

First recursion steps

Assume, we know: $p_1 = 3, p_2 = 5$

$$\Rightarrow \text{I. } \bar{n}_1 = 3(2k_1 + 1) - 2\Delta_1, \Delta_1 \in \{1, 2\}, k_1, k_2 \in \mathbb{N}$$

$$\text{II. } \bar{n}_2 = 5(2k_2 + 1) - 2\Delta_2, \Delta_2 \in \{1, 2, 3, 4\}$$

$$n_1: 9, 15, 21, 27, 33, 39, \dots$$

$$n_2: 15, 25, 35, 45, 55, 65, \dots$$

$$n_3 = (2(x_2 + 1) + 1) = 2 \cdot 3 + 1 = 7: 21, 35, 49, 63, 77, 91, \dots$$

* \Rightarrow From the knowledge of $p_1 = 3$ and $p_2 = 5 \in \mathbb{P}$, (\mathbb{P} set of Primes)

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, \dots$$

$$\Rightarrow \text{our allowed values: } k_1 = 1, 2, \Delta_1 \in \{1, 2\}$$

$$k_2 = 1, \Delta_2 \in \{1, 2, 3, 4\}$$

$$\Rightarrow \left\{ \begin{array}{l} k_1 = 1, \Delta_1 = \{1\} \\ k_1 = 2, \Delta_1 = \{1, 2\} \\ k_2 = 1, 2, \Delta_2 = \{1, 2, 3, 4\} \end{array} \right.$$

$$\rightarrow \text{Since } \Delta x_{1,2} = 1 \Rightarrow \bar{n}_1' = \bar{n}_1 \text{ and } \bar{n}_2' = \bar{n}_2$$

$$\Rightarrow \bar{n}_1' = \bar{n}_2'$$

$$\bar{n}_1 = \bar{n}_2$$

$$3(2k_1 + 1) - 2\Delta_1 = 5(2k_2 + 1) - 2\Delta_2$$

$$\Leftrightarrow 0 = 5(2k_2 + 1) - 3(2k_1 + 1) - 2\Delta_2 + 2\Delta_1$$

$$\Rightarrow \left\{ \begin{array}{l} k_1 = (2x_2 + 1)z_{1,2} + (-\Delta_2 + \Delta_1 + 1)x_2 \\ k_2 = (2x_1 + 1)z_{1,2} + (-\Delta_2 + \Delta_1 + 1)x_1 \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} k_1 = 5z_{1,2} + (-\Delta_2 + \Delta_1 + 1) \cdot 2 \\ k_2 = 3z_{1,2} + (-\Delta_2 + \Delta_1 + 1) \cdot 1 \end{array} \right.$$

$$\Rightarrow \bar{n}_1 = 3(2k_1 + 1) - 2\Delta_1$$

$$= 3[2 \cdot (5z_{1,2} + (-\Delta_2 + \Delta_1 + 1) \cdot 2) + 1] - 2\Delta_1$$

$$\bar{n}_1' = \bar{n}_1 = 3 \cdot 2 \cdot 5 \cdot z_{1,2} + 3 \cdot 2 \cdot 2 \cdot (-\Delta_2 + \Delta_1 + 1) + 3 - 2\Delta_1$$

$$u_1 = (2x_2 + 1)z_{1,2} + (-\Delta_2 + \Delta_1 + 1)x_2$$

$$u_2 = (2x_1 + 1)z_{1,2} + (-\Delta_2 + \Delta_1 + 1)x_1$$

$$\Rightarrow z_{1,2} = \frac{1}{(2x_2 + 1)} [u_1 - (-\Delta_2 + \Delta_1 + 1)x_2] \quad (*)^1$$

$$\Leftrightarrow z_{1,2} = \frac{1}{(2x_1 + 1)} [u_2 - (-\Delta_2 + \Delta_1 + 1)x_1] \quad (*)^2$$

with: $u_1 = 1$ $\Delta_1 = \{1\}$ and $u_1 = 2$ $\Delta_1 = \{1, 2\}$

$$(*)^1: z_{1,2}(u_1=1, \Delta_1=1) = \frac{1}{5} [1 - (-\Delta_2 + 1 + 1) \cdot 2]$$

