# co-**RE** and Reducibility

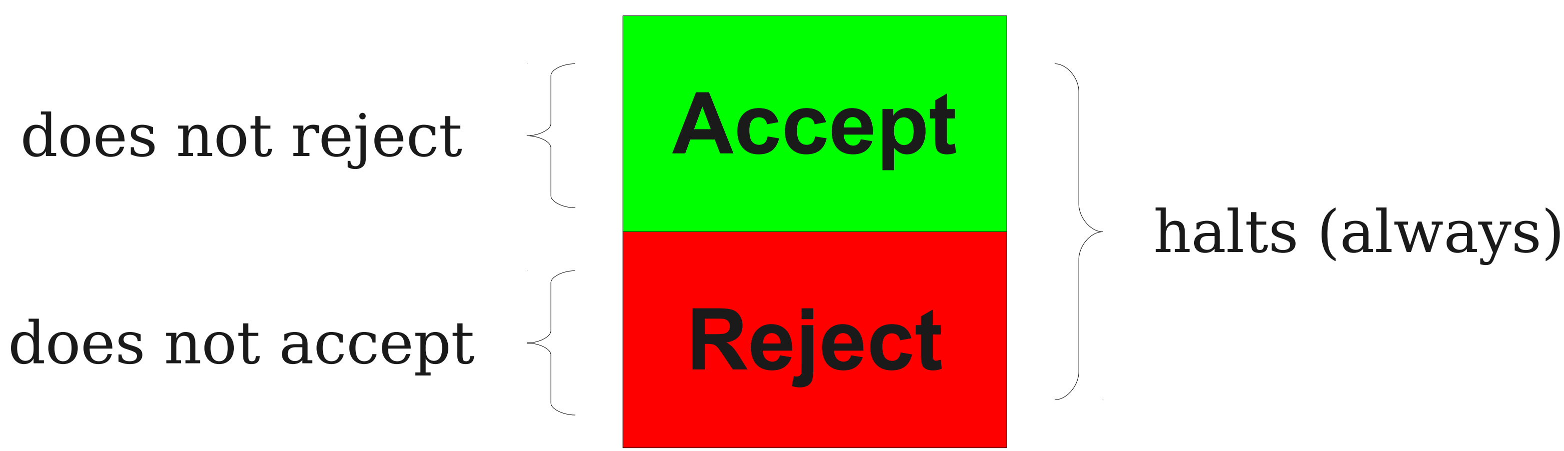
Friday Four Square!
Today at 4:15PM, Outside Gates

# Announcements

- Problem Set 6 graded, will be returned at end of lecture.
  - Late submissions will be graded by Monday.
- Problem Set 7 due this Monday, March 4 at the start of lecture.
  - We are working on shuffling around OH for this weekend; we'll send out an email with updates.

# Major Ideas from Last Time

- Some Turing machines always halt; they never go into an infinite loop.
- Turing machines of this sort are called **deciders**.
- For deciders, accepting is the same as not rejecting and rejecting is the same as not accepting.



# Major Ideas from Last Time

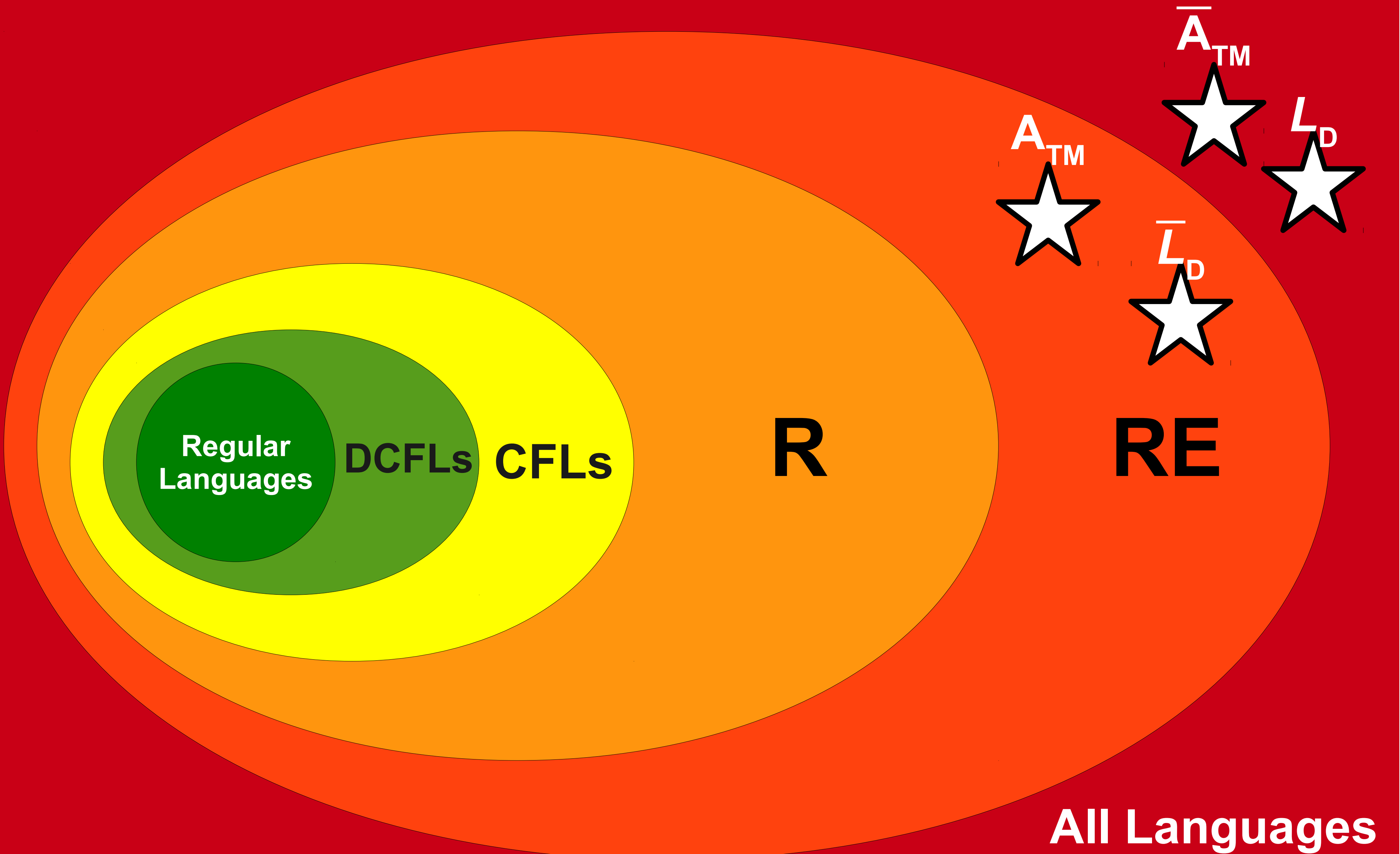
- A language $L$ is called **decidable** iff there is a decider $M$ such that $\mathscr{L}(M) = L$.
- Given a decider $M$, you *can* learn whether or not a string $w \in \mathscr{L}(M)$.
  - Run $M$ on $w$.
  - Although it might take a staggeringly long time, $M$ will eventually accept or reject $w$.
- The set $\mathbf{R}$ is the set of all decidable languages.

$$L \in \mathbf{R} \quad \text{iff} \quad L \text{ is decidable}$$

# **R** and **RE** Languages

- Intuitively, a language is in **RE** if there is some way that you could exhaustively search for a proof that $w \in L$.
  - If you find it, accept!
  - If you don't find one, keep looking!
- Intuitively, a language is in **R** if there is a concrete algorithm that can determine whether $w \in L$.
  - It tends to be *much* harder to show that a language is in **R** than in **RE**.

# The Limits of Computability



# Outline for Today

- **The Halting Problem**
  - An important problem about TMs.
- **co-RE Languages**
  - Resolving a fundamental asymmetry.
- **Mapping Reductions**
  - A tool for finding unsolvable problems.

# The Halting Problem

# The Halting Problem

- The **halting problem** is the following problem:

**Given a TM *M* and string *w*,**
**does *M* halt on *w*?**

- Note that *M* doesn't have to *accept* *w*; it just has to *halt* on *w*.
- As a formal language:

$$\boldsymbol{HALT = \{ \langle M, w \rangle \mid M \textbf{ is a TM and } M \textbf{ halts on } w. \}}$$

- Is $HALT \in \mathbf{R}$? Is $HALT \in \mathbf{RE}$?

# *HALT* is Recognizable

- Consider this Turing machine:

  $H$ = "On input $\langle M, w \rangle$:

  Run $M$ on $w$.

  If $M$ accepts, accept.

  If $M$ rejects, accept."

- Then $H$ accepts $\langle M, w \rangle$ iff $M$ halts on $w$.
- Thus $\mathscr{L}(H) = \mathit{HALT}$, so $\mathit{HALT} \in \mathbf{RE}$.

***Theorem:*** $HALT \notin \mathbf{R}$.

(The halting problem is undecidable)

# Proving $HALT \notin \mathbf{R}$

- Our proof will work as follows:
  - Suppose that $HALT \in \mathbf{R}$.
  - Using a decider for $HALT$, construct a decider for $A_{TM}$.
  - Reach a contradiction, since there is no decider for $A_{TM}$ ($A_{TM} \notin \mathbf{R}$).
  - Conclude, therefore, that $HALT \notin \mathbf{R}$.

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Conclude,

This is the creative step of the proof. How exactly are we going to do this?

# Accepting, Rejecting, and Looping

- Suppose we have a TM $M$ and a string $w$.
- Then $M$ either
  - **Accepts**, or
  - **Does not accept** (by rejecting or looping).
- What if $M$ never rejects?
- Then $M$ either
  - **Accepts**, or
  - **Does not accept** (by looping).

# The Key Insight

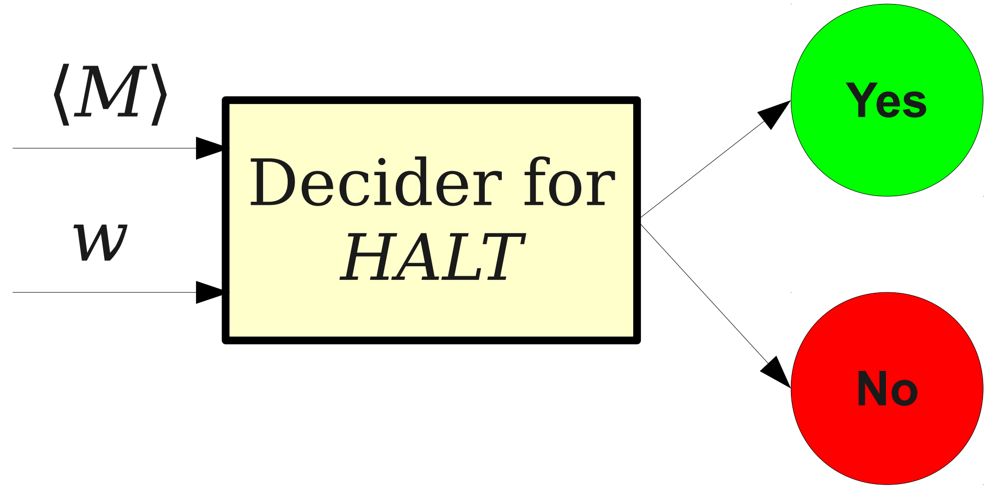
- If $M$ never rejects, then

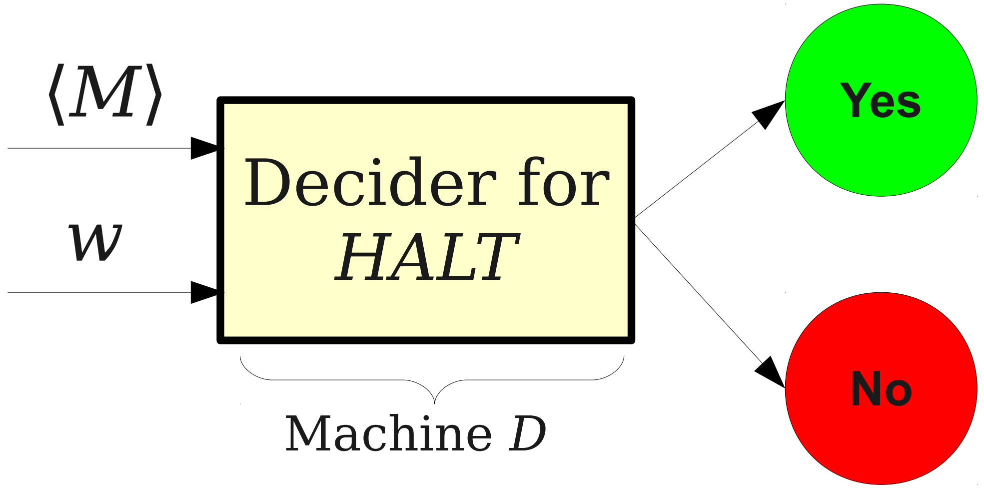
  **$M$ accepts $w$ iff $M$ halts on $w$**

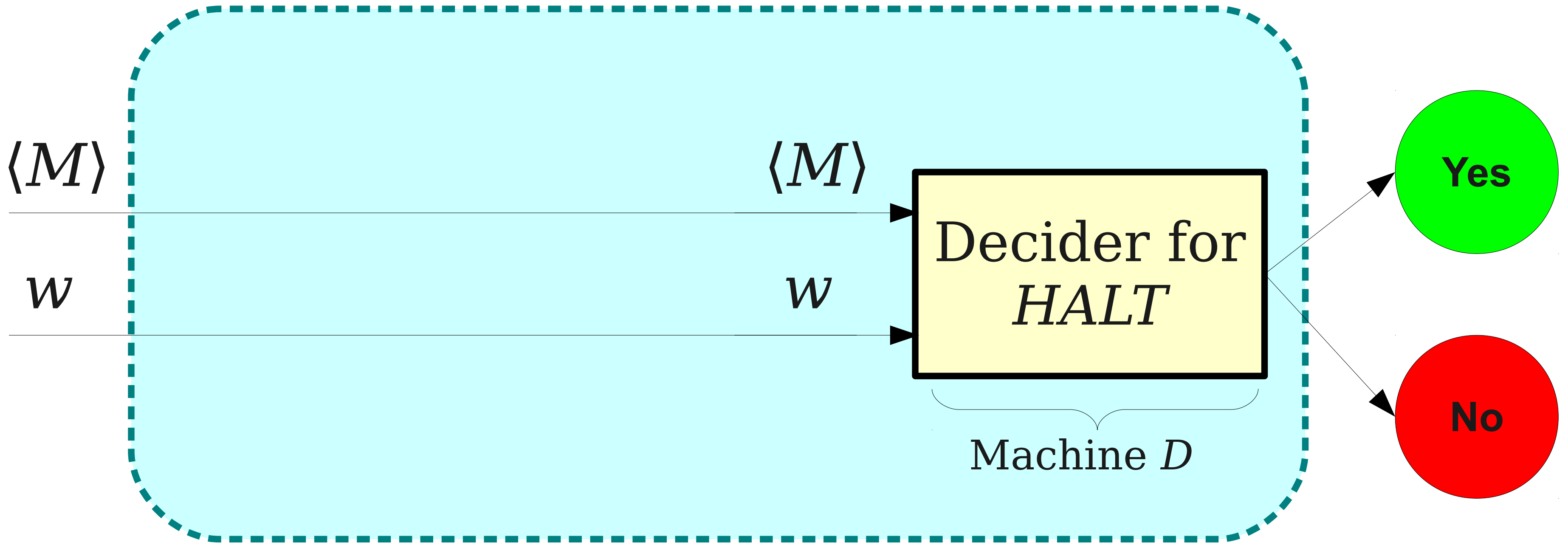
- In other words, if $M$ never rejects, then

  **$\langle M, w\rangle \in A_{TM}$ iff $\langle M, w\rangle \in HALT$**

- If we can modify an arbitrary TM $M$ so that $M$ never rejects, then a decider for *HALT* can be made to decide $A_{TM}$.
  - Since $A_{TM} \notin \mathbf{R}$, this is a contradiction!


⟨M⟩
w
Decider for HALT
Yes
No


⟨M⟩
w
Decider for HALT
Machine D
Yes
No



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w
⟨M⟩
w
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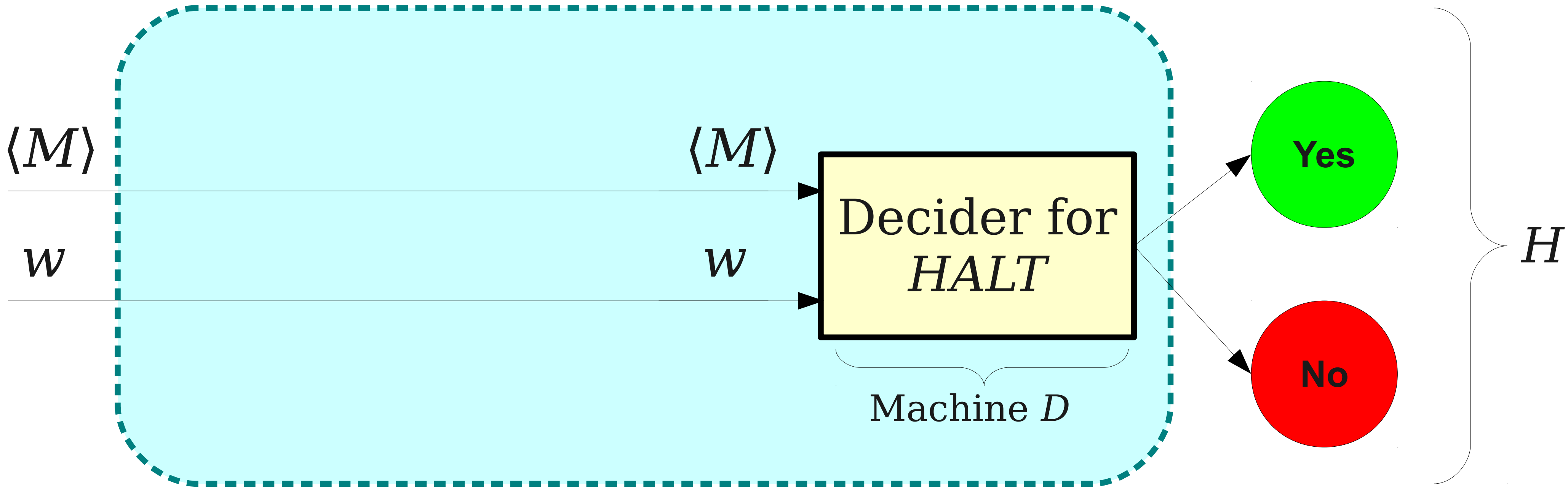
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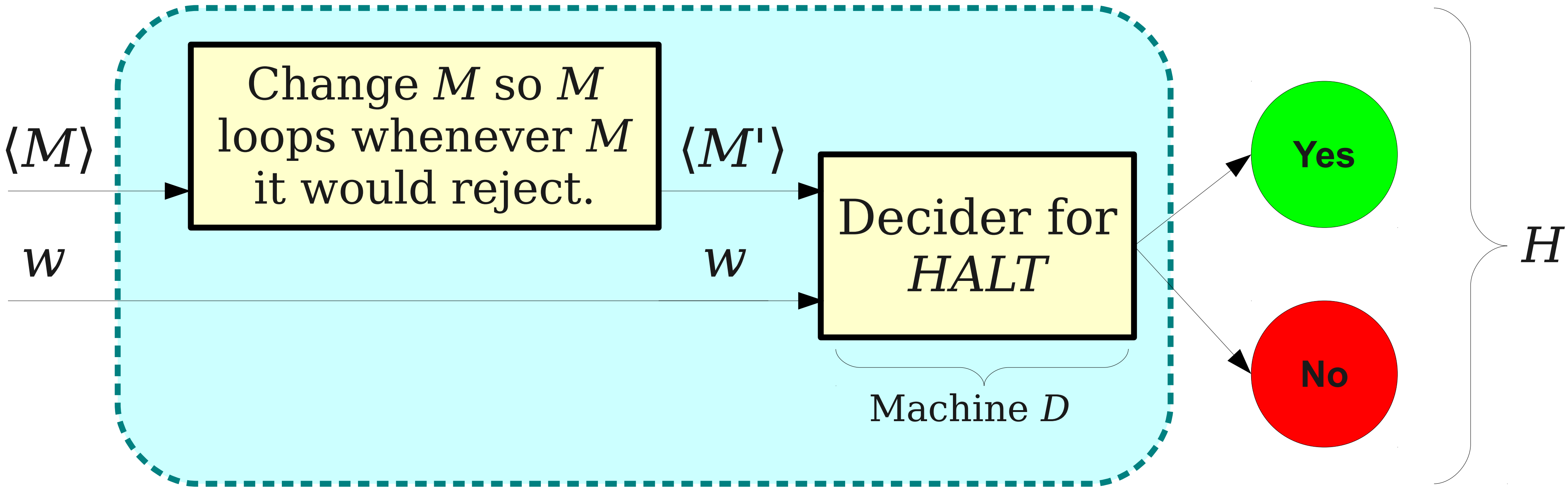
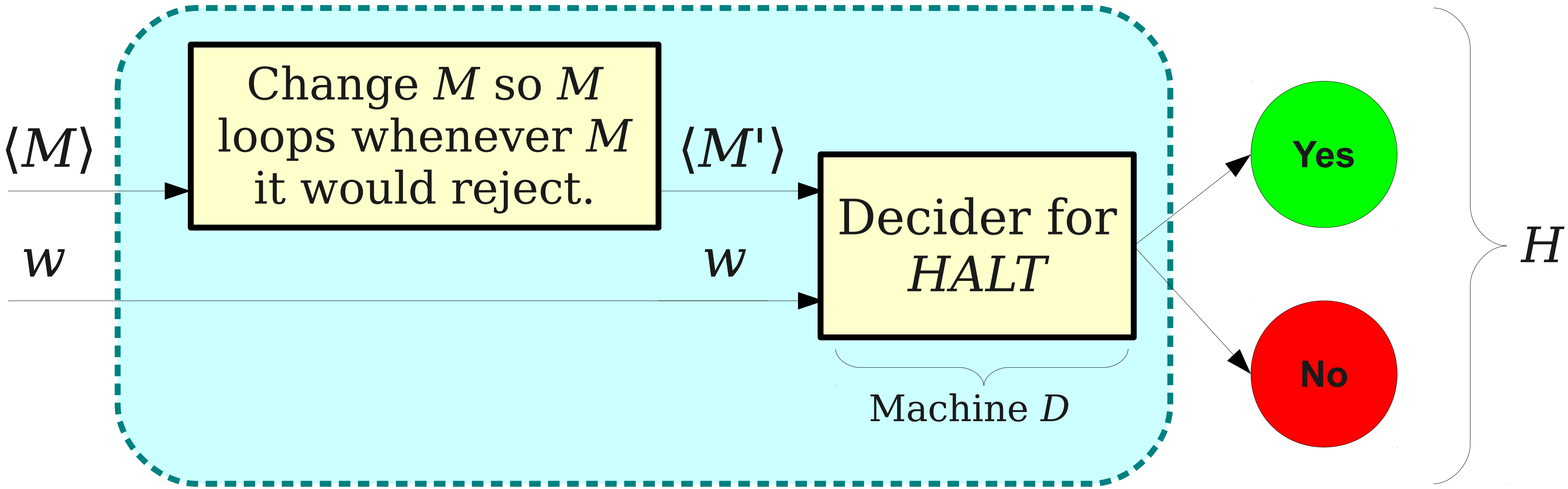
H


⟨M⟩
w
Change M so M loops whenever M it would reject.
⟨M'⟩
w
Decider for HALT
Machine D
Yes
No
H

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- Run $D$ on $\langle M', w \rangle$.
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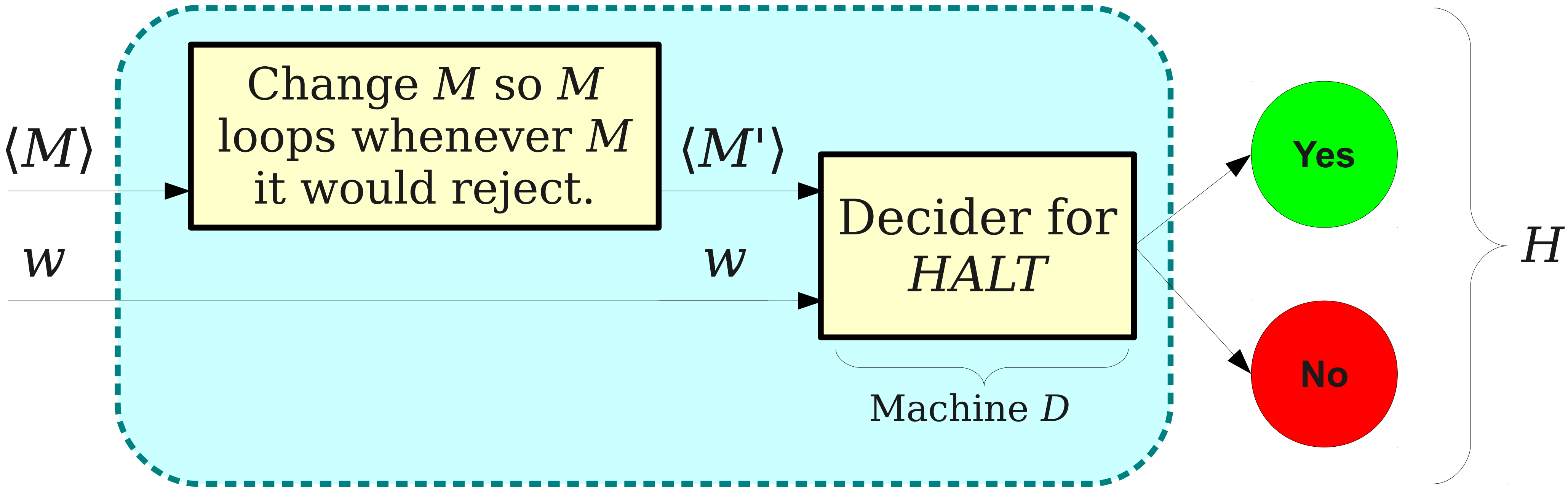


