Notes: Distribution constraints of prime numbers

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July 24, 2020

DRAFT

Abstract

We examine the numbers of particular multiplication tables with the help of their belonging binary representation. With this approach, we are able to set constraints for the distribution of prime numbers.

Keywords: XXX, YYY, ZZZ
2010 Mathematics Classification: XXX, YYY, ZZZ

Preamble

The following content is a sketch for discussion purposes only, without warranty for mathematical completeness.

1 Introduction

In this short notes, we will deduce some constraints for the distribution of prime numbers.

Definition 1 (Prime numbers)**.** *The set of prime numbers* $\mathbb{P}$ *is given by all natural numbers* p *greater than one which are only integer divisible by one and themselves. A natural number which is not a prime number is called a composite number.*

2 Preliminaries

Some preliminaries regarding prime numbers and binary representations.

2.1 Basic prime properties

At first, we have to recap some basic natural and prime number properties.

Lemma 2.1 (Number two is the only even prime)**.** *It exists exact one even prime number and this is number* 2*.*

Proof. Trivial. □

*The author believes in the importance of the independence of research and is funded by the public community. If you also believe in this values, you can find ways for supporting the author's work here: `https://research.carolin-zoebelein.de/funding.html`, Email: `contact@carolin-zoebelein.de`, PGP: D4A7 35E8 D47F 801F 2CF6 2BA7 927A FD3C DE47 E13B, `https://research.carolin-zoebelein.de`

Lemma 2.2 (Odd composite numbers)**.** *All odd composite numbers $n \in \mathbb{N}_{\geq 3}$ can be written in the following way*

$$n = 2\left(2k_ik_j + k_i + k_j\right) + 1 \tag{1}$$

with $k_i, k_j \in \mathbb{N}$.

Proof. By definition of odd numbers, an odd composite number has at least two factors. All of its factors always have to be odd, too. Hence, if we define the two odd factors $2k_i + 1$ and $2k_j + 1$, $k_i, k_j \in \mathbb{N}$, we can write a composite numbers n also as

$$\begin{aligned} n :&= \left(2k_i + 1\right)\left(2k_j + 1\right) \\ &= 2^2k_ik_j + 2k_i + 2k_j + 1 \\ &= 2\left(2k_ik_j + k_i + k_j\right) + 1. \end{aligned} \tag{2}$$

□

2.2 Binary representation

We want to explain how we can describe the distribution of numbers of a given multiplication table in binary representation of decimal integer numbers and its basic properties.

Definition 2 (Multiplication table binary representation)**.** *All natural numbers $h \in \mathbb{N}$ which are element of a specific multiplication table T_m, $m \in \mathbb{N}$, with $h := mn$, $n \in \mathbb{N}$, can be described by the binary representation $B_m = b_{m(N-1)}b_{m(N-1)-1}\dots b_1b_0$ of a decimal number D_m given by*

$$D_m := \sum_{n=1}^{N-1} 2^{mn}. \tag{3}$$

Each bit b_i of B_m which represents a multiple of m has the value 1, else it has the value 0.

For our construction we will need the toggle (B) function on a bit string B

Definition 3 (Bit toggle(b))**.** *The toggle function* toggle (B) *inverts the bits b_i of the bit string B bitwise by $0 \mapsto 1$ and $1 \mapsto 0$.*

and also the two's complement B^{-1} of a given bit string B.

Definition 4 (Two's complement B^{-1})**.** *The two's complement B^{-1} of a bit string B with highest bit index $N-1$ is defined as its complement of B so that $B + B^{-1} = 2^N$. The two's complement B^{-1} of B can be determined by $B^{-1} = \mathrm{toggle}\,(B) + 1$ and the two's complement B of B^{-1} by $B = \mathrm{toggle}\left(B^{-1}\right) + 1$.*

From definition 4 follows directly the way, how we can determine the belonging binary toggled decimal number D^{toggle} of a given decimal number D.