$$\Delta_2 = \{1, 2, 3, 4\}$$

$\Gamma_{1, L}$ solutions

$$= \frac{1}{5} [1 - (-\Delta_2 + 2) \cdot 2]$$

$$= \frac{1}{5} [1 + 2\Delta_2 - 4]$$

$$\left. \begin{array}{l} z_{1,2}=0: \Delta_2=1 \\ z_{1,2}=0: \Delta_2=2 \\ z_{1,2}=0: \Delta_2=3 \\ z_{1,2}=1: \Delta_2=4 \end{array} \right\}$$

$$\boxed{z_{1,2}(u_1=1, \Delta_1=1) = \frac{1}{5} [2\Delta_2 - 3]}$$

$$\left\{ \begin{array}{l} \Delta_2=1: z_{1,2} = -\frac{1}{5} \\ \Delta_2=2: z_{1,2} = \frac{1}{5} \\ \Delta_2=3: z_{1,2} = \frac{3}{5} \\ \Delta_2=4: z_{1,2} = \frac{5}{5} = 1 \end{array} \right.$$

$$z_{1,2}(u_1=2, \Delta_1=1) = \frac{1}{5} [2 - (-\Delta_2 + 1 + 1) \cdot 2]$$

$$= \frac{1}{5} [2 + 2\Delta_2 - 4]$$

$$\left. \begin{array}{l} z_{1,2}=0: \Delta_2=1 \\ z_{1,2}=0: \Delta_2=2 \\ z_{1,2}=0: \Delta_2=3 \\ z_{1,2}=1: \Delta_2=4 \end{array} \right\}$$

$$\boxed{z_{1,2}(u_1=2, \Delta_1=1) = \frac{1}{5} [2\Delta_2 - 2]}$$

$$\left\{ \begin{array}{l} \Delta_2=1: z_{1,2} = \frac{0}{5} = 0 \\ \Delta_2=2: z_{1,2} = \frac{2}{5} \\ \Delta_2=3: z_{1,2} = \frac{4}{5} \\ \Delta_2=4: z_{1,2} = \frac{6}{5} \end{array} \right.$$

$$z_{1,2}(u_1=2, \Delta_1=2) = \frac{1}{5} [2 - (-\Delta_2 + 2 + 1) \cdot 2]$$

$$= \frac{1}{5} [2 + 2\Delta_2 - 6]$$

$$\left. \begin{array}{l} z_{1,2}=0: \Delta_2=1 \\ z_{1,2}=0: \Delta_2=2 \\ z_{1,2}=0: \Delta_2=3 \\ z_{1,2}=1: \Delta_2=4 \end{array} \right\}$$

$$\boxed{z_{1,2}(u_1=2, \Delta_1=2) = \frac{1}{5} [2\Delta_2 - 4]}$$

$$\left\{ \begin{array}{l} \Delta_2=1: z_{1,2} = -\frac{2}{5} \\ \Delta_2=2: z_{1,2} = \frac{0}{5} = 0 \\ \Delta_2=3: z_{1,2} = \frac{2}{5} \\ \Delta_2=4: z_{1,2} = \frac{4}{5} \end{array} \right.$$

with $u_2=1$ $\Delta_2 = \{1, 2, 3, 4\}$ and $u_2=2$ $\Delta_2 = \{1, 2, 3, 4\}$

$$(*)^2: z_{1,2}(u_2=1, \Delta_2=1) = \frac{1}{3} [1 - (-1 + \Delta_1 + 1) \cdot 1]$$

$$\left. \begin{array}{l} z_{1,2}=0: \Delta_1=1 \\ z_{1,2}=0: \Delta_1=2 \end{array} \right\} \left[z_{1,2}(u_2=1, \Delta_2=1) = \frac{1}{3} [1 - \Delta_1] \right] \left\{ \begin{array}{l} \Delta_1=1: z_{1,2} = \frac{0}{3} = 0 \\ \Delta_1=2: z_{1,2} = -\frac{1}{3} \end{array} \right.$$

$$z_{1,2}(u_2=1, \Delta_2=2) = \frac{1}{3} [1 - (-2 + \Delta_1 + 1) \cdot 1]$$

$$= \frac{1}{3} [1 + 1 - \Delta_1]$$

$$\left. \begin{array}{l} z_{1,2}=0: \Delta_1=1 \\ z_{1,2}=0: \Delta_1=2 \end{array} \right\} \left[z_{1,2}(u_2=1, \Delta_2=2) = \frac{1}{3} [2 - \Delta_1] \right] \left\{ \begin{array}{l} \Delta_1=1: z_{1,2} = \frac{1}{3} \\ \Delta_1=2: z_{1,2} = \frac{0}{3} = 0 \end{array} \right.$$

