Private research project

The recursively calculation of prime numbers.

Draft/(Working) paper

Carolin Zöbelein[1]

State: October 8, 2017

Keywords: Prime numbers, Primes, Recursive, Number Theory
Subjclass: 2010 *Mathematics Subject Classification.* Primary XX.

[1] *E-mail address:* contact@carolin-zoebelein.de, *PGP Fingerprint*: D4A7 35E8 D47F 801F 2CF6 2BA7 927A FD3C DE47 E13B; *URL:* http://www.carolin-zoebelein.de

Abstract.

Roadmap

- ...

Contents

1 Introduction

In the following paper, I will show that prime numbers can be calculated recursively. I will start with the suggestion of descriptions itself, over different perspectives on this problem, until the final explanation of caculating prime numbers in the most efficient way, as a result from this considerations.

Let's start with the definition of prime numbers itself.

Definition 1.0.1 (Prime numbers) *Every natural number greater than one which has no positive integer divisors apart from one and itself is called Prime Number or just only Prime.*

Be $\mathcal{P}$ the set of all prime numbers p. So we can write

$$\mathcal{P} := \{p \in \mathbb{N}_{>1} \mid \forall n \in \mathbb{N}_{>1} \setminus \{p\} : \ n \nmid p\}.$$

Hence, the first prime numbers are $\mathcal{P} := \{2, 3, 5, 7, 11, 13, 17, 19, 23, \dots\}$.

2 Odd-Divisible Numbers

Contents

At first, for the description of prime numbers, we have to look at the set of divisible numbers. Since, apart from 2, all prime numbers are odd, we will only analyze this numbers. In the whole paper, we will ignore the prime number 2, because we will see, this makes a lot easier.

2.1 Basic description: Odd-Numbers

Be given the set of all odd natural numbers $y \in \mathbb{N}_{>1}$ through

$$y_i\left(x_i\right) := 2x_i + 1, \tag{2.1}$$

with $x, i \in \mathbb{N}$. If we expand the definition set of x to $\mathbb{Z}$, we also know

$$\begin{aligned} y\left(0\right) &= 1 && (2.2) \\ \text{and} \quad y(-x) &= 2\left(-x\right) + 1 \\ &= -\left(2x - 1\right) \\ &= -\left(2\left(x - 1\right) + 1\right) \\ &= -y\left(x - 1\right). && (2.3) \end{aligned}$$

Later, we will see that this properties can be very useful.

2.2 Basic description: Odd-Divisible Numbers

Next, we look at all odd-divisible numbers. We know, they can't have a factor which is a multiple of 2. Hence, we get an equation which describes all odd-divisible numbers by

$$\begin{aligned} y_{i,j}\left(x_i, x_j\right) &= y_i\left(x_i\right) \cdot y_j\left(x_j\right) \\ &= \left(2x_i + 1\right)\left(2x_j + 1\right) \\ &= 2^2 x_i x_j + 2x_i + 2x_j + 1 \\ &= 2\left(\underbrace{2x_i x_j + x_i + x_j}_{=:x_{i,j}}\right) + 1 \\ &= y_{i,j}\left(x_{i,j}\right). \end{aligned} \tag{2.4}$$

If we expand again our sets to $\mathbb{Z}$, we receive additional cases. At first, assume at one factor is $y(0) = 1$. We see directly

$$\begin{aligned} y_{0,j}\left(0, x_j\right) &= y_0\left(0\right) \cdot y_j\left(x_j\right) \\ &= 1 \cdot \left(2x_j + 1\right) \\ &= 2x_j + 1 \\ &= y_j\left(x_j\right) && (2.5) \\ \text{respectively} \quad y_{i,0}\left(x_i, 0\right) &= y_i\left(x_i\right). && (2.6) \end{aligned}$$

Next, assume we have one factor with $y(-x)$.

$$\begin{aligned} y_{i,j}\left(-x_i, x_j\right) &= y_i\left(-x_i\right) \cdot y_j\left(x_j\right) \\ &= \left(2\left(-x_i\right) + 1\right)\left(2x_j + 1\right) \\ &= -2^2 x_i x_j - 2x_i + 2x_j + 1 \\ &= -\left(2\left(2x_i x_j + x_i - x_j - 1\right) + 1\right) \\ &= -\left(2\left(2x_i x_j + x_i - 2x_j + x_j - 1\right) + 1\right) \\ &= -\left(2\left(2\left(x_i - 1\right)x_j + \left(x_i - 1\right) + x_j\right) + 1\right) \\ &= -y_i\left(x_i - 1\right) \cdot y_j\left(x_j\right) && (2.7) \\ \text{respectively} \quad y_{i,j}\left(x_i, -x_j\right) &= -y_i\left(x_i\right) \cdot y_j\left(x_j - 1\right) && (2.8) \end{aligned}$$

In the case of two negative factors, we have

$$\begin{aligned}
y_{i,j}\left(-x_i,-x_j\right) &= y_i\left(-x_i\right)\cdot y_j\left(-x_j\right)\\
&= \left(2\left(-x_i\right)+1\right)\left(2\left(-x_j\right)+1\right)\\
&= \left(2x_i-1\right)\left(2x_j-1\right)\\
&= 2^2x_ix_j-2x_i-2x_j+1\\
&= 2\left(2x_ix_j-x_i-x_j\right)+1\\
&= \left(2x_i-2+1\right)\left(2x_j-2+1\right)\\
&= \left(2\left(x_i-1\right)+1\right)\left(2\left(x_j-1\right)+1\right)\\
&= \left(-1\right)y_i\left(x_i-1\right)\left(-1\right)y_j\left(x_j-1\right)\\
&= \left(-1\right)^2 y_{i,j}\left(x_i-1,x_j-1\right). \qquad (2.9)
\end{aligned}$$

2.3 Odd-Divisible Numbers: Different perspectives

Finally, we see the different possible perspectives for odd-divisible numbers.

$$\begin{aligned}
y_{i,j}\left(x_i,x_j\right) &= 2\left(2x_ix_j+x_i+x_j\right)+1\\
&= 2\left(\left(2x_i+1\right)x_j+x_i\right)+1 \qquad (2.10)\\
\text{respectively} \quad &= 2\left(\left(2x_j+1\right)x_i+x_j\right)+1 \qquad (2.11)
\end{aligned}$$

We will use (2.10) respectively (2.11) in the next step, for the description of odd numbers which are not divisible by a particular other odd number.
From previous sections, we know, we get an index shift by one and a negative sign for each negative value of x_i respectively x_j.

