Mapping Reductions

Announcements

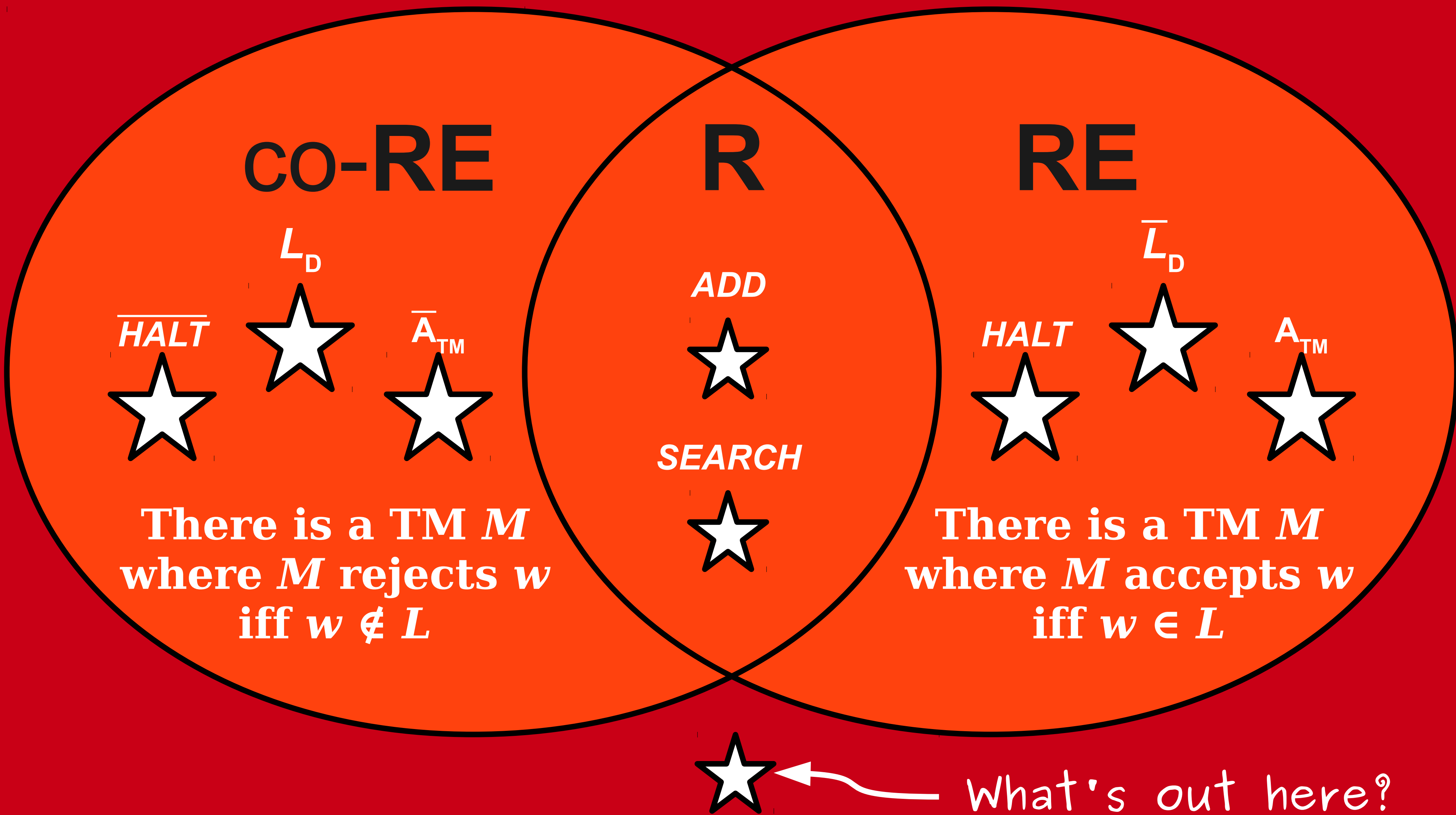
- Casual CS Dinner for Women Studying Computer Science: **Thursday, March 7** at **6PM** in **Gates 219**!
- RSVP through the email link sent out earlier today.

Announcements

- All Problem Set 6's are graded, will be returned at end of lecture.
- Problem Set 7 due right now, or due at Thursday at 12:50PM with a late day.
  - **Please submit no later than 12:50PM**; we're hoping to get solutions posted then. This is a hard deadline.
- Problem Set 8 out, due next Monday, March 11 at 12:50PM.
  - Explore the limits of computation!

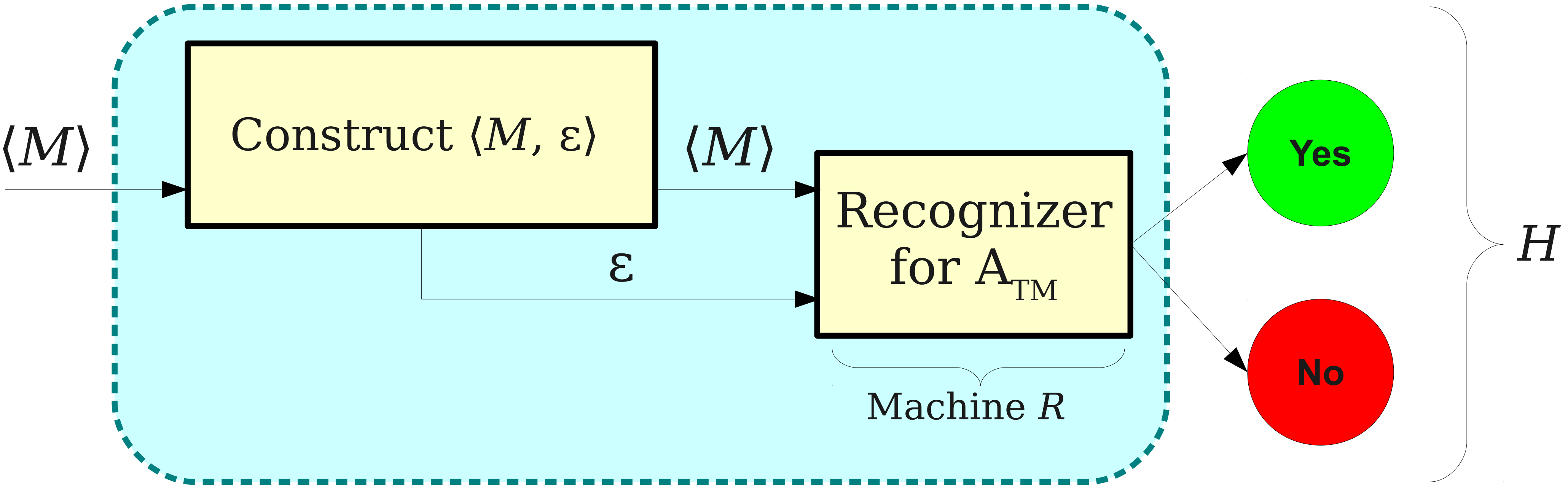
Recap from Last Time

The Limits of Computability



A Repeating Pattern

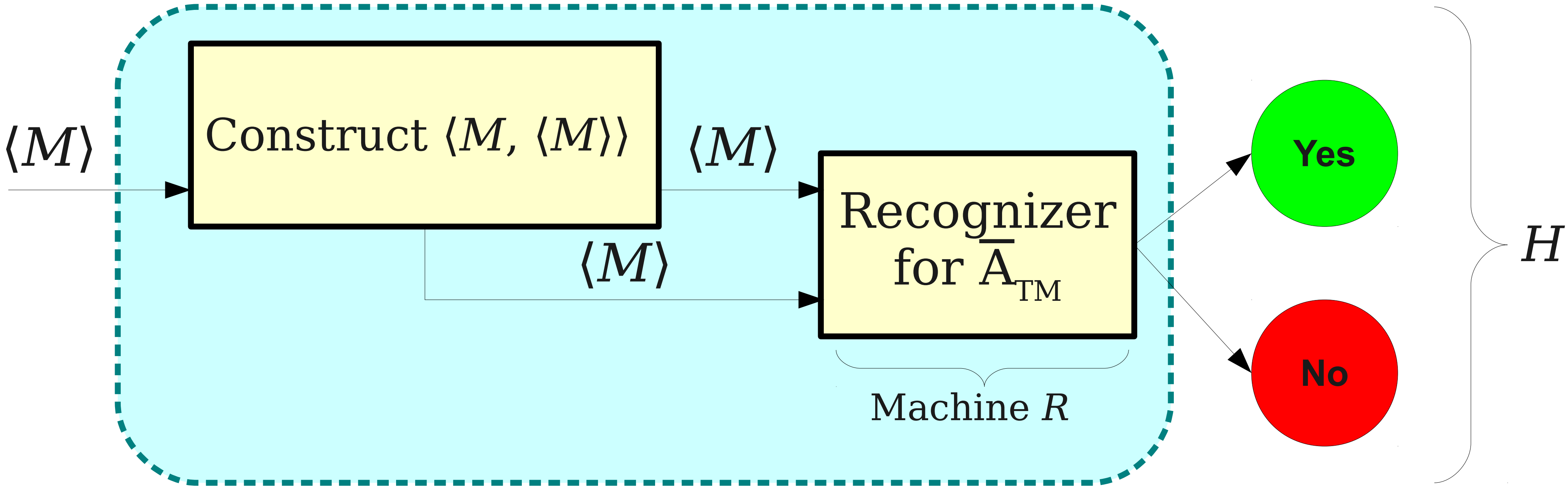
$L = \{ \langle M \rangle \mid M \text{ is a TM that accepts } \varepsilon \}$



H = "On input $\langle M \rangle$:

- Construct the string $\langle M, \varepsilon \rangle$.
- Run R on $\langle M, \varepsilon \rangle$.
- If R accepts $\langle M, \varepsilon \rangle$, then H accepts $\langle M, \varepsilon \rangle$.
- If R rejects $\langle M, \varepsilon \rangle$, then H rejects $\langle M, \varepsilon \rangle$."

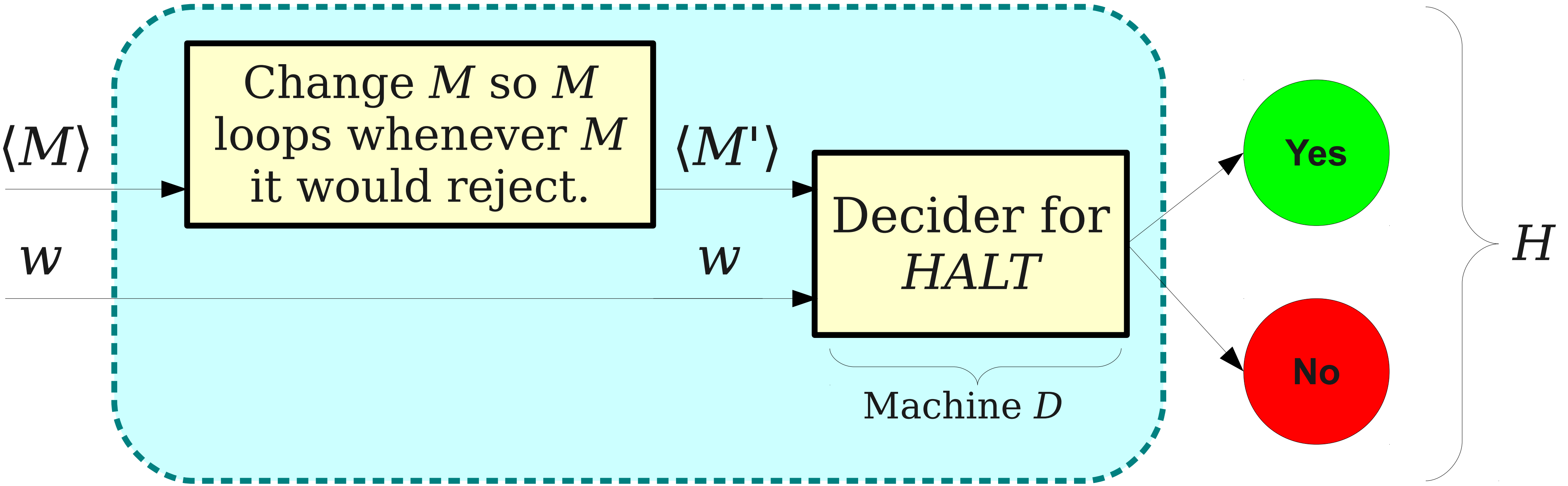
From $\overline{\mathrm{A}}_{\mathrm{TM}}$ to L_{D}



H = “On input $\langle M \rangle$:

- Construct the string $\langle M, \langle M \rangle\rangle$.
- Run R on $\langle M, \langle M \rangle\rangle$.
- If R accepts $\langle M, \langle M \rangle\rangle$, then H accepts $\langle M, \langle M \rangle\rangle$.
- If R rejects $\langle M, \langle M \rangle\rangle$, then H rejects $\langle M, \langle M \rangle\rangle$.”

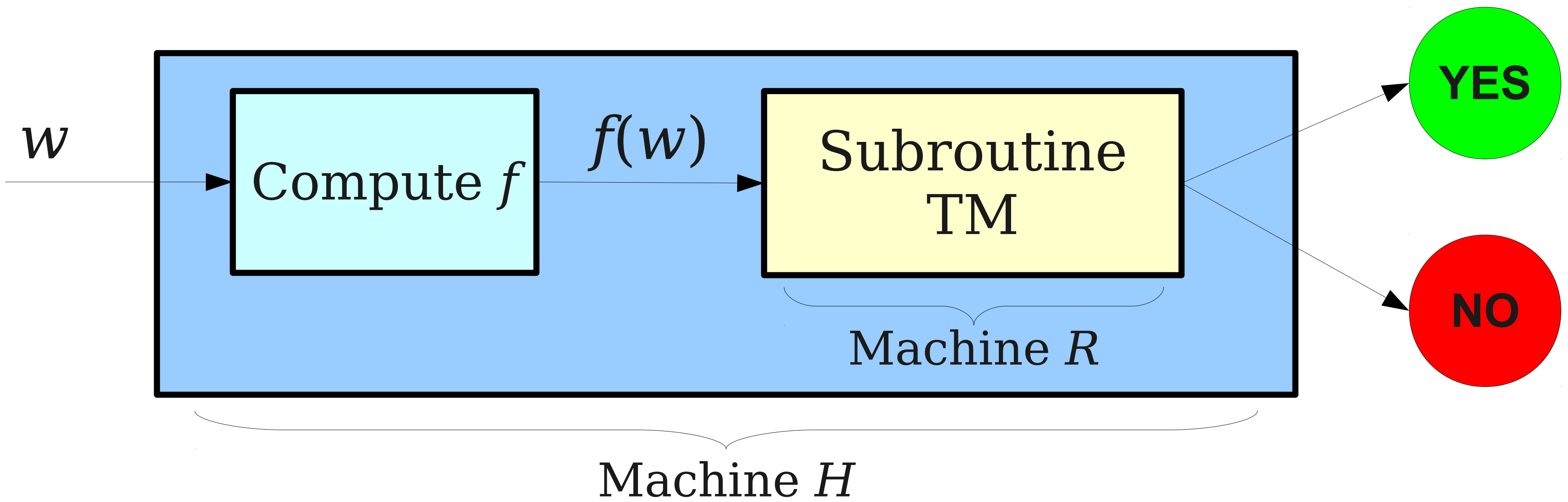
From *HALT* to A_{TM}



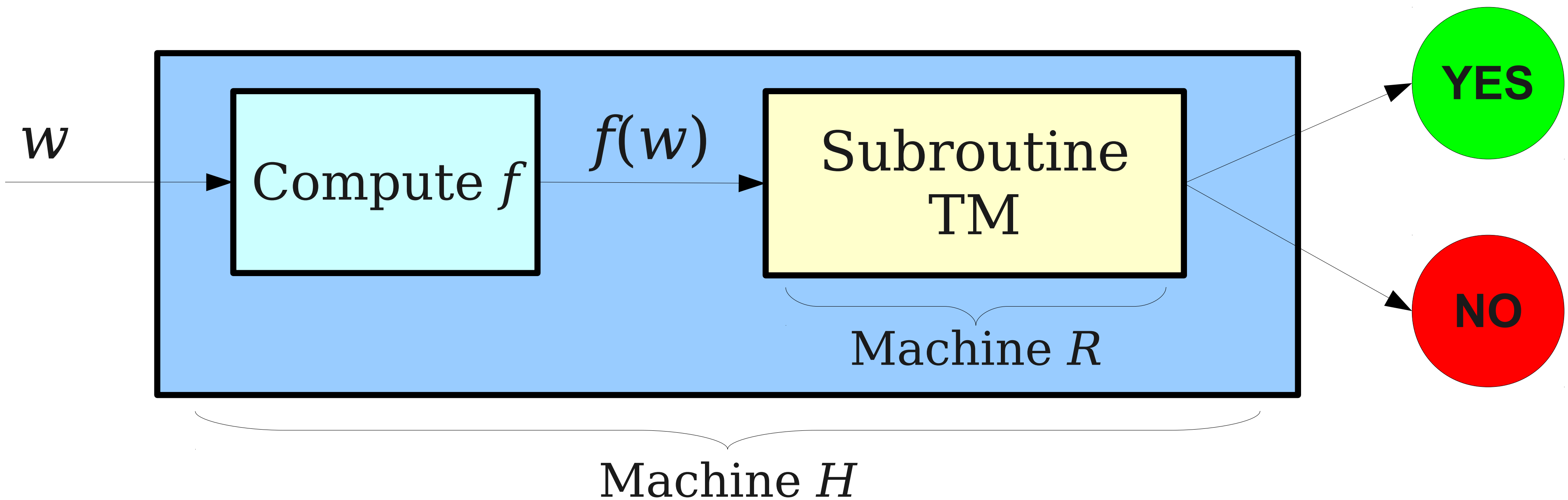
H = "On input $\langle M, w \rangle$:

- Build M into M' so M' loops when M rejects.
- Run D on $\langle M', w \rangle$.
- If D accepts $\langle M', w \rangle$, then H accepts $\langle M, w \rangle$.
- If D rejects $\langle M', w \rangle$, then H rejects $\langle M, w \rangle$."

The General Pattern



The General Pattern

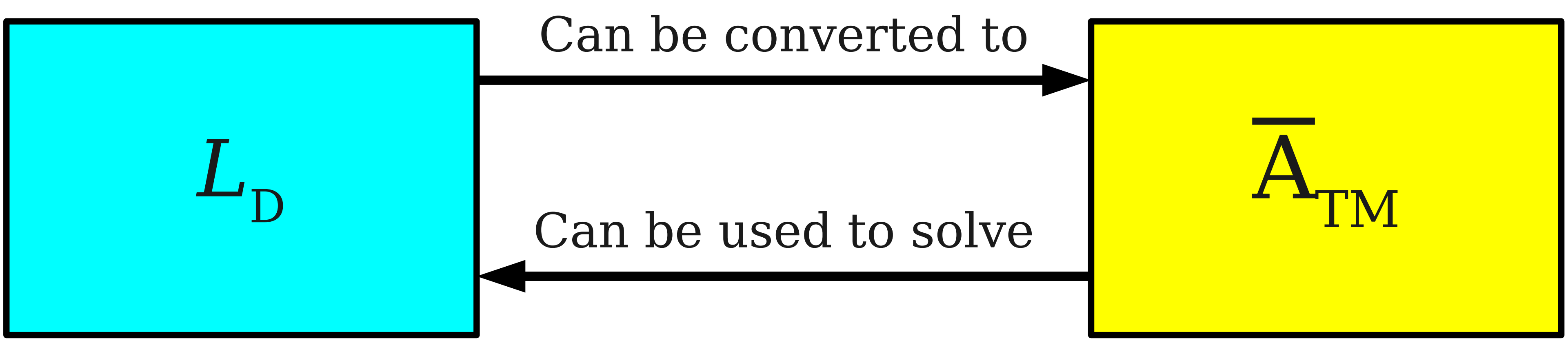


H = "On input w:

- Transform the input w into $f(w)$.
- Run machine R on $f(w)$.
- If R accepts $f(w)$, then H accepts w.
- If R rejects $f(w)$, then H rejects w."

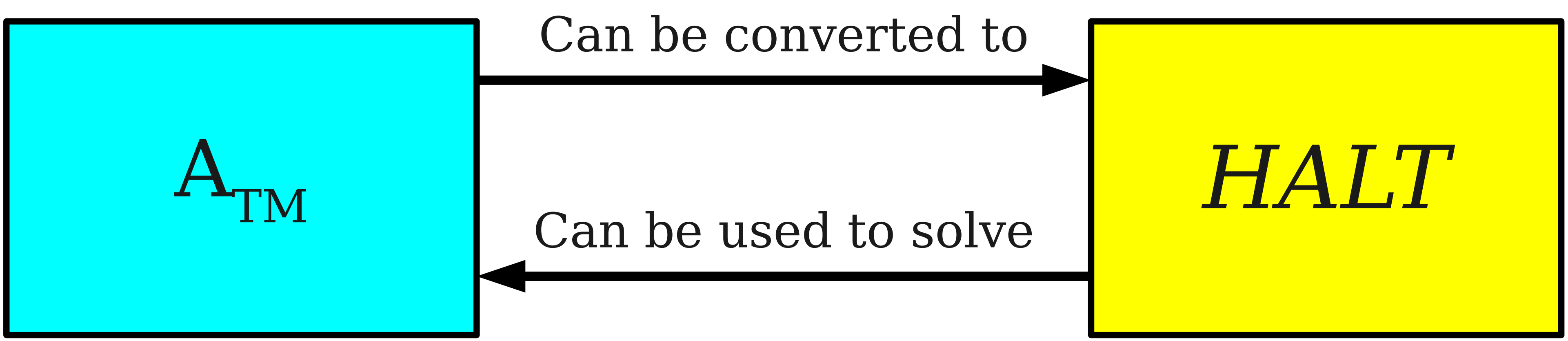
Reductions

- Intuitively, problem A **reduces** to problem B iff a solver for B can be used to solve problem A.



Reductions

- Intuitively, problem *A* **reduces** to problem *B* iff a solver for *B* can be used to solve problem *A*.



Reductions

- Intuitively, problem A **reduces** to problem B iff a solver for B can be used to solve problem A.
- Reductions can be used to show certain problems are "solvable:"

If *A* reduces to *B* and *B* is "solvable," then *A* is "solvable."

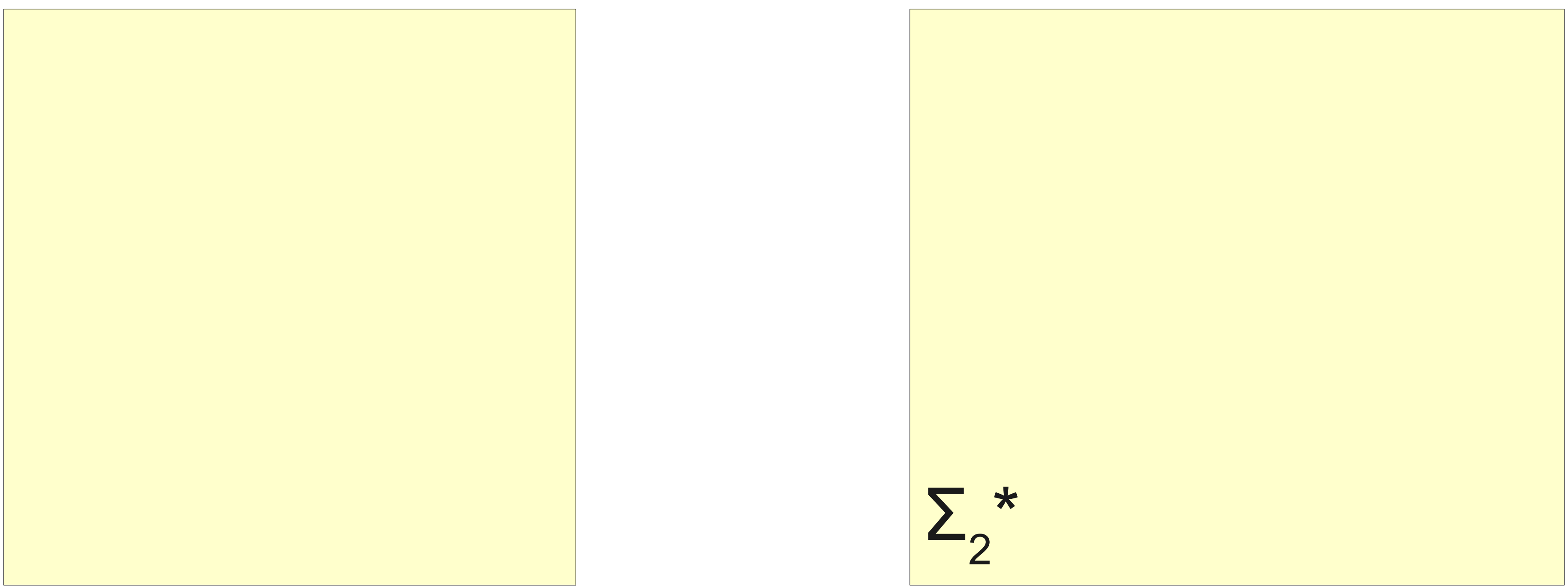
- Reductions can be used to show certain problems are "*unsolvable*:"

If *A* reduces to *B* and *A* is "unsolvable," then *B* is "unsolvable."

Defining Reductions

- A **reduction** from A to B is a function $f : \Sigma_1^* \to \Sigma_2^*$ such that

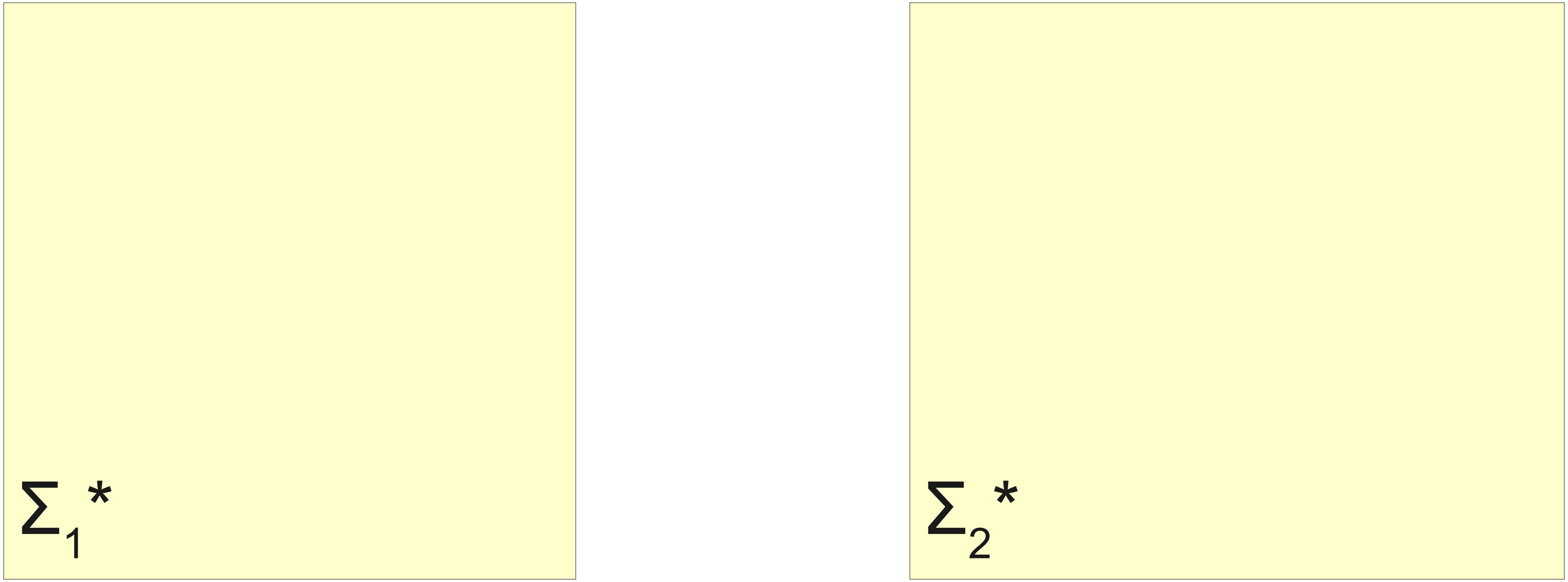
For any $w \in \Sigma_1^*$, $w \in A$ iff $f(w) \in B$



Defining Reductions

- A **reduction** from A to B is a function $f : \Sigma_1^* \to \Sigma_2^*$ such that

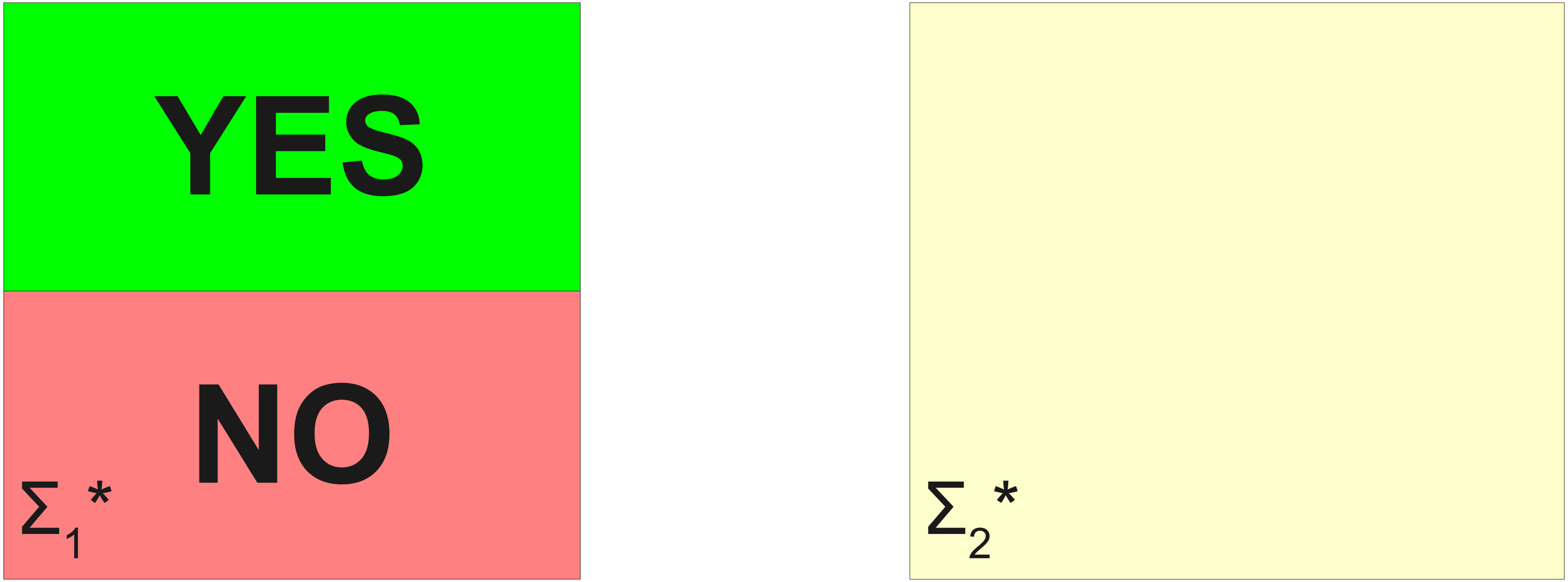
For any $w \in \Sigma_1^*$, $w \in A$ iff $f(w) \in B$



Defining Reductions

- A **reduction** from A to B is a function $f : \Sigma_1^* \to \Sigma_2^*$ such that

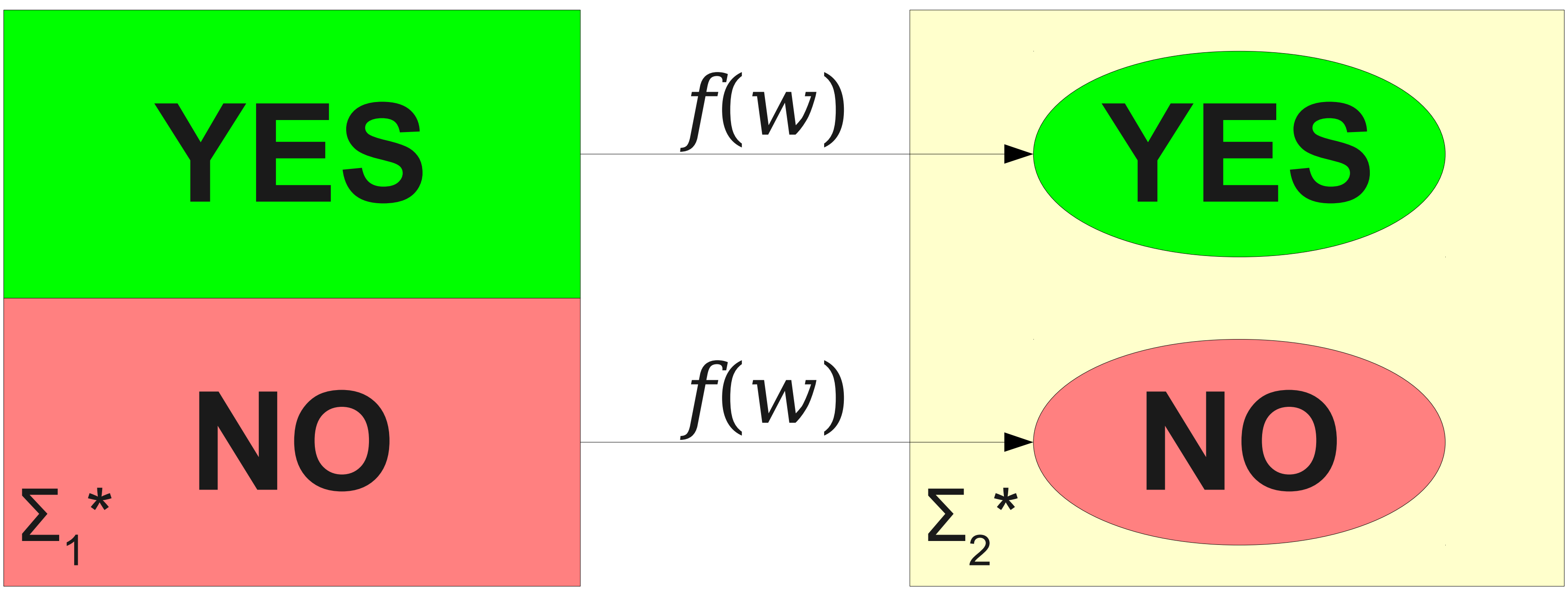
For any $w \in \Sigma_1^*$, $w \in A$ iff $f(w) \in B$



Defining Reductions

- A **reduction** from A to B is a function $f : \Sigma_1^* \rightarrow \Sigma_2^*$ such that

For any $w \in \Sigma_1^*$, $w \in A$ iff $f(w) \in B$



Defining Reductions

- A **reduction** from A to B is a function $f : \Sigma_1^* \to \Sigma_2^*$ such that

For any $w \in \Sigma_1^*$, $w \in A$ iff $f(w) \in B$

- Every $w \in A$ maps to some $f(w) \in B$.
- Every $w \notin A$ maps to some $f(w) \notin B$.
- f does not have to be injective or surjective.

Computable Functions

- Not all mathematical functions can be computed by Turing machines.
- A function $f : \Sigma_1^* \rightarrow \Sigma_2^*$ is called a **computable function** if there is some TM M with the following behavior:

  "On input w:

  Compute $f(w)$ and write it on the tape.

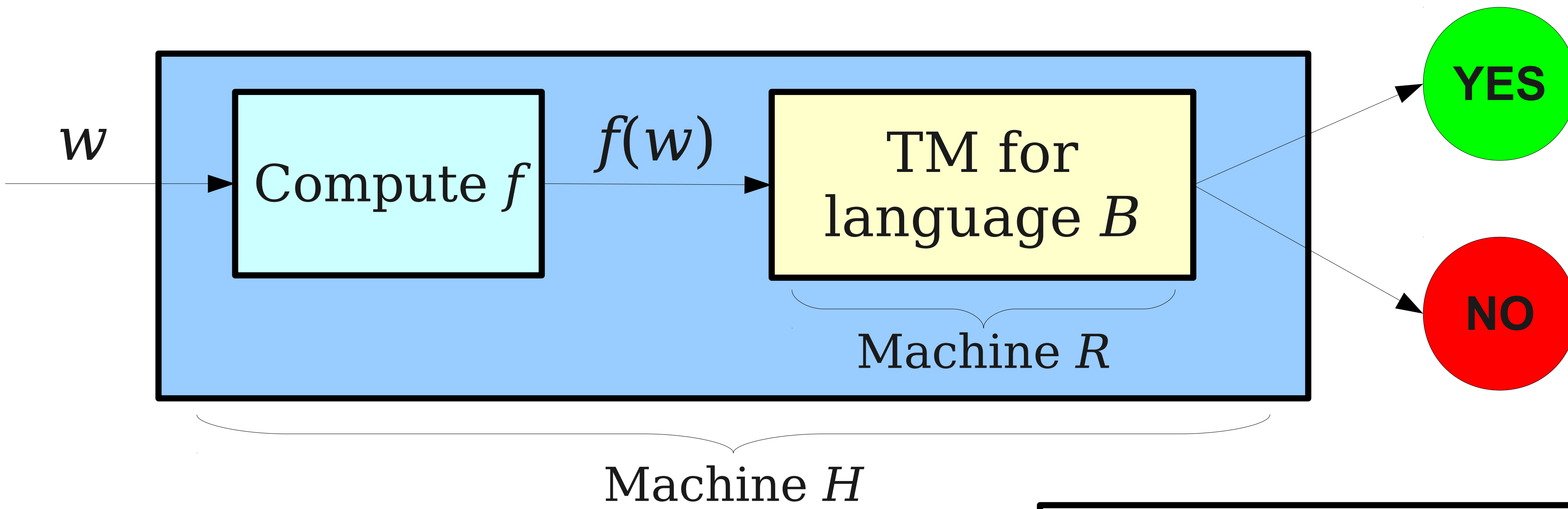
  Move the tape head to the start of $f(w)$.

  Halt."

Mapping Reductions

- A function $f : \Sigma_1^* \to \Sigma_2^*$ is called a **mapping reduction** from A to B iff
  - For any $w \in \Sigma_1^*$, $w \in A$ iff $f(w) \in B$.
  - f is a computable function.
- Intuitively, a mapping reduction from A to B says that a computer can transform any instance of A into an instance of B such that the answer to B is the answer to A.

$$w \in A \quad \text{iff} \quad f(w) \in B$$



H = "On input w:

- Transform the input w into $f(w)$.
- Run machine R on $f(w)$.
- If R accepts $f(w)$, then H accepts w.
- If R rejects $f(w)$, then H rejects w."

