Basic Count based Probability

Terminologies

Probability

Probability is a numerical value that describes the chance that something will happen.

A value between zero and one, inclusive, describing the relative possibility (chance or likelihood) an event will occur.

Example :

- Probability of getting head in when we toss a coin = 0.5
- Probability of getting ‘6’ when we roll a dice = 1/6

Experiment

A process that leads to the occurrence of one and only one of several possible results.

Example :

- Flipping a coin.
- Rolling a dice.

Terminologies

Outcome

A particular result of an experiment is called as an outcome.

Example :

- Getting head when we toss a coin.
- Getting '1' on face when we roll a dice.

Sample space

The set of all possible outcomes is called as sample space.

If we toss a coin then
Sample space : {H,T}

Event

A collection of one or more outcomes of an experiment is considered as an event.

Example :

- Getting head or getting a tail when we toss a coin.
- Getting odd number or getting even number on face when we roll a dice.

Classical and Empirical Probability

Classical Probability

Classical probability is based on the assumption that the outcomes of an experiment **are equally likely.**

The probability of an event happening is computed by dividing the number of favourable outcomes by the number of possible outcomes.

Classical probability refers to a probability that is based on formal reasoning.

$$\text{Probability of an event} = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}}$$

Empirical Probability

Empirical probability is number of times an event occurs as a proportion of a known number of trials.

$$\text{Empirical probability} = \frac{\text{Number of times the event occurs}}{\text{Total number of observations}}$$

Empirical probability is based on the results which we obtain from an experiment

Subjective Probability

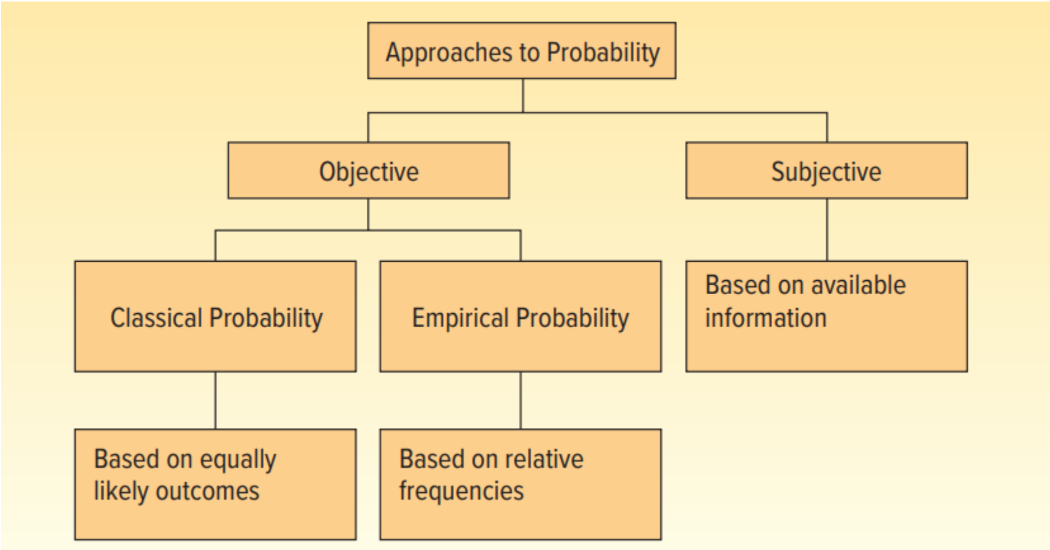
Subjective Probability

If there is little or no experience or information on which to base a probability, it is estimated subjectively. Essentially, this means an individual evaluates the available opinions and information and then estimates or assigns the probability. This probability is called a subjective probability.

The likelihood (probability) of a particular event happening that is assigned by an individual based on whatever information is available.

Examples:

- Probability that it will rain tomorrow is 0.65.
- Probability that you will win lottery is 0.02.



Mutually Exclusive and exhaustive events

Mutually Exclusive Events

Two events A and B are said to be mutually exclusive if they can not occur simultaneously i.e. if one event occurs then other can not occur.

Example :

If we take an examination then either we fail or pass.

Exhaustive Events

A set of events are called exhaustive events if at least one of them necessarily occurs whenever the experiment is performed.

Also, the union of all these events constitutes the sample space of that experiment.