3 Odd-Not-Divisible Numbers

Contents

After we spent time with the set of all odd-divisible numbers, now, we switch to the set of all odd numbers which are not divisible by a particular other odd number.

3.1 Representation: Odd-Divisible Numbers

Let's look again at (2.10)

$$y_{i,j}\left(x_i, x_j\right) = 2\left(\left(2x_i + 1\right)x_j + x_i\right) + 1,$$

and its belonging values.

- Be $x_1 = 1$:

$$y_{1,j}\left(1, x_j\right) = 2\left(3x_j + 1\right) + 1, \quad x_{1,j} = 3x_j + 1 \tag{3.1}$$

Table 3.1: The first ten values for (3.1).

x_j	1	2	3	4	5	6	7	8	9	10
$x_{1,j}$	4	7	10	13	16	19	22	25	28	31
$y_{1,j}$	9	15	21	27	33	39	45	51	57	63

- Be $x_2 = 2$:

$$y_{2,j}\left(2, x_j\right) = 2\left(5x_j + 2\right) + 1, \quad x_{2,j} = 5x_j + 2 \tag{3.2}$$

Table 3.2: The first ten values for (3.2).

x_j	1	2	3	4	5	6	7	8	9	10
$x_{2,j}$	7	12	17	23	28	33	38	43	48	53
$y_{2,j}$	15	25	35	47	57	67	77	87	97	107

- Be $x_3 = 3$:

$$y_{3,j}(3, x_j) = 2(7x_j + 3) + 1, \quad x_{3,j} = 7x_j + 3 \tag{3.3}$$

Table 3.3: The first ten values for (3.3).

x_j	1	2	3	4	5	6	7	8	9	10
$x_{3,j}$	10	17	24	31	38	45	52	59	66	73
$y_{3,j}$	21	35	49	63	77	91	105	119	133	147

- Be $x_i = \ldots$: ....

Now, let us also have a look at the extension to $\mathbb{Z}$. At first, we do the change $x_i \to -x_i$.

$$y_{i,j}(x_i, x_j) = -(2((2(x_i - 1) + 1)x_j + (x_i - 1)) + 1) \tag{3.4}$$

To have attention on this case will still play an role in the next sections. Now, we do the change $x_j \to -x_j$.

$$y_{i,j}(x_i, x_j) = -(2((2x_i + 1)(x_j - 1) + x_i) + 1) \tag{3.5}$$

That's a simple case. We don't have to do anymore.

3.2 Representation: Odd-Not-Divisible Numbers

Now, we take again $y_{i,j}(x_i, x_j) = 2((2x_i + 1)x_j + x_i) + 1$ and rephrase it into an equation which descripes all odd numbers which are not divisible by $2x_i + 1$.
That's not very hard. We can write

$$y_{i,j}(x_i, x_j) = 2((2x_i + 1)x_j + x_i - \mu(x_i)) + 1, \tag{3.6}$$

with

$$\mu(x_i) = 1, \ldots, 2x_i, \quad \mu(x_i) \in \mathbb{N}. \tag{3.7}$$

Let's have a short look at the first values for $x_i = 1, 2, 3$.

- Be $x_1 = 1$:

$$y_{1,j}(1, x_j) = 2(3x_j + 1 - \mu(1)) + 1, \quad \mu(1) = 1, 2, \quad x_{1,j} = 3x_j + 1 \tag{3.8}$$

Table 3.4: The first values for (3.8).

x_j	1		2		3		4		5	
$\mu(1)$	1	2	1	2	1	2	1	2	1	2
$x_{1,j}$	2	3	5	6	8	9	11	12	14	15
$y_{1,j}$	5	7	11	13	17	19	23	25	29	31

- Be $x_2 = 2$:

$$y_{2,j}\left(2, x_j\right) = 2\left(5x_j + 2 - \mu\left(2\right)\right) + 1, \quad \mu\left(2\right) = 1, \ldots, 4 \quad x_{2,j} = 5x_j + 1 \tag{3.9}$$

Table 3.5: The first values for (3.9).

x_j	1				2				3	
$\mu(1)$	1	2	3	4	1	2	3	4	1	2
$x_{2,j}$	3	4	5	6	8	9	10	11	13	14
$y_{2,j}$	7	9	11	13	17	19	21	23	27	29

- Be $x_3 = 3$:

$$y_{3,j}\left(3, x_j\right) = 2\left(7x_j + 1 - \mu\left(3\right)\right) + 1, \quad \mu\left(3\right) = 1, \ldots, 6 \quad x_{3,j} = 7x_j + 1 \tag{3.10}$$

Table 3.6: The first values for (3.10).

x_j	1						1			
$\mu(1)$	1	2	3	4	5	6	1	2	3	4
$x_{3,j}$	4	5	6	7	8	9	11	12	13	14
$y_{3,j}$	9	11	13	15	17	19	23	25	17	29

- Be $x_i = \ldots$: ...

Remark 3.2.1 (Value set) *You can see, the valid value set start not till $x_{i,j} = x_i + 1$.*

3.3 Odd-Not-Divisible Numbers: Intersection

Now we look at the intersection of two equations of the type (3.6) with (3.7). Hence, we start with

$$\begin{aligned} y_{i,j}^{(1)}\left(x_i^{(1)}, x_j^{(1)}\right) &= 2\left(\left(2x_i^{(1)} + 1\right)x_j^{(1)} + x_i^{(1)} - \mu\left(x_i^{(1)}\right)\right) + 1 \\ \mu\left(x_i^{(1)}\right) &= 1, \ldots, 2x_i^{(1)} \end{aligned} \tag{3.11}$$

$$\begin{aligned} \text{and} \quad y_{i,j}^{(2)}\left(x_i^{(2)}, x_j^{(2)}\right) &= 2\left(\left(2x_i^{(2)} + 1\right)x_j^{(2)} + x_i^{(2)} - \mu\left(x_i^{(2)}\right)\right) + 1 \\ \mu\left(x_i^{(2)}\right) &= 1, \ldots, 2x_i^{(2)}. \end{aligned} \tag{3.12}$$

