PRIVATE RESEARCH PROJECT

The recursively calculation of prime numbers.

Draft/(Working) paper

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Abstract.

Roadmap

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1 Introduction

In the following paper, I will show that prime numbers can be calculated recursively. I will start with the suggestion of descriptions itself, over different perspectives on this problem, until the final explanation of caculating prime numbers in the most efficient way, as a result from this considerations.

Let's start with the definition of prime numbers itself.

Definition 1.0.1 (Prime numbers) *Every natural number greater than one which has no positive integer divisors apart from one and itself is called Prime Number or just only Prime.*

Be $\mathcal{P}$ the set of all prime numbers p. So we can write

$$\mathcal{P} := \{p \in \mathbb{N}_{>1} \mid \forall n \in \mathbb{N}_{>1} \setminus \{p\} : \ n \nmid p\}.$$

Hence, the first prime numbers are $\mathcal{P} := \{2, 3, 5, 7, 11, 13, 17, 19, 23, \dots\}$.

2 Odd-Divisible Numbers

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At first, for the description of prime numbers, we have to look at the set of divisible numbers. Since, apart from 2, all prime numbers are odd, we will only analyze this numbers. In the whole paper, we will ignore the prime number 2, because we will see, this makes a lot easier.

2.1 Basic description: Odd-Numbers

Be given the set of all odd natural numbers $y \in \mathbb{N}_{>1}$ through

$$y_i\left(x_i\right) := 2x_i + 1, \tag{2.1}$$

with $x, i \in \mathbb{N}$. If we expand the definition set of x to $\mathbb{Z}$, we also know

$$\begin{aligned} y\left(0\right) &= 1 \qquad (2.2) \\ \text{and} \quad y(-x) &= 2\left(-x\right) + 1 \\ &= -\left(2x - 1\right) \\ &= -\left(2\left(x-1\right)+1\right) \\ &= -y\left(x-1\right). \qquad (2.3) \end{aligned}$$

Later, we will see that this properties can be very useful.

2.2 Basic description: Odd-Divisible Numbers

Next, we look at all odd-divisible numbers. We know, they can't have a factor which is a multiple of 2. Hence, we get an equation which describes all odd-divisible numbers by

$$\begin{aligned} y_{i,j}\left(x_i, x_j\right) &= y_i\left(x_i\right) \cdot y_j\left(x_j\right) \\ &= \left(2x_i + 1\right)\left(2x_j + 1\right) \\ &= 2^2 x_i x_j + 2x_i + 2x_j + 1 \\ &= 2\left(\underbrace{2x_i x_j + x_i + x_j}_{=:x_{i,j}}\right) + 1 \\ &= y_{i,j}\left(x_{i,j}\right). \end{aligned} \tag{2.4}$$

If we expand again our sets to $\mathbb{Z}$, we receive additional cases. At first, assume at one factor is $y\left(0\right) = 1$. We see directly

$$\begin{aligned} y_{0,j}\left(0, x_j\right) &= y_0\left(0\right) \cdot y_j\left(x_j\right) \\ &= 1 \cdot \left(2x_j + 1\right) \\ &= 2x_j + 1 \\ &= y_j\left(x_j\right) \end{aligned} \tag{2.5}$$

$$\text{respectively} \quad y_{i,0}\left(x_i, 0\right) = y_i\left(x_i\right). \tag{2.6}$$

Next, assume we have one factor with $y\left(-x\right)$.

$$\begin{aligned} y_{i,j}\left(-x_i, x_j\right) &= y_i\left(-x_i\right) \cdot y_j\left(x_j\right) \\ &= \left(2\left(-x_i\right) + 1\right)\left(2x_j + 1\right) \\ &= -2^2 x_i x_j - 2x_i + 2x_j + 1 \\ &= -\left(2\left(2x_i x_j + x_i - x_j - 1\right) + 1\right) \\ &= -\left(2\left(2x_i x_j + x_i - 2x_j + x_j - 1\right) + 1\right) \\ &= -\left(2\left(2\left(x_i - 1\right)x_j + \left(x_i - 1\right) + x_j\right) + 1\right) \\ &= -y_i\left(x_i - 1\right) \cdot y_j\left(x_j\right) \end{aligned} \tag{2.7}$$

$$\text{respectively} \quad y_{i,j}\left(x_i, -x_j\right) = -y_i\left(x_i\right) \cdot y_j\left(x_j - 1\right) \tag{2.8}$$

