CSE2500 - Discrete Mathematics November 5, 2018

Samuel Secondo

Problem Set 4

Note: o7 = QED... it's a salute symbol

1 Exercise Set 4.2, Question 10

Assume that m and n are both integers and that $n \neq 0$. Explain why $\frac{5m + 12n}{4n}$ mus be a rational number.

Definition of rational number:

A real number r is rational iff it can be expressed as the quotient of two integers with a non-zero denominator. That is to say $\exists a, b \in \mathbb{Z}, r = \frac{a}{b} \land b \neq 0$.

In this example, the numerator is an integer because the sums and products of integers are integers. Similarly, the denominator follows the same path because it is just the product of two integers. Therefore the numerator and the denominator are both integers.

Now, all that is needed is to show the the denominator will not be 0. In the given information, it is said to be that $n \neq 0$. Therefore, it is logical to say that $4 * n$, where n is never 0, will never result in 0. Therefore, the denominator will not be 0.

By this, what is left is an integer numerator and denominator, where the denominator is never 0. By the definition of a rational number, this expression has to be a rational number.

o7

2 Exercise Set 4.2, Question 28

Suppose a, b, c, d are integers and $a \neq c$. Suppose also that x is a real number that satisfies the equation:

$$\frac{ax + b}{cx + d} = 1$$

Must x be rational? If so, express x as a ratio of two integers.

Definition of rational number:

A real number r is rational iff it can be expressed as the quotient of two integers with a non-zero denominator. That is to say $\exists a, b \in \mathbb{Z}, r = \frac{a}{b} \land b \neq 0$.

To start, the equation can be modified to solve for x as such:

$$\frac{ax + b}{cx + d} = 1$$

$ax + b = 1 * (cx + d)$

$ax + b = cx + d$

$ax - cx = d - b$

$x(a - c) = d - b$

$x = \frac{d - b}{a - c}$

With this algebraically simplified form, it can be seen as the ratio of two integers, $d-b$ and $a - c$. This portions are both integers because the difference between integers are integers. If one were so inclined, this COULD be written as:

$z = d - b$, where z is an integer (difference of integers are integers)

$w = a - c$, where w is an integer (difference of integers are integers)

$x = \frac{z}{w}$, by Substitution

However, because $a \neq c$, the difference between a and c will never be 0. Therefore, w will never be 0. Hence by the definition of a rational number, x must be rational because it is the division of two integers where the denominator will not be 0.

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3 Exercise Set 4.2, Question 30

Prove that if one solution for a quadratic equation of the form $ax^2 + bx + c = 0$ is rational (where b and c are rational), then the other solution is also rational. Use the fact that if the solutions of the equation are r and s then $x^2 + bx + c = (x - r)(x - s)$.

Definition of rational number:

A real number r is rational iff it can be expressed as the quotient of two integers with a non-zero denominator. That is to say $\exists a, b \in \mathbb{Z}, r = \frac{a}{b} \land b \neq 0$.

Working backwards, we can achieve an expression for the original equation as such:

$x^2 + bx + c = (x - r)(x - s)$

$x^2 + bx + c = x^2 - xs - xr + sr$

$x^2 + bx + c = x^2 + x(-s - r) + sr$

Here, it is simple to see where these variables fall into line with the original equation. This is demonstrated as $b = -s - r$. This is because the expression $(-s - r)$ can be substituted for b. Additionally, the given information includes that b is rational. Furthermore, it is stated that one of the solutions is rational, and the task is to prove the other is also rational. By this we can split this into cases:

Case 1(r is rational)

By this, the expression of b can be manipulated to: $s = -r - b$. As such, it can be stated that s is rational because the difference of rational numbers, $-r - b$, is rational.

Case 2(s is rational)

Similarly, the expression of b can be manipulated to: $r = -s - b$. As such, it can be stated that r is rational because the difference of rational numbers, $-s - b$, is rational.

These cases demonstrate the possibilities, and that if one solution is rational, the other will be as well.

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4 Exercise Set 4.3, Question 5

Is $6m(2m + 10)$ divisible by 4? Where m is an integer.

Definition of divisibility:

For some integers n and d, where d does not equal 0, n is divisible by d iff n equals d times some other integer k.

With this definition in mind, we can simplify the given expression as:

$6m(2m + 10)$

$6m * 2(m + 5)$

$12m(m + 5)$

$4 * 3m(m + 5)$

With this, we can let $3m(m + 5)$ be an integer n. This is because m is an integer and the sum and product of integers are integers. Therefore, the expression simplifies to:

$4 * n$

By the definition of divisibility, the expression is divisible by 4 because it is equal to the product of 4 and some other integer n.

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5 Exercise Set 4.3, Question 29

For all integers a and b, if $a|b$ then $a^2|b^2$

Definition of divisibility:

For some integers n and d, where d does not equal 0, n is divisible by d iff n equals d times some other integer k.

Using this we can say that, $b = a * c$ by the definition of divisibility, and that a divides b from the given information.