H accepts w

iff

R accepts $f(w)$

iff

$f(w) \in B$

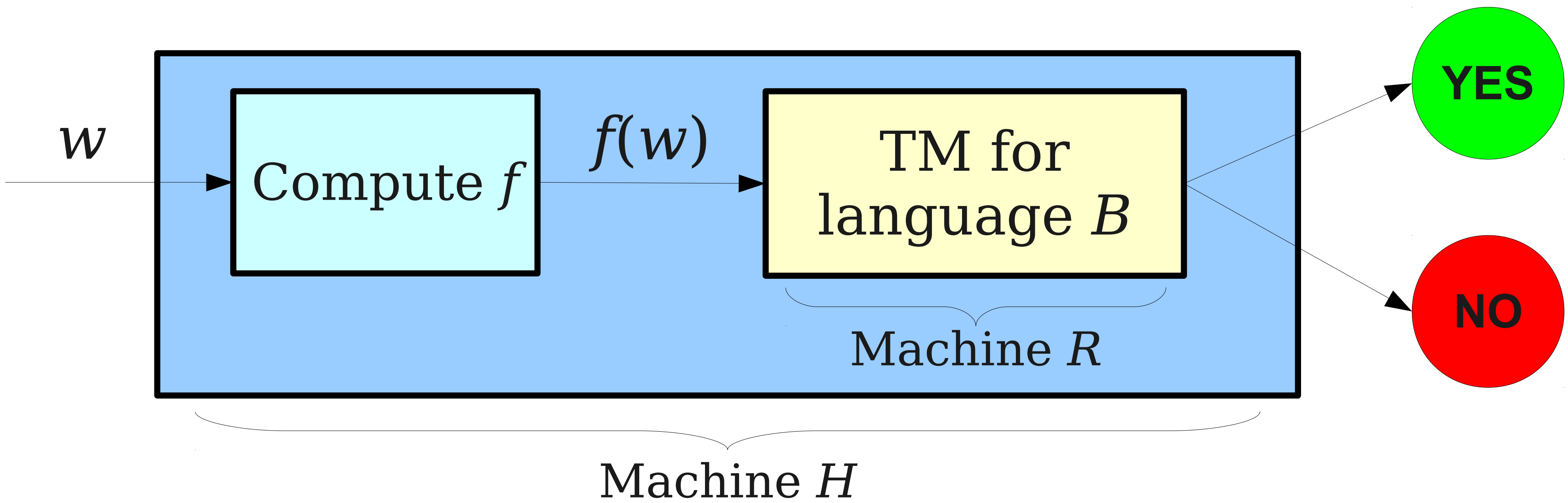
iff

$w \in A$

Mapping Reducibility

- If there is a mapping reduction from language *A* to language *B*, we say that language *A* is **mapping reducible** to language *B*.
- Notation: $\boldsymbol{A \leq_M B}$ iff language *A* is mapping reducible to language *B*.
- Note that we reduce ***languages***, not ***machines***.
- Interesting exercise: Show $\leq_M$ is reflexive and transitive, but not antisymmetric.

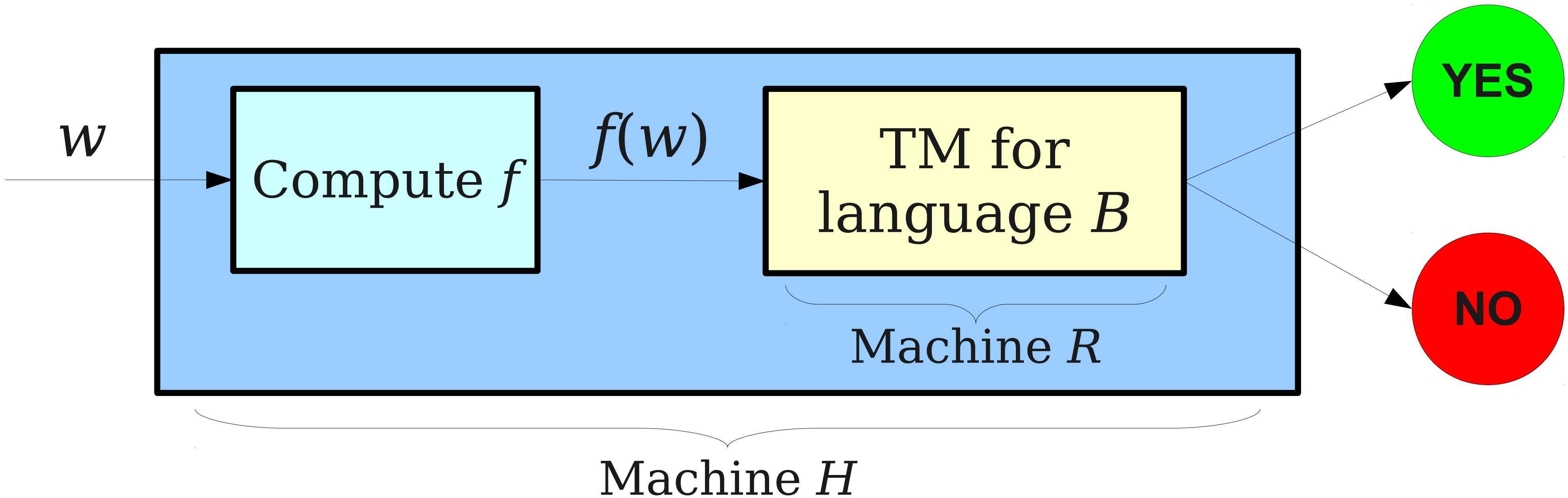
$A \leq_M B$



H = "On input w:

- Compute $f(w)$.
- Run machine R on $f(w)$.
- If R accepts $f(w)$, then H accepts w.
- If R rejects $f(w)$, then H rejects w."

$A \leq_M B$

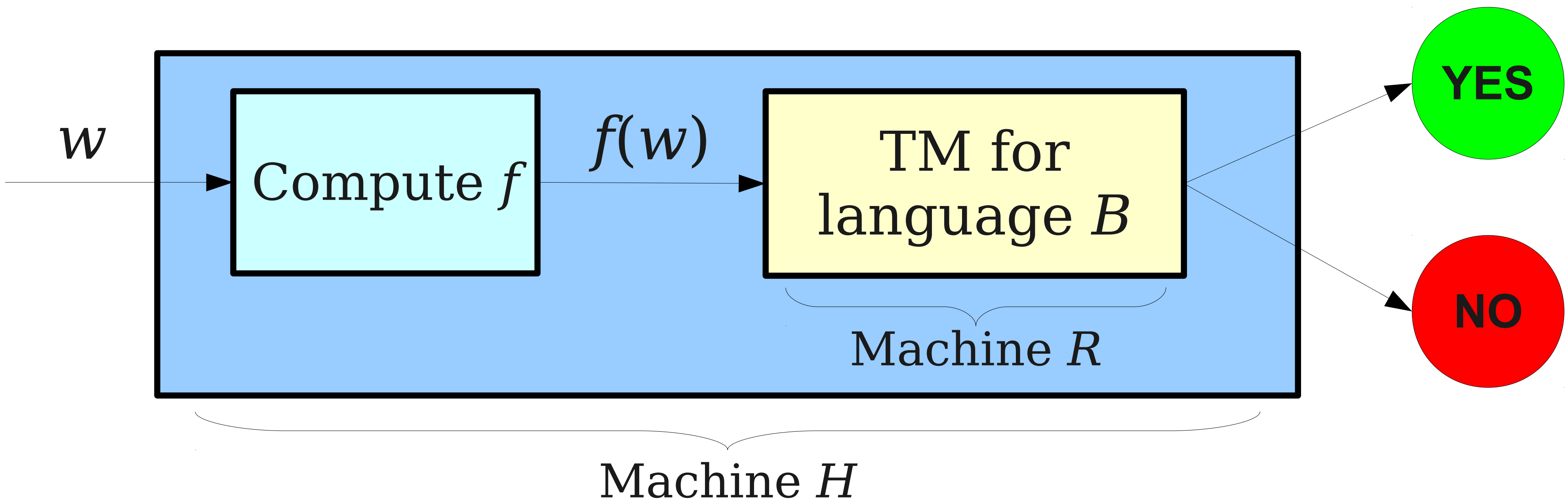


H = "On input w:

- Compute $f(w)$.
- Run machine R on $f(w)$.
- If R accepts $f(w)$, then H accepts w.
- If R rejects $f(w)$, then H rejects w."

If R is a decider for B,
then H is a decider for A.

$A \leq_M B$



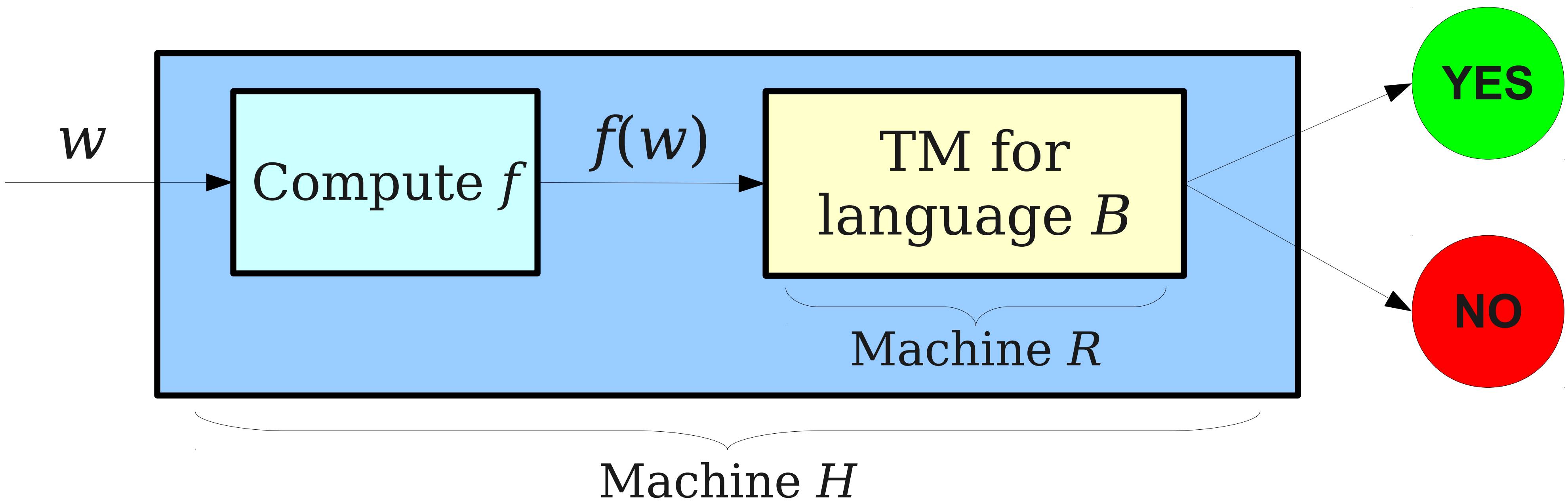
H = “On input w:

- Compute $f(w)$.
- Run machine R on $f(w)$.
- If R accepts $f(w)$, then H accepts w.
- If R rejects $f(w)$, then H rejects w.”

If R is a decider for B, then H is a decider for A.

If R is a recognizer for B, then H is a recognizer for A.

$A \leq_M B$



H = "On input w:

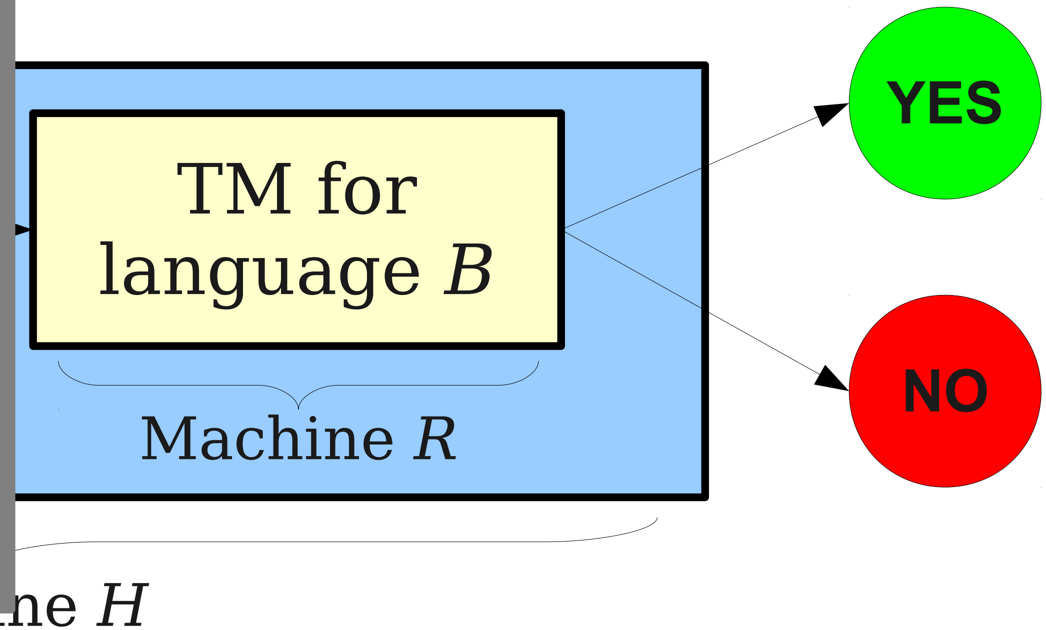
- Compute $f(w)$.
- Run machine R on $f(w)$.
- If R accepts $f(w)$, then H accepts w.
- If R rejects $f(w)$, then H rejects w."

If R is a decider for B, then H is a decider for A.

If R is a recognizer for B, then H is a recognizer for A.

If R is a co-recognizer for B, then H is a co-recognizer for A.

$\leq_{\mathrm{M}} B$



H = "On input w:

- Compute $f(w)$.
- Run machine R on $f(w)$.
- If R accepts $f(w)$, then H accepts w.
- If R rejects $f(w)$, then H rejects w."

If R is a decider for B, then H is a decider for A.

If R is a recognizer for B, then H is a recognizer for A.

If R is a co-recognizer for B, then H is a co-recognizer for A.

Why Mapping Reducibility Matters

- **Theorem**: If $B \in \mathbf{R}$ and $A \leq_{\mathrm{M}} B$, then $A \in \mathbf{R}$.
- **Theorem**: If $B \in \mathbf{RE}$ and $A \leq_{\mathrm{M}} B$, then $A \in \mathbf{RE}$.
- **Theorem**: If $B \in \text{co-}\mathbf{RE}$ and $A \leq_{\mathrm{M}} B$, then $A \in \text{co-}\mathbf{RE}$.
- *Intuitively:* $A \leq_{\mathrm{M}} B$ means "A is not harder than B."

Why Mapping Reducibility Matters

- **Theorem**: If $A \notin \mathbf{R}$ and $A \leq_{\mathrm{M}} B$, then $B \notin \mathbf{R}$.
- **Theorem**: If $A \notin \mathbf{RE}$ and $A \leq_{\mathrm{M}} B$, then $B \notin \mathbf{RE}$.
- **Theorem**: If $A \notin \text{co-}\mathbf{RE}$ and $A \leq_{\mathrm{M}} B$, then $B \notin \text{co-}\mathbf{RE}$.
- *Intuitively:* $A \leq_{\mathrm{M}} B$ means "B is at at least as hard as A."

Why Mapping Reducibility Matters

If this one is "easy" (R, RE, co-RE)…

$$A \leq_{\mathrm{M}} B$$

… then this one is "easy" (R, RE, co-RE) too.

Why Mapping Reducibility Matters

If this one is "hard" (not **R**, not **RE**, or not co-**RE**)…

$$A \leq_{\mathrm{M}} B$$

… then this one is "hard" (not **R**, not **RE**, or not co-**RE**) too.

Using Mapping Reductions

Revisiting our Proofs

- Consider the language

$$L = \{ \langle M \rangle \mid M \text{ is a TM and } M \text{ accepts } \varepsilon \}$$

- We have already proven that this language is **RE** by building a TM for it.
- Let's repeat this proof using mapping reductions.
- Specifically, we will prove

$$L \leq_{\mathrm{M}} \mathrm{A_{TM}}$$

$L = \{ \langle M \rangle \mid M$ is a TM and M accepts ε $\}$

- To prove $L \leq_M A_{TM}$, we will need to find a computable function f such that

$$\boldsymbol{\langle M \rangle \in L \quad \textbf{iff} \quad f(\langle M \rangle) \in A_{TM}}$$

- Since A_{TM} is a language of TM/string pairs, let's assume $f(\langle M \rangle) = \langle N, w \rangle$ for some TM N and string w (which we'll pick later):

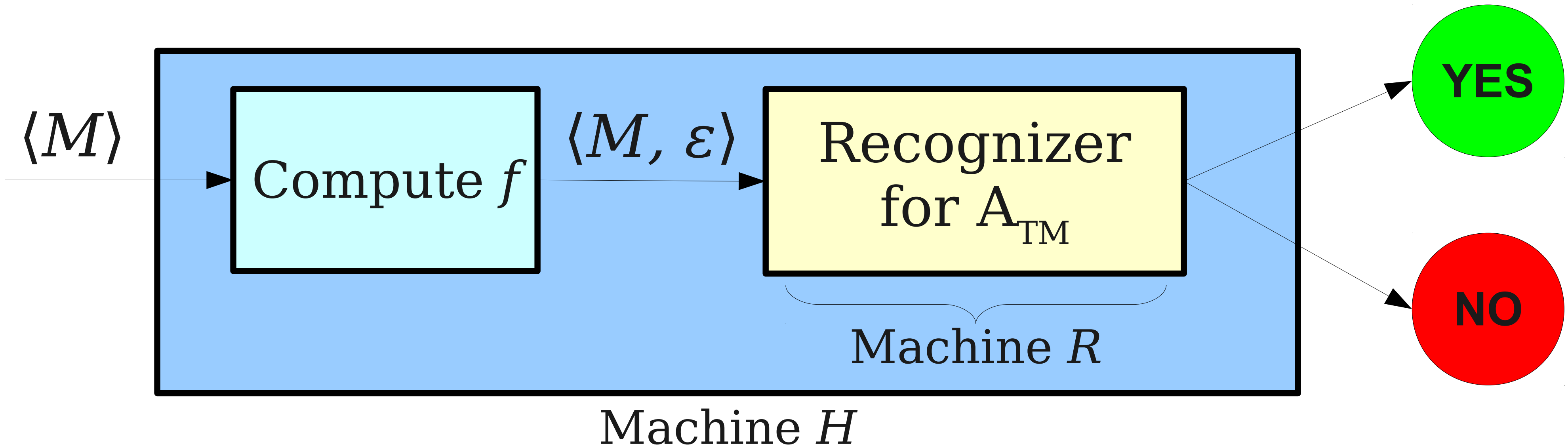
$$\boldsymbol{\langle M \rangle \in L \quad \textbf{iff} \quad \langle N, w \rangle \in A_{TM}}$$

- Substituting definitions:

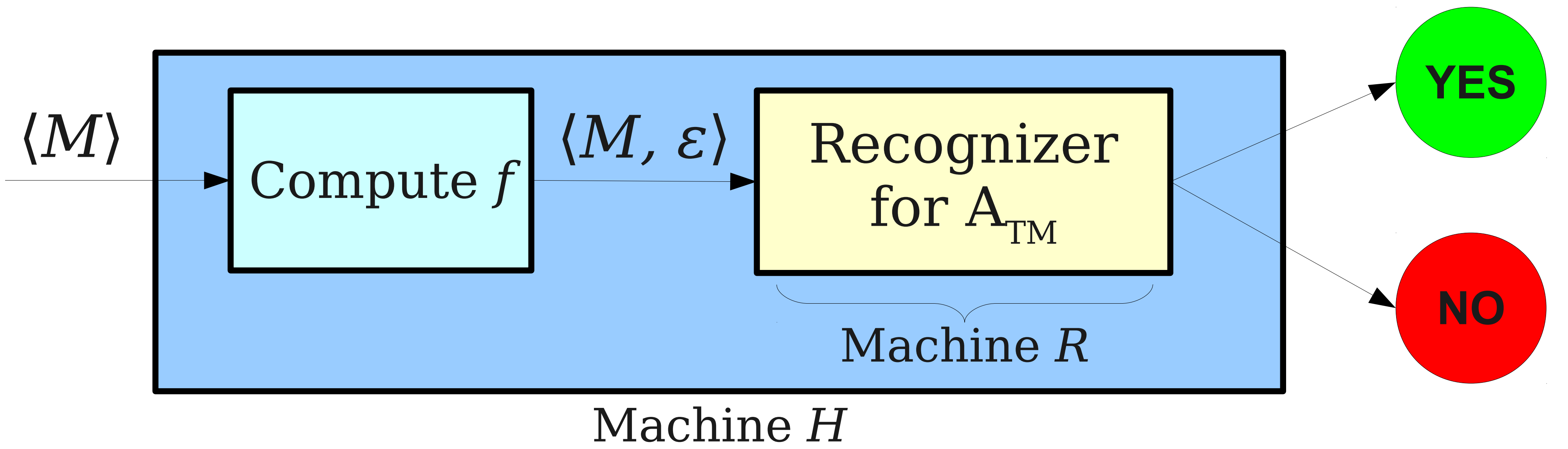
$$\boldsymbol{M \textbf{ accepts } \varepsilon \quad \textbf{iff} \quad N \textbf{ accepts } w}$$

- Choose $\boldsymbol{N = M}$, $\boldsymbol{w = \varepsilon}$. So $\boldsymbol{f(\langle M \rangle) = \langle M, \varepsilon \rangle}$.

One Interpretation of the Reduction



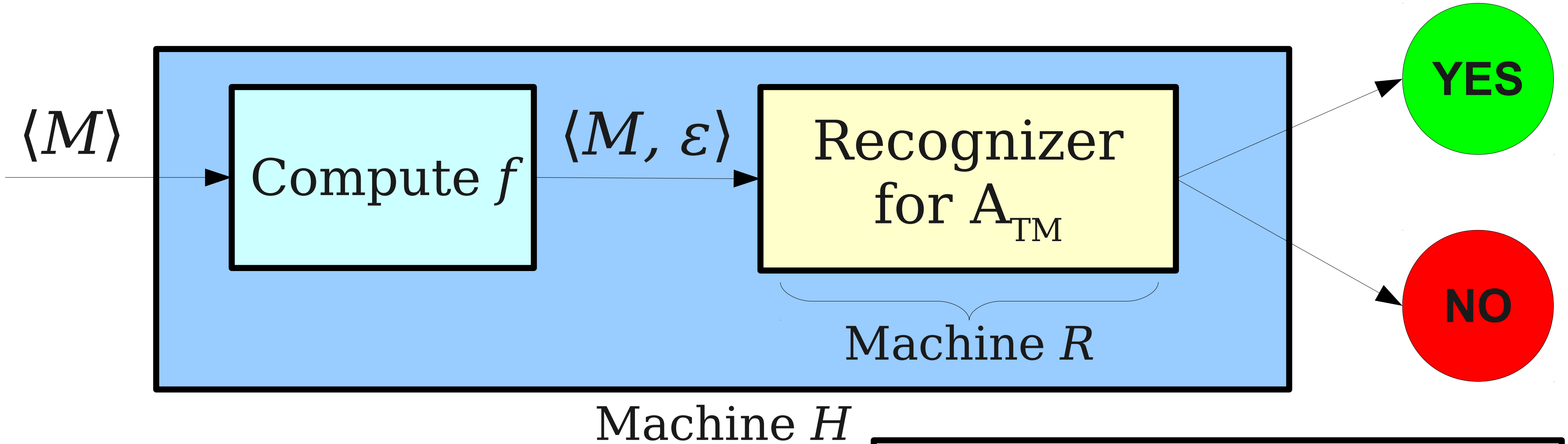
One Interpretation of the Reduction



H = "On input $\langle M \rangle$:

- Run machine R on $\langle M, \varepsilon \rangle$.
- If R accepts $\langle M, \varepsilon \rangle$, then H accepts w.
- If R rejects $\langle M, \varepsilon \rangle$, then H rejects w."

One Interpretation of the Reduction

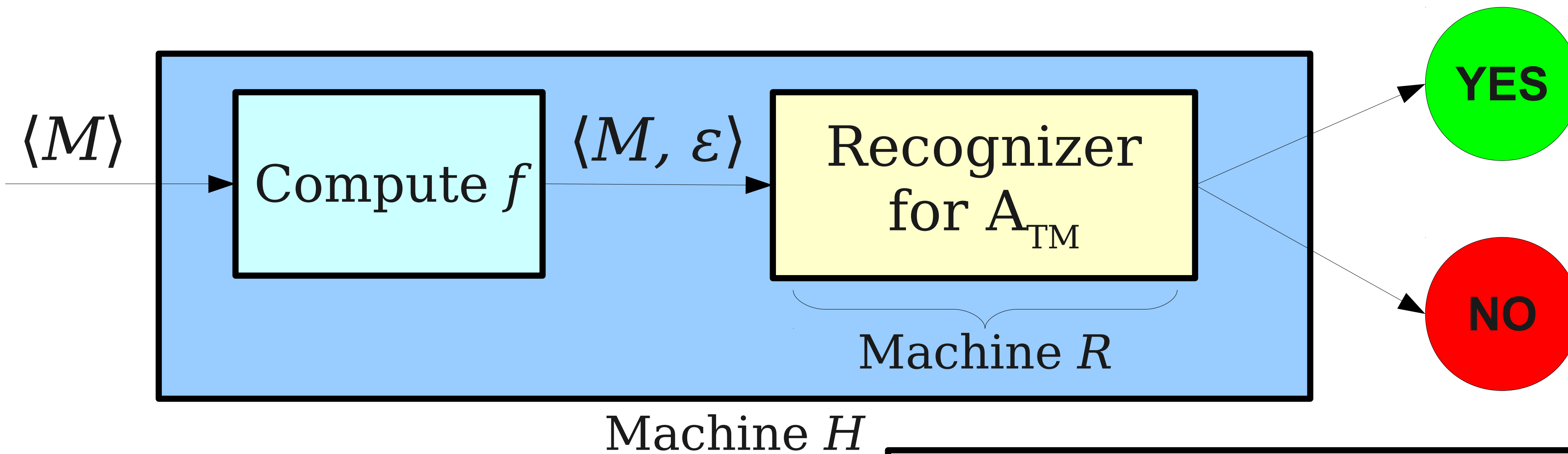


H = "On input ⟨M⟩:

- Run machine R on ⟨M, ε⟩.
- If R accepts ⟨M, ε⟩, then H accepts w.
- If R rejects ⟨M, ε⟩, then H rejects w."

H accepts ⟨M⟩

One Interpretation of the Reduction



H = "On input $\langle M \rangle$:

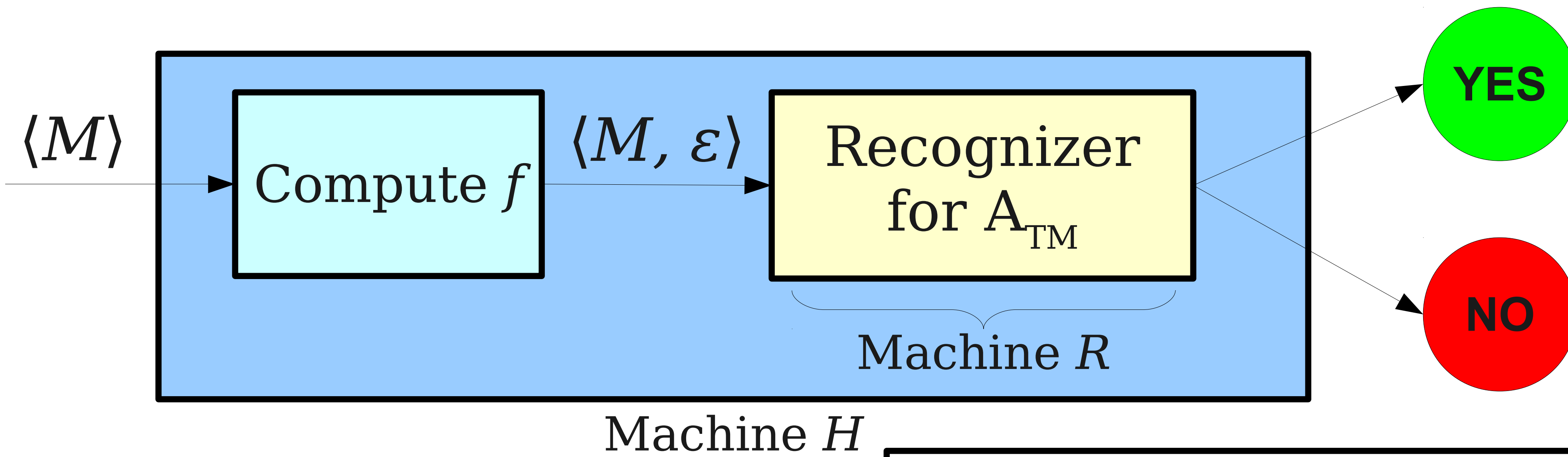
- Run machine R on $\langle M, \varepsilon \rangle$.
- If R accepts $\langle M, \varepsilon \rangle$, then H accepts w.
- If R rejects $\langle M, \varepsilon \rangle$, then H rejects w."

H accepts $\langle M \rangle$

iff

R accepts $\langle M, \varepsilon \rangle$

One Interpretation of the Reduction



H = "On input $\langle M \rangle$:

- Run machine R on $\langle M, \varepsilon \rangle$.
- If R accepts $\langle M, \varepsilon \rangle$, then H accepts w.
- If R rejects $\langle M, \varepsilon \rangle$, then H rejects w."

H accepts $\langle M \rangle$

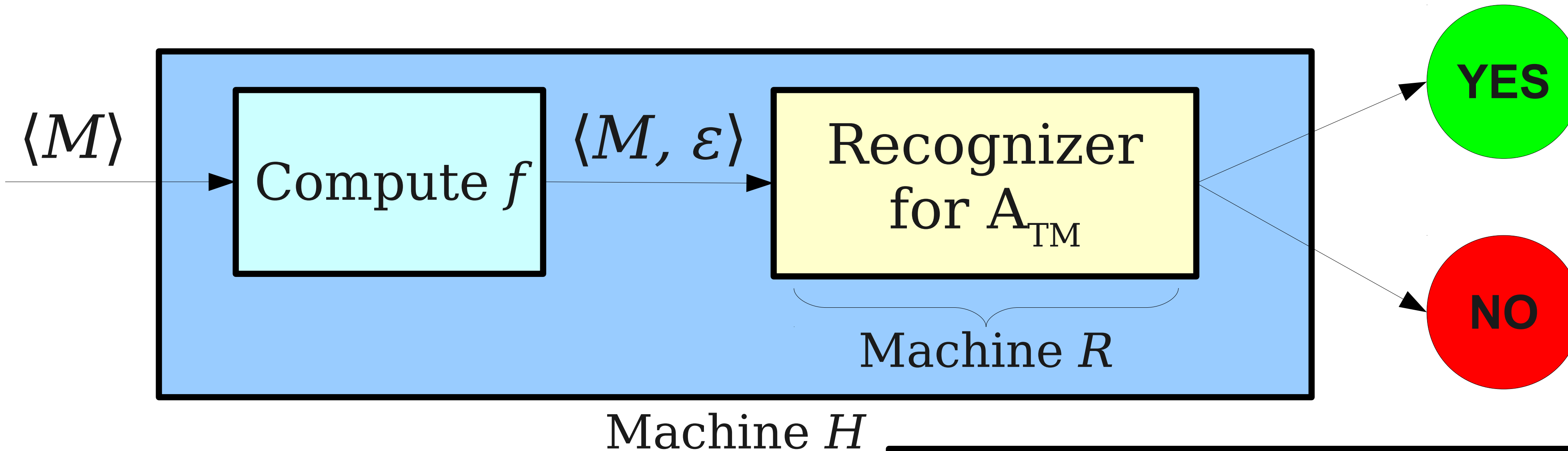
iff

R accepts $\langle M, \varepsilon \rangle$

iff

M accepts ε

One Interpretation of the Reduction



H = "On input $\langle M \rangle$:

- Run machine R on $\langle M, \varepsilon \rangle$.
- If R accepts $\langle M, \varepsilon \rangle$, then H accepts w.
- If R rejects $\langle M, \varepsilon \rangle$, then H rejects w."

H accepts $\langle M \rangle$

iff

R accepts $\langle M, \varepsilon \rangle$

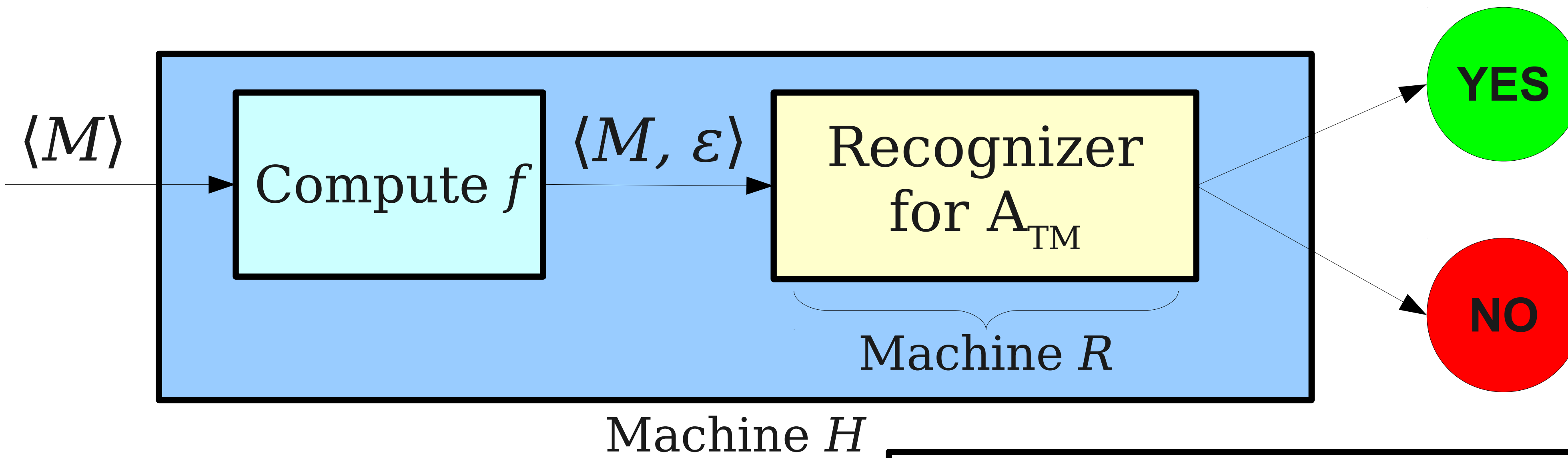
iff

M accepts ε

iff

$\langle M \rangle \in L$

One Interpretation of the Reduction



H = "On input $\langle M \rangle$:

- Run machine R on $\langle M, \varepsilon \rangle$.
- If R accepts $\langle M, \varepsilon \rangle$, then H accepts w.
- If R rejects $\langle M, \varepsilon \rangle$, then H rejects w."

H accepts $\langle M \rangle$

iff

R accepts $\langle M, \varepsilon \rangle$

iff

M accepts ε

iff

$\langle M \rangle \in L$

$$L = \{ \langle M \rangle \mid M \text{ is a TM that accepts } \varepsilon \}$$

Theorem: $L \in \mathbf{RE}$.

$$L = \{ \langle M \rangle \mid M \text{ is a TM that accepts } \varepsilon \}$$

Theorem: $L \in \mathbf{RE}$.

Proof: We will prove that $L \leq_M A_{TM}$.

$$L = \{ \langle M \rangle \mid M \text{ is a TM that accepts } \varepsilon \}$$

Theorem: $L \in \mathbf{RE}$.

Proof: We will prove that $L \leq_M A_{TM}$. Since $A_{TM} \in \mathbf{RE}$, this proves $L \in \mathbf{RE}$ as well.

$$L = \{ \langle M \rangle \mid M \text{ is a TM that accepts } \varepsilon \}$$

Theorem: $L \in$ **RE**.

Proof: We will prove that $L \leq_M A_{TM}$. Since $A_{TM} \in$ **RE**, this proves $L \in$ **RE** as well.

Consider the function $f(\langle M \rangle) = \langle M, \varepsilon \rangle$.

$$L = \{ \langle M \rangle \mid M \text{ is a TM that accepts } \varepsilon \}$$

Theorem: $L \in \mathbf{RE}$.

Proof: We will prove that $L \leq_M A_{TM}$. Since $A_{TM} \in \mathbf{RE}$, this proves $L \in \mathbf{RE}$ as well.

Consider the function $f(\langle M \rangle) = \langle M, \varepsilon \rangle$. We state without proof that this function is computable and claim that f is a mapping reduction from L to A_{TM}.

$$L = \{ \langle M \rangle \mid M \text{ is a TM that accepts } \varepsilon \}$$

Theorem: $L \in$ **RE**.

Proof: We will prove that $L \leq_M A_{TM}$. Since $A_{TM} \in$ **RE**, this proves $L \in$ **RE** as well.

Consider the function $f(\langle M \rangle) = \langle M, \varepsilon \rangle$. We state without proof that this function is computable and claim that f is a mapping reduction from L to A_{TM}. To see this, note that $f(\langle M \rangle) = \langle M, \varepsilon \rangle \in A_{TM}$ iff M accepts ε

$L = \{ \langle M \rangle \mid M$ is a TM that accepts ε $\}$

Theorem: $L \in \mathbf{RE}$.

Proof: We will prove that $L \leq_M A_{TM}$. Since $A_{TM} \in \mathbf{RE}$, this proves $L \in \mathbf{RE}$ as well.

Consider the function $f(\langle M \rangle) = \langle M, \varepsilon \rangle$. We state without proof that this function is computable and claim that f is a mapping reduction from L to A_{TM}. To see this, note that $f(\langle M \rangle) = \langle M, \varepsilon \rangle \in A_{TM}$ iff M accepts ε iff $\langle M \rangle \in L$

$$L = \{ \langle M \rangle \mid M \text{ is a TM that accepts } \varepsilon \}$$

Theorem: $L \in$ **RE**.

Proof: We will prove that $L \leq_M A_{TM}$. Since $A_{TM} \in$ **RE**, this proves $L \in$ **RE** as well.

Consider the function $f(\langle M \rangle) = \langle M, \varepsilon \rangle$. We state without proof that this function is computable and claim that f is a mapping reduction from L to A_{TM}. To see this, note that $f(\langle M \rangle) = \langle M, \varepsilon \rangle \in A_{TM}$ iff M accepts ε iff $\langle M \rangle \in L$, so $\langle M \rangle \in L$ iff $f(\langle M \rangle) \in A_{TM}$.

$$L = \{ \langle M \rangle \mid M \text{ is a TM that accepts } \varepsilon \}$$

Theorem: $L \in \mathbf{RE}$.

Proof: We will prove that $L \leq_M A_{TM}$. Since $A_{TM} \in \mathbf{RE}$, this proves $L \in \mathbf{RE}$ as well.

Consider the function $f(\langle M \rangle) = \langle M, \varepsilon \rangle$. We state without proof that this function is computable and claim that f is a mapping reduction from L to A_{TM}. To see this, note that $f(\langle M \rangle) = \langle M, \varepsilon \rangle \in A_{TM}$ iff M accepts ε iff $\langle M \rangle \in L$, so $\langle M \rangle \in L$ iff $f(\langle M \rangle) \in A_{TM}$.

Since f is a mapping reduction from L to A_{TM}, we have $L \leq_M A_{TM}$, and thus $L \in \mathbf{RE}$.

$$L = \{ \langle M \rangle \mid M \text{ is a TM that accepts } \varepsilon \}$$

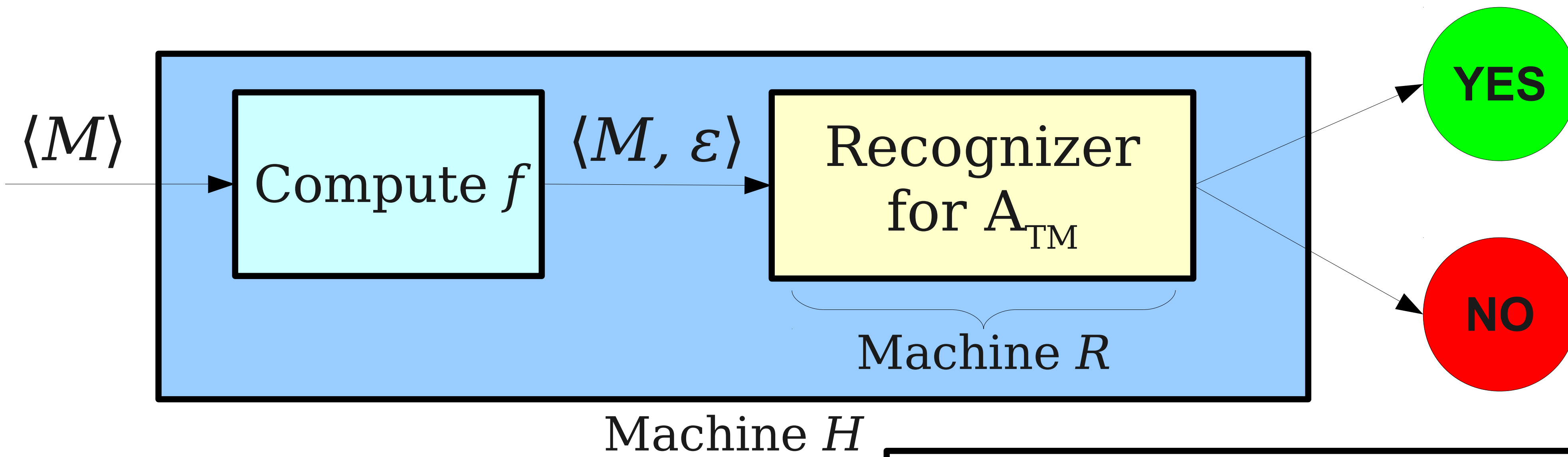
Theorem: $L \in \mathbf{RE}$.

