Z Tests

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Overview of Z Tests

We will begin with the Z-test.

The Z-test for proportions is a statistical method used to compare observed proportions, such as the proportion of successes in two groups, to see if there is a significant difference between them. This test is particularly useful when dealing with large sample sizes.

Depending on the research question and the type of data being analyzed, different types of Z-tests are frequently employed in statistics. Variations of the Z-test can be used to compare sample means or sample proportions. In this chapteer, we will confine our attention to the Z-tests used to analyze **categorical** data and **sample proportions**.

The following are three prevalent types of Z-tests for Categorical data:

1. **One-sample Z-test:** This test determines whether a sample proportion significantly deviates from a known or hypothesized population mean or proportion.
2. **Two-sample Z-test:** This test is used to determine whether there is a statistically significant difference between the proportions of two independent groups.
3. **Paired Z-test for proportions:** This test determines whether there is a statistically significant difference in proportions between two related groups, such as before and after a treatment or intervention. [1]

Business Applications of Z Tests for Categorical Data

Marketing

In marketing research and analysis, Z-tests for categorical data are useful tools for testing hypotheses and determining the significance of relationships between categorical variables. Here are some possible marketing applications of Z-tests for categorical data:

1. **Brand loyalty:** A marketer may wish to determine whether customers who purchase a particular brand are more loyal to the brand based on repeat purchase, than those who purchase a competing brand. To compare the proportion of loyal customers between the two brands, a two-sample Z-test for proportions could be utilized.
2. **Customer satisfaction:** A marketer may wish to determine whether a new product or service is more satisfying to customers than its predecessor. Before and after the introduction of the new product or service, the proportion of satisfied customers could be compared.
3. **Market segmentation:** A marketer may use market segmentation to determine whether there are significant differences in customer demographics or behavior across market segments. A test could be utilized to determine if there is a significant relationship between the demographic variables and market segments.
4. **Advertising effectiveness:** A marketer may wish to determine if one advertising campaign is more effective than another at reaching the target audience. A two-sample Z-test for proportions could be used to compare the proportion of viewers who remember the message or take action following exposure to each campaign.
5. **Product preference:** A marketer may wish to determine whether a particular product feature or characteristic is more preferred by customers than another. A one-sample Z-test for proportions could be utilized to determine if the proportion of customers who prefer the feature significantly deviates from the null hypothesis. [2]

Finance

Z tests can be helpful in studying different aspects of Finance such as default rates, investment decisions, stock market analysis, risk analysis.

1. **Credit risk assessment:** A financial institution assessing credit risk may wish to determine whether the proportion of delinquent loans varies significantly across risk categories. A two-sample Z-test for proportions could be utilized to compare the proportion of delinquent loans across various risk categories.
2. **Investment portfolio analysis:** An investor may wish to determine whether the proportion of winning trades varies significantly between different types of investments, including stocks and bonds. Using a two-sample Z-test for proportions, it is possible to compare the proportion of profitable trades for the various types of investments.
3. **Customer behavior analysis:** A financial institution may conduct an analysis of customer behavior to determine whether the proportion of customers who use online banking varies significantly by age group. A Chi-square test could be utilized to determine if there is a significant relationship between the demographic variable (age group) and the proportion of customers who use online banking.

4. **Market analysis:** An investment bank may wish to determine if there is a significant correlation between the market type (bullish or bearish) and the proportion of investors who sell or purchase stocks. A Chi-square test could be used to determine if there is a significant relationship between the type of market and the proportion of investors who sell or purchase stocks.

5. **Fraud detection:** A financial institution may wish to determine whether the proportion of fraudulent transactions varies significantly between different types of transactions, such as ATM withdrawals, online purchases, and wire transfers, for the purpose of detecting fraud. A one-sample Z-test for proportions could be used to determine if the proportion of fraudulent transactions in a specific type of transaction significantly deviates from a null hypothesis. [3]

Organizational Behavior:

In organizational behavior research, Z-tests for categorical data can be useful tools for analyzing data and making informed decisions. Here are some potential organizational behavior applications of Z-tests for categorical data:

1. **Employee satisfaction:** An organization may wish to determine whether the proportion of satisfied employees varies significantly across departments. Using a two-sample Z-test for proportions, it is possible to compare the percentage of satisfied employees across departments.