Lemma 2.3 (Decimal number toggle(D))**.** *The toggle $D^{toggled} = \mathrm{toggled}\,(D)$ of the binary representation of a given decimal number D can be determined by $D^{toggled} = (-1)\,D - 1$.*

Proof. From the definition of two's complement follows that $-D = D^{toggled} + 1$ and hence $D^{toggled} = -D - 1$. □

Next, we want to define the bit shift function on the binary string B which shifts all the bits b_i of B to the right, so that bit b_p get shifted to the position of bit b_0. All bits b_i with bit indices $i < p$ are getting moved into the negative range.

Definition 5 (Zero position bit shift zshift(B,p))**.** *The zero position bit shift* zshift (B,p) *of bit* b_p *of the bit string* B *with highest index* $N-1$ *from position* p *to position* 0 *is defined by*
zshift $(B,p) = b_{N-1-p}b_{N-2-p}\dots b_1 b_0 b_{-1}\dots b_{1-p}b_{-p}$.

We want to consider zshift (D,p) on the decimal number D of the bit string B.

Definition 6 (Decimal number zshift(D,p))**.** *For a decimal number* $D = \sum_{n=0}^{N-1} a_n 2^n$, $a_n \in \{0,1\}$, *the* p *bit shift* $D^{zshifted(p)}$ *is given by* $D^{zshifted(p)} = \sum_{n=0}^{N-1} a_n 2^{n-p}$.

Remark 2.1. We recognize that the amount of bits with negative index for zshift (B,p) equals p.

Finally, we want to define a special kind bit switch function on the binary string B which switches the position of bit b_p with the position of b_0 with each other by moving bit indices into the negative range.

Definition 7 (Zero position bit switch zswitch(B,p))**.** *The zero position bit switch* zswitch (B,p) *of bit* b_p *of the bit string* B *with highest index* $N-1$ *with bit* b_0 *is defined by*
zswitch $(B,p) = b_{p-(N-1)}b_{p-(N-2)}\dots b_{-1}b_0 b_1\dots b_{p-1}b_p$.

If we want to read this in the classical bit representation going from right to left for positiv indices and from left to right for negative indices, we can also write it by zswitch $(B,p) = b_p b_{p-1}\dots b_1 b_0 b_{-1}\dots b_{p-(N-2)}b_{p-(N-1)}$.

We also look at zswitch (D,p) on the decimal number D of the bit string B.

Definition 8 (Deciaml number zswitch(D,p))**.** *For a decimal number* $D = \sum_{n=0}^{N-1} a_n 2^n$, $a_n \in \{0,1\}$, *the* p *bit switch* $D^{zswitched(p)}$ *is given by* $D^{zswitched(p)} = \sum_{n=0}^{N-1} a_n 2^{p-n}$.

Remark 2.2. We recognize that the amount of bits with negative index for zswitch (B,p) equals $|p-(N-1)|$.

3 Prime number constraints

Now, we will use our definitions to receive the constraints regarding prime number distribution.

3.1 The Construction

We give the construction for the constraints for the distribution of prime numbers which consists of several steps.

Step 1: Multiplication table

In our first step, we use the multiplication table binary representation of definition 2 together with equation 2.1 for the multiplication table T_{2k_1+1}, $k_1 \in \mathbb{N}$

$$\begin{aligned} D_{2k_1+1}^{M_2-1} &:= \sum_{k_2=1}^{M_2-1} 2^{(2k_1+1)(2k_2+1)} \\ &= \frac{2^{(2k_1+1)(2M_2+1)} - 8^{2k_1+1}}{4^{2k_1+1}-1} \end{aligned} \tag{4}$$

with highest bit index $i_{max} := (2k_1+1)(2(M_2-1)+1) = (2k_1+1)(2M_2-1)$.

