

# Notes: Complex boolean functions

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DRAFT

## Abstract

We examine the boolean functions **NEG**, **AND** and **OR** on the field of complex numbers  $\mathbb{C}$  with boolean values  $\{i, 1\}$  and the map between them and the classical definition with  $\{0, 1\}$ .

**Keywords:** XXX, YYY, ZZZ

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## Preamble

The following content is a sketch for discussion purposes only, without warranty for mathematical completeness.

## 1 Introduction

In this short notes, we will do an analysis of boolean functions on the field of complex numbers  $\mathbb{C}$  for a specific defined case regarding the values *True* and *False*, based on the work of Ryan ODonnell [1], Zilong Wang and Guang Gong [2] and Zhiwei Zhang [3] [4].

## 2 Real and complex boolean functions

We examine the representation of the boolean functions **NEG**, **AND** and **OR** on the field of real numbers and on the field of complex numbers as well as the connections between this two definitions.

### 2.1 Boolean functions

Given are integer boolean values  $x \in \{0, 1\}$  and complex values  $z \in \mathbb{C}$ . For the boolean cases **NEG**, **AND** and **OR** we will use the two complex boolean values  $\{i, 1\}$ .

We will denote the inverse boolean value of  $x$  by  $\bar{x}$  respectively for  $z$  by  $\bar{z}$ . For the complex conjugate of  $z$ , we will write  $z^c$ .

In table 1, you can see the values for **NEG** of  $x$  and  $z$ , which gives us several statements.

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Table 1: NEG

$x$	$-x$	$\bar{x}$	$z$	$-z$	$\bar{z}$
0	0	1	i	-i	1
1	-1	0	1	-1	i

**Theorem 2.1** (NEG of  $x$ ). *The inverse boolean value of  $x$ , generated by  $NEG(x)$ , is given by  $\bar{x} = 1 - x$ .*

**Theorem 2.2** (NEG of  $z$ ). *The inverse boolean value of  $z$ , generated by  $NEG(z)$ , is given by  $\bar{z} = (1 + i) - z$  and by  $\bar{z} = z^3 i$ .*

For AND and OR we need an helper boolean function, called *generator*.

**Theorem 2.3** (Generator  $g$  of  $(x_i, x_j)$ ). *The generator  $g$  of  $(x_i, x_j)$ , which maps  $(0, 0) \mapsto 1$ ,  $(0, 1) \mapsto 0$ ,  $(1, 0) \mapsto 0$  and  $(1, 1) \mapsto 0$ , is given by*

$$\begin{aligned} g(x_i, x_j) &= (-2) \left( \frac{1}{2} (x_i x_j + (x_i + x_j) - 1) - x_i x_j \right) \\ &= x_i x_j - (x_i + x_j) + 1. \end{aligned} \quad (1)$$

**Theorem 2.4** (Generator  $g$  of  $(z_i, z_j)$ ). *The generator  $g$  of  $(z_i, z_j)$ , which maps  $(i, i) \mapsto i$ ,  $(i, 1) \mapsto 0$ ,  $(1, i) \mapsto 0$  and  $(1, 1) \mapsto 0$ , is given by*

$$\begin{aligned} g(z_i, z_j) &= \frac{1}{2} (z_i z_j + (z_i + z_j) - 1) - z_i z_j \\ &= \frac{1}{2} (-z_i z_j + (z_i + z_j) - 1). \end{aligned} \quad (2)$$

In table 2, you can see the values for AND of  $(x_i, x_j)$  and  $(z_i, z_j)$ .

Table 2: AND

$x_i$	$x_j$	$x_i x_j$	$x_i \wedge x_j$	$z_i$	$z_j$	$z_i z_j$	$z_i \wedge z_j$
0	0	0	0	i	i	-1	i
0	1	0	0	i	1	i	i
1	0	0	0	1	i	i	i
1	1	1	1	1	1	1	1

**Theorem 2.5** (AND of  $(x_i, x_j)$ ). *The boolean value of  $x_i \wedge x_j$ , generated by  $AND(x_i, x_j)$ , is given by*

$$\begin{aligned} x_i \wedge x_j &= x_i x_j \\ &= g(\bar{x}_i, \bar{x}_j). \end{aligned} \quad (3)$$

**Theorem 2.6** (AND of  $(z_i, z_j)$ ). *The boolean value of  $z_i \wedge z_j$ , generated by  $AND(z_i, z_j)$ , is given by*

$$\begin{aligned} z_i \wedge z_j &= z_i z_j + (1 - i) g(z_i, z_j) \\ &= i - (1 + i) g(\bar{z}_i, \bar{z}_j). \end{aligned} \quad (4)$$

Table 3: OR

$x_i$	$x_j$	$x_i + x_j$	$x_i \vee x_j$	$z_i$	$z_j$	$z_i + z_j$	$z_i \vee z_j$
0	0	0	0	i	i	2i	i
0	1	1	1	i	1	1 + i	1
1	0	1	1	1	i	1 + i	1
1	1	2	1	1	1	2	1

Finally, in table 3, you can see the values for OR of  $(x_i, x_j)$  and  $(z_i, z_j)$ .