Proof: We will prove that $L \leq_M A_{TM}$. Since $A_{TM} \in \mathbf{RE}$, this proves $L \in \mathbf{RE}$ as well.

Consider the function $f(\langle M \rangle) = \langle M, \varepsilon \rangle$. We state without proof that this function is computable and claim that f is a mapping reduction from L to A_{TM}. To see this, note that $f(\langle M \rangle) = \langle M, \varepsilon \rangle \in A_{TM}$ iff M accepts ε iff $\langle M \rangle \in L$, so $\langle M \rangle \in L$ iff $f(\langle M \rangle) \in A_{TM}$.

Since f is a mapping reduction from L to A_{TM}, we have $L \leq_M A_{TM}$, and thus $L \in \mathbf{RE}$. ■

What Did We Prove?



H = "On input $\langle M \rangle$:

- Run machine R on $\langle M, \varepsilon \rangle$.
- If R accepts $\langle M, \varepsilon \rangle$, then H accepts w.
- If R rejects $\langle M, \varepsilon \rangle$, then H rejects w."

H accepts $\langle M \rangle$

iff

R accepts $\langle M, \varepsilon \rangle$

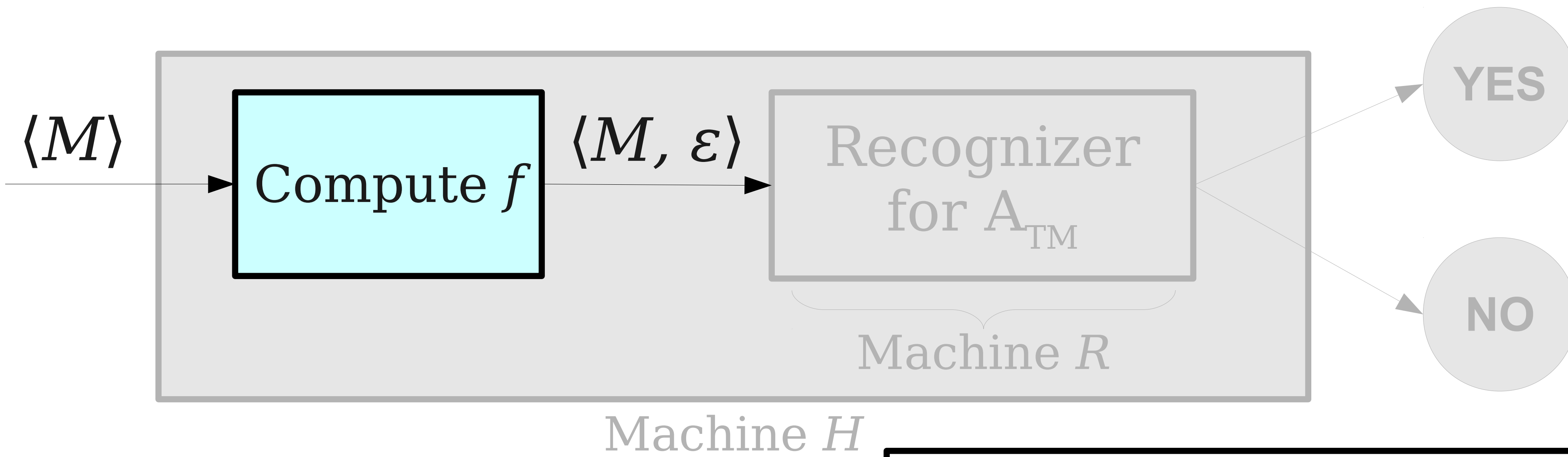
iff

M accepts ε

iff

$\langle M \rangle \in L$

What Did We Prove?



H = "On input $\langle M \rangle$:

- Run machine R on $\langle M, \varepsilon \rangle$.
- If R accepts $\langle M, \varepsilon \rangle$, then H accepts w.
- If R rejects $\langle M, \varepsilon \rangle$, then H rejects w."

H accepts $\langle M \rangle$

iff

R accepts $\langle M, \varepsilon \rangle$

iff

M accepts ε

iff

$\langle M \rangle \in L$

Interpreting Mapping Reductions

- If $A \leq_M B$, there is a known construction to turn a TM for B into a TM for A.
- When doing proofs with mapping reductions, you do *not* need to show the overall construction.
- You just need to prove that
  - f is a computable function, and
  - $w \in A$ iff $f(w) \in B$.

Another Mapping Reduction

L_{D} and $\overline{\text{A}}_{\text{TM}}$

- Earlier, we proved $\overline{\text{A}}_{\text{TM}} \notin \textbf{RE}$ by proving that

$$\text{If } \overline{\text{A}}_{\text{TM}} \in \textbf{RE}\text{, then } L_{\text{D}} \in \textbf{RE}.$$

- The proof constructed this TM, assuming R was a recognizer for $\overline{\text{A}}_{\text{TM}}$.

H = "On input $\langle M \rangle$:
  - Construct the string $\langle M, \langle M \rangle \rangle$.
  - Run R on $\langle M, \langle M \rangle \rangle$.
  - If R accepts $\langle M, \langle M \rangle \rangle$, then H accepts $\langle M \rangle$.
  - If R rejects $\langle M, \langle M \rangle \rangle$, then H rejects $\langle M \rangle$."

- Let's do another proof using mapping reductions.

$L_D \leq_M \overline{A}_{TM}$

- To prove that $\overline{A}_{TM} \notin \mathbf{RE}$, we will prove

$$\boldsymbol{L_D \leq_M \overline{A}_{TM}}$$

- By our earlier theorem, since $L_D \notin \mathbf{RE}$, we have that $\overline{A}_{TM} \notin \mathbf{RE}$.
- *Intuitively*: $\overline{A}_{TM}$ is "at least as hard" as L_D, and since $L_D \notin \mathbf{RE}$, this means $\overline{A}_{TM} \notin \mathbf{RE}$.

$L_D \leq_M \overline{A}_{TM}$

- Goal: Find a computable function f such that

$$\boldsymbol{\langle M \rangle \in L_D \quad \textbf{iff} \quad f(\langle M \rangle) \in \overline{A}_{TM}}$$

- Simplifying this using the definition of L_D

$$\boldsymbol{M \text{ does not accept } \langle M \rangle \quad \textbf{iff} \quad f(\langle M \rangle) \in \overline{A}_{TM}}$$

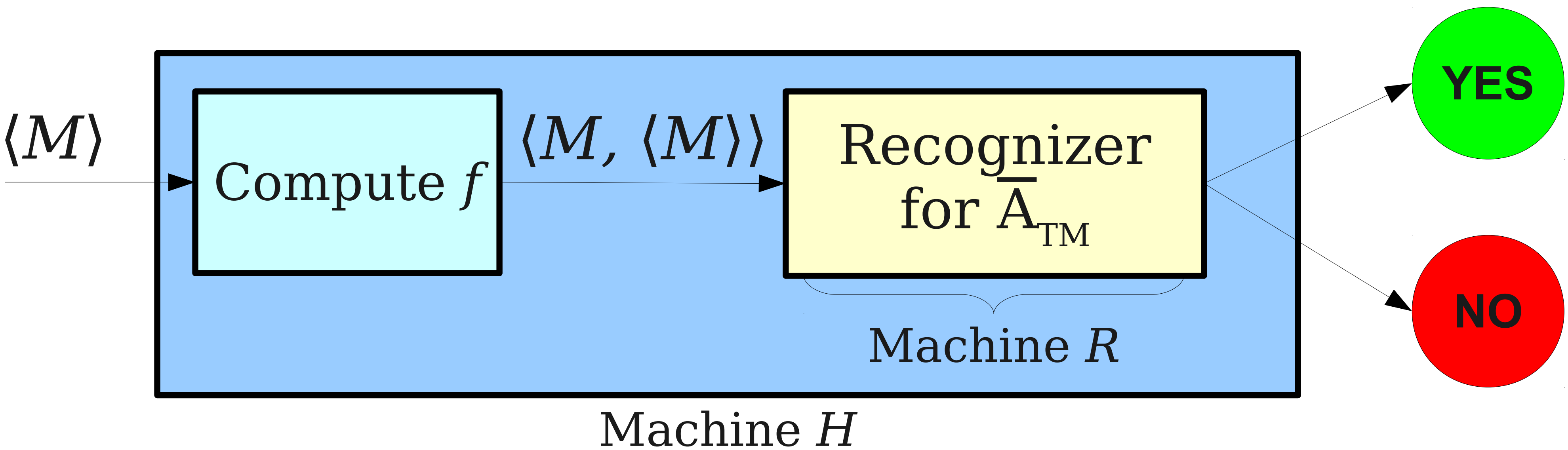
- Let's assume that $f(\langle M \rangle)$ has the form $\langle N, w \rangle$ for some TM N and string w. This means that

$$\boldsymbol{M \text{ does not accept } \langle M \rangle \quad \textbf{iff} \quad \langle N, w \rangle \in \overline{A}_{TM}}$$

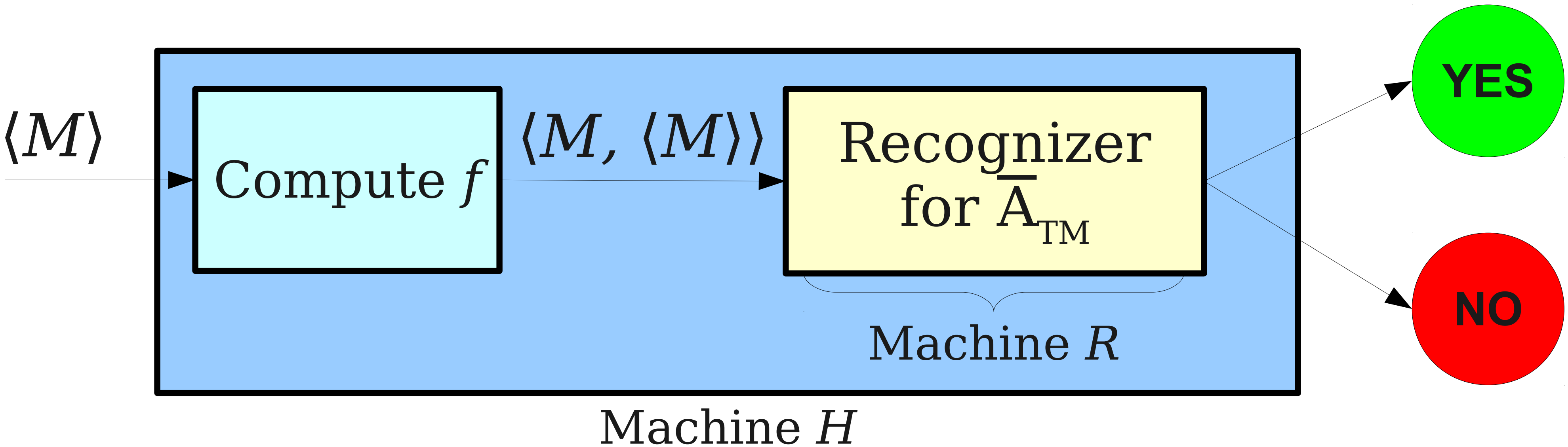
$$\boldsymbol{M \text{ does not accept } \langle M \rangle \quad \textbf{iff} \quad N \text{ does not accept } w}$$

- If we can choose w and N such that the above is true, we will have our reduction from L_D to $\overline{A}_{TM}$.
- Choose $\boldsymbol{N = M}$ and $\boldsymbol{w = \langle M \rangle}$.

One Interpretation of the Reduction



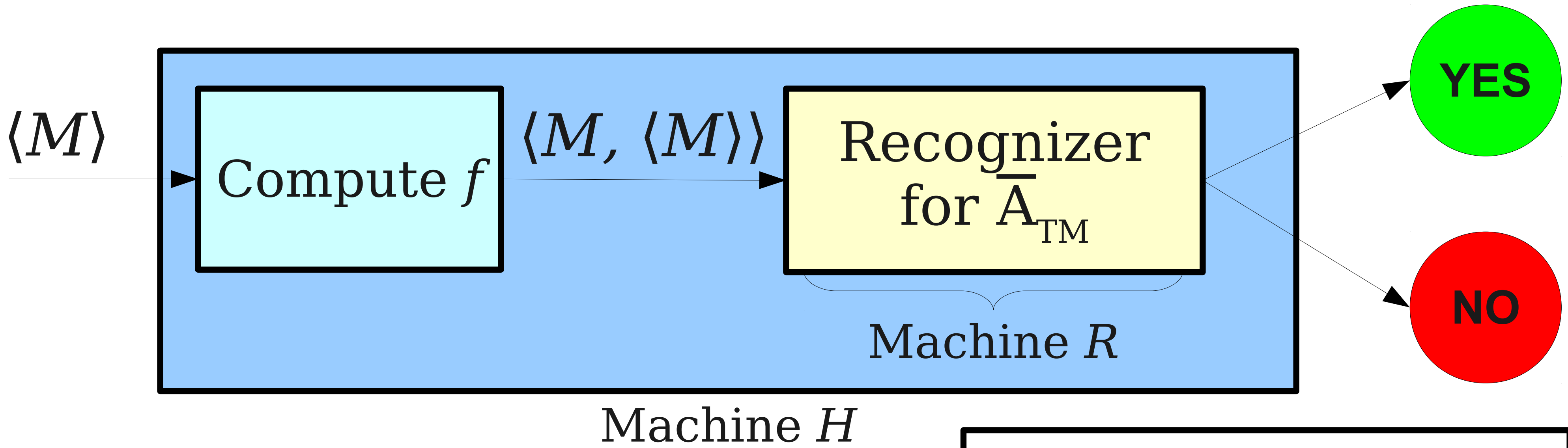
One Interpretation of the Reduction



H = "On input $\langle M \rangle$:

- Run machine R on $\langle M, \langle M \rangle \rangle$.
- If R accepts $\langle M, \langle M \rangle \rangle$, then H accepts w.
- If R rejects $\langle M, \langle M \rangle \rangle$, then H rejects w."

One Interpretation of the Reduction

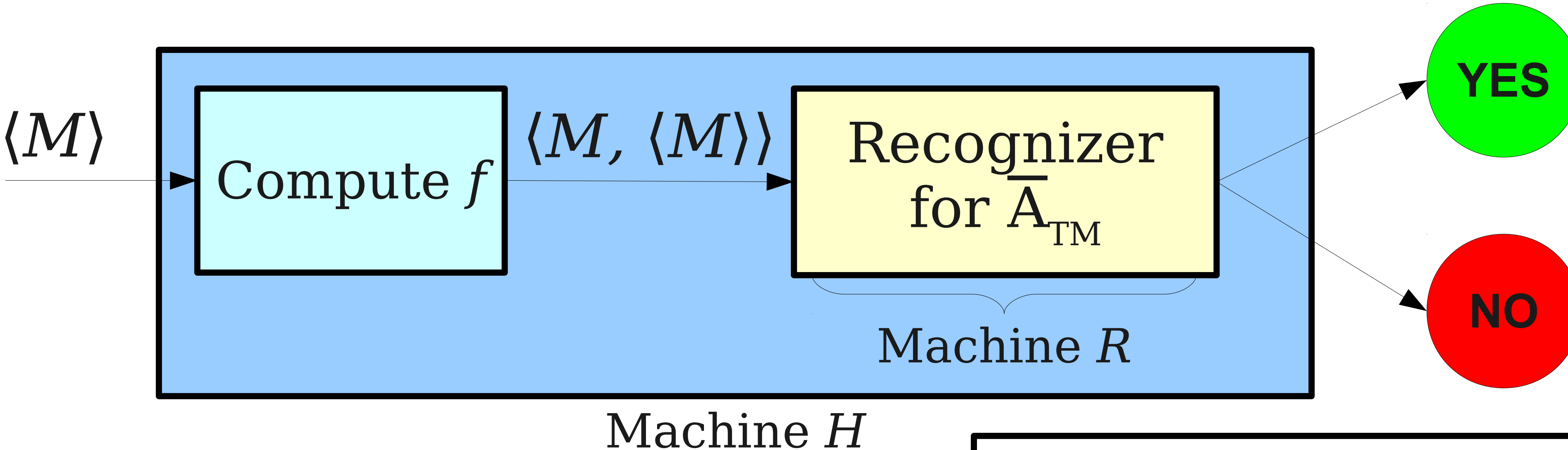


H = "On input $\langle M \rangle$:

- Run machine R on $\langle M, \langle M \rangle \rangle$.
- If R accepts $\langle M, \langle M \rangle \rangle$, then H accepts w.
- If R rejects $\langle M, \langle M \rangle \rangle$, then H rejects w."

H accepts $\langle M \rangle$

One Interpretation of the Reduction



H = "On input $\langle M \rangle$:

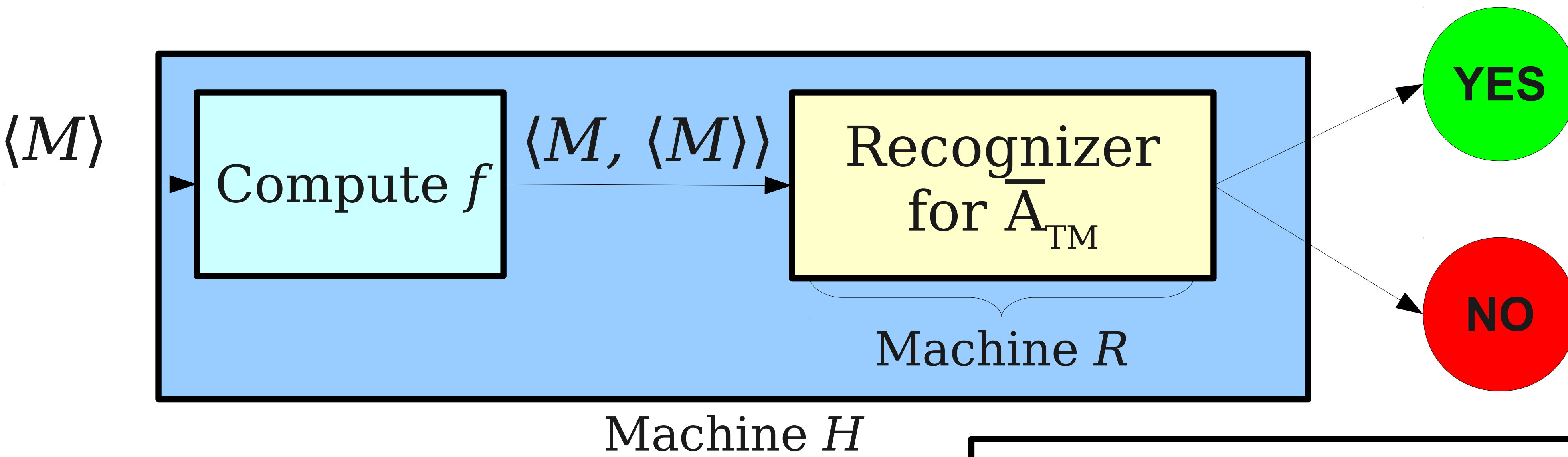
- Run machine R on $\langle M, \langle M \rangle \rangle$.
- If R accepts $\langle M, \langle M \rangle \rangle$, then H accepts w.
- If R rejects $\langle M, \langle M \rangle \rangle$, then H rejects w."

H accepts $\langle M \rangle$

iff

R accepts $\langle M, \langle M \rangle \rangle$

One Interpretation of the Reduction



H = "On input $\langle M \rangle$:

- Run machine R on $\langle M, \langle M \rangle \rangle$.
- If R accepts $\langle M, \langle M \rangle \rangle$, then H accepts w.
- If R rejects $\langle M, \langle M \rangle \rangle$, then H rejects w."

H accepts $\langle M \rangle$

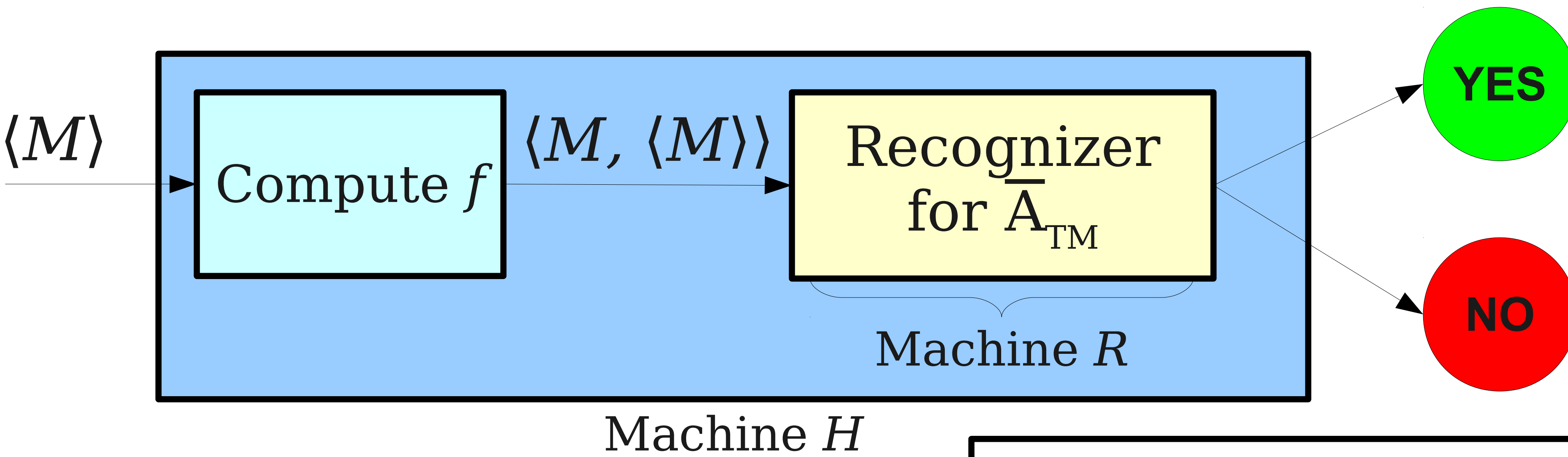
iff

R accepts $\langle M, \langle M \rangle \rangle$

iff

M does not accept $\langle M \rangle$

One Interpretation of the Reduction



H = "On input $\langle M \rangle$:

- Run machine R on $\langle M, \langle M \rangle \rangle$.
- If R accepts $\langle M, \langle M \rangle \rangle$, then H accepts w.
- If R rejects $\langle M, \langle M \rangle \rangle$, then H rejects w."

H accepts $\langle M \rangle$

iff

R accepts $\langle M, \langle M \rangle \rangle$

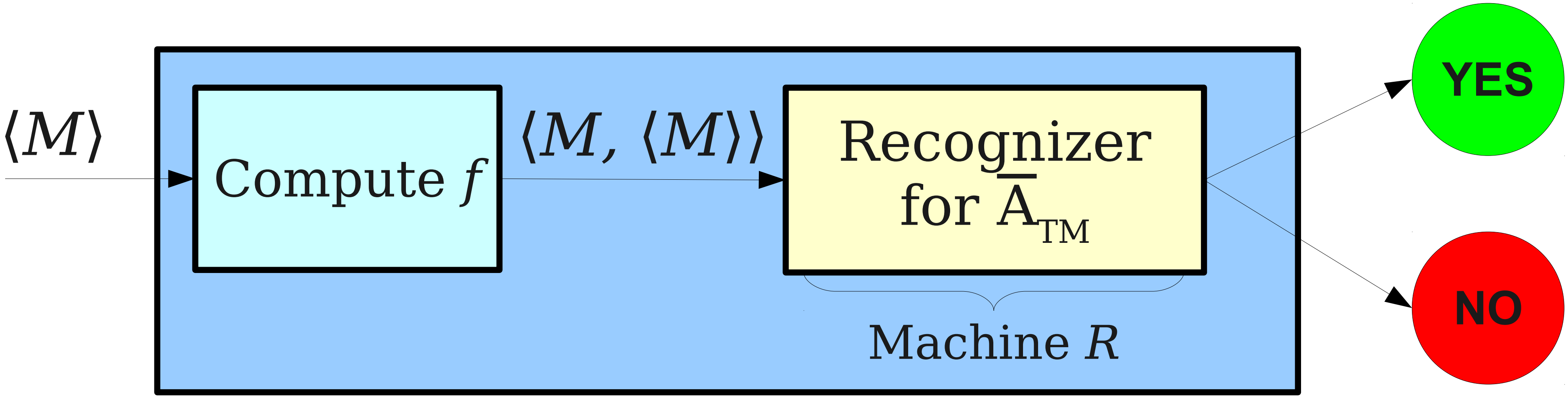
iff

M does not accept $\langle M \rangle$

iff

$\langle M \rangle \in L_D$

One Interpretation of the Reduction



H = "On input $\langle M \rangle$:

- Run machine R on $\langle M, \langle M \rangle\rangle$.
- If R accepts $\langle M, \langle M \rangle\rangle$, then H accepts w.
- If R rejects $\langle M, \langle M \rangle\rangle$, then H rejects w."

H accepts $\langle M \rangle$

iff

R accepts $\langle M, \langle M \rangle\rangle$

iff

M does not accept $\langle M \rangle$

iff

$\langle M \rangle \in L_D$

Theorem: $\overline{\mathrm{A}}_{\mathrm{TM}} \notin \mathbf{RE}$.

Proof: We exhibit a mapping reduction f from L_{D} to $\overline{\mathrm{A}}_{\mathrm{TM}}$.

Consider the function f defined as follows:

$$f(\langle M \rangle) = \langle M, \langle M \rangle \rangle$$

We claim that f can be computed by a TM and omit the details from this proof. We will prove that $\langle M \rangle \in L_{\mathrm{D}}$ iff $f(\langle M \rangle) \in \overline{\mathrm{A}}_{\mathrm{TM}}$. Note that $f(\langle M \rangle) = \langle M, \langle M \rangle \rangle$, so $f(\langle M \rangle) \in \overline{\mathrm{A}}_{\mathrm{TM}}$ iff $\langle M, \langle M \rangle \rangle \in \overline{\mathrm{A}}_{\mathrm{TM}}$. By definition of $\overline{\mathrm{A}}_{\mathrm{TM}}$, $\langle M, \langle M \rangle \rangle \in \overline{\mathrm{A}}_{\mathrm{TM}}$ iff $\langle M \rangle \notin \mathscr{L}(M)$. Finally, note that $\langle M \rangle \notin \mathscr{L}(M)$ iff $\langle M \rangle \in L_{\mathrm{D}}$. Thus $f(\langle M \rangle) \in \overline{\mathrm{A}}_{\mathrm{TM}}$ iff $\langle M \rangle \in L_{\mathrm{D}}$, so f is a mapping reduction from L_{D} to $\overline{\mathrm{A}}_{\mathrm{TM}}$.

Since f is a mapping reduction from L_{D} to $\overline{\mathrm{A}}_{\mathrm{TM}}$, we have $L_{\mathrm{D}} \leq_{\mathrm{M}} \overline{\mathrm{A}}_{\mathrm{TM}}$. Since $L_{\mathrm{D}} \notin \mathbf{RE}$ and $L_{\mathrm{D}} \leq_{\mathrm{M}} \overline{\mathrm{A}}_{\mathrm{TM}}$, this means $\overline{\mathrm{A}}_{\mathrm{TM}} \notin \mathbf{RE}$, as required. ■

Another Example of Mapping Reductions

A More Elaborate Reduction

- Since $\overline{\mathrm{A}}_{\mathrm{TM}} \notin$ **RE**, there is no algorithm for determining whether a TM will not accept a given string.
- Could we check instead whether a TM *never* accepts a string?
- Consider the language

$$L_e = \{ \langle M \rangle \mid M \text{ is a TM and } M \text{ never accepts } \}$$

- How "hard" is L_e? Is it **R**, **RE**, co-**RE**, or none of these?

Building an Intuition

- Before we even try to prove how “hard” this language is, we should build an intuition for its difficulty.
- L_e is *probably* not in **RE**, since if we were convinced a TM never accepted, it would be hard to find positive evidence of this.
- L_e is *probably* in co-**RE**, since if we were convinced that a TM *did* accept some string, we could exhaustively search over all strings and try to find the string it accepts.
- Best guess: $L_e \in$ co-**RE** – **R**.

$\overline{\mathrm{A}}_{\mathrm{TM}} \leq_{\mathrm{M}} L_{\mathrm{e}}$

- We will prove that $L_{\mathrm{e}} \notin \mathbf{RE}$ by showing that $\overline{\mathrm{A}}_{\mathrm{TM}} \leq_{\mathrm{M}} L_{\mathrm{e}}$. (This also proves $L_{\mathrm{e}} \notin \mathbf{R}$).
- We want to find a function f such that

$$\boldsymbol{\langle M, w\rangle \in \overline{\mathrm{A}}_{\mathrm{TM}} \quad \text{iff} \quad f(\langle M, w\rangle) \in L_{\mathrm{e}}}$$

- Since L_{e} is a language of TM descriptions, let's assume $f(\langle M, w\rangle) = \langle N\rangle$ for some TM N. Then

$$\boldsymbol{\langle M, w\rangle \in \overline{\mathrm{A}}_{\mathrm{TM}} \quad \text{iff} \quad \langle N\rangle \in L_{\mathrm{e}}}$$

- Expanding out definitions, we get

***M* doesn't accept *w* iff *N* doesn't accept any strings**

- How do we pick the machine N?

The Reduction

- Find a TM N such that N does not accept any strings iff M does not accept w.
- **Key idea:** Build N such that running N on any input runs M on w.
- Here is one choice of N:

  N = "On input x:

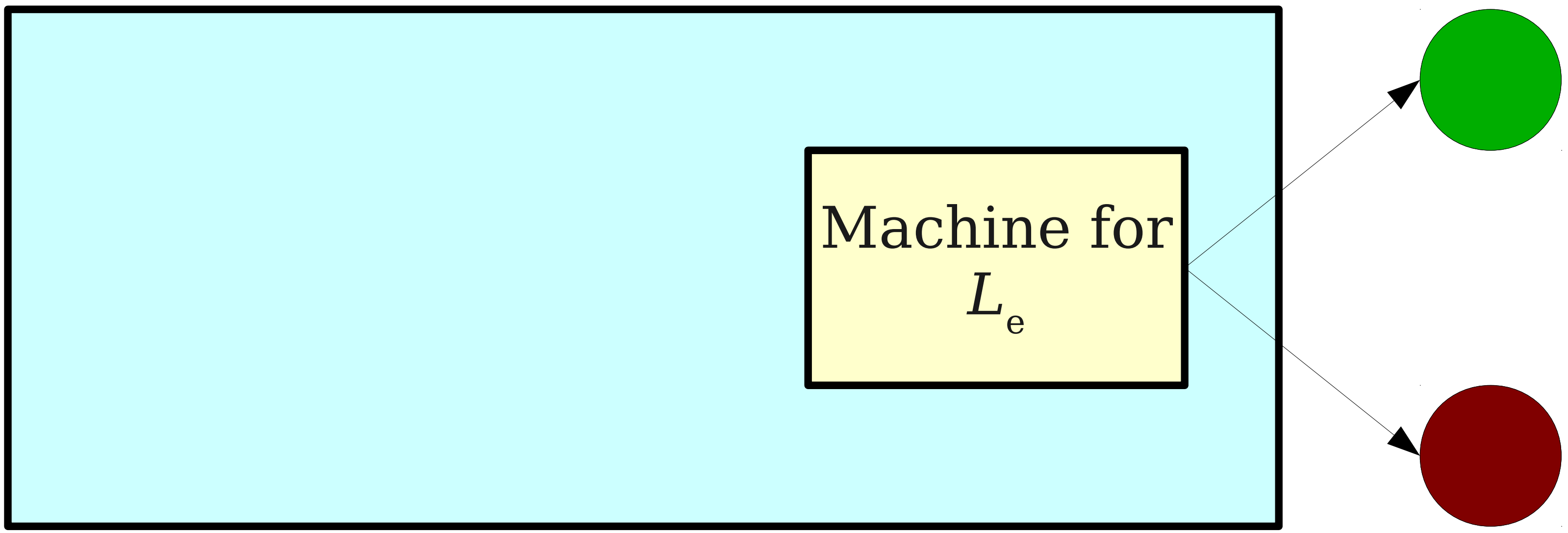
  Ignore x.

  Run M on w.

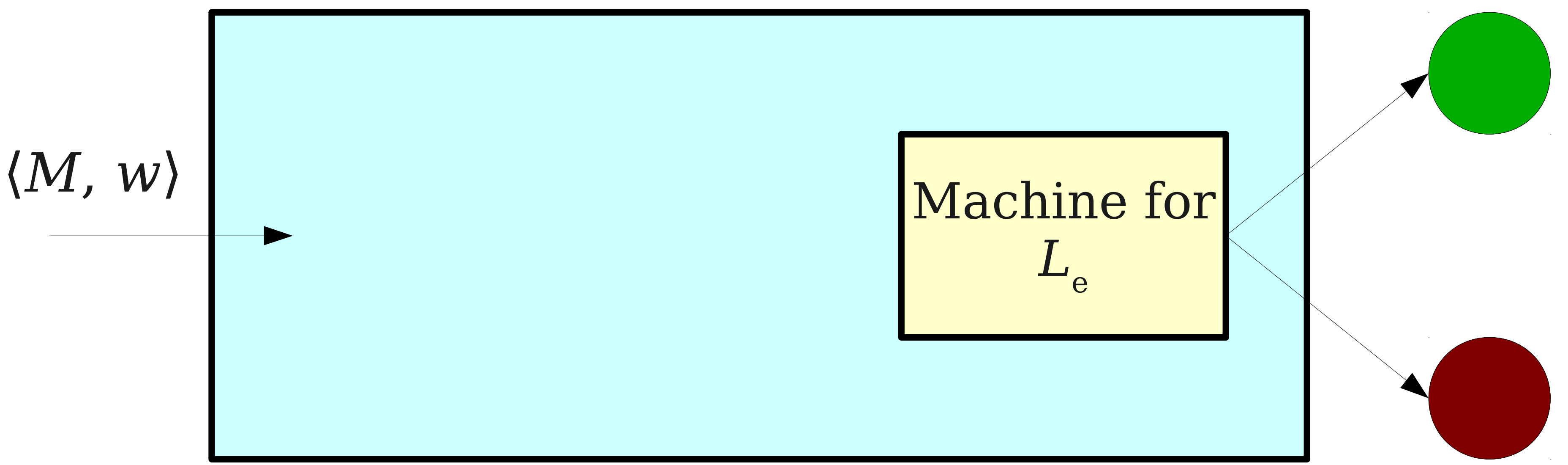
  If M accepts w, then N accepts x.

  If M rejects w, then N rejects x."
- Notice that N "amplifies" what M does on w:
  - If M does not accept w, N does not accept anything.
  - If M does accept w, N accepts everything.