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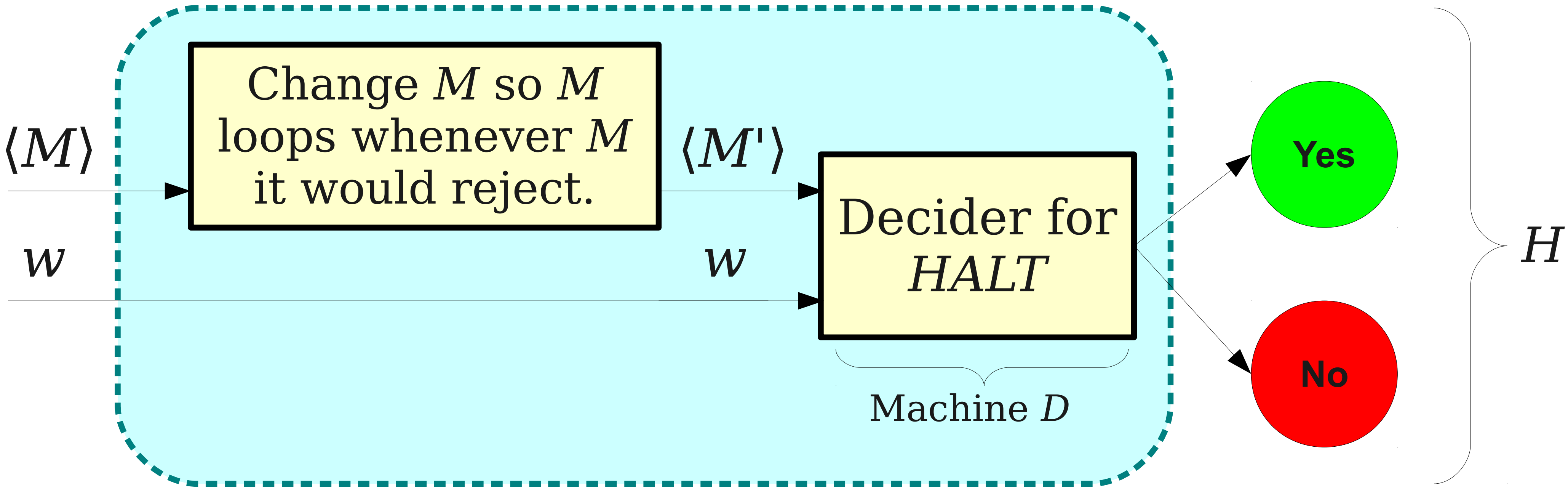


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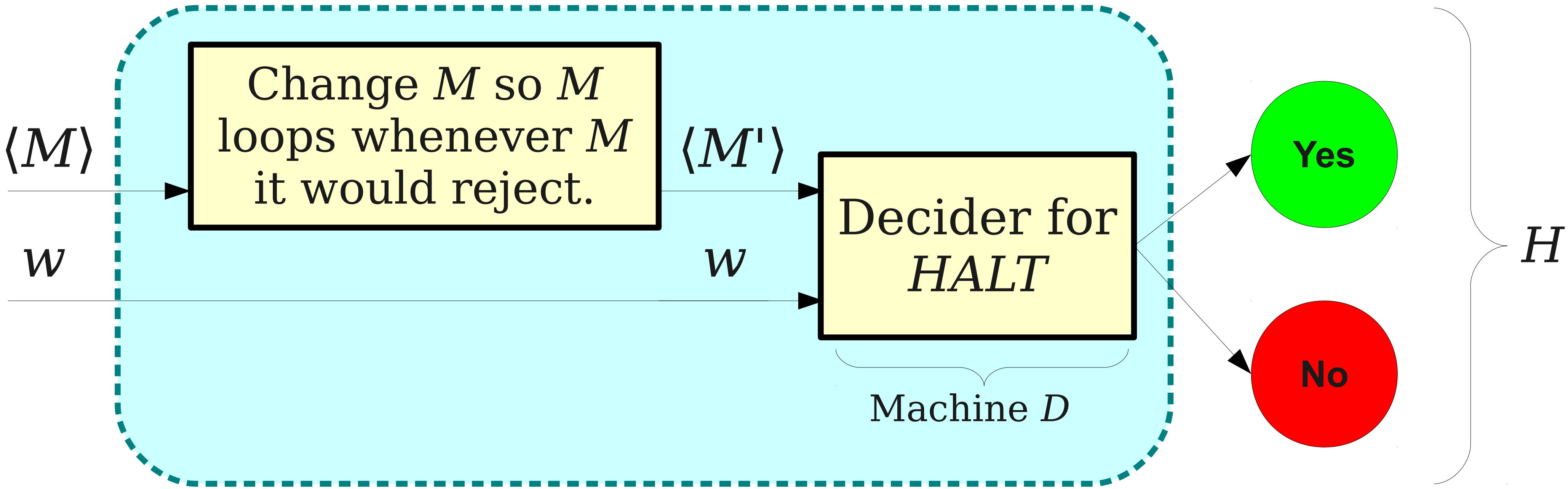


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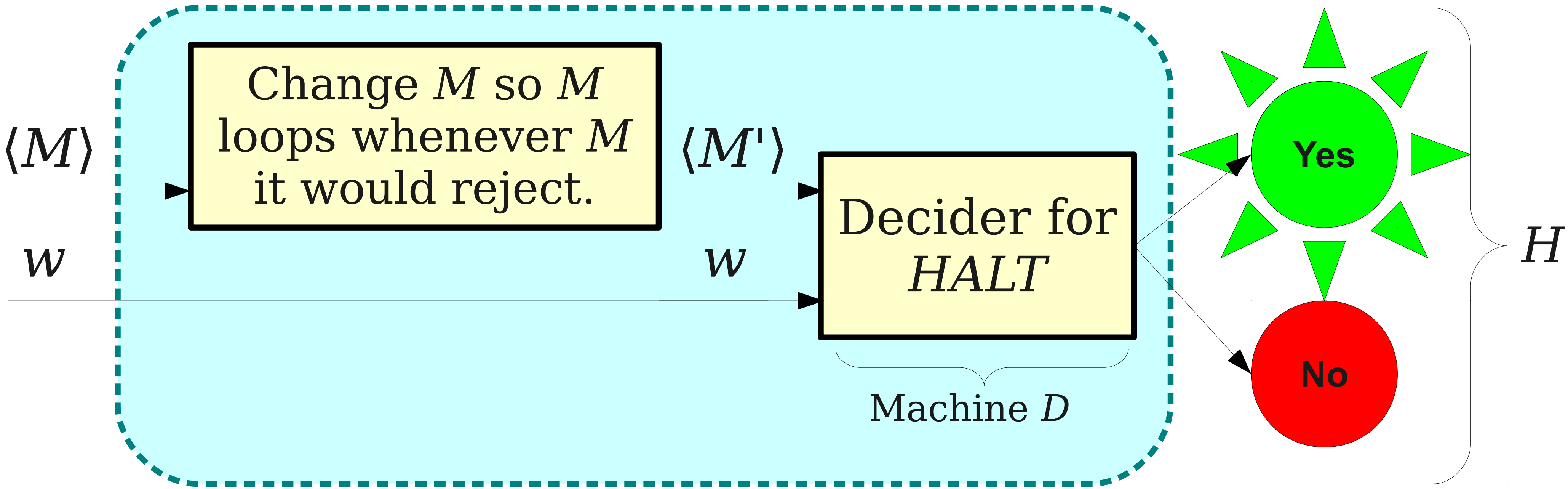
What happens if…

$M$ accepts $w$?

Machine $M$ accepts $w$, so $M'$ accepts $w$.

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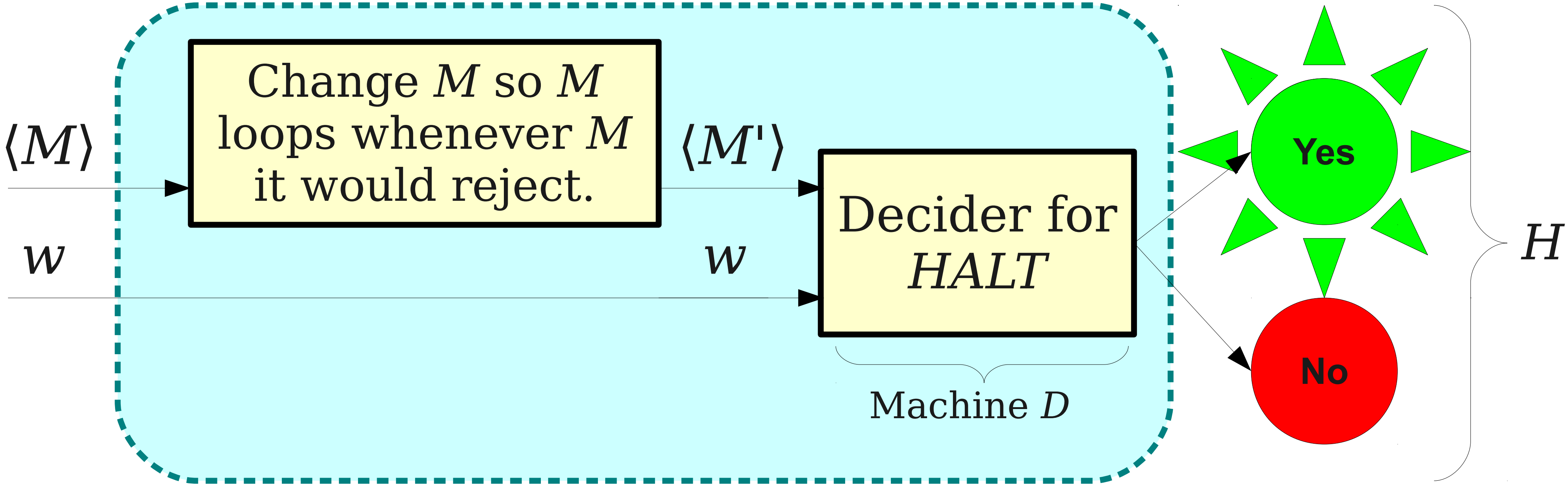


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$M$ accepts $w$? **Accept**

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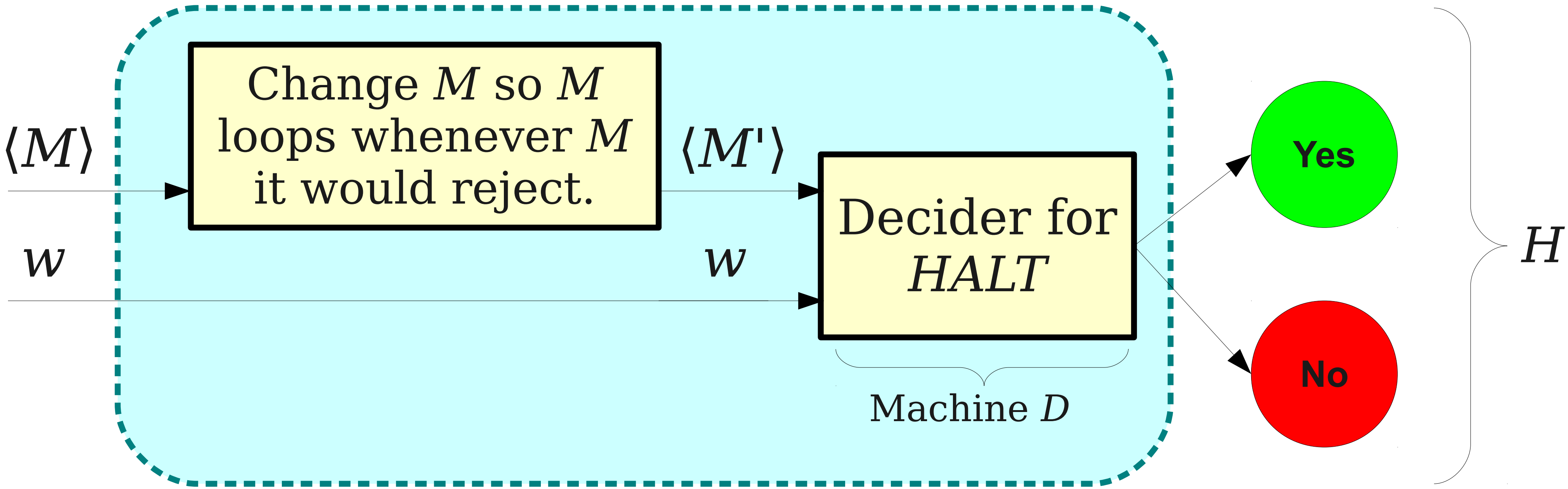


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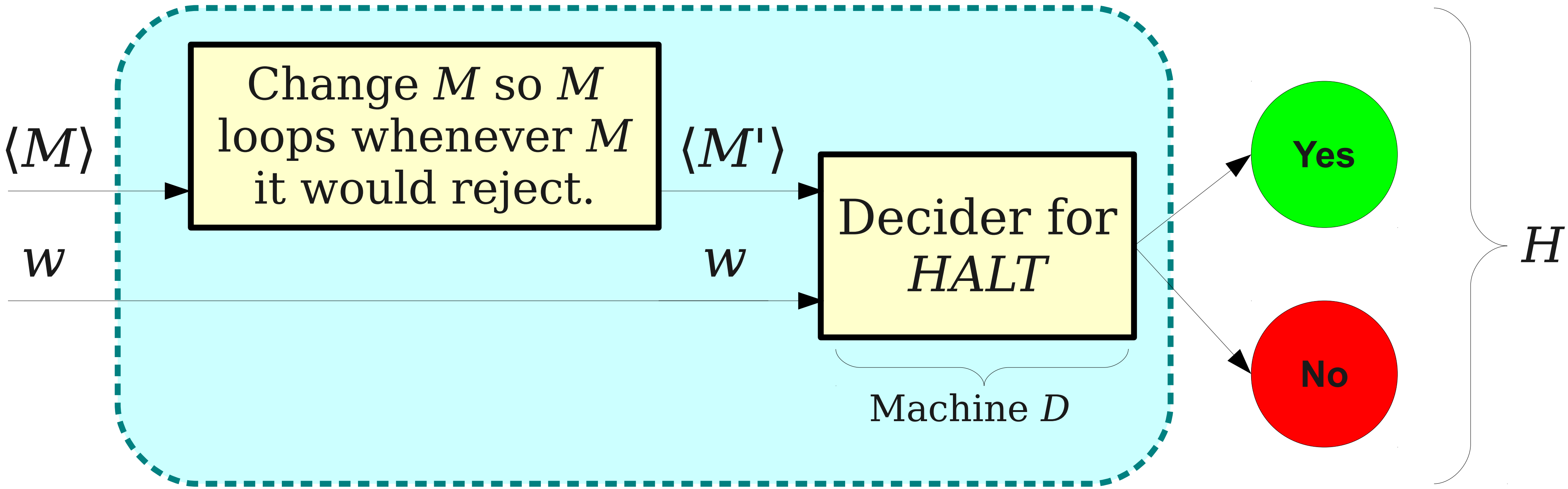


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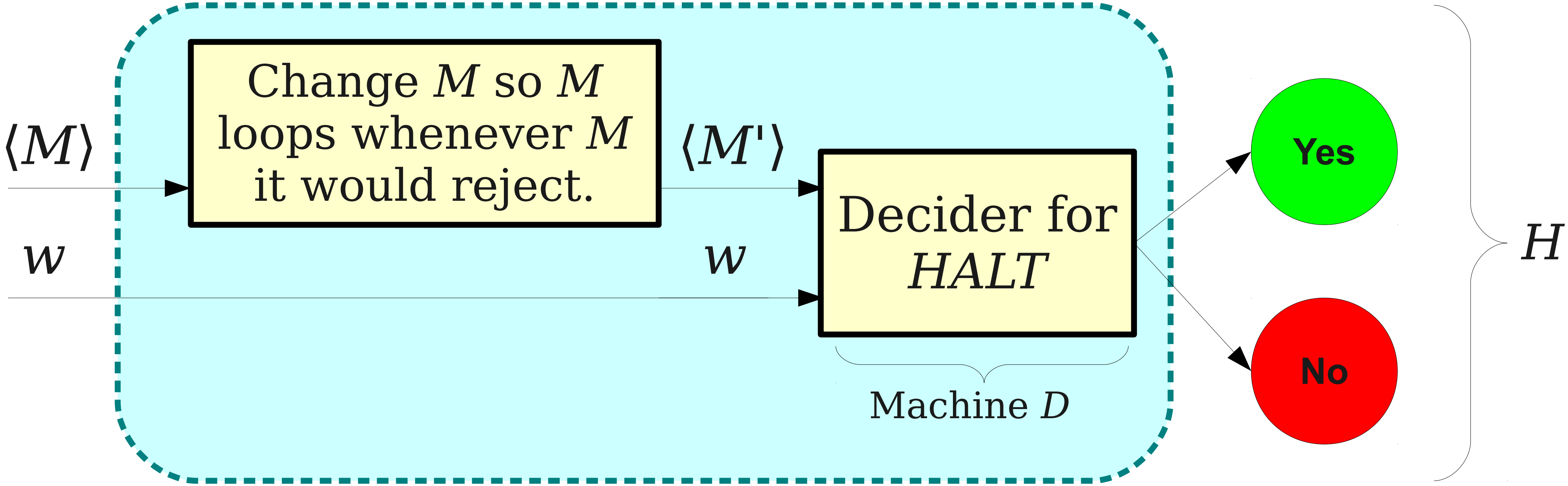


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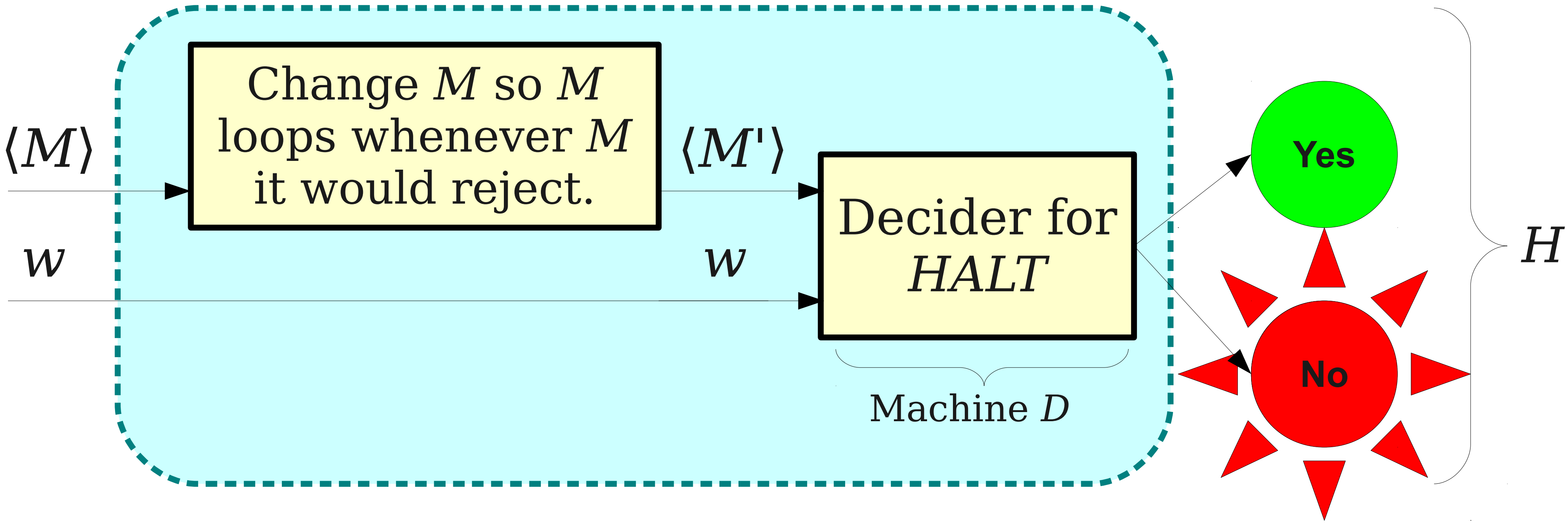


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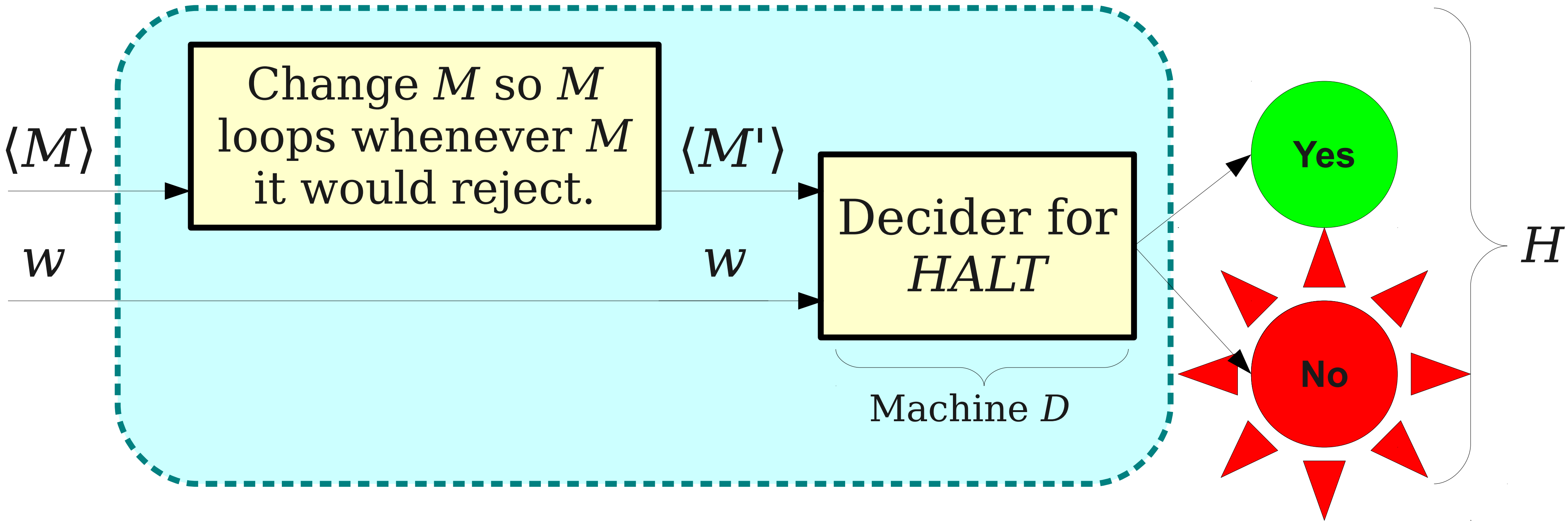


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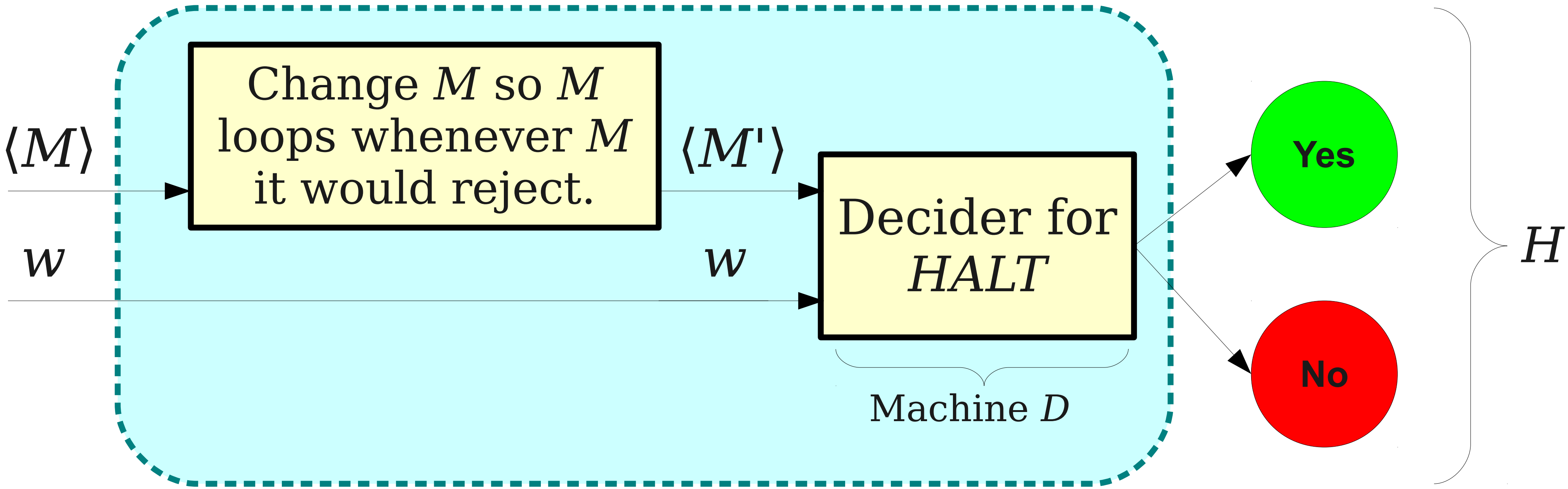