In the case of two negative factors, we have

$$\begin{aligned}
y_{i,j}\left(-x_i,-x_j\right) &= y_i\left(-x_i\right)\cdot y_j\left(-x_j\right)\\
&= \left(2\left(-x_i\right)+1\right)\left(2\left(-x_j\right)+1\right)\\
&= \left(2x_i-1\right)\left(2x_j-1\right)\\
&= 2^2x_ix_j-2x_i-2x_j+1\\
&= 2\left(2x_ix_j-x_i-x_j\right)+1\\
&= \left(2x_i-2+1\right)\left(2x_j-2+1\right)\\
&= \left(2\left(x_i-1\right)+1\right)\left(2\left(x_j-1\right)+1\right)\\
&= \left(-1\right)y_i\left(x_i-1\right)\left(-1\right)y_j\left(x_j-1\right)\\
&= \left(-1\right)^2 y_{i,j}\left(x_i-1,x_j-1\right). \qquad (2.9)
\end{aligned}$$

2.3 Odd-Divisible Numbers: Different perspectives

Finally, we see the different possible perspectives for odd-divisible numbers.

$$\begin{aligned}
y_{i,j}\left(x_i,x_j\right) &= 2\left(2x_ix_j+x_i+x_j\right)+1\\
&= 2\left(\left(2x_i+1\right)x_j+x_i\right)+1 \qquad (2.10)\\
\text{respectively} \quad &= 2\left(\left(2x_j+1\right)x_i+x_j\right)+1 \qquad (2.11)
\end{aligned}$$

We will use (2.10) respectively (2.11) in the next step, for the description of odd numbers which are not divisible by a particular other odd number.
From previous sections, we know, we get an index shift by one and a negative sign for each negative value of x_i respectively x_j.

3 Odd-Not-Divisible Numbers

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After we spent time with the set of all odd-divisible numbers, now, we switch to the set of all odd numbers which are not divisible by a particular other odd number.

3.1 Representation: Odd-Divisible Numbers

Let's look again at (2.10)

$$y_{i,j}\left(x_i, x_j\right) = 2\left(\left(2x_i + 1\right)x_j + x_i\right) + 1,$$

and its belonging values.

- Be $x_1 = 1$:
$$y_{1,j}\left(1, x_j\right) = 2\left(3x_j + 1\right) + 1, \quad x_{1,j} = 3x_j + 1 \tag{3.1}$$

Table 3.1: The first ten values for (3.1).

x_j	1	2	3	4	5	6	7	8	9	10
$x_{1,j}$	4	7	10	13	16	19	22	25	28	31
$y_{1,j}$	9	15	21	27	33	39	45	51	57	63

- Be $x_2 = 2$:
$$y_{2,j}\left(2, x_j\right) = 2\left(5x_j + 2\right) + 1, \quad x_{2,j} = 5x_j + 2 \tag{3.2}$$

Table 3.2: The first ten values for (3.2).

x_j	1	2	3	4	5	6	7	8	9	10
$x_{2,j}$	7	12	17	23	28	33	38	43	48	53
$y_{2,j}$	15	25	35	47	57	67	77	87	97	107

- Be $x_3 = 3$:

$$y_{3,j}\left(3, x_j\right) = 2\left(7x_j + 3\right) + 1, \quad x_{3,j} = 7x_j + 3 \tag{3.3}$$

Table 3.3: The first ten values for (3.3).

x_j	1	2	3	4	5	6	7	8	9	10
$x_{3,j}$	10	17	24	31	38	45	52	59	66	73
$y_{3,j}$	21	35	49	63	77	91	105	119	133	147

- Be $x_i = \ldots$: ....

Now, let us also have a look at the extension to $\mathbb{Z}$. At first, we do the change $x_i \to -x_i$.

$$y_{i,j}\left(x_i, x_j\right) = -\left(2\left(\left(2\left(x_i - 1\right) + 1\right)x_j + \left(x_i - 1\right)\right) + 1\right) \tag{3.4}$$

To have attention on this case will still play an role in the next sections. Now, we do the change $x_j \to -x_j$.

$$y_{i,j}\left(x_i, x_j\right) = -\left(2\left(\left(2x_i + 1\right)\left(x_j - 1\right) + x_i\right) + 1\right) \tag{3.5}$$

That's a simple case. We don't have to do anymore.