Example :

when throwing an unbiased six-sided die, the outcomes 1, 2, 3, 4, 5, and 6 are collectively exhaustive.

Example:

If we roll a dice and assume the following events

A : Getting Odd numbers {1,3,5}

B : Getting Even numbers {2,4,6}

C : Getting 2 or 4 {2,4}

Event A and B are mutually exclusive and also exhaustive.

Event A and C are mutually exclusive but not exhaustive.

Addition rule

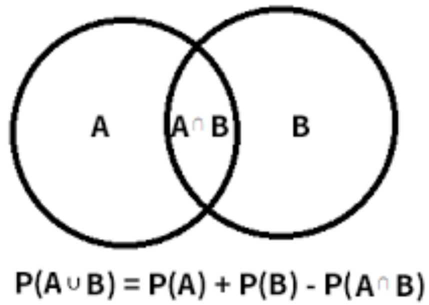
Addition rule of probability

When we perform an experiment which results in two events A and B
If A and B are not mutually exclusive

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

If A and B are mutually exclusive events

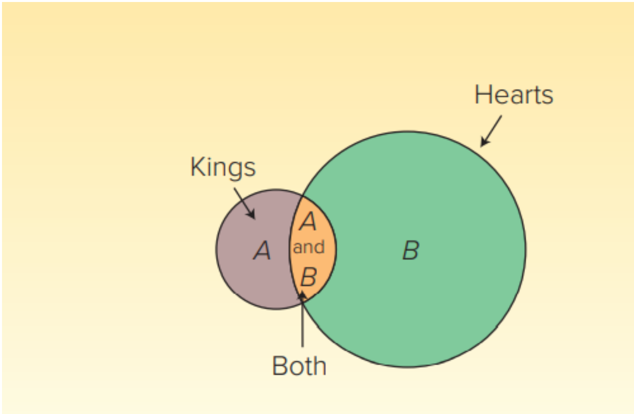
$$P(A \text{ or } B) = P(A) + P(B)$$



Example :

What is the probability that a card chosen at random from a standard deck of cards will be either a king or a heart?

Card	Probability	Explanation
King	$P(A) = 4/52$	4 kings in a deck of 52 cards
Heart	$P(B) = 13/52$	13 hearts in a deck of 52 cards
King of Hearts	$P(A \text{ and } B) = 1/52$	1 king of hearts in a deck of 52 cards



$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ &= 4/52 + 13/52 - 1/52 \\ &= 16/52, \text{ or } .3077 \end{aligned}$$

Independent and Dependent events

Independent events

Two events A and B are said to be independent if occurrence or non occurrence of one event is not affected by occurrence or non occurrence of another event.

Example:

Choosing a marble from a jar and then landings on heads after tossing a coin.

Dependent Events

Two events A and B are said to be dependent if occurrence or non occurrence of one event is affected by occurrence or non occurrence of another event.

Example:

Choosing a marble from a jar and choosing another marble without replacing the first marble.

Conditional Probability

Multiplication Rule

Conditional Probability

Conditional probability is the probability of a particular event occurring, given that another event has occurred.
If there are two events A and B then condition probability is denoted as
P(A/B) : Probability that event A will occur when event B has already occurred OR
P(B/A) : Probability that event B will occur when event A has already occurred

Multiplication Rule

If two events A and B are dependent

P(A and B) = P(A)*P(B/A)

If two events A and B are independent

P(A and B) = P(A)*P(B)

Multiplication Rule

Examples

Conditional probability is the probability of a particular event occurring, given that another event has occurred.
If there are two events A and B then condition probability is denoted as
P(A/B) : Probability that event A will occur when event B has already occurred OR
P(B/A) : Probability that event B will occur when event A has already occurred

Multiplication Rule

If two events A and B are dependent

P(A and B) = P(A)*P(B/A)

If two events A and B are independent

P(A and B) = P(A)*P(B)

Conditional Probability Formula

$$\underbrace{P(A \mid B)}_{\text{Probability of } A \text{ given } B} = \frac{\overbrace{P(A \cap B)}^{\text{Probability of } A \text{ and } B}}{\underbrace{P(B)}_{\text{Probability of } B}}$$

Conditional Probability Examples

Examples

Two dies are thrown simultaneously and the sum of the numbers obtained is found to be 7. What is the probability that the number 3 has appeared at least once?