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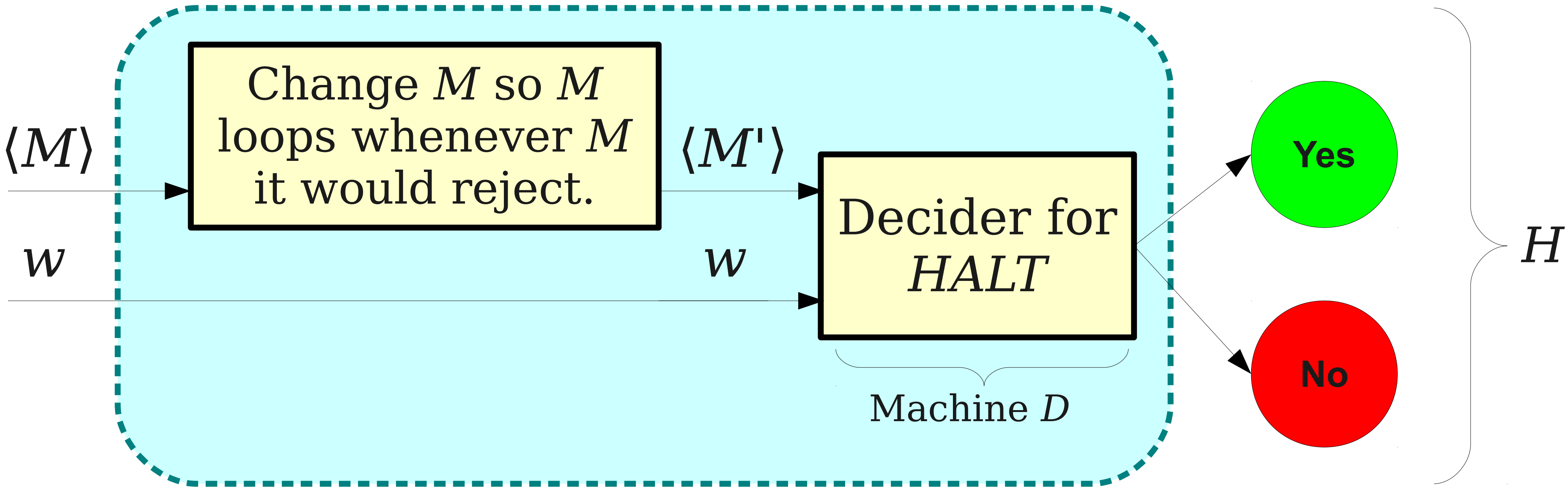


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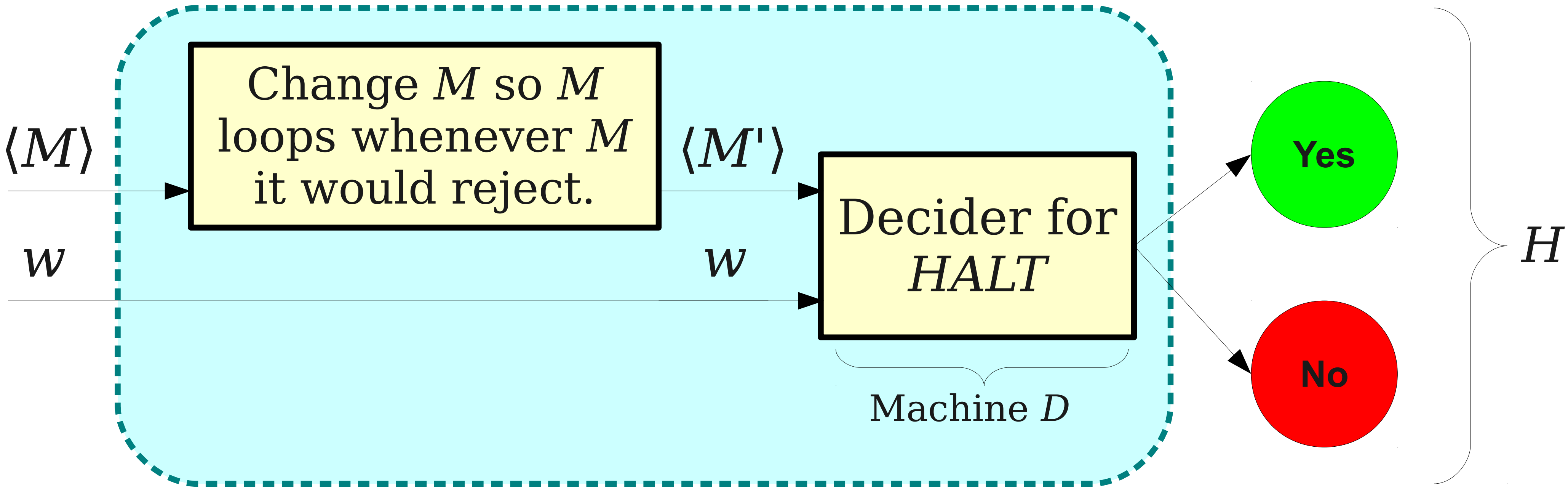


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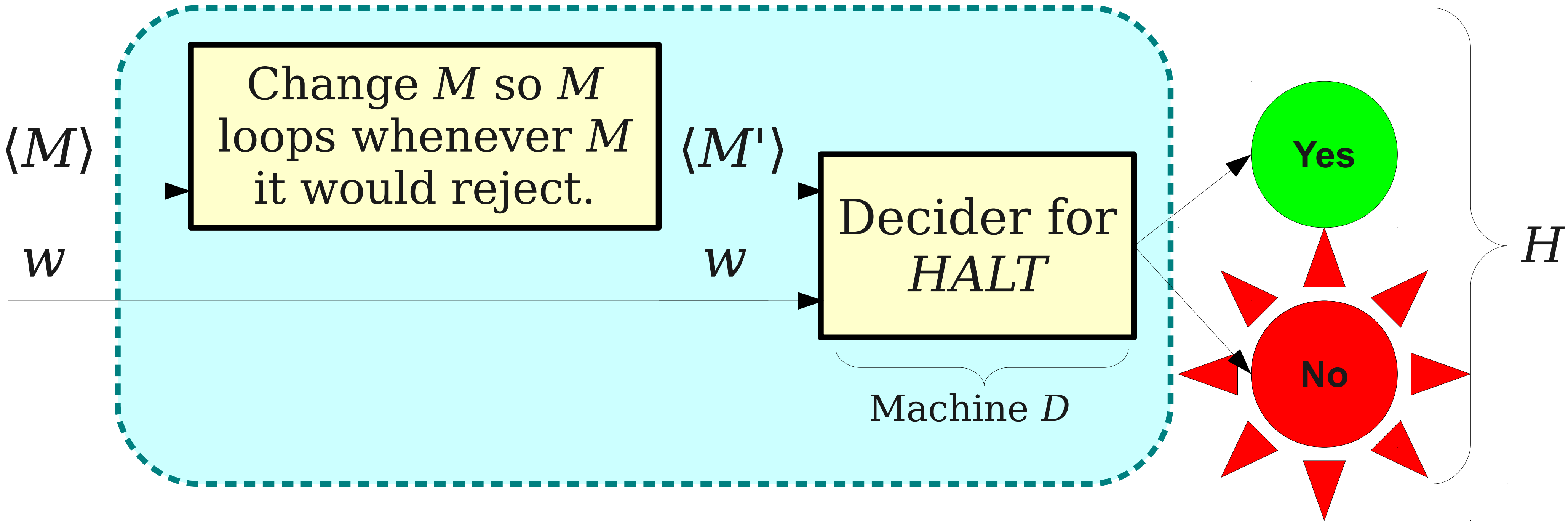


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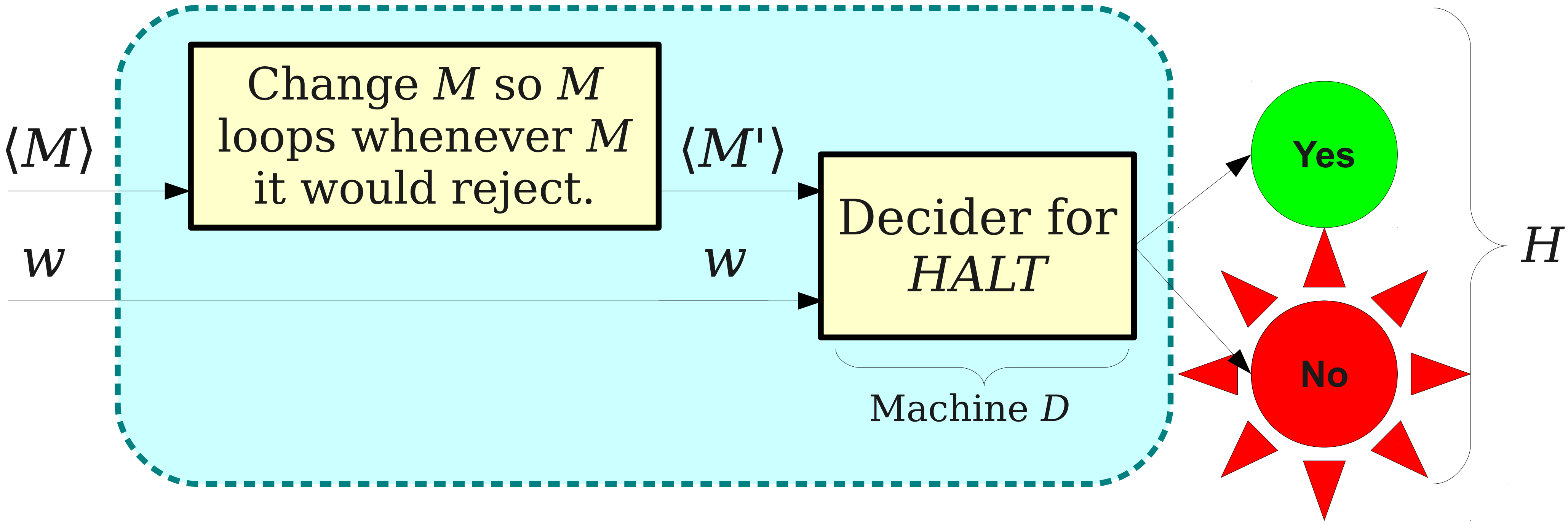


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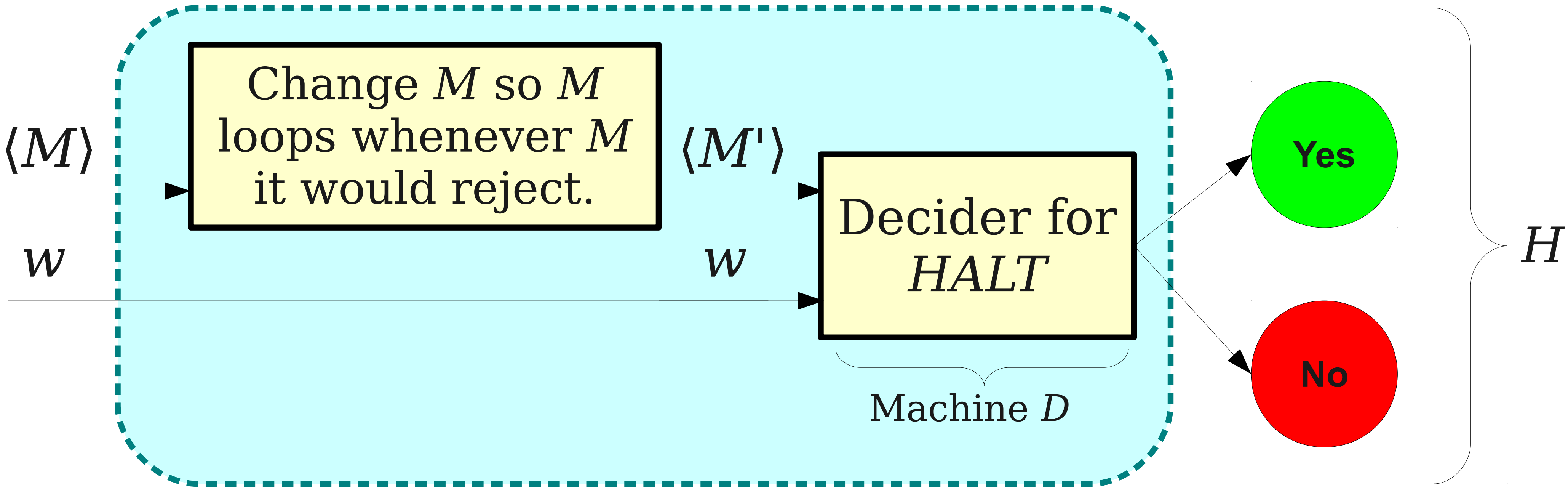


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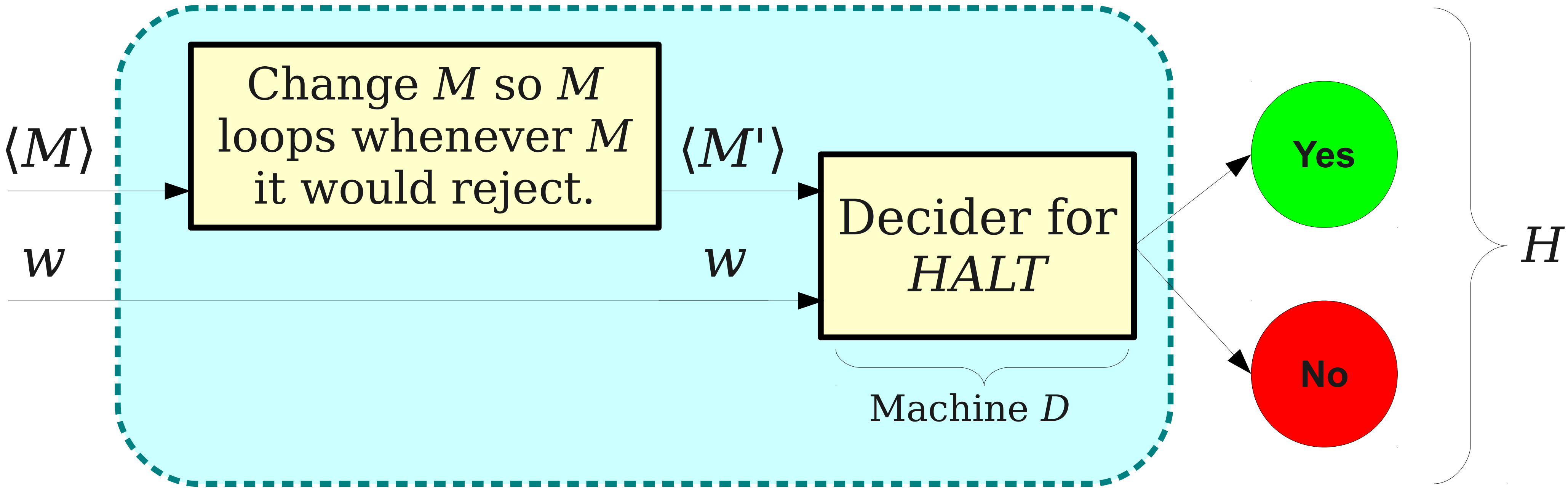


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$M$ accepts $w$? **Accept**
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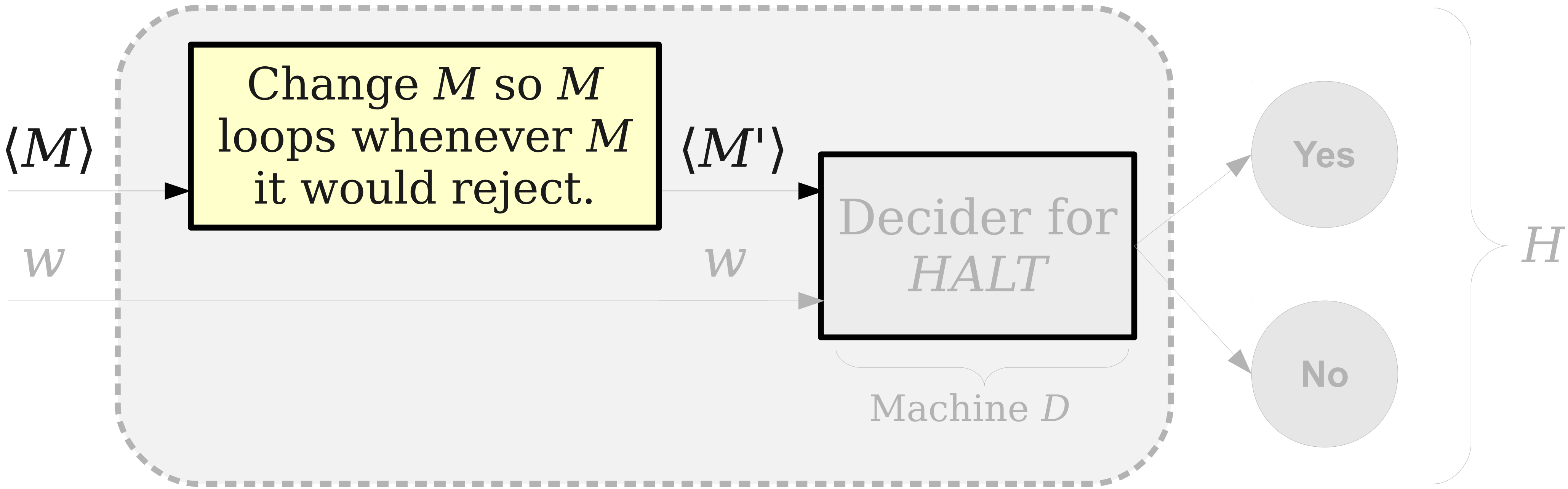
Machine $H$ is a decider for $A_{TM}$!

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What happens if...

| | |
|---|---|
| $M$ accepts $w$? | **Accept** |
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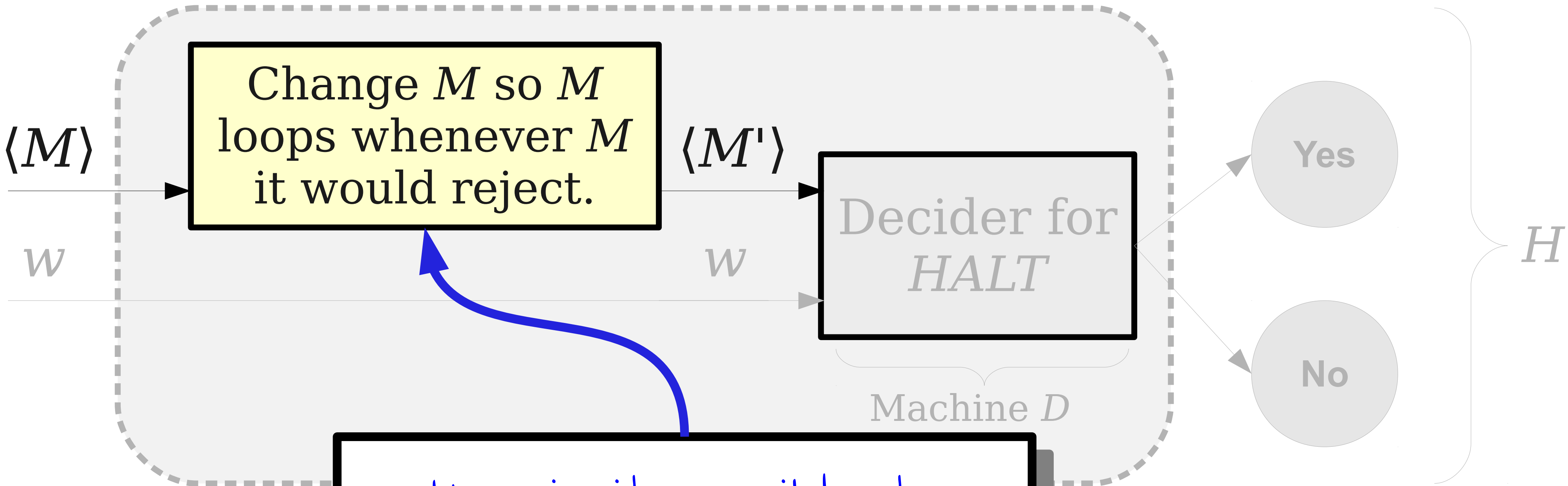


How is it possible to build this part of the machine?

$H$ = “On inpu

- Trans
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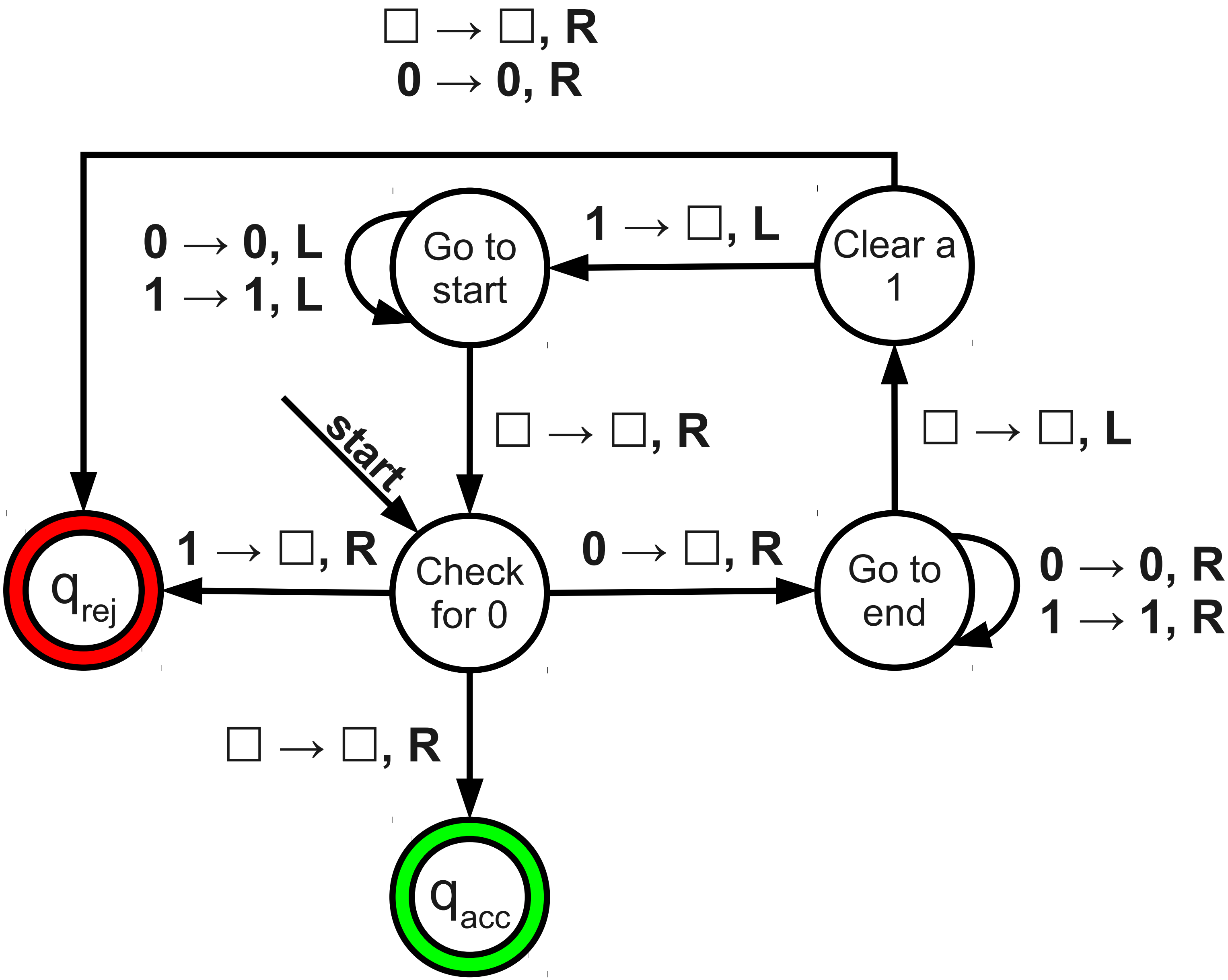
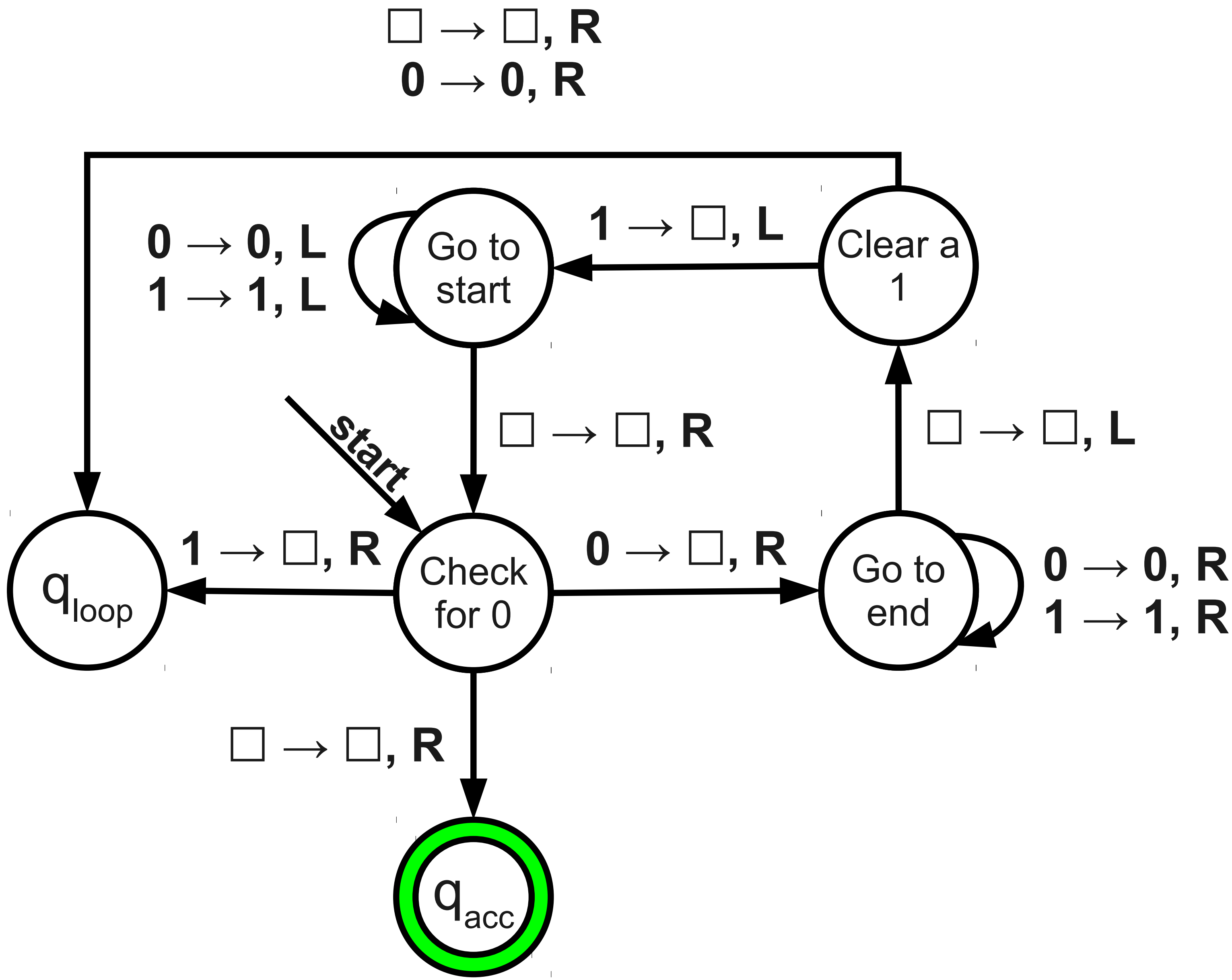