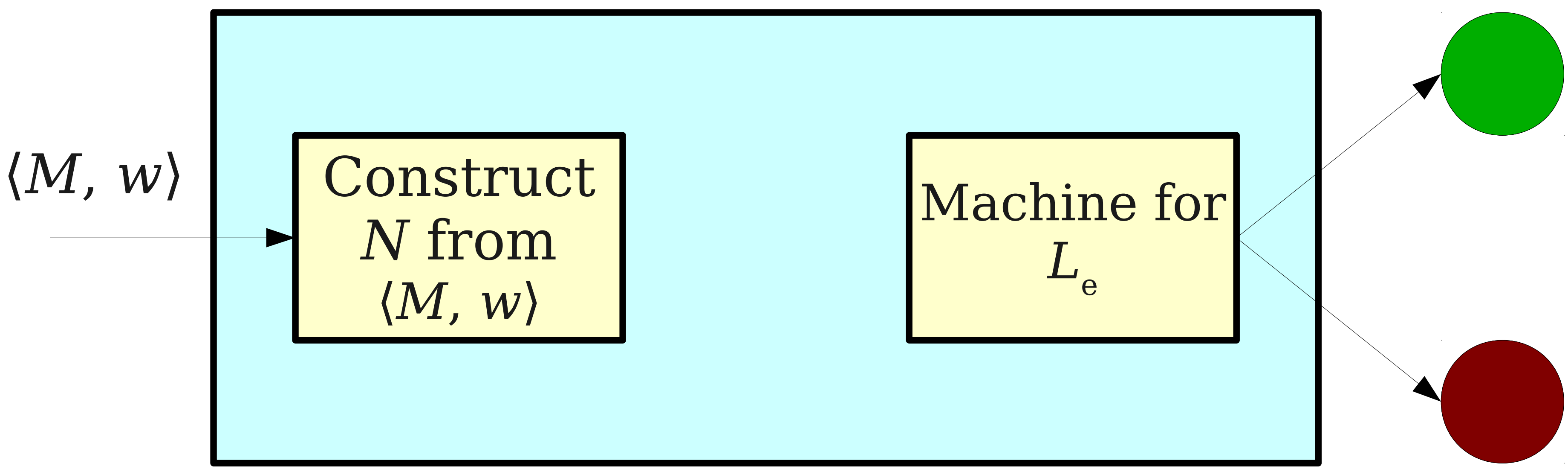
The Reduction



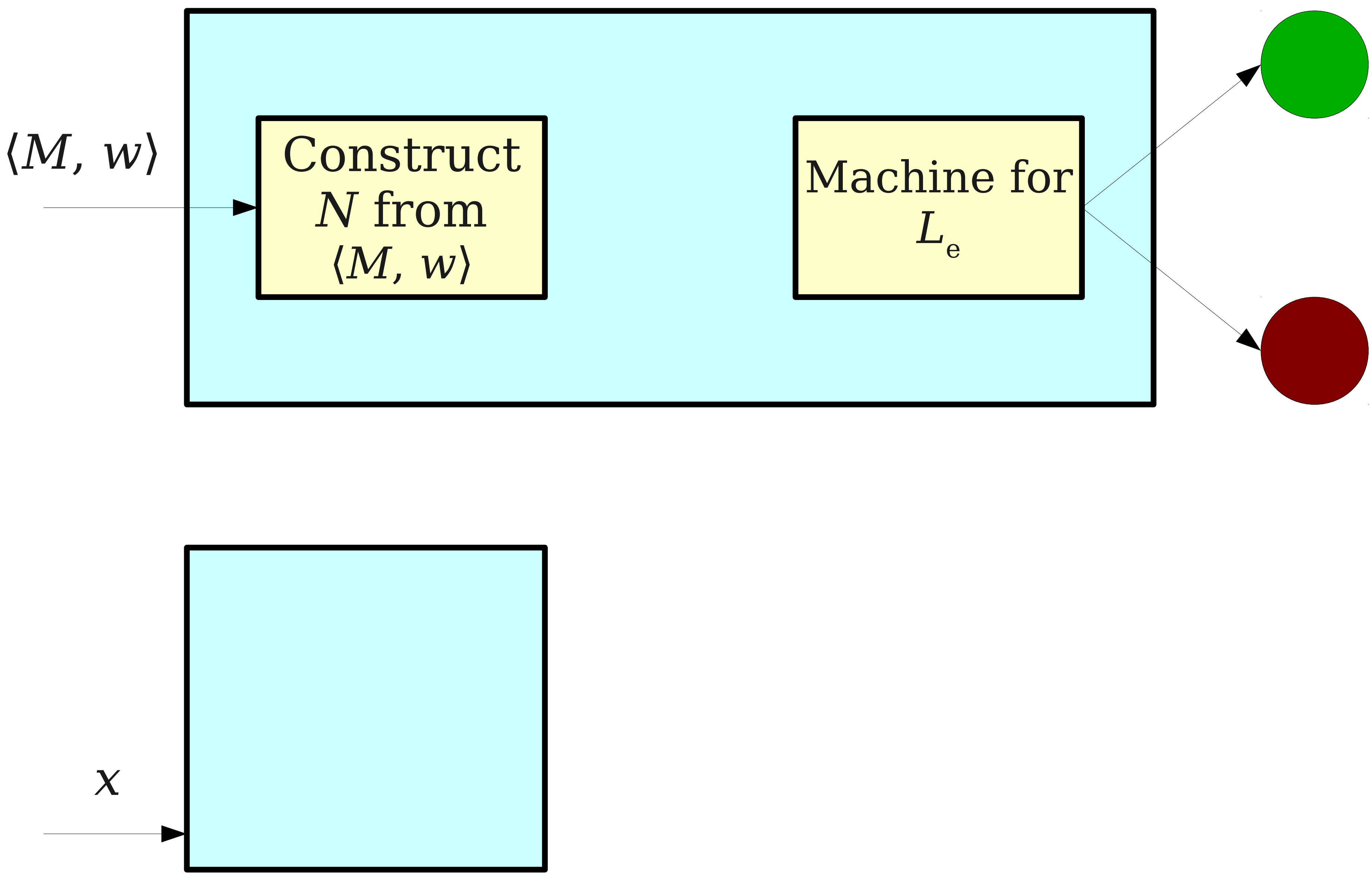
The Reduction



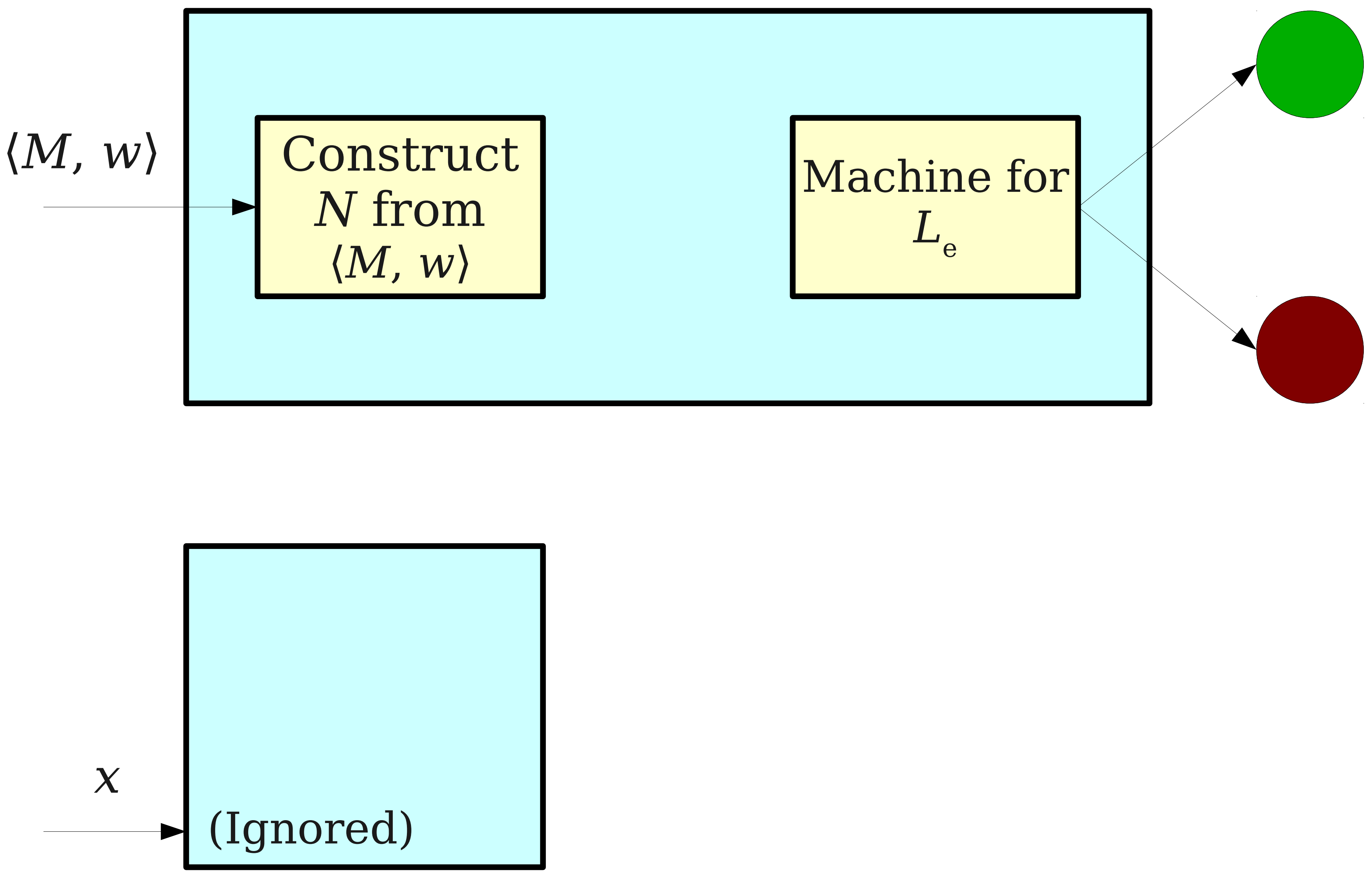
The Reduction



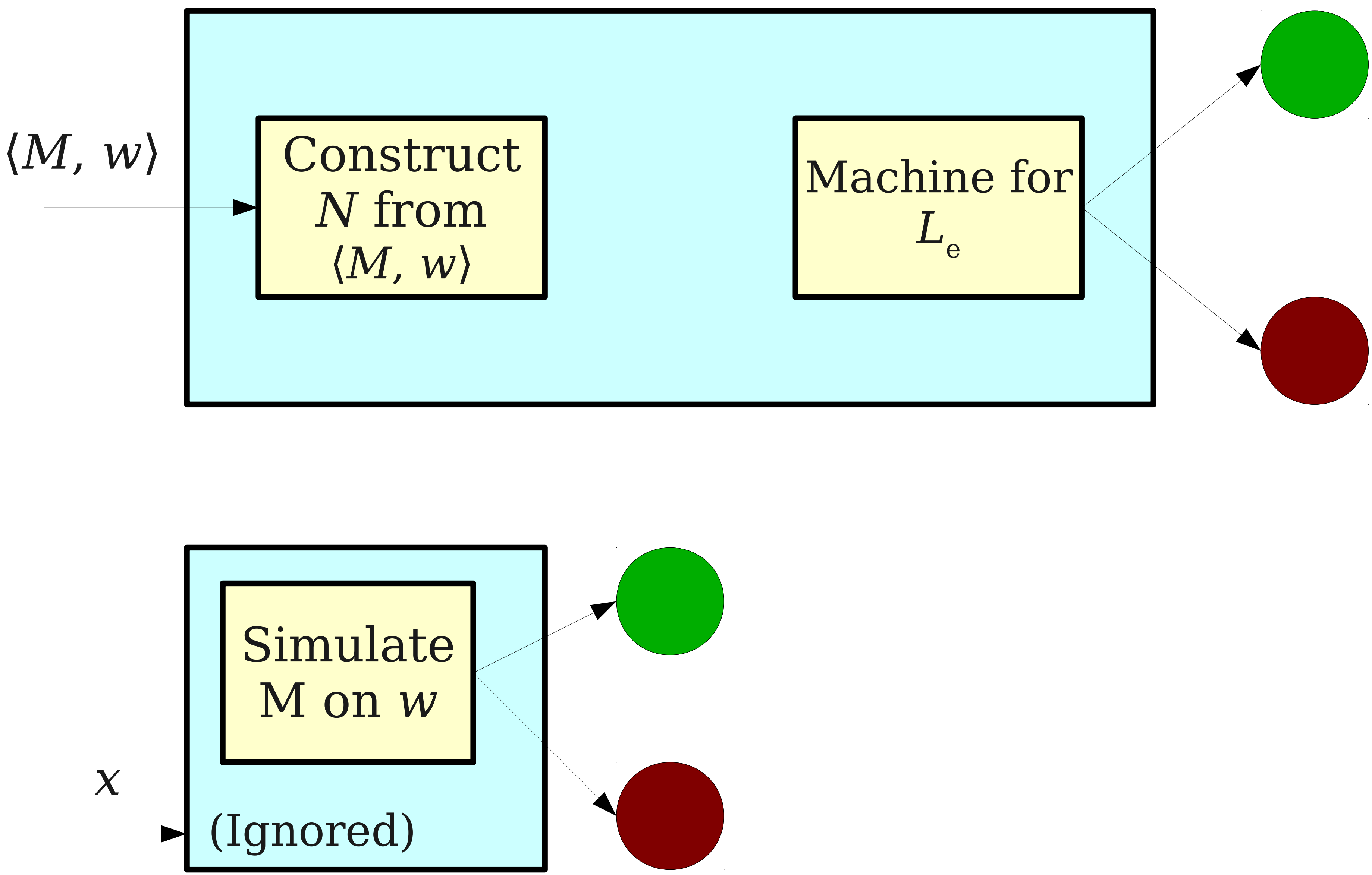
The Reduction



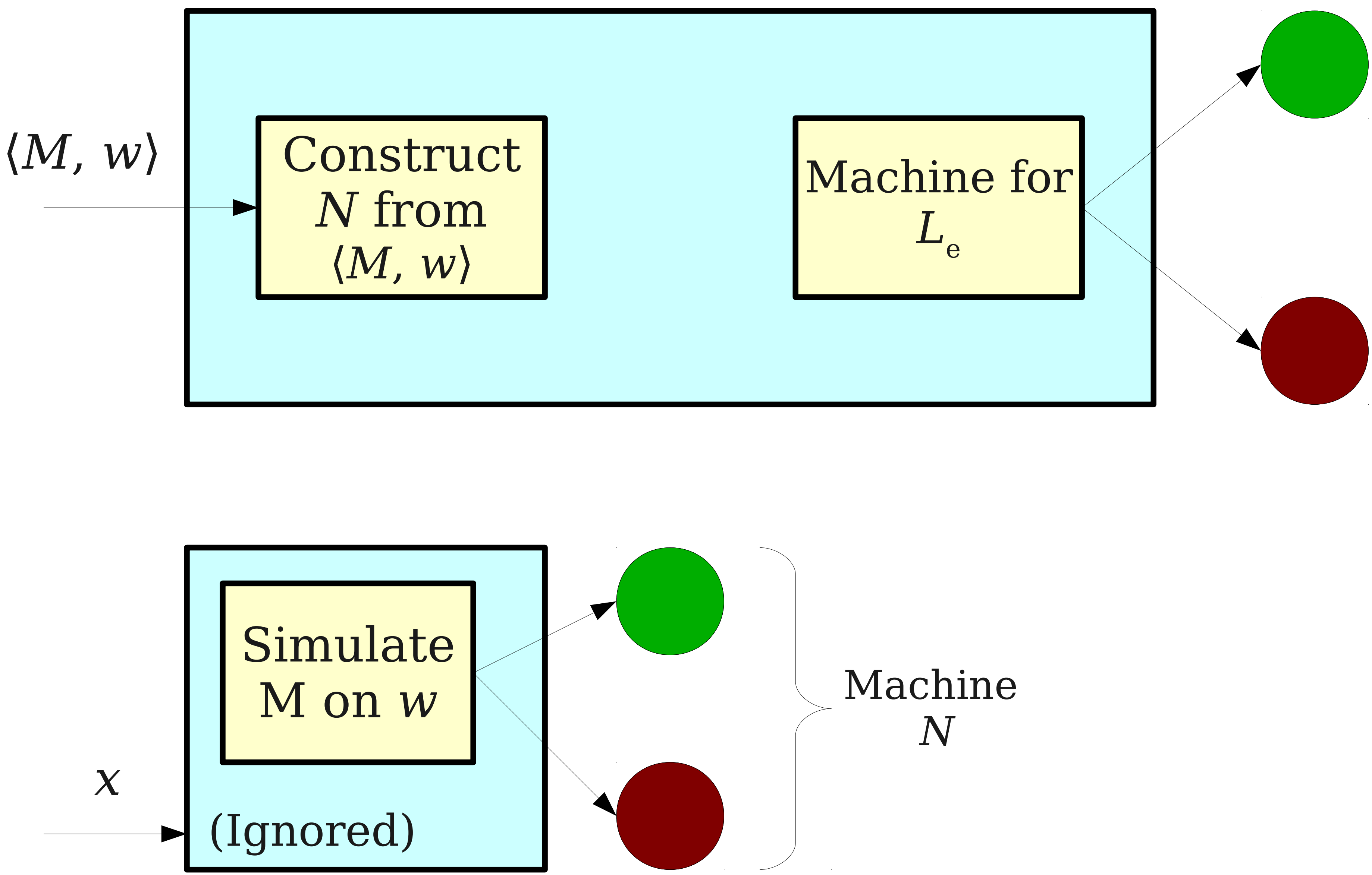
The Reduction



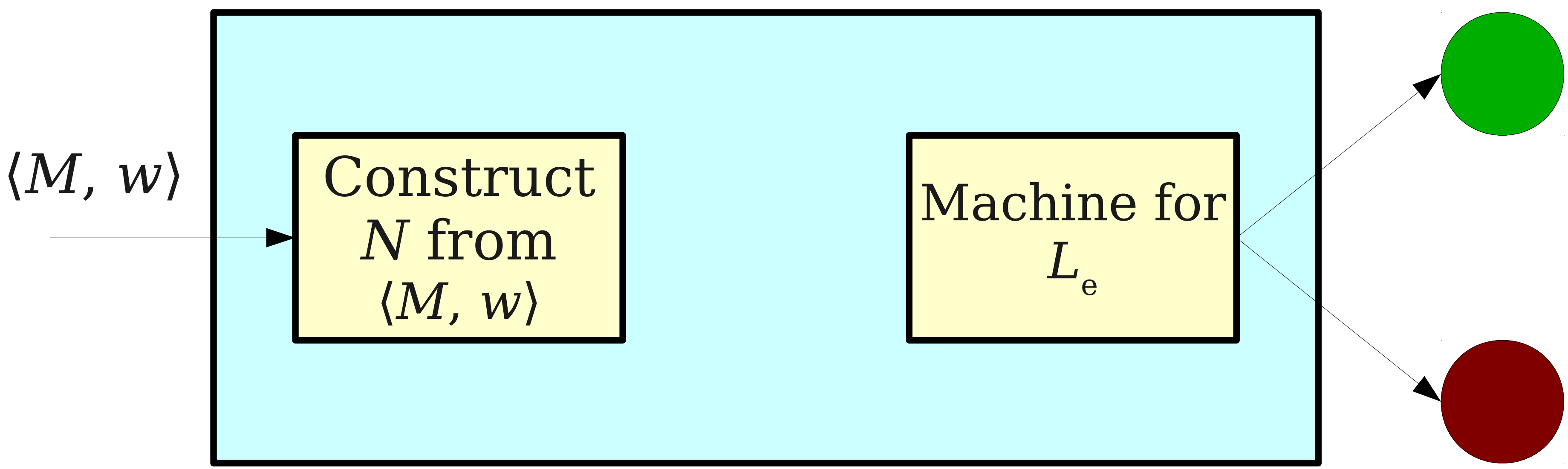
The Reduction

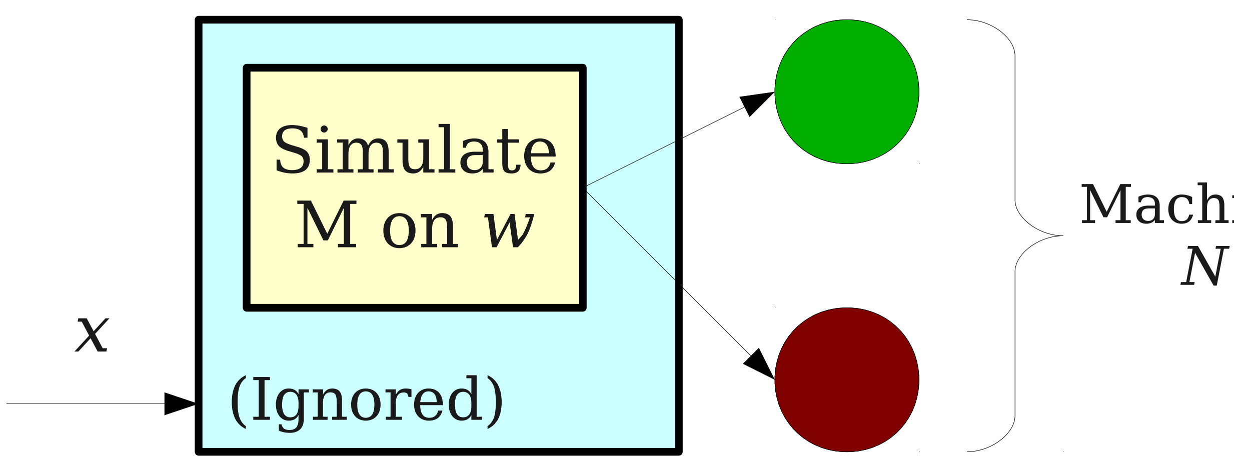


The Reduction



The Reduction

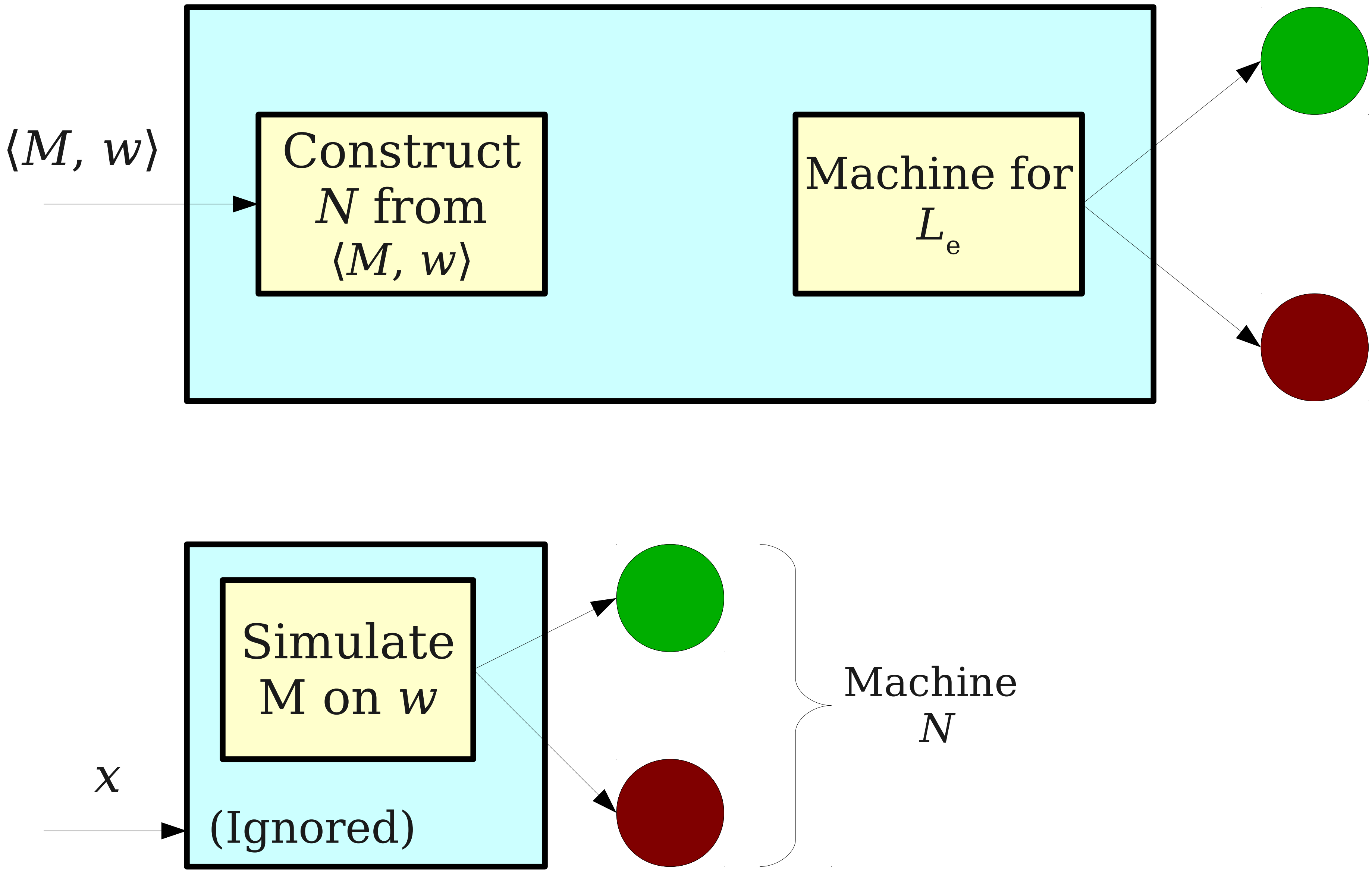




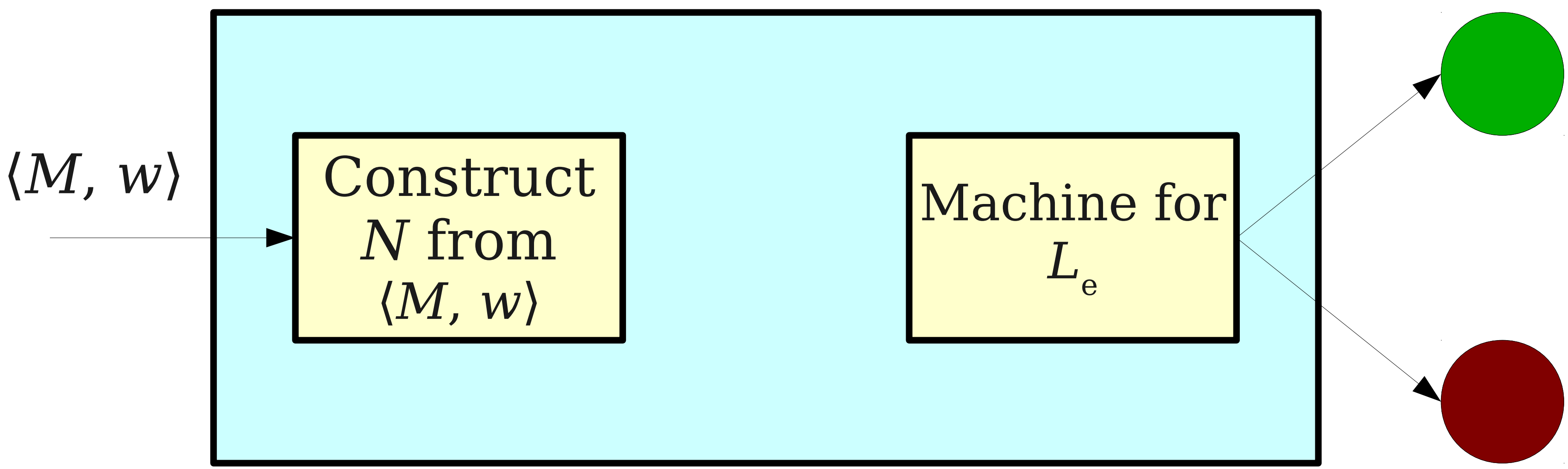
N = "On input x:

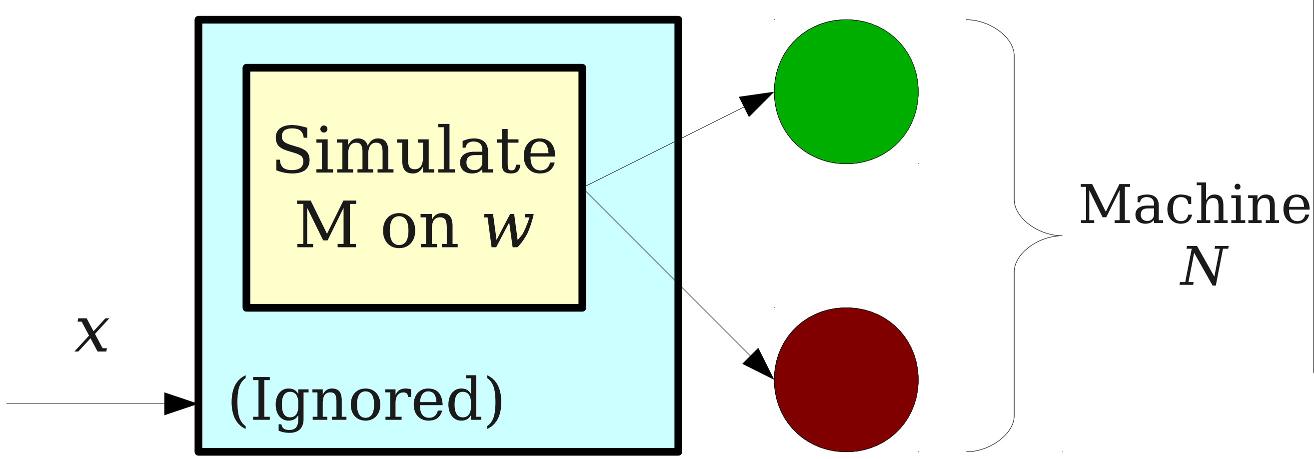
- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

The Reduction



The Reduction

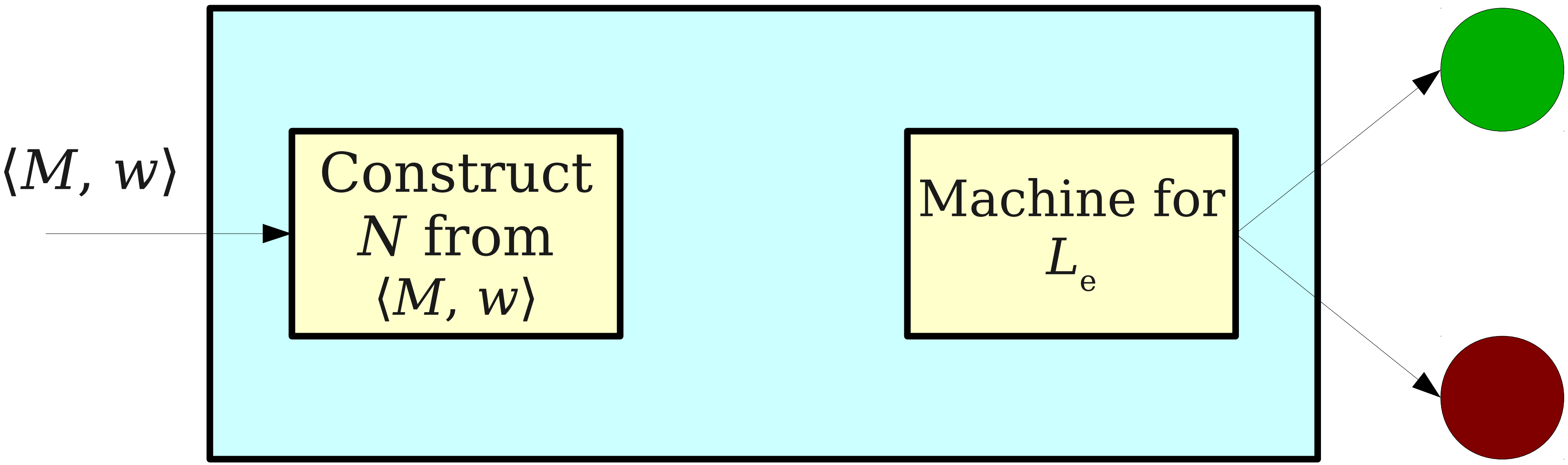


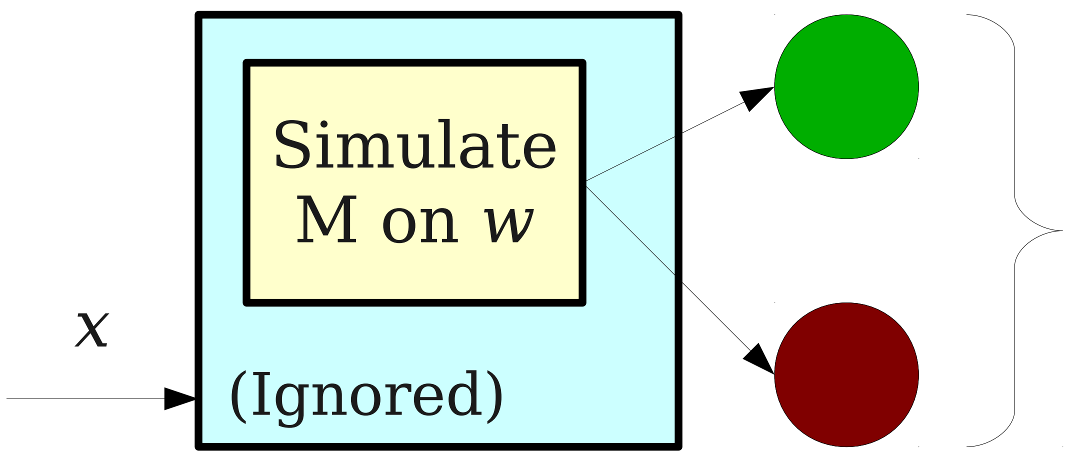


What does N do on *any* input if…

M accepts w?

