Notes: Complex boolean functions

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DRAFT

Abstract

We examine the boolean functions `NEG`, `AND` and `OR` on the field of complex numbers $\mathbb{C}$ with boolean values $\{i, 1\}$ and the map between them and the classical definition with $\{0, 1\}$.

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Preamble

The following content is a sketch for discussion purposes only, without warranty for mathematical completeness.

1 Introduction

In this short notes, we will do an analysis of boolean functions on the field of complex numbers $\mathbb{C}$ for a specific defined case regarding the values *True* and *False*, based on the work of Ryan ODonnell [1], Zilong Wang and Guang Gong [2] and Zhiwei Zhang [3] [4].

2 Real and complex boolean functions

We examine the representation of the boolean functions `NEG`, `AND` and `OR` on the field of real numbers and on the field of complex numbers as well as the connections between this two definitions.

2.1 Boolean functions

Given are integer boolean values $x \in \{0, 1\}$ and complex values $z \in \mathbb{C}$. For the boolean cases `NEG`, `AND` and `OR` we will use the two complex boolean values $\{i, 1\}$.

We will denote the inverse boolean value of x by $\bar{x}$ respectively for z by $\bar{z}$. For the complex conjungate of z, we will write z^c.

In table 1, you can see the values for `NEG` of x and z, which gives us several statements.

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Table 1: NEG

x	$-x$	$\bar{x}$	z	$-z$	$\bar{z}$
0	0	1	i	-i	1
1	-1	0	1	- 1	i

Theorem 2.1 (NEG of x). *The inverse boolean value of x, generated by NEG(x), is given by $\bar{x} = 1 - x$.*

Theorem 2.2 (NEG of z). *The inverse boolean value of z, generated by NEG(z), is given by $\bar{z} = (1 + i) - z$ and by $\bar{z} = z^3 i$.*

For AND and OR we need an helper boolean function, called *generator*.

Theorem 2.3 (*Generator g* of (x_i, x_j)). *The generator g of (x_i, x_j), which maps $(0,0) \mapsto 1$, $(0,1) \mapsto 0$, $(1,0) \mapsto 0$ and $(1,1) \mapsto 0$, is given by*

$$\begin{aligned} g(x_i, x_j) &= (-2)\left(\frac{1}{2}(x_i x_j + (x_i + x_j) - 1) - x_i x_j\right) \\ &= x_i x_j - (x_i + x_j) + 1. \end{aligned} \tag{1}$$

Theorem 2.4 (*Generator g* of (z_i, z_j)). *The generator g of (z_i, z_j), which maps $(i,i) \mapsto i$, $(i,1) \mapsto 0$, $(1,i) \mapsto 0$ and $(1,1) \mapsto 0$, is given by*

$$\begin{aligned} g(z_i, z_j) &= \frac{1}{2}(z_i z_j + (z_i + z_j) - 1) - z_i z_j \\ &= \frac{1}{2}(-z_i z_j + (z_i + z_j) - 1). \end{aligned} \tag{2}$$

In table 2, you can see the values for AND of (x_i, x_j) and (z_i, z_j).

Table 2: AND

x_i	x_j	$x_i x_j$	$x_i \wedge x_j$	z_i	z_j	$z_i z_j$	$z_i \wedge z_j$
0	0	0	0	i	i	-1	i
0	1	0	0	i	1	i	i
1	0	0	0	1	i	i	i
1	1	1	1	1	1	1	1

Theorem 2.5 (AND of (x_i, x_j)). *The boolean value of $x_i \wedge x_j$, generated by AND(x_i, x_j), is given by*

$$\begin{aligned} x_i \wedge x_j &= x_i x_j \\ &= g(\bar{x}_i, \bar{x}_j). \end{aligned} \tag{3}$$

Theorem 2.6 (AND of (z_i, z_j)). *The boolean value of $z_i \wedge z_j$, generated by AND(z_i, z_j), is given by*

$$\begin{aligned} z_i \wedge z_j &= z_i z_j + (1 - i)\, g(z_i, z_j) \\ &= i - (1 + i)\, g(\bar{z}_i, \bar{z}_j). \end{aligned} \tag{4}$$

Table 3: OR

x_i	x_j	$x_i + x_j$	$x_i \vee x_j$	z_i	z_j	$z_i + z_j$	$z_i \vee z_j$
0	0	0	0	i	i	2i	i
0	1	1	1	i	1	1 + i	1
1	0	1	1	1	i	1 + i	1
1	1	2	1	1	1	2	1

Finally, in table 3, you can see the values for `OR` of (x_i, x_j) and (z_i, z_j).