We do the intersection:

$$\begin{aligned} 0 &= \left(2x_i^{(1)} + 1\right)x_j^{(1)} - \left(2x_i^{(2)} + 1\right)x_j^{(2)} + x_i^{(1)} - x_i^{(2)} - \mu\left(x_i^{(1)}\right) + \mu\left(x_i^{(2)}\right) && (3.13) \\ &= \left(2x_i^{(1)} + 1\right)\left(x_j^{(1)} - x_j^{(2)}\right) - 2\Delta x_i^{(1,2)}x_j^{(2)} - \Delta x_i^{(1,2)} - \mu\left(x_i^{(1)}\right) + \mu\left(x_i^{(2)}\right) && (3.14) \\ &= \left(2x_i^{(1)} + 1\right)\left(x_j^{(1)} - x_j^{(2)}\right) - \left(2x_j^{(2)} + 1\right)\Delta x_i^{(1,2)} - \mu\left(x_i^{(1)}\right) + \mu\left(x_i^{(2)}\right) && (3.15) \end{aligned}$$

For the second one, we used $x_i^{(2)} = x_i^{(1)} + \Delta x_i^{(1,2)}$, $x_i^{(2)} > x_i^{(1)}$ and $\Delta x_i^{(1,2)} \in \mathbb{N}$.
To solve (3.13) respectively (3.14), we recognize that we have the boundary constraint, that $\left(2x_i^{(1)}+1\right)$ and $\left(2x_i^{(2)}+1\right)$ must not have any common factors.

Let's look at the case $\Delta x_i^{(1,2)} = 1$.

$$\begin{aligned} 0 &= \left(2x_i^{(1)}+1\right)\left(x_j^{(1)}-x_j^{(2)}\right) - 2x_j^{(2)} - 1 - \mu\left(x_i^{(1)}\right) + \mu\left(x_i^{(2)}\right) && (3.16) \\ &= \left(x_j^{(1)}-x_j^{(2)}\right) - \left(2x_i^{(1)}+1\right)^{-1}\left(2x_j^{(2)}+1+\mu\left(x_i^{(1)}\right)-\mu\left(x_i^{(2)}\right)\right) && (3.17) \end{aligned}$$

Now, let be

$$x_j^{(2)} = \left(1+\mu\left(x_i^{(1)}\right)-\mu\left(x_i^{(2)}\right)\right)x_i^{(1)}. \tag{3.18}$$

$$\begin{aligned} 0 &= x_j^{(1)} - \left(1+\mu\left(x_i^{(1)}\right)-\mu\left(x_i^{(2)}\right)\right)x_i^{(1)} - \left(1+\mu\left(x_i^{(1)}\right)-\mu\left(x_i^{(2)}\right)\right) && (3.19) \\ &= x_j^{(1)} - \left(1+\mu\left(x_i^{(1)}\right)-\mu\left(x_i^{(2)}\right)\right)\left(x_i^{(1)}+1\right). && (3.20) \end{aligned}$$

It follows

$$x_j^{(1)} = \left(1+\mu\left(x_i^{(1)}\right)-\mu\left(x_i^{(2)}\right)\right)\left(x_i^{(1)}+1\right). \tag{3.21}$$

The equations (3.18) and (3.21) give us one particular solution. It's trivial to see, that we receive all solutions on $\mathbb{Z}$ for

$$\begin{aligned} x_j^{(1)} &= \left(2\left(x_i^{(1)}+1\right)+1\right)z^{(1,2)} + \left(1+\mu\left(x_i^{(1)}\right)-\mu\left(x_i^{(2)}\right)\right)\left(x_i^{(1)}+1\right) && (3.22) \\ x_j^{(2)} &= \left(2x_i^{(1)}+1\right)z^{(1,2)} + \left(1+\mu\left(x_i^{(1)}\right)-\mu\left(x_i^{(2)}\right)\right)x_i^{(1)}, && (3.23) \end{aligned}$$

with $z^{(1,2)} \in \mathbb{Z}$.

Now, we switch to the general case for $\Delta x_i^{(1,2)}$. We start again with

$$\begin{aligned} 0 &= \left(2x_i^{(1)}+1\right)\left(x_j^{(1)}-x_j^{(2)}\right) - 2\Delta x_i^{(1,2)}x_j^{(2)} - \Delta x_i^{(1,2)} - \mu\left(x_i^{(1)}\right) + \mu\left(x_i^{(2)}\right) \\ &= \left(x_j^{(1)}-x_j^{(2)}\right) - \left(2x_i^{(1)}+1\right)^{-1}\left(2\Delta x_i^{(1,2)}x_j^{(2)} + \Delta x_i^{(1,2)} + \mu\left(x_i^{(1)}\right)-\mu\left(x_i^{(2)}\right)\right). \end{aligned} \tag{3.24}$$

We can see that our Ansatz from the case $\Delta x_i^{(1,2)} = 1$ not works anymore, since we find $\Delta x_i^{(1,2)}$ only in two of our four terms in the second part of equation (3.24), and hence it is not possible to factor out it.
But we will see, that it only seems to be not possible. In fact, it is possible!

Let's look at $\mu(x_i)$ for our considerations. We know our belonging equation

$$x_{i,j} = (2x_i+1)x_j + x_i - \mu(x_i), \quad \mu(x_i) = 1,\ldots,2x_i, \quad \mu(x_i) \in \mathbb{N}.$$

Now, we will assume, we have $\mu(x_i) \to \Delta x_i^{(1,2)}\mu(x_i)$ instead. With this transition, we get two questions, which are important to be answered.

1. For which values of $\Delta x_i^{(1,2)}$ we receive divisible solutions for $y_{i,j}(x_{i,j})$ and hence, which values for $\Delta x_i^{(1,2)}$ are prohibited?