Solution: The sample space S would consist of all the numbers possible by the combination of two dies. Therefore S consists of 6 × 6 i.e. 36 events.

Event A indicates the combination in which 3 has appeared at least once.

Event B indicates the combination of the numbers which sum up to 7.

A = {(3, 1), (3, 2), (3, 3)(3, 4)(3, 5)(3, 6)(1, 3)(2, 3)(4, 3)(5, 3)(6, 3)}

B = {(1, 6)(2, 5)(3, 4)(4, 3)(5, 2)(6, 1)}

P(A) = 11/36

P(B) = 6/36

A ∩ B = 2

P(A ∩ B) = 2/36

Applying the conditional probability formula we get,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{36}}{\frac{6}{36}} = \frac{1}{3}$$

Bayes Rule

Bayes theorem

Prior Probability

Prior probability is the probability of an event that is calculated before considering the new information obtained. It is the probability of an outcome that is determined based on current knowledge before the experiment is performed.

Posterior Probability

Posterior probability is the probability of an event that is calculated after all the information related to the event has been accounted for. It is also known as conditional probability.

Bayes theorem

Bayes theorem, in simple words, determines the conditional probability of an event A given that event B has already occurred. Bayes theorem is also known as the Bayes Rule or Bayes Law.

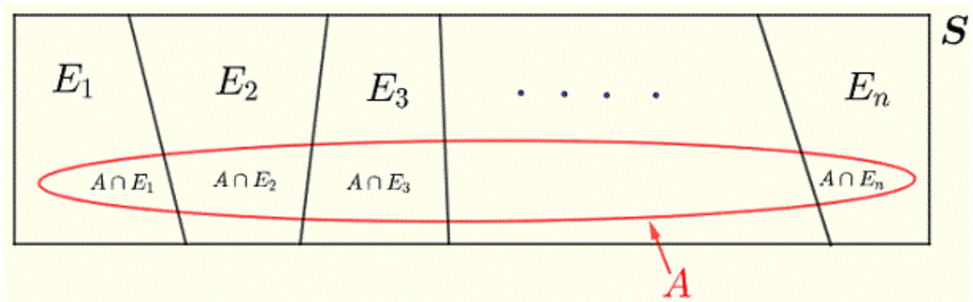
$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

A, B = events

$P(A|B)$ = probability of A given B is true

$P(B|A)$ = probability of B given A is true

$P(A), P(B)$ = the independent probabilities of A and B



Bayes theorem Example

Example

A bag I contains 4 white and 6 black balls while another Bag II contains 4 white and 3 black balls. One ball is drawn at random from one of the bags, and it is found to be black. Find the probability that it was drawn from Bag I.

Answer

Let E1 be the event of choosing bag I, E2 the event of choosing bag II, and A be the event of drawing a black ball.

$$P(E_1) = P(E_2) = \frac{1}{2}$$

Also, $P(A|E_1)$ = P(drawing a black ball from Bag I) = 6/10 = 3/5

$P(A|E_2)$ = P(drawing a black ball from Bag II) = 3/7

By using Bayes' theorem, the probability of drawing a black ball from bag I out of two bags,

$$P(E_1|A) = \frac{P(E_1)P(A|E_1)}{P(E_1)P(A\,|\,E_1)+P(E_2)P(A|E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{3}{5}}{\frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{3}{7}}$$

$$= \frac{7}{12}$$

Discrete Probability Distributions

Probability distribution

What is Probability Distribution?

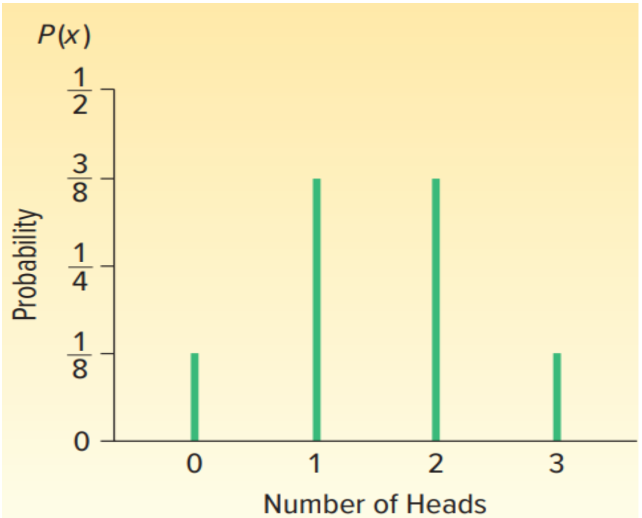
The probability distribution gives the possibility of each outcome of a random experiment or event. It provides the probabilities of different possible occurrences.

