DS JULY 2022 Batch
Module 6 : Mathematics Basics

Topics

- Functions
- Derivatives
- Derivative as slope of curve
- Optimality conditions
- Integration as area under the curve
- Vectors, Matrices and their multiplication
- Eigen Values and Eigen Vectors

Functions

Functions

Functions

A function relates an input to an output. It is a mapping that maps an input to an output.

It is denoted by f(x) where x is the input for the function.

For example :

f(x) = 2x is our function

f(6) = 12 then 6 is our input and 12 is output.

Domain : Possible values of input

Range : Possible values of output

Example : For the above function if x is in [1,2,3,…100] then this is our domain.

and [2,4,6,…,200] is our range.

Derivatives

Derivatives

Derivatives

Derivatives in mathematics is rate at which one quantity changes with respect to another.
It is also called as instantaneous rate of change.

Example :

In next 5 years how many data scientists will be required globally?

It is represented as $\frac{dy}{dx}$

Here y represents the quantity whose rate of change we want to estimate.

x represents the variable with respect to we are calculating rate of change in y.

Generally We try to find rate of change of a function with respect to its input.

So here y will generally be our function ($y = f(x)$)

Some Important Results

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$

(ii) $\frac{d}{dx}(\sin x) = \cos x$

(iii) $\frac{d}{dx}(\cos x) = -\sin x$

(iv) $\frac{d}{dx}(\tan x) = \sec^2 x$

(v) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(vi) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(vii) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

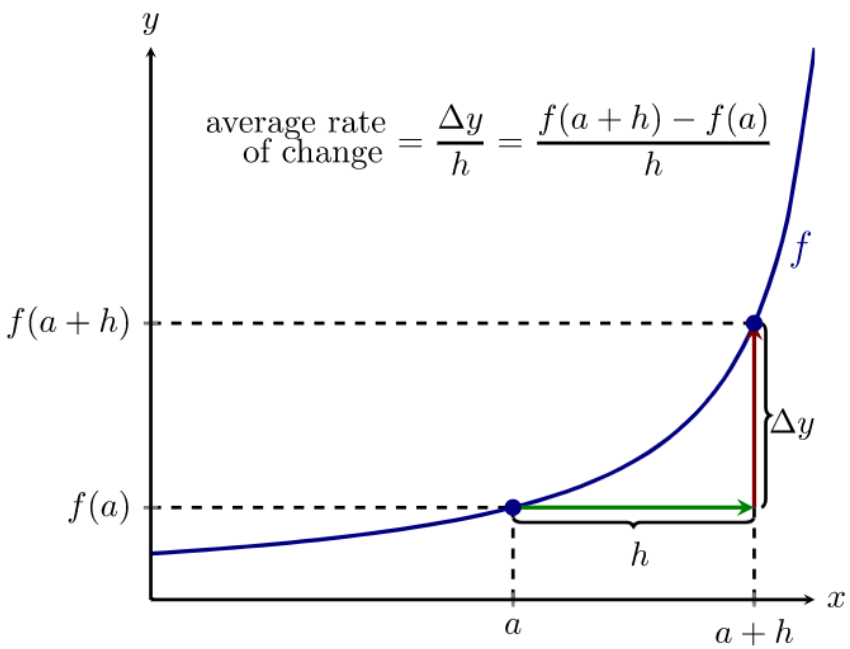
(viii) $\frac{d}{dx}(a^x) = a^x \log_e a$

(ix) $\frac{d}{dx}(e^x) = e^x$

(x) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

Derivatives as Slope of Curve

Derivatives as Slope of Curve



Explanation:

- X takes value f(a) at x=a and f(a+h) at x=a+h
- As x goes from a to a+h it increases it’s value by h (a+h-a).
- When x goes from a to a+h y goes from f(a) to f(a+h).
- Therefore we can see that rate of changes in function f is as shown in the above image.

Optimality Conditions

Optimality Conditions

Maxima and Minima

Maxima and minima are the peaks and valleys in the curve of a function. There can be any number of maxima and minima for a function.