3.2 Representation: Odd-Not-Divisible Numbers

Now, we take again $y_{i,j}\left(x_i, x_j\right) = 2\left(\left(2x_i + 1\right)x_j + x_i\right) + 1$ and rephrase it into an equation which descripes all odd numbers which are not divisible by $2x_i + 1$.
That's not very hard. We can write

$$y_{i,j}\left(x_i, x_j\right) = 2\left(\left(2x_i + 1\right)x_j + x_i - \mu\left(x_i\right)\right) + 1, \tag{3.6}$$

with

$$\mu\left(x_i\right) = 1, \ldots, 2x_i, \quad \mu\left(x_i\right) \in \mathbb{N}. \tag{3.7}$$

Let's have a short look at the first values for $x_i = 1, 2, 3$.

- Be $x_1 = 1$:

$$y_{1,j}\left(1, x_j\right) = 2\left(3x_j + 1 - \mu\left(1\right)\right) + 1, \quad \mu\left(1\right) = 1, 2, \quad x_{1,j} = 3x_j + 1 \tag{3.8}$$

Table 3.4: The first values for (3.8).

x_j	1		2		3		4		5	
$\mu(1)$	1	2	1	2	1	2	1	2	1	2
$x_{1,j}$	2	3	5	6	8	9	11	12	14	15
$y_{1,j}$	5	7	11	13	17	19	23	25	29	31

- Be $x_2 = 2$:

$$y_{2,j}\left(2, x_j\right) = 2\left(5x_j + 2 - \mu\left(2\right)\right) + 1, \quad \mu\left(2\right) = 1, \ldots, 4 \quad x_{2,j} = 5x_j + 1 \tag{3.9}$$

Table 3.5: The first values for (3.9).

x_j	1				2				3	
$\mu(1)$	1	2	3	4	1	2	3	4	1	2
$x_{2,j}$	3	4	5	6	8	9	10	11	13	14
$y_{2,j}$	7	9	11	13	17	19	21	23	27	29

- Be $x_3 = 3$:

$$y_{3,j}\left(3, x_j\right) = 2\left(7x_j + 1 - \mu\left(3\right)\right) + 1, \quad \mu\left(3\right) = 1, \ldots, 6 \quad x_{3,j} = 7x_j + 1 \tag{3.10}$$

Table 3.6: The first values for (3.10).

x_j	1						1			
$\mu(1)$	1	2	3	4	5	6	1	2	3	4
$x_{3,j}$	4	5	6	7	8	9	11	12	13	14
$y_{3,j}$	9	11	13	15	17	19	23	25	17	29

- Be $x_i = \ldots$: ...

Remark 3.2.1 (Value set) *You can see, the valid value set start not till $x_{i,j} = x_i + 1$.*

3.3 Odd-Not-Divisible Numbers: Intersection

Now we look at the intersection of two equations of the type (3.6) with (3.7). Hence, we start with

$$\begin{aligned} y_{i,j}^{(1)}\left(x_i^{(1)}, x_j^{(1)}\right) &= 2\left(\left(2x_i^{(1)} + 1\right)x_j^{(1)} + x_i^{(1)} - \mu\left(x_i^{(1)}\right)\right) + 1 \\ \mu\left(x_i^{(1)}\right) &= 1, \ldots, 2x_i^{(1)} \end{aligned} \tag{3.11}$$

$$\begin{aligned} \text{and} \quad y_{i,j}^{(2)}\left(x_i^{(2)}, x_j^{(2)}\right) &= 2\left(\left(2x_i^{(2)} + 1\right)x_j^{(2)} + x_i^{(2)} - \mu\left(x_i^{(2)}\right)\right) + 1 \\ \mu\left(x_i^{(2)}\right) &= 1, \ldots, 2x_i^{(2)}. \end{aligned} \tag{3.12}$$