The Reduction

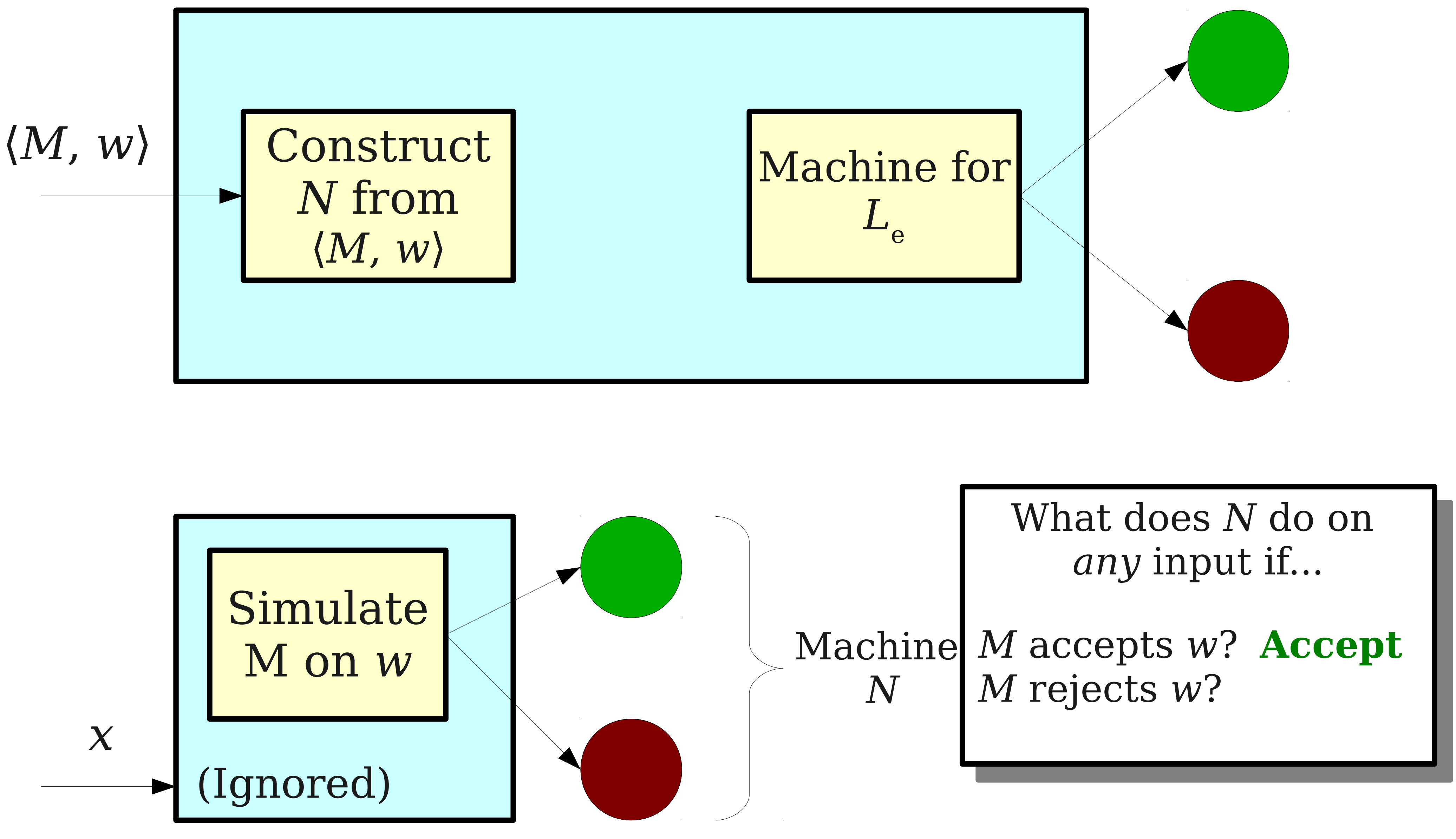




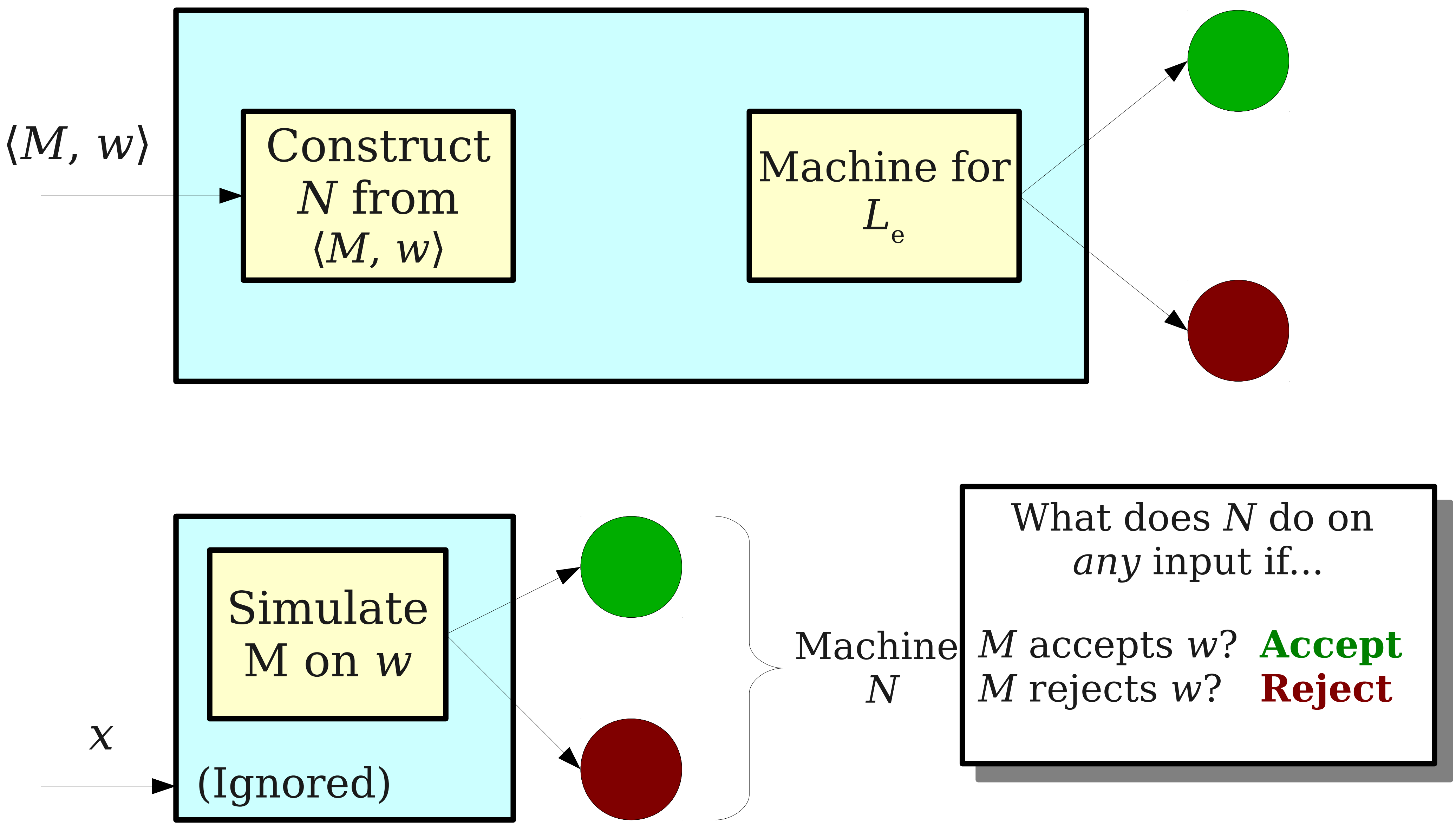
What does N do on *any* input if…

M accepts w? **Accept**

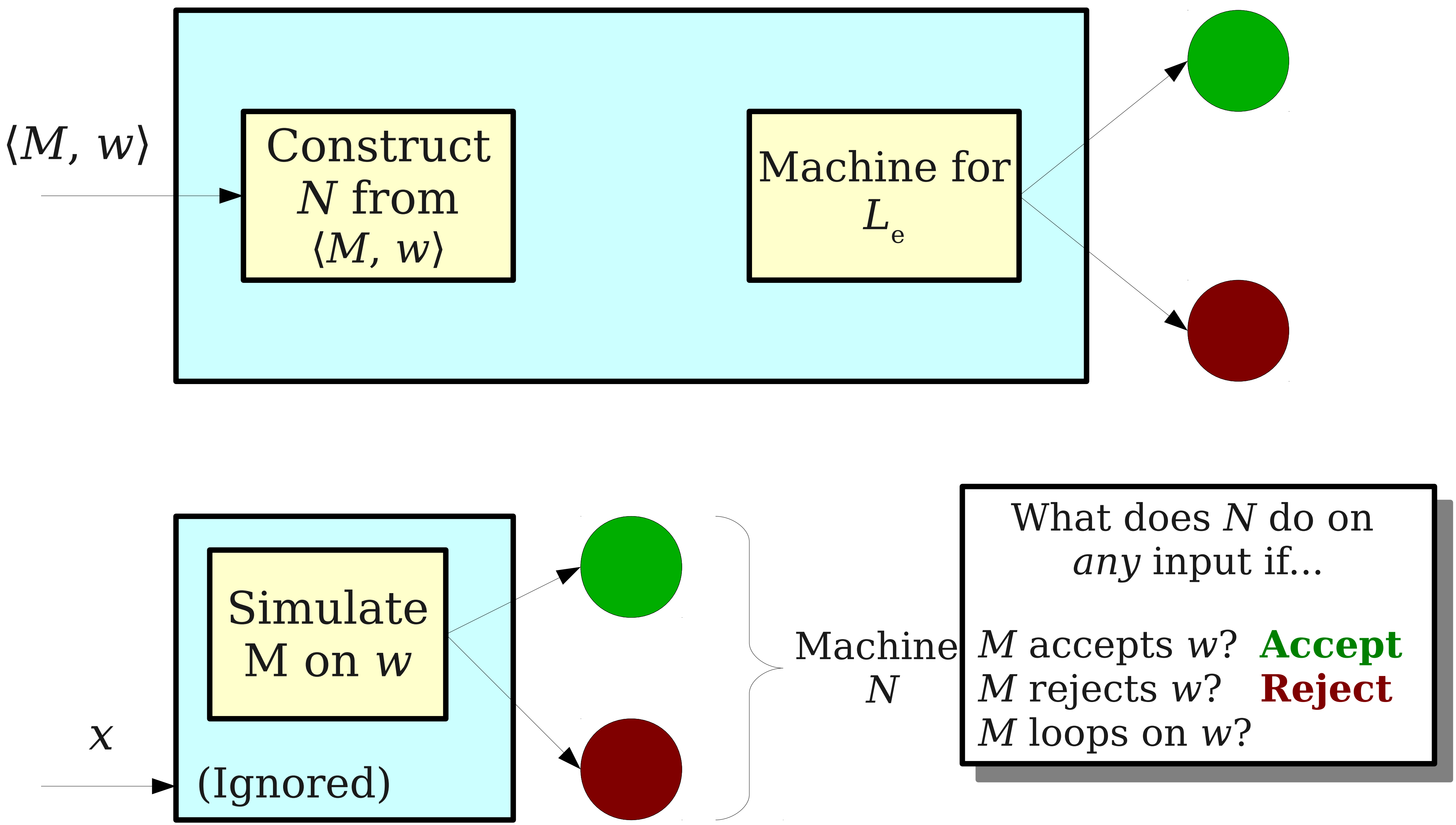
The Reduction



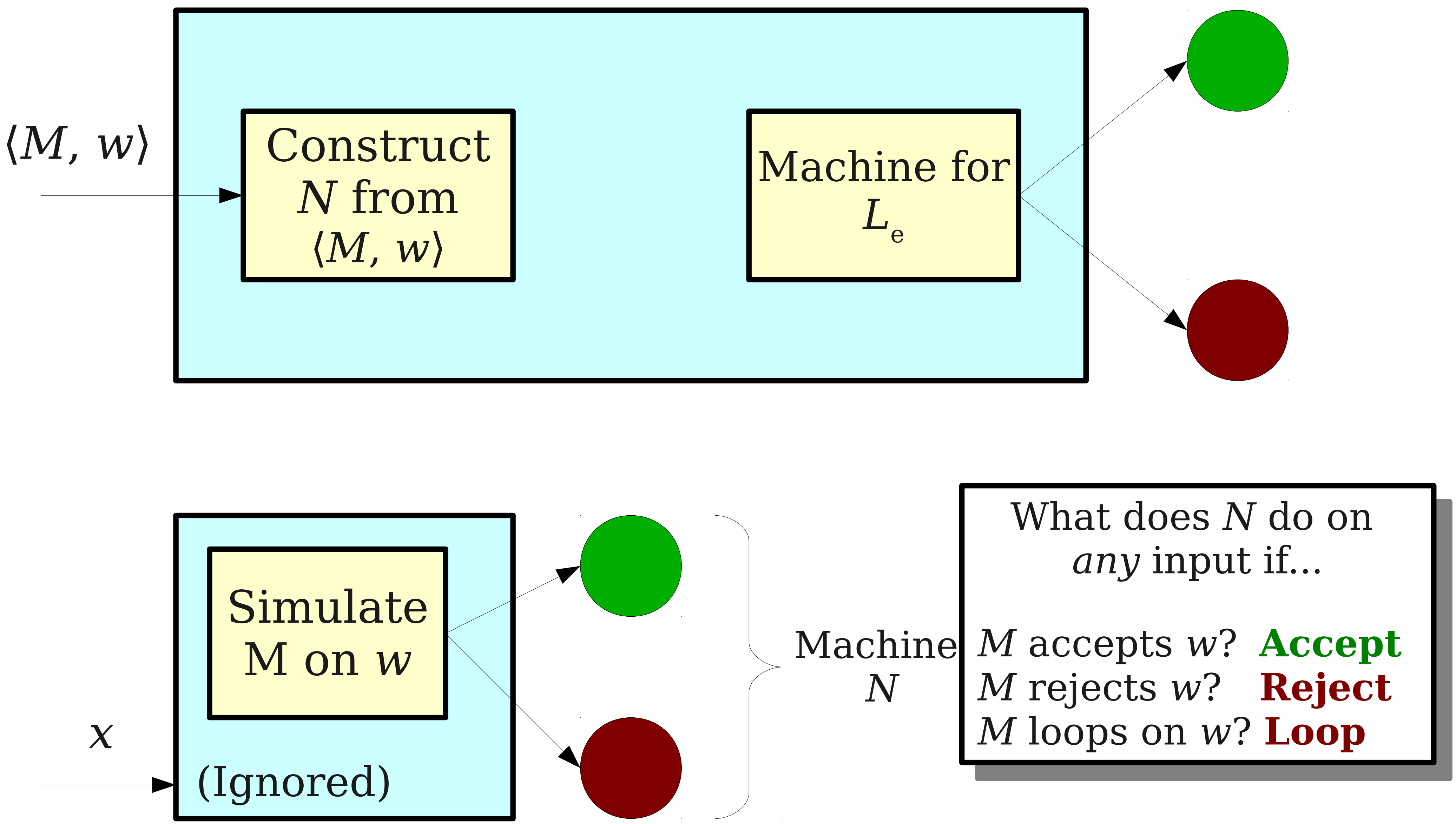
The Reduction



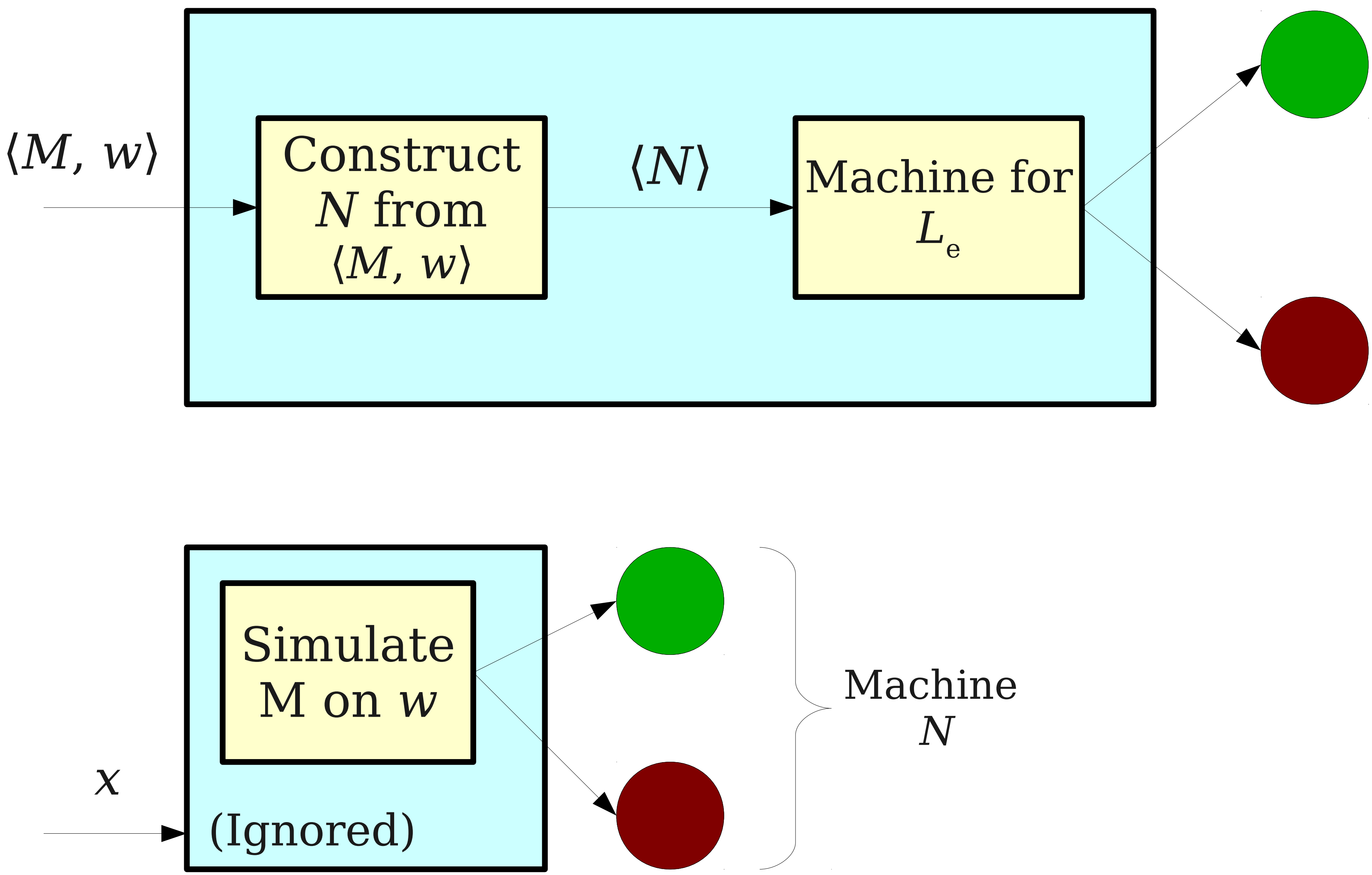
The Reduction



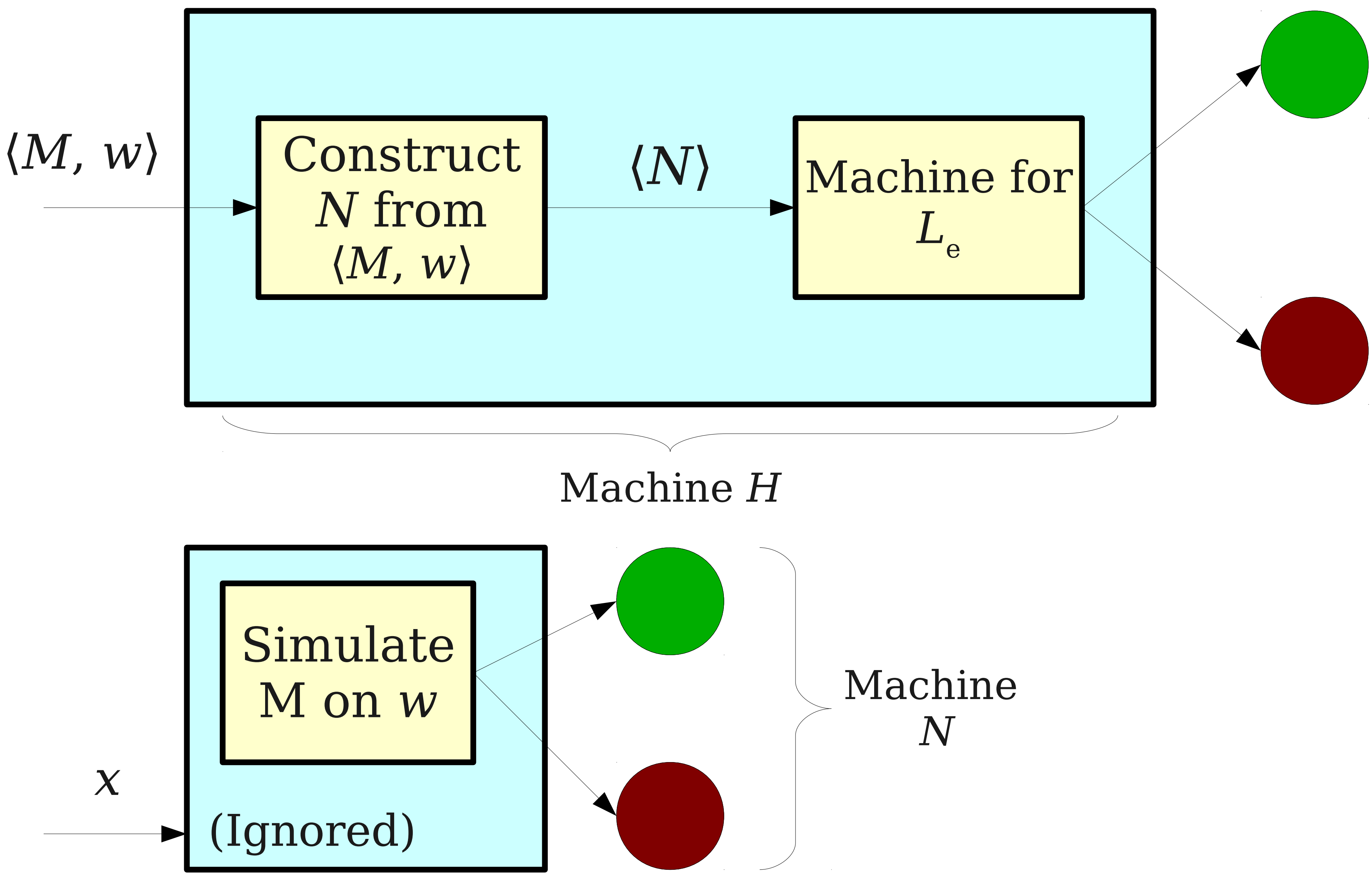
The Reduction



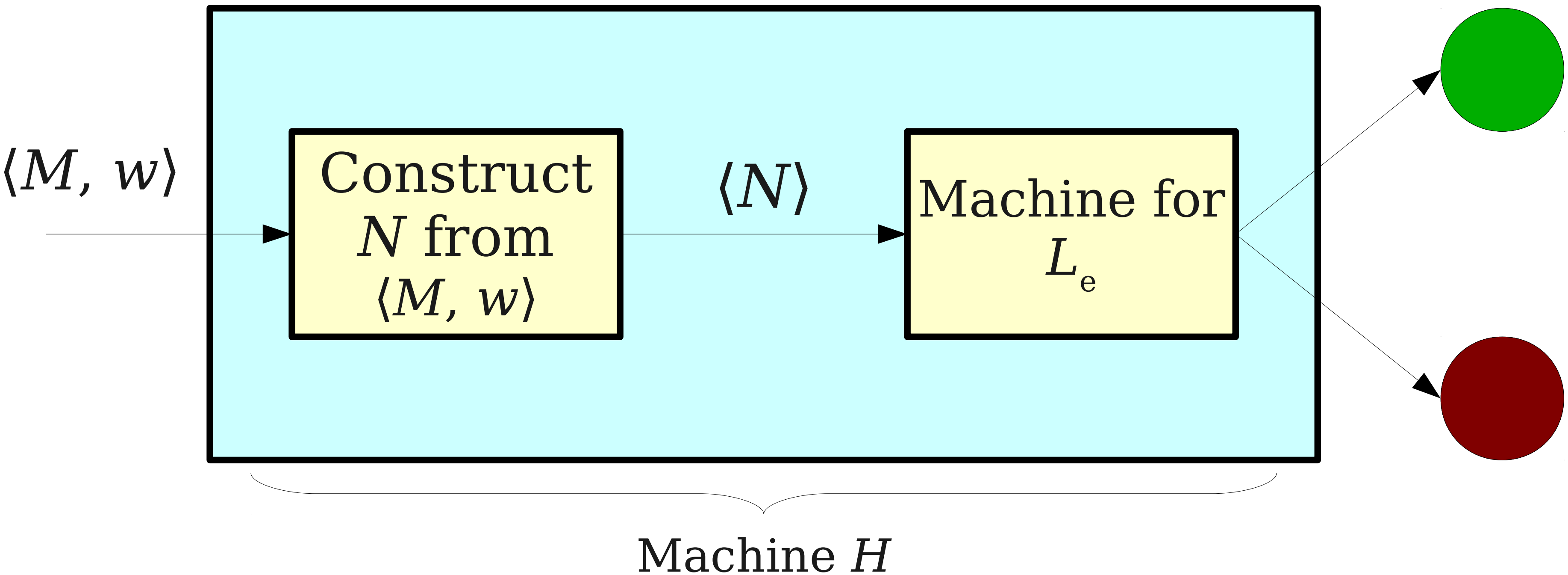
The Reduction

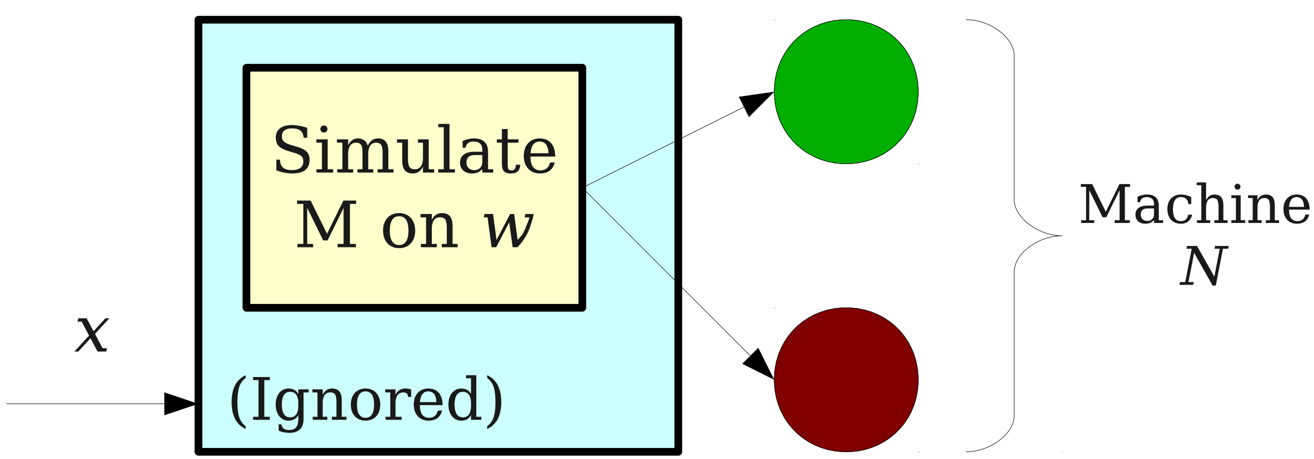


The Reduction



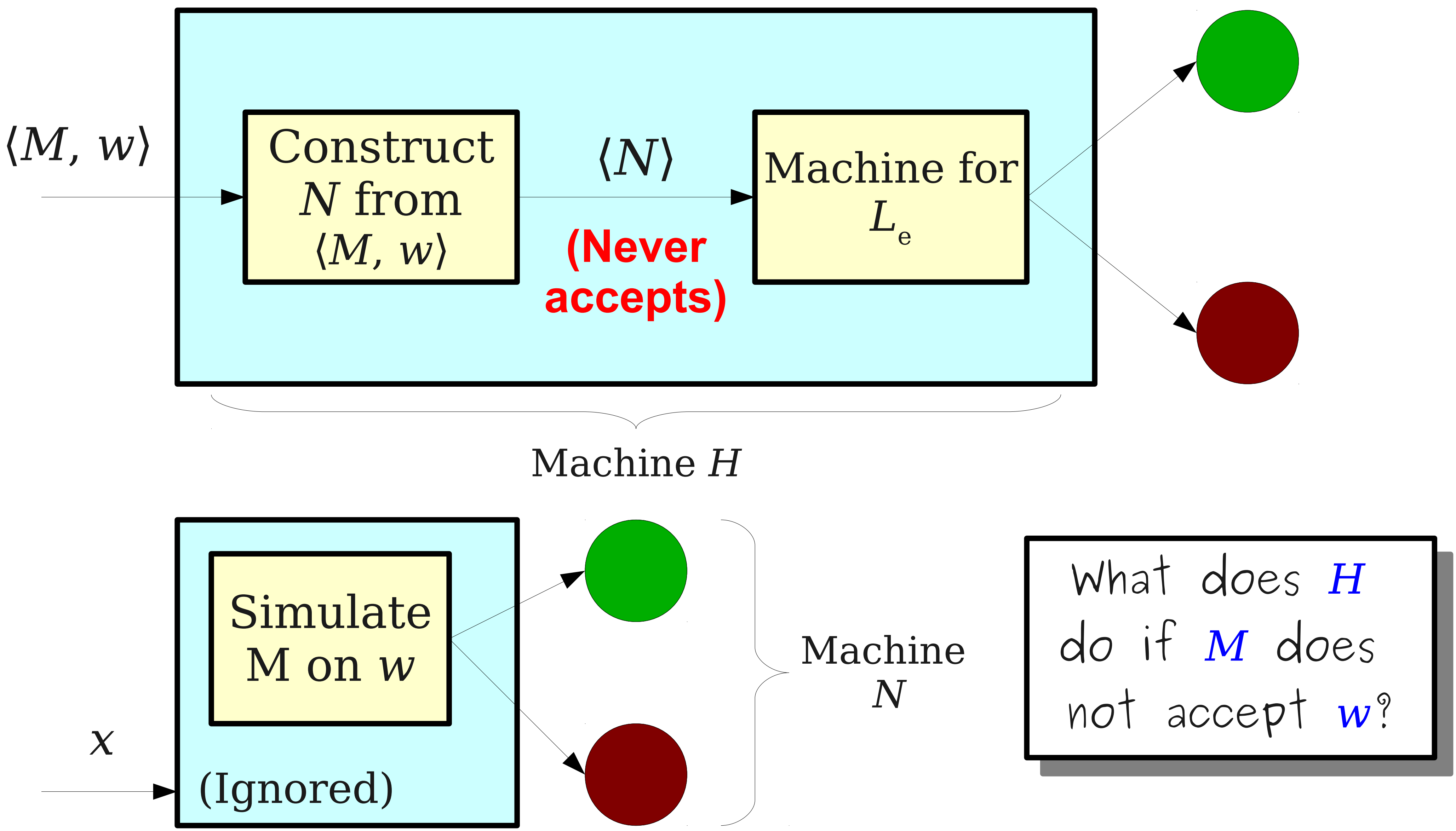
The Reduction



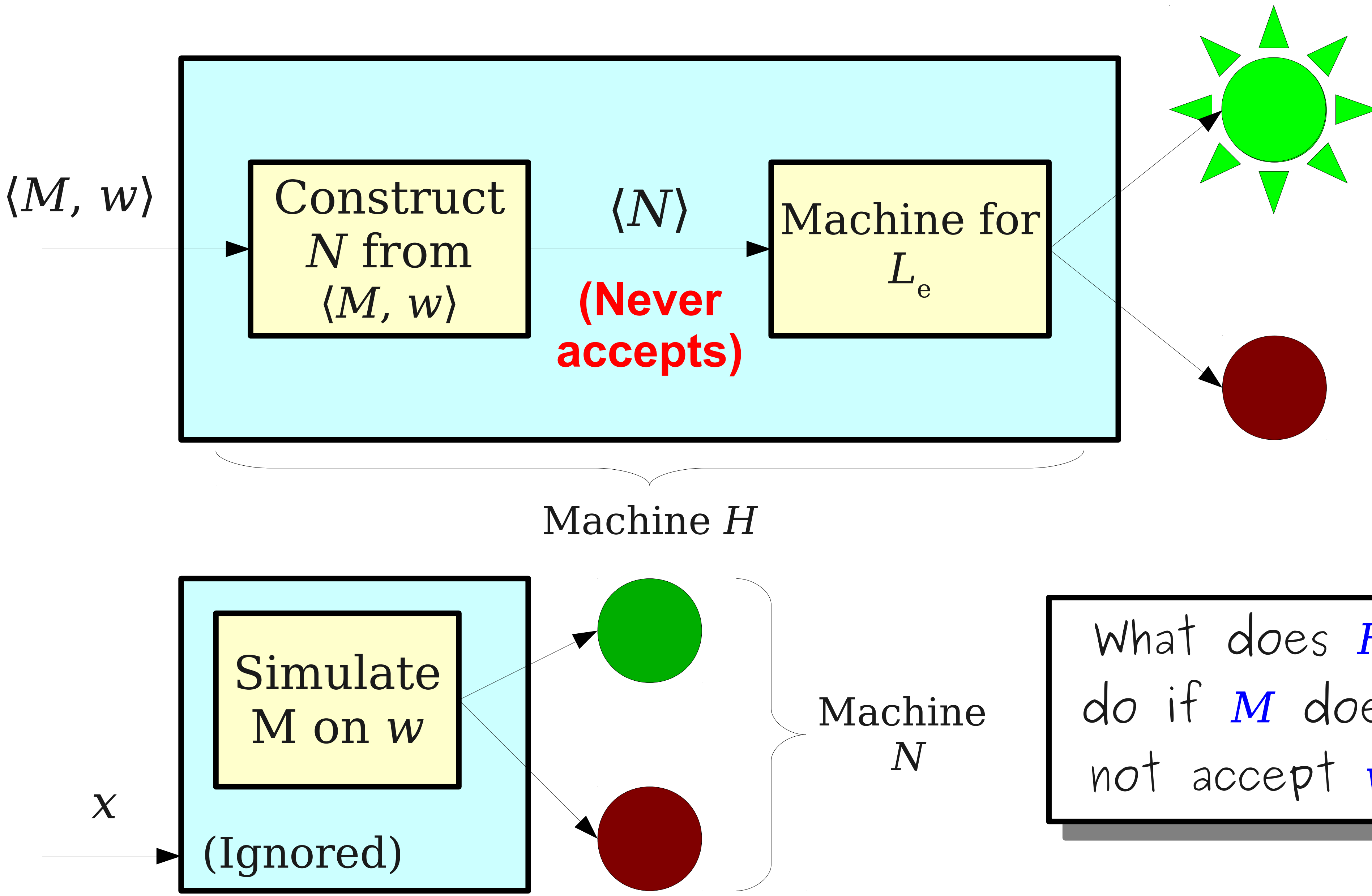


What does H do if M does not accept w?

The Reduction

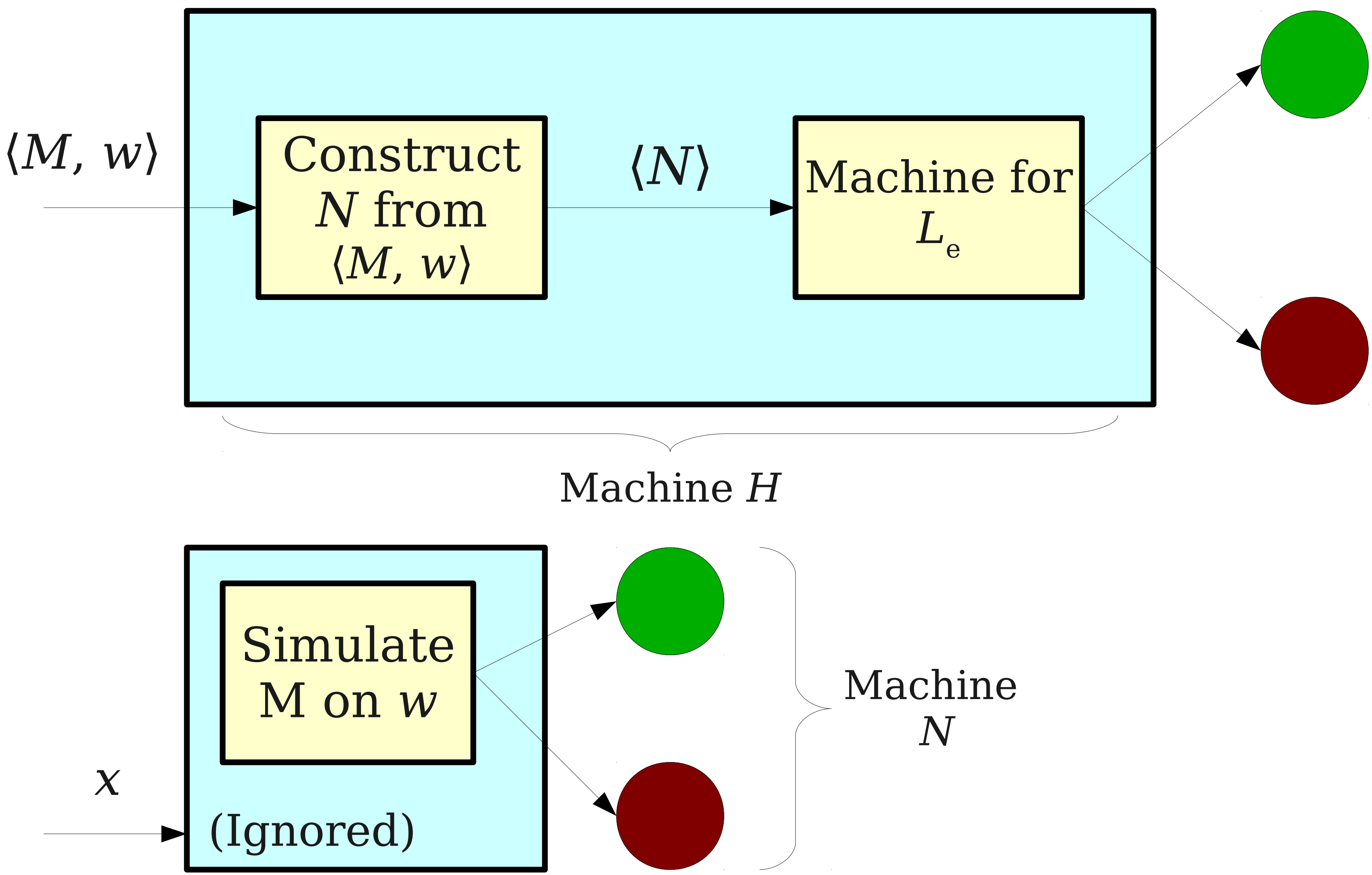


The Reduction

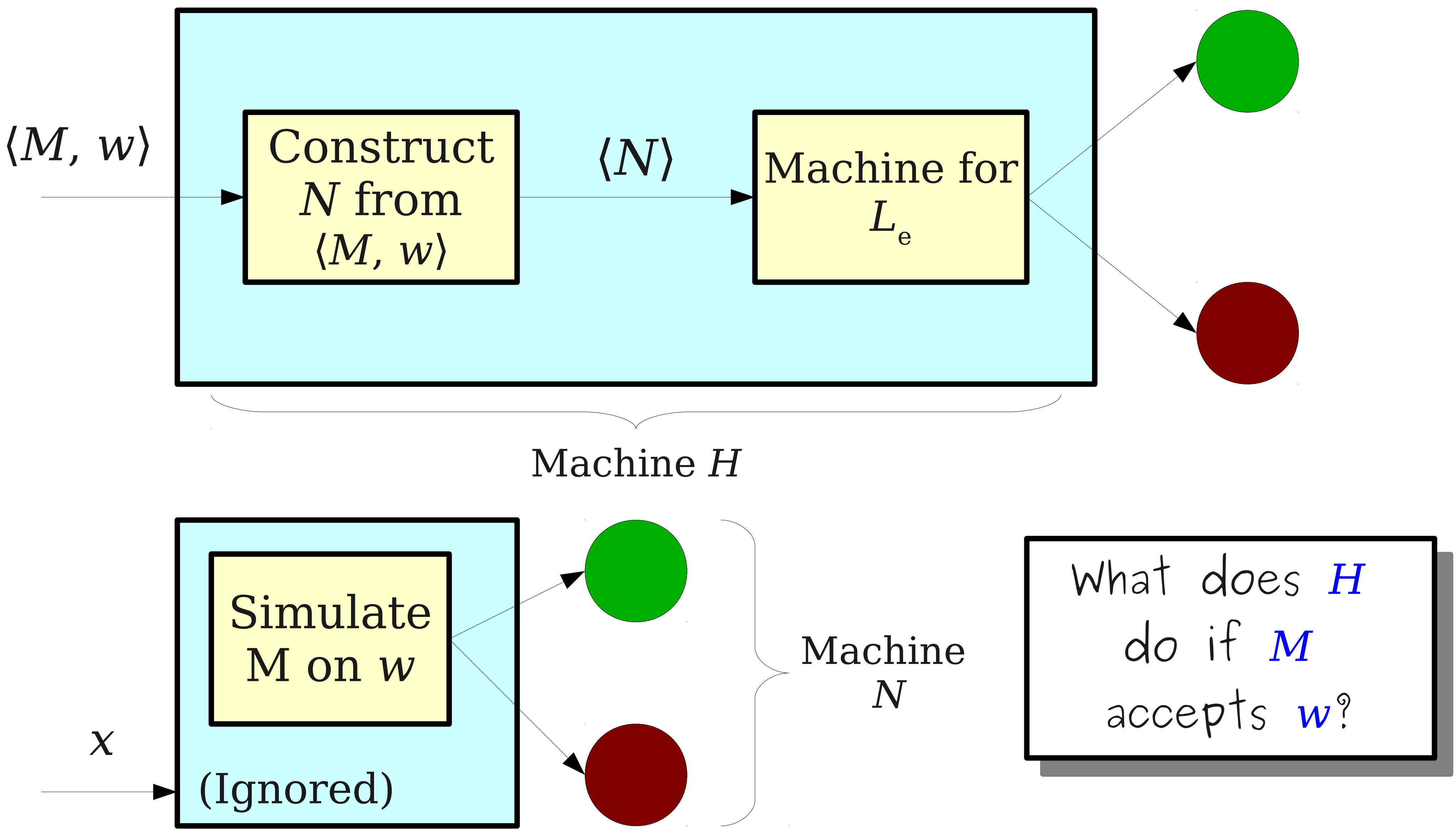


What does H do if M does not accept w?

