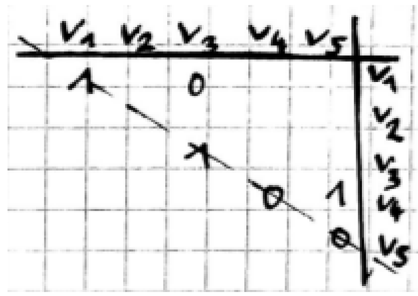

Figure 5: Discussion of case 4, item 1

We get a MIS: $\{v_4\}$ and $\{v_5\}$
$\Rightarrow$ We can derive the set $\{v_4, v_5\}$ by having an additional look at our original weighted tensor $T_{dim'}^{V-\mathcal{A}}$, by taking v_4 and v_5 and seeing that $v_4 v_5 = -1$, so just a reducing by -1 by taking an addiontal vertex to v_4 (respectively taking an additional vertex to v_5). $\Rightarrow$ So, we get the same total size solution of $|I|$, $G(V)$, by $\{v_4, v_5\}$

$\rightarrow$ $\{v_1, v_3\}$ would be not find by a MIS solution of $G(V-\mathcal{A})$. We only get it, if we notice that $v_1 v_3 = 0$ and that all diagonal entries of the participating vertices are -1 in $T_{dim'}^{V-\mathcal{A}}$. So, we would also get this set of vertices.

$\rightarrow$ If we put all together, we get totally: $\{v_1, v_3\} \cup \{v_4\}$ and $\{v_1, v_3\} \cup \{v_5\}$ and $\{v_1, v_3\} \cup \{v_4, v_5\}$, all with the same total MIS size of $G(V-\mathcal{A})$ by $|I| = |I'| + 1 + 1 = |I'| + 2$.

2. Assume, we take the standardized adjacence tensor $T_{dim'}^{(V-\mathcal{A})'}$, but set all diagonal entries to 0 (see figure 6). We get the MIS $\{v_1, v_3\}$ with $|I| = |I'| + 2$ which is obvousily wrong.

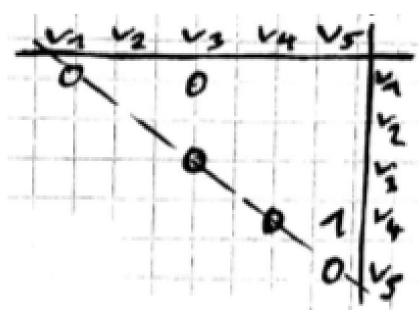

Figure 6: Discussion of case 4, item 2

$\Rightarrow$ So it is necessary still to check the results in the original tensor $T_{dim'}^{(V-\mathcal{A})'}$.

$\Rightarrow$ Furthermore, with an MIS algorithm on this tensor, here, we would not find $\{v_4\}$, $\{v_5\}$ and $\{v_4, v_5\}$!

We assume, that we are able to identify the MIS in $G(\mathcal{A})$, which we denote by I'.

We look again at the weighted adjacence tensor $T_{dim'}^{V-\mathcal{A}}$.

1. Case:
   The MIS of $G(V-\mathcal{A})$ is given by a set of vertice entries in $T_{dim'}^{V-\mathcal{A}}$ in which all entries of the vertices of this set have value 0.

   $\Rightarrow$ We denote this MIS by I'' and recognize that the already known MIS of $G(\mathcal{A})$ by I' **doesn't** get reduced in its size.

   $\Rightarrow$ The total MIS of $G(V)$ is given by: $I = I' \cup I''$ and I'' in $T_{dim'}^{V-\mathcal{A}}$ consists of only 0 entries, so we have no participating edge in I'' (similar to the case: $I'' \cap C = \emptyset$ in [**?**]).

2. Case:
   The MIS of $G(V-\mathcal{A})$ is given by a set of vertices in $T_{dim'}^{V-\mathcal{A}}$, so that at least one vertex of this set of vertices does have an entry value in $T_{dim'}^{V-\mathcal{A}}$ of $\neq 0$.