Thus, we can square both sides to get the desired form:

$b^2 = (a * c)^2$

$b^2 = a^2 * c^2$

Here, this states that there exists an integer (c^2) such that $c^2 * a^2$ is b^2. (Note: c^2 is still an integer because the product of integers are integers). By the definition of divisibility, a^2 divides b^2

6 Exercise Set 4.3, Question 45

Prove that if n is any non-negative integer whose decimal representation ends in 5, then $5|n$.

Definition of divisibility:

For some integers n and d, where d does not equal 0, n is divisible by d iff n equals d times some other integer k.

Because this is a series of integers, n, it can be represented as the following:

$n = d_k * 10^k + d_{k-1} * 10^{k-1} + ... + d_1 * 10 + 5$, where 5 is the last digit, and d is a digit at index k.

Furthermore, this can be factored as such:

$n = d_k * (2 * 5)^k + d_{k-1} * (2 * 5)^{k-1} + ... + d_1 * (2 * 5) + 5$

$n = d_k * 2^k * 5^k + d_{k-1} * 2^{k-1} * 5^{k-1} + ... + d_1 * 2 * 5 + 5$

Naturally, we can factor out a 5 from the entire expression

$n = 5 * (d_k * 2^k * 5^{k-1} + d_{k-1} * 2^{k-1} * 5^{k-2} + ... + d_1 * 2 * +1)$

Let $p = d_k * 2^k * 5^{k-1} + d_{k-1} * 2^{k-1} * 5^{k-2} + ... + d_1 * 2 * +1$

$n = 5 * p$

However, since p is simply the sum and product of integers, p is an integer. Thus, the number n, where the last digit of which is a 5, can be expressed as 5 times an integer p. Therefore, by the definition of divisibility, 5 divides n.

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7 Exercise Set 4.4, Question 17

Prove that the product of any two consecutive integers is even.

Suppose n is a particular but arbitrarily chosen integer. Naturally, n will either be an even or odd integer. With this, we can split into different cases to account for this.

7.1 Case 1 (n is odd)

$n = 2k + 1$, for some integer k by the definition of odd

$n + 1 = 2k + 2$

$n * (n + 1) = (2k + 1) * (2k + 2)$

$n * (n + 1) = 4k^2 + 4k + 2k + 2$

$n * (n + 1) = 4k^2 + 6k + 2$

$n * (n + 1) = 2 * (2k^2 + 3k + 1)$

Let $p = 2k^2 + 3k + 1$, where p is an integer since the product and sums of integers are integers.

$n * (n + 1) = 2 * p$

With this, the product of two consecutive integers, starting with an odd integer, is proven to be 2 times another integer, p, which happens to be the definition of an even number. Thus, the product results in an even integer.

7.2 Case 2 (n is even)

$n = 2k$, for some integer k by the definition of even.

$n + 1 = 2k + 1$

$n * (n + 1) = 2k * (2k + 1)$

$n * (n + 1) = 2 * (2k^2 + k)$

Let $p = 2k^2 + k$

$n * (n + 1) = 2 * p$

With this, the product of two consecutive integers, starting with an even integer, is proven to be 2 times another integer, p, which happens to be the definition of an even number. Thus, the product results in an even integer.

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8 Exercise Set 4.4, Question 35

Prove the fourth power of any integer, n, has the form $8m$ or $8m+1$. Naturally, n will either be even or odd. As such this can be divided into 2 cases.

8.1 Case 1 (n is odd)

$n^4 = (2k+1)^4$, for some integer k by the definition of odd
$n^4 = (2k+1)^2 * (2k+1)^2$
$n^4 = (2k+1) * (2k+1) * (2k+1) * (2k+1)$
$n^4 = (4k^2 + 4k + 1) * (4k^2 + 4k + 1)$
$n^4 = 16^4 + 16k^3 + 4k^2 + 16k^3 + 16k^2 + 4k + 4k^2 + 4k + 1$
$n^4 = 16k^4 + 32k^3 + 24k^2 + 8k + 1$
$n^4 = 8 * (2k^4 + 4k^3 + 3k^2 + k) + 1$
Let $m = 2k^4 + 4k^3 + 3k^2 + k$, where m is an integer because the sum and product of integers are integers.
$n^4 = 8m + 1$

8.2 Case 2 (n is even)

$n^4 = (2k)^4$, for some integer k by the definition of even
$n^4 = 2^4 * k^4$
Let $m = k^4$
$n^4 = 8 * m$
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9 Exercise Set 4.4, Question 53

Prove that if m, d, and k are integers and $d > 0$, then $(m + dk) \bmod d = m \bmod d$.

The Quotient Remainder Theorem:
Given any n and positive integer d, there exists unique integers q and r such that, $n = qd + r$ and $0 \leq r < d$

Let $r = (m + dk) \bmod d$
By the Quotient Remainder Theorem, there exists an integer q such that:
$m = dq + r$
$m + dk$, from the given information
$dq + r + dk$, by substitution
$d(q + k) + r$, by distributive
Let $h = q + k$, where h is an integer because the sum of integers are integers.
$m + dk = dh + r$

Let $e = m \bmod d$
Therefore: $m = gd + e$, for some integer g
$m + dk = dh + r$
$gd + e + dk = dh + r$
$d(g + k) + e \bmod d = e$
$dh + r \bmod d = r$
Therefore, the remainder of the two expressions are equivalent
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