□ → □, R
0 → 0, R
0 → 0, L
1 → 1, L
Go to start
1 → □, L
Clear a 1
start
□ → □, R
□ → □, L
q_rej
1 → □, R
Check for 0
0 → □, R
Go to end
0 → 0, R
1 → 1, R
□ → □, R
q_acc



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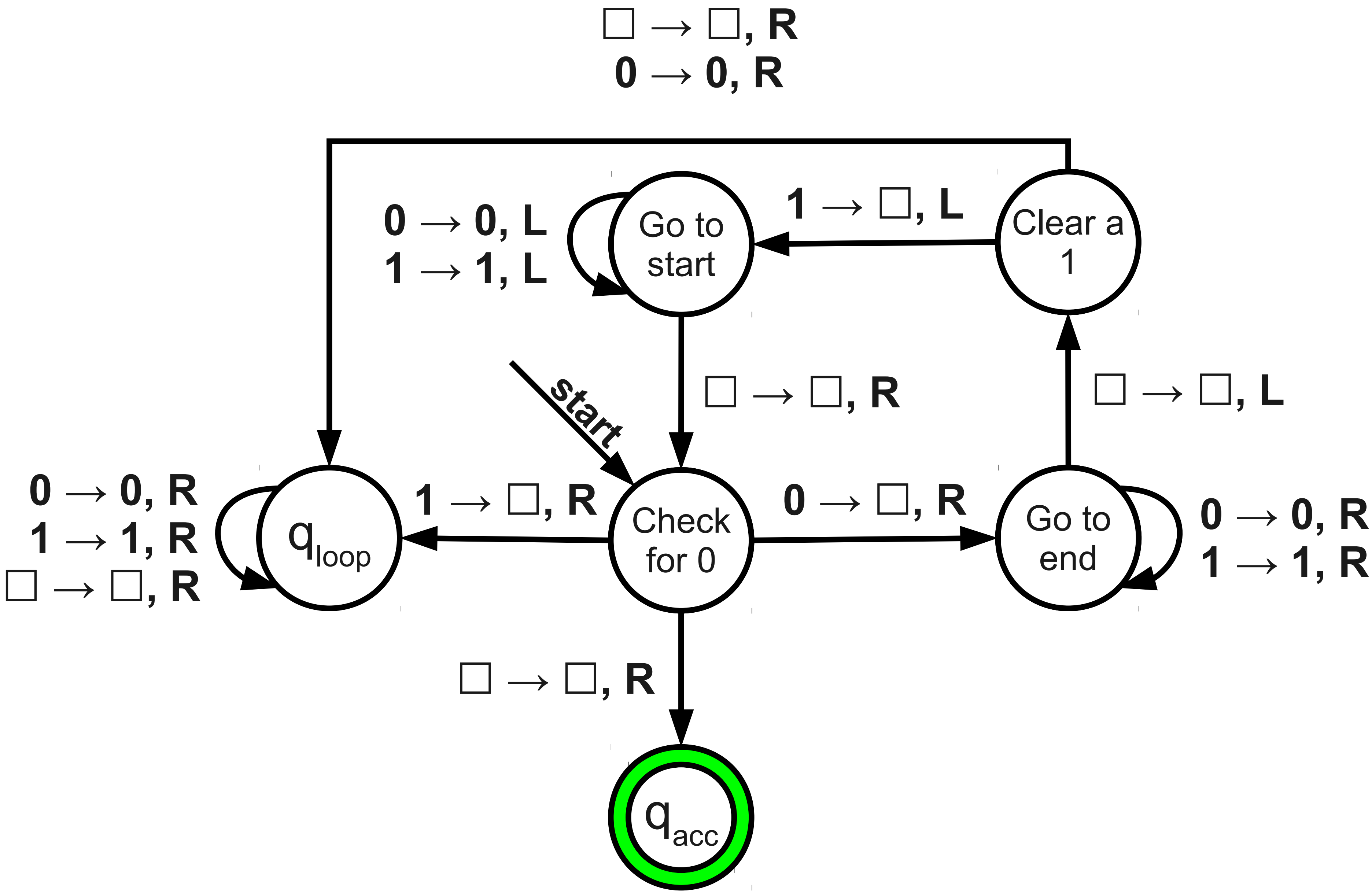
□ → □, R
□ → □, L
q_loop
1 → □, R
Check for 0
0 → □, R
Go to end
0 → 0, R
1 → 1, R
□ → □, R
q_acc



□ → □, R
0 → 0, R
0 → 0, L
1 → 1, L
Go to start
1 → □, L
Clear a 1
start
□ → □, R
□ → □, L
0 → 0, R
1 → 1, R
□ → □, R
$q_{loop}$
1 → □, R
Check for 0
0 → □, R
Go to end
0 → 0, R
1 → 1, R
□ → □, R
$q_{acc}$

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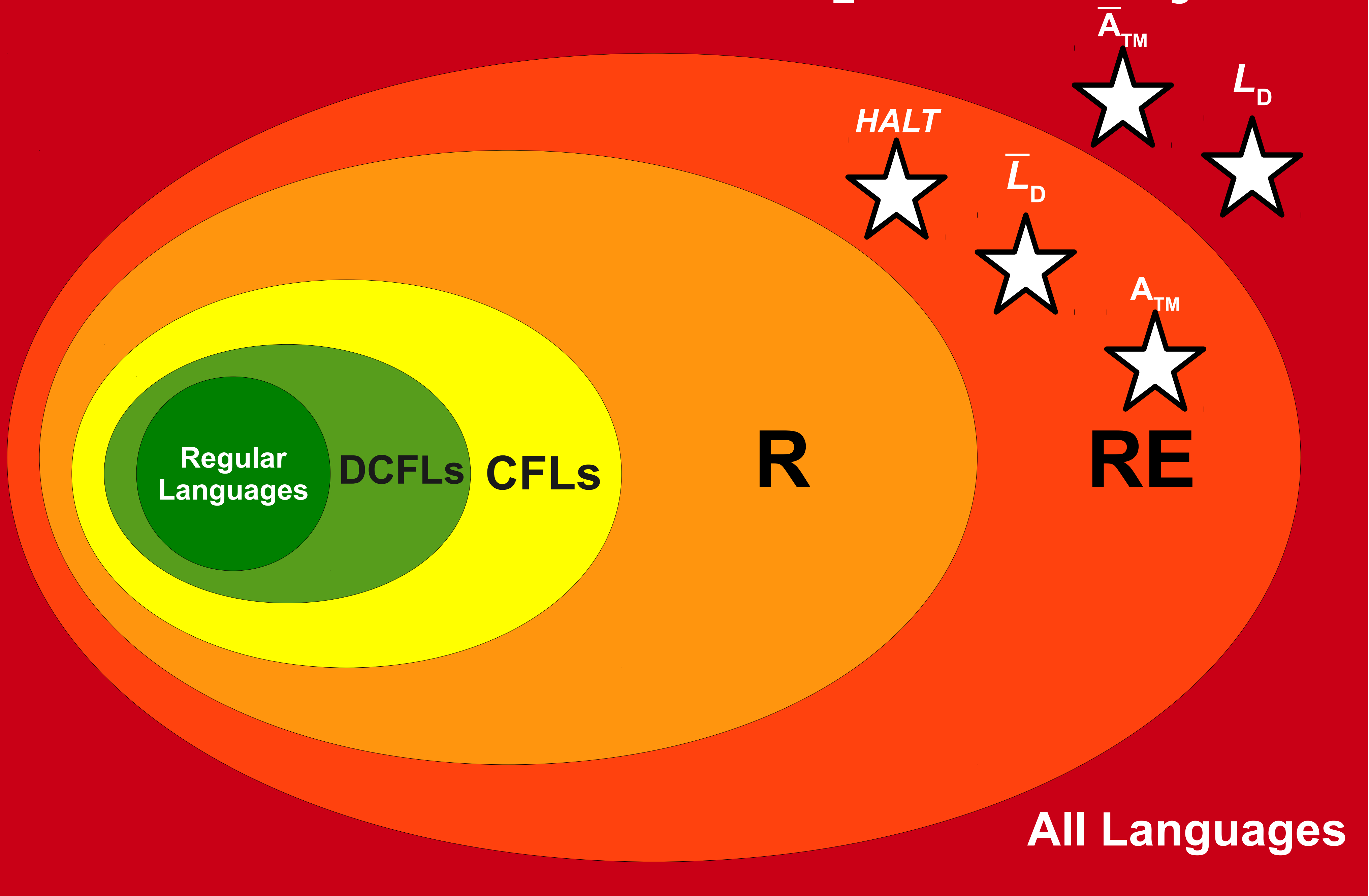
# The Limits of Computability



# $A_{TM}$ and *HALT*

- Both $A_{TM}$ and *HALT* are undecidable.
  - There is no way to decide whether a TM will accept or eventually terminate.
- However, both $A_{TM}$ and *HALT* are recognizable.
  - We can always run a TM on a string *w* and accept if that TM accepts or halts.
- **Intuition:** The only general way to learn what a TM will do on a given string is to run it and see what happens.

# Resolving an Asymmetry

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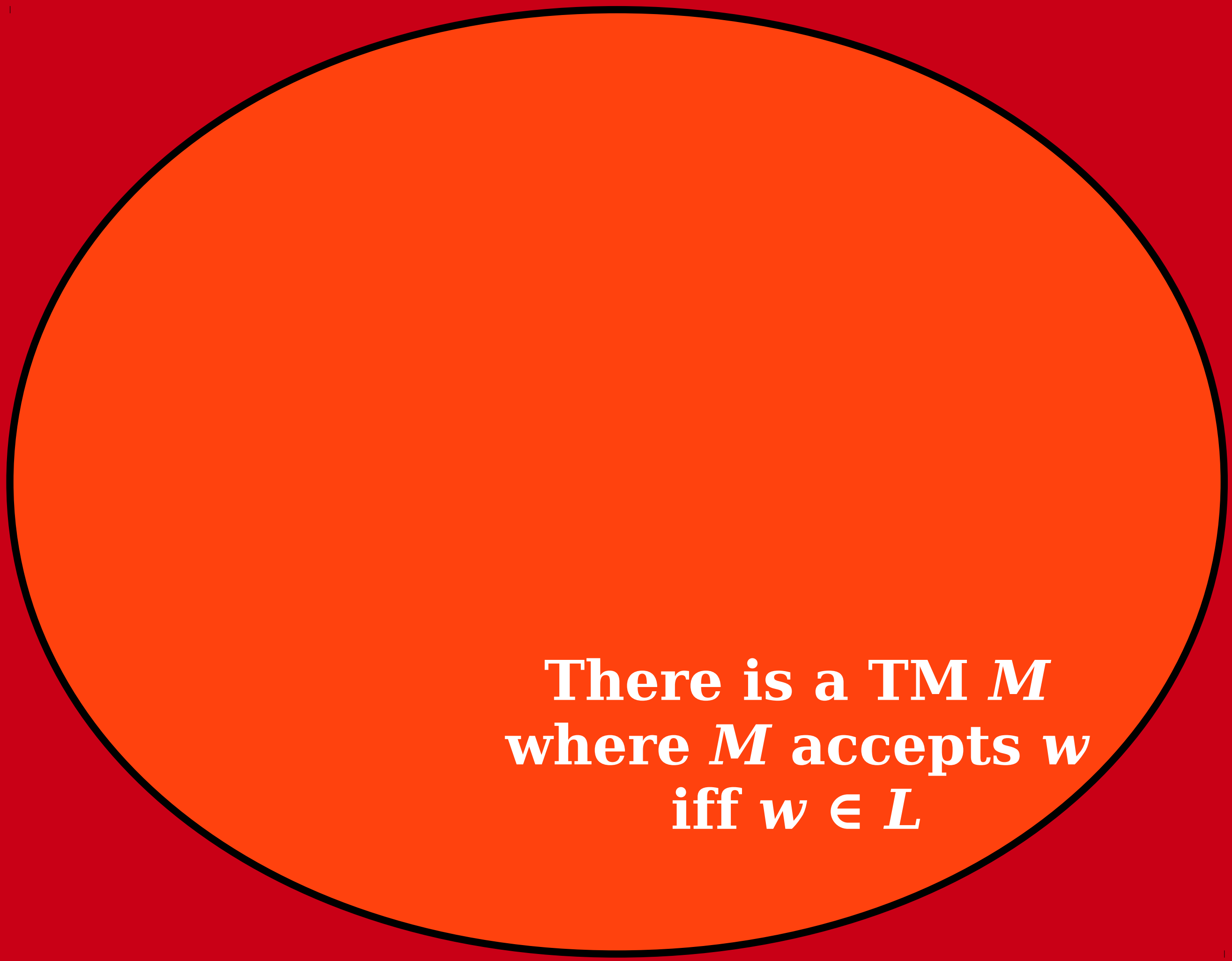

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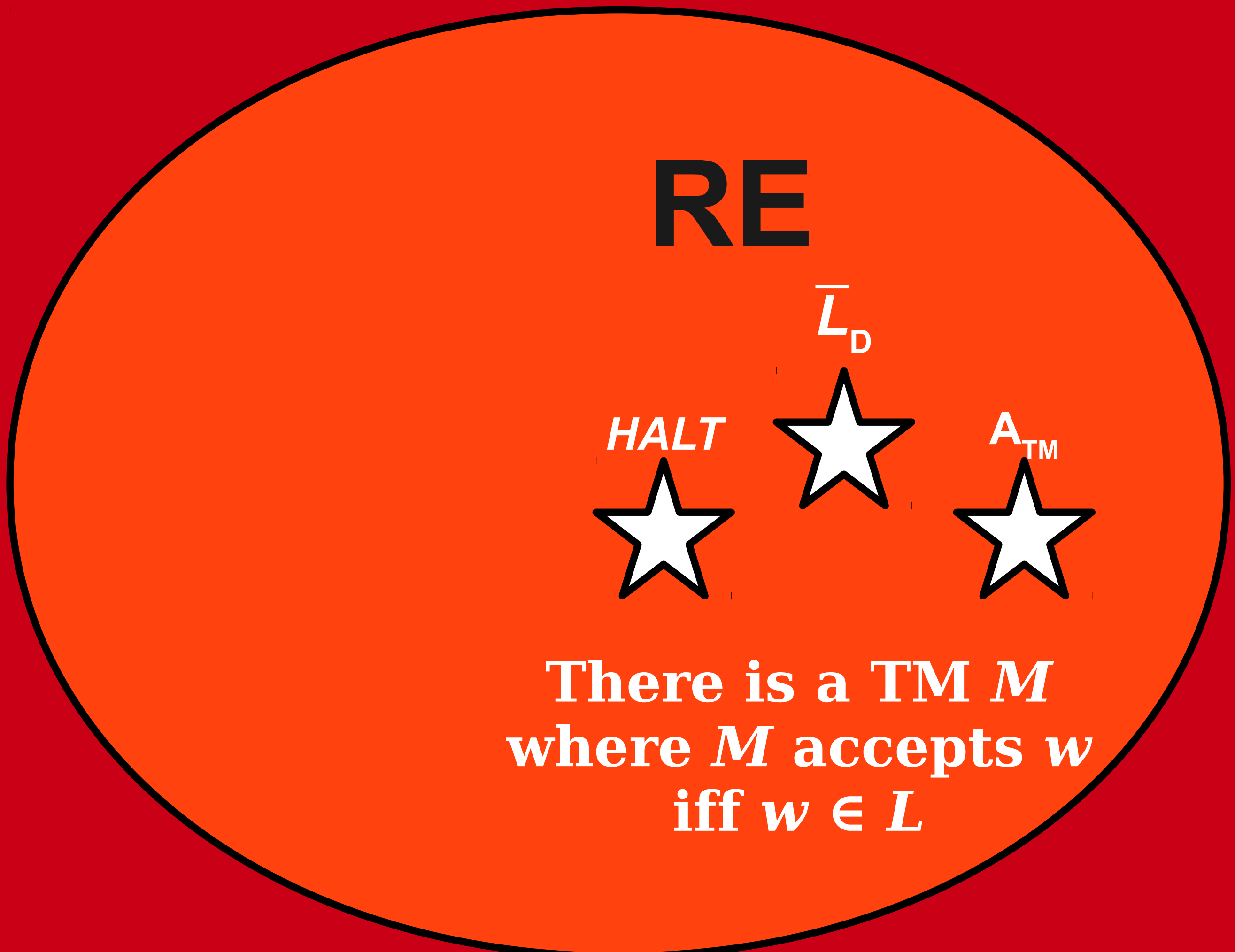

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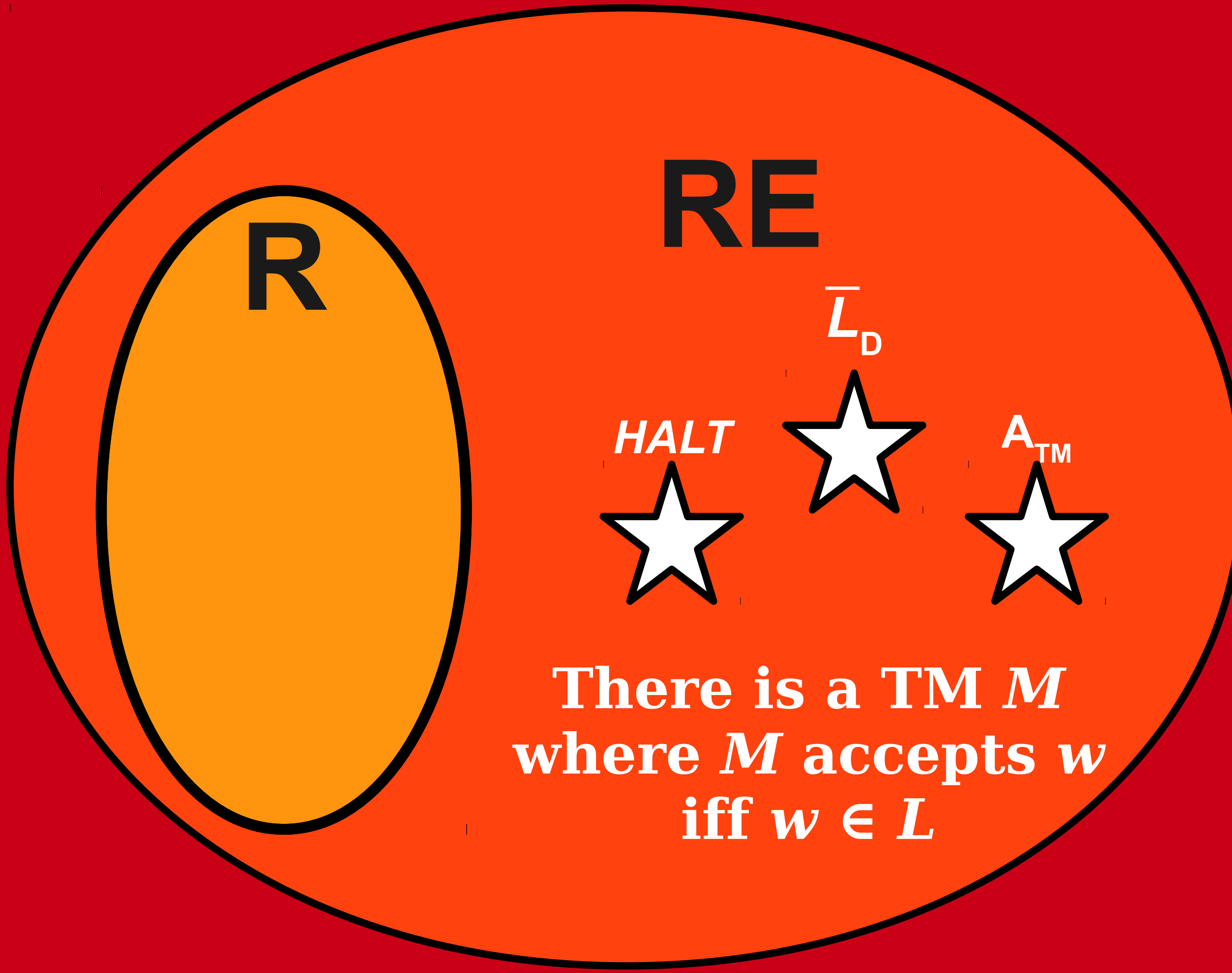

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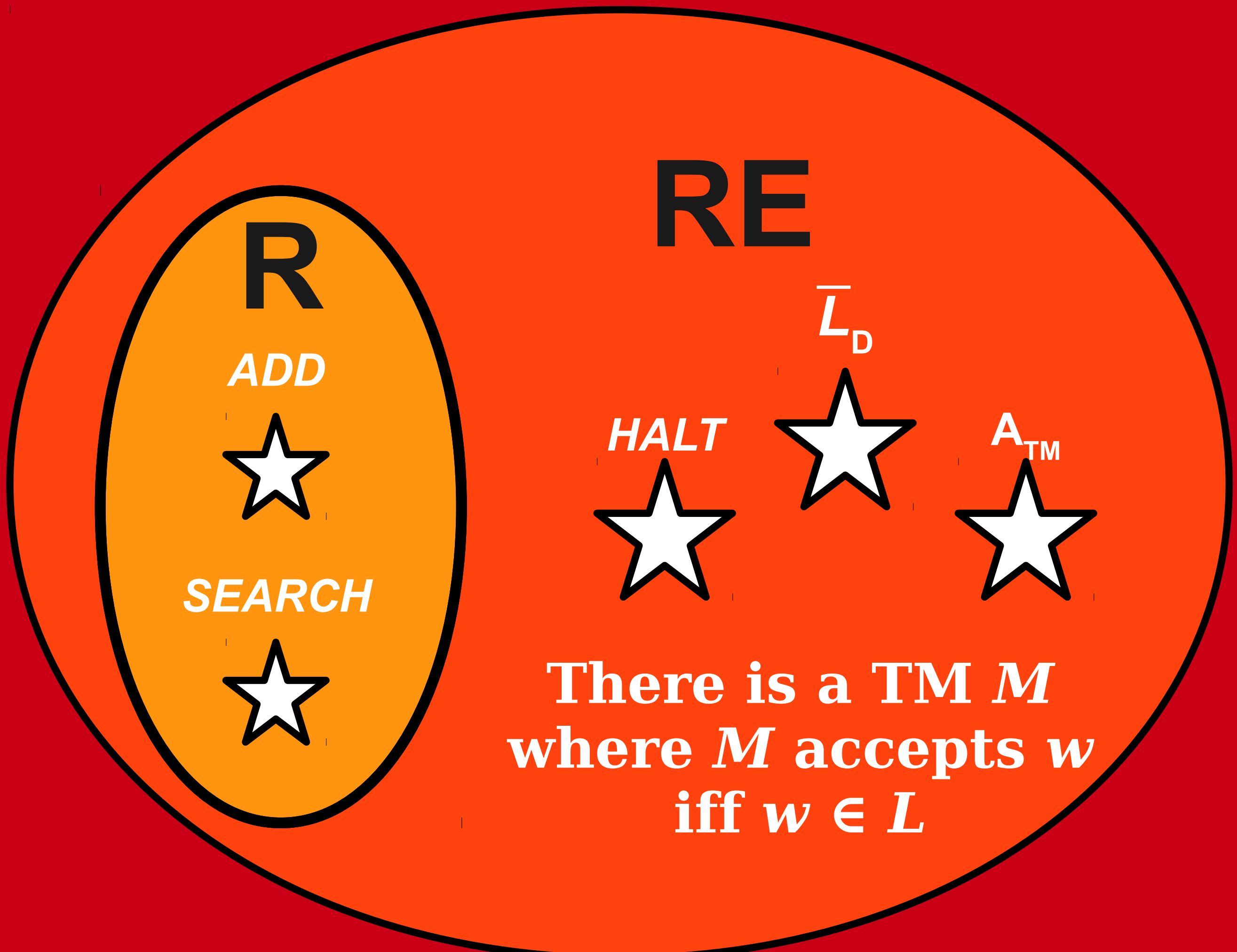

