Log-Linear Regression

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Overview

We use log-linear regression as a statistical technique to model the relationship between a dependent variable and one or more independent variables. This is done by applying a logarithmic transformation to the dependent variable. This approach is especially useful when dealing with **non-linear** relationships or when the dependent variable is strictly positive and can vary over a wide range.

Purpose

- **Handle Asymmetry:** It's often utilized when the dependent variable has a skewed distribution. The logarithmic transformation can help stabilize variance and make the relationship more linear.
- **Multiplicative Effects:** It models the relationship between the variables in a multiplicative manner rather than an additive one.

Model Form

The general form of a log-linear regression model can be expressed as follows:

$$\log(Y) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_n X_n + \epsilon \tag{0.1}$$

Where: - Y is the dependent variable (after applying the logarithm).

- $X_1, X_2, ..., X_n$ are the independent variables.
- $\beta_0, \beta_1, ..., \beta_n$ are the coefficients that the regression model will estimate.
- ε is the error term.

Assumptions

In log-linear regression, like other forms of regression analysis, we make several key assumptions:

- **Linearity:** The relationship between the transformed dependent variable and the independent variables is linear.
- **Independence:** Observations are independent from each other.
- **Homoscedasticity:** The variance of the error terms is constant across the values of the independent variables.
- **Normal Distribution of Errors:** The errors are normally distributed, which is especially important for hypothesis testing.

Advantages

- **Flexibility:** Can handle a wide range of dependent variable values.
- **Interpretability:** Coefficients can be interpreted as percentage changes, which makes it easier for us to understand the impact of independent variables.

Disadvantages

- **Transformation Bias:** The transformation of the dependent variable might introduce bias.
- **Outliers:** Logarithmic transformation can make the model sensitive to outliers in the data.

MODEL 1: Base Model

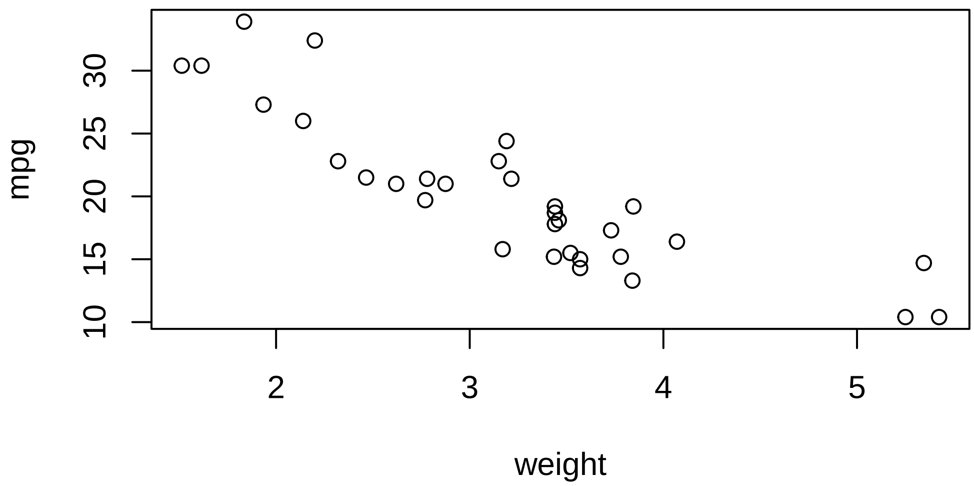
1. Reading the data

```
data(mtcars)
attach(mtcars)
```