   We have the following two equations

$$x_{i,j\prime} = (2x_i + 1)\, x_{j\prime} + x_i \tag{3.25}$$

$$x_{i,j} = (2x_i + 1)\, x_j + x_i - \Delta x_i^{(1,2)} \mu(x_i)\,. \tag{3.26}$$

   $x_{i,j\prime}$ gives us all divisible numbers and $x_{i,j}$ all not-divisible numbers for one particular multiplication table given by x_i. We make the intersection and receive

$$0 = (2x_i + 1)\,(x_j - x_{j\prime}) - \Delta x_i^{(1,2)} \mu(x_i)\,. \tag{3.27}$$

   Later, in our recursive calculation steps, we will see, that $(2x_i + 1)$ is always a prime. Since also $\mu(x_i) = 1, \ldots, 2x_i$, it follows from this two conditions and the uniqueness of prime factorization, that (3.27) is only be fulfilled for $\Delta x_i^{(1,2)} = n\,(2x_i + 1)$, $n \in \mathbb{Z}$.

   Thankfully, this case can never happen during our recursive prime number calculation! If we would choose $\Delta x_i^{(1,2)} = n\,(2x_i + 1)$, we would land on our divisible numbers for this multiplication table, and that is exactly NOT, what we want. Hence, we don't have this case.

2. Is the result of the transition $\mu(x_i) \to \Delta x_i^{(1,2)} \mu(x_i)$ still surjective?

   We take

$$x_{i,j} = (2x_i + 1)\, x_j + x_i - \mu(x_i) \tag{3.28}$$

$$x_{i,j\prime} = (2x_i + 1)\, x_{j\prime} + x_i - \Delta x_i^{(1,2)} \mu\prime(x_i)\,. \tag{3.29}$$

   $x_{i,j}$ is the equation before and $x_{i,j\prime}$ after the transition. For the intersection, we receive

$$0 = \underbrace{(2x_i + 1)\,(x_j - x_{j\prime})}_{*} - \underbrace{\mu(x_i)}_{**} + \underbrace{\Delta x_i^{(1,2)} \mu\prime(x_i)}_{***} \tag{3.30}$$

$$\Leftrightarrow \quad \mu(x_i) = (2x_i + 1)\,(x_j - x_{j\prime}) + \Delta x_i^{(1,2)} \mu\prime(x_i)\,. \tag{3.31}$$

   We know $(2x_i + 1)$ is prime, $\mu(x_i), \mu\prime(x_i) \in \{1, \cdots, 2x_i\}$ and $\Delta x_i^{(1,2)} \neq 2x_i + 1$. From $*$ follows that $** + ***$ has to be an integer multiple from $2x_i + 1$. Since neither $\Delta x_i^{(1,2)}$ nor $\mu\prime(x_i)$ can be an integer multiple of $2x_i + 1$, we know $***$ is also never an integer multiple of $2x_i + 1$. But also, caused by $*$ we know the maximum gap between two generated numbers can only be $2x_i$. Caused by the definition range of $**$ it follows, the trueness of the given assumption, that the result of the transition is still surjective.

After getting our answers to this two questions, let's go back to our original problem.

The results of our questions show that it isn't a problem, to apply the transition $\mu(x_i) \to \Delta x_i^{(1,2)} \mu(x_i)$ on our equation (3.24)

$$0 = \left(x_j^{(1)} - x_j^{(2)}\right) - \left(2x_i^{(1)} + 1\right)^{-1} \Delta x_i^{(1,2)} \left(2x_j^{(2)} + 1 + \mu\left(x_i^{(1)}\right) - \mu\left(x_i^{(2)}\right)\right). \tag{3.32}$$

Because we know that $\Delta x_i^{(1,2)} \neq 2x_i^{(1)} + 1$, we can do this without concerns. If we compare (3.32) with (3.17) from our $\Delta x_i^{(1,2)} = 1$ case, we see that our new equation differs from the old one, only by the factor $\Delta x_i^{(1,2)}$ in front of the second term.

Now, let be again

$$x_j^{(2)} = \left(1 + \mu\left(x_i^{(1)}\right) - \mu\left(x_i^{(2)}\right)\right) x_i^{(1)}. \tag{3.33}$$

After analog calculation of our prior case, we receive

$$x_j^{(1)} = \left(1 + \mu\left(x_i^{(1)}\right) - \mu\left(x_i^{(2)}\right)\right)\left(x_i^{(1)} + \Delta x_i^{(1,2)}\right). \tag{3.34}$$

For all solutions on $\mathbb{Z}$ we have

$$x_j^{(1)} = \left(2\left(x_i^{(1)} + \Delta x_i^{(1,2)}\right) + 1\right) z^{(1,2)} + \left(1 + \mu\left(x_i^{(1)}\right) - \mu\left(x_i^{(2)}\right)\right)\left(x_i^{(1)} + \Delta x_i^{(1,2)}\right) \tag{3.35}$$

$$x_j^{(2)} = \left(2x_i^{(1)} + 1\right) z^{(1,2)} + \left(1 + \mu\left(x_i^{(1)}\right) - \mu\left(x_i^{(2)}\right)\right) x_i^{(1)}, \tag{3.36}$$

with $z^{(1,2)} \in \mathbb{Z}$.

But have attention, that this is the solution for

$$x_{i,j}^{(1)}\left(x_i^{(1)}, x_j^{(1)}\right) = \left(2x_i^{(1)} + 1\right) x_j^{(1)} + x_i^{(1)} - \Delta x_i^{(1,2)} \mu\left(x_i^{(1)}\right) \tag{3.37}$$

$$x_{i,j}^{(2)}\left(x_i^{(2)}, x_j^{(2)}\right) = \left(2x_i^{(2)} + 1\right) x_j^{(2)} + x_i^{(2)} - \Delta x_i^{(1,2)} \mu\left(x_i^{(2)}\right) \tag{3.38}$$

now! $\mu\left(x_i^{(1)}\right)$ and $\mu\left(x_i^{(2)}\right)$ are the same like before!