2. **Diversity and inclusion:** An organization may wish to determine whether the proportion of employees from diverse backgrounds varies significantly across management levels. A Chi-square test could be used to determine whether there is a significant relationship between the demographic variable (race, gender, etc.) and the management level.

3. **Employee engagement:** An organization may wish to determine whether the proportion of engaged employees differs significantly across teams or work units. Using a two-sample Z-test for proportions, it is possible to compare the proportion of engaged employees among various teams or work units.

4. **Training effectiveness:** An organization may wish to determine whether the percentage of employees who pass a certification exam differs significantly between those who have completed a training program and those who have not. The proportion of employees who pass the certification exam could be compared between the two groups using a two-sample Z-test for proportions.

5. **Employee turnover:** An organization may wish to determine whether the proportion of employees who voluntarily leave the organization varies significantly across job functions or departments. A Chi-square test could be used to determine whether there is a significant relationship between the job function or department and the percentage of employees who leave the organization voluntarily. [4]

One-sample Z-test for Proportions

1. This type of Z-test is used to examine a population proportion hypothesis.
2. For instance, a business may wish to determine whether the proportion of customers who are satisfied with their product exceeds 0.5.
3. The company could collect a sample of customers, determine the proportion of satisfied customers in the sample, and use a one-sample Z-test to determine if the proportion is significantly different from 0.5.

One-sample Z-test for Proportions - Concepts

1. The general steps to perform a one-sample Z-test are:
    1. **State the null hypothesis and alternative hypothesis:** Typically, the null hypothesis states that the sample proportion matches the hypothesized population proportion, whereas the alternative hypothesis states that the sample proportion differs from the hypothesized population proportion.
    2. **Determine the level of significance ( ):** This is the maximum probability of rejecting the null hypothesis when it is true. Typical values for are 0.05 or 0.01.
    3. **Collect sample data:** Obtain a random sample from the population and calculate the sample proportion (p).
    4. **Calculate the test statistic:** The test statistic is calculated as follows: $z = (p_0 - p)/\sqrt{(p * (1 - p)/n)}$, where p is the population proportion assumed under the null hypothesis, and n is the sample size.
    5. **Determine the critical value:** The critical value is the value of z that corresponds to the chosen level of significance and degrees of freedom ($df = n - 1$).
    6. **Compare the test statistic to the critical value:** If the absolute value of the test statistic is greater than the critical value, we reject the null hypothesis. Otherwise, we fail to reject the null hypothesis.
    7. **Interpret the results:** If the null hypothesis is rejected, one can conclude that the proportion of the sample is significantly different from that of the population. If the null hypothesis is not rejected, it can be concluded that the sample proportion is not significantly different from the population proportion.
    8. **Determine the p-value** using a Z-table or calculator, and compare the p-value to the level of significance (alpha) in order to reach a conclusion.
    9. **Draw a conclusion.** [5]

Numerical Example for One-sample Z-test for Proportions (Car Ownership)

1. **Research Objective**:
   A researcher wants to determine if the proportion of adults in a population who own a car significantly deviates from 70%. A random sample of 200 adults shows that 150 of them own a car.

2. **Hypotheses**:

   - Null Hypothesis (`H0`): `p = 0.7`
   - Alternative Hypothesis (`Ha`): `p   0.7`
   - Here, `p` is the population proportion.

3. **Significance Level**:
   The significance level is set at 0.05.

4. **Critical Value**:
   The critical value for a two-tailed test at the 0.05 significance level is ± 1.96.

5. **Test Statistic Formula**:
   The test statistic (Z) is calculated using the formula:

$$Z = \frac{p_0 - p}{\sqrt{p \cdot (1 - p)/n}}$$

   - `p` is the hypothesized population proportion.
   - `p_0` is the sample proportion.
   - `n` is the sample size.