2. Predict a mpg of car from their weight

Plot mpg and wt

```r
plot(wt
     , mpg
     , xlab="weight", ylab="mpg")
```



Simple Linear Regression

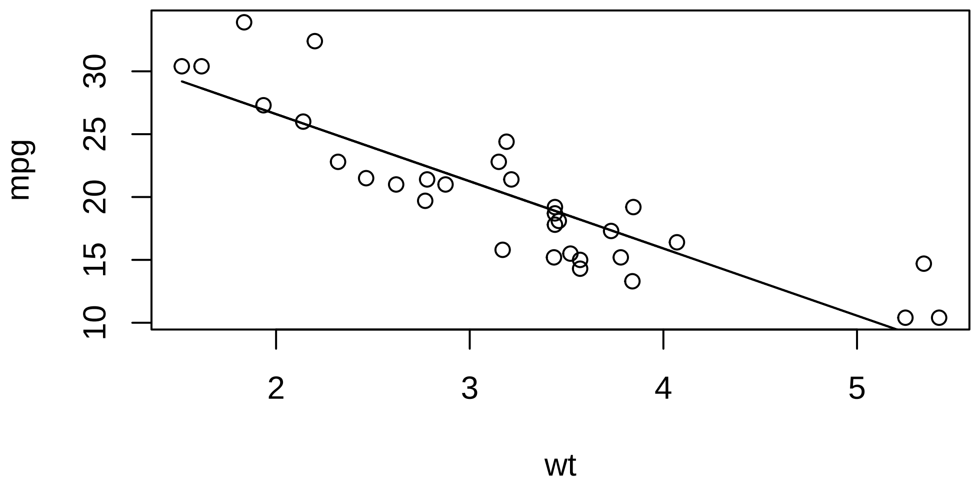
```r
fit <- lm(mpg ~ wt, data = mtcars)
summary(fit)
```

```
Call:
lm(formula = mpg ~ wt, data = mtcars)

Residuals:
    Min      1Q  Median      3Q     Max
-4.5432 -2.3647 -0.1252  1.4096  6.8727

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  37.2851     1.8776  19.858  < 2e-16 ***
wt           -5.3445     0.5591  -9.559 1.29e-10 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
Residual standard error: 3.046 on 30 degrees of freedom
Multiple R-squared:  0.7528,	Adjusted R-squared:  0.7446
F-statistic: 91.38 on 1 and 30 DF,  p-value: 1.294e-10
```

4. Plot

```r
plot(mtcars$wt,
     mtcars$mpg,
     main="",
     xlab="wt",
     ylab="mpg")
lines(mtcars$wt, fitted(fit))
```



Beta coefficients

```r
fit$coefficients
```

```
(Intercept)          wt
  37.285126   -5.344472
```

Predict

We can list the predicted values in a fitted model

```r
fitted(fit)
```

```
          Mazda RX4       Mazda RX4 Wag          Datsun 710      Hornet 4 Drive
          23.282611           21.919770           24.885952           20.102650
  Hornet Sportabout             Valiant          Duster 360           Merc 240D
          18.900144           18.793255           18.205363           20.236262
           Merc 230            Merc 280           Merc 280C          Merc 450SE
          20.450041           18.900144           18.900144           15.533127
         Merc 450SL         Merc 450SLC  Cadillac Fleetwood Lincoln Continental
          17.350247           17.083024            9.226650            8.296712
  Chrysler Imperial            Fiat 128         Honda Civic      Toyota Corolla
           8.718926           25.527289           28.653805           27.478021
      Toyota Corona    Dodge Challenger         AMC Javelin          Camaro Z28
          24.111004           18.472586           18.926866           16.762355
   Pontiac Firebird           Fiat X1-9       Porsche 914-2        Lotus Europa
          16.735633           26.943574           25.847957           29.198941
     Ford Pantera L        Ferrari Dino       Maserati Bora          Volvo 142E
          20.343151           22.480940           18.205363           22.427495
```

Residuals

Residuals are the vertical distances between the data and the fitted line. The Ordinary Least Squares (OLS) method minimizes the residuals. In OLS, the accuracy of a line through the sample data points is measured by the sum of squared residuals, and the goal is to make this sum as small as possible.

```r
residuals(fit)
```

```
          Mazda RX4       Mazda RX4 Wag          Datsun 710      Hornet 4 Drive
         -2.2826106          -0.9197704          -2.0859521           1.2973499
  Hornet Sportabout             Valiant          Duster 360           Merc 240D
         -0.2001440          -0.6932545          -3.9053627           4.1637381
           Merc 230            Merc 280           Merc 280C          Merc 450SE
          2.3499593           0.2998560          -1.1001440           0.8668731
         Merc 450SL         Merc 450SLC  Cadillac Fleetwood Lincoln Continental
         -0.0502472          -1.8830236           1.1733496           2.1032876
  Chrysler Imperial            Fiat 128         Honda Civic      Toyota Corolla
          5.9810744           6.8727113           1.7461954           6.4219792
      Toyota Corona    Dodge Challenger         AMC Javelin          Camaro Z28
         -2.6110037          -2.9725862          -3.7268663          -3.4623553
```

```
Pontiac Firebird            Fiat X1-9       Porsche 914-2        Lotus Europa
       2.4643670            0.3564263           0.1520430           1.2010593
  Ford Pantera L         Ferrari Dino       Maserati Bora          Volvo 142E
      -4.5431513           -2.7809399          -3.2053627          -1.0274952
```

Statistical Significance and p-values

- The regression coefficient (3.45) is significantly dfferent from zero ($p < 0.001$)
- There is an expected increase of 3.45 lbs of weight for every 1 inch increases in height.