**Theorem 2.7** (OR of  $(x_i, x_j)$ ). *The boolean value of  $x_i \vee x_j$ , generated by  $OR(x_i, x_j)$ , is given by*

$$\begin{aligned} x_i \vee x_j &= (x_i + x_j) - g(\bar{x}_i, \bar{x}_j) \\ &= 1 - g(x_i, x_j). \end{aligned} \quad (5)$$

**Theorem 2.8** (OR of  $(z_i, z_j)$ ). *The boolean value of  $z_i \vee z_j$ , generated by  $OR(z_i, z_j)$ , is given by*

$$\begin{aligned} z_i \vee z_j &= (z_i + z_j) - i + (1 + i)g(\bar{z}_i, \bar{z}_j) \\ &= 1 + (1 + i)g(z_i, z_j). \end{aligned} \quad (6)$$

*Proof of theorems 2.1 - 2.8.* The correctness of the given statements can be easily checked with the help of the given tables 1, 2 and 3.  $\square$

## 2.2 Inter-field connections

We want to determine some useful inter-fiel connections between the boolean functions for  $x$  and the boolean functions for  $z$ . Since, we can also express OR with the help of NEG and AND by  $y_i \vee y_j = \overline{\bar{y}_i \wedge \bar{y}_j}$ , we will only consider AND connections.

At first we consider the connection between  $x_i \wedge x_j$  and  $z_i \wedge z_j$ .

**Theorem 2.9** (Inter-field  $x_i \wedge x_j$  with  $z_i \wedge z_j$  connection). *The inter-field connection between  $x_i \wedge x_j$  and  $z_i \wedge z_j$  is given by*

$$(x_i \wedge x_j)(1 - i) = (z_i \wedge z_j) - i. \quad (7)$$

Next, we consider the connection between  $\bar{x}_i \wedge \bar{x}_j$ ,  $z_i \wedge z_j$  and  $z_i z_j$ .

**Theorem 2.10** (Inter-field  $\bar{x}_i \wedge \bar{x}_j$  with  $z_i \wedge z_j$  and  $z_i z_j$  connection). *The inter-field connection between  $\bar{x}_i \wedge \bar{x}_j$ ,  $z_i \wedge z_j$  and  $z_i z_j$  is given by*

$$(\bar{x}_i \wedge \bar{x}_j)(1 + i) = (z_i \wedge z_j) - z_i z_j. \quad (8)$$

*Proof of theorem 2.9 and 2.10.* The correctness of the given statements can be easily checked with the help of the given tables 1, 2 and 3.  $\square$

Finally, we consider the connection between  $x_i \wedge x_j$  and  $z_i z_j$ .

**Theorem 2.11** (Inter-field  $x_i \wedge x_j$  with  $z_i z_j$  connection). *The inter-field connection between  $x_i \wedge x_j$  and  $z_i z_j$  is given by*

$$x_i \wedge x_j = \frac{1}{4} \left( z_i z_j + (z_i z_j)^2 + (z_i z_j)^3 + 1 \right). \quad (9)$$

*Proof of theorem 2.11.* In table 4 you can see the powers of  $z_i z_j$ . One possibility to proof the statement is simply to check this for each possible value. Additionally, we can also interpret  $z_i z_j$  as a 4'th root of unity  $\zeta$ . For n'th roots of unity it is known that

$$1 + \zeta + \zeta^2 + \dots + \zeta^{n-1} = \begin{cases} n & \text{if } \zeta = 1 \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

This equals to a classical AND given by  $4 \cdot (x_i \wedge x_j)$  with  $\{0, 1\}^2 \rightarrow \{0, 4\}$  for  $n = 4$ .  $\square$

Table 4: Powers of  $z_i z_j$

$z_i$	$z_j$	$z_i z_j$	$(z_i z_j)^2$	$(z_i z_j)^3$	$(z_i z_j)^4$
i	i	-1	1	-1	1
i	1	i	-1	-i	1
1	i	i	-1	-i	1
1	1	1	1	1	1

From the properties of 4'th roots of unity, we can follow

$$\begin{aligned} \bigwedge_j x_j &= \frac{1}{4} \left( \prod_j z_j + \prod_j z_j^2 + \prod_j z_j^3 + 1 \right) \\ &= \frac{1}{4} \sum_{k=1}^4 \prod_j z_j^k. \end{aligned} \quad (11)$$

### 3 Conclusion

We will see, that our results will be helpful in future works.

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