S3S: $(\bar{n})_{123,4}$ $(n=4)$

x_2

$$\vartheta_{(4)} = \prod_{e_1=1}^{n-1}(2x_{e_1}+1)$$

$$\cdot\Big\{-\Delta_{n-0}-\prod_{e_2=1}^{n-2}(2x_{e_2}+1)\Big\{-\Delta_{n-1}-\prod_{e_3=1}^{n-3}(2x_{e_3}+1)(-\Delta_{n-2}+\Delta_{n-3})x_{n-2}$$

$$+\Delta x_{1..n-3,n-2}\,\Delta_{n-3}\Big\}x_{n-1}$$

$$+\Delta x_{1..n-2,n-1}\Big\{-\prod_{e_4=1}^{n-3}(2x_{e_4}+1)(-\Delta_{n-2}+\Delta_{n-3})x_{n-2}$$

$$+\Delta x_{1..n-3,n-2}\,\Delta_{n-3}\Big\}\Big\}$$

$$-\Delta x_{1...n-1,n}\Big\{-\prod_{e_5=1}^{n-2}(2x_{e_5}+1)\Big\{-\Delta_{n-1}-\prod_{e_6=1}^{n-3}(2x_{e_6}+1)(-\Delta_{n-2}$$

$$+\Delta_{n-3})\cdot x_{n-2}$$

$$+\Delta_{1..n-3,n-2}\,\Delta_{n-3}\Big\}x_{n-1}$$

$$+\Delta x_{1..n-2,n-1}\Big\{-\prod_{e_7=1}^{n-3}(2x_{e_7}+1)(-\Delta_{n-2}+\Delta_{n-3})$$

$$\cdot x_{n-2}$$

$$+\Delta x_{1..n-3,n-2}\,\Delta_{n-3}\Big\}\Big\}$$

$$\cancel{\vartheta_{(n)}=\prod_{e_1=1}^{n-1}(2x_{e_1}+1)\cdot\Big\{-\Delta_{n-0}-\sum_{i=2}^{n-1}\prod_{e_i=1}^{n-i}(2x_{e_i}+1)\Big(-\sum_{k=i-1}^{(n-i)}\Delta_{n-k}}$$

$$\cancel{-\Delta_n-\sum\prod\prod(2x+1)}$$

S1S: $$\vartheta_{(2)}=\prod_{e_1=1}^{n-1}(2x_{e_1}+1)\Big(\underbrace{\sum_{k_1=1}^{n}(-1)^{k_1+1}\Delta_{k_1}}\Big)x_n-\underbrace{\Delta x_{1,2}\Delta_1}$$

$$=\Delta x_{1,n}\Delta_1$$

$$=\Delta x_{1..n-1,n}\Delta_1$$

S2S:

$$\vartheta_{(3)}=\Big\{(2x_1+1)(2x_2+1)(-\Delta_3)-\big((2x_1+1)(2x_2+1)\big)(2x_1+1)(-\Delta_2+\Delta_1)x_2$$

$$+(2x_1+1)(2x_2+1)\cdot\Delta x_{1,2}\Delta_1\Big\}x_3-\Delta x_{12,3}\Big\{-(2x_1+1)(-\Delta_2+\Delta_1)x_2$$

$$+\Delta x_{1,2}\Delta_1\Big\}$$