We do the intersection:

$$0 = \left(2x_i^{(1)} + 1\right)x_j^{(1)} - \left(2x_i^{(2)} + 1\right)x_j^{(2)} + x_i^{(1)} - x_i^{(2)} - \mu\left(x_i^{(1)}\right) + \mu\left(x_i^{(2)}\right) \tag{3.13}$$

$$= \left(2x_i^{(1)} + 1\right)\left(x_j^{(1)} - x_j^{(2)}\right) - 2\Delta x_i^{(1,2)} x_j^{(2)} - \Delta x_i^{(1,2)} - \mu\left(x_i^{(1)}\right) + \mu\left(x_i^{(2)}\right) \tag{3.14}$$

$$= \left(2x_i^{(1)} + 1\right)\left(x_j^{(1)} - x_j^{(2)}\right) - \left(2x_j^{(2)} + 1\right)\Delta x_i^{(1,2)} - \mu\left(x_i^{(1)}\right) + \mu\left(x_i^{(2)}\right) \tag{3.15}$$

For the second one, we used $x_i^{(2)} = x_i^{(1)} + \Delta x_i^{(1,2)}$, $x_i^{(2)} > x_i^{(1)}$ and $\Delta x_i^{(1,2)} \in \mathbb{N}$.
To solve (3.13) respectively (3.14), we recognize that we have the boundary constraint, that $\left(2x_i^{(1)} + 1\right)$ and $\left(2x_i^{(2)} + 1\right)$ must not have any common factors.

Let's look at the case $\Delta x_i^{(1,2)} = 1$.

$$\begin{aligned} 0 &= \left(2x_i^{(1)} + 1\right)\left(x_j^{(1)} - x_j^{(2)}\right) - 2x_j^{(2)} - 1 - \mu\left(x_i^{(1)}\right) + \mu\left(x_i^{(2)}\right) && (3.16) \\ &= \left(x_j^{(1)} - x_j^{(2)}\right) - \left(2x_i^{(1)} + 1\right)^{-1}\left(2x_j^{(2)} + 1 + \mu\left(x_i^{(1)}\right) - \mu\left(x_i^{(2)}\right)\right) && (3.17) \end{aligned}$$

Now, let be

$$x_j^{(2)} = \left(1 + \mu\left(x_i^{(1)}\right) - \mu\left(x_i^{(2)}\right)\right)x_i^{(1)}. \tag{3.18}$$

$$\begin{aligned} 0 &= x_j^{(1)} - \left(1 + \mu\left(x_i^{(1)}\right) - \mu\left(x_i^{(2)}\right)\right)x_i^{(1)} - \left(1 + \mu\left(x_i^{(1)}\right) - \mu\left(x_i^{(2)}\right)\right) && (3.19) \\ &= x_j^{(1)} - \left(1 + \mu\left(x_i^{(1)}\right) - \mu\left(x_i^{(2)}\right)\right)\left(x_i^{(1)} + 1\right). && (3.20) \end{aligned}$$

It follows

$$x_j^{(1)} = \left(1 + \mu\left(x_i^{(1)}\right) - \mu\left(x_i^{(2)}\right)\right)\left(x_i^{(1)} + 1\right). \tag{3.21}$$

The equations (3.18) and (3.21) give us one particular solution. It's trivial to see, that we receive all solutions on $\mathbb{Z}$ for

$$\begin{aligned} x_j^{(1)} &= \left(2\left(x_i^{(1)} + 1\right) + 1\right)z^{(1,2)} + \left(1 + \mu\left(x_i^{(1)}\right) - \mu\left(x_i^{(2)}\right)\right)\left(x_i^{(1)} + 1\right) && (3.22) \\ x_j^{(2)} &= \left(2x_i^{(1)} + 1\right)z^{(1,2)} + \left(1 + \mu\left(x_i^{(1)}\right) - \mu\left(x_i^{(2)}\right)\right)x_i^{(1)}, && (3.23) \end{aligned}$$

with $z^{(1,2)} \in \mathbb{Z}$.

4 The recursive calculation

Contents

In this section, now, we do the final recursive calculation. To understand the deep structure we will do this by discussing the first steps by manually calculation. We will see, it exists different ways how you can look at each situation/problem in each step, hence this will mainly a discussion of this different ways.

4.1 Recursion step: $n^{(0)} = 0$

4.1.1 Calculation

Let's start with our step zero. It's called zero, since we start with the simplest first possible prime generation. Since we know 3 is the smallest odd number at all, we know 3 has to be a prime. Hence we have our first prime.

4.1.2 Results

- Known prime $x_{i,j}$: 1
- Known prime $y_{i,j}$: 3
- Known prime number range for $x_{i,j}$: $[1, 1]$
- Known prime number range for $y_{i,j}$: $[1, 3]$

That's of course not much and trivial, but hey, it's a beginning!