Fit

We can get the Multiple R-squared

```r
# Summary of the model to get various statistics
model_summary <- summary(fit)

# Extracting Multiple R-squared value
model_summary$r.squared
```

```
[1] 0.7528328
```

We can also get the Adjusted R-squared

```r
# Extracting Adjusted R-squared value
model_summary$adj.r.squared
```

```
[1] 0.7445939
```

Confidence Intervals

We can compute Confidence Interval for a prediction.

```r
newdata = data.frame(wt = 4)
predict(fit,
        newdata,
        interval = "confidence")
```

```
       fit      lwr      upr
1 15.90724 14.49018 17.32429
```

F-Statistic

The F-statistic tests whether the predictor variables, taken together,predict the response variable.

MODEL 2: Log-Linear Model

Model 2 employs a log-linear regression approach to understand the relationship between the weight (`wt`) of cars and their fuel efficiency (`mpg`) in the `mtcars` dataset. Unlike a simple linear regression model which predicts `mpg` directly from `wt`, this model predicts the logarithm of `mpg` based on `wt`. This transformation allows us to model a multiplicative relationship between the dependent and independent variables, which can be more appropriate for certain types of data and relationships.

```
fit2 <- lm(log(mpg) ~ wt, data = mtcars)
summary(fit2)
```

```

Call:
lm(formula = log(mpg) ~ wt, data = mtcars)

Residuals:
      Min        1Q    Median        3Q       Max 
-0.210346 -0.085932 -0.006136  0.061335  0.308623 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  3.83191    0.08396   45.64  < 2e-16 ***
wt          -0.27178    0.02500  -10.87 6.31e-12 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1362 on 30 degrees of freedom
Multiple R-squared:  0.7976,	Adjusted R-squared:  0.7908 
F-statistic: 118.2 on 1 and 30 DF,  p-value: 6.31e-12
```

Model Output and Interpretation

The output of the model can be summarized and interpreted as follows:

Residuals

The residuals, or differences between the observed and predicted values of `log(mpg)`, have a median close to zero (-0.006136), indicating that the model's predictions are, on average, accurate. The range of residuals from the minimum (-0.210346) to the maximum (0.308623) suggests that most predictions are within this range of the actual log values.

Coefficients

- **Intercept (3.83191):** This value indicates the expected value of `log(mpg)` when `wt` is 0. It's a theoretical intercept since a car's weight cannot be zero, but it helps anchor the regression line.
- **Weight (wt) Coefficient (-0.27178):** This coefficient represents the expected change in `log(mpg)` for a one-unit increase in car weight. Specifically, for each one-unit increase in weight, `log(mpg)` is expected to decrease by 0.27178 units. This negative relationship suggests that heavier cars tend to have lower fuel efficiency.

Significance

The p-values for both the intercept and weight coefficient are well below the standard thresholds (e.g., 0.05, 0.01), indicating that these coefficients are statistically significant and that weight is a meaningful predictor of `log(mpg)`.

Model Fit

- **Residual Standard Error (RSE): 0.1362** on 30 degrees of freedom indicates the average amount by which the predicted values deviate from the actual values.
- **Multiple R-squared (0.7976):** This statistic indicates that approximately 79.76% of the variability in `log(mpg)` can be explained by the model. It's a measure of the model's goodness of fit.
- **Adjusted R-squared (0.7908):** This adjusts the R-squared value for the number of predictors in the model and suggests that after adjustment, about 79.08% of the variability in `log(mpg)` is explained by the model.
- **F-statistic (118.2):** This value tests the overall significance of the model. The very low p-value (6.31e-12) associated with the F-statistic indicates that the model is statistically significant.

Interpretation of the Beta Coefficient

The beta coefficient for `wt` in our log-linear regression model is -0.27178. This statistic tells us about the relationship between a car's weight and its fuel efficiency (mpg), with the mpg being logarithmically transformed.

In practical terms, a one-unit increase in a car's weight is associated with a decrease of 0.27178 in the log of its mpg. This relationship is multiplicative due to the log transformation of the mpg variable.

To understand the impact of this coefficient in more intuitive terms, we can use the mathematical operation `exp(-0.27178)`:

```
exp(-0.27178)
```

```
[1] 0.7620219
```

This calculation gives us approximately 0.762, or 76.2%.

This figure can be interpreted as follows: for every one-unit increase in weight, the fuel efficiency of a car is expected to be about 76.2% of what it would be if it were one unit lighter, all else being equal.

This means that **the heavier car's fuel efficiency is 76.2% of the lighter car's efficiency**, representing a significant decrease in efficiency due to the increase in weight.

Hence, the beta coefficient of -0.27178 signifies a strong negative impact of weight on a car's fuel efficiency, illustrating that as a car's weight increases, its fuel efficiency substantially decreases, when other factors are held constant.

Another Log-Linear Model

We present two regression models, Model 3a (Linear-Linear Model) and Model 3b (Log-Linear Model), using the `mtcars` dataset. Both models aim to predict the miles per gallon (`mpg`) of cars based on their weight (`wt`) and transmission type (`am`, with levels "Automatic" and "Manual").

