UNIT - 1

TAYLOR'S SERIES

A Taylor's Series is a series expansion of a function about a point.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

$\Downarrow$ (special case) $\begin{bmatrix} x=0 \\ h=x \end{bmatrix}$

$$\boxed{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots}$$

for Instance : Taylor expansion of $f(x) = e^x$.

$$e^x = e^0 + x e^0 + \frac{x^2}{2!} e^0 + \frac{x^3}{3!} e^0 + \cdots$$

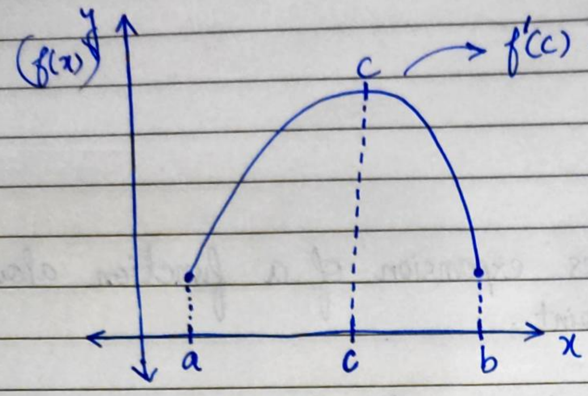
$$\Rightarrow \boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$$

ROLLE'S THEOREM

1.) If $f(x)$ is continuous on $[a, b]$.
2.) If $f(x)$ is differentiable on (a, b).
3.) $f(a) = f(b)$ holds,

then, $\exists$ at least one $c \in (a, b)$ such that $f'(c) = 0$.

Graphical Representation of Rolle's Theorem.



Ques.) $f(x) = 3 + (x-2)^{2/3}$ on $[1, 3]$ existance of Rolle's Theorem.

Ans.) $f'(x) = \frac{2}{3} \frac{1}{(x-2)^{\frac{1}{3}}}$

So,

$x - 2 = 0$

$x = 2$

So, derivative doesn't exist on $x = 2$ and $2 \in (1, 3)$.

$\therefore$ Rolle's Theorem doesn't hold.

Note: Polynomial, exponential, logarithmic, trigonometric, Inverse Trigonometric functions are always continuous & differentiable on their domain.

* Rolle's Theorem for POLYNOMIAL

1.) If $f(x)$ Continuous on $[a, b]$.
2.) If $f(x)$ is differentiable on (a, b).
3.) $f(a) = f(b) = 0$ i.e. a & b are roots.

then, ∃ at least one root of $f'(c)$ with 2 roots of $f(x)$ (a & b) included.

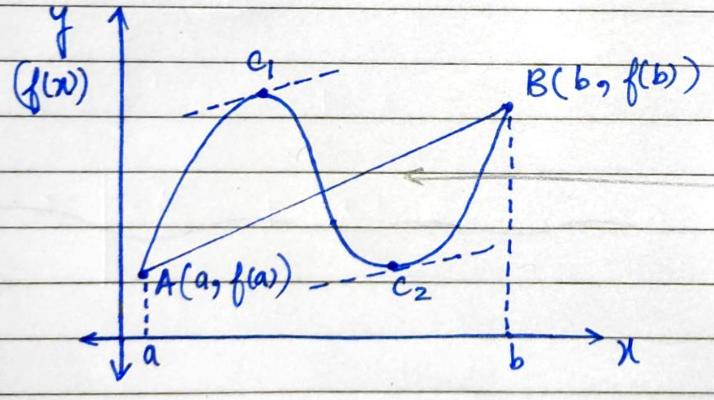
MEAN - VALUE THEOREM

let $f(x)$ be a function which is :
1. continuous on $[a, b]$.
2. differentiable on (a, b).

then, ∃ at least one point $c \in (a, b)$ such that :-

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

- Graphical Representation.



secant made by end points (A & B) of the curve

→ Tangents made on points c_1 & c_2 are parallel to the secant.

SIGNIFICANT FIGURES

Rules

- Non-zero digits are significant.
- Zeroes b/w non-zero digits are significant.
- Final zeroes in the decimal portion are significant.
- Final zeroes are non-significant if there's no decimal point.