Properties of Probability distribution

1. The probability of a particular outcome is between 0 and 1 inclusive.

2. The outcomes are mutually exclusive.

3. The list of outcomes is exhaustive. So the sum of the probabilities of the outcomes is equal to 1.



Random Variables

What is Random Variable?

A random variable is a variable whose value is unknown, or a function that assigns values to each of an experiment's outcomes.

Random variables could be either discrete or continuous.

Examples :

- Number of items sold at a store on a certain day.
- Interest rate of loans in a certain country.
- Height of a certain species of plant.

Discrete random Variables

A discrete random variable is one which may take on only a countable number of distinct values such as 0,1,2,3,4.

Examples:

- The number of children in a family
- The number of defective light bulbs in a box

Discrete Probability Distribution

Discrete Probability Distribution

A discrete probability distribution can be defined as a probability distribution giving the probability that a discrete random variable will have a specified value.

Examples :

Suppose a fair dice is rolled. the discrete probability distribution table for a dice roll can be given as follows:

x	1	2	3	4	5	6
P(X = x)	1 / 6	1 / 6	1 / 6	1 / 6	1 / 6	1 / 6

PMF

The probability mass function can be defined as a function that gives the probability of a discrete random variable.
The formula is given as follows: $f(x) = P(X = x)$

CDF

The cumulative distribution function gives the probability that a discrete random variable will be lesser than or equal to a particular value.
Its formula is given as follows: $F(x) = P(X \leq x)$

Expectation and Variance for Discrete Probability Distribution

Expectation

Expected value or mean of a discrete probability distribution is given by

$$\mu_x = \sum[x * P(x)]$$

Variance

Variance of a discrete probability distribution is given by

$$\sigma^2{}_x = \sum[x^2 * P(x)] - \mu_x^2$$

Example

x	P(x)	x^2	x^2 * P(x)
1	0.10	1*1 = 1	1 * 0.10 = 0.10
2	0.30	2*2 = 4	4 * 0.30 = 1.20
3	0.45	3*3 = 9	9 * 0.45 = 4.05
4	0.15	4*4 = 16	16 * 0.15 = 2.40

Mean Formula:

$$\mu_x = \sum[x * P(x)]$$

$$\mu_x = 2.65$$

Variance Formula:

$$\sigma^2{}_x = \sum[x^2 * P(x)] - \mu_x^2$$

Uniform Distribution

Uniform distribution is a probability distribution that States that all the outcomes for a discrete set of data have the same probability.

PMF

If X is a random variable which follows discrete uniform distribution then its denoted as **$X \sim U(n)$** its PMF is given by

P(X=x) = $\frac{1}{n}$ **X = 1,2,…,n**

Mean

the mean or expected value of a discrete uniform distribution is given by

E[X] = $\frac{n+1}{2}$

Variance

the variance of a Bernoulli distribution is **Var[X] = $\frac{n^2-1}{12}$**

Bernoulli Distribution

- In Bernoulli distribution the random variable can only have 2 possible outcomes.
- If in a Bernoulli trial the random variable takes on the value of 1, it means that this is a success. The probability of success is given by p.
- Similarly, if the value of the random variable is 0, it indicates failure. The probability of failure is q or 1 - p.(q = 1-p)

PMF

If X is a random variable which follows Bernoulli distribution then its denoted as **X ~ Bernoulli(p)** and its PMF is given by

$f(x, p) = p^x (1 - p)^{1 - x}$ **X = 0,1**

Mean

the mean or expected value of a Bernoulli distribution is given by

E[X] = p

Variance

the variance of a Bernoulli distribution is **Var[X]**

= p(1 - p)

= p . q

Binomial Distribution

- In Binomial distribution also, the random variable can only have 2 possible outcomes.
- In binomial distribution the random variable is number of successes in a fixed number of trials.
- The probability of success is the same for each trial and is given by **p**. The probability of failure is **q** or **1 - p**. **(q = 1-p)**
- The probability of success is the same for each trial.