Theorem 2.7 (`OR` of (x_i, x_j))**.** *The boolean value of $x_i \vee x_j$, generated by `OR`(x_i, x_j), is given by*

$$\begin{aligned} x_i \vee x_j &= (x_i + x_j) - g\left(\bar{x}_i, \bar{x}_j\right) \\ &= 1 - g\left(x_i, x_j\right). \end{aligned} \tag{5}$$

Theorem 2.8 (`OR` of (z_i, z_j))**.** *The boolean value of $z_i \vee z_j$, generated by `OR`(z_i, z_j), is given by*

$$\begin{aligned} z_i \vee z_j &= (z_i + z_j) - i + (1 + i)\, g\left(\bar{z}_i, \bar{z}_j\right) \\ &= 1 + (1 + i)\, g\left(z_i, z_j\right). \end{aligned} \tag{6}$$

Proof of theorems 2.1 - 2.8. The correctness of the given statements can be easily checked with the help of the given tables 1, 2 and 3. ☐

2.2 Inter-field connections

We want to determine some useful inter-fiel connections between the boolean functions for x and the boolean functions for z. Since, we can also express `OR` with the help of `NEG` and `AND` by $y_i \vee y_j = \overline{\overline{y_i} \wedge \overline{y_j}}$, we will only consider `AND` connections.

At first we consider the connection between $x_i \wedge x_j$ and $z_i \wedge z_j$.

Theorem 2.9 (Inter-field $x_i \wedge x_j$ with $z_i \wedge z_j$ connection)**.** *The inter-field connection between $x_i \wedge x_j$ and $z_i \wedge z_j$ is given by*

$$\left(x_i \wedge x_j\right)(1 - i) = \left(z_i \wedge z_j\right) - i. \tag{7}$$

Next, we consider the connection between $\bar{x}_i \wedge \bar{x}_j$, $z_i \wedge z_j$ and $z_i z_j$.

Theorem 2.10 (Inter-field $\bar{x}_i \wedge \bar{x}_j$ with $z_i \wedge z_j$ and $z_i z_j$ connection)**.** *The inter-field connection between $\bar{x}_i \wedge \bar{x}_j$, $z_i \wedge z_j$ and $z_i z_j$ is given by*

$$\left(\bar{x}_i \wedge \bar{x}_j\right)(1 + i) = \left(z_i \wedge z_j\right) - z_i z_j. \tag{8}$$

Proof of theorem 2.9 and 2.10. The correctness of the given statements can be easily checked with the help of the given tables 1, 2 and 3. ☐

Finally, we consider the connection between $x_i \wedge x_j$ and $z_i z_j$.

Theorem 2.11 (Inter-field $x_i \wedge x_j$ with $z_i z_j$ connection)**.** *The inter-field connection between $x_i \wedge x_j$ and $z_i z_j$ is given by*

$$x_i \wedge x_j = \frac{1}{4}\left(z_i z_j + (z_i z_j)^2 + (z_i z_j)^3 + 1\right). \tag{9}$$

Proof of theorem 2.11. In table 4 you can see the powers of $z_i z_j$. One possibility to proof the statement is simply to check this for each possible value. Additionally, we can also interpret $z_i z_j$ as a 4'th root of unity ζ. For n'th roots of unity it is known that

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} = \begin{cases} n & \text{if } \zeta = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

This equals to a classical `AND` given by $4 \cdot (x_i \wedge x_j)$ with $\{0,1\}^2 \to \{0,4\}$ for $n = 4$. ∎

Table 4: Powers of $z_i z_j$

z_i	z_j	$z_i z_j$	$(z_i z_j)^2$	$(z_i z_j)^3$	$(z_i z_j)^4$
i	i	-1	1	-1	1
i	1	i	-1	-i	1
1	i	i	-1	-i	1
1	1	1	1	1	1

From the properties of 4'th roots of unity, we can follow

$$\bigwedge_j x_j = \frac{1}{4} \sum_{k=1}^{4} \prod_j z_j^k. \tag{11}$$

2.3 Basic exponential functions

For applications of our results from above we will use the complex exponential function $z(z') := \exp(z')$, $z' \in \mathbb{C}$. Our both cases of interest are given by $0 + i \cdot 1 = z(i(\pi/2 + 2\pi r))$ and $1 + i \cdot 0 = z(i(2\pi r))$, $r \in \mathbb{Z}$.