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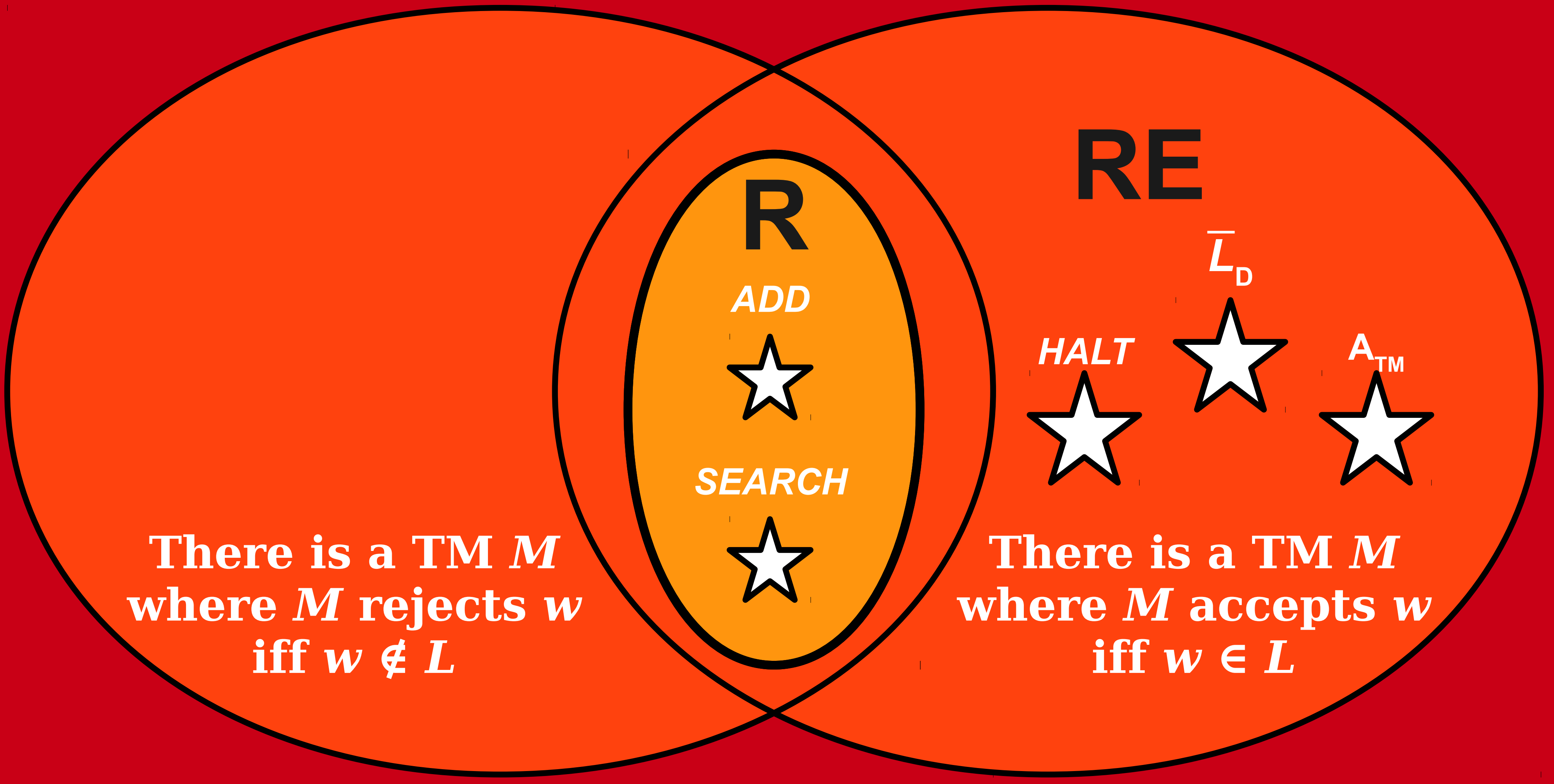

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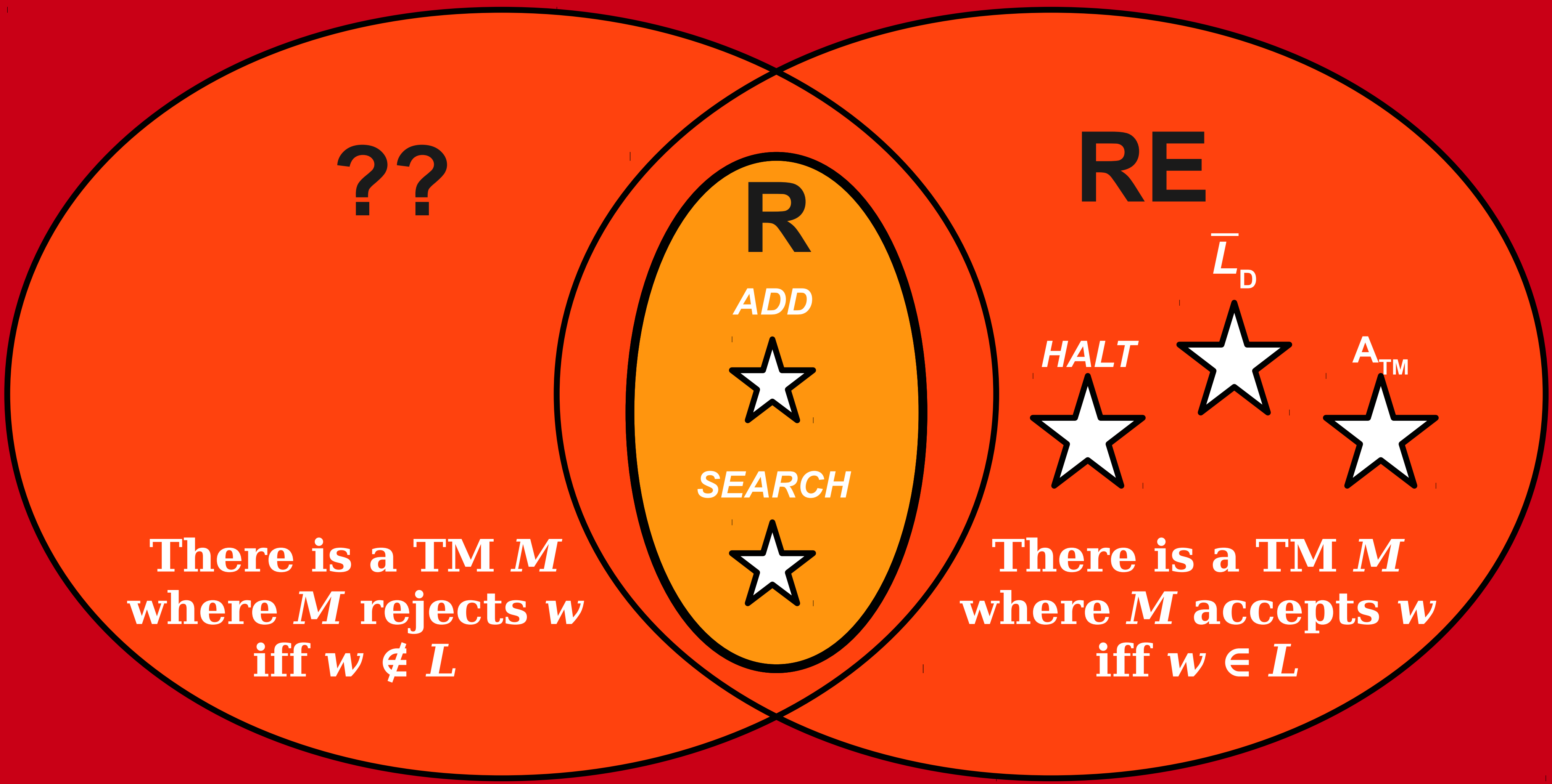

# A New Complexity Class

- A language $L$ is in **RE** iff there is a TM $M$ such that
  - if $w \in L$, then $M$ accepts $w$.
  - if $w \notin L$, then $M$ does not accept $w$.
- A TM $M$ of this sort is called a *recognizer*, and $L$ is called *recognizable*.
- A language $L$ is in co-**RE** iff there is a TM $M$ such that
  - if $w \in L$, then $M$ does not reject $w$.
  - if $w \notin L$, then $M$ rejects $w$.
- A TM M of this sort is called a ***co-recognizer***, and $L$ is called ***co-recognizable***.

# **RE** and co-**RE**

- Intuitively, **RE** consists of all problems where a TM can exhaustively search for proof that $w \in L$.
  - If $w \in L$, the TM will find the proof.
  - If $w \notin L$, the TM cannot find a proof.
- Intuitively, co-**RE** consists of all problems where a TM can exhaustively search for a disproof that $w \in L$.
  - If $w \in L$, the TM cannot find the disproof.
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# **RE** and co-**RE** Languages

- $A_{TM}$ is an **RE** language:
  - Simulate the TM $M$ on the string $w$.
  - If you find that $M$ accepts $w$, accept.
  - If you find that $M$ rejects $w$, reject.
  - (If $M$ loops, we implicitly loop forever)
- $\overline{A}_{TM}$ is a co-**RE** language:
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# **RE** and co-**RE** Languages

- $\overline{L}_{\mathrm{D}}$ is an **RE** language.
  - Simulate $M$ on $\langle M \rangle$.
  - If you find that $M$ accepts $\langle M \rangle$, accept.
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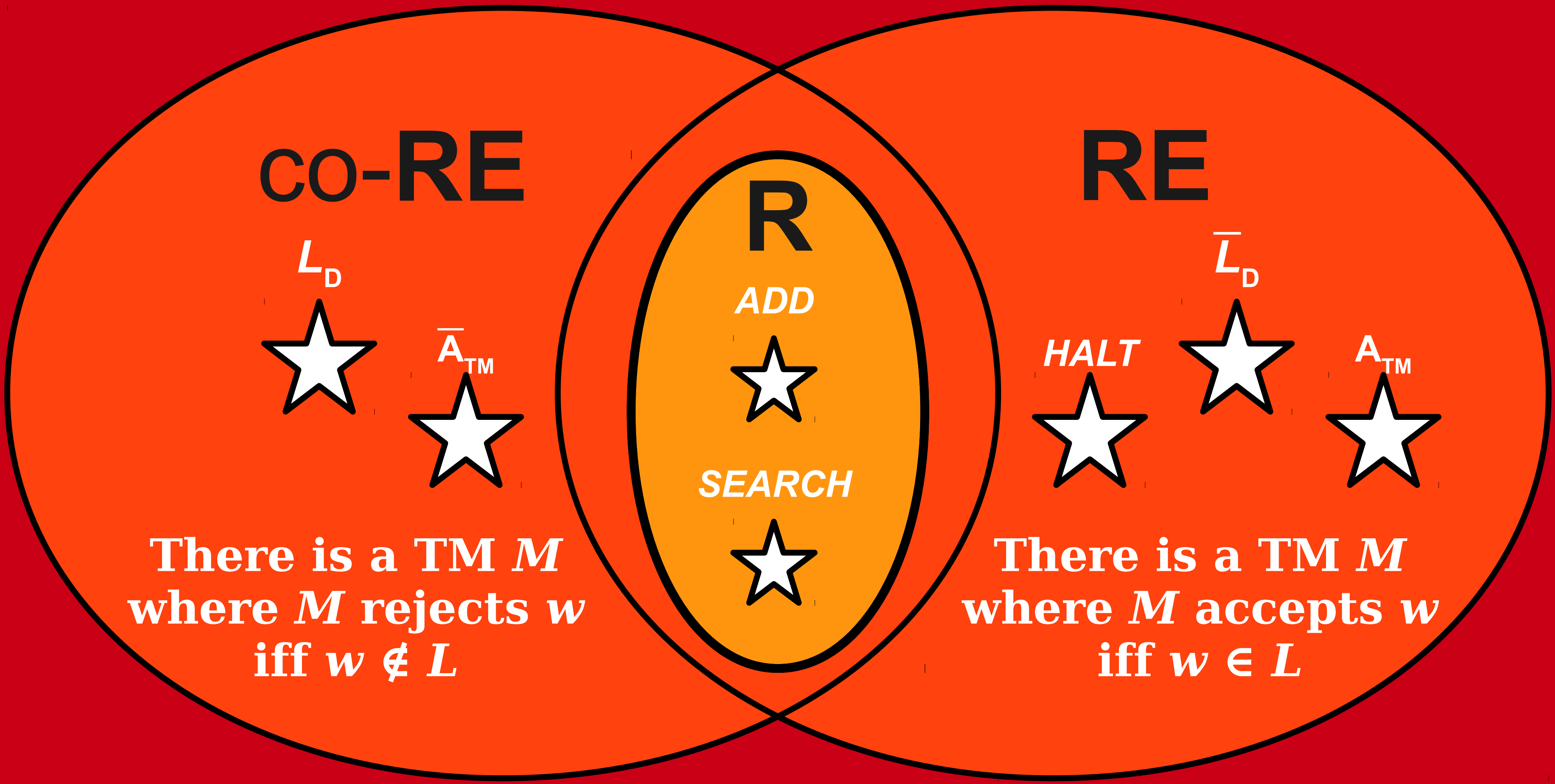
# The Limits of Computability



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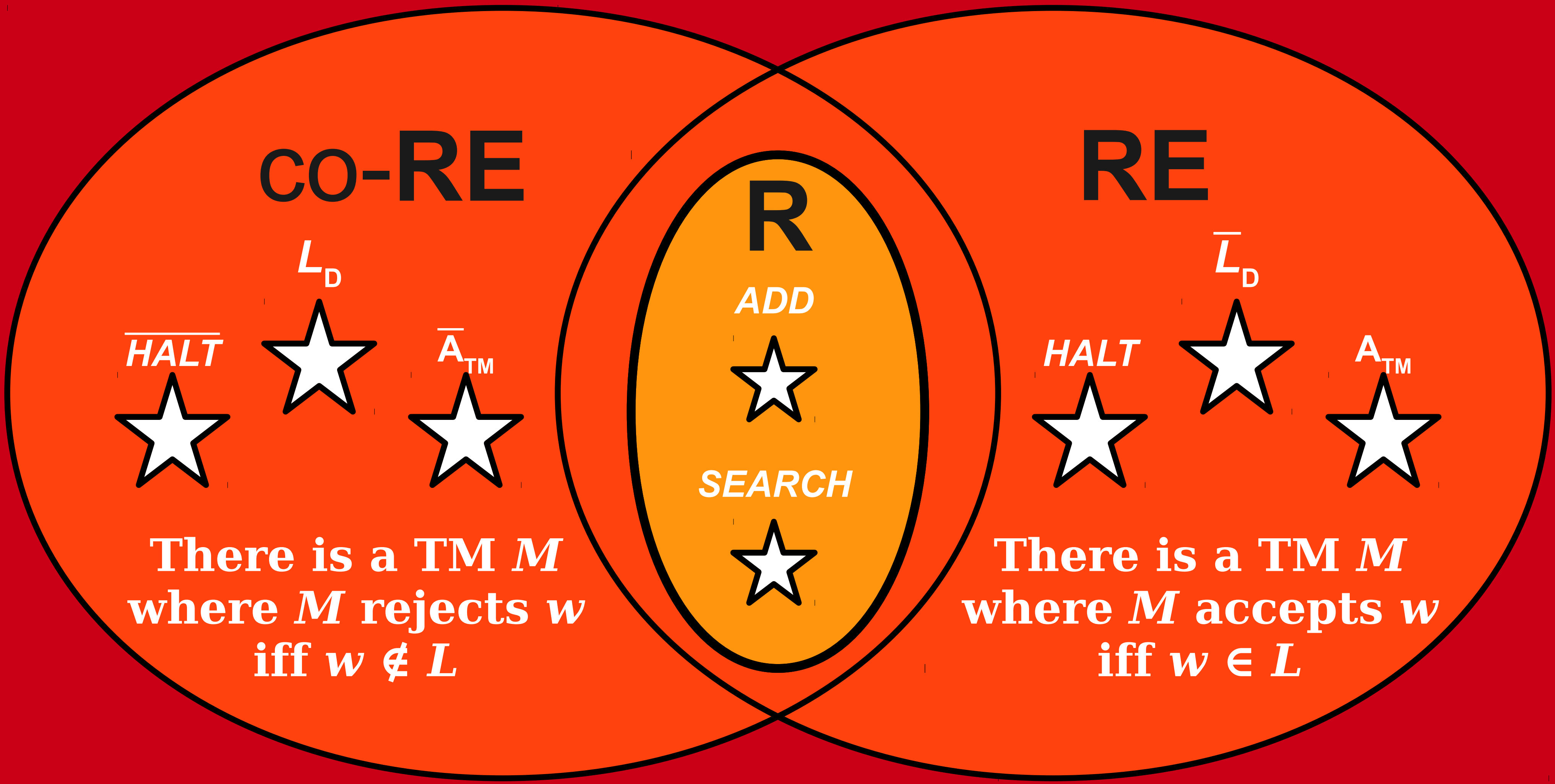
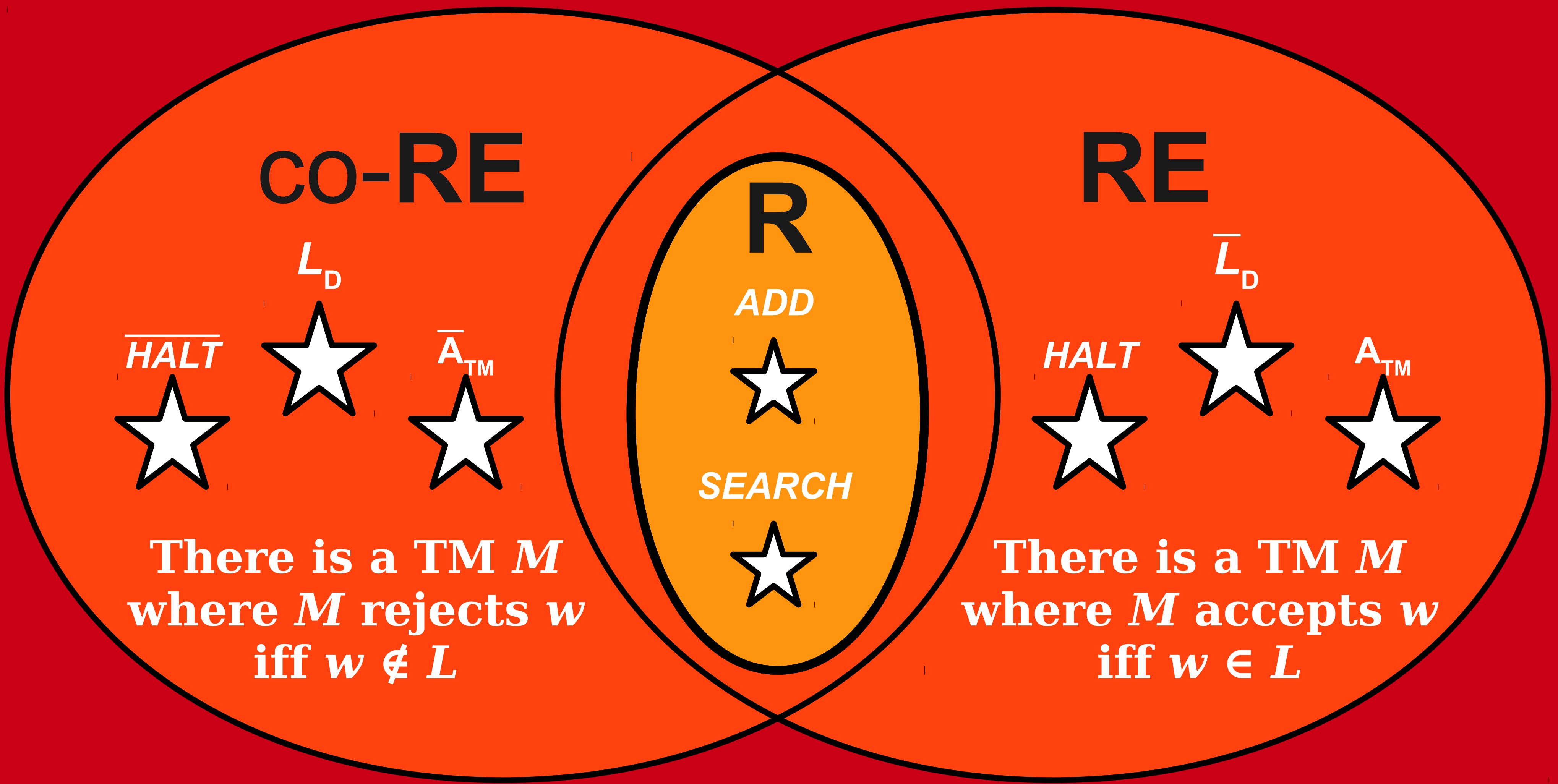
# The Limits of Computability



# **R**, **RE**, and co-**RE**

- Every language in **R** is in both **RE** and co-**RE**.
- Why?
  - A decider for $L$ accepts all $w \in L$ and rejects all $w \notin L$.
- In other words, $\mathbf{R} \subseteq \mathbf{RE} \cap \text{co-}\mathbf{RE}$.
- **Question:** Does $\mathbf{R} = \mathbf{RE} \cap \text{co-}\mathbf{RE}$?