PMF

If X is a random variable which follows Binomial distribution then it is denoted as **X ~ Binomial(n,p)** its PMF is given by

$P(x{:}n,p) = {}^{n}C_{x}\, p^{x}\, (1-p)^{n-x}$ **X = 0,1,…,n**

Mean

the mean or expected value of a Bernoulli distribution is given by

E[X] = np

Variance

the variance of a Bernoulli distribution is **Var[X]**

= np(1 - p)

= npq

Formulae for Reference

$$n! = n \times (n-1) \times \cdots \times 1$$

$$1! = 1 = 1$$

$$2! = 2 \times 1 = 2$$

$$3! = 3 \times 2 \times 1 = 6$$

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

$${}_{n}C_{r} = \frac{n!}{r!(n-r)!}$$

Binomial Distribution Examples

Question : If a coin is tossed 5 times, find the probability of:
(a) Exactly 2 heads

Solution:

According to the problem:
Number of trials: n=5
Probability of head: p= 1/2 = 0.5
Therefore probability of tail : q = 1 – p = 1 - 0.5 = 0.5

P(Exactly 2 heads)
= $\binom{5}{2}0.5^2 0.5^{5-2}$

$$= \frac{5!}{2!3!} \times (0.5)^5$$

= 10 * 0.03125
= 0.3125

Difference between Bernoulli and Binomial Distribution

Bernoulli Distribution	Binomial Distribution
Bernoulli distribution is used when we want to model the outcome of a single trial of an event.	If we want to model the outcome of multiple trials of an event, Binomial distribution is used.
It is represented as X ~~ Bernoulli (p). Here, p is the probability of success.	It is denoted as X ~~ Binomial (n, p). Where n is the number of trials.
Mean E[X] = p	Mean E[X] = np
Variance Var[X] = p(1-p)	Variance Var[X]= np(1-p)
Example: Suppose the probability of passing an exam is 80% and failing is 20%. Then the Bernoulli distribution can be used to model the passing or failing in such an exam.	Example: Suppose the probability of passing an exam is 80% and failing is 20%. Then if we want to find the probability that a student will pass in exactly 4 out of 5 exams, we use the Binomial Distribution.

Poisson Distribution

The Poisson probability distribution describes the number of times some event occurs during a specified interval. Examples of an interval may be time, distance, area, or volume.

PMF

If X is a random variable which follows Poisson distribution then its denoted as **$X \sim \text{Poisson}(\mu)$** its PMF is given by

$$P(x) = \frac{\mu^x e^{-\mu}}{x!}$$

X = 0,1,2,…

Mean

the mean or expected value of a Poisson distribution is given by

E[X] = μ

Variance

the variance of a Poisson distribution is **Var[X] = μ**

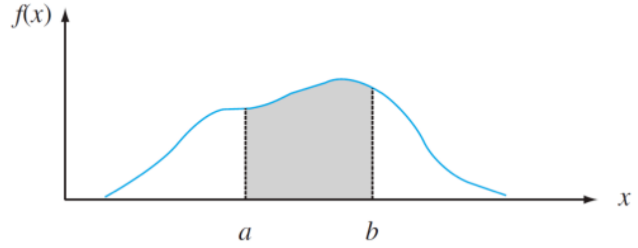
Continuous Probability Distributions

Continuous Probability Distribution

Continuous Probability Distribution

A continuous probability distribution can be defined as a probability distribution giving the probability that a continuous random variable will lie between two numbers a and b where a ≤ b and its also called as probability density function.
The probability that X takes on a value in the interval [a, b] is the area shown in the graph of the density function:

$$P(a \le X \le b) = \int_a^b f(x)dx$$



$P(a \le X \le b)$ = the area under the density curve between a and b

Properties of Continuous Probability distribution function

1. $f(x) \ge 0$ for all x

2. $\int_{-\infty}^{\infty} f(x)dx$ = area under the entire graph of $f(x)$ = 1

Continuous Probability Distribution

Cumulative distribution function

The cumulative distribution function F(x) for a continuous rv X is defined for every number x by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\,dy$$