   $\Rightarrow$ We also denote MIS by I'' and recognize that the already known MIS of $G(\mathcal{A})$ by I' get reduced in its size by the entries $\neq 0$ in $T_{dim'}^{V-\mathcal{A}}$, which belonging vertices are building a clique on $e^w := \left(e_{dim-1}^w, d_j\right)$, dim', edges [3]. We denote the reduced MIS of $G(\mathcal{A})$ by $I(\{v\})$, $\{v\}$ being the set of vertices with entries $\neq 0$.

   $\Rightarrow$ The total MIS of $G(V)$ is given by: $I = I(\{v\}) \cup I''$ and I'' in $T_{dim'}^{V-\mathcal{A}}$ consists of at least one $\neq 0$ entry, so we have at least one participating edge in I'' (similar to the case: $v \in I'' \cap C$ in [**?**]).

   Remark 2.2. MIS vertex entries $\neq 0$ of I'' in $T_{dim'}^{V-\mathcal{A}}$ are given by the virtual weighted edges which we introduced and are in fact 'connections' between $d_i \in D_{\mathcal{A}}$ vertices.

Until now, we assumed that $G(V)$ and $G(V-\mathcal{A})$ be always connected graphs.

Given a not connected graph G. Then there exists $G_1, G_2, G_3, \dots$ part graphs such that $G(V) = \bigcup_i G_i$ with $V := \bigcup_i V_i$ and $E := \bigcup_i E_i$ for all $G_i(V_i, E_i)$. The MIS of $G(V)$ is given by the union of the MIS_i of $G_i(V_i, E_i)$, for all G_i with $MIS_V := \bigcup_i MIS_i$.

Until now, we assumed that $G(V)$ and $G(V-\mathcal{A})$ be always Eulerian graphs.

Why we use Eulerian graphs(?)
The efficiency of the proposed MIS approach is mostly influenced by the choice of vertices for the subgraphs A_i of $\mathcal{A}$ (their vertex edge structure and degrees).
The easiest ways of choosing the most favorite subgraph with certain characteristics is given by going through a known Eulerian cycle sequence from which we can grab A_i sets with $\mathcal{O}(length(S_E))$ time complexity.

In the beginning we talked about Eulerian graphs and Hierholzer's algorithm for graphs $G(V)$ which can be represented by a classical adjacence matrix, our $T_{dim=2}^V$ tensor on $\mathbb{R}^2$.

Without loss of generality, we can assume that there exists for each graph on $\mathbb{R}^{dim>2}$ manifold regarding its belonging T_{dim}^V representation, there exists a recursive mapping for $e_{dim} = (e_{dim-1}, v_{dim})$ with $e_{dim-1} \mapsto e_{dim}$. Hence, there exists a valid representation mapping of an Eulerian cycle of a $T_{dim=2}^V$ representation by an edge concatenation to an Eulerian cycle of a $T_{dim>2}^V$ representation and vice versa.

$\Rightarrow$ It exists an Hierholzer's algorithm equivalent algorithm for graphs with a $T_{dim>2}^V$ representation.

[3] Not sure. I think the case of having only one vertex with an edge on themselve is also possible

Given a connected graph $G(V)$. Assume G is not Eulerian, which means it exists vertices $v_i' \in V$ with $deg(v_i') = odd$. From basic graph theory, we know that for every graph G, we have $2|E| = \sum_{v_i \in V} deg(v_i)$.

$\Rightarrow$ We know that we always have an even number of vertices with odd degree.

$\Rightarrow$ We can map G to an Eulerian graph by connecting always two of the odd degree vertices $\{u, v\}$ by one new introduced helper vertex $h \in H$ (be H the set of helper vertices) and hence we get two new edges (u, h) and (h, v).

Consequences of adding helper vertices regarding MIS's:

1. Case: The helper vertex h connects two vertices u and v which belong to the same independent set S, which means $u \in S \wedge v \in S$ and hence $(u, v) \notin E$.

   $\Rightarrow$ Since u and v are still not directly connected with each other, they will still belong to the same independet set. Since, each of them have a connection to h, h will become a member of an other, second, independent set. So, we will get our original independent set S unchanged and any other second independent set with $S' \cup \{h\}$.

2. Case: The helper vertex h connects two vertices u and v which belong to two different independent sets S_i, S_j with $S_i \nsubseteq S_j$, which means $u \in S_i \wedge v \in S_j$ and hence $(u, v) \in E$.