Ques.) find no. of significant figures :-

1.) 58 – 2
2.) 603 – 3
3.) 1008 – 4
4.) 95080 – 4
5.) 70 – 1
6.) 0.80 – 2
7.) 18.00 – 4
8.) 0.0003 – 1
9.) 0.0030 – 2

ROUNDING OFF

1.) 39.72431 (3D) – 39.724
2.) 79.87893 (3D) – 79.879
3.) 2.589951 (2D) – 2.59
4.) 59.4865 (3D) – 59.486
5.) 47.6975 (3D) – 47.698

UNIT – 1

BISECTION METHOD

→ This method is based on the repeated application of the intermediate value theorem to obtain an approximate root of the equation $f(x) = 0$.

Choose 'a' & 'b' such that : $f(a) \to +ve$ $f(b) \to -ve$

$\Downarrow$ i.e.

$f(a) \cdot f(b) < 0$

then, a root lies b/w a and b. The mean value $\boxed{x_0 = \frac{a+b}{2}}$ is taken as the first approximate value of the root.

- If, $f(x_0) = 0$, then obviously $x = x_0$ is a root of the equation $f(x) = 0$.
- But if, $f(a) \cdot f(x_0) < 0$ i.e. $f(a) \to +ve$ & $f(x_0) \to -ve$

then, the next approx. root lies in the interval (a, x_0).

OR if, $f(b) \cdot f(x_0) < 0$ i.e. $f(b) \to -ve$ & $f(x_0) \to +ve$. then, the next approx. root lies in the interval (x_0, b).

Note : The new interval is of the length $\left(\frac{b-a}{2}\right)$.

Now,

- If the root lie in the interval (a, x_0), then the new approximate root will be :-

$$\boxed{x_1 = \frac{a + x_0}{2}}$$

- If the root lie in the interval (x_0, b), then the new approx. root will be :-

$$\boxed{x_1 = \frac{x_0 + b}{2}}$$

⊙ If $f(x_1) \cdot f(a) < 0$ i.e. $f(a) \to +ve$ & $f(x_1) \to -ve$, then the next approx. root (x_2) will lie in the interval (a, x_1).
Otherwise, the interval will be (x_1, x_0).

Note : The length of this interval will be $\left(\frac{b-a}{2^2}\right)$.

→ This process is repeated until the interval (which contains the root) is as small as desired, say numerically less than some given small number $\epsilon > 0$.

In General :-

The length of the interval at the n^{th} step will be $= \frac{b-a}{2^n}$

Dt.:
Pg.:

- Number of Iteration required to find desired solution by BISECTION METHOD.

$$\frac{b-a}{2^n} < \epsilon$$

$$\Rightarrow \frac{2^n}{b-a} > \frac{1}{\epsilon}$$

$$\Rightarrow 2^n > \frac{b-a}{\epsilon}$$

$$\Rightarrow \boxed{n > \log_2\left(\frac{b-a}{\epsilon}\right)}$$

Ques. 1.) find real root of $f(x) = x^3 - 2x - 5$.

Ans. 1.) (Hit & Trial)

$f(0) \rightarrow -ve.$ $f(2) \rightarrow -ve.$
$f(1) \rightarrow -ve.$ $f(3) \rightarrow +ve.$

So,

$a = 2$
$b = 3$ $\longrightarrow [a, b] = [2, 3]$

• 1st Iteration

$x_0 = \frac{a+b}{2} = \frac{2+3}{2}$

$\Rightarrow x_0 = 2.5$

Now,

$f(x_0) = 5.625 > 0$ (+ve)

So,

Next approx. root will lie in interval $[2, 2.5]$.

• 2nd Iteration

$x_1 = \frac{a+x_0}{2} = \frac{2+2.5}{2}$

$\Rightarrow x_1 = 2.25$

Now,

$f(x_1) = 1.89062 > 0$ (+ve)

So,

Next root in interval : $[2, 2.25]$

• 3rd Iteration

$x_2 = \frac{a+x_1}{2} = \frac{2+2.25}{2}$

$\Rightarrow x_2 = 2.125$

Now,

$f(x_2) \longrightarrow +ve$

So,

Next root in interval : $[2, 2.125]$

• 4th Iteration

$x_3 = \frac{2+2.125}{2} = 2.0625$

Now, $f(x_3) = -ve$

So,

Next approx. root in the interval : $[2.0625, 2.125]$

• 5th Iteration

$x_4 = \frac{2.0625 + 2.125}{2}$

$\Rightarrow x_4 = 2.09375$

Now, $f(x_4) \rightarrow -ve$

So,

Next approx. root in interval : $[2.09375, 2.125]$

• 6th Iteration

$x_5 = \frac{2.09375 + 2.125}{2}$

$\Rightarrow x_5 = 2.109375$

Ques.2.) $f(x) = x \sin x + \cos x$. Using Bisection method find root of $f(x)$.