Expected Value or Mean

The expected (or mean) value of a continuous rv X with the pdf f(x) is:

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx$$

Variance and Standard Deviation

The variance of a continuous random variable X with pdf f(x) and mean value μ is

$$\begin{aligned}\sigma_X^2 = V(X) &= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\,dx \\ &= E[(X-E(X))^2] \\ &= E(X^2) - E(X)^2\end{aligned}$$

The standard deviation (SD) of X is

$$\sigma_X = \sqrt{V(X)}$$

Normal Distribution

Normal Distribution

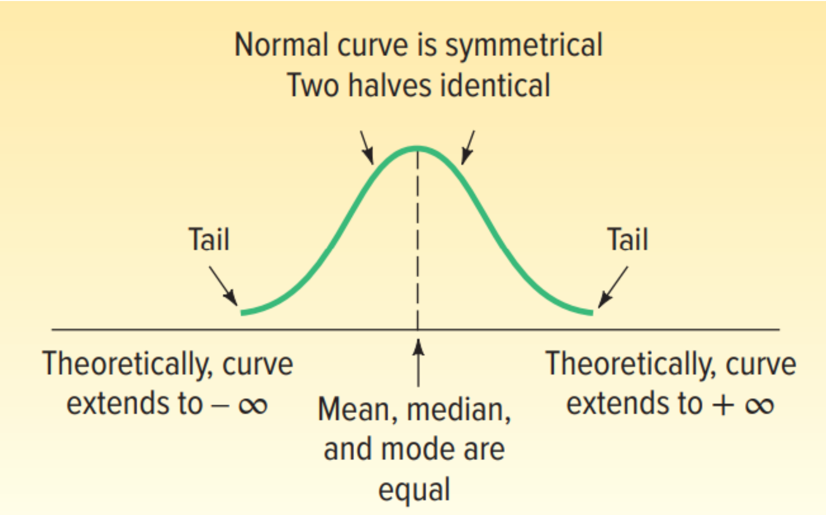
Normal Distribution

If X is a continuous random variable which follows normal distribution with parameters (mean = μ, Standard deviation = σ) then its denoted as **X ~ Normal(μ, σ)** and its pdf is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad -\infty \leq X \leq \infty$$

Properties of Normal Distribution

- Bell Shaped and has single peak at the center.
- The mean, mode and median are all equal.
- The curve is symmetric at the center (i.e. around the mean, μ).
- Exactly half of the values are to the left of center and exactly half the values are to the right.
- The total area under the curve is 1.



Standard normal variable

Standard Normal Distribution

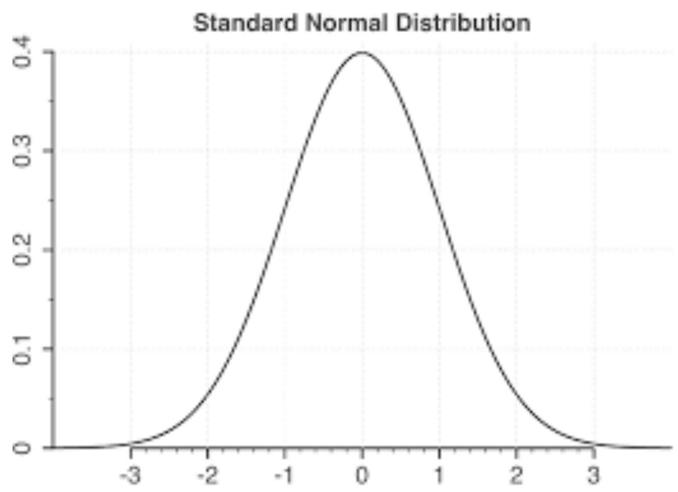
- Any normal probability distribution can be converted into a standard normal probability distribution by subtracting the mean from each observation and dividing this difference by the standard deviation.
- So, a z value is the distance from the mean, measured in units of the standard deviation. The formula for this conversion is:

$$z = \frac{x - \mu}{\sigma}$$

where:
x is the value of any particular observation or measurement.
μ is the mean of the distribution.
σ is the standard deviation of the distribution.