3.4 Intersection of an arbitrary number of equations

Until now, we only had the intersection of two equations. Now we will switch to the case of an arbitrary number of equations. So, now, we have

$$x_{i,j}^{(k)}\left(x_i^{(k)}, x_j^{(k)}\right) = \left(2x_i^{(k)} + 1\right) x_j^{(k)} + x_i^{(k)} - \Delta x_i^{(k,k\prime)} \mu\left(x_i^{(k)}\right), \tag{3.39}$$

with $k \in \mathbb{N}$, $k \neq k\prime$. Ok. What is the general solution of the intersection of n equations? Let's start again with two equations. We receive

$$x_j^{(1)} = \left(2x_i^{(2)} + 1\right) z^{(1,2)} + \left(1 + \mu\left(x_i^{(1)}\right) - \mu\left(x_i^{(2)}\right)\right) x_i^{(2)} \tag{3.40}$$

$$x_j^{(2)} = \left(2x_i^{(1)} + 1\right) z^{(1,2)} + \left(1 + \mu\left(x_i^{(1)}\right) - \mu\left(x_i^{(2)}\right)\right) x_i^{(1)} \tag{3.41}$$

and finally

$$\begin{aligned}
x_{i,j}^{(1,2)} =& x_{i,j}^{(1)}\left(x_i^{(1)}, x_j^{(1)}\right) \\
=& \left(2x_i^{(1)} + 1\right)\left(2x_i^{(2)} + 1\right) z^{(1,2)} + \left(2x_i^{(1)} + 1\right)\left(1 + \mu\left(x_i^{(1)}\right) - \mu\left(x_i^{(2)}\right)\right) x_i^{(2)} \\
& + x_i^{(1)} - \left(x_i^{(2)} - x_i^{(1)}\right)\mu\left(x_i^{(1)}\right) \\
=& \underbrace{\left(2x_i^{(1)} + 1\right)\left(2x_i^{(2)} + 1\right)}_{=:\, 2x_i^{(1,2)}+1} z^{(1,2)} + \underbrace{2x_i^{(1)}x_i^{(2)} + x_i^{(1)} + x_i^{(2)}}_{=:\, x_i^{(1,2)}} \\
& - \left(\underbrace{-\mu\left(x_i^{(1)}\right)x_i^{(1)}\left(2x_i^{(2)} + 1\right) + \mu\left(x_i^{(2)}\right)x_i^{(2)}\left(2x_i^{(1)} + 1\right)}_{=:\, \mu\left(x_i^{(1,2)}\right)}\right) \\
=& \left(2x_i^{(1,2)} + 1\right) z^{(1,2)} + x_i^{(1,2)} - \mu\left(x_i^{(1,2)}\right).
\end{aligned} \tag{3.42}$$

In (3.42), we make the transition $\mu\left(x_i^{(1,2)}\right) \rightarrow \Delta x_i^{((1,2),3)}\mu\left(x_i^{(1,2)}\right)$ for the intersection with the third equation. We have

$$z^{(1,2)} = \left(2x_i^{(3)} + 1\right) z^{((1,2),3)} + \left(1 + \mu\left(x_i^{(1,2)}\right) - \mu\left(x_i^{(3)}\right)\right) x_i^{(3)} \tag{3.43}$$

$$x_j^{(3)} = \left(2x_i^{(1,2)} + 1\right) z^{((1,2),3)} + \left(1 + \mu\left(x_i^{(1,2)}\right) - \mu\left(x_i^{(3)}\right)\right) x_i^{(1,2)} \tag{3.44}$$

and finally

$$
\begin{aligned}
x_{i,j}^{((1,2),3)} =& x_{i,j}^{(1,2)}\left(x_i^{(1,2)}, z^{(1,2)}\right)\\
=& \left(2x_i^{(1,2)}+1\right)\left(2x_i^{(3)}+1\right)z^{((1,2),3)}+\left(2x_i^{(1,2)}+1\right)\left(1+\mu\left(x_i^{(1,2)}\right)-\mu\left(x_i^{(3)}\right)\right)x_i^{(3)}\\
&+x_i^{(1,2)}-\left(x_i^{(3)}-x_i^{(1,2)}\right)\mu\left(x_i^{(1,2)}\right)\\
=& \left(2x_i^{(1)}+1\right)\left(2x_i^{(2)}+1\right)\left(2x_i^{(3)}+1\right)z^{((1,2),3)}+2x_i^{(1,2)}x_i^{(3)}+x_i^{(1,2)}+x_i^{(3)}\\
&-\left(-\mu\left(x_i^{(1,2)}\right)x_i^{(1,2)}\left(2x_i^{(3)}+1\right)+\mu\left(x_i^{(3)}\right)x_i^{(3)}\left(2x_i^{(1,2)}+1\right)\right)\\
=& \left(2x_i^{(1)}+1\right)\left(2x_i^{(2)}+1\right)\left(2x_i^{(3)}+1\right)z^{((1,2),3)}\\
&+\ 2^2x_i^{(1)}x_i^{(2)}x_i^{(3)}+2^1\left(x_i^{(1)}x_i^{(3)}+x_i^{(2)}x_i^{(3)}+x_i^{(1)}x_i^{(2)}\right)+2^0\left(x_i^{(1)}+x_i^{(2)}+x_i^{(3)}\right)\\
&-\left(\mu\left(x_i^{(1)}\right)x_i^{(1)}\left(2x_i^{(2)}+1\right)-\mu\left(x_i^{(2)}\right)x_i^{(2)}\left(2x_i^{(1)}+1\right)\right)\\
&\cdot\left(2x_i^{(1)}x_i^{(2)}+x_i^{(1)}+x_i^{(2)}\right)\left(2x_i^{(3)}+1\right)\\
&-\left(\mu\left(x_i^{(3)}\right)x_i^{(3)}\left(2x_i^{(1)}+1\right)\left(2x_i^{(2)}+1\right)\right)\\
=& \left(2x_i^{(1)}+1\right)\left(2x_i^{(2)}+1\right)\left(2x_i^{(3)}+1\right)z^{((1,2),3)}\\
&+\frac{1}{2}\left(\left(2x_i^{(1)}+1\right)\left(2x_i^{(2)}+1\right)\left(2x_i^{(3)}+1\right)-1\right)\\
&-\left(\mu\left(x_i^{(1)}\right)x_i^{(1)}\left(2x_i^{(2)}+1\right)\frac{1}{2}\left(\left(2x_i^{(1)}+1\right)\left(2x_i^{(2)}+1\right)-1\right)\left(2x_i^{(3)}+1\right)\right)\\
&-\left(-\mu\left(x_i^{(2)}\right)x_i^{(2)}\left(2x_i^{(1)}+1\right)\frac{1}{2}\left(\left(2x_i^{(1)}+1\right)\left(2x_i^{(2)}+1\right)-1\right)\left(2x_i^{(3)}+1\right)\right)\\
&-\left(\mu\left(x_i^{(3)}\right)x_i^{(3)}\left(2x_i^{(1)}+1\right)\left(2x_i^{(2)}+1\right)\right).
\end{aligned}
\tag{3.45}
$$

