The Soup Can

Aperture Laboratories

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Introduction

Dear Campbell's Soup Company,

We are Aperture Laboratories, and we are here to inform you of the solution of your company's can. According to your request, you company wanted a can with a volume of 1 liter and you wish to find the dimensions that would minimize its cost where the materials cost 0.02 cents per square centimeter and the seam cost 0.25 cents per centimeter square. This proposal will explain and guide you to your solution.

0.1 Who We Are

Aperture Laboratories is grateful to have been chosen to solve this cylindrical puzzle, as presented by Campbell's Soup Company. Aperture Laboratories strives to find the most optimal solutions to our clients' problems. Whether it be industrial farmers needing to fence their pens, or optimizing the inter-dimensional portals to the moon, Aperture is here to solve your problems. There is no doubt that there will be no greater solution than the one provided through Aperture, for we have very extensive and rigorous testing processes to ensure the best for our clients. No matter the cost.

0.2 The Tools

0.2.1 Maple 2018

Aperture wishes to ensure accurate and consistent solutions. As such, a trusted and well known tool named Maple. Our company is partnered to receive this tool through outside servers to ensure that the tool is valid and legitimate. The purpose of this software is to compute different calculations and manipulate formulas that could take a person hours to do correctly. In addition to manipulation, Maple offers the capability to plot both 2 dimensional curves and 3 dimensional shapes. As such, all plot and shape figures in this report were created through maple unless specified otherwise.

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Chapter 1

The Soup Can

1.1 The Problem

Aforementioned, your company has consulted us to develop a cylindrical can with the dimensions that would minimize its cost. From the information given, here is what is known:

- The shape of the can is a cylinder
- The volume of the can is exactly 1 liter. However, in the interest of keeping consistent units, 1000 cubic centimeters will be used instead, for the two are equivalent.
- There are 3 seams on the can
  - 1 seam down the height of the can to seal the material
  - 1 seam on the circumference of the top of the can
  - 1 seam on the circumference of the bottom of the can
- The cost of material is 0.02 cents per square centimeter
- The cost of a seam is 0.25 cents per centimeter

The goal is to find the dimensions of a cylinder that fit these constraints and create the most cost effective product to store excellent tasting soup.

1.2 The Approach

Here, the general approach of this situation will be summarized on an abstract level. The aim is to create a cost function that will show the cost of the can with respect to its dimensions.

However, since there are 2 variables at play the expression for the cylinder's cost must be brought down to just one variable. This is necessary because having multiple variables for this simple of a problem would only add extra complexity that would not be beneficial. Thus, Maple will be used to manipulate the given information and formulas in order to create an expression between the radius and the height of the cylinder. For this, it was decided to use the radius as the single variable.

With this, all of the elements of the cylinder will be built upon its radius alone. As such, a cost function will be created in order to represent the cost of the can with any given radius. This is crucial because this function can be used to find the radius that has the lowest cost. Additionally, because an expression was created between the radius and height the numerical value of its height can be calculated by this new found radius.

This will all be gone into further detail later in the report.

Chapter 2

Breaking Down the Cylinder

As mentioned, the shape of the can has been expressed as a cylinder. This geometric shape has many geometric properties that will be visited upon throughout this report. However, in the spirit of being thorough, we wish to explain them here.

2.1 The Dimensions of a Cylinder: Height and Radius

There are two distinct characteristics that differ two cylinders, the height and radius. As Figure 2.1 shows, the radius refers to the radius of the circles on the top and bottom of the closed cylinder. In addition, the height refers to the measurement of the shape from top to bottom.

2.2 The Shapes of the Cylinder

At its core, a cylinder is merely a rectangle and two circles. To explain, if the seam that runs down the side of the cylinder were cut, the material would unwrap into a rectangle. This much should be known, as all the material on the can is printed in a rectangular manner. However, it is important to note that the side is just a rectangle with height that is equal to the cylinder's height, and a base that is equal to the circumference of the base of the rectangle. In addition, it can be visually seen that the two caps of the cylinder are circles with the specified radius.