Properties of standard normal variable

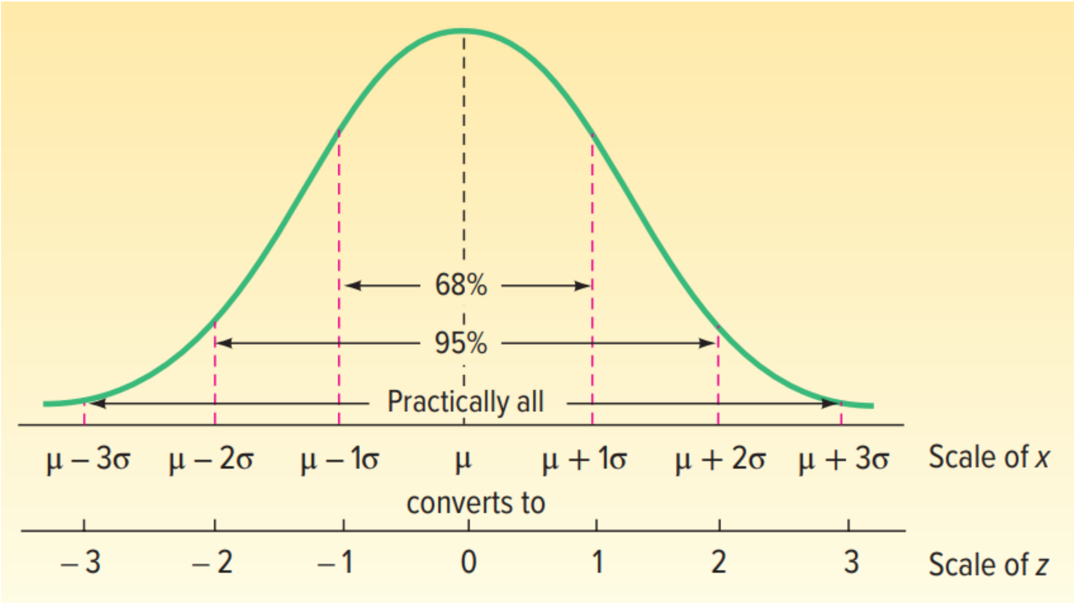
1) The graph of standard normal distribution is bell shaped.
2) The curve is symmetric about mean (μ =0)
3) Standard deviation is equal to 1



The Empirical Rule

Normal Distribution

1) Approximately 68% of the observations will lie within 1 standard deviation of the mean.
2) About 95% of the observations will lie within 2 standard deviations of the mean.
3) Practically all, or 99.7% of the observations will lie within 3 standard deviations of the mean.



Normal Distribution Example

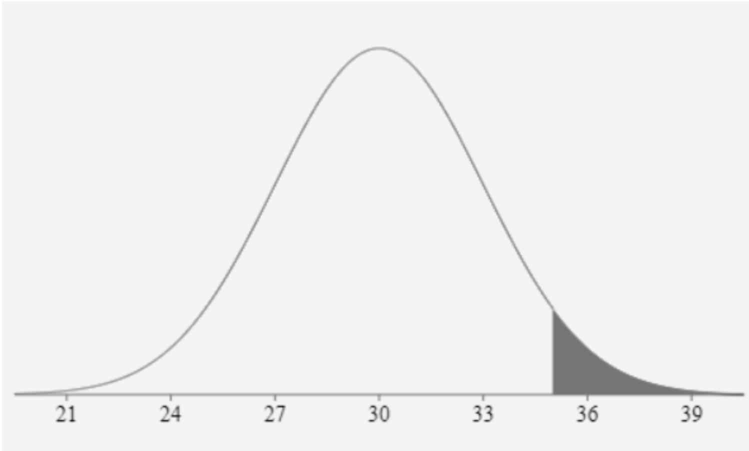
Normal Distribution

The lifetime of a bulbs in a certain region are normally distributed with a mean of 30 days and standard deviation of 3 days. What is the probability that bulbs will working even after 35?

Solution:

Let X be the random variable denoting the lifetime of bulbs.

$X \sim \text{Normal}(\mu = 30, \sigma = 3)$



Probability that bulb is working for more than 35 days

$= P(X > 35)$

$= P(Z > \frac{35-30}{3})$

$= P(Z > 1.67)$

$= 0.0478$


Thank you!