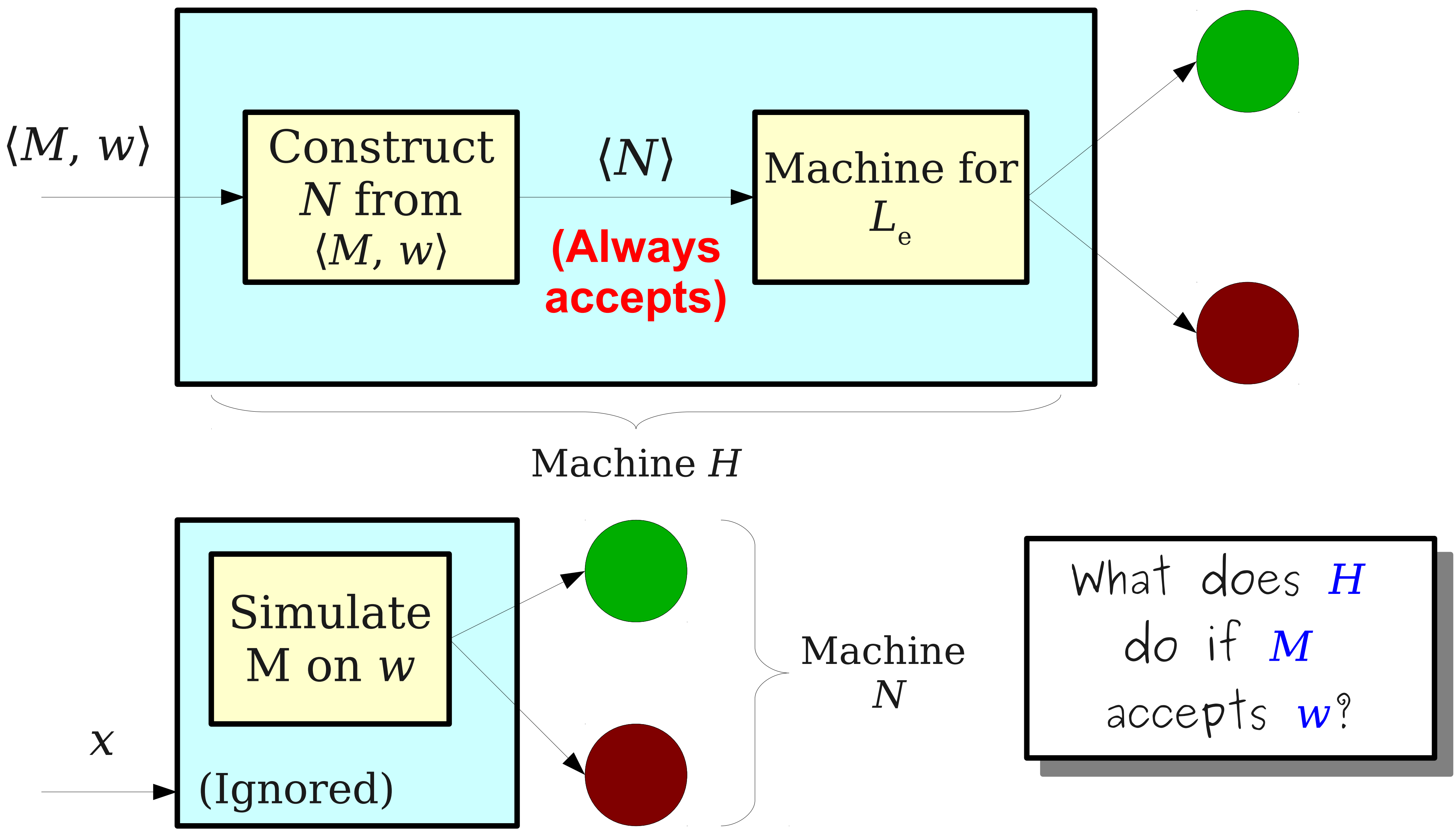
The Reduction



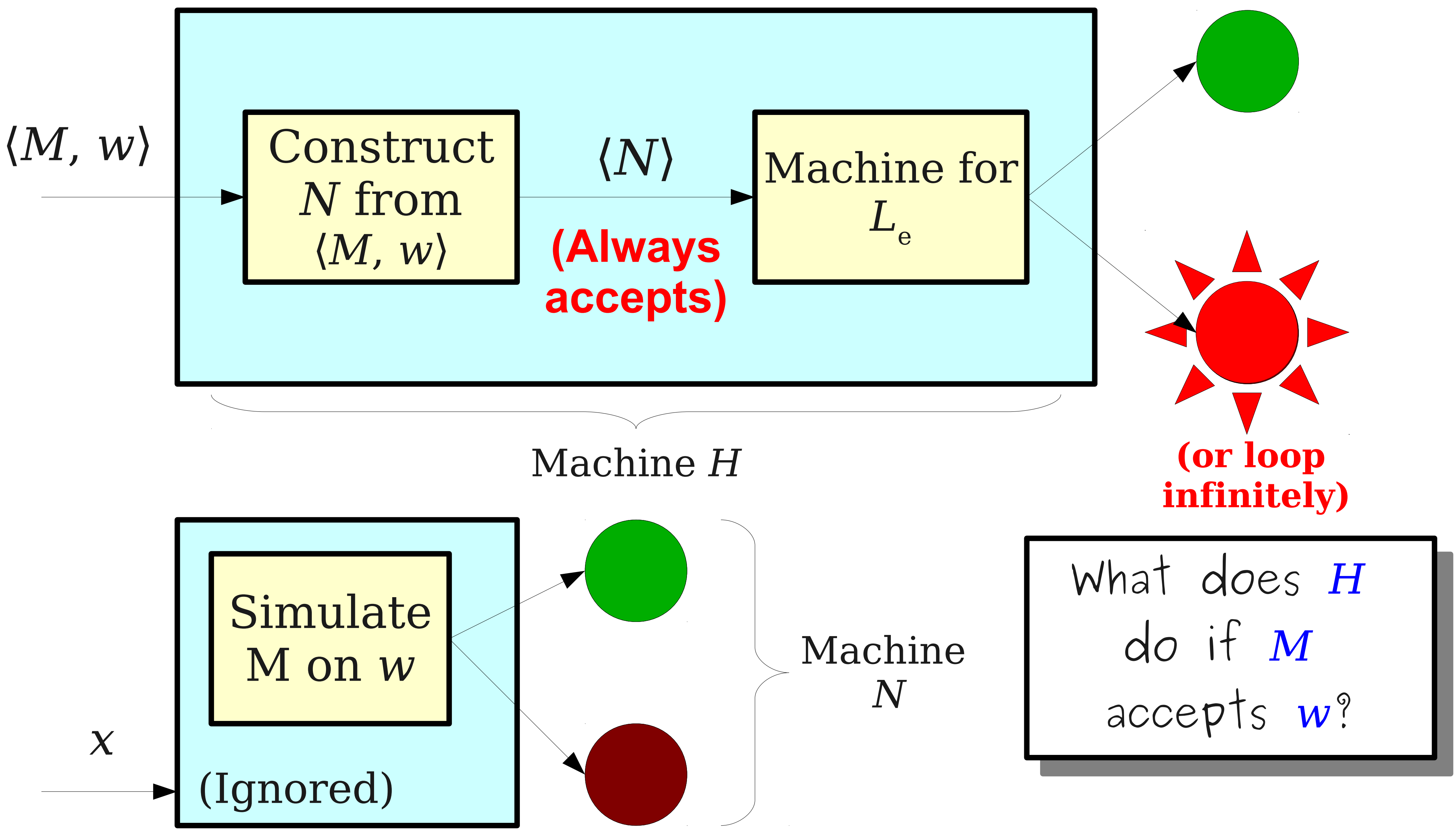
The Reduction



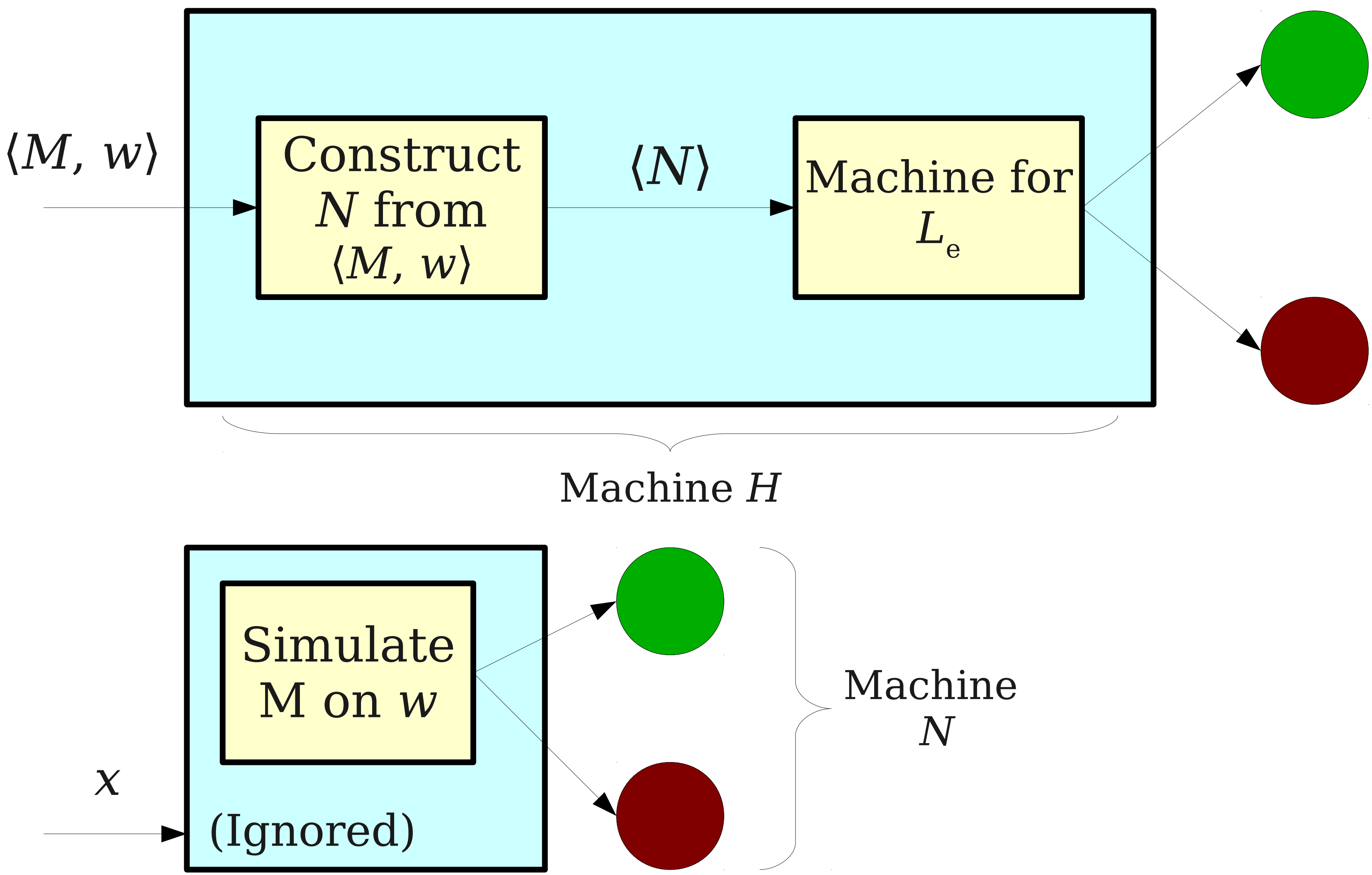
The Reduction



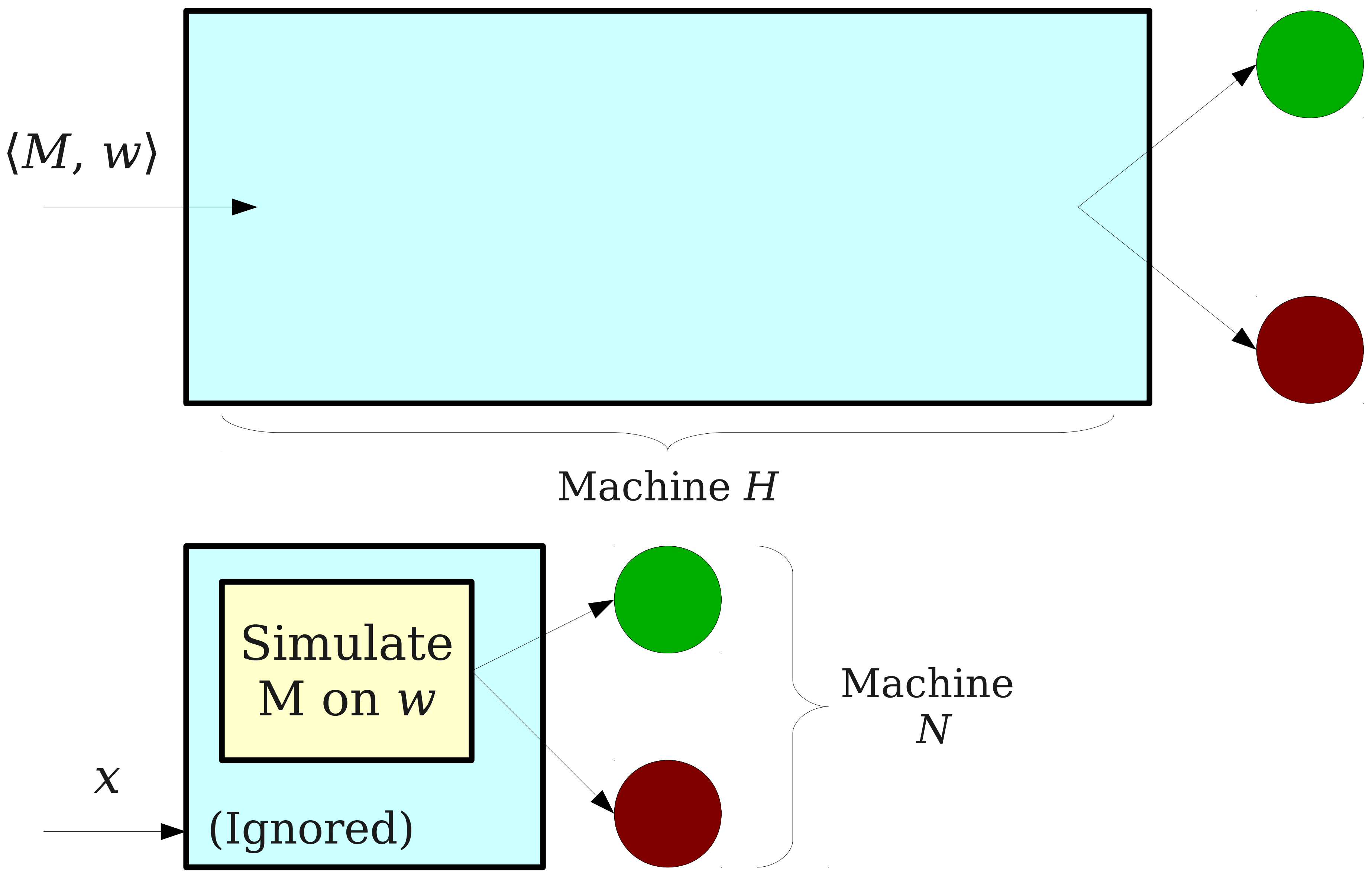
The Reduction



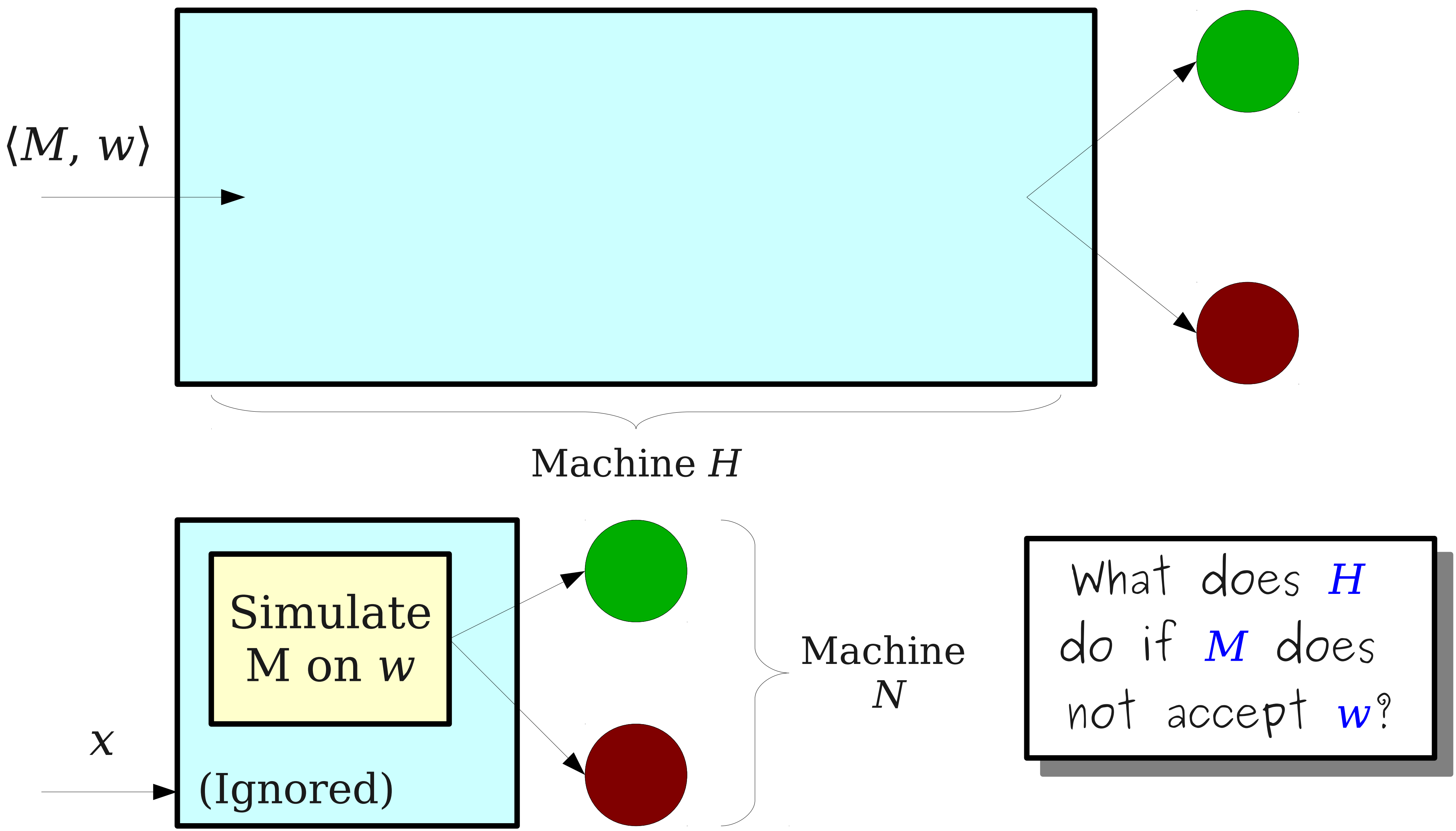
The Reduction



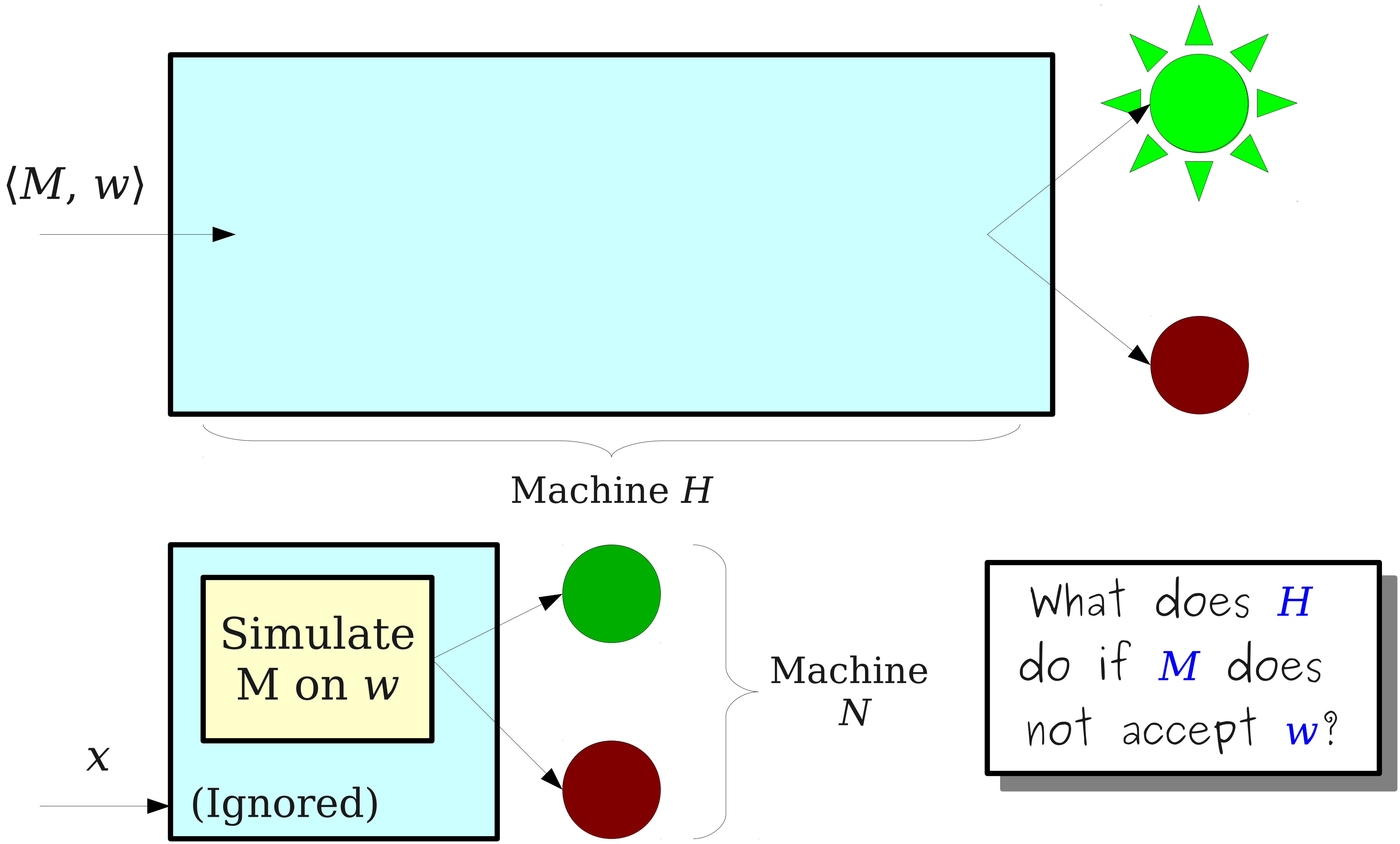
The Reduction



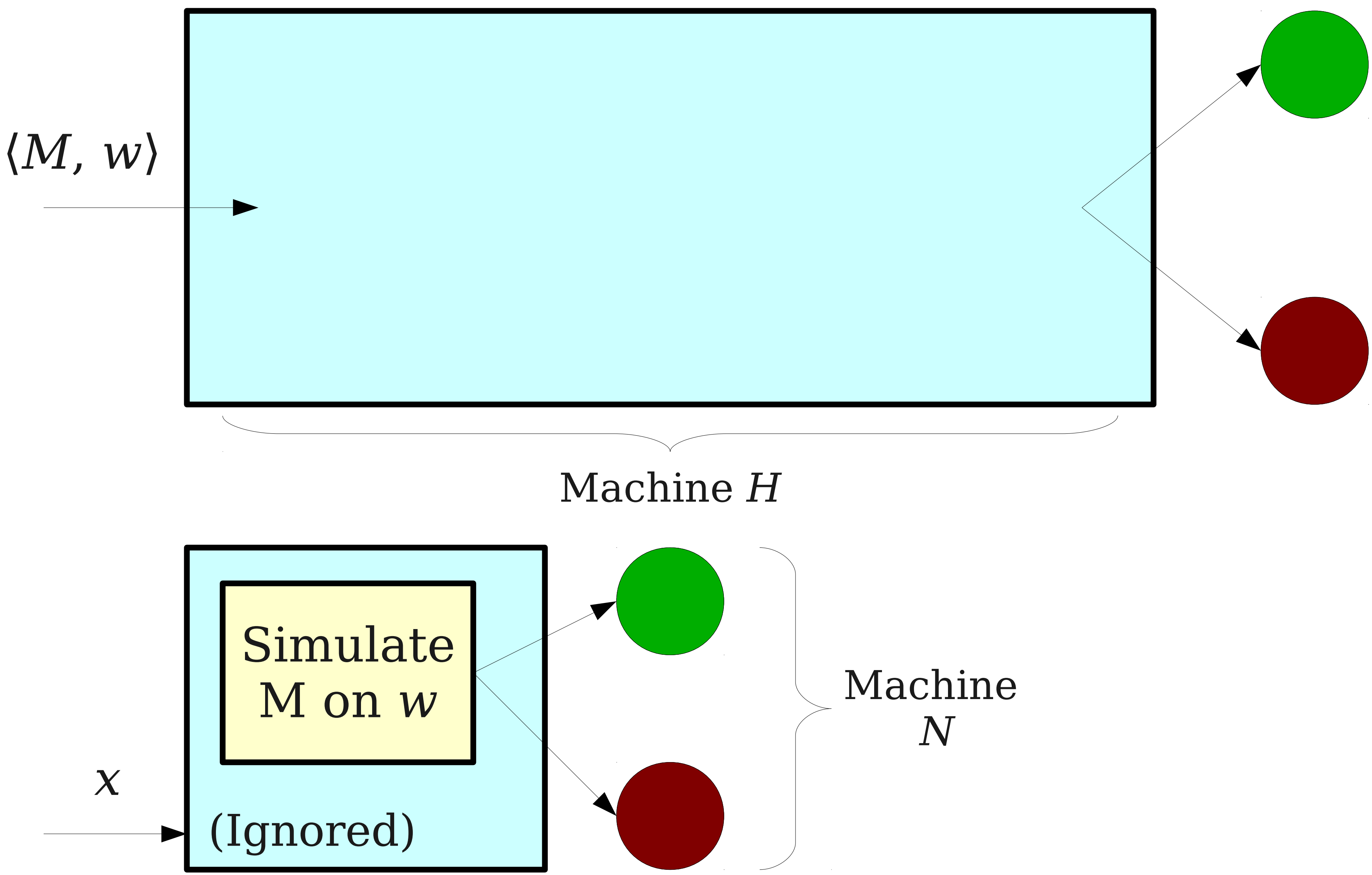
The Reduction



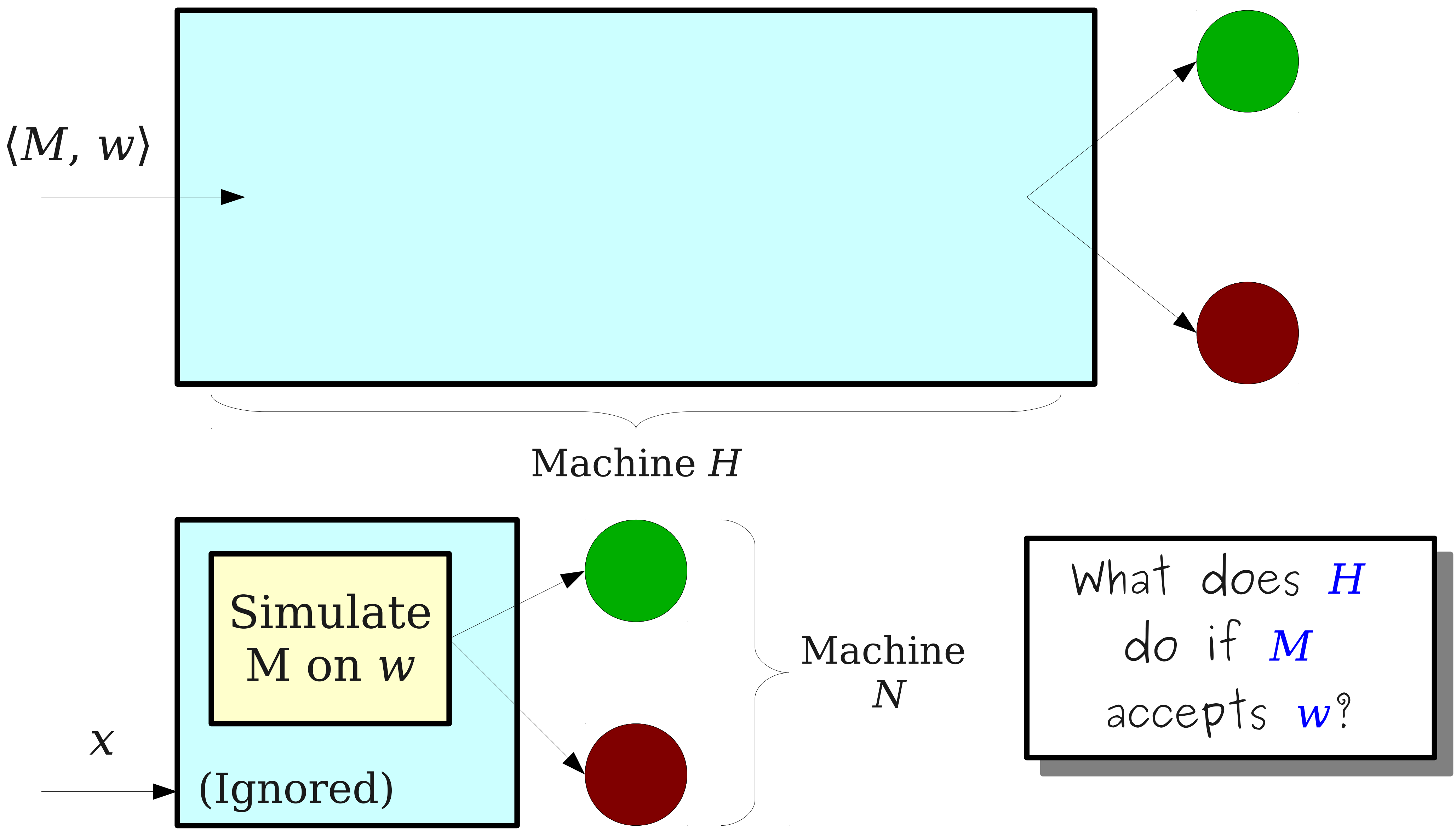
The Reduction



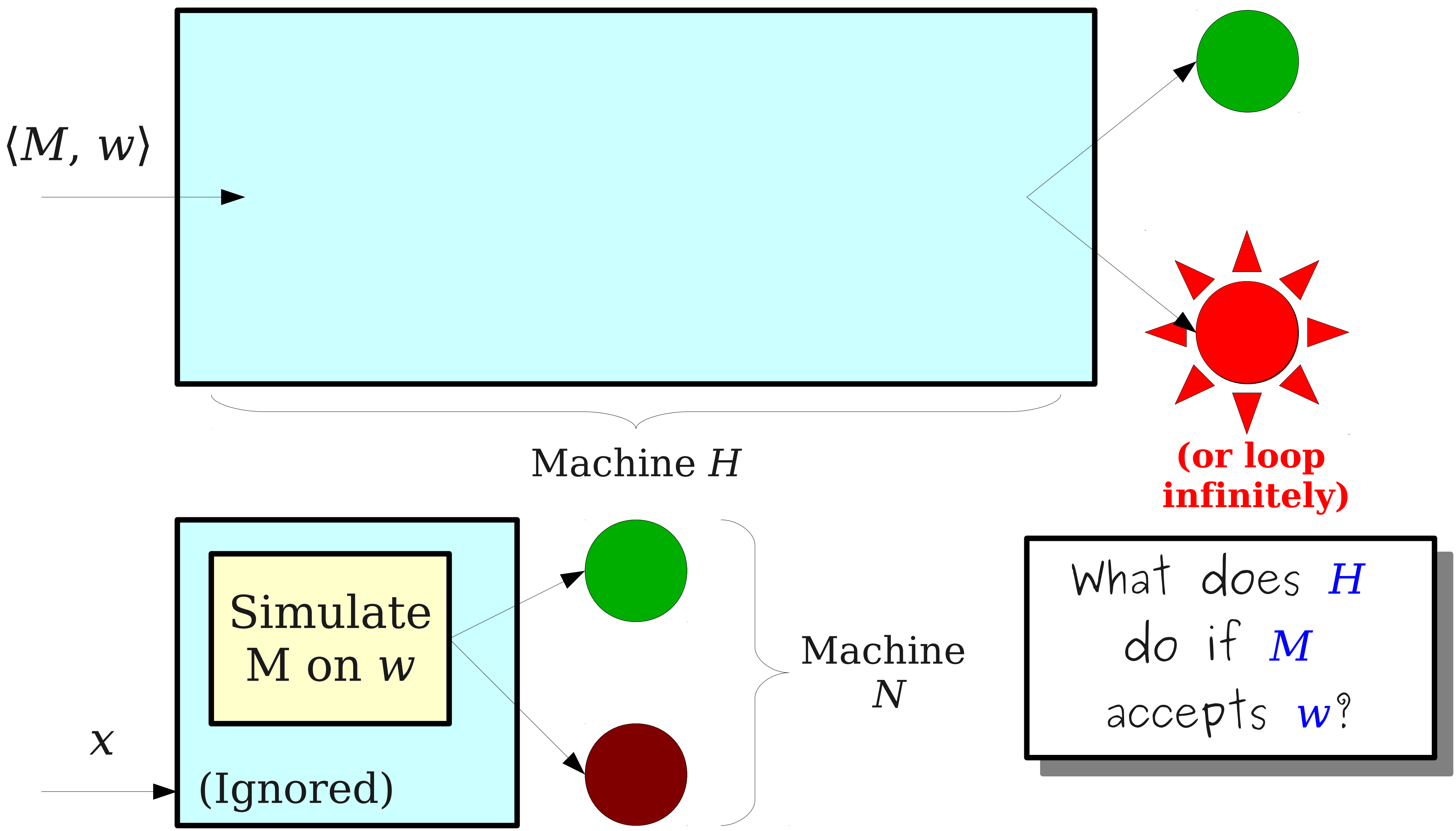
The Reduction



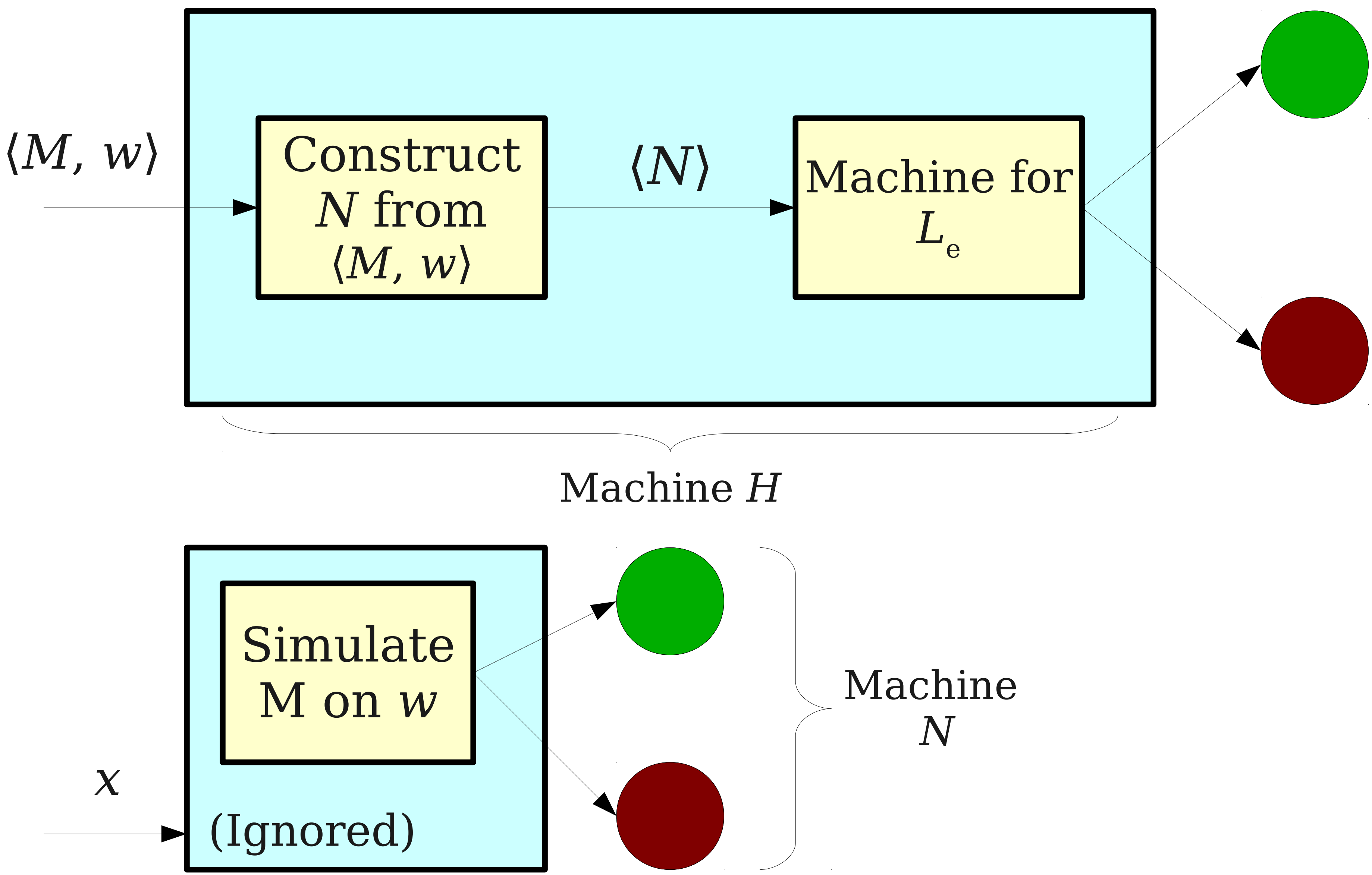
The Reduction



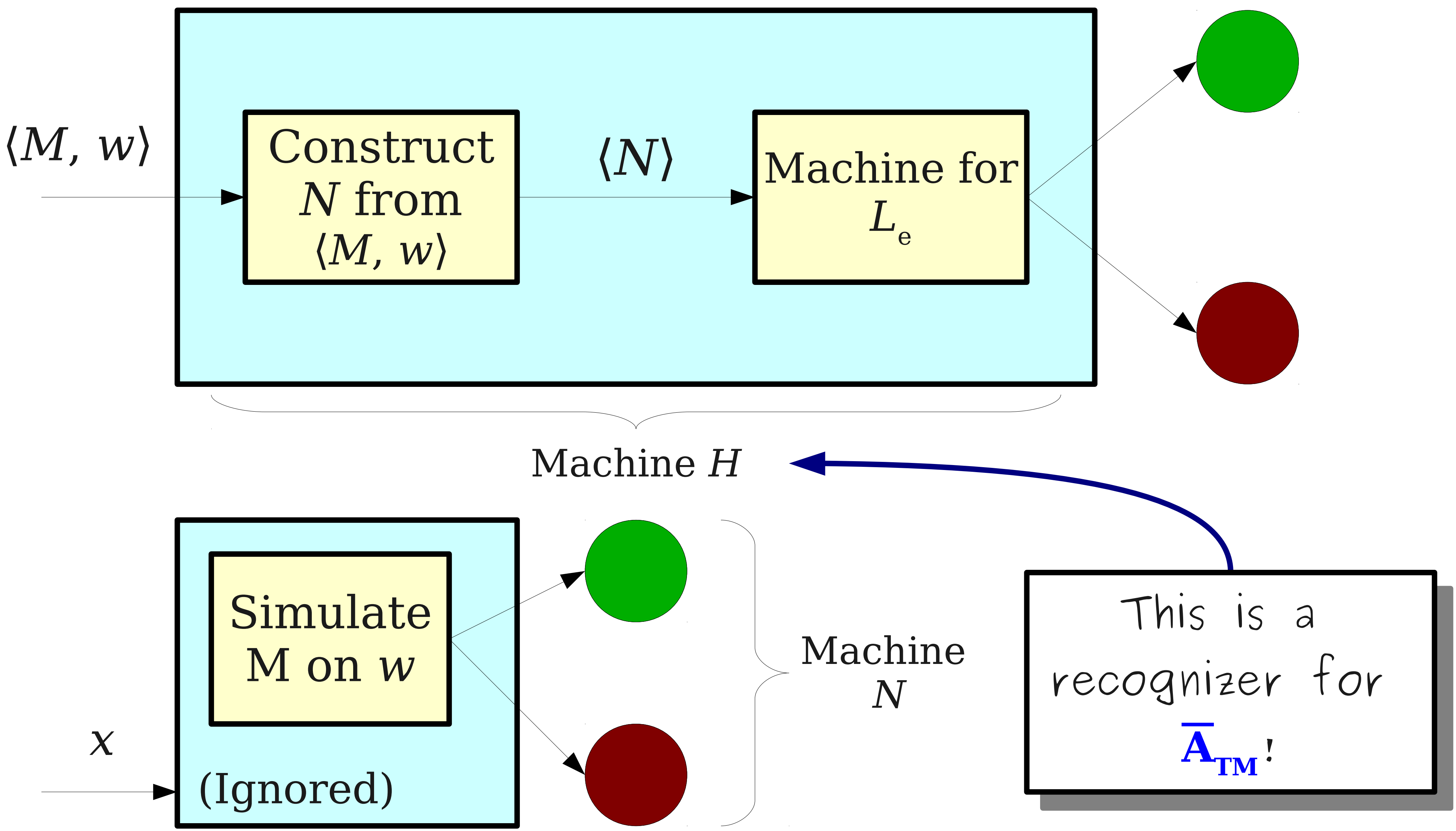
The Reduction



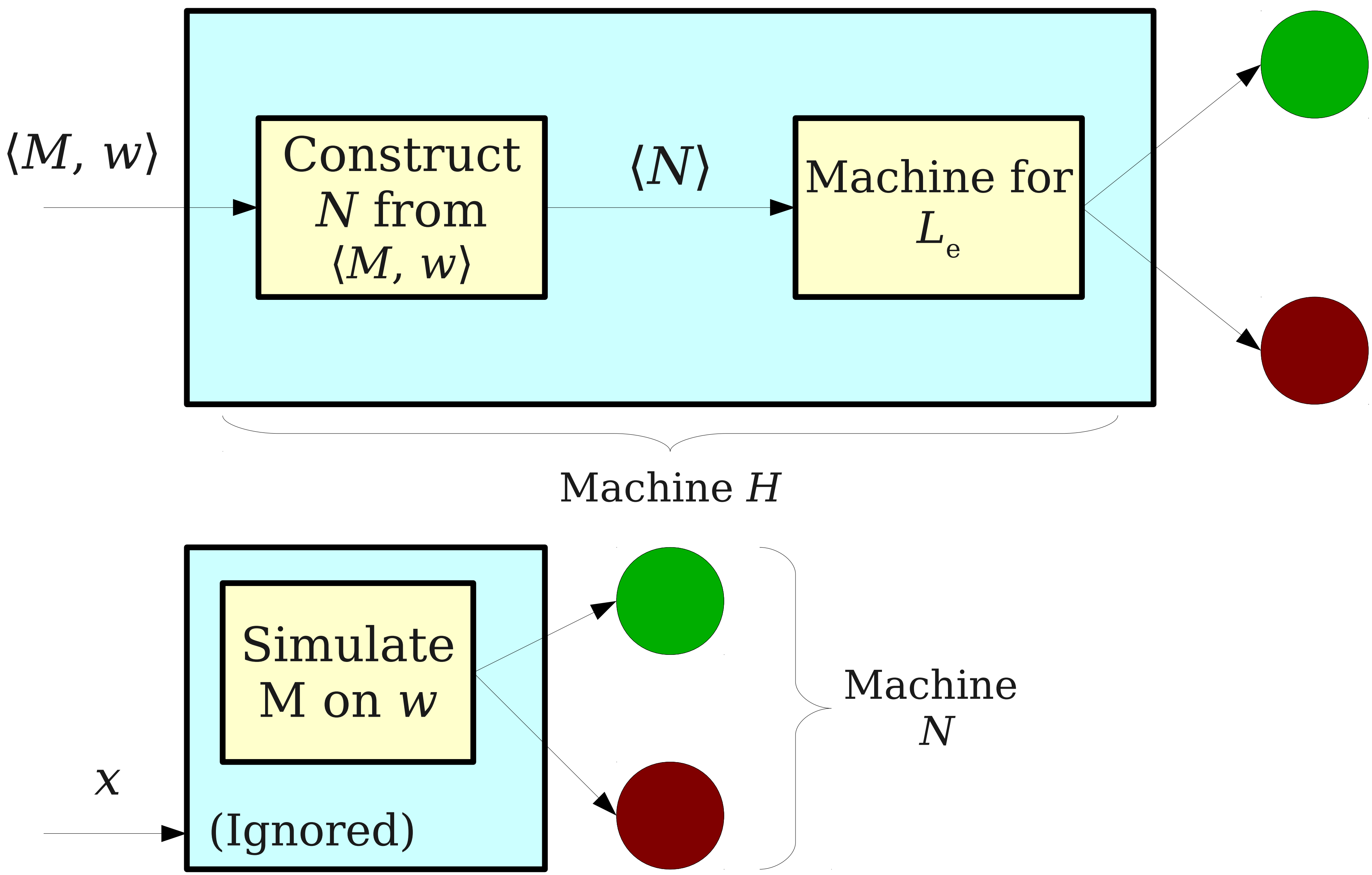
The Reduction



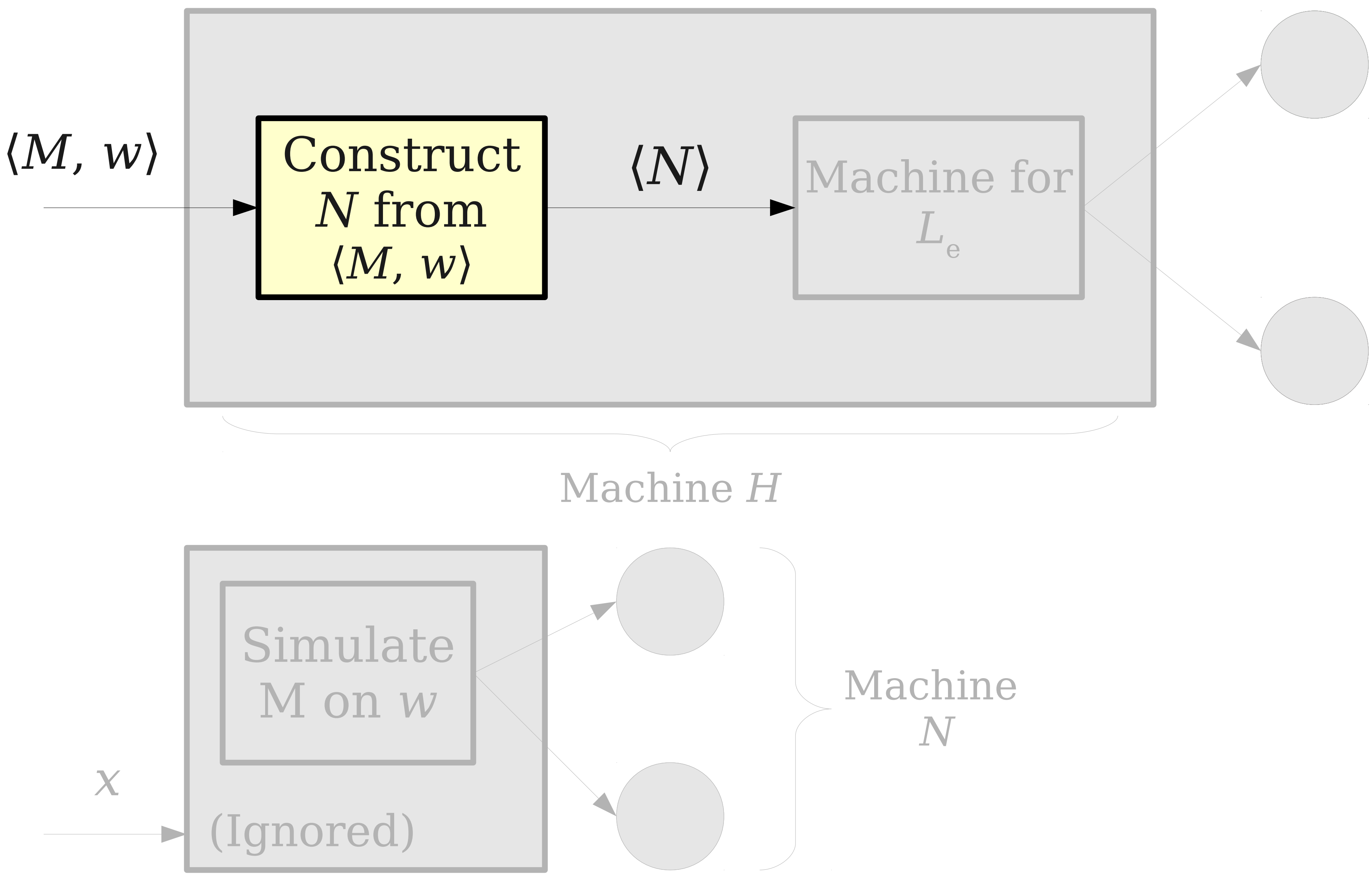
The Reduction



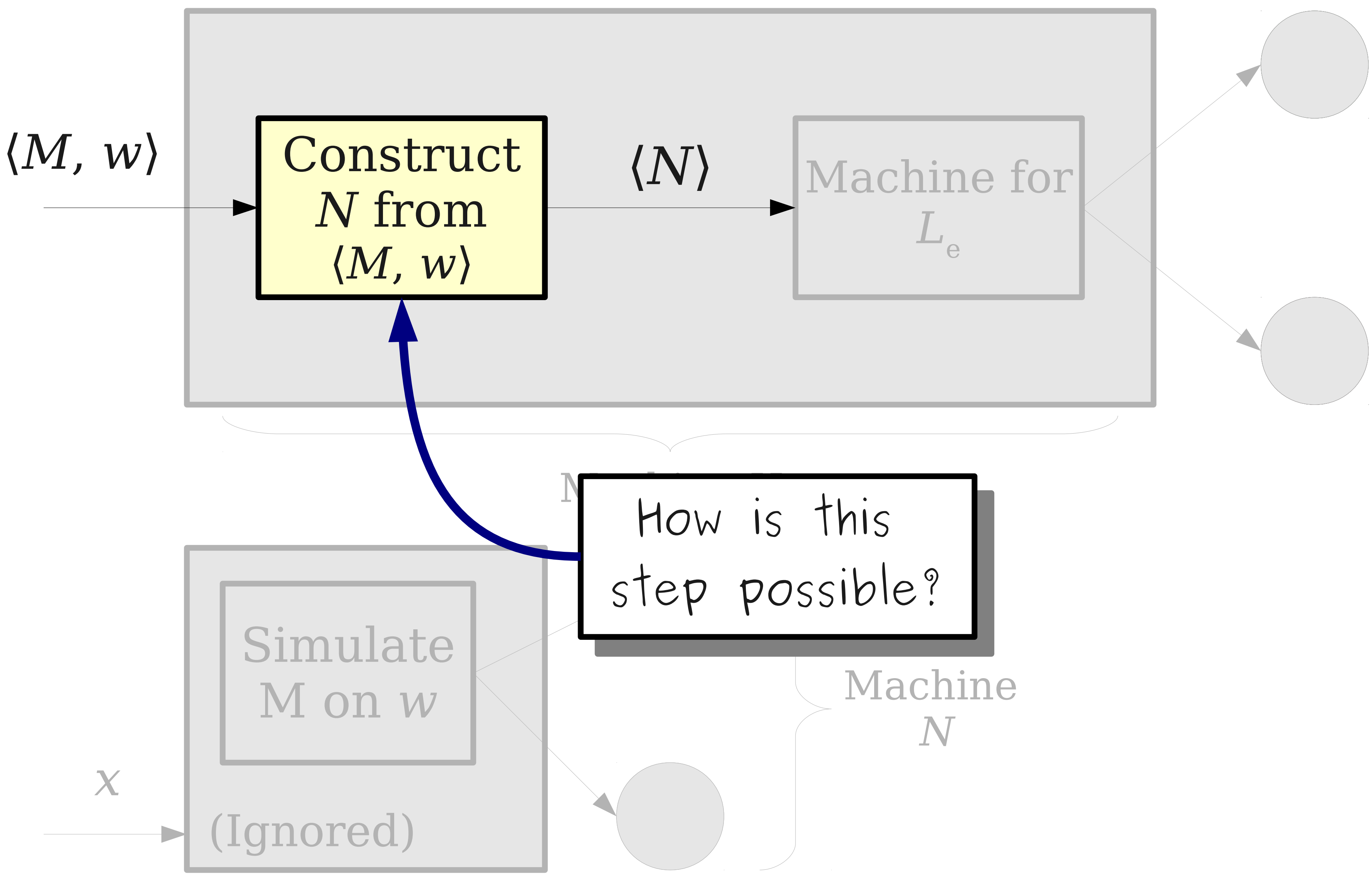
The Reduction



The Reduction



The Reduction



Justifying N

- Notice that our machine N has the machine M and string w built into it!
- This is different from the machines we have constructed in the past.
- How do we justify that it's possible for some TM to construct a new TM at all?

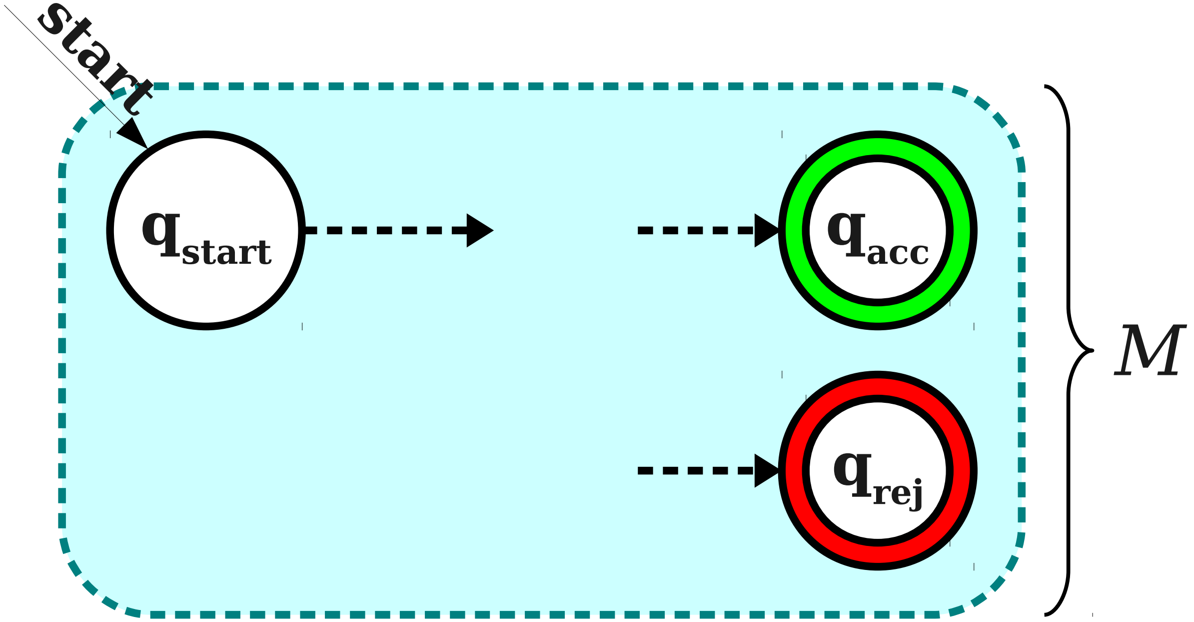
N = "On input x:

Ignore x.

Run M on w.

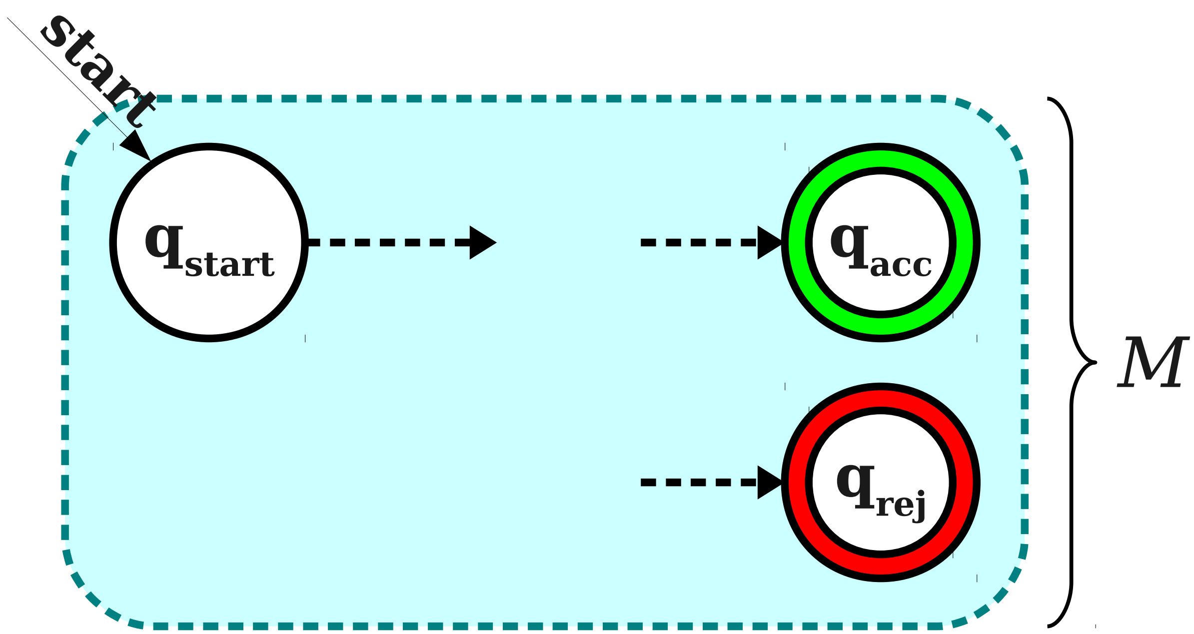
If M accepts w, accept.

If M rejects w, reject."


start
$\mathbf{q_{start}}$
$\mathbf{q_{acc}}$
$\mathbf{q_{rej}}$
M

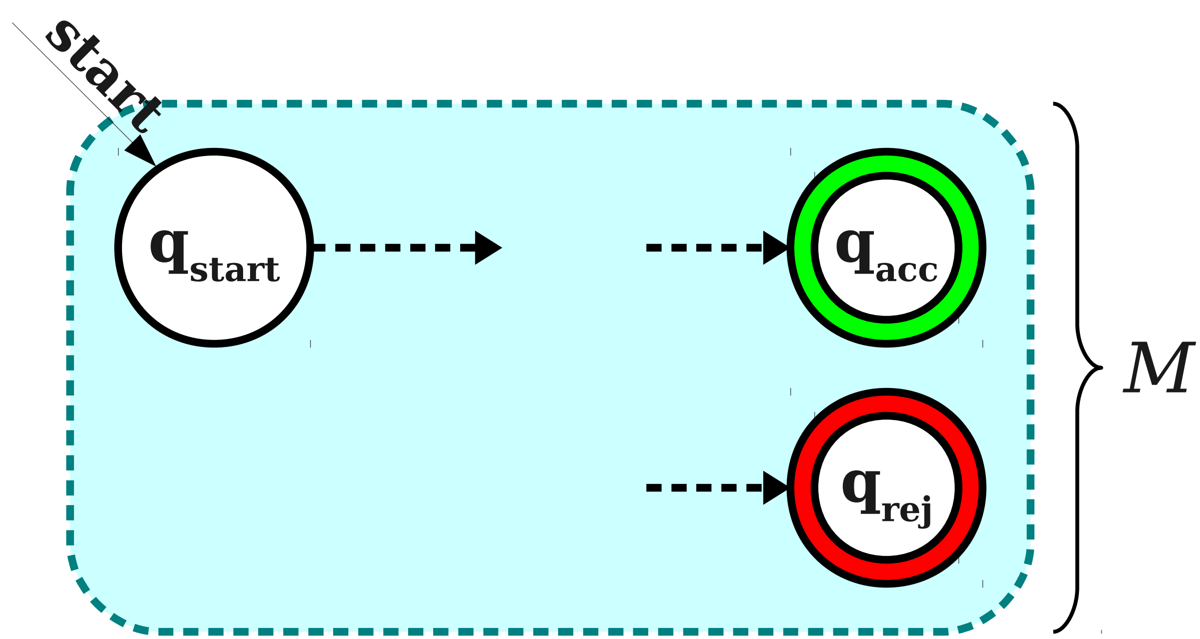