Model 3a - Linear-Linear Model

```r
# Convert am to a factor variable
mtcars$am <- factor(mtcars$am, labels = c("Automatic", "Manual"))

fit3a <- lm((mpg) ~ wt + am, data = mtcars)
summary(fit3a)
```

```
Call:
lm(formula = (mpg) ~ wt + am, data = mtcars)

Residuals:
    Min      1Q  Median      3Q     Max 
-4.5295 -2.3619 -0.1317  1.4025  6.8782 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 37.32155    3.05464  12.218 5.84e-13 ***
wt          -5.35281    0.78824  -6.791 1.87e-07 ***
amManual    -0.02362    1.54565  -0.015    0.988    
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.098 on 29 degrees of freedom
Multiple R-squared:  0.7528,	Adjusted R-squared:  0.7358 
F-statistic: 44.17 on 2 and 29 DF,  p-value: 1.579e-09
```

This model directly predicts `mpg` from `wt` and the type of transmission (`am`), treating `am` as a factor variable.

Output Interpretation:

- **Coefficients:**
  - The intercept, 37.32155, suggests the expected `mpg` for an automatic transmission car (baseline category) with a weight of 0, which is a hypothetical scenario serving as a reference point.
  - The `wt` coefficient of -5.35281 indicates that for each additional unit of weight, `mpg` decreases by approximately 5.35, holding the type of transmission constant.

  - The `amManual` coefficient of -0.02362 suggests a negligible and statistically insignificant change in `mpg` for manual cars compared to automatic ones, holding weight constant.

- **Model Fit:** The Multiple R-squared value of 0.7528 suggests that approximately 75.28% of the variability in `mpg` can be explained by the model. The Adjusted R-squared value of 0.7358 adjusts this for the number of predictors and indicates a strong model fit.

MODEL 3b: Another Log-Linear Model

```r
fit3b <- lm(log(mpg) ~ wt + am, data = mtcars)
summary(fit3b)
```

```
Call:
lm(formula = log(mpg) ~ wt + am, data = mtcars)

Residuals:
     Min       1Q   Median       3Q      Max 
-0.21351 -0.08281  0.00304  0.04962  0.32349 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  3.89834    0.13567  28.734  < 2e-16 ***
wt          -0.28699    0.03501  -8.198 4.87e-09 ***
amManual    -0.04307    0.06865  -0.627    0.535    
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1376 on 29 degrees of freedom
Multiple R-squared:  0.8003,	Adjusted R-squared:  0.7865 
F-statistic:  58.1 on 2 and 29 DF,  p-value: 7.186e-11
```

This model predicts the log of `mpg` from `wt` and the type of transmission (`am`), again treating `am` as a factor variable.

Output Interpretation:

- **Coefficients:**

- The intercept, 3.89834, is the expected value of log(`mpg`) for an automatic transmission car with a weight of 0.
- The `wt` coefficient of -0.28699 suggests that for each additional unit of weight, the log of `mpg` decreases by approximately 0.287, indicating a strong negative relationship between weight and `mpg`, after controlling for transmission type.
- The `amManual` coefficient of -0.04307 is not statistically significant (p=0.535), indicating that, after controlling for weight, the difference in log(`mpg`) between manual and automatic cars is negligible.

- **Model Fit:** The model has a Multiple R-squared of 0.8003, showing that it explains about 80.03% of the variability in log(`mpg`), which is an improvement over Model 3a. The Adjusted R-squared of 0.7865 indicates a strong fit to the data.

Exponential Interpretations:

- **exp(-0.04307) = 0.9578443**:

This value implies that, all else being equal, the `mpg` of automatic cars is expected to be approximately 95.78% of that of manual cars.

Given the statistical insignificance of the `amManual` coefficient, this difference is not considered meaningful in the context of this model.

- **exp(-0.2869) = 0.7505868**:

This exponentiated coefficient translates to a more intuitive interpretation for the `wt` variable. It suggests that for each one-unit increase in weight, the `mpg` is expected to be multiplied by approximately 0.75, holding the transmission type constant. This represents a substantial decrease in fuel efficiency with increased weight.

Comparison

Both models provide valuable insights into the factors affecting `mpg`.

Model 3a highlights a strong negative relationship between `mpg` and `wt`, with the type of transmission (`am`) showing no significant effect when modeled linearly.

Model 3b, with its log-linear approach, not only confirms the strong negative impact of weight on `mpg` but also offers a slightly better fit to the data as indicated by the higher adjusted R-squared value.

The lack of a significant effect from the transmission type is consistent across both models. The log-linear model's ability to better account for the variability suggests that the relationship between `mpg` and `wt` might be more appropriately modeled on a multiplicative (logarithmic) scale.