For the case of the intersection of n $(n>1)$ equations, we can write

$$
\begin{aligned}
x_{i,j}^{(1\ldots n)} =& \prod_{k=1}^{n}\left(2x_i^{(k)}+1\right)z^{(1\ldots n)}\\
&+\frac{1}{2}\left(\prod_{k=1}^{n}\left(2x_i^{(k)}+1\right)-1\right)\\
&-\left((-1)^{n+1}\mu\left(x_i^{(1)}\right)x_i^{(1)}\prod_{l\neq 1}^{n}\left(2x_i^{(l)}+1\right)\prod_{f=1}^{n-2,n>2}\left(\frac{1}{2}\left(\prod_{m=1}^{f+1}\left(2x_i^{(m)}+1\right)-1\right)\right)\right)\\
&-\left(\sum_{k=2}^{n}(-1)^{n+k}\mu\left(x_i^{(k)}\right)x_i^{(k)}\prod_{l\neq k}^{n}\left(2x_i^{(l)}+1\right)\prod_{f=1}^{n-k,n>2}\left(\frac{1}{2}\left(\prod_{m=1}^{f+1}\left(2x_i^{(m)}+1\right)-1\right)\right)\right).
\end{aligned}
\tag{3.46}
$$

Here we used the definition

$$\prod_{f=1}^{0} A_f := 1. \tag{3.47}$$

Like in the section before, we have to take a look at the $\Delta x_i^{((1,2),\dots)}$ values from our transition. Now, for the intersection of an arbitrary number of equations, we have the case of pre-factors of z which are the product of the already calculated primes. Given be

$$x_{i,j}^{(1\dots n)} = \left(2x_i^{(1)}+1\right)\left(2x_i^{(2\dots n)}+1\right) z^{(1\dots n)} + x_i^{(1,\dots n)} - \Delta x_i^{((1\dots n),n+1)} \mu\left(x_i^{(1\dots n)}\right) \tag{3.48}$$

with

$$\Delta x_i^{((1\dots n),n+1)} = \left(2x_i^{(1)}+1\right) \quad \text{and} \quad \mu\left(x_i^{(1\dots n)}\right) = 1,\dots,2x_i^{(1\dots n)}. \tag{3.49}$$

It's easy to see that this could lead to problems, since now, also the case $\mu\left(x_i^{1\dots n}\right) = \left(2x_i^{(2\dots n)}+1\right)$ is possible and we would receive again the equation for divisible, instead of not-divisible numbers again.

But thankfully, during our recursion for prime number generation, we will never have this case!

Assume we would have given (3.49)

$$\begin{aligned} x_{i,j}^{(1\dots n)} = & \left(2x_i^{(1)}+1\right)\left(2x_i^{(2\dots n)}+1\right) z^{(1\dots n)} \\ & + \left(\left(2x_i^{(1)}+1\right)\left(2x_i^{(2\dots n)}+1\right)-1\right)\frac{1}{2} \\ & - \left(2x_i^{(1)}+1\right)\mu\left(x_i^{(1\dots n)}\right). \end{aligned} \tag{3.50}$$

Now, assume we would have $x_i^{(n+1)} = x_i^{(1\dots n)} \pm \left(2x_i^{(1)}+1\right)$. With this, we would receive

$$
\begin{aligned}
x_{i,j}^{(n+1)} = &\left(2\left(\left(\left(2x_i^{(1)}+1\right)\left(2x_i^{(2\dots n)}+1\right)-1\right)\frac{1}{2}\pm\left(2x_i^{(1)}+1\right)\right)+1\right)x_j^{(n+1)}\\
&+\left(\left(2x_i^{(1)}+1\right)\left(2x_i^{(2\dots n)}+1\right)-1\right)\frac{1}{2}\pm\left(2x_i^{(1)}+1\right)\\
&\pm\left(2x_i^{(1)}+1\right)\mu\left(x_i^{(1\dots n)}\pm\left(2x_i^{(1)}+1\right)\right)\\
= &\left(2x_i^{(1)}+1\right)\left(\left(2x_i^{(2\dots n)}+1\right)\pm 2\right)x_j^{(n+1)}\\
&+\left(2x_i^{(1)}+1\right)\left(\left(2x_i^{(2\dots n)}+1\right)\frac{1}{2}\pm 1\right)-\frac{1}{2}\\
&\pm\left(2x_i^{(1)}+1\right)\mu\left(x_i^{(1\dots n)}\pm\left(2x_i^{(1)}+1\right)\right)\\
= &\left(2x_i^{(1)}+1\right)\left(\left(2x_i^{(2\dots n)}+1\right)\pm 2\right)x_j^{(n+1)}\\
&+\ x_i^{(1)}+\left(2x_i^{(1)}+1\right)\left(x^{(2\dots n)}\pm 1\right)\\
&\pm\left(2x_i^{(1)}+1\right)\mu\left(x_i^{(1\dots n)}\pm\left(2x_i^{(1)}+1\right)\right).
\end{aligned}
\tag{3.51}
$$

We can see that we get an equation for a multiplication table for which we have already done our recursion before. Since, the case $\Delta x_i^{((1\dots n),n+1)} = \left(2x_i^{(1)}+1\right)$ would make no sense for our recursion.

4 Approximation methods

We will look at ways of approximation for different parts of our prime number calculation.

5 The recursive calculation

Contents

In this section, we do the final recursive calculation. To understand the deep structure we will do this by discussing the first steps by manually calculation. We will see, it exists different ways how you can look at each situation/problem in each step, hence this will mainly a discussion of this different ways.

5.1 Recursion step: $n^{(0)} = 0$

5.1.1 Calculation

Let's start with our step zero. It's called zero, since we start with the simplest first possible prime generation. Since we know 3 is the smallest odd number at all, we know 3 has to be a prime. Hence we have our first prime.