N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."



N = "On input x:
- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

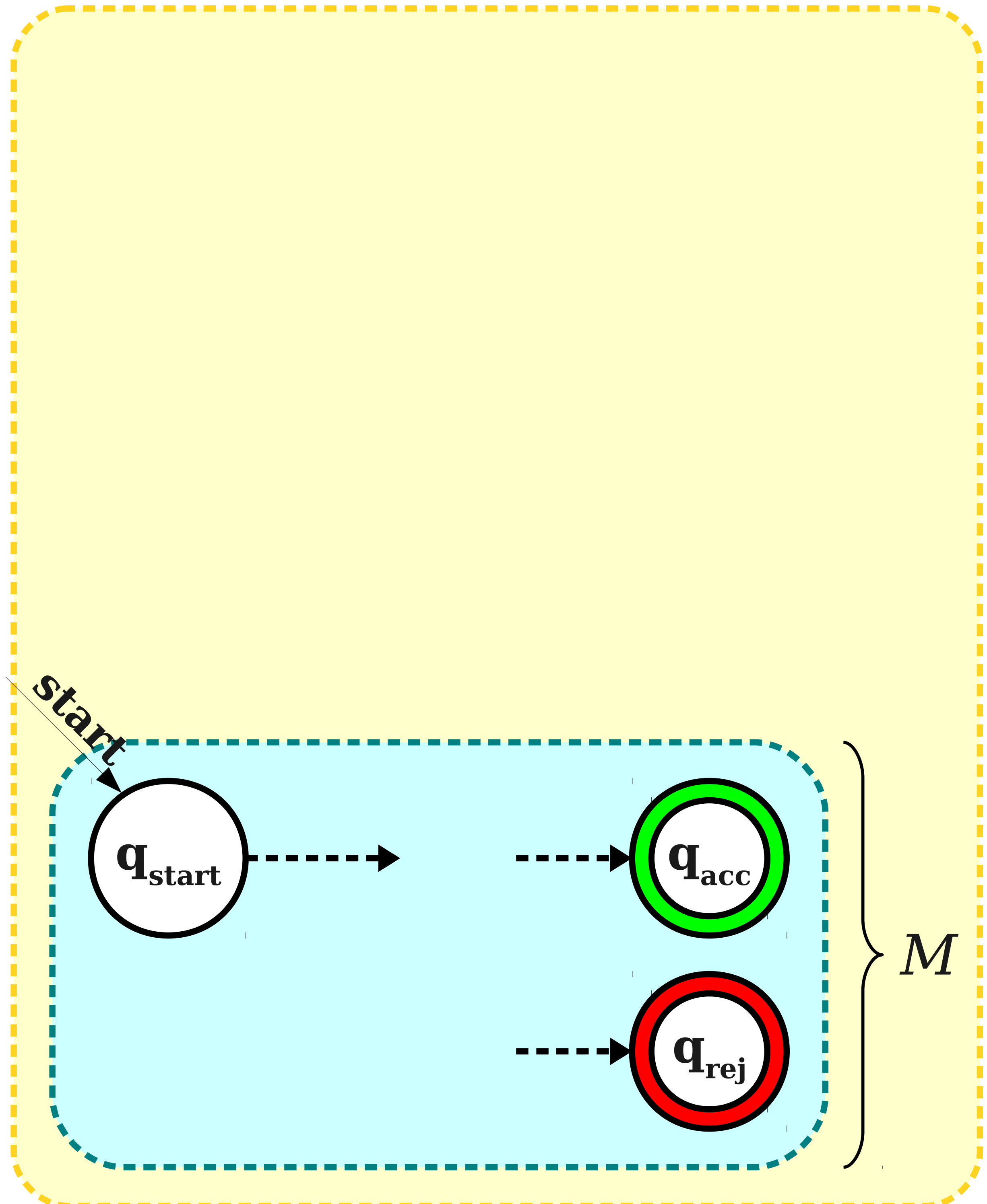
Hypothetically, assume that w is the string **11011**.



N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

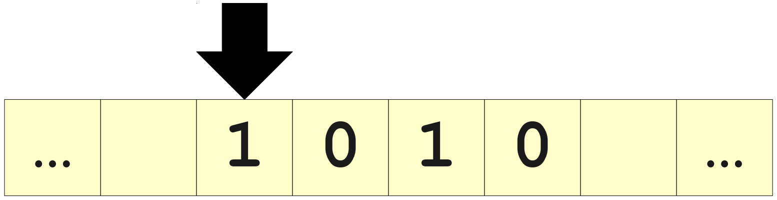
Hypothetically, assume that w is the string **11011**.

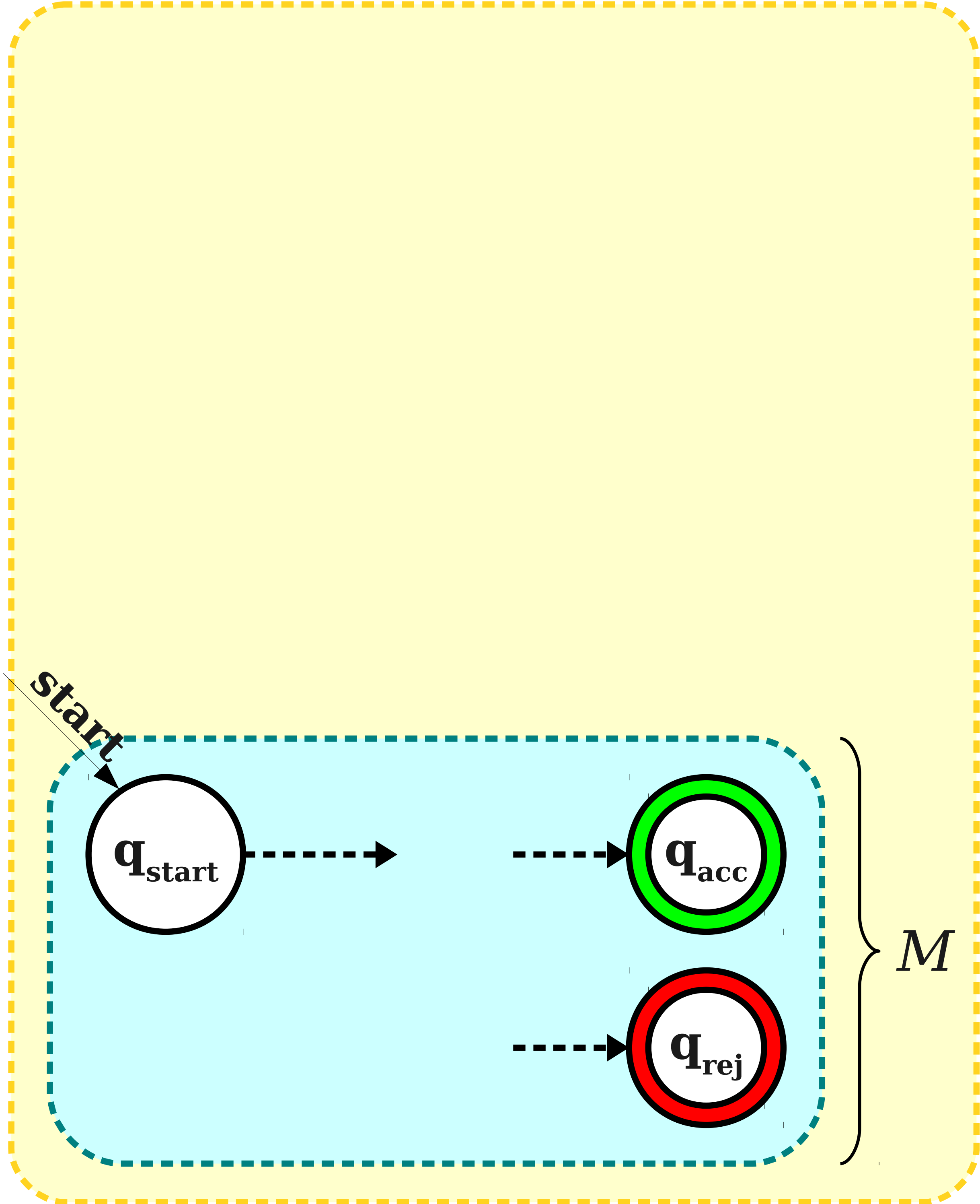


N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

Hypothetically, assume that w is the string **11011**.

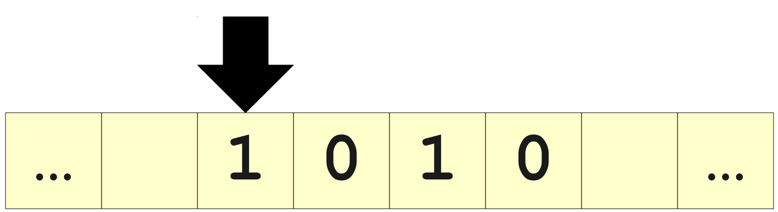


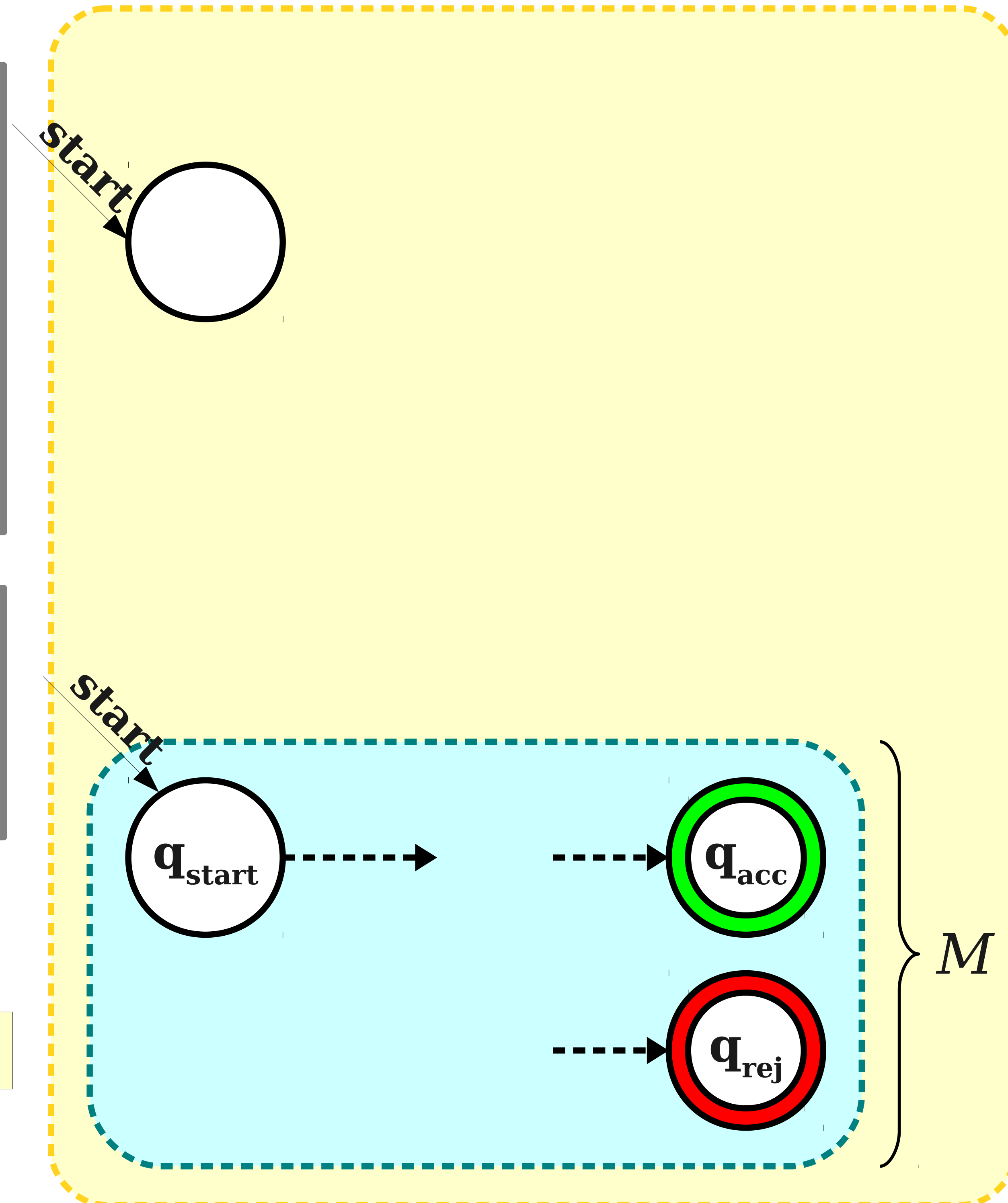


N = “On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x.”

Hypothetically, assume that w is the string `11011`.

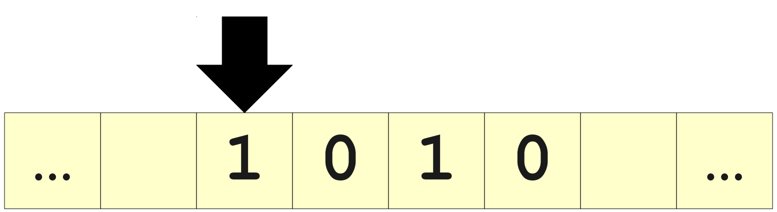


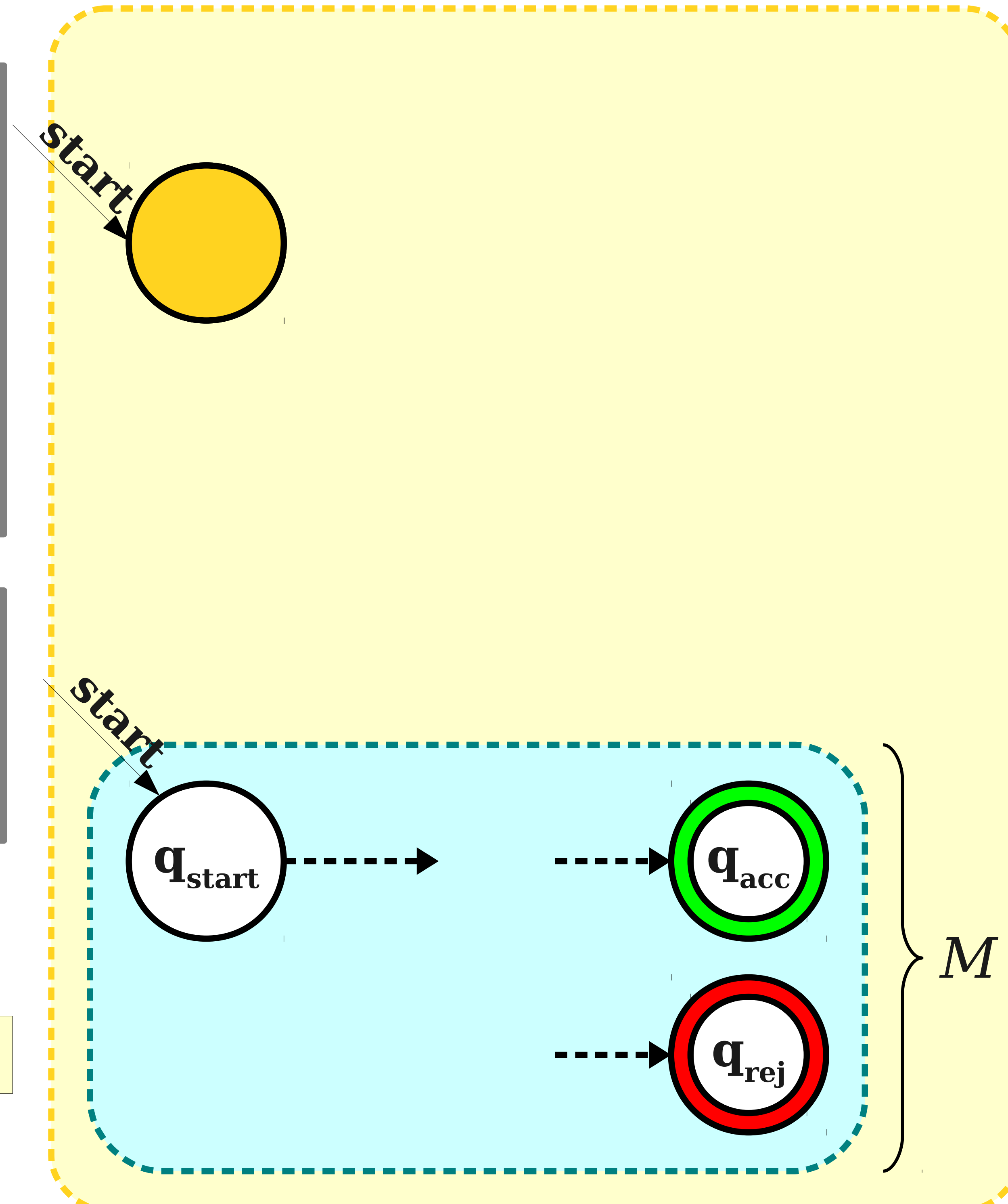


N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

Hypothetically, assume that w is the string 11011.

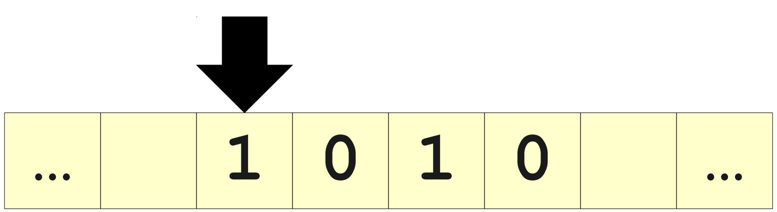


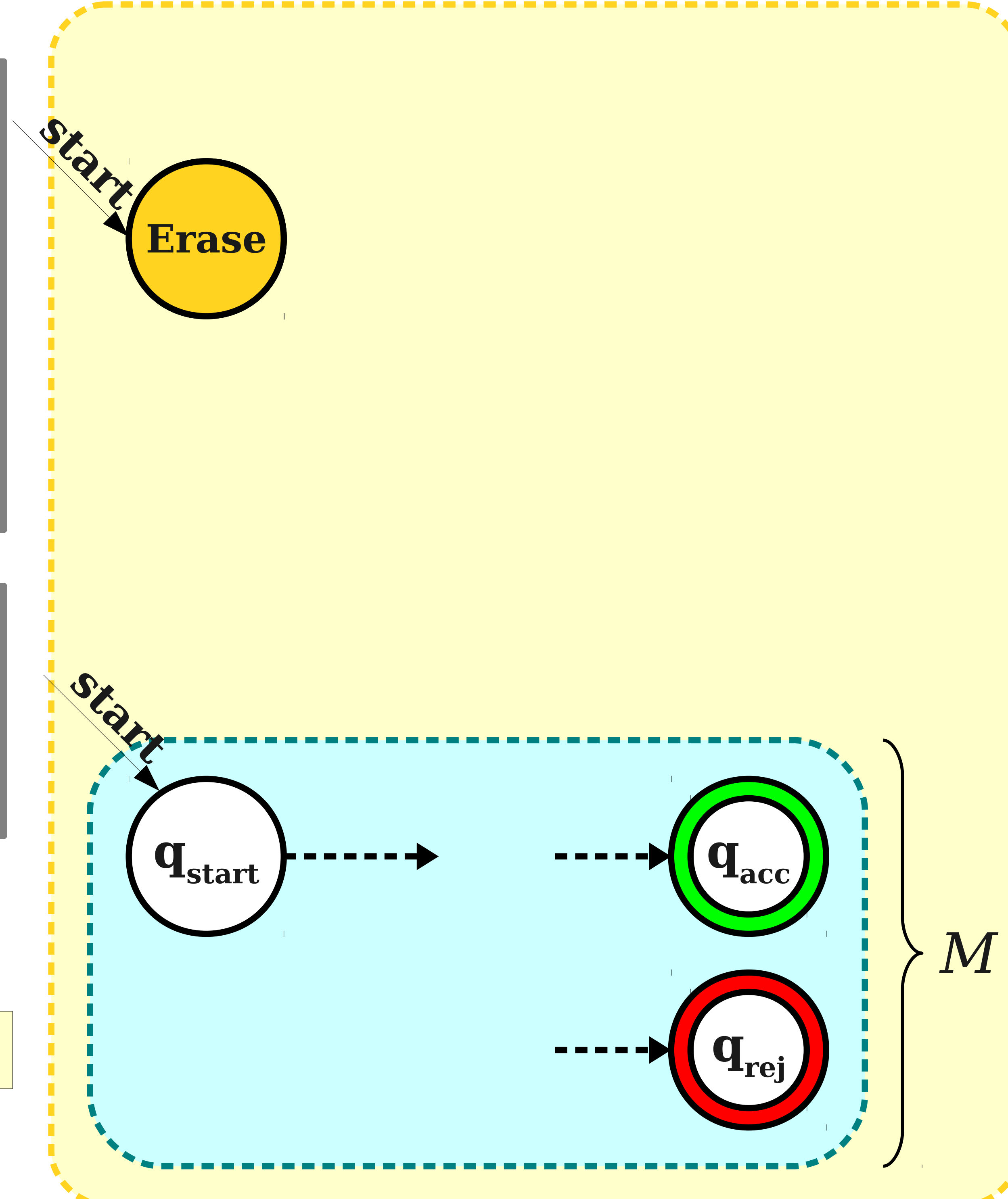


N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

Hypothetically, assume that w is the string **11011**.

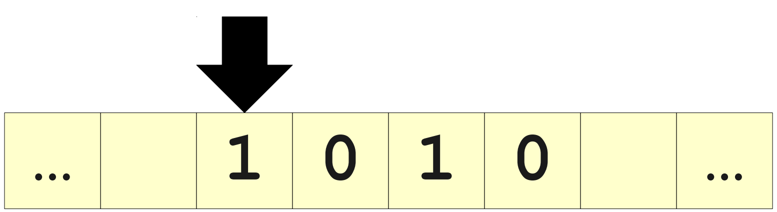


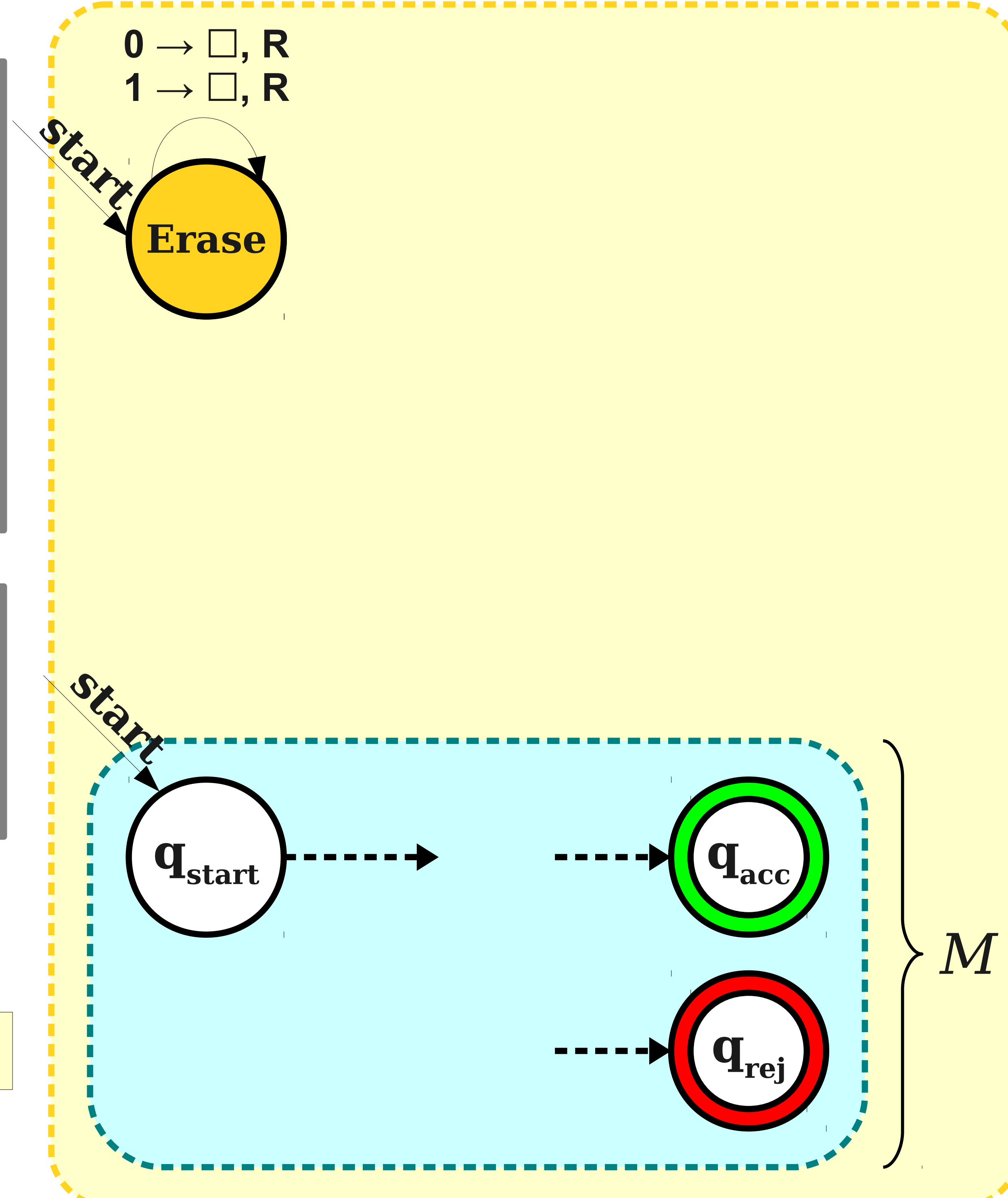


N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

Hypothetically, assume that w is the string `11011`.