$$z_{1,2}(u_2=1, \Delta_2=3) = \frac{1}{3} [1 - (-3 + \Delta_1 + 1) \cdot 1]$$

$$= \frac{1}{3} [1 + 2 - \Delta_1]$$

$$\left. \begin{array}{l} z_{1,2}=0: \Delta_1=1 \\ z_{1,2}=0: \Delta_1=2 \end{array} \right\} \left[z_{1,2}(u_2=1, \Delta_2=3) = \frac{1}{3} [3 - \Delta_1] \right] \left\{ \begin{array}{l} \Delta_1=1: z_{1,2} = \frac{2}{3} \\ \Delta_1=2: z_{1,2} = \frac{1}{3} \end{array} \right.$$

$$z_{1,2}(u_2=1, \Delta_2=4) = \frac{1}{3} [1 - (-4 + \Delta_1 + 1) \cdot 1]$$

$$= \frac{1}{3} [1 + 3 - \Delta_1]$$

$$\left. \begin{array}{l} z_{1,2}=1: \Delta_1=1 \\ z_{1,2}=0: \Delta_1=2 \end{array} \right\} \left[z_{1,2}(u_2=1, \Delta_2=4) = \frac{1}{3} [4 - \Delta_1] \right] \left\{ \begin{array}{l} \Delta_1=1: z_{1,2} = \frac{3}{3} = 1 \\ \Delta_1=2: z_{1,2} = \frac{2}{3} \end{array} \right.$$

$$z_{1,2}(u_2=2, \Delta_2=1) = \frac{1}{3} [2 - (-1 + \Delta_1 + 1) \cdot 1]$$

$$\left. \begin{array}{l} z_{1,2}=0: \Delta_1=1 \\ z_{1,2}=0: \Delta_1=2 \end{array} \right\} \left[z_{1,2}(u_2=2, \Delta_2=1) = \frac{1}{3} [2 - \Delta_1] \right] \left\{ \begin{array}{l} \Delta_1=1: z_{1,2} = \frac{1}{3} \\ \Delta_1=2: z_{1,2} = \frac{0}{3} = 0 \end{array} \right.$$

$$z_{1,2}(u_2=2, \Delta_2=2) = \frac{1}{3} [2 - (-2 + \Delta_1 + 1) \cdot 1]$$

$$= \frac{1}{3} [2 + 1 - \Delta_1]$$

$$\left. \begin{array}{l} z_{1,2}=0: \Delta_1=1 \\ z_{1,2}=0: \Delta_1=2 \end{array} \right\} \left[z_{1,2}(u_2=2, \Delta_2=2) = \frac{1}{3} [3 - \Delta_1] \right] \left\{ \begin{array}{l} \Delta_1=1: z_{1,2} = \frac{2}{3} \\ \Delta_1=2: z_{1,2} = \frac{1}{3} \end{array} \right.$$

$$z_{1,2}(u_2=2, \Delta_2=3) = \frac{1}{3} [2 - (-3 + \Delta_1 + 1) \cdot 1]$$

$$= \frac{1}{3} [2 + 2 - \Delta_1]$$

$$\left. \begin{array}{l} z_{1,2}=1: \Delta_1=1 \\ z_{1,2}=0: \Delta_1=2 \end{array} \right\} \left[z_{1,2}(u_2=2, \Delta_2=3) = \frac{1}{3} [4 - \Delta_1] \right] \left\{ \begin{array}{l} \Delta_1=1: z_{1,2} = \frac{3}{3} = 1 \\ \Delta_1=2: z_{1,2} = \frac{2}{3} \end{array} \right.$$

$$z_{1,2}(u_2=2, \Delta_2=4) = \frac{1}{3} [2 - (-4 + \Delta_1 + 1) \cdot 1]$$

$$= \frac{1}{3} [2 + 3 - \Delta_1]$$

$$\left. \begin{array}{l} \Delta_1=1 \\ z_{1,2}=1 \\ \Delta_1=2 \\ z_{1,2}=-1 \end{array} \right\}$$

$$z_{1,2}(u_2=2, \Delta_2=4) = \frac{1}{3} [5 - \Delta_1]$$

$$\left. \begin{array}{l} \Delta_1=1: z_{1,2}=\frac{4}{3} \\ \Delta_1=2: z_{1,2}=\frac{3}{3} \\ =1 \end{array} \right\}$$