5.1.2 Results

- Known prime $x_{i,j}$: 1
- Known prime $y_{i,j}$: 3
- Known prime number range for $x_{i,j}$: $[1, 1]$
- Known prime number range for $y_{i,j}$: $[1, 3]$

That's of course not much and trivial, but hey, it's a beginning!

5.2 Recursion step: $n^{(0)} = 1$

5.2.1 Calculation

Now, we can start with our first step. From $n^{(0)} = 0$, we know 3(1).

Notation Note: From now on I will write $y\,(x)$ for shortness. For example, 3(1) means $3 = 2 \cdot 1 + 1$. If I only write 1, I mean $x = 1$.

So, let's look again at the values for $x_{1,j}$ from

$$y_{1,j}\,(1, x_j) = 2\,(3x_j + 1) + 1, \quad x_{1,j} = 3x_j + 1. \tag{5.1}$$

Table 5.1: The first values for $x_{1,j}$ are marked in bold. Italic values are within the not describable range.

x_j	–	–	–	1	–	–	2	–	–	3	–	–	4	–	–	5	–	–	6
$x_{1,j}$	*1*	2	3	**4**	5	6	**7**	8	9	**10**	11	12	**13**	14	15	**16**	17	18	**19**

What we can see from our table 5.1 are the next two sure primes, the numbers 5(2) and 7(3). All larger numbers could theoretically still have a divider, hence we only know this two additional numbers surely, at the moment.

We can describe this two numbers with (3.6) and (3.7) by

$$x_{1,1}\,(1, 1) = 3 \cdot 1 + 1 - \mu\,(1)\,, \tag{5.2}$$

with

$$\mu\,(1) = 1, 2. \tag{5.3}$$

and a maximum value for $x_j = 1$.

5.2.2 Results

Only from step $n^{(0)} = 1$:

- Known prime $x_{i,j}$: $2, 3$
- Known primes $y_{i,j}$: $5, 7$
- Known prime number range for $x_{i,j}$: $[2, 4]$
- Known prime number range for $y_{i,j}$: $[5, 9]$
- Describing prime equation: $x_{1,1}\,(1, 1) = 3 \cdot 1 + 1 - \mu\,(1)$, $\mu\,(1) = 1, 2$

From all steps until now:

- Known prime $x_{i,j}$: $1, 2, 3$

- Known primes $y_{i,j}$: $3, 5, 7$
- Known prime number range for $x_{i,j}$: $[1, 4]$
- Known prime number range for $y_{i,j}$: $[1, 9]$
- Describing prime equation for $x_{i,j} \in [2, 4]$: $x_{1,1}(1, 1) = 3 \cdot 1 + 1 - \mu(1)$, $\mu(1) = 1, 2$

Our first prime equation for $x_{i,j} \in [2, 4]$:

$$x_{1,1}(1, 1, \mu(1)) = (2 \cdot 1 + 1)\, 1 + 1 - \mu(1) \tag{5.4}$$

5.3 Recursion step: $n^{(0)} = 2$

5.3.1 Calculation

Now, from our steps above we have (5.4) and hence

$$\begin{aligned}
x_{1,j}\left(1, x_j^{(1)}, \mu(1)\right) &= 3x_j^{(1)} + 1 - \Delta x_i^{(a,b)} \mu(1) \\
&= (2 \cdot 1 + 1)\, x_j^{(1)} + 1 - \Delta x_i^{(a,b)} \mu(1) && (5.5) \\
x_{2,j}\left(2, x_j^{(2)}, \mu(2)\right) &= 5x_j^{(2)} + 2 - \Delta x_i^{(a,b)} \mu(2) \\
&= (2 \cdot 2 + 1)\, x_j^{(2)} + 2 - \Delta x_i^{(a,b)} \mu(2) \\
&= (2 \cdot ((2 \cdot 1 + 1)\, 1 + 1 - 2) + 1)\, x_j^{(2)} \\
&\quad + ((2 \cdot 1 + 1)\, 1 + 1 - 2) - \Delta x_i^{(a,b)} \mu(2) && (5.6) \\
x_{3,j}\left(3, x_j^{(3)}, \mu(3)\right) &= 7x_j^{(3)} + 3 - \Delta x_i^{(a,b)} \mu(3) \\
&= (2 \cdot 3 + 1)\, x_j^{(3)} + 3 - \Delta x_i^{(a,b)} \mu(3) \\
&= (2 \cdot ((2 \cdot 1 + 1)\, 1 + 1 - 1) + 1)\, x_j^{(3)} \\
&\quad + ((2 \cdot 1 + 1)\, 1 + 1 - 1) - \Delta x_i^{(a,b)} \mu(3), && (5.7)
\end{aligned}$$

with $\mu(1) = 1, 2$, $\mu(2) = 1, 2, 3, 4$, $\mu(3) = 1, 2, 3, 4, 5, 6$ and $a, b \in \{1, 2, 3\}$, $a \neq b$. Let's have a look at the values (including $\mu = 0$).

Table 5.2: The first values for $x_{i,j}$ are marked in bold. Italic values are within the not describable range.

$x_{1,j}$	*1*	2	3	**4**	5	6	**7**	8	9	**10**	11	12	**13**	14	15	**16**	17	18	**19**
$x_{2,j}$	*1*	*2*	3	4	5	6	**7**	8	9	10	11	**12**	13	14	15	16	**17**	18	19
$x_{3,j}$	*1*	*2*	*3*	4	5	6	7	8	9	**10**	11	12	13	14	15	16	**17**	18	19

Now, at this point, we have different possible ways, how we can calculate more primes from this. We will discuss this different ways step by step.