If we want a representation with one entries at these bit indices which not belong to the multiplication table T_{2k_1+1}, which we will denote by $\overline{T}_{2k_1+1}$, we can get this by toggling of (4).

$$
\begin{aligned}
\overline{D}_{2k_1+1}^{M_2-1} &:= \text{toggle}\left(D_{2k_1+1}^{M_2-1}\right) \\
&= -1 - D_{2k_1+1}^{M_2-1} \\
&= -1 - \sum_{k_2=1}^{M_2-1} 2^{(2k_1+1)(2k_2+1)} \\
&= -1 + \frac{-2^{(2k_1+1)(2M_2+1)} + 8^{2k_1+1}}{4^{2k_1+1}-1} \\
&= \frac{-2^{(2k_1+1)(2M_2+1)} + 8^{2k_1+1} - 2^{2(2k_1+1)} + 1}{4^{2k_1+1}-1}
\end{aligned} \tag{5}
$$

Step 2: Bit switch for $i = p$

Now, let's say we are interested in the value of bit b_p and want to figure out if our decimal number $D_{2k_1+1}^{M_2-1}$ does have a 0 or an 1 value at bit position p. For this we switch this bit to position $i = 0$ by zswitch (D_{2k_1+1}, p).

$$
\begin{aligned}
D_{2k_1+1}^{M_2-1,zswitched(p)} &= \sum_{k_2=1}^{M_2-1} 2^{p-(2k_1+1)(2k_2+1)} \\
&= \frac{-2^{p-(2k_1+1)(2M_2+1)} + 2^{p-(2k_1+1)}}{4^{2k_1+1}-1}
\end{aligned} \tag{6}
$$

The bit of interesting, b_p, is now at position $i = 0$, means the new bit b_0. All bits which were before at positions larger than p, have now negative powers. Remember that all bits which represent the elements of the mutliplication table T_{2k_1+1} do have the value 1, else 0. Since, we shifted bit b_p to position $i = 0$, we can now check if the bit value is 1, means composite number, by checking if our decimal number $D_{2k_1+1}^{M_2-1,zswitched(p)}$ is an odd number.

Additional, we have negative powers, the former bits b_i, with $i > p$, now. It's easy to see that we reach this negative values if $p < (2k_1+1)(2M_2+1)$. The best solution for further examinations of our equation (6) would be, to completely eliminate this negative power numbers. Sadly, we couldn't figure out an easy analytical way for doing this. On a computer, this would be trivial by simple shifting bitwise the original integer bit string to right by p bit shifts and we wouldn't need the zswitch function instead. Since, we want to do an analytical analysis here, we choose zswitch instead of zshift, since it gives us the possibility to move the larger part of B, the one with bit positions $i > p$, instead of the smaller part with bit positions $i < p$ for $M_2 \to \infty$ into the negative range. Because of the limit $\lim_{N\to\infty} \sum_{i=1}^{N} 2^{-i} = 1$, this approach leads to some benefits and we can consider some possible limit properties.

We can also consider to determine the maximal allowed value of M_2 for having only positive powers. We could look of course at $(p \mod (2k_1+1))$, but this would mean, that we already know the property of p if it is element of the multiplication table T_{2k_1+1} or not. But this is exactly the information we try to get from our equation (6) means our expected result, so this approach doesn't make sense.

In equation (7) you can see an other way for the determination of the maximal allowed value M_2^{max} for a fixed k_1 and $(2k_1+1)(2(M_2'-1)+1) \geq p \geq (2k_1+1)(2(M_2^{max}-1)+1)$ which has to be satisfied $\forall x \in [1,p]$.

$$
\begin{aligned}
2\mathbb{N} &= 2^{-(x-1)}\left(D_{2k_1+1}^{M_2'-1} + \text{toggle}\left(D_{2k_1+1}^{M_2^{max}-1}\right) - \sum_{i=0}^{2(M_2^{max}-1)+1} 2^i\right) \\
2^x\mathbb{N} &= \frac{2^{(2k_1+1)\left(2M_2'+1\right)} - 8^{2k_1+1}}{4^{2k_1+1}-1} \\
&\quad - \frac{2^{(2k_1+1)(2M_2^{max}+1)} - 8^{2k_1+1}}{4^{2k_1+1}-1} - 1 \\
&\quad - 2^{2M_2^{max}} - 1 \\
&= \frac{2^{(2k_1+1)\left(2M_2'+1\right)} - 2^{(2k_1+1)(2M_2^{max}+1)} - \left(2^{2M_2^{max}}+2\right)\left(4^{2k_1+1}+1\right)}{4^{2k_1+1}-1}
\end{aligned} \tag{7}
$$