Ans.2.) $f(0) = 1$ (+ve) $f(2) \to 1.40244$ (+ve)
$f(1) = 1.38177$ (+ve) $f(3) \to -0.5666$ (−ve)

So, $a = 2$, $b = 3$ $\longrightarrow$ interval : $[2, 3]$

• 1st Iteration

$x_0 = \frac{a+b}{2} = \frac{2+3}{2}$

$\Rightarrow x_0 = 2.5$

Now,
$f(x_0) \to +ve$

So,
Next Interval of approx. root : $(2.5, 3)$

• 2nd Iteration

$x_1 = \frac{2.5+3}{2}$

$\Rightarrow x_1 = 2.75$

Now,
$f(x_1) \Rightarrow +ve$

So,
Next approx. root in the interval : $(2.75, 3)$

• 3rd Iteration

$x_2 = \frac{2.75+3}{2}$

$\Rightarrow x_2 = 2.875$

Now,
$f(x_2) \Rightarrow -ve$

So,
Next approx. root in the interval : $(2.75, 2.875)$

• 4th Iteration

$x_3 = \frac{2.75 + 2.875}{2}$

$\Rightarrow x_3 = 2.8125$

Now,
$f(x_3) \longrightarrow -ve$

So,
Next Interval of approx. root : $(2.75, 2.8125)$

• 5th Iteration

$x_4 = \frac{2.75 + 2.8125}{2}$

$\Rightarrow x_4 = 2.78125$

Now,
$f(x)_4 \to +ve$

So,
Next approx. root in the interval : $(2.78125, 2.8125)$

• 6th Iteration

$x_5 = \frac{2.78125 + 2.8125}{2}$

$\Rightarrow x_5 = 2.796875$

CONVERGENCE ANALYSIS OF BISECTION METHOD

- $f(x) = 0$ (given function)
- $x = \alpha$ is actual root of $f(x)$.
- $x_n \rightarrow$ is n^{th} approx. root / successive approximation of actual root $x = \alpha$.

→ x_n is said to converge to actual root $x = \alpha$ with order $q \geq 1$ if $|x_{n+1} - \alpha| \leq C|x_n - \alpha|^q$

$$\boxed{x_{n+1} - \alpha \leq C|x_n - \alpha|^q}$$

[$C \rightarrow$ Rate of Convergence.
$q \rightarrow$ Order of Convergence.]

Note: When $q = 1$ & $0 < C < 1$, then convergence is said to be of 1^{st} Order.

- Order of Convergence $\rightarrow q$.
- Rate of Convergence $\rightarrow C$.

BISECTION METHOD ALWAYS CONVERGES.

→ let 'α' be actual root of $f(x) = 0 \Rightarrow f(\alpha) = 0$.

→ let x_n is successive approximation of actual root α.

$$x_n = \epsilon_n + \alpha$$

($\epsilon \rightarrow$ Error (gap b/w actual & approx. root))

Dt.:
Pg.:

and, $x_{n-1} = \epsilon_{n-1} + \alpha$ —— ①

so,

$$x_{n+1} = \frac{x_{n-1} + x_n}{2}$$

$$\Rightarrow x_{n+1} = \frac{\epsilon_{n-1} + \alpha + \epsilon_n + \alpha}{2}$$

$$\Rightarrow \epsilon_{n+1} + \cancel{\alpha} = \frac{\epsilon_{n-1} + \epsilon_n}{2} + \cancel{\alpha}$$

$$\Rightarrow \epsilon_{n+1} = \frac{\epsilon_n}{2}\left[1 + \frac{\epsilon_{n-1}}{\epsilon_n}\right]$$