   $\Rightarrow$ Since u and v are already directly connected, also with a connection to h, they will still both belong to two different sets. Since, each of them have a connection to h, h will become a member of an other, third, independent set. So, we will get our original two independent sets S_i and S_j unchanged and any other third independent set with $S_k \cup \{h\}$.

3. It's clear, that an helper vertex is not allowed to be element of a MIS solution, else this solution wouldn't be anymore, automatically the MIS solution of th real, original, given graph $G(V)$.

   $\Rightarrow$ After generating the Eulerian cycle sequence S_E, we have to take care that helper vertices are not be considered for finding the MIS soutions of the graphs $G(\mathcal{A})$ and $G(V - \mathcal{A})$.

   $\Rightarrow$ In the disruptive bit string representation S_E^{N-1}, the belonging bits of helper vertices have to be set to 0 mandatory.

   $\Rightarrow$ To be sure, that helper vetices are not considered for the IS solutions of $G(V - \mathcal{A})$, all entries of $T_{dim'}^{V-\mathcal{A}}$ with $v_{i_1} v_{i_2} \cdots v_{i_{dim'}}$, with at least one v_{i_j} being a helper vertex are set to '$-\infty$' (Marked as not allowed vertices to choose).

   $\Rightarrow$ The identification of known helper vertices within the Eulerian cycle sequence can be done in $\mathcal{O}(length(S_E))$ time complexity.

Finally at this point, we want still give a sketchy discussion regarding the choice of $\mathcal{A}$ respectively a single subgraph vertice set A_i.

Remember the general property of $\mathcal{A}$ with A_i building connected subgraphs $G(A_i)$ of $G(\mathcal{A})$ and $A_i \cap A_j = \emptyset$, $\mathcal{A} \neq \emptyset$. Furthermore, there exists **no** *edge* (a_i, a_j), $a_i \in A_i$ and $a_j \in A_j$ between subsets.

Let's look at a single subset A_i, its belonging subgraph $G(A_i)$ and the influencers of A_i, D_{A_i}.

We define $n := |D_{A_i}|$.

If the edge influence inf introduces a new weighted virtual edge of the form $e^{w}_{dim'} := \left(e^{w}_{dim'-1}, d_{i_1}\right)$, $e^{w}_{dim'-1} := \left(e^{w}_{dim'-2}, d_{i_2}\right)$, $\ldots$, $e^{w}_{dim'-(dim'-1)} := \left(d_0, d_{i_{dim'-1}}\right)$, $d_0 = d_{i_1} = d_{i_2} = \cdots = d_{i_{dim'-1}} \in D_{A_i}$, we call the edge $e^{w}_{dim'}$ an $(1, dim') - edge$ (this vertex is only connected to themselve).

If $d_0 = d_{i_2} = d_{i_3} = \cdots = d_{i_{dim'-1}}$ and $d_0 \neq d_{i_1} \in D_{A_i}$, we call the edge $e^{w}_{dim'}$ an $(2, dim'-1, 1) - edge$, and so on ... .

The tuple $(x, a, b, c, \ldots)$ means, we have x different vertices. The first of them appears a times, the second of them appears b times and so on ... .

Assume now , we have a subgraph $G(A_i)$ which only introduces edges of the $(1, dim')$ form on $T^{V-A_i}_{dim'}$. This means we only get new edges on themselves for $d_i \in D_{A_i}$. All the entries, appart from the tensor diagonal entries, stay the same means $T^{V-A_i}_{dim'} = T^{V}_{dim}$ for non-diagonal entries. So, we have 'mainly' the same tensor like before, just reduced by the vertices of set A_i.

If we consider for example $(2, dim'-1, 1)$ edges instead, we now can have edges between two $d_i, d_k \in D_{A_i}$ vertices, which didn't have an edge before.

$\Rightarrow$ Hence, the kind of choice we make regarding the decision which subgraphs A_i we choose, has a strong influence on the characteristics of $T^{V-\mathcal{A}}_{dim'}$ tensors and hence also on our algorithm approach for finding (M)IS's.

License

References