There are two types of maxima and minima that exist in a function, which are:

•Local Maxima and Minima

•Absolute or Global Maxima and Minima

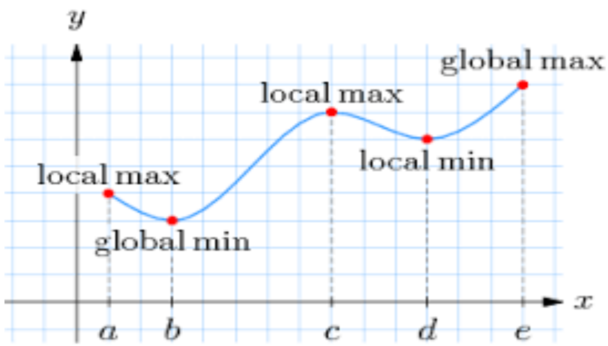
Local Maxima and Minima

Local maxima and minima are the maxima and minima of the function which arise in a particular interval.

Local maxima would be the value of a function at a point in a particular interval for which the values of the function near that point are always less than the value of the function at that point.

Absolute Maxima and Minima

The highest point of a function within the entire domain is known as the absolute maxima of the function whereas the lowest point of the function within the entire domain of the function, is known as the absolute minima of the function.



Optimality Conditions

First and second derivative:

Let f(x) is a function of our interest

If we differentiate f(x) with respect to x then it is called first order derivative and it is denoted by $f'(x)$

If we differentiate f(x) twice with respect to x then it is called second order derivative and it is denoted by $f''(x)$

Second Derivative test

This test is used to find maxima and minima of a function.

Rules :

If f(x) is our function of interest and assume x can take any arbitrary value a within its domain.

1) f(x) will have a minimum at a if $f'(a) = 0$ and $f''(a) > 0$
2) f(x) will have a maximum at a if $f'(a) = 0$ and $f''(a) < 0$

Integration as area under the curve

Integration

We have seen the process of differentiation where we know the function and out interest is in rate of change of that function with respect to a variable.

What if we know the rate of change and we want to know the value of a function at a particular point ?

Mathematical integration helps us in that.

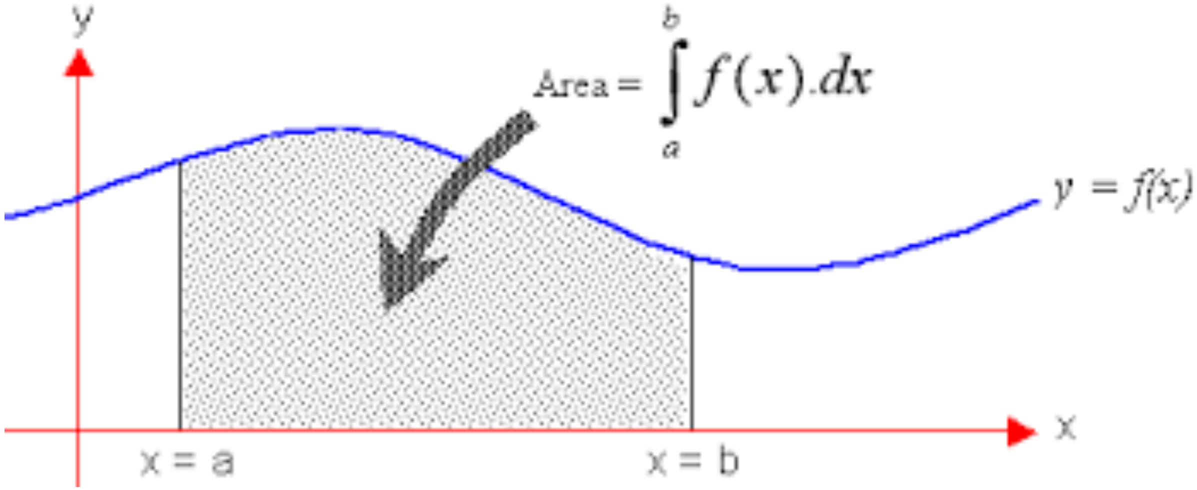
Integration

- Integration is a method of adding or summing up the parts to find the whole.
- It is a reverse process of differentiation, where we reduce the functions into parts.
- Here we already know f’(x) and we want to know f(x).