6. **Calculation**:

   - Sample proportion (`p_0`): `0.75` (150 out of 200)
   - Hypothesized population proportion (`p`): `0.7`
   - Sample size (`n`): `200`
   - The Z-score is calculated as approximately `1.54`.

Verify by plugging in the values:

$$Z = \frac{0.75 - 0.7}{\sqrt{0.7 \cdot (1 - 0.7)/200}}$$

The calculated Z-score is approximately `1.54`.

7. **Decision Rule**:

   - Compare the absolute value of the test statistic with the critical value. If it's greater than 1.96, reject the null hypothesis.

8. **Conclusion**:

   - Since the absolute value of the test statistic (1.54) is less than the critical value (1.96), we **fail to reject the null** hypothesis. There is not enough statistical evidence to conclude that the proportion of adults who own a car in the population is significantly different from 70%. [6]

Running One-sample Z-test for Proportions in R

1. Consider the `mtcars` data set. Suppose we want to test whether the proportion of cars with a **6-cylinder engine** (`cyl=6`) is significantly different from a hypothesized value of **50%**.

- **Null hypothesis**: The proportion of 6-cylinder engine cars in the `mtcars` dataset is 0.5 (`p = 0.5`).
- **Alternative hypothesis:** The proportion of 6-cylinder engine cars in the `mtcars` dataset is not equal to 0.5 (`p   0.5`).

```
# Load the mtcars dataset
data(mtcars)

# Create a binary variable indicating whether a car has a 6-cylinder engine or not
mtcars$is_cyl6 <- ifelse(mtcars$cyl == 6, 1, 0)

# Count the number of 6 cylinder cars
n6 <- sum(mtcars$is_cyl6)

# Count the total number of cars
n <- length(mtcars$is_cyl6)
  # alternate code
  # n <- nrow(mtcars)

# Calculate the sample proportion of cars with a 6-cylinder engine
prop_cyl6 <- mean(mtcars$is_cyl6)

# Conduct a one-sample Z-test for proportions using the prop.test function
prop.test(n6,
          n,
          p = 0.5)
```

```
	1-sample proportions test with continuity correction
```

```
data:  n6 out of n, null probability 0.5
X-squared = 9.0312, df = 1, p-value = 0.002654
alternative hypothesis: true p is not equal to 0.5
95 percent confidence interval:
 0.09944097 0.40441815
sample estimates:
      p
0.21875
```

2. The sample proportion is **0.21875**, indicating that only **21.875%** of the cars in the dataset are equipped with a six-cylinder engine.
3. Using the `prop.test` function, we can now perform a one-sample Z-test on proportions:
4. The `prop.test` function requires three arguments:

   the number of successes (in this case, the number of cars with a 6-cylinder engine),

   the total number of trials (in this case, the total number of cars), and

   the population proportion hypothesis (0.5 in this case)
5. The output reveals that the test statistic is X-squared = 9.0312 with a p-value of 0.002654 and one degree of freedom.
6. Since the p-value is less than the significance level (0.05), we reject the null hypothesis and conclude that the proportion of cars in the population with a 6-cylinder engine is significantly different from 50%.
7. The sample proportion estimate is 0.21875, with a 95% confidence interval encompassing this value (0.099, 0.404). [7]

Independent Samples Z-Test for Proportions (2)

1. **Introduction**: The independent samples Z-test for proportions is used to determine if there's a significant difference between two sample proportions drawn from independent populations.
2. **Example Scenario**: A company might want to know if the proportion of male customers who purchase their product is significantly different from the proportion of female customers.
3. **Application**: The company collects separate samples of male and female customers, calculates the purchase proportions, and uses an independent samples Z-test to assess the significance of any difference. [8]

Independent Samples Z-Test for Proportions - Concepts

Conducting the Test:

1. **Hypotheses**:
   - Null Hypothesis (H0): The two population proportions are equal.
   - Alternative Hypothesis (Ha): The two population proportions are not equal.
2. **Significance Level**: Typically set at 0.05, it's the acceptable risk level for incorrectly rejecting the null hypothesis.
3. **Test Statistic**: The test statistic is calculated using the formula:

$$Z = \frac{p1 - p2}{\sqrt{p \cdot (1-p) \cdot (\frac{1}{n1} + \frac{1}{n2})}}$$

   where (p1) and (p2) are sample proportions, (n1) and (n2) are sample sizes, and (p) is the pooled sample proportion:

$$p = \frac{x1 + x2}{n1 + n2}$$

   (x1) and (x2) are the number of successes in each sample.