4.2 Recursion step: $n^{(0)} = 1$

4.2.1 Calculation

Now, we can start with our first step. From $n^{(0)} = 0$, we know 3(1).

Notation Note: From now on I will write $y\,(x)$ for shortness. For example, 3(1) means $3 = 2 \cdot 1 + 1$. If I only write 1, I mean $x = 1$.

So, let's look again at the values for $x_{1,j}$ from

$$y_{1,j}\,(1, x_j) = 2\,(3x_j + 1) + 1, \quad x_{1,j} = 3x_j + 1. \tag{4.1}$$

Table 4.1: The first values for $x_{1,j}$ are marked in bold. Italic values are within the not describable range.

x_j	–	–	–	1	–	–	2	–	–	3	–	–	4	–	–	5	–	–	6
$x_{1,j}$	*1*	2	3	**4**	5	6	**7**	8	9	**10**	11	12	**13**	14	15	**16**	17	18	**19**

What we can see from our table 4.1 are the next two sure primes, the numbers 5(2) and 7(3). All larger numbers could theoretically still have a divider, hence we only know this two additional numbers surely, at the moment.

We can describe this two numbers with (3.6) and (3.7) by

$$x\,(1, 1) = 3 \cdot 1 + 1 - \mu\,(1)\,, \tag{4.2}$$

with

$$\mu\,(1) = 1, 2. \tag{4.3}$$

and a maximum value for $x_j = 1$.

4.2.2 Results

Only from step $n^{(0)} = 1$:

- Known prime $x_{i,j}$: $2, 3$
- Known primes $y_{i,j}$: $5, 7$
- Known prime number range for $x_{i,j}$: $[2, 4]$
- Known prime number range for $y_{i,j}$: $[5, 9]$

From all steps until now:

- Known prime $x_{i,j}$: $1, 2, 3$
- Known primes $y_{i,j}$: $3, 5, 7$
- Known prime number range for $x_{i,j}$: $[1, 4]$
- Known prime number range for $y_{i,j}$: $[1, 9]$

4.3 Recursion step: $n^{(0)} = 2$

4.3.1 Calculation

Now, from our steps above we have

$$x\left(1, x_j^{(1)}\right) = 3 \cdot x_j^{(1)} + 1 - \mu(1) \tag{4.4}$$

$$x\left(2, x_j^{(2)}\right) = 5 \cdot x_j^{(2)} + 2 - \mu(2) \tag{4.5}$$

$$x\left(3, x_j^{(3)}\right) = 7 \cdot x_j^{(3)} + 3 - \mu(3), \tag{4.6}$$

with $\mu(1) = 1, 2$, $\mu(2) = 1, 2, 3, 4$ and $\mu(3) = 1, 2, 3, 4, 5, 6$. Let's have a look at the values (including $\mu = 0$).

Table 4.2: The first values for $x_{i,j}$ are marked in bold. Italic values are within the not describable range.

$x_{1,j}$	*1*	2	3	**4**	5	6	**7**	8	9	**10**	11	12	**13**	14	15	**16**	17	18	**19**
$x_{2,j}$	*1*	*2*	3	4	5	6	**7**	8	9	10	11	**12**	13	14	15	16	**17**	18	19
$x_{3,j}$	*1*	*2*	*3*	4	5	6	7	8	9	**10**	11	12	13	14	15	16	**17**	18	19

Now, at this point we have different possible ways, how we can calculate more primes from this. We will discuss this different ways step by step.
At first, we make an intersection between (4.4) and (4.5)

$$\begin{aligned} 0 &= 3 \cdot x_j^{(1)} - 5 \cdot x_j^{(2)} + 1 - 2 - \mu(1) + \mu(2) \\ &= 3 \cdot x_j^{(1)} - 5 \cdot x_j^{(2)} - 1 - \mu(1) + \mu(2). \end{aligned} \tag{4.7}$$

From the main section prior to this, we know our solutions with (3.22) and (3.23).

$$x_j^{(1)} = 5z^{(1,2)} + (1 + \mu(1) - \mu(2))\, 2 \tag{4.8}$$

$$x_j^{(2)} = 3z^{(1,2)} + (1 + \mu(1) - \mu(2))\, 1, \tag{4.9}$$

$z^{(1,2)} \in \mathbb{Z}$.

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