It is denoted as

$$\int f'(x) = f(x)$$

Integration as area under the curve



Suppose f(x) is a function of x.
f(x) takes value f(a) at x =a and f(b) at x=b.

If we integrate f(x) between x=a and x=b we will get the area under the curve of f(x) as shown in the above image.

Matrix

Matrix

A **matrix** is a rectangular array of numbers arranged in rows and columns. The array of numbers below is an example of a matrix

$$\begin{bmatrix} 21 & 62 & 33 & 93 \\ 44 & 95 & 66 & 13 \\ 77 & 38 & 79 & 33 \end{bmatrix}$$

The number of rows and columns that a matrix has is called its **dimension** or its **order**. By convention, rows are listed first; and columns, second. Thus, we would say that the dimension (or order) of the above matrix is 3 x 4, meaning that it has 3 rows and 4 columns.

Numbers that appear in the rows and columns of a matrix are called **elements** of the matrix. In the above matrix, the element in the first column of the first row is 21; the element in the second column of the first row is 62; and so on.

Another approach for representing matrix **A** is:

$$\mathbf{A} = [\, A_{ij} \,] \text{ where } i = 1, 2,3 \text{ and } j = 1, 2, 3, 4$$

Types of Matrices

Transpose Matrix

The **transpose** of one matrix is another matrix that is obtained by using rows from the first matrix as columns in the second matrix.

$$\mathbf{A} = \begin{bmatrix} 111 & 222 \\ 333 & 444 \\ 555 & 666 \end{bmatrix} \qquad \mathbf{A'} = \begin{bmatrix} 111 & 333 & 555 \\ 222 & 444 & 666 \end{bmatrix}$$

Note that the order of a matrix is reversed after it has been transposed. Matrix **A** is a 3 x 2 matrix, but matrix **A'** is a 2 x 3 matrix.

Vectors

Vectors are a type of matrix having only one column or one row.
For example, matrix **a** is a column vector, and matrix **a'** is a row vector.

$$\mathbf{a} = \begin{bmatrix} 11 \\ 12 \\ 33 \end{bmatrix} \qquad \mathbf{a'} = \begin{bmatrix} 11 & 12 & 33 \end{bmatrix}$$

Square Matrix

A square matrix is an n x n matrix; that is, a matrix with the same number of rows as columns.

Determinant of Matrices

If A is a square matrix, then the determinant of a matrix A is generally represented using det (A) or |A|.

Finding Determinants for 2×2 matrix:

Let us assume a 2×2 square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

, then

|A| =

$$\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}$$

$$|A| = a_{11}a_{22} - a_{21}a_{12}$$

Finding Determinants for 3×3 Matrix

Now, assume the 3×3 matrix, say

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

, then

$$|A| = \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

$$|A| = a_{11} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Matrix Addition and Subtraction

Two matrices may be added or subtracted only if they have the same dimension; that is, they must have the same number of rows and columns.

Addition or subtraction is accomplished by adding or subtracting corresponding elements.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5 & 6 & 7 \\ 3 & 4 & 5 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1+5 & 2+6 & 3+7 \\ 7+3 & 8+4 & 9+5 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 6 & 8 & 10 \\ 10 & 12 & 14 \end{bmatrix}$$

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 1-5 & 2-6 & 3-7 \\ 7-3 & 8-4 & 9-5 \end{bmatrix}$$

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} -4 & -4 & -4 \\ 4 & 4 & 4 \end{bmatrix}$$

Matrix Multiplication

In matrix algebra, there are two kinds of matrix multiplication: multiplication of a matrix by a number and multiplication of a matrix by another matrix.

