Notes: Complex boolean functions

Carolin Zöbelein*

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DRAFT

Abstract

We examine the boolean functions `NEG`, `AND` and `OR` on the field of complex numbers $\mathbb{C}$ with boolean values $\{i, 1\}$ and the map between them and the classical definition with $\{0, 1\}$.

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Preamble

The following content is a sketch for discussion purposes only, without warranty for mathematical completeness.

1 Introduction

In this short notes, we will do an analysis of boolean functions on the field of complex numbers $\mathbb{C}$ for a specific defined case regarding the values *True* and *False*, based on the work of Ryan ODonnell [1], Zilong Wang and Guang Gong [2] and Zhiwei Zhang [3] [4].

2 Real and complex boolean functions

We examine the representation of the boolean functions `NEG`, `AND` and `OR` on the field of real numbers and on the field of complex numbers as well as the connections between this two definitions.

2.1 Boolean functions

Given are integer boolean values $x \in \{0, 1\}$ and complex values $z \in \mathbb{C}$. For the boolean cases `NEG`, `AND` and `OR` we will use the two complex boolean values $\{i, 1\}$.

We will denote the inverse boolean value of x by $\bar{x}$ respectively for z by $\bar{z}$. For the complex conjungate of z, we will write z^c.

In table 1, you can see the values for `NEG` of x and z, which gives us several statements.

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Table 1: NEG

x	$-x$	$\bar{x}$	z	$-z$	$\bar{z}$
0	0	1	i	-i	1
1	-1	0	1	- 1	i

Theorem 2.1 (`NEG` of x)**.** *The inverse boolean value of x, generated by `NEG`(x), is given by $\bar{x} = 1 - x$.*

Theorem 2.2 (`NEG` of z)**.** *The inverse boolean value of z, generated by `NEG`(z), is given by $\bar{z} = (1 + i) - z$ and by $\bar{z} = z^3 i$.*

For `AND` and `OR` we need an helper boolean function, called *generator*.

Theorem 2.3 (*Generator g* of (x_i, x_j))**.** *The generator g of (x_i, x_j), which maps $(0,0) \mapsto 1$, $(0,1) \mapsto 0$, $(1,0) \mapsto 0$ and $(1,1) \mapsto 0$, is given by*

$$\begin{aligned} g(x_i, x_j) &= \left((x_i x_j + (x_i + x_j) - 1) \frac{1}{2} - x_i x_j \right)(-2) \\ &= x_i x_j - (x_i + x_j) + 1. \end{aligned} \tag{1}$$

Theorem 2.4 (*Generator g* of (z_i, z_j))**.** *The generator g of (z_i, z_j), which maps $(i,i) \mapsto i$, $(i,1) \mapsto 0$, $(1,i) \mapsto 0$ and $(1,1) \mapsto 0$, is given by*

$$\begin{aligned} g(z_i, z_j) &= (z_i z_j + (z_i + z_j) - 1) \frac{1}{2} - z_i z_j \\ &= (-z_i z_j + (z_i + z_j) - 1) \frac{1}{2}. \end{aligned} \tag{2}$$

In table 2, you can see the values for `AND` of (x_i, x_j) and (z_i, z_j).

Table 2: AND

x_i	x_j	$x_i x_j$	$x_i \wedge x_j$	z_i	z_j	$z_i z_j$	$z_i \wedge z_j$
0	0	0	0	i	i	-1	i
0	1	0	0	i	1	i	i
1	0	0	0	1	i	i	i
1	1	1	1	1	1	1	1

Theorem 2.5 (`AND` of (x_i, x_j))**.** *The boolean value of $x_i \wedge x_j$, generated by `AND`(x_i, x_j), is given by*

$$\begin{aligned} x_i \wedge x_j &= x_i x_j \\ &= g(\bar{x}_i, \bar{x}_j). \end{aligned} \tag{3}$$

Theorem 2.6 (`AND` of (z_i, z_j))**.** *The boolean value of $z_i \wedge z_j$, generated by `AND`(z_i, z_j), is given by*

$$\begin{aligned} z_i \wedge z_j &= z_i z_j + (1 - i)\, g(z_i, z_j) \\ &= i - (1 + i)\, g(\bar{z}_i, \bar{z}_j). \end{aligned} \tag{4}$$

Table 3: OR

x_i	x_j	$x_i + x_j$	$x_i \vee x_j$	z_i	z_j	$z_i + z_j$	$z_i \vee z_j$
0	0	0	0	i	i	2i	i
0	1	1	1	i	1	1 + i	1
1	0	1	1	1	i	1 + i	1
1	1	2	1	1	1	2	1

Finally, in table 3, you can see the values for `OR` of (x_i, x_j) and (z_i, z_j).

Theorem 2.7 (`OR` of (x_i, x_j))**.** *The boolean value of $x_i \vee x_j$, generated by `OR`(x_i, x_j), is given by*

$$\begin{aligned} x_i \vee x_j &= (x_i + x_j) - g(\bar{x}_i, \bar{x}_j) \\ &= 1 - g(x_i, x_j)\,. \end{aligned} \tag{5}$$

Theorem 2.8 (`OR` of (z_i, z_j))**.** *The boolean value of $z_i \vee z_j$, generated by `OR`(z_i, z_j), is given by*

$$\begin{aligned} z_i \vee z_j &= (z_i + z_j) - i + (1 + i)\, g(\bar{z}_i, \bar{z}_j) \\ &= 1 + (1 + i)\, g(z_i, z_j)\,. \end{aligned} \tag{6}$$

Proof of theorems 2.1 - 2.8. The correctness of the given statements can be easily checked with the help of the given tables 1, 2 and 3. □

3 Conclusion

We will see, that our results will be helpful in future works.

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