# Which Picture is Correct?



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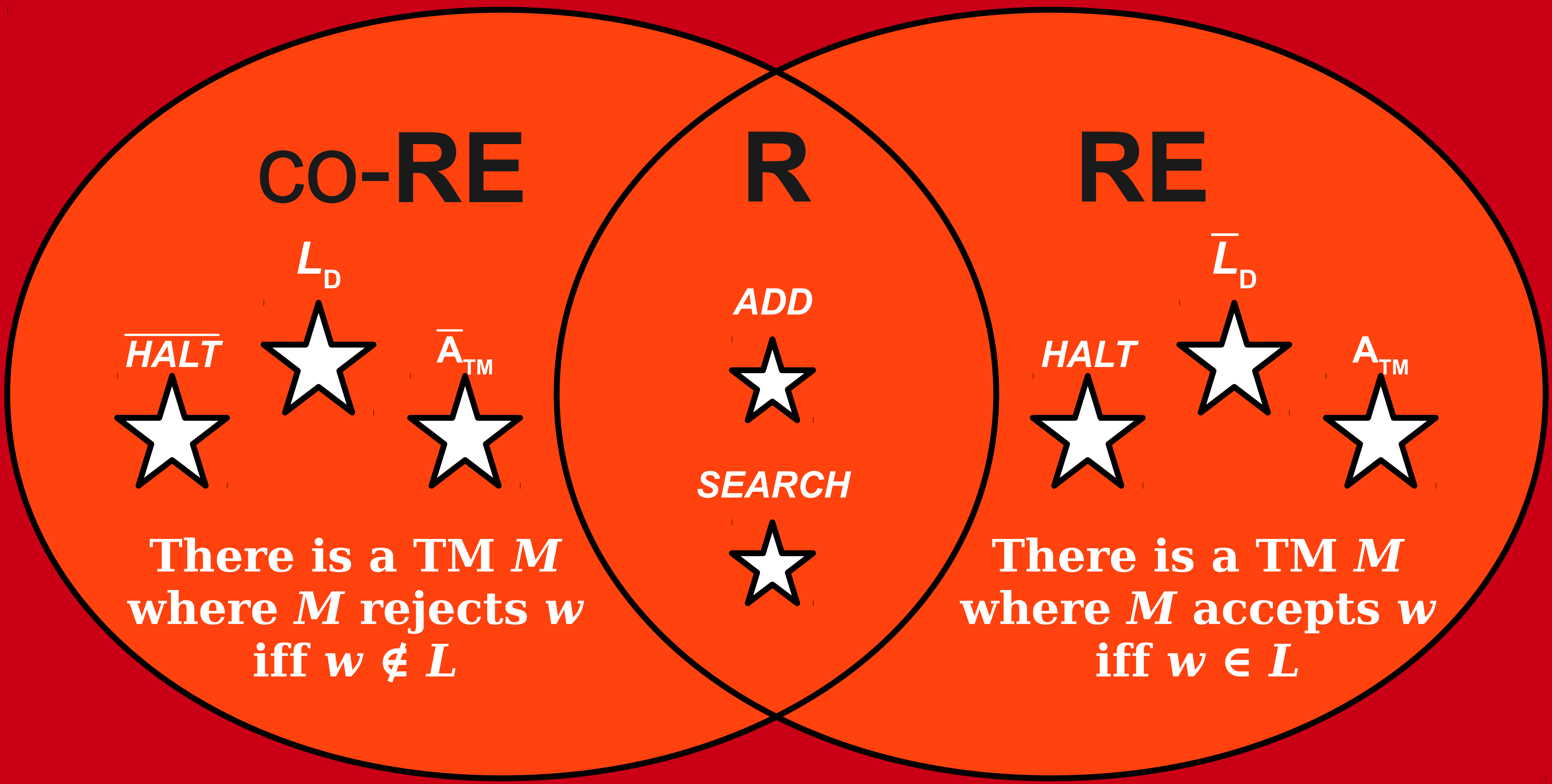

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  This TM $D$ is a decider for $L$:

# $\mathbf{R}$, $\mathbf{RE}$, and co-$\mathbf{RE}$

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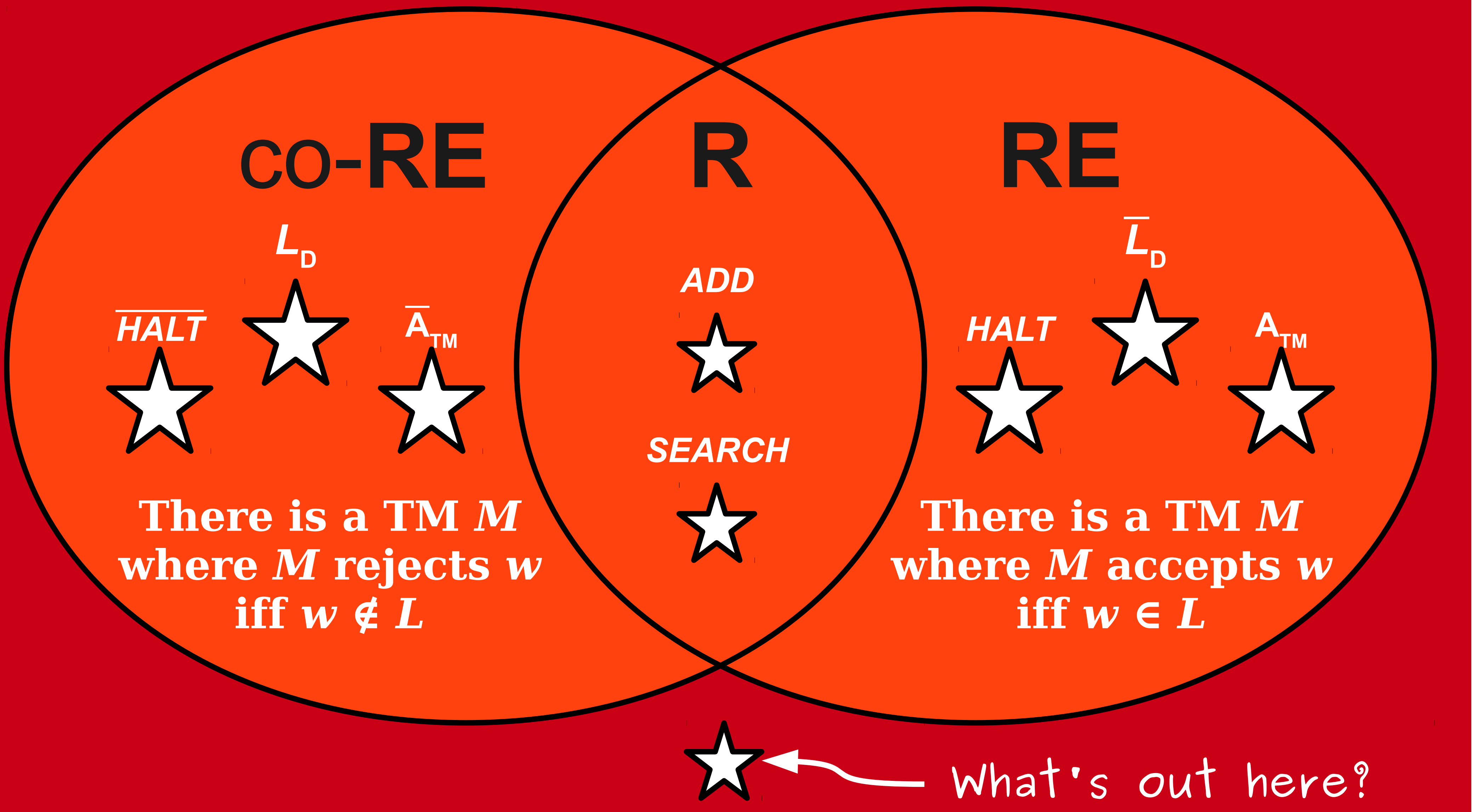
  This TM $D$ is a decider for $L$:

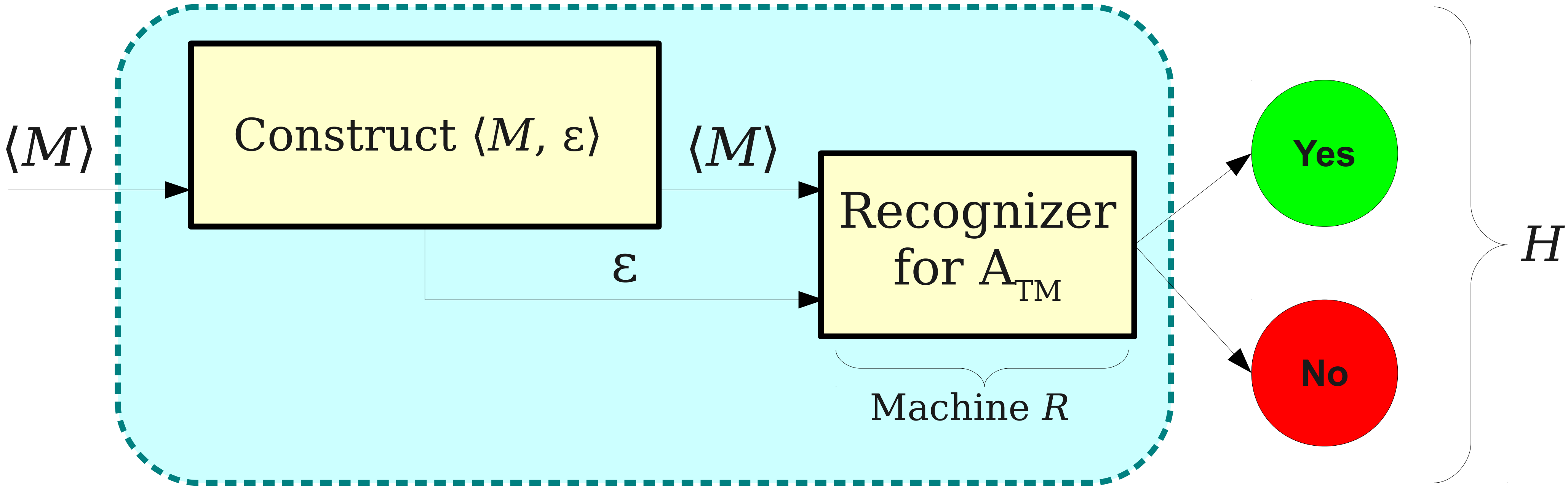
  > $D$ = "On input $w$:
  > Run $M$ on $w$ and $\overline{M}$ on $w$ in parallel.
  > If $M$ accepts $w$, accept.
  > If $\overline{M}$ rejects $w$, reject.

# The Limits of Computability



# A Repeating Pattern

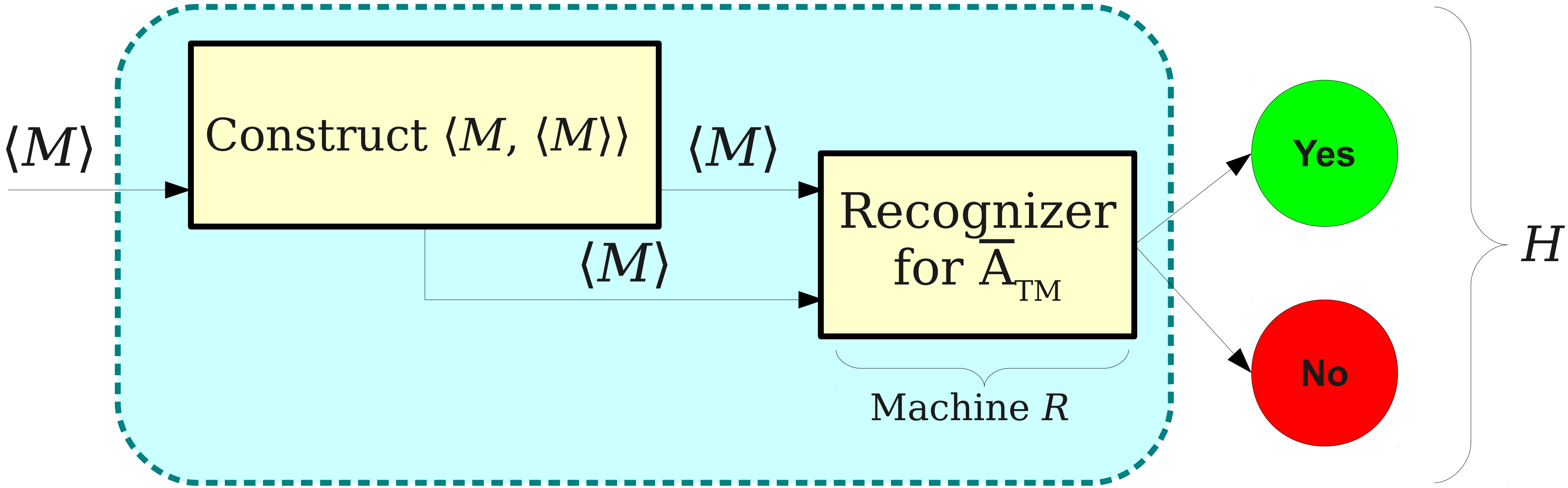
$$L = \{ \langle M \rangle \mid M \text{ is a TM that accepts } \varepsilon \}$$




$H$ = "On input $\langle M \rangle$:

- Construct the string $\langle M, \varepsilon \rangle$.
- Run $R$ on $\langle M, \varepsilon \rangle$.
- If $R$ accepts $\langle M, \varepsilon \rangle$, then $H$ accepts $\langle M, \varepsilon \rangle$.
- If $R$ rejects $\langle M, \varepsilon \rangle$, then $H$ rejects $\langle M, \varepsilon \rangle$."

# From $\overline{\mathrm{A}}_{\mathrm{TM}}$ to $L_{\mathrm{D}}$

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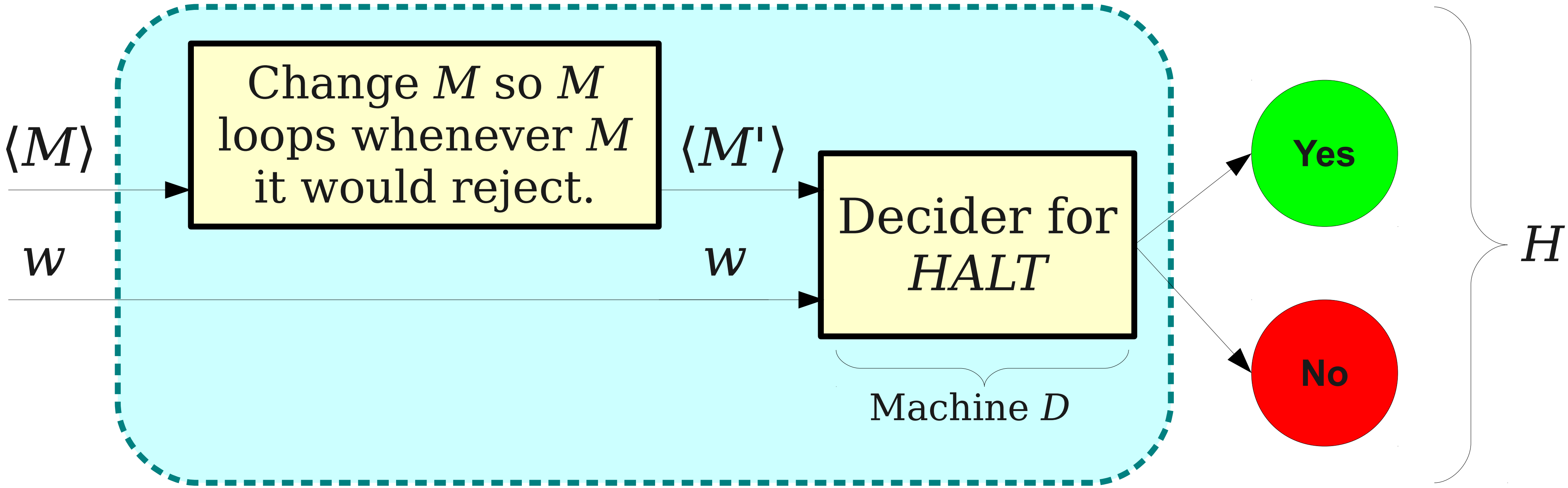
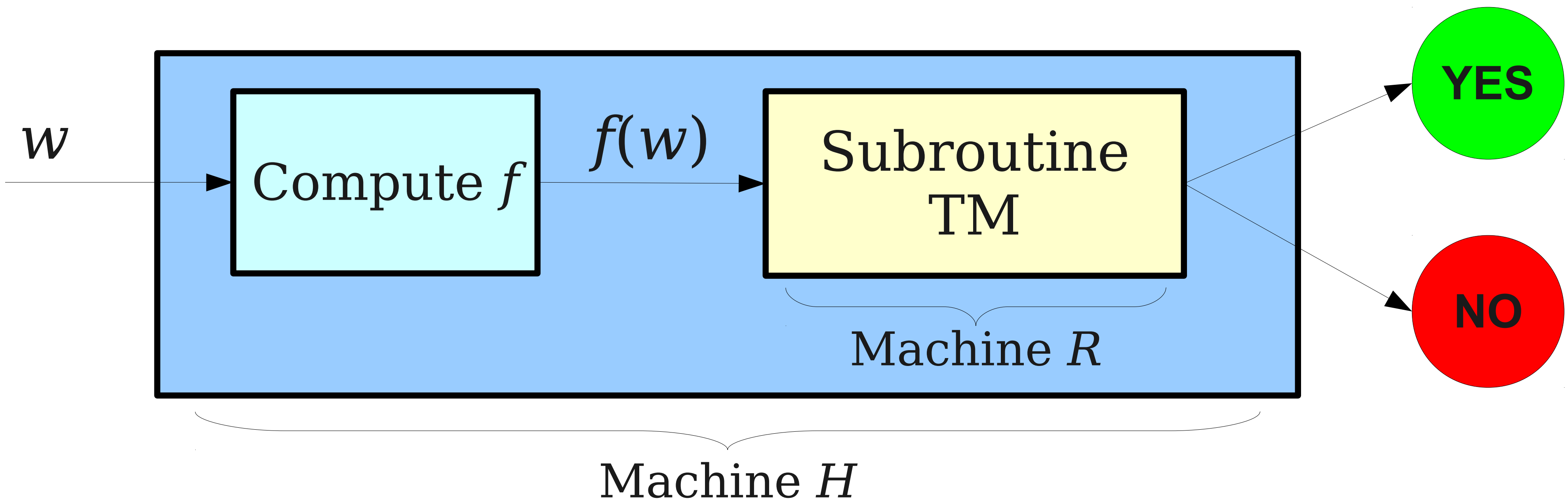
# From *HALT* to $A_{TM}$




$H$ = "On input ⟨$M$, $w$⟩:

- Build $M$ into $M'$ so $M'$ loops when $M$ rejects.
- Run $D$ on ⟨$M'$, $w$⟩.
- If $D$ accepts ⟨$M'$, $w$⟩, then $H$ accepts ⟨$M$, $w$⟩.
- If $D$ rejects ⟨$M'$, $w$⟩, then $H$ rejects ⟨$M$, $w$⟩."

# The General Pattern



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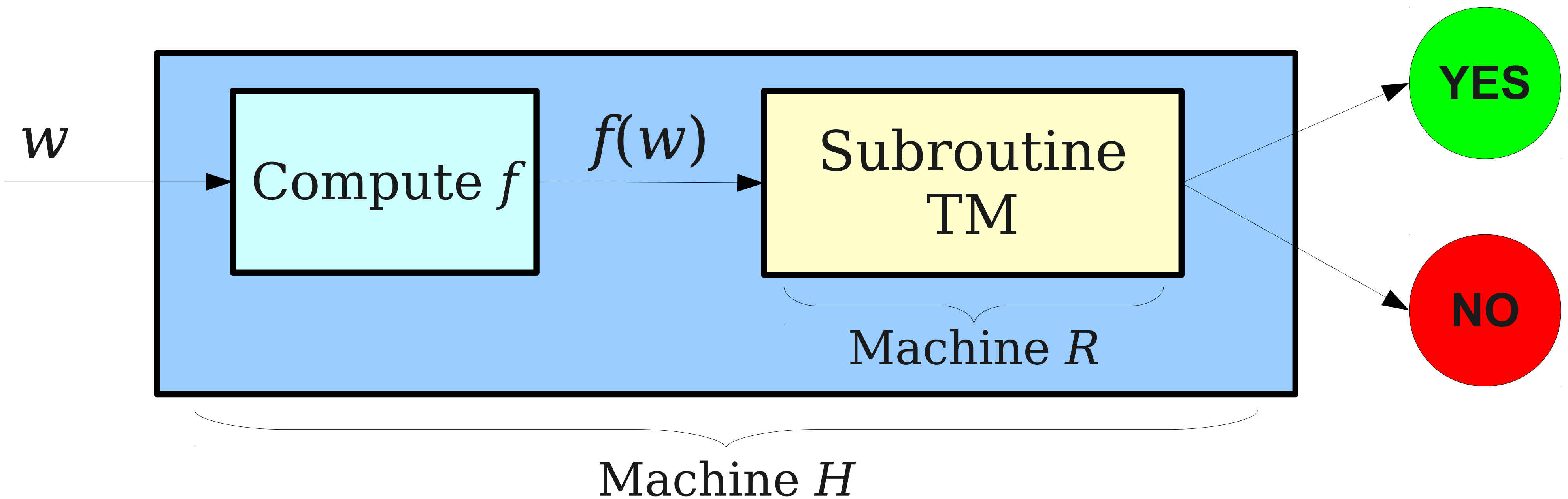
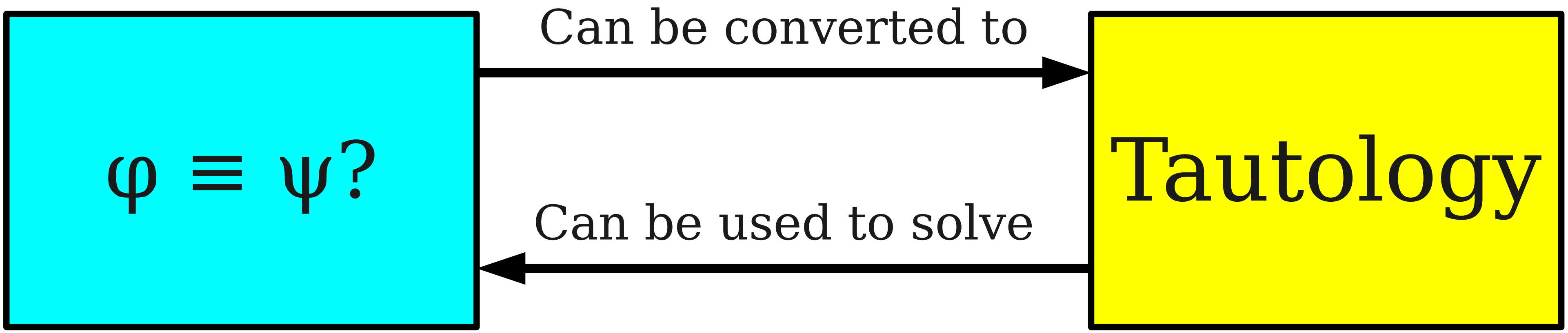



$H$ = "On input $w$:

- Transform the input $w$ into $f(w)$.
- Run machine $R$ on $f(w)$.
- If $R$ accepts $f(w)$, then $H$ accepts $w$.
- If $R$ rejects $f(w)$, then $H$ rejects $w$."

# Reductions

- Intuitively, problem $A$ **reduces** to problem $B$ iff a solver for $B$ can be used to solve problem $A$.



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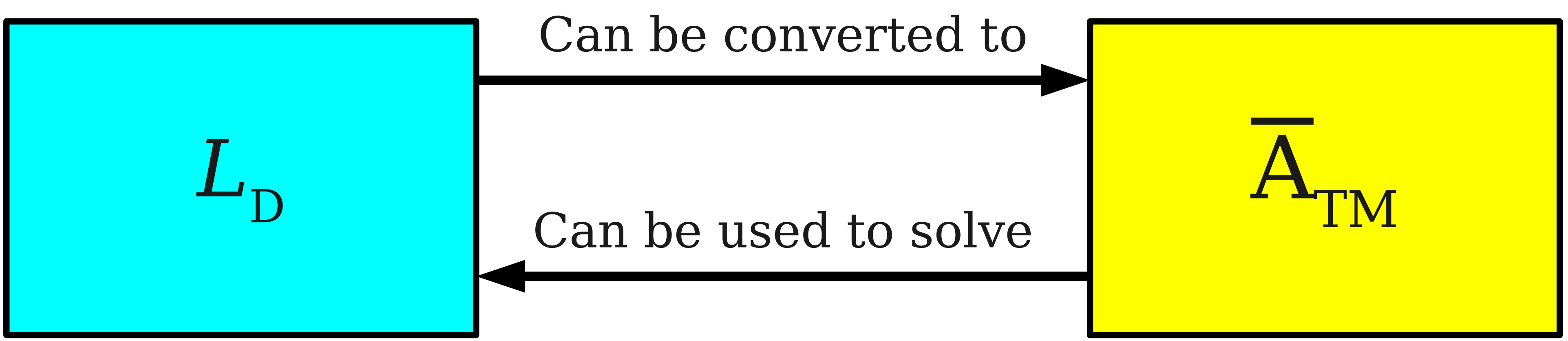

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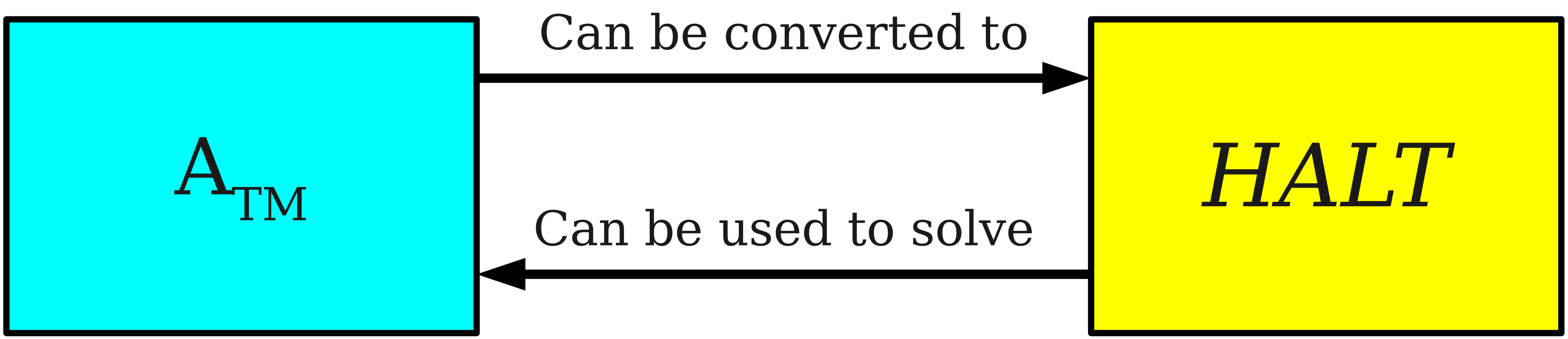


# Reductions

- Intuitively, problem $A$ **reduces** to problem $B$ iff a solver for $B$ can be used to solve problem $A$.
- Reductions can be used to show certain problems are "solvable:"

**If $A$ reduces to $B$ and $B$ is "solvable," then $A$ is "solvable."**

- Reductions can be used to show certain problems are "*unsolvable*:"

**If $A$ reduces to $B$ and $A$ is "unsolvable," then $B$ is "unsolvable."**

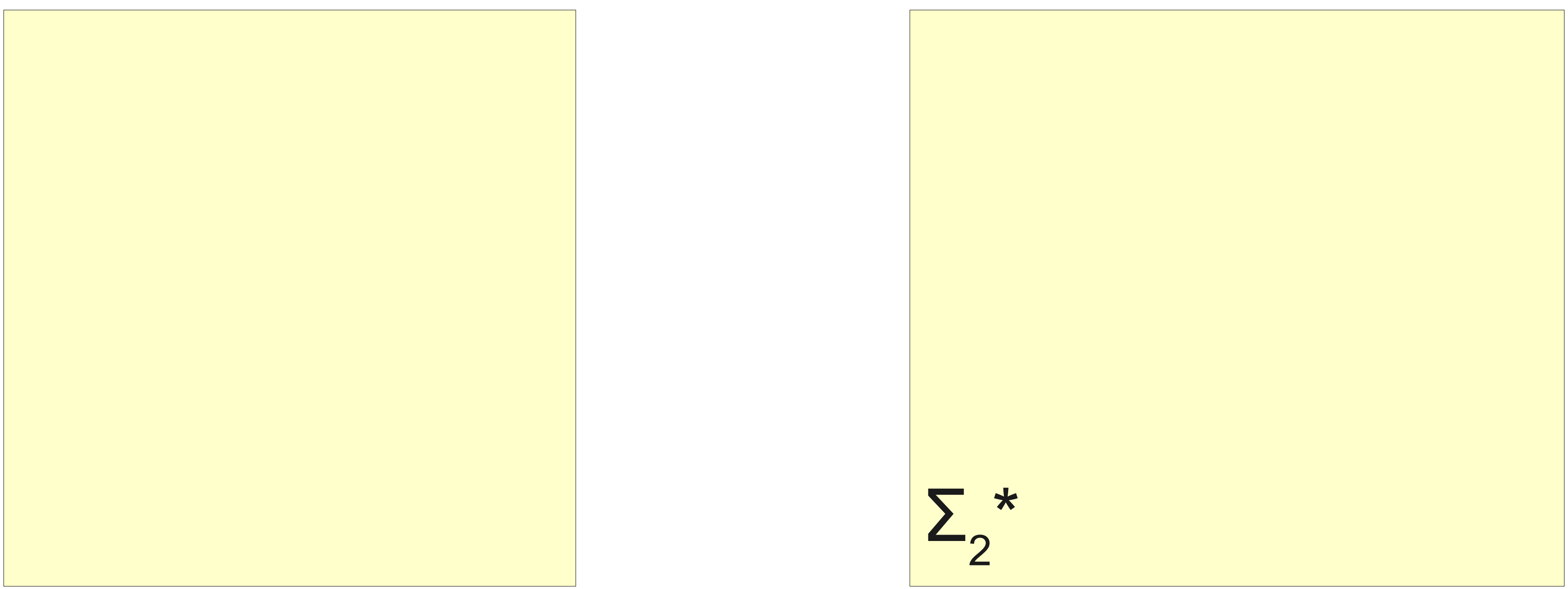
# Formalizing Reductions

- In order to make the previous intuition more rigorous, we need to formally define reductions.
- There are many ways to do this; we'll explore two:
  - **Mapping reducibility** (today / Monday), and
  - **Polynomial-time reducibility** (next week).