$\left(\because \frac{\epsilon_{n-1}}{\epsilon_n}\right.$ is v.v. small, we can neglect it.$\left.\right)$

$$\Rightarrow \epsilon_{n+1} \approx \frac{\epsilon_n}{2}$$

$$\Rightarrow \boxed{\epsilon_{n+1} = \frac{1}{2}(\epsilon_n)^1}$$

so,

Rate of Convergence = 1/2
Order of Convergence = 1

NEWTON - RAPHSON METHOD

→ Let x_0 be approximate root of f^n. $f(x)$. And, $x_1 = x_0 + h$ be the correct root, then :-

$$f(x_1) = 0$$

$$\Rightarrow f(x_0 + h) = 0$$

$$\Rightarrow f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0$$

$$\Rightarrow f(x_0) + h f'(x_0) = 0 \quad \text{(Neglecting higher powers)}$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

So,

$$x_1 = x_0 + h$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

So, In General,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad (n \to 0, 1, 2, \cdots)$$

Ques. 1.) Use Newton - Raphson method to find approx. root of $(30)^{-1/5}$.

Ans.) $x = (30)^{-1/5}$

$$\Rightarrow x = \left(\frac{1}{30}\right)^{1/5}$$

Dt.:
Pg.:

$\Rightarrow \quad x^5 = \frac{1}{30}$

Soln

$f(x) = x^5 - \frac{1}{30}$

$\rightarrow \quad f(x) = 30x^5 - 1$

Soln

let, $x_0 = \frac{1}{2}$ (Choose x_0 such that $f(x)$ is nearly equal to zero)

Soln

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (n \rightarrow 0, 1, 2, \ldots)$$

$n=0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow x_1 = \frac{1}{2} - \frac{30x_0^5 - 1}{150x_0^4}$$

$$\Rightarrow x_1 = \frac{1}{2} - \frac{30(1/2)^5 - 1}{150(1/2)^4}$$

$$\Rightarrow x_1 = \frac{1}{2} + \frac{0.0625}{9.375}$$

$$\Rightarrow x_1 = 0.50667$$

$n=1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\Rightarrow x_2 = 0.50667 - \frac{30(0.50667)^5 - 1}{150(0.50667)^4}$$

$$\Rightarrow x_2 = 0.50667 - \frac{0.001689}{9.885}$$

Dt.:
Pg.:

$\Rightarrow x_2 = 0.50649$

$n = 2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\Rightarrow x_3 = 0.50649 - \frac{30(0.50649)^5 - 1}{150(0.50649)^4}$$

$$\Rightarrow x_3 = 0.50649 + \frac{0.0005611}{9.87131}$$

$$\Rightarrow x_3 = 0.50654$$

Ques. 2) Use N-R method to find approximate root of $(24)^{1/3}$.

Ans.) let, $x = (24)^{1/3}$

$$\Rightarrow x^3 = 24$$

So, $f(x) = x^3 - 24$

let, $x_0 = 3$

We know :- $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$n = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x^3 - 24}{3x^2}$$

$$\Rightarrow x_1 = 3 - \frac{(3)^3 - 24}{3(3)^2}$$

$$\Rightarrow x_1 = 2.888889$$

$n=1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\Rightarrow x_2 = 2.8889 - \left(\frac{(2.8889)^3 - 24}{3(2.8889)^2}\right)$$

$$\Rightarrow x_2 = 2.8889 - \frac{0.10974}{25.037}$$

$$\Rightarrow x_2 = 2.8845$$

$n=2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\Rightarrow x_3 = 2.8845 - \left(\frac{(2.8845)^3 - 24}{3(2.8845)^2}\right)$$

$$\Rightarrow x_3 = 2.884499$$

Ques:) Use N-R method to derive formula for $N^{1/q}$, $N > 0$ & hence find value of $18^{1/3}$ correct to 4 decimal places.