4. **Decision Rule**: The null hypothesis is rejected if the absolute value of (Z) is greater than the critical value (e.g., 1.96 for a two-tailed test at 0.05 significance level).
5. **Conclusion**: Based on the test results, a conclusion is drawn about the difference in population proportions. [8]

Numerical Example for Independent Samples Z-Test

Scenario:

1. A researcher compares the purchasing behavior of men and women for a product. A sample of 200 men and 300 women reveals 100 men and 150 women purchased the product.
2. **Hypotheses**:
   - (H0: p1 = p2)
   - (Ha: p1 not equal to p2) where (p1) is the proportion of men, and (p2) is the proportion of women purchasing the product.
3. **Significance Level**: Set at 0.05.

4. **Critical Value**: For a two-tailed test at 0.05, it's 1.96.
5. **Test Statistic Calculation**:

$$Z = \frac{p1 - p2}{\sqrt{p \cdot (1-p) \cdot (\frac{1}{n1} + \frac{1}{n2})}}$$

   where (p) is the pooled proportion:

$$p = \frac{x1 + x2}{n1 + n2}$$

6. **Using the Data**:

$$p1 = \frac{100}{200} = 0.5$$

$$p2 = \frac{150}{300} = 0.5$$

$$p = \frac{100 + 150}{200 + 300} = 0.4167$$

7. **Test Statistic**:

$$Z = \frac{0.5 - 0.5}{\sqrt{0.4167 \cdot (1 - 0.4167) \cdot (\frac{1}{200} + \frac{1}{300})}} = 0$$

8. **Conclusion**: Since (|Z| < 1.96), we fail to reject the null hypothesis, suggesting no significant difference in the purchase proportions of men and women.
9. **Interpretation**: We are 95% confident that the purchasing behavior does not differ significantly between men and women.

Running Independent Samples Z-Test in R

Example with mtcars Dataset:

1. **Objective**: Compare the proportion of cars with automatic transmission (am = 1) to those with V-shaped engines (vs = 1) in the `mtcars` dataset.

```
# Load the mtcars dataset
data(mtcars)

# Create vectors for the two samples
sample1 <- mtcars$am  # Cars with automatic transmission
```

```r
sample2 <- mtcars$vs  # Cars with V-shaped engines

# Conduct the Z-test for proportions
prop.test(x = c(sum(sample1 == 1), sum(sample2 == 1)),
          n = c(length(sample1), length(sample2)),
          alternative = "two.sided")
```

```
	2-sample test for equality of proportions with continuity correction

data:  c(sum(sample1 == 1), sum(sample2 == 1)) out of c(length(sample1), length(sample2))
X-squared = 0, df = 1, p-value = 1
alternative hypothesis: two.sided
95 percent confidence interval:
 -0.3043652  0.2418652
sample estimates:
 prop 1  prop 2 
0.40625 0.43750 
```

2. **Explanation**: `sample1` and `sample2` represent cars with automatic transmissions and V-shaped engines, respectively. The `prop.test` function in R is used for the test, where `x` and `n` are vectors of success counts and total counts.

3. **Interpretation**: The test results include the test statistic, p-value, and confidence interval. A p-value less than 0.05 indicates a significant difference between the proportions.

4. **Output Analysis**: The output provides the test name, sample proportions, test statistic, degrees of freedom, p-value, alternative hypothesis, and 95% confidence interval. A significant p-value suggests a difference in proportions between the two groups.

5. **Conclusion**: Based on the p-value, if less than 0.05, we conclude that the proportions of cars with automatic transmission and V-shaped engines differ significantly. The confidence interval provides a range within which the true difference in proportions lies with 95% confidence. [8]

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