Multiply a Matrix by a Number

When you multiply a matrix by a number, you multiply every element in the matrix by the same number. This operation produces a new matrix, which is called a **scalar multiple**.

$$\mathbf{A} = \begin{bmatrix} 100 & 200 \\ 300 & 400 \end{bmatrix}$$

$$5\mathbf{A} = 5 \begin{bmatrix} 100 & 200 \\ 300 & 400 \end{bmatrix}$$

$$5\mathbf{A} = \begin{bmatrix} 5 * 100 & 5 * 200 \\ 5 * 300 & 5 * 400 \end{bmatrix}$$

$$5\mathbf{A} = \begin{bmatrix} 500 & 1000 \\ 1500 & 2000 \end{bmatrix}$$

Matrix Multiplication

Multiply a Matrix by a Matrix

The matrix product **AB** is defined only when the number of columns in **A** is equal to the number of rows in **B**. Similarly, the matrix product **BA** is defined only when the number of columns in **B** is equal to the number of rows in **A**.

Suppose that **A** is an *i* x *j* matrix, and **B** is a *j* x *k* matrix. Then, the matrix product **AB** results in a matrix **C**, which has *i* rows and *k* columns; and each element in **C** can be computed according to the following formula.

$$C_{ik} = \Sigma_j A_{ij}B_{jk}$$

where

C_{ik} = the element in row *i* and column *k* from matrix **C**

A_{ij} = the element in row *i* and column *j* from matrix **A**

B_{jk} = the element in row *j* and column *k* from matrix **B**

Σ_j = summation sign, which indicates that the $a_{ij}b_{jk}$ terms should be summed over *j*

Matrix Multiplication

Example:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 6 & 7 \\ 8 & 9 \\ 10 & 11 \end{bmatrix}$$

Let **AB** = **C**. Because **A** has 2 rows, we know that **C** will have two rows; and because **B** has 2 columns, we know that **C** will have 2 columns. To compute the value of every element in the 2 x 2 matrix **C**, we use the formula $C_{ik} = \Sigma_j A_{ij}B_{jk}$, as shown below.

- $C_{11} = \Sigma A_{1j}B_{j1} = 0*6 + 1*8 + 2*10 = 0 + 8 + 20 = 28$
- $C_{12} = \Sigma A_{1j}B_{j2} = 0*7 + 1*9 + 2*11 = 0 + 9 + 22 = 31$
- $C_{21} = \Sigma A_{2j}B_{j1} = 3*6 + 4*8 + 5*10 = = 18 + 32 + 50 = 100$
- $C_{22} = \Sigma A_{2j}B_{j2} = 3*7 + 4*9 + 5*11 = 21 + 36 + 55 = 112$

Based on the above calculations, we can say

$$\mathbf{AB} = \mathbf{C} = \begin{bmatrix} 28 & 31 \\ 100 & 112 \end{bmatrix}$$

Vector Matrix Multiplication

We have seen how to multiply a matrix by another matrix.
Similarly we can multiply a vector by a matrix but the condition is that number of columns in the left object(Vector, Matrix) should be equal to number of rows in right object(Vector, Matrix).

Example:

$$\begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} \times \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2a + 4b + 6c \\ 2x + 4y + 6z \end{bmatrix} = 2\begin{bmatrix} a \\ x \end{bmatrix} + 4\begin{bmatrix} b \\ y \end{bmatrix} + 6\begin{bmatrix} c \\ z \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \times \begin{bmatrix} 6 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 8 & 6 \\ 18 & 12 & 9 \\ 24 & 16 & 12 \end{bmatrix}$$

Eigen Values and Eigen Vectors

Eigen Vector

If V is a vector that is not zero than it is an eigenvector of a square matrix A if AV is a scaler multiple of V i.e.

$$AV = \lambda V$$

Eigen Value

In above equation λ is a scaler known as eigenvalue or characteristic value associated with eigenvector V.

We can find the eigenvalues by determining the roots of the characteristic equation

$$|A-\lambda I| = 0$$

Eigen Values and Eigen Vectors

Example:

Ex.1 Find the eigenvalues and eigenvectors of matrix A.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Taking the determinant to find characteristic polynomial A-

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 3 - 4\lambda + \lambda^2 = 0$$

It has roots at $\lambda = 1$ and $\lambda = 3$, which are the two eigenvalues of A.

Please refer this link if you want to learn the process of calculating roots of quadratic equation
https://www.youtube.com/watch?v=vlHeeseehZ8


Thank you!