2.3 The Surface Area

As mentioned, the cylinder is essentially the combination of a rectangle and two circles. With this, the surface area can be expressed as the sum of the rectangle's area and the two circles' area. Given the height as h and the radius as r, the area of the rectangle is as follows:

$$2\pi r * h$$

In addition, there are two circles on the cylinder. As such, the expression to calculate their area is as follows:

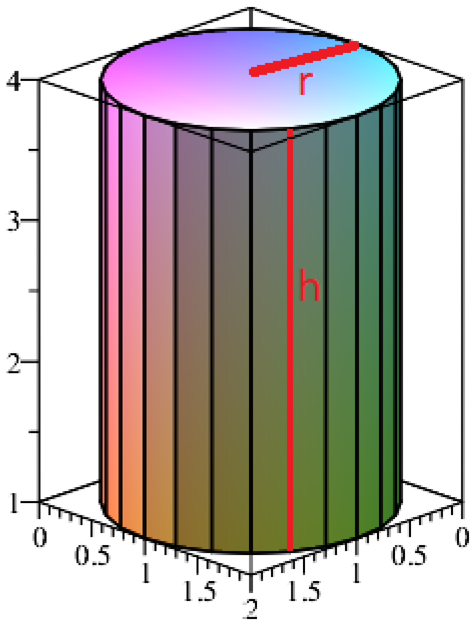


Figure 2.1: The radius and height of a cylinder

$$2 * \pi r^2$$

Combining these terms, the total surface area of the cylinder is concluded as:

$$2\pi r * h + 2 * \pi r^2$$

2.4 The Volume

The volume of the cylinder can be thought of as the area of the base of the cylinder distributed throughout its height. Thus, given the radius and height as r and h respectively, the formula for the cylinder's volume becomes:

$$\pi r^2 * h$$

Chapter 3

Solving

3.1 Isolating the Height

As detailed before, this problem needs to be narrowed down into just one single variable. Thus, this can begin with the given volume, 1000 cubic centimeters. In addition to this, the volume of a cylinder was shown to be the expression $\pi r^2 * h$. By this, the height can be related to the radius variable by simple manipulations of the equation. As seen in Figure 3.1a, h is related to r by the following equation:

$$h = \frac{1000}{\pi r^2}$$

3.2 Modifying Surface Area

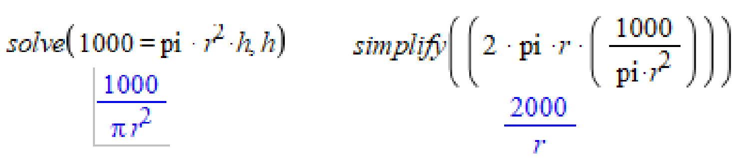
With this new expression for height, the variable h can be substituted such that the surface area of the cylinder is entirely in terms of r. This is done by the following, starting with the formula for surface area:

$$2\pi r h + 2\pi r^2$$

$$(2\pi r * \frac{1000}{\pi r^2}) + 2\pi r^2, \text{ by substitution}$$

However, as seen in Figure 3.1b, the first term can be simplified. Thus the new expression for the surface area is as follows:

$$\frac{2000}{r} + 2\pi r^2$$



(a) Solving for height (b) Simplifying the first term

3.3 Creating the Cost Function

The next objective is to create a cost function that demonstrates the relationship between the radius of the cylinder and the cost to make the can. Thus, the individual components of the cylinder must be revisited and attached to a cost.

3.3.1 The Caps and Side

As proven and simplified, the total surface area of the cylinder in terms of the radius is as follows:

$$\frac{2000}{r} + 2\pi r^2$$

Additionally, the given information stated that the material cost is 0.02 cents per square centimeter. Therefore, the cost of the material in cents, in terms of the radius, is as follows:

$$0.02 * (\frac{2000}{r} + 2\pi r^2)$$

$$\frac{40}{r} + 0.04\pi r^2, \text{ by the Distributive Property}$$

3.3.2 The Seams

In the given information, the seams cost .25 cents per centimeter. In addition, the locations of the seams were detailed to be the height of the cylinder, and the two circumferences of the cylinder. Mathematically, this can be represented, without consideration of cost, as the following:

$$2 * (2\pi r) + h$$

$$4\pi r + h$$

However, with consideration of cost, this can be expressed in cents by multiplying by the cost of every centimeter:

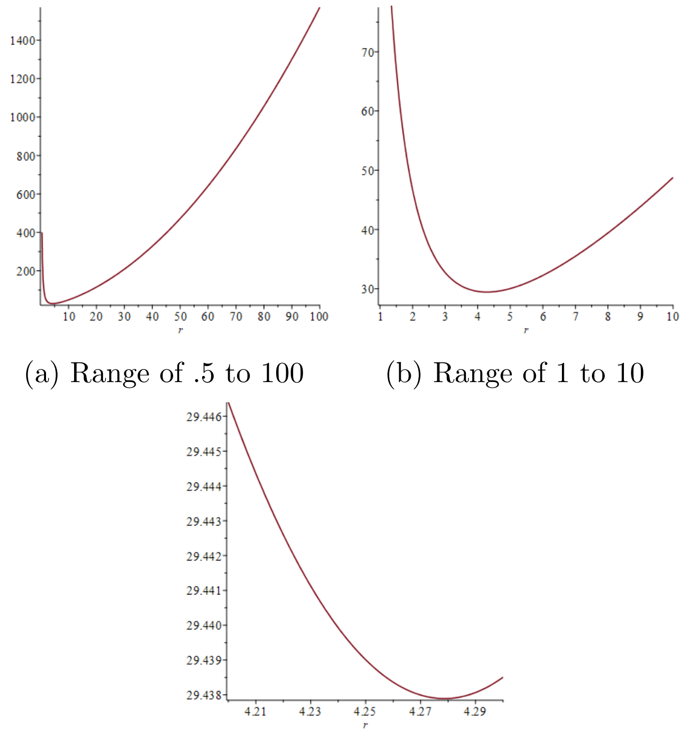
$$0.25 * (4\pi r + h)$$

$$\pi r + 0.25h$$

Consequently however, the expressions all need to be in terms of the radius. Thus, the height can be substituted with the earlier expression of, $h = \frac{1000}{\pi r^2}$.

$$\pi r + 0.25 * \frac{1000}{\pi r^2}$$

$$\pi r + \frac{250}{\pi r^2}$$



(a) Range of .5 to 100 (b) Range of 1 to 10

(c) Range of 4.2 to 4.3

Figure 3.2: Graphs of the cost (cents) function with respect to radius (centimeters)

3.3.3 The Final Function

With all of this, the final cost function can be determined to find the amount of cents associated with a radius in centimeters. This function is the sum of the two previously proven expressions:

$$\frac{40}{r} + 0.04\pi r^2 + \pi r + \frac{250}{\pi r^2}$$

3.4 Solving with a graph

To find the radius that would be the most cost effective, one method would be to graph this function and see where the cost is at the lowest. This is the visual approach that demonstrates the relationship between the radius and the cost. As seen in Figure 3.2 as the range is narrowed in on the lowest cost, the radius seems to approach about 4.28 centimeters (as seen in Figure 3.2c). This visual approach allows a simple and nice answer to how big the radius should be. In addition, the graph visually demonstrates that there is no other minimum to this equation. Although, it should be noted that all negative values for the radius are ignored because a negative radius is not possible.

3.5 Solving with Derivatives

The following methods are alternative solutions to find the best and cost efficient radius for the cylinder. The first and second derivatives of the cost function will be used in order to test where there are maximums and minimums in the cost function. The minimums are of interest because a minimum in this cost function represents a possible solution to the problem. Once all of these minimums are gathered, they can be compared to find the absolute minimum of the cost function.

Both of these methods will use what is called critical numbers. Put simply, the critical number is the point where the first derivative is equal to 0. As seen in Figure 3.3, the only non-negative and real answer is 4.278623553. While this is the answer from before, it has yet to be proven.

$$C := r \rightarrow \frac{40}{r} + \left(0.04\,\pi \cdot r^2\right) + \left(\pi \cdot r\right) + \frac{250}{\pi \cdot r^2}$$

$$C := r \mapsto \frac{40}{r} + 0.04\,\pi\,r^2 + \pi\,r + \frac{250}{\pi\,r^2}$$

$$F := \mathit{diff}(C(r), r)$$

$$F := -\frac{40}{r^2} + 0.2513274124\,r + \pi - \frac{500}{\pi\,r^3}$$

$$\mathit{solve}(F = 0)$$

$$4.278623553,\ -2.521182608 + 2.500910110\,\mathrm{I},\ -11.73625833,\ -2.521182608 - 2.500910110\,\mathrm{I}$$

Figure 3.3: Critical Numbers of the Cost Function

3.5.1 First Derivative Test

The first derivative test aims to prove the presence of a minimum by investigating the behavior of the first derivative. To do so, values around the the critical point are evaluated in order to inspect how the derivative changes through the critical point. If the derivative begins negative, then 0 at the critical point, and then positive after, then the critical point must have been a minimum. This is similarly true for the maximum where it goes from positive, to 0, and then to a negative slope.

