GNR Assignment

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Why the Difference of Two Convex Functions Is Not Necessarily Convex

1. Definition of Convexity

A function f is called *convex* if, for all x, y and for all $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \ \leq \ \lambda f(x) + (1-\lambda)f(y).$$

When f is twice differentiable, a sufficient condition for convexity is that its second derivative is nonnegative everywhere:

$$f''(x) \geq 0 \quad forallx.$$

2. Two Convex Functions

Consider the following functions:

$$f_1(x) = x^2, \quad f_2(x) = x^4.$$

- For $f_1(x) = x^2$, we have $f_1''(x) = 2$, which is always positive. Hence, f_1 is convex.
- For $f_2(x) = x^4$, we have $f_2''(x) = 12x^2$, which is nonnegative for all x. Hence, f_2 is convex.

3. Forming the Difference

Define

$$h(x) = f_1(x) - f_2(x) = x^2 - x^4.$$

We want to check whether $h(x)$ is convex.

4. Derivatives of hh

Compute the first and second derivatives of h:

$$h'(x) = 2x - 4x^3,$$

$$h''(x) = 2 - 12x^2.$$

5. Sign of the Second Derivative

For $h(x)$ to be convex, we need $h''(x) \geq 0$ for all x. However,

$$h''(x) = 2 - 12x^2.$$

This is nonnegative only if

$$2 - 12x^2 \geq 0 \quad \iff \quad 12x^2 \leq 2 \quad \iff \quad x^2 \leq \frac{1}{6}.$$

For $x > \sqrt{\frac{1}{6}}$, we have $h''(x) < 0$. Thus, $h''(x)$ is not nonnegative everywhere, so $h(x)$ is not convex.

6. Conclusion

We have:

$$f_1(x) = x^2 \quad (convex), \quad f_2(x) = x^4 \quad (convex), \quad but \quad h(x) = x^2 - x^4 \quad (notconvex).$$

This shows that while each of f_1 and f_2 is convex, their difference $f_1 - f_2$ may fail to be convex. Hence, *the difference of two convex functions is not necessarily convex.*