Let's say, we want a $z(z')$ which gets $0 + i \cdot 1$ if a number y, $y \in \mathbb{R}$, belongs to a specific times table C, $C \in \mathbb{N}_{\geq 2}$, with $y = Cs$, $s \in \mathbb{Z}$. One possibility to reach this aim is

$$z\left(i\left(\frac{\pi}{2} + 2\pi\frac{y}{C}\right)\right) = \exp\left(i\left(\frac{\pi}{2} + 2\pi\frac{y}{C}\right)\right). \tag{12}$$

Now, we assume that $y \in \mathbb{Z}$ instead of $y \in \mathbb{R}$. In this case, we can also reach this aim by

$$z\left(i\left(\frac{\pi}{2} + Cs - y\right)\right) = \exp\left(i\left(\frac{\pi}{2} + Cs - y\right)\right). \tag{13}$$

Since $y \in \mathbb{Z}$, $C \in \mathbb{N}_{\geq 2}$ and $s \in \mathbb{Z}$, we can always be sure that $Cs - y \neq 2\pi r$, so this equation only gets 0 for $y = Cs$.

In the same way, we can do this for the case of getting $1 + i \cdot 0$ for $z(z')$ if a number y belongs to a specific times table C. In the case of $y \in \mathbb{R}$, we have to take

$$z\left(i\left(2\pi\frac{y}{C}\right)\right) = \exp\left(i\left(2\pi\frac{y}{C}\right)\right). \tag{14}$$

If we move from $y \in \mathbb{R}$ to $y \in \mathbb{Z}$, we can write

$$z\left(i\left(2\pi + Cs - y\right)\right) = \exp\left(i\left(2\pi + Cs - y\right)\right). \tag{15}$$

Now, with this, we have a function which gives one certain value for exactly one defined parameter and else, something else on the whole rest of the value set of $\exp(x)$.

In the next step, we want to determine a function with exactly two certain values for two given input values. Let's assume, we want a function $z(x)$ which becomes -1 for the input value $-N$ and $+1$ for the input value $+N$, with $N \in \mathbb{Z} \setminus \{0\}$. In the context of complex exponential functions, this means we need a function which becomes π for $-N$ and 2π for $+N$ regarding their argument $\arg(z)$. We can reach this in a simple way, if we remember boolean values and how we build inverse values. We can write with $0 = N - x$ for $x = +N$ and $0 = N + x$ for $x = -N$

$$\begin{aligned} \arg(z) : &= \pi \frac{N - x}{-2N} + \frac{N + x}{2N} 2\pi \\ &= \pi \frac{x - N}{2N} + \frac{N + x}{2N} 2\pi. \end{aligned} \tag{16}$$

We can interpret this function also as a kind of exponential sign function for N.

Definition 1 (Exponential sign function)**.** *For $N \in \mathbb{Z} \setminus \{0\}$ the exponential sign function defined as follows*

$$\mathrm{sgn}_N(x) := \exp\left(i\left(\pi \frac{x - N}{2N} + \frac{N + x}{2N} 2\pi\right)\right), \tag{17}$$

gives us $+1$ for $N > 0$ and -1 for $N < 0$.

In a more general case of $N_1, N_2 \in \mathbb{Z} \setminus \{0\}$, we can write the argument as

$$\begin{aligned} \arg(z) : &= \pi \frac{N_1 - x}{N_1 - N_2} + \frac{N_2 - x}{N_2 - N_1} 2\pi \\ &= \pi \frac{N_1 - x}{N_1 - N_2} - \frac{N_2 - x}{N_1 - N_2} 2\pi. \end{aligned} \tag{18}$$

Now, we introduce two additional certain cases $x = +\omega$ with function value i, hence $\pi/2$, and $x = -\omega$ with function value $-i$, $3\pi/2$ as function argument, $\omega \in \mathbb{Z} \setminus \{0\}$. We can describe this function with the argument

$$\arg(z) : = \left(\frac{\pi}{2} \frac{\omega + x}{2\omega} + \frac{\omega - x}{2\omega} \frac{3\pi}{2}\right) \frac{(x - N)(x + N)}{\omega^2 - N^2} + \frac{(\omega + x)(\omega - x)}{\omega^2 - N^2} \left(\pi \frac{x - N}{2N} + \frac{N + x}{2N} 2\pi\right). \tag{19}$$

Finally, the following expression will be useful in some situations, too.

$$\sum_{s=1}^{N} \exp(iCs) = \frac{\exp(iC)\left(-1 + \exp(iCN)\right)}{-1 + \exp(iC)} \tag{20}$$

3 Conclusion

We will see, that our results will be helpful in future works.

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License

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