Ans:) let, $x = N^{1/q}$

$$\Rightarrow x^q = N$$

$$\Rightarrow x^q - N = 0$$

so

$$f(x) = x^q - N$$

$$\Rightarrow f(x_n) = x_n^q - N$$

so

Dt.:
Pg.:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \boxed{x_{n+1} = x_n - \frac{x_n^q - N}{q x_n^{q-1}}}$$

(General formula)

Now,

$x = 18^{1/3}$

$\Rightarrow x^3 = 18$

$\Rightarrow x^3 - 18 = 0$

So,

$f(x) = x^3 - 18$

$\Rightarrow f(x_n) = x_n^3 - 18$

So,

$$x_{n+1} = x_n - \frac{x_n^3 - 18}{3x_n^2}$$

$n=0$

$$x_1 = x_0 - \frac{x_0^3 - 18}{3x_0^2}$$

(let, $x_0 = 2.5$)

$$\Rightarrow x_1 = 2.5 - \left(\frac{(2.5)^3 - 18}{3(2.5)^2}\right)$$

$\Rightarrow x_1 = 2.5 + 0.12667$

$\Rightarrow x_1 = 2.62667$

$n=1$

$$x_2 = x_1 - \left(\frac{x_1^3 - 18}{3x_1^2}\right)$$

$$\Rightarrow x_2 = (2.62667) - \left(\frac{(2.62667)^3 - 18}{3(2.62667)^2}\right)$$

$\Rightarrow x_2 = 2.62093$

$n=2$

$$x_3 = x_2 - \left(\frac{(x_1)^3 - 18}{3x_1^2}\right)$$

$$\Rightarrow x_3 = 2.62093 - \left(\frac{(2.62093)^3 - 18}{3(2.62093)^2}\right)$$

$$\Rightarrow x_3 = 2.62074$$

$n=3$

$$x_4 = x_3 - \frac{x_3^3 - 18}{3x_3^2}$$

$$\Rightarrow x_4 = (2.62074) - \left(\frac{(2.62074)^3 - 18}{3(2.62074)^2}\right)$$

$$\Rightarrow x_4 = 2.62074$$

CONVERGENCE ANALYSIS OF NEWTON-RAPHSON METHOD

we know : $\epsilon_n = x_n - \alpha \quad \longleftarrow (1)$ ($\alpha \rightarrow$ Actual root, $f(\alpha) = 0$)

so,

$$\epsilon_{n+1} = x_{n+1} - \alpha$$

$$\Rightarrow \epsilon_{n+1} = \left(x_n - \frac{f(x_n)}{f'(x_n)}\right) - \alpha$$

($x_n = \epsilon_n + \alpha$ (from eqⁿ (1)))

$$\Rightarrow \epsilon_{n+1} = \left(\epsilon_n + \cancel{\alpha} - \frac{f(\epsilon_n + \alpha)}{f'(\epsilon_n + \alpha)}\right) - \cancel{\alpha}$$

$$\Rightarrow \epsilon_{n+1} = \epsilon_n - \frac{f(\alpha+\epsilon_n)}{f'(\alpha+\epsilon_n)}$$

(Using Taylor's expansion, we get,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots\cdots$$
)

$$\Rightarrow \epsilon_{n+1} = \epsilon_n - \left(\frac{f(\alpha) + \epsilon_n f'(\alpha) + \epsilon_n^2/2!\, f''(\alpha) + \cdots\cdots}{f'(\alpha) + \epsilon_n f''(\alpha) + \frac{\epsilon_n^2}{2!} f'''(\alpha) + \cdots\cdots} \right)$$

(neglecting higher power as they are v.v. small)

$$\Rightarrow \epsilon_{n+1} = \epsilon_n - \left(\frac{f(\alpha) + \epsilon_n f'(\alpha) + \epsilon_n^2/2!\, f''(\alpha)}{f'(\alpha)} \right)$$

$$\Rightarrow \epsilon_{n+1} = \epsilon_n - \cancel{\frac{f(\alpha)}{f'(\alpha)}}^{\,0} - \frac{\epsilon_n \cancel{f'(\alpha)}}{\cancel{f'(\alpha)}} - \frac{\epsilon_n^2}{2!} \frac{f''(\alpha)}{f'(\alpha)}$$

$$\Rightarrow \epsilon_{n+1} = \cancel{\epsilon_n} - \cancel{\epsilon_n} - \epsilon_n^2 \frac{f''(\alpha)}{2!\, f'(\alpha)} \rightarrow \text{(constant} = K)$$

$$\Rightarrow \epsilon_{n+1} = K \epsilon_n^2$$

So, NR method has 2^{nd} order convergence.