Approximations for floor function: (from different Internet sources)

1. $$\lfloor x \rfloor = -\frac{1}{2} + x + \frac{\arctan(\cot(\pi x))}{\pi} = -\frac{1}{2} + x - \frac{\arctan(\tan(\pi(x-\frac{1}{2})))}{\pi}$$

(eq.24)

2. Fourier series of the Floor function

$$\lfloor x \rfloor = -\frac{1}{2} + x + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{k}$$

(eq.25) which converges to

$$= \underbrace{-\frac{1}{2} + x - \frac{i\left(\ln(1-e^{-2\pi i x}) - \ln(1-e^{2\pi i x})\right)}{2\pi}}_{\text{real}} \quad (\mathbb{C})$$

3. $$\lfloor x \rfloor = \lim_{n\to\infty} \left(\sum_{k=-n}^{n} \mu(x-k) \right) - n - 1$$

(eq.26) μ step function: $\mu(x) = \lim_{n\to\infty} f(nx)$

with $f(x) = e^{-x^2} + \frac{2}{\pi} \mathrm{Si}(x)$.

So

$$f(x) = 1 + \frac{2}{\pi} x + \sum_{k=1}^{\infty} \frac{(-1)^k}{\prod_{j=1}^{k} j} x^{2k} + \frac{2}{\pi} \frac{\left(\prod_{j=1}^{2k} j\right)(-1)^k}{\left(\prod_{j=1}^{2k+1} j\right)^2} x^{2k+1}$$

Taylor series converges for every $x \in \mathbb{R}$.

4. $$\lfloor x \rfloor = \sum_{n=-\infty}^{\infty} n\,\Theta(x-n)\,\Theta(n+1-x), \quad x \notin \mathbb{Z}$$

(eq.27)

$$\lfloor x \rfloor = \lim_{x \to x^+} \lfloor z \rfloor$$