The simplest way to do this is create a table of values that expresses the evaluation of the first derivative. For this table, let r be the radius in centimeters, Let C(r) be the cost in cents for a given radius r. Let C'(r) be the derivative of the cost function, at a radius r. The evaluations of these inputs can be found in Figure 3.4.

r	4.2	4.278623553	4.3
C(r)	29.44640098	29.43789428	29.43850725
C'(r)	-.218593782	0	0.57196285e-1

Table 3.1: Cost and First Derivative

In this table, the first derivative starts at a negative value, becomes 0, and then goes up to a positive value. Therefore, the critical number 4.278623553 is in fact a minimum of the cost function.

$$C := r \rightarrow \frac{40}{r} + (0.04\,\pi \cdot r^2) + (\pi \cdot r) + \frac{250}{\pi \cdot r^2}$$

$$C := r \mapsto \frac{40}{r} + 0.04\,\pi r^2 + \pi r + \frac{250}{\pi r^2}$$

$C(4.2)$

29.44640098

$C(4.278623553)$

29.43789428

$C(4.3)$

29.43850725

(a) Evaluating Cost at Radius Values

$F := diff(C(r), r)$

$$F := -\frac{40}{r^2} + 0.2513274124\,r + \pi - \frac{500}{\pi r^3}$$

$eval(F, r = 4.2)$

-0.218593782

$eval(F, r = 4.3)$

0.057196285

(b) Evaluating Slope at Radius Values

Figure 3.4: Evaluations of Cost and the first derivative

3.5.2 Second Derivative Test

The rules of the second derivative test are much more simple. If the value of the second derivative, with the critical point as the x input, is positive then there is a minimum. On the other hand, if the value is negative, then that means there is a maximum at that point. This works because if the derivative is 0 at the critical point, but is still increasing, then the slope must be going from negative to positive. Conceptually, if the slope changes from negative to positive, the curve must have hit a lowest value. Therefore, if the second derivative is positive at the critical point, then the curve has a minimum at that point. This is similarly true for a negative value and a maximum.

Figure 3.5 demonstrates Maple solving for the second derivative, and then evaluating the result for the critical number. With this, it shows that the second derivative is positive at the critical point. This means that there is in fact a minimum at this point. Furthermore, because this is the only minimum, this must be the lowest cost out of any other valid radius. Therefore, 4.278623553 centimeters is proven to be the correct answer by the second derivative test.

$$\frac{1000}{\mathrm{Pi} \cdot \left(4.278623553^2\right)}$$

$$17.38769334$$

Figure 3.6: Evaluating Height

$$S := \mathit{diff}\left(-\frac{40}{r^2} + 0.2513274124\, r + \pi - \frac{500}{\pi r^3}, r\right)$$

$$S := \frac{80}{r^3} + 0.2513274124 + \frac{1500}{\pi r^4}$$

$$\mathit{eval}(S, r = 4.278623553)$$

$$2.697390265$$

Figure 3.5: Evaluating Second Derivative at Critical Point

3.6 Finding The Height

The previous demonstrated that the most optimal radius is about 4.2786 centimeters. This has a cost of about 29.4379 cents per can. However this is not yet complete, for height still needs to be solved for. Thus, the relation equation of height and radius will make its use once more:

$$h = \frac{1000}{\pi r^2}$$

$$r = 4.278623553$$

$$h = \frac{1000}{\pi 4.278623553^2}$$

$$h = 17.38769334, \text{ by Figure 3.6}$$

Thus, it can be concluded that the radius is about 4.2786 centimeters, the height is about 17.38769 centimeters, and the cost is about 29.4379 cents per can.

Chapter 4

The Model

Included with this report is a prototype of the can with the same dimensions as concluded in this report. This is to demonstrate that the can itself is a reasonable size, and is capable of storing up to 1 liter of delicious soup. Observe the model in Figure 4.1. In addition, this has been attached to the report as well. Aperture Laboratories is proud to present Campbell's Soup Company the next, and most cost effective, soup container for all soup containing needs!

Figure 4.1: The Soup Can Model

Chapter 5

Conclusion

This section aims to verify and conclude the results of this report.

5.1 Volume Check

Part of the given information was that the volume of the can must be 1000 cubic centimeters. This can be checked by plugging in the concluded dimensions into the volume formula as follows:

$$1000 = \pi r^2 h$$

$$h = 17.38769334$$

$$r = 4.278623553$$

$$1000 = \pi * 4.278623553^2 * 17.38769334$$

$$1000 = 999.999999744$$

It is important to note that it is not exactly 1000 due to rounding. This is because the values for height and radius expand much further than provided here. However, as seen in the calculation, the error is barely noticeable, but should not be forgotten. With this, it has been demonstrated that the determined dimensions do result in a volume of 1000 cubic centimeters, as given in the request for this product.

5.2 The Units

It is relevant to note that the units of this product are reasonable. By this, it would be unreasonable for a can to be several feet tall. However, the results from this report give realistic dimensions that is convenient for the company and the consumers. In addition, the cost to make this is about 29 cents per can, which is a reasonable cost for a can of this size.