At first, we make an intersection between (5.5) and (5.6), $\Delta x_i^{(1,2)} = 1$,

$$\begin{aligned} 0 &= 3x_j^{(1)} - 5x_j^{(2)} + 1 - 2 - \mu(1) + \mu(2) \\ &= 3x_j^{(1)} - 5x_j^{(2)} - 1 - \mu(1) + \mu(2). \end{aligned} \tag{5.8}$$

From the main section prior to this, we know our solutions with (3.22) and (3.23).

$$x_j^{(1)} = 5z^{(1,2)} + (1 + \mu(1) - \mu(2))\, 2 \tag{5.9}$$
$$x_j^{(2)} = 3z^{(1,2)} + (1 + \mu(1) - \mu(2))\, 1, \tag{5.10}$$

$z^{(1,2)} \in \mathbb{Z}$ and finally

$$x_{1,j}^{(1,2)} = 3 \cdot 5z^{(1,2)} + 3(1 + \mu(1) - \mu(2))\, 2 + 1 - \mu(1) \tag{5.11}$$
$$= 3 \cdot 5z^{(1,2)} + 7 + 5\mu(1) - 6\mu(2) \tag{5.12}$$
$$x_{2,j}^{(1,2)} = 5 \cdot 3z^{(1,2)} + 5(1 + \mu(1) - \mu(2))\, 1 + 2 - \mu(2) \tag{5.13}$$
$$= 3 \cdot 5z^{(1,2)} + 7 + 5\mu(1) - 6\mu(2).$$

Ok, we have our equation which generates all solutions for our intersection. But since we only want to generate primes with this equation, we have to restrict the valid range. So, what is the maximum valid range for $x_{i,j}$ after the intersection above?

1. *Lower bound*: This bound is easy to see. We know for each original equation of the form $x_{i,j}(x_i, x_j, \mu(x_i)) = (2x_i + 1)\, x_j + x_i - \Delta x_i^{(a,b)} \mu(x_i)$ that its lower bound is given by

$$x_{i,j}(x_i, 0, 0) = (2x_i + 1) \cdot 0 + x_i - 0 + 1 \tag{5.14}$$
$$= x_i + 1 \tag{5.15}$$
$$\Leftrightarrow \quad x_{i,j}(x_i, 1, 2x_i) = (2x_i + 1) \cdot 1 + x_i - 2x_i. \tag{5.16}$$

   If follows, for the intersection of n equations for the lower bound

$$\begin{aligned} x_{i,j}^{(1\ldots n),\text{ lower bound}}(x_i^{\max}, 1, 2x_i^{\max}) &= (2x_i^{\max} + 1) \cdot 1 + x_i^{\max} - 2x_i^{\max} \\ &= x_i^{\max} + 1, \end{aligned} \tag{5.17}$$

   with $x_i^{\max} := \max\{x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(n-1)}, x_i^{(n)}\}$.

2. *Upper bound*: We have again the intersection of n equations. As first assumption we can set

$$x_{i,j}(x_i^{\max} + 1, 1, 0) = (2(x_i^{\max} + 1) + 1) \cdot 1 + (x_i^{\max} + 1), \tag{5.18}$$

with $x_i^{\max} := \max\{x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(n-1)}, x_i^{(n)}\}$. But if we look closer on our multiplication tables, we see,

$$\begin{aligned} x_{i,j}\left(x_i^{\max}+1,1,0\right) &= \left(2\left(x_i^{\max}+1\right)+1\right)\cdot 1 + \left(x_i^{\max}+1\right) \\ &= \left(2\cdot 1+1\right)\left(x_i^{\max}+1\right)+1, \end{aligned} \tag{5.19}$$

that this values are also always an element of the multiplication table of three. Since we always use the three multiplication table equation for intersection calculation, we have this first values of every other multipliation table already given. This means, we can look for an higher upper bound.

Let's go to our next assumption and take the second value

$$\begin{aligned} x_{i,j}\left(x_i^{\max}+1,2,0\right) &= \left(2\left(x_i^{\max}+1\right)+1\right)\cdot 2 + \left(x_i^{\max}+1\right) \\ &= \left(2\cdot 2+1\right)\left(x_i^{\max}+1\right)+2. \end{aligned} \tag{5.20}$$

Assume we already know all primes in the range $x_i \in [1, x_i^{\mathrm{s}}]$ after the s'th, $s \in \mathbb{N}_0$, recursion step, with n equations. From this assumptions we can follow our valid maximum upper bound for the generation of the next primes

$$\begin{aligned} x_{i,j}^{(1\ldots n),\text{ upper bound}}\left(x_i^{\max}+1, x_i^{\max}+1, 0\right) = &\ \left(2\left(x_i^{\max}+1\right)+1\right)\left(x_i^{\max}+1\right) \\ &+ \left(x_i^{\max}+1\right) - 0 - 1 \\ = &\ 2\left(x_i^{\max}\right)^2 + 6x_i^{\max} + 3, \end{aligned} \tag{5.21}$$

with $x_i^{\max} = x_i^{\mathrm{s}}$. Finally, we determine the value of x_j for an arbitrary multiplication table x_i for our upper bound.

$$\begin{aligned} \left(2x_i+1\right)x_j + x_i &= 2\left(x_i^{\max}\right)^2 + 6x_i^{\max} + 3 \\ \left(2x_i+1\right)x_j &= 2\left(x_i^{\max}\right)^2 + 6x_i^{\max} + 3 - x_i \\ x_j^{\text{upper bound}} &:= \left\lfloor \frac{1}{2x_i+1}\left(2\left(x_i^{\max}\right)^2 + 6x_i^{\max} + 3 - x_i\right)\right\rfloor \end{aligned} \tag{5.22}$$

Since we know our maximum valid range for our first intersection, we can think how we can receive the belonging valid z values as a function of all μ's, now.

Notation: Given by (5.21), we see that our true maximum upper bound $x_{i,j}^{\text{upper bound}}$ in the recursion step $n^{(0)} = 1$ is already $x_{1,j}^{(1),\text{ upper bound}} = 2\cdot 1^2 + 6\cdot 1 + 3 = 11$ and not 4, what we claimed before.

What we will need additionally, is the connection between the z-values for the equations of the same multiplication table x_i, but with different $\Delta x_i^{(a,b)}$-values.

$$
\begin{aligned}
x_j &= (2x_i + 1)\, z + x_i + \Delta x_i^{(a,b)} \mu\,(x_i) \\
x_j &= (2x_i + 1)\, z' + x_i + \left(\Delta x_i^{(a,b)}\right)' \mu\,(x_i) \\
\Rightarrow \quad 0 &= (2x_i + 1)\left(z - z'\right) + \Delta x_i^{(a,b)} \mu\,(x_i) - \left(\Delta x_i^{(a,b)}\right)' \mu\prime\,(x_i)
\end{aligned} \tag{5.23}
$$

List of Figures

List of Tables

Listings

Bibliography

Changelog