The idea behind this approach is simple. We know that $D_{2k_1+1}^{M-1}+\text{toggle}\left(D_{2k_1+1}^{M-1}\right) = \sum_{i=0}^{(2k_1+1)(2(M-1)+1)} 2^i$. So let's assume that $(2k_1+1)\left(2\left(M_2'-1\right)+1\right) \geq p \geq (2k_1+1)\left(2\left(M_2^{max}-1\right)+1\right)$ and $(2k_1+1)\left(2\left(M_2^{max}-1\right)+1\right)$ is the maximal closest multiple of $(2k_1+1)$ to p, which means that $(2k_1+1)\left(2M_2^{max}+1\right) \geq p \geq (2k_1+1)\left(2\left(M_2^{max}-1\right)+1\right)$. Then we know that the bit values of the bit positions $\left[p-1, (2k_1+1)\left(2\left(M_2^{max}-1\right)+1\right)\right)$ always have to be 0. Hence, the expression in equation (7) has to have p zeros and following from this, it has to be integer divisible by 2 for all $x \in [1,p]$.

We can do of course the same step for $\overline{D}_{2k_1+1}^{M_2-1}$

$$
\begin{aligned}
\overline{D}_{2k_1+1}^{M_2-1,zswitched(p)} &= -1 - \sum_{k_2=1}^{M_2-1} 2^{p-(2k_1+1)(2k_2+1)} \\
&= -1 + \frac{2^{p-(2k_1+1)(2M_2+1)} - 2^{p-(2k_1+1)}}{4^{2k_1+1}-1} \\
&= \frac{2^{p-(2k_1+1)(2M_2+1)} - 2^{p-(2k_1+1)} - 2^{2(2k_1+1)} + 1}{4^{2k_1+1}-1}
\end{aligned} \tag{8}
$$

and also consider M_2^{max} properties.

$$
\begin{aligned}
2\mathbb{N} &= 2^{-(x-1)}\left(\text{toggle}\left(D_{2k_1+1}^{M_2'-1}\right) + \text{toggle}\left(\text{toggle}\left(D_{2k_1+1}^{M_2^{max}-1}\right)\right) - \sum_{i=0}^{2(M_2^{max}-1)+1} 2^i\right) \\
&= 2^{-(x-1)}\left(\text{toggle}\left(D_{2k_1+1}^{M_2'-1}\right) + D_{2k_1+1}^{M_2^{max}-1} - \sum_{i=0}^{2(M_2^{max}-1)+1} 2^i\right) \\
2^x\mathbb{N} &= -\frac{2^{(2k_1+1)\left(2M_2'+1\right)} - 8^{2k_1+1}}{4^{2k_1+1}-1} - 1 \\
&\quad + \frac{2^{(2k_1+1)(2M_2^{max}+1)} - 8^{2k_1+1}}{4^{2k_1+1}-1} \\
&\quad - 2^{2M_2^{max}} - 1 \\
&= \frac{-2^{(2k_1+1)\left(2M_2'+1\right)} + 2^{(2k_1+1)(2M_2^{max}+1)} - \left(2^{2M_2^{max}}+2\right)\left(4^{2k_1+1}+1\right)}{4^{2k_1+1}-1}
\end{aligned} \tag{9}
$$