# Defining Reductions

- A **reduction** from $A$ to $B$ is a function $f : \Sigma_1^* \to \Sigma_2^*$ such that

**For any $w \in \Sigma_1^*$, $w \in A$ iff $f(w) \in B$**



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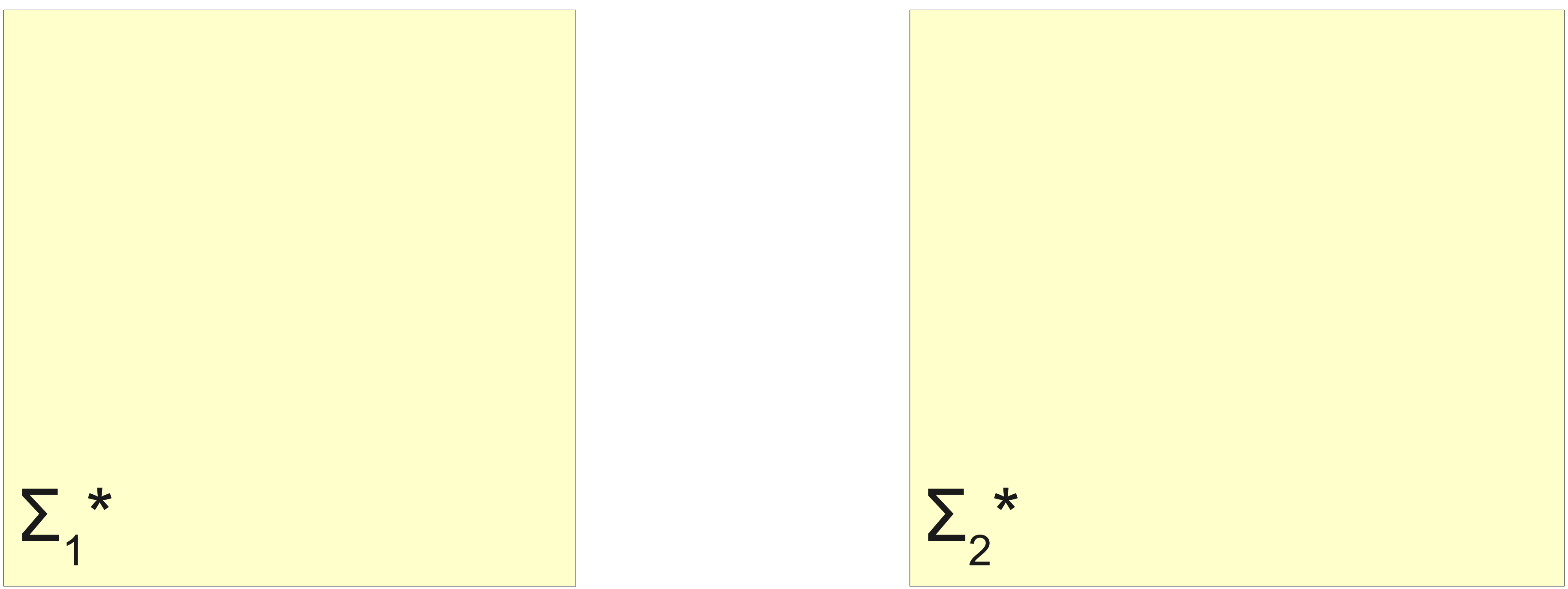

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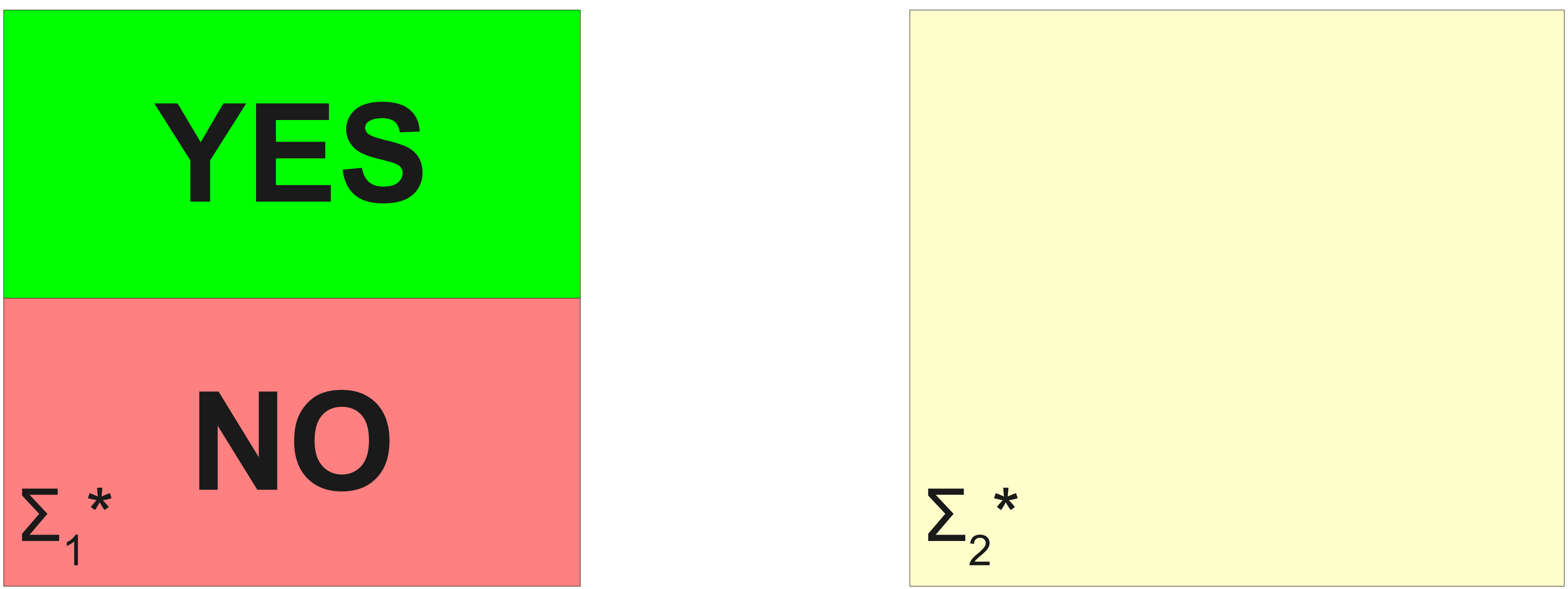

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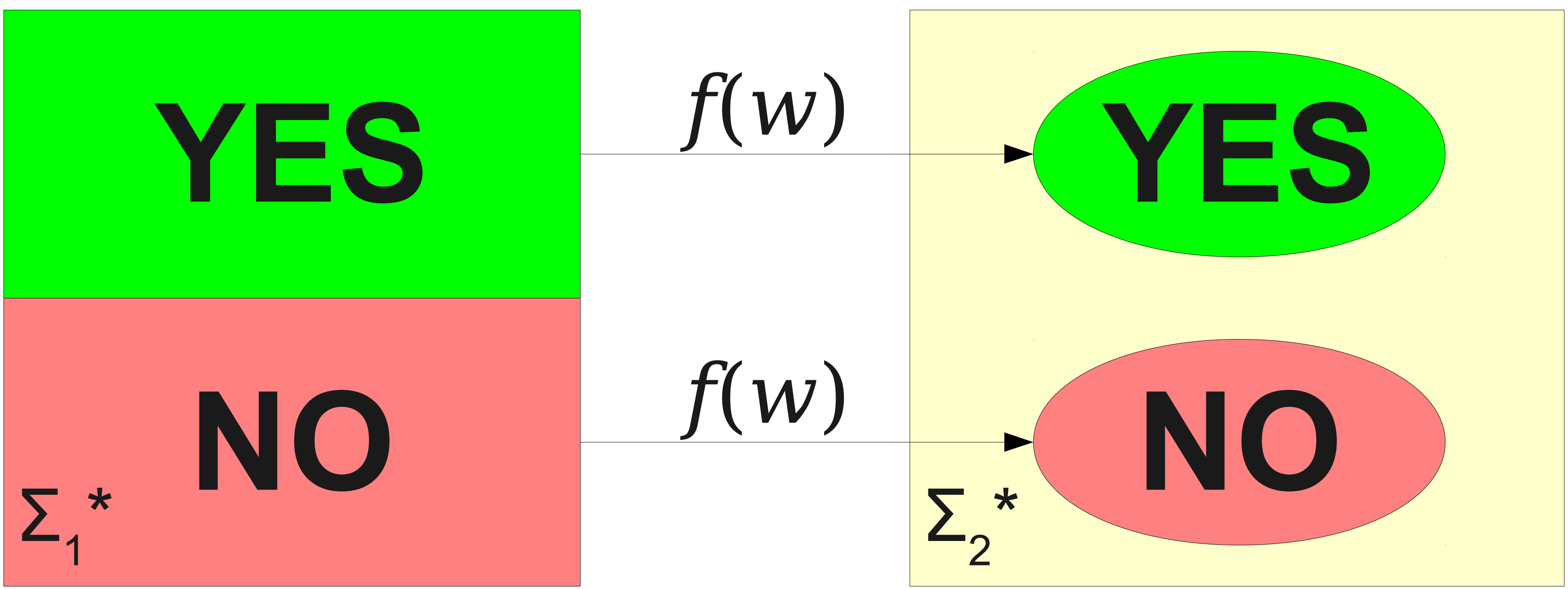

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**For any $w \in \Sigma_1^*$, $w \in A$ iff $f(w) \in B$**

- Every $w \in A$ maps to some $f(w) \in B$.
- Every $w \notin A$ maps to some $f(w) \notin B$.
- $f$ does not have to be injective or surjective.

# Why Reductions Matter

- If language $A$ reduces to language $B$, we can use a recognizer / co-recognizer / decider for $B$ to recognize / co-recognize / decide problem $A$.
  - (There's a slight catch – we'll talk about this in a second).
- How is this possible?

$$w \in A \quad \text{iff} \quad f(w) \in B$$

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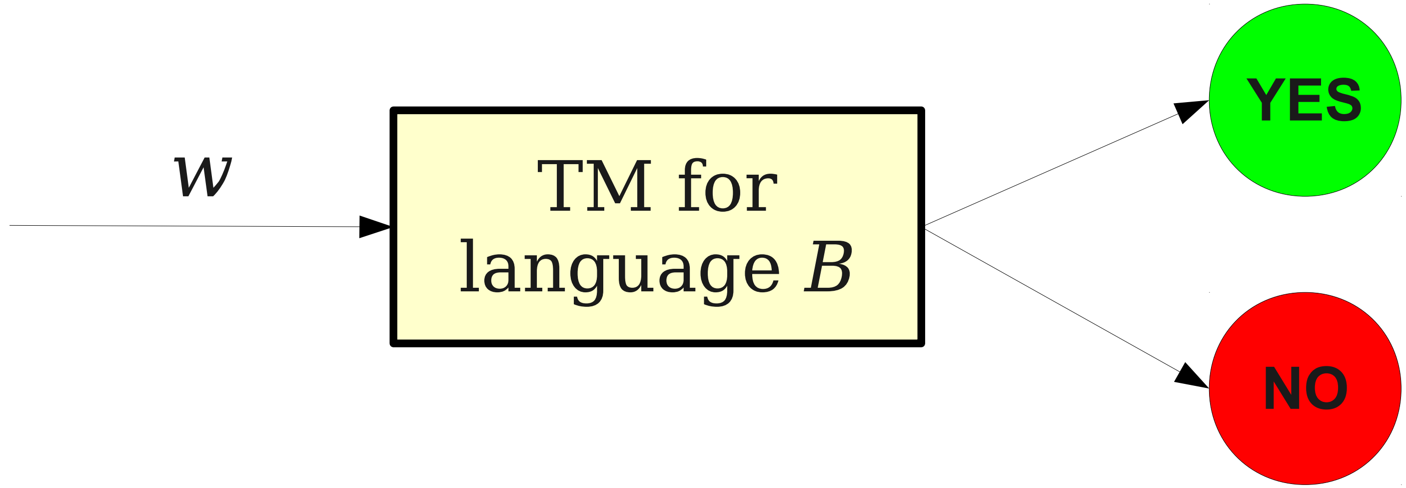

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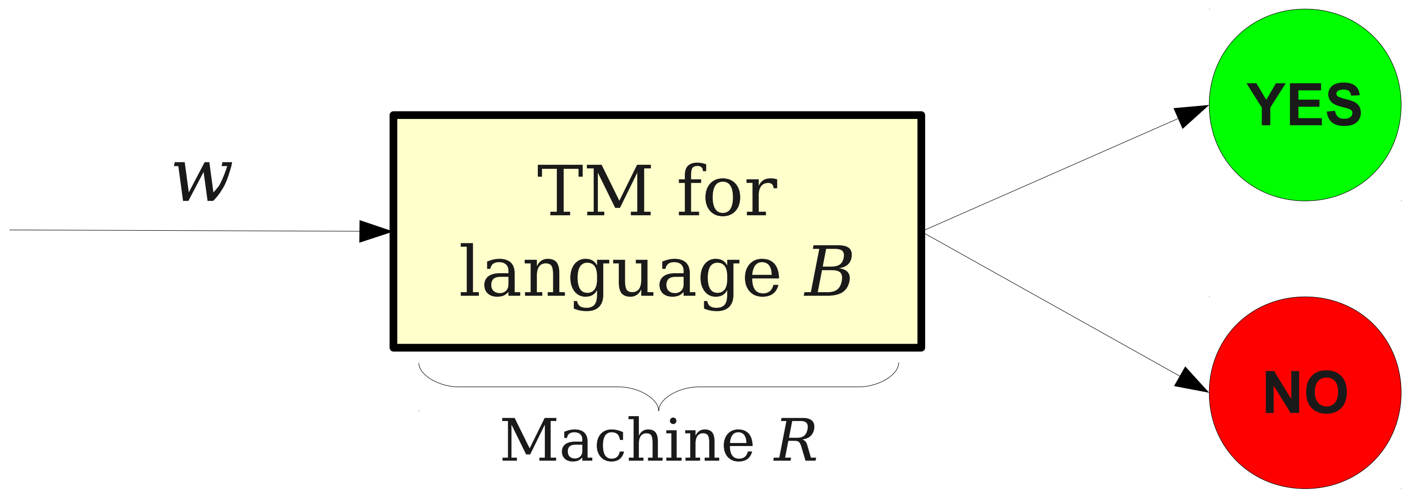

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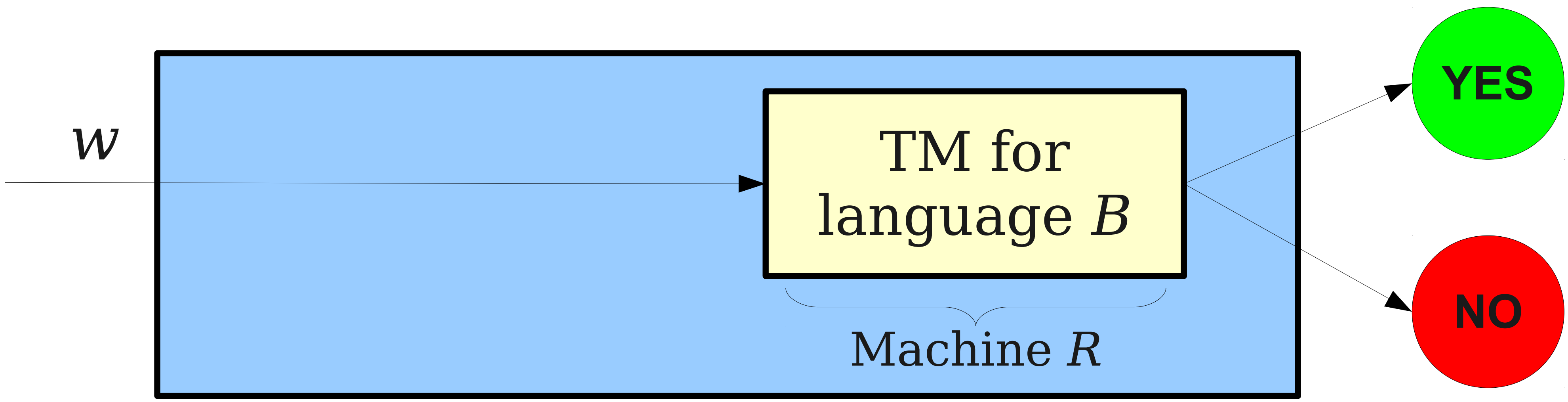
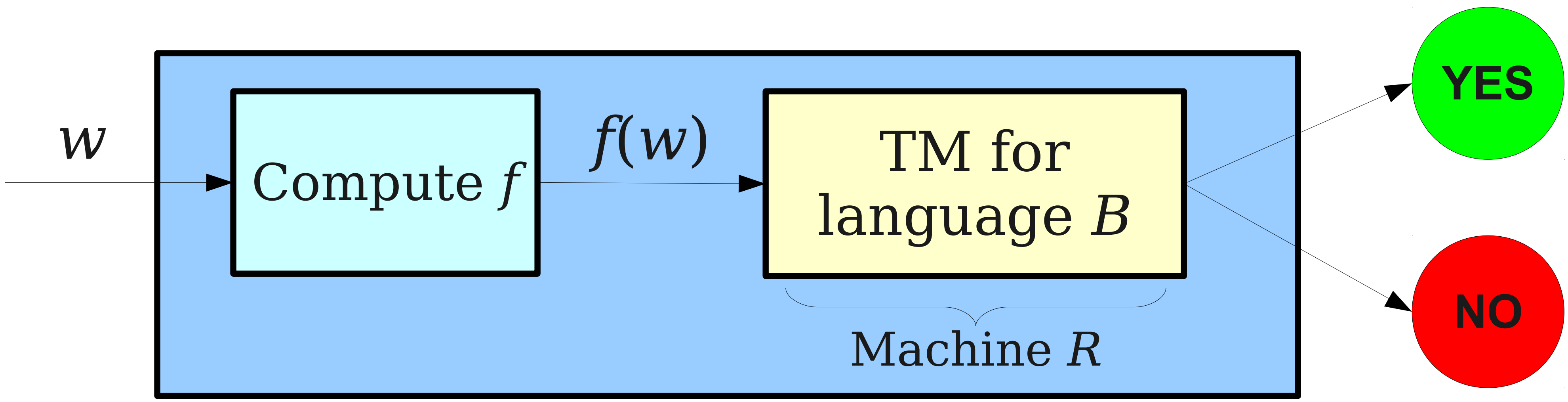


## $w \in A$ iff $f(w) \in B$



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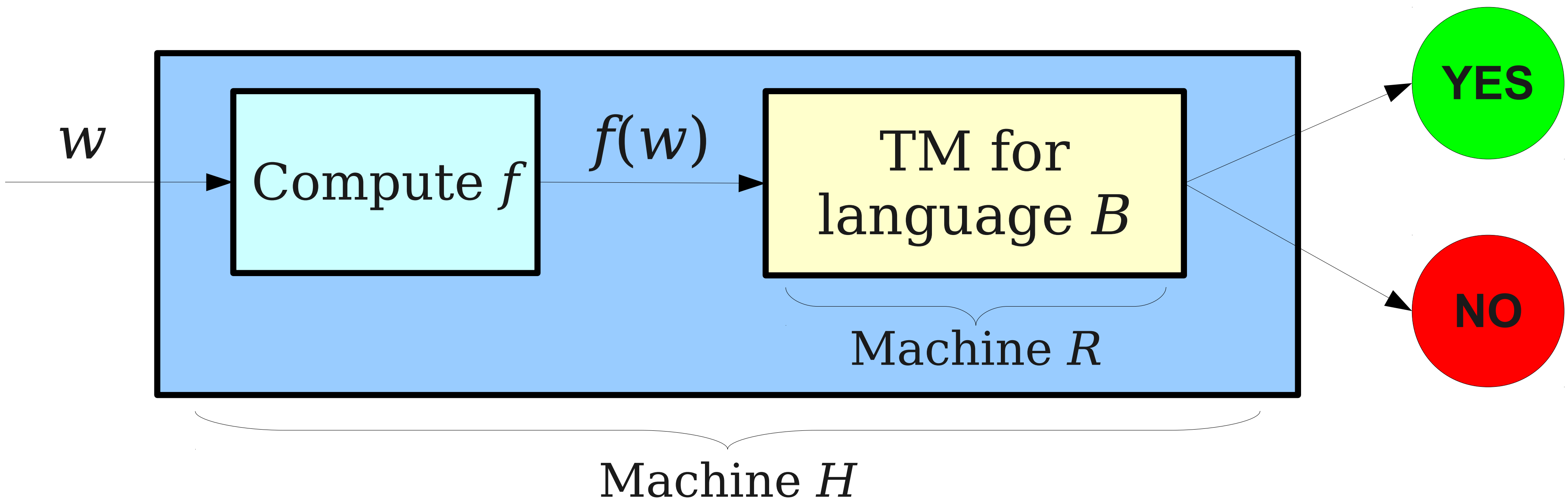

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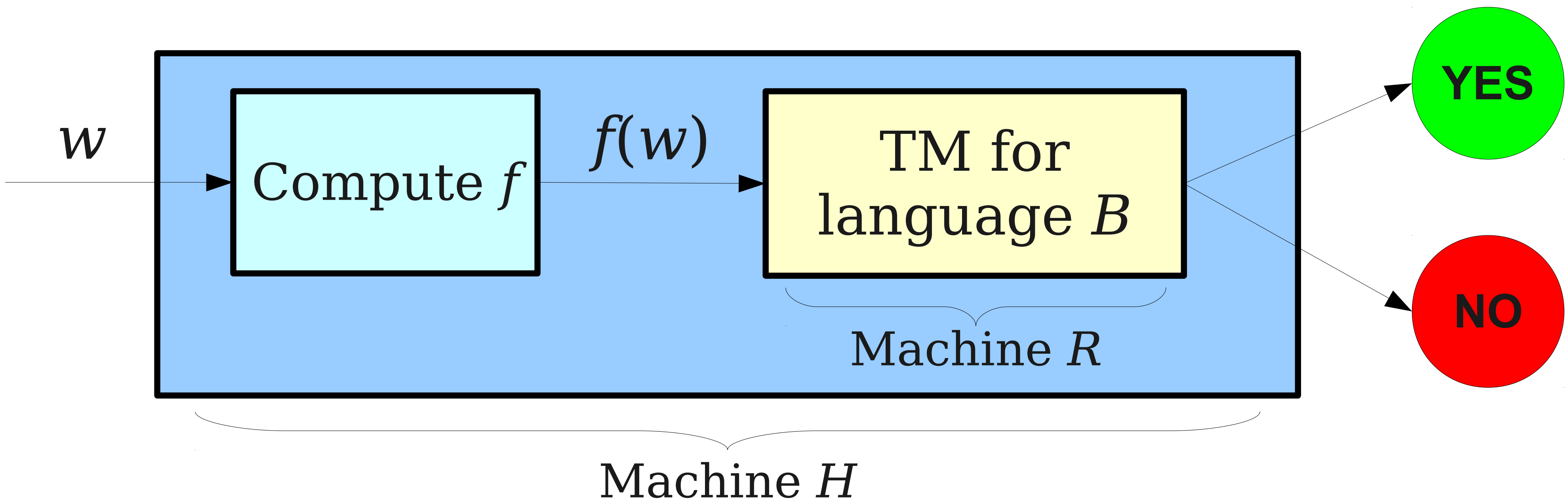