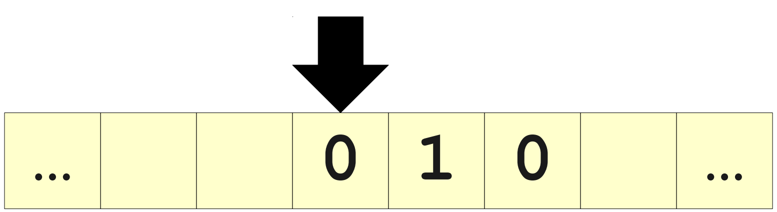


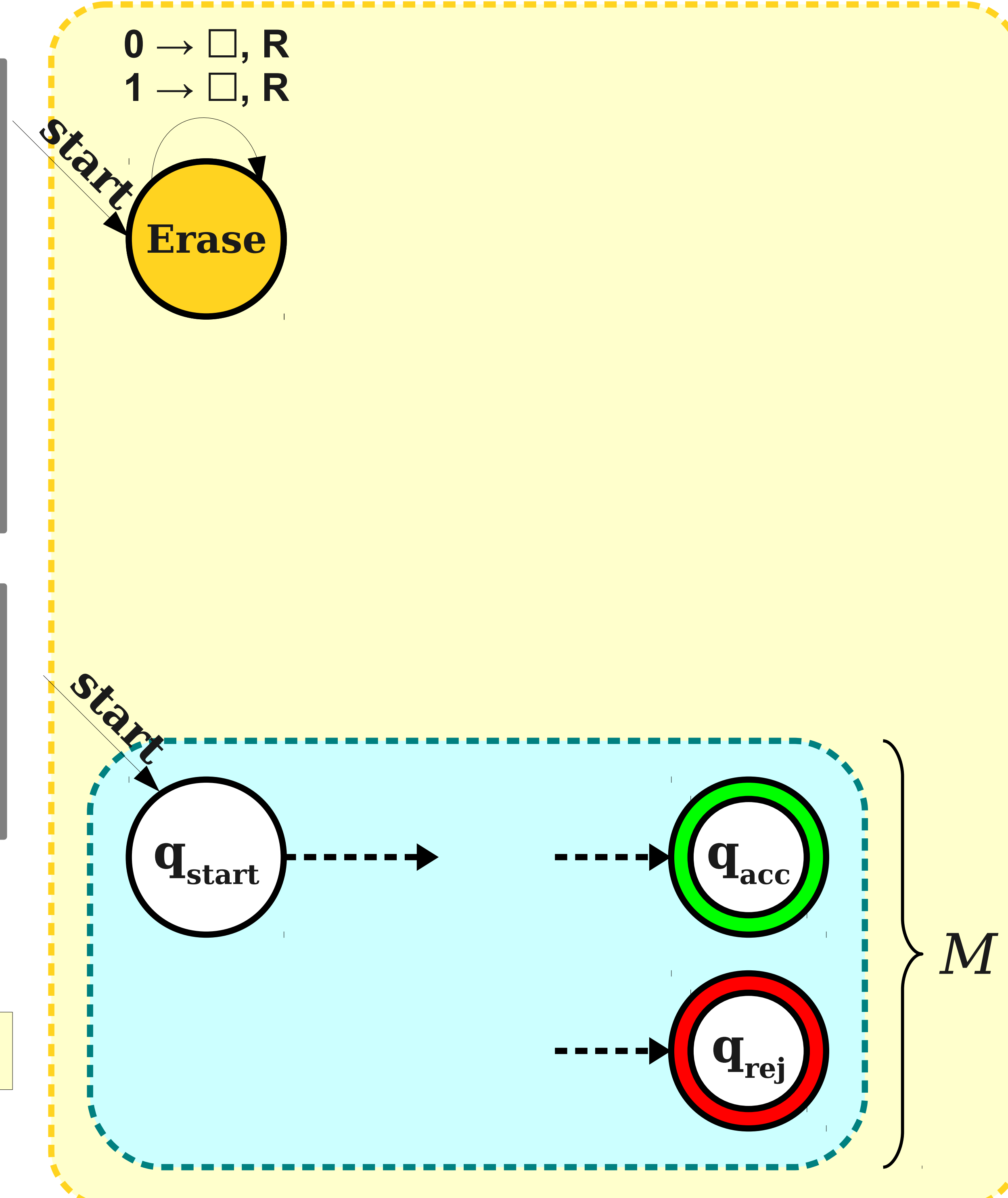


N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

Hypothetically, assume that w is the string `11011`.

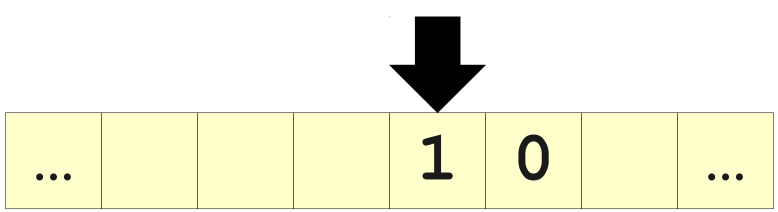


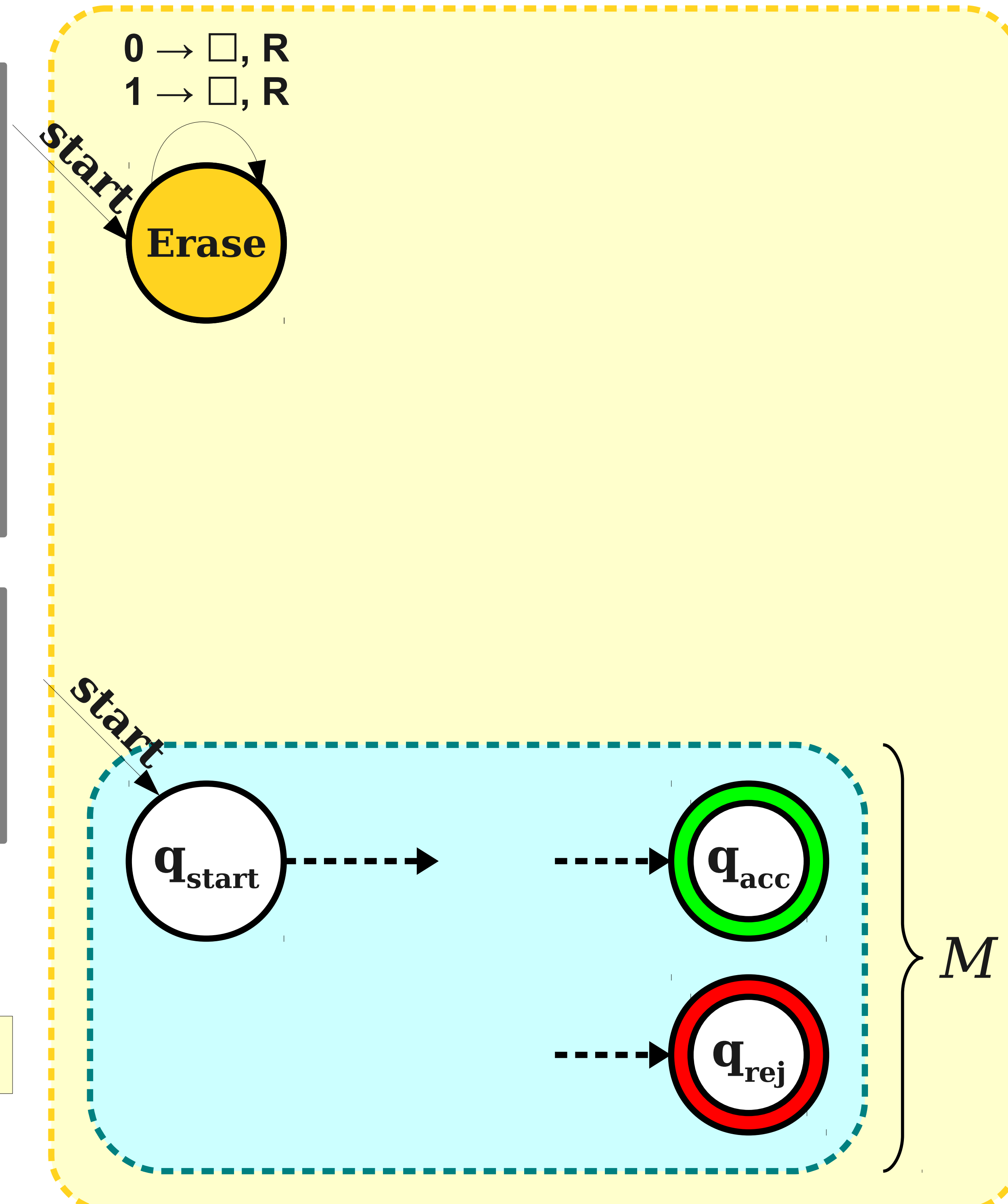


N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

Hypothetically, assume that w is the string **11011**.

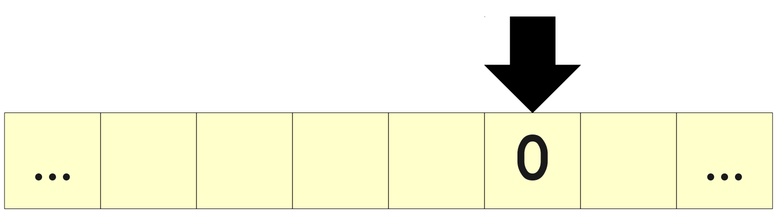


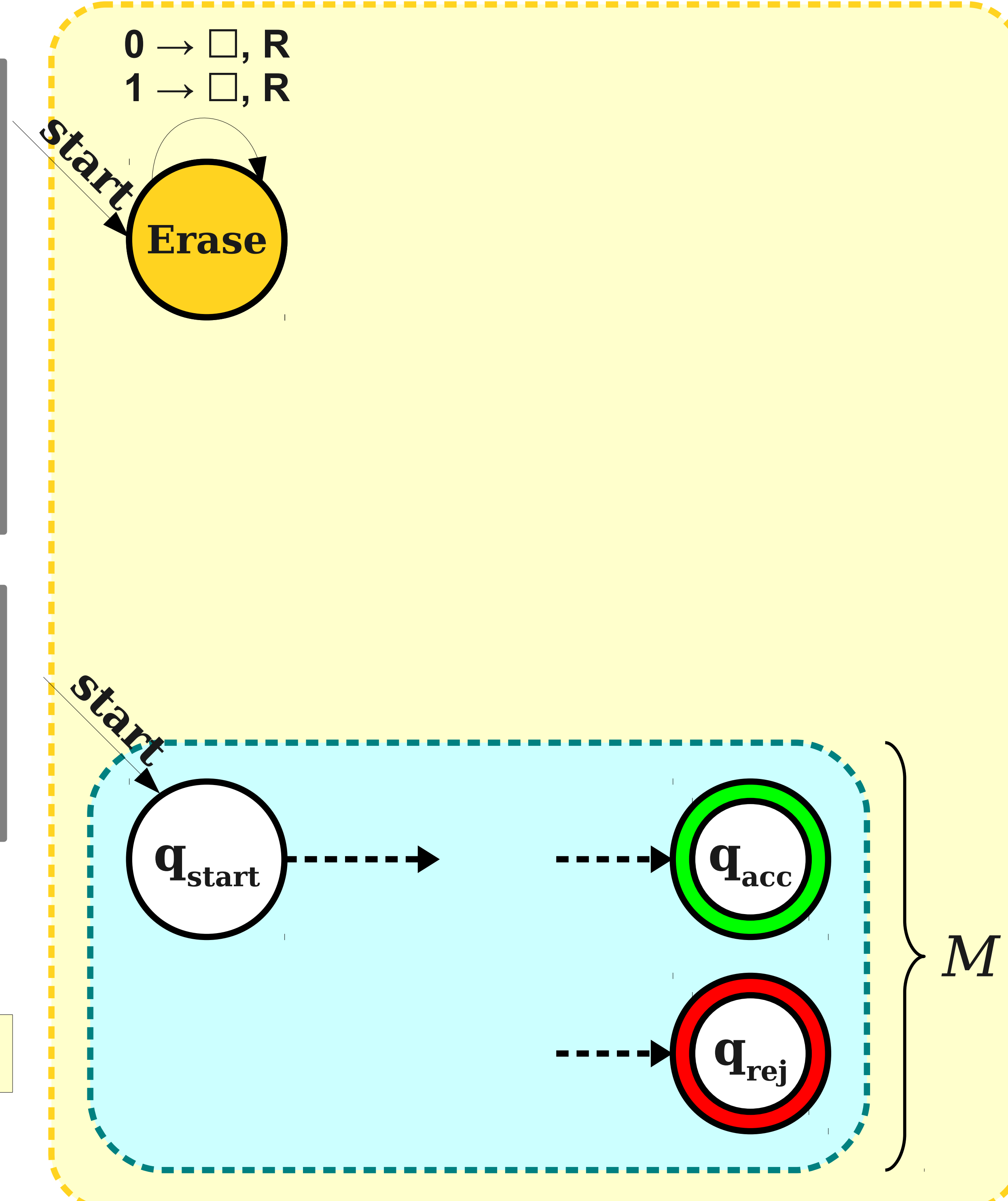


N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

Hypothetically, assume that w is the string **11011**.

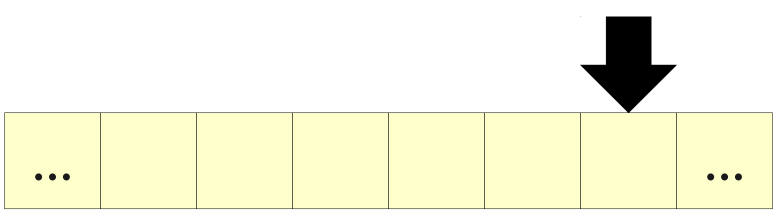


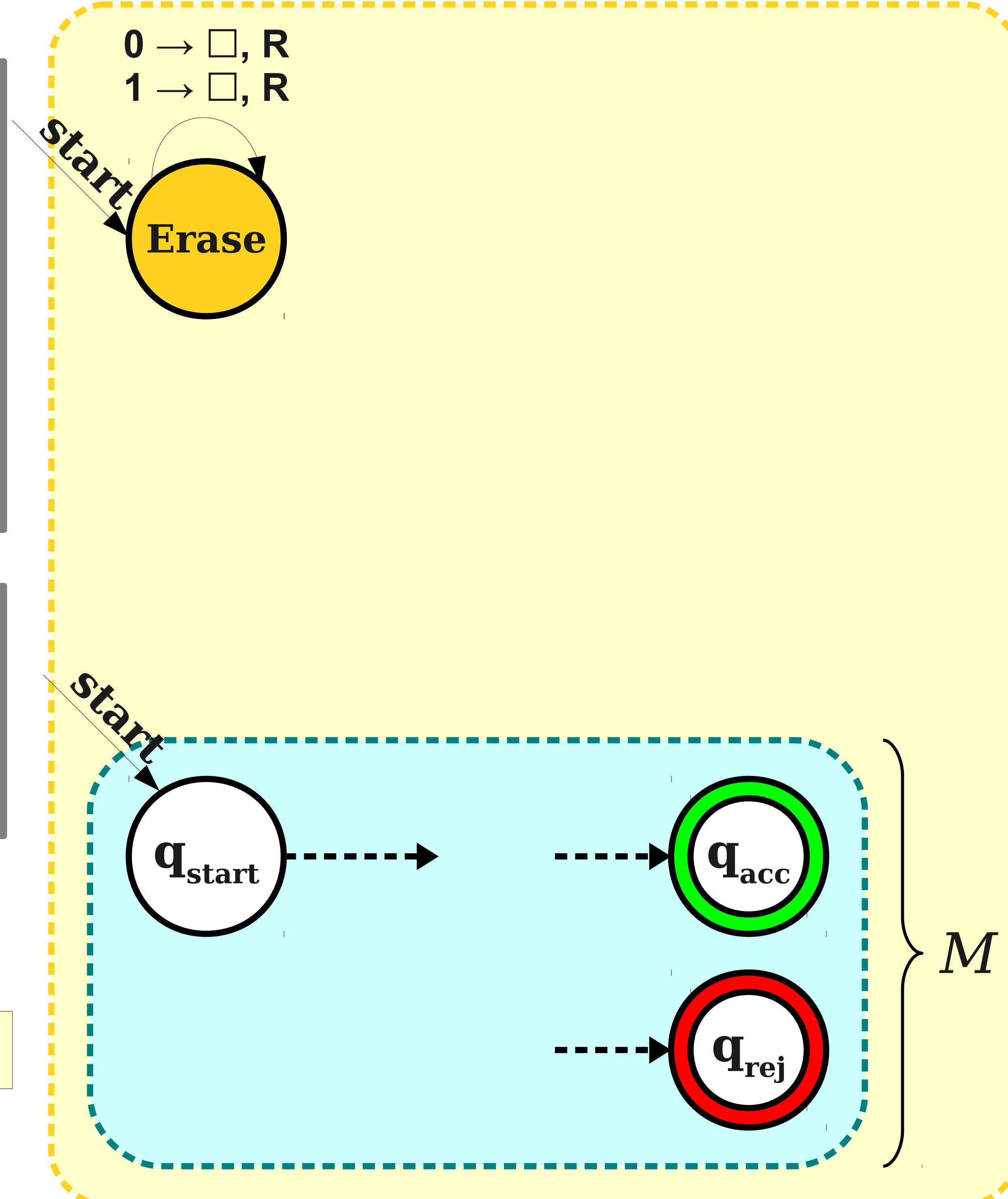


N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

Hypothetically, assume that w is the string **11011**.

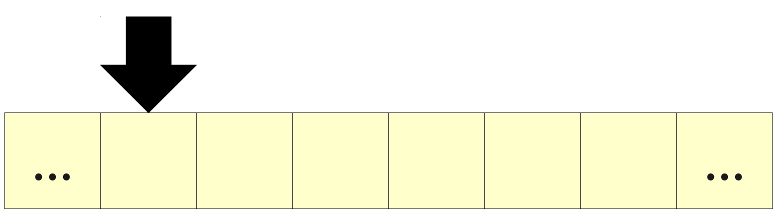


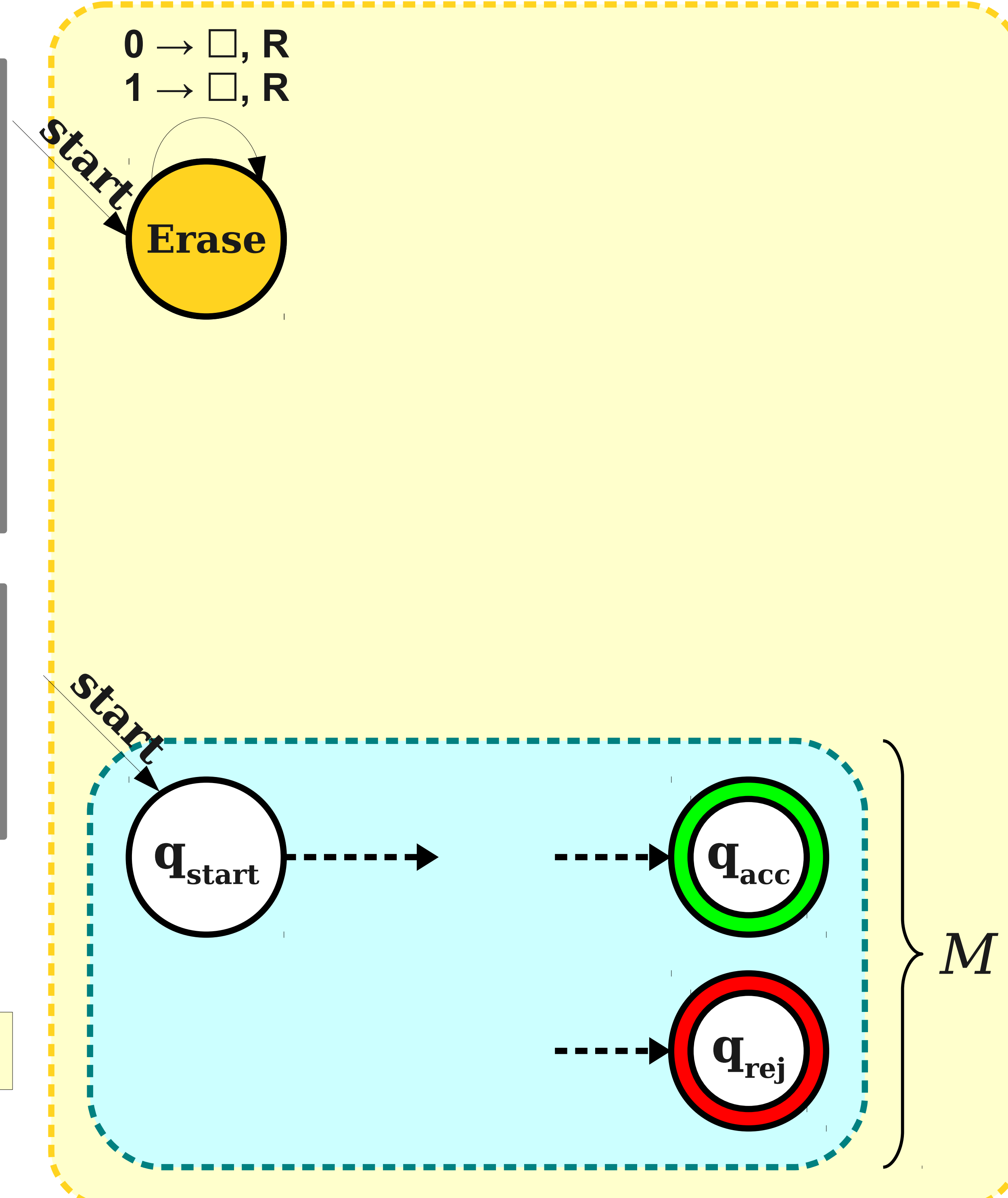


N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

Hypothetically, assume that w is the string **11011**.

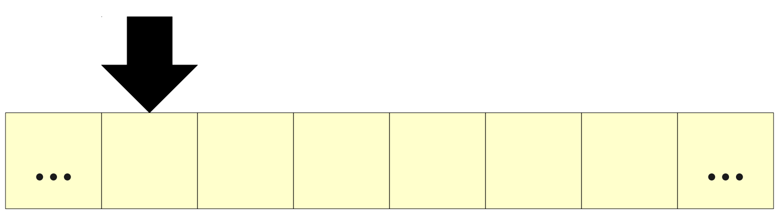


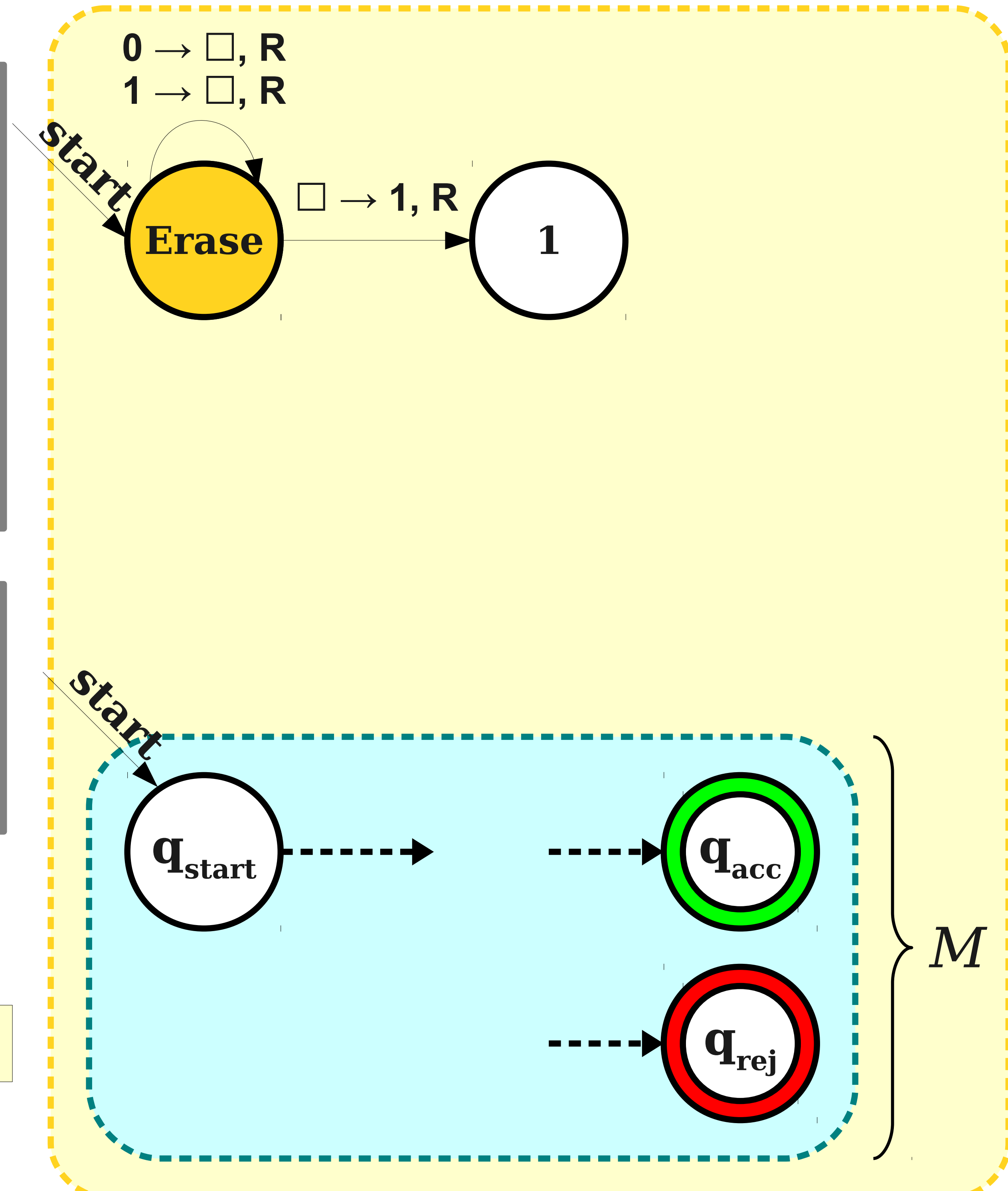


N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

Hypothetically, assume that w is the string `11011`.



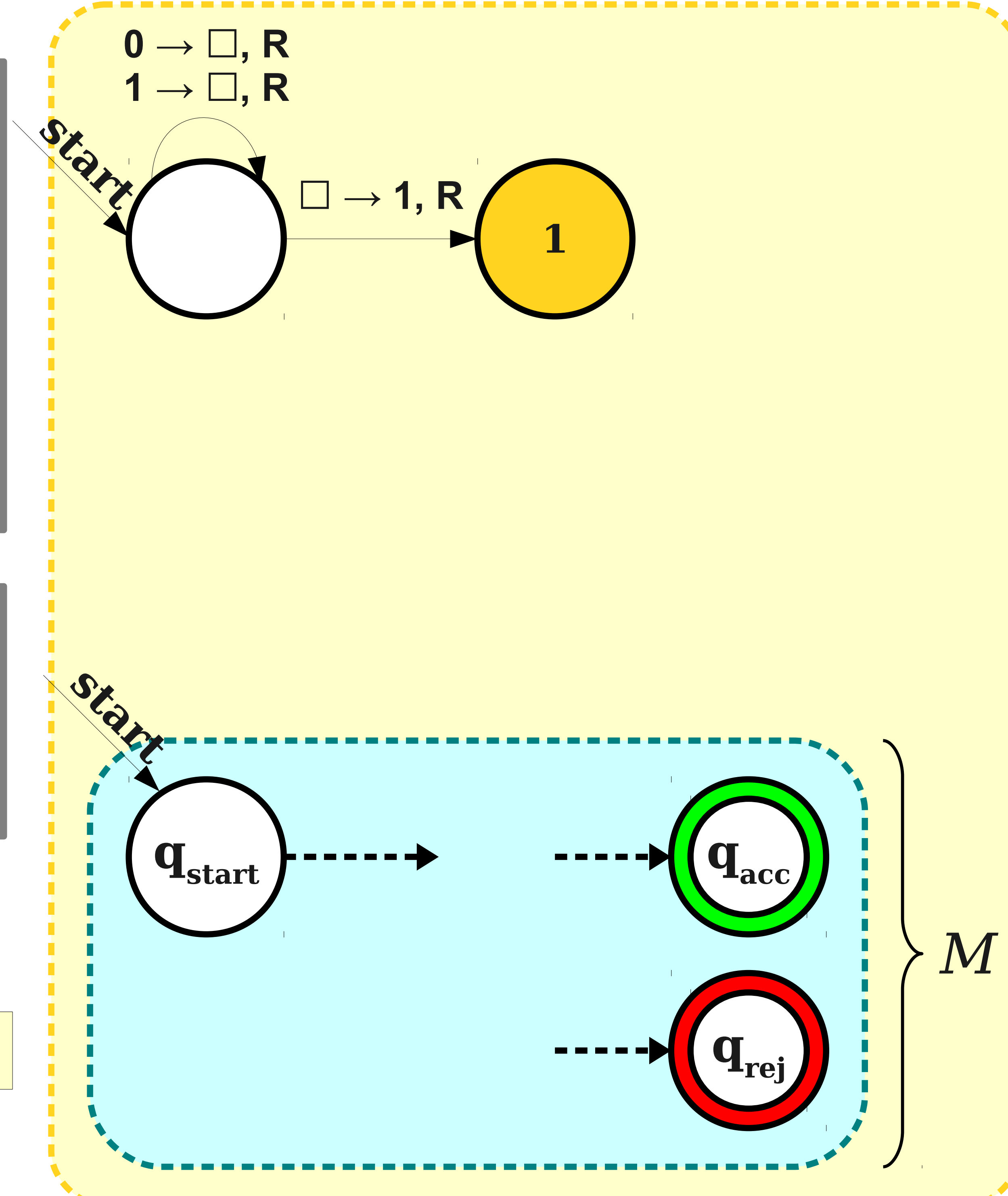


N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

Hypothetically, assume that w is the string **11011**.

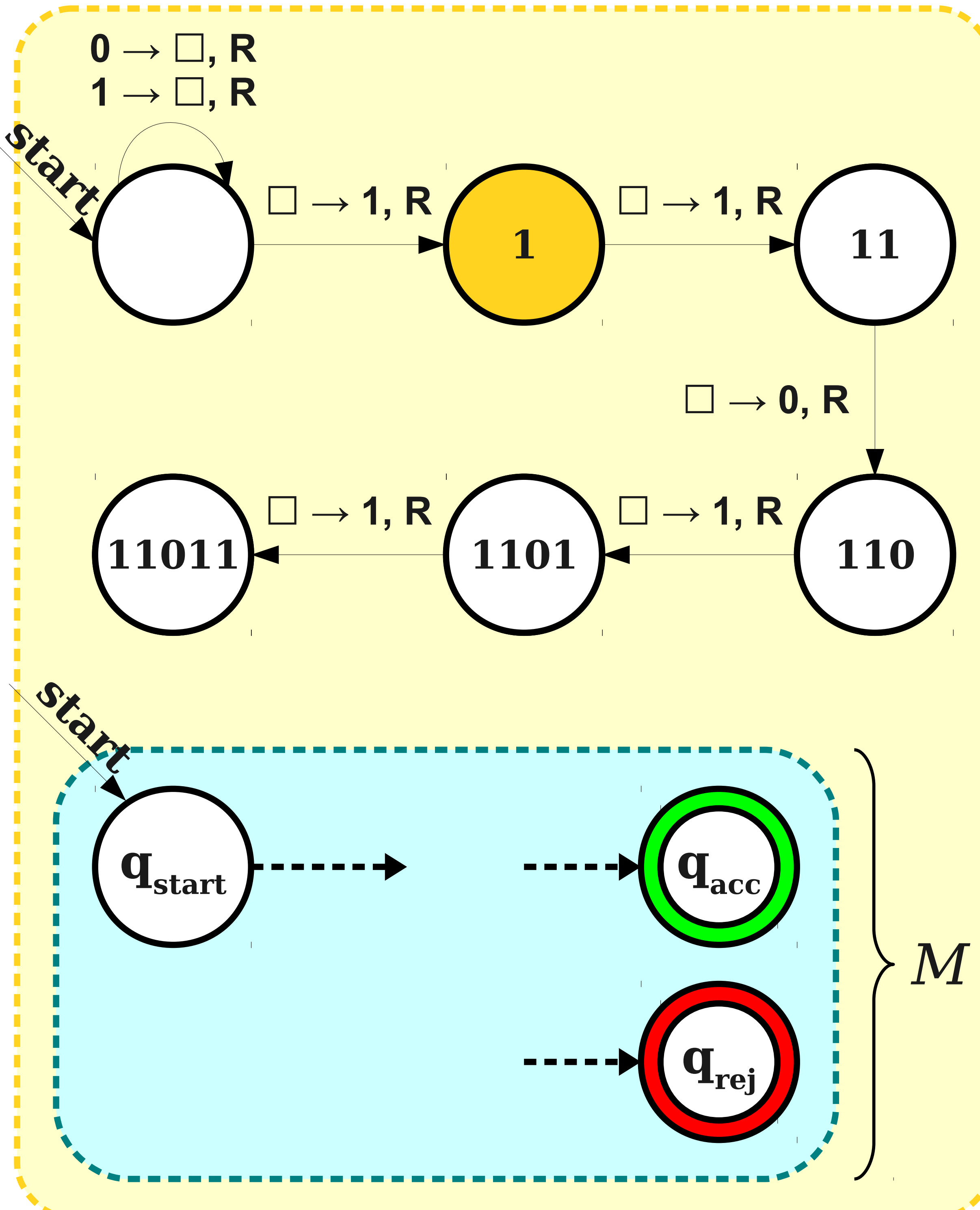
...	1						...

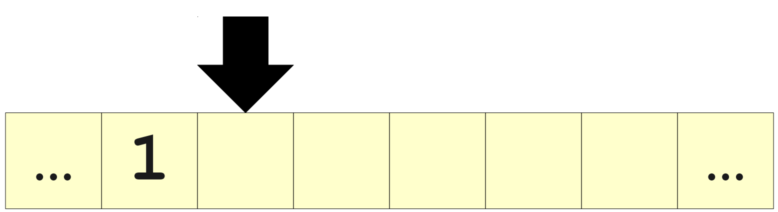


N = “On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x.”

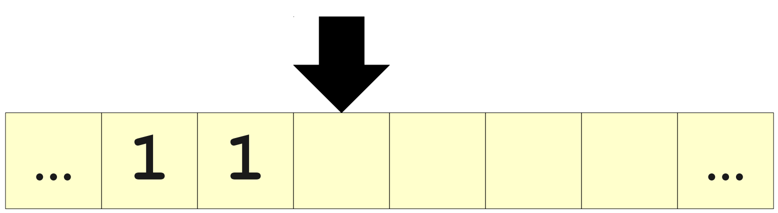
Hypothetically, assume that w is the string **11011**.

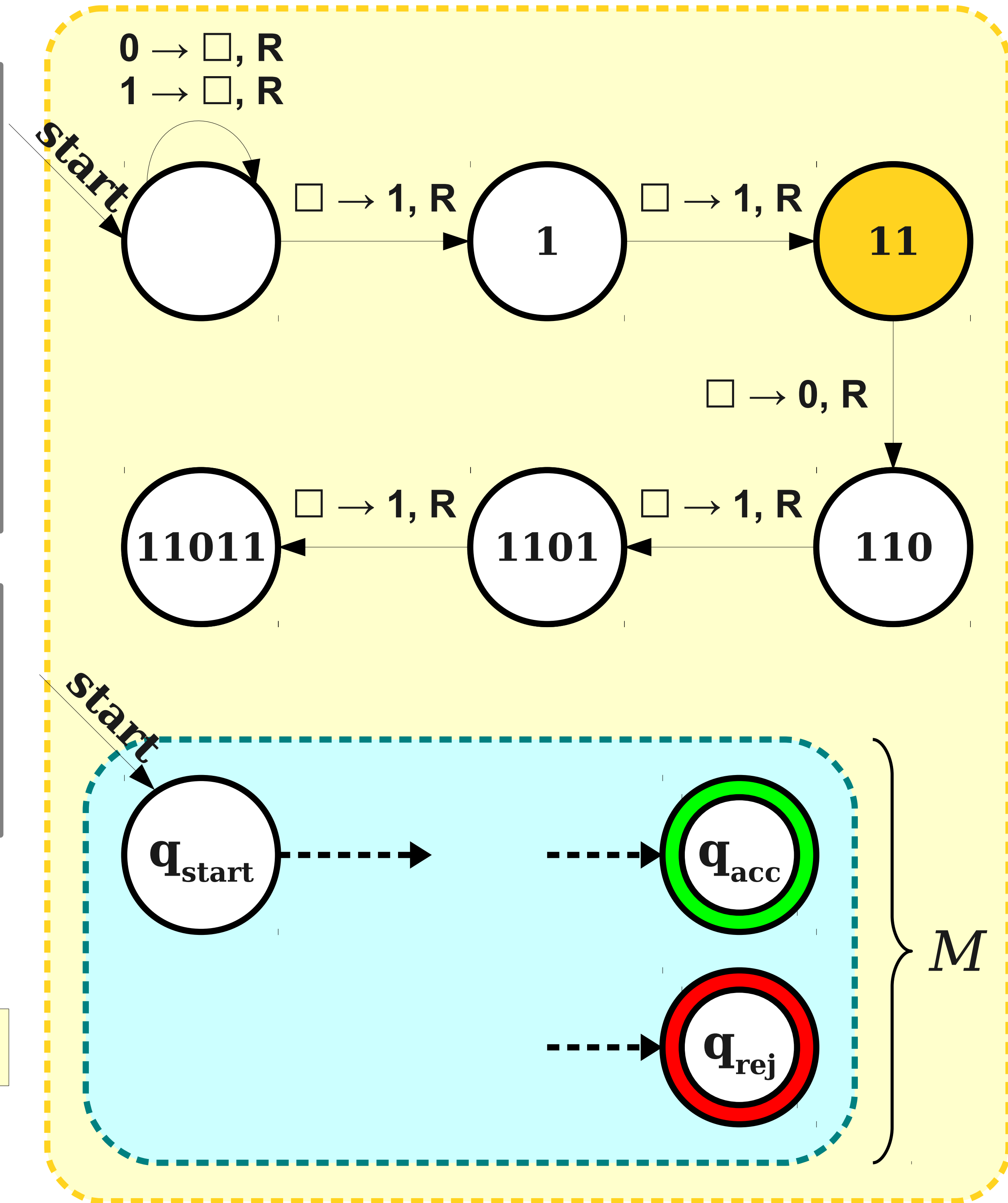




N = "On input x:
- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

Hypothetically, assume that w is the string `11011`.

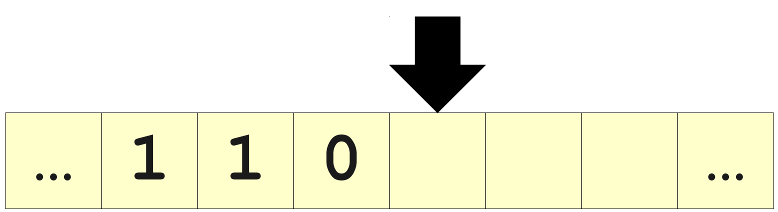