Step 3: Summation for k_1

After considering M_2, we will look at k_1 and M_1, now. In equation (6), we determined the binary representation for a certain multiplication table T_{2k_1+1} and went through all possible multiple of $2k_1+1$ by letting go our sum for k_2 from 1 (which equals the multiplication with 3) to M_2-1. If we let run $M_2 \to \infty$ our equation represents all possible multiple of $2k_1+1$. So, we can describe all numbers of a particular multiplication table until now.
We know that equation (6), always has to have integer solutions. If we are interested in compose numbers, this integer solutions has to be odd, else even. Since, the denominator of (6) is always odd, the numerator has to be odd, if we are interested in compose numbers, else it has to be even.
To get additional information regarding the prime and compose number distribution, we consider the sum over the numerator. If we are interested in the case that for all $k_1 \in [1, M_1-1]$ with $M_2 = M_2^{max}$ or $(2k_1+1)(2M_2+1) \geq p$, for all k_1, the numerator has to be odd, then we know that the sum over all the numerators has to alternate between odd numbers and even numbers for consecutive values of M_1, starting with an odd case for $M_1 = 2$. If we are interested in the case of having an even numerator for all M_1, then the sum has also always to be even for consecutive values of M_1. In equation (10), you can see this sum.

$$
\begin{aligned}
S_{M_1-1}^{M_2-1} &= \sum_{k_1=1}^{M_1-1}\left(-2^{p-(2k_1+1)(2M_2+1)} + 2^{p-(2k_1+1)}\right) \\
&= \frac{2^{p-(2M_1+1)(2M_2+1)} - 2^{p-(2M_2+1)}}{4^{2M_2+1}-1} + \frac{1}{3}\left(4^{M_1}-4\right)2^{p-(2M_1+1)} \\
&= \frac{2^{p-(2M_1+1)(2M_2+1)} - 2^{p-(2M_2+1)} + \left(2^{p-1} - 2^{p-(2M_1-1)}\right)\left(2^{2(2M_2+1)}-1\right)}{3\left(2^{2(2M_2+1)}-1\right)}
\end{aligned} \tag{10}
$$

In this summation, we assumed to have the same M_2 for all k_1. In the case of $(2k_1+1)(2M_2+1) > p$ for all k_1, it's no problem to take this approach, else we have to dive into deeper analysis.

Finally, we also look for the summation at the toggled case. In equation (8), we are also only interested in the integer solutions, but here we are interested in potential prime numbers, for which the integer solutions has to be odd and else even, now. The denominator of (8) is always odd, the numerator has to be odd, if we are interested in potential prime numbers, else it has to be even.
So, for getting additional information regarding the appearing of potential prime numbers, we consider the sum over the enumerator and are interested in the case of alternating between odd numbers and even numbers for consecutive values of M_1, starting with an odd case for $M_1 = 2$, in equation (11).

$$
\begin{aligned}
\overline{S}_{M_1-1}^{M_2-1} &= \sum_{k_1=1}^{M_1-1}\left(2^{p-(2k_1+1)(2M_2+1)} - 2^{p-(2k_1+1)} - 2^{2(2k_1+1)} + 1\right) \\
&= -\frac{2^{p-(2M_1+1)(2M_2+1)} - 2^{p-(2M_2+1)} + \left(2^{p-1} - 2^{p-(2M_1-1)}\right)\left(2^{2(2M_2+1)}-1\right)}{3\left(2^{2(2M_2+1)}-1\right)} \\
&\quad - \frac{2^2}{15}\left(4^{2M_1} - 4^2\right) + M_1 - 1 \\
&= \frac{-5\cdot 2^{p-(2M_1+1)(2M_2+1)} + 5\cdot 2^{p-(2M_2+1)} - 5\left(2^{p-1} - 2^{p-(2M_1-1)}\right)\left(2^{2(2M_2+1)}-1\right)}{3\cdot 5\left(2^{2(2M_2+1)}-1\right)} \\
&\quad + \frac{2^2\left(2^{2(2M_1)} - 2^{2\cdot 2}\right)\left(2^{2(2M_2+1)}-1\right) + 3\cdot 5\left(M_1-1\right)\left(2^{2(2M_2+1)}-1\right)}{3\cdot 5\left(2^{2(2M_2+1)}-1\right)}
\end{aligned} \tag{11}
$$

3.2 The Analysis

TODO

4 Conclusion

TODO

Acknowledgement

Thanks to the private donators who financially support this work.

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References

A Appendix