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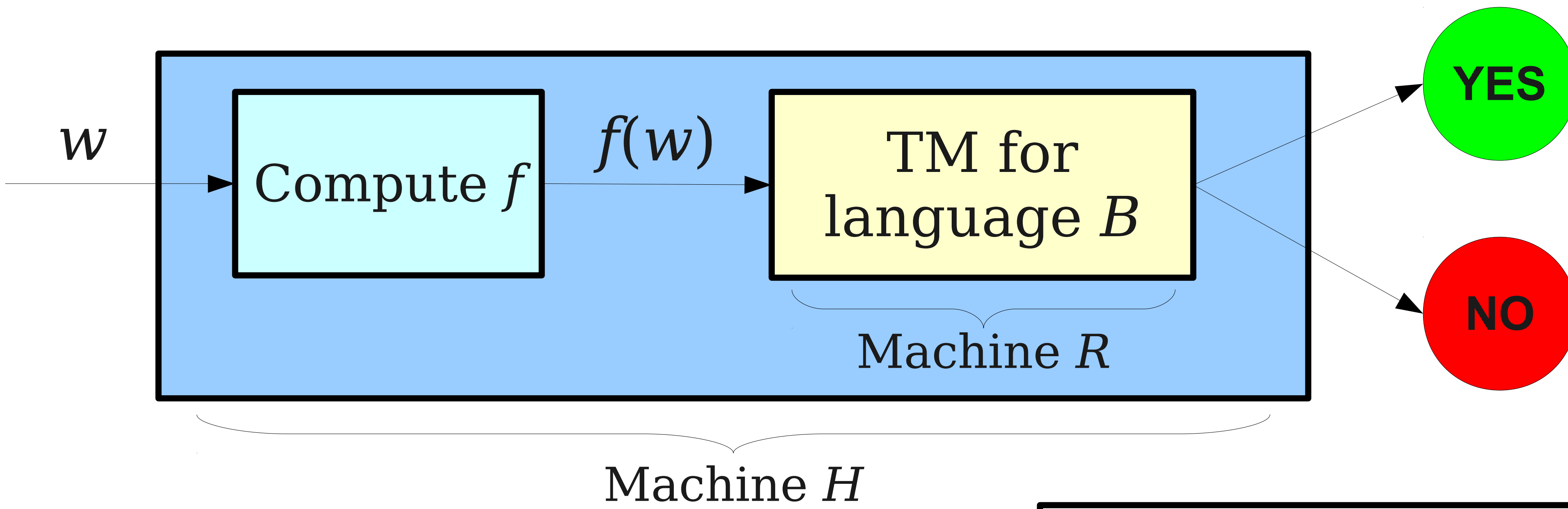


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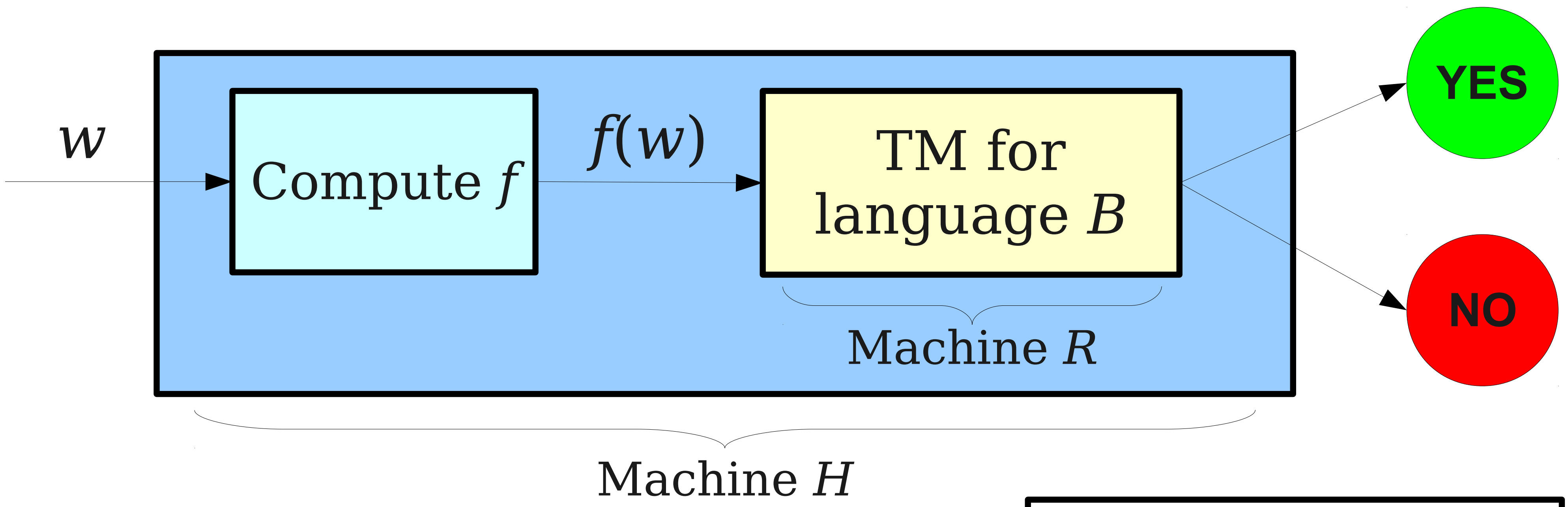
**$H$ accepts $w$**

## $w \in A$ iff $f(w) \in B$

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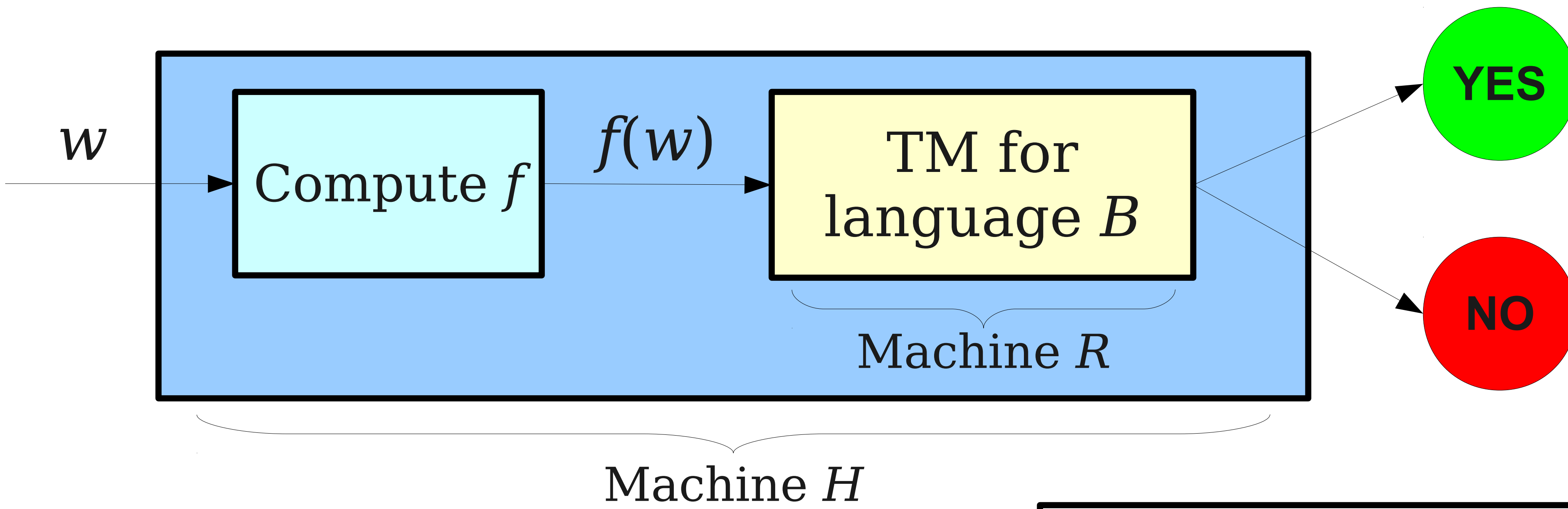
***H* accepts *w***
iff
***R* accepts *f*(*w*)**

$$w \in A \quad \text{iff} \quad f(w) \in B$$

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- If $R$ rejects $f(w)$, then $H$ rejects $w$."

***H* accepts *w***

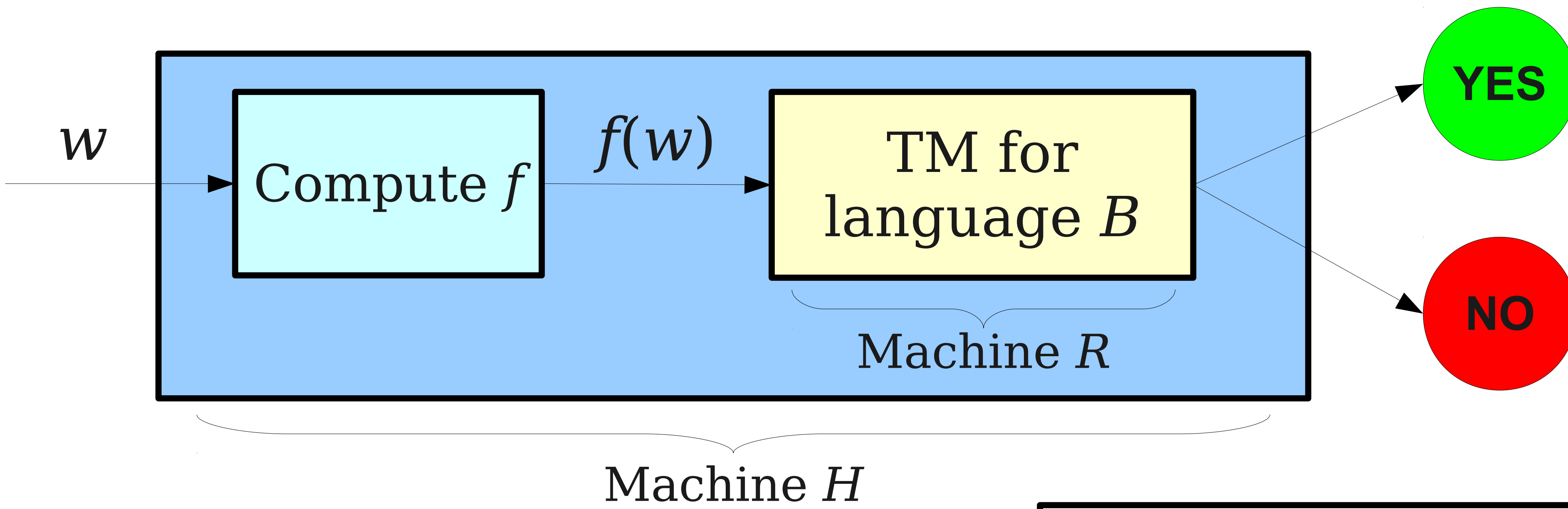
iff

***R* accepts *f*(*w*)**

iff

$\boldsymbol{f(w) \in B}$

## $w \in A$ iff $f(w) \in B$




$H$ = “On input $w$:

- Transform the input $w$ into $f(w)$.
- Run machine $R$ on $f(w)$.
- If $R$ accepts $f(w)$, then $H$ accepts $w$.
- If $R$ rejects $f(w)$, then $H$ rejects $w$.”

***H* accepts *w***

iff

***R* accepts *f(w)***

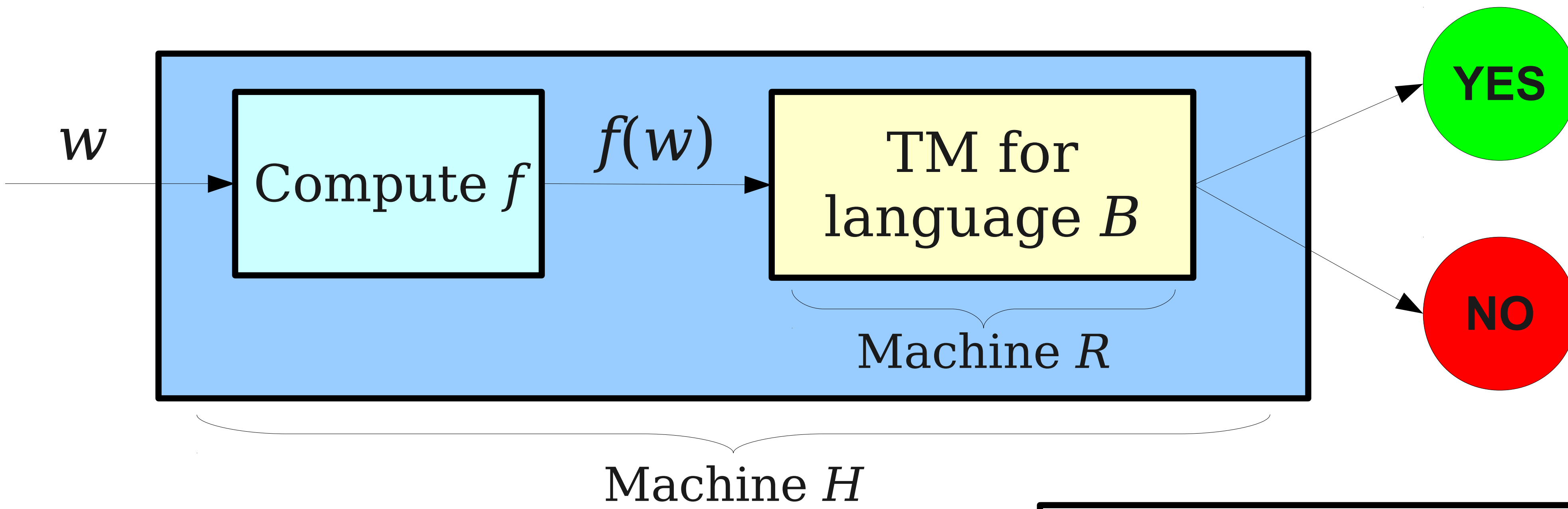
iff

$f(w) \in B$

iff

$w \in A$

$$w \in A \quad \text{iff} \quad f(w) \in B$$

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$$\mathscr{L}(H) = A$$

# A Problem

- Recall: $f$ is a reduction from $A$ to $B$ iff

$$\boldsymbol{w \in A \quad \text{iff} \quad f(w) \in B}$$

- Under this definition, *any* language $A$ reduces to *any* language $B$ unless $B = \emptyset$ or $\Sigma^*$.
- Since $B \neq \emptyset$ and $B \neq \Sigma^*$, there is some $w_{yes} \in B$ and some $w_{no} \notin B$.
- Define $f : \Sigma_1^* \to \Sigma_2^*$ as follows:

$$\textbf{If } \boldsymbol{w \in A}\textbf{, then } \boldsymbol{f(w) = w_{yes}}$$

$$\textbf{If } \boldsymbol{w \notin A}\textbf{, then } \boldsymbol{f(w) = w_{no}}$$

- Then $f$ is a reduction from $A$ to $B$.

# A Problem

- Example: let's reduce $L_D$ to `0`*`1`*.
- Take $w_{yes}$ = `01`, $w_{no}$ = `10`.
- Then $f(w)$ is defined as
  - If $w \in L_D$, $f(w)$ = `01`.
  - If $w \notin L_D$, $f(w)$ = `10`.
- There is no TM that can actually evaluate the function $f(w)$ on all inputs, since no TM can decide whether or not $w \in L_D$.


- If $w \notin L_D$, $f(w) =$ `10`.

- There is no TM that can actually evaluate the function $f(w)$ on all inputs, since no TM can decide whether or not $w \in L_D$.

# Computable Functions

- This general reduction is mathematically well-defined, but might be impossible to actually compute!
- To fix our definition, we need to introduce the idea of a computable function.
- A function $f : \Sigma_1^* \to \Sigma_2^*$ is called a **computable function** if there is some TM $M$ with the following behavior:

“On input $w$:

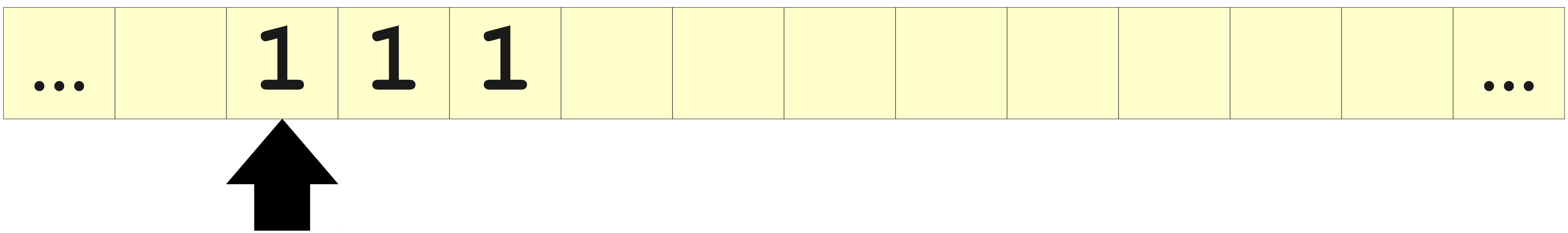
Compute $f(w)$ and write it on the tape.

Move the tape head to the start of $f(w)$.

Halt.”

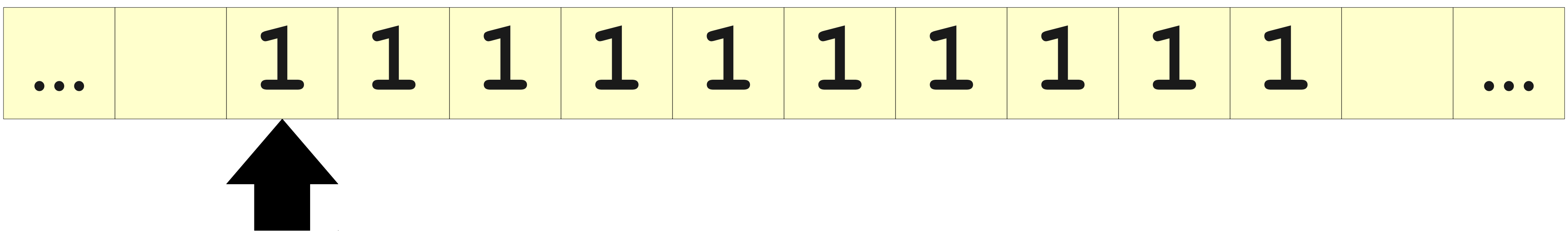
# Computable Functions

$$f(\mathbf{1}^n) = \mathbf{1}^{3n+1}$$



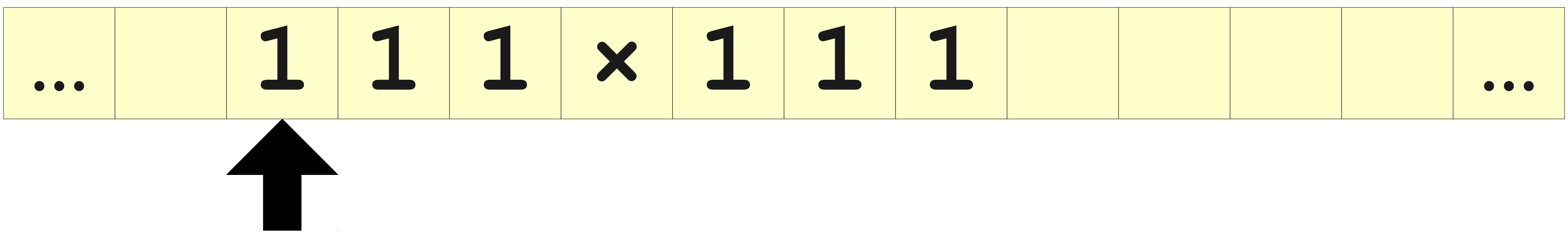
# Computable Functions

$$f(\mathbf{1}^n) = \mathbf{1}^{3n+1}$$



# Computable Functions

$$f(w) = \begin{cases} 1^{mn} & \textbf{if } w = 1^n \times 1^m \\ \varepsilon & \textbf{otherwise} \end{cases}$$



# Computable Functions

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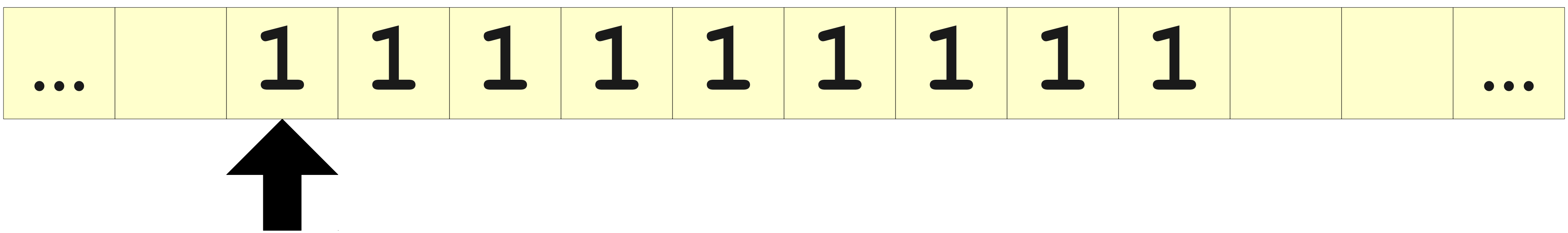
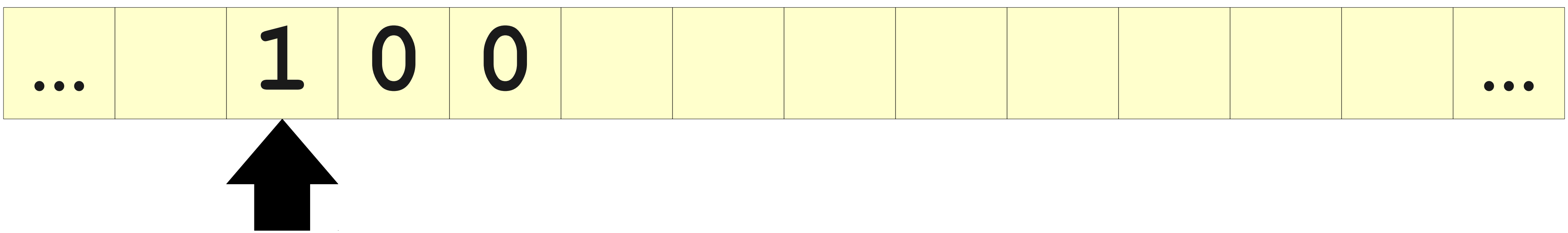


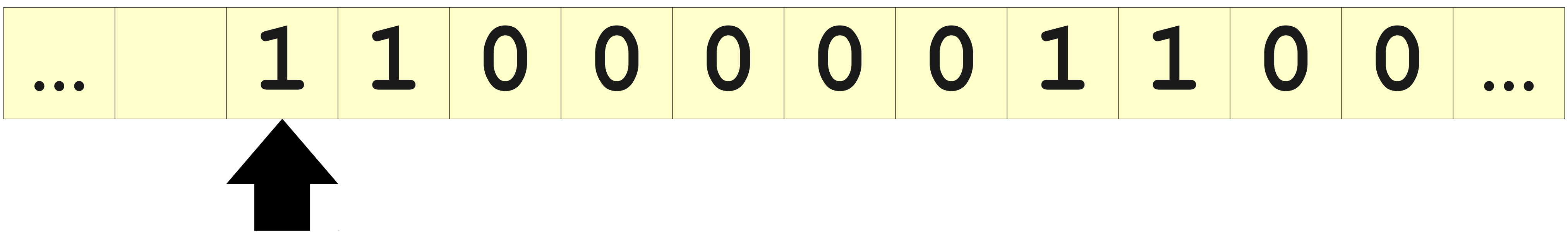
# Computable Functions

$$f(\langle M \rangle) = \langle M, \langle M \rangle \rangle$$



# Computable Functions

$$f(\langle M \rangle) = \langle M, \langle M \rangle \rangle$$



# Mapping Reductions

- A function $f : \Sigma_1^* \to \Sigma_2^*$ is called a **mapping reduction** from $A$ to $B$ iff
  - For any $w \in \Sigma_1^*$, $w \in A$ iff $f(w) \in B$.
  - $f$ is a computable function.
- Intuitively, a mapping reduction from $A$ to $B$ says that a computer can transform any instance of $A$ into an instance of $B$ such that the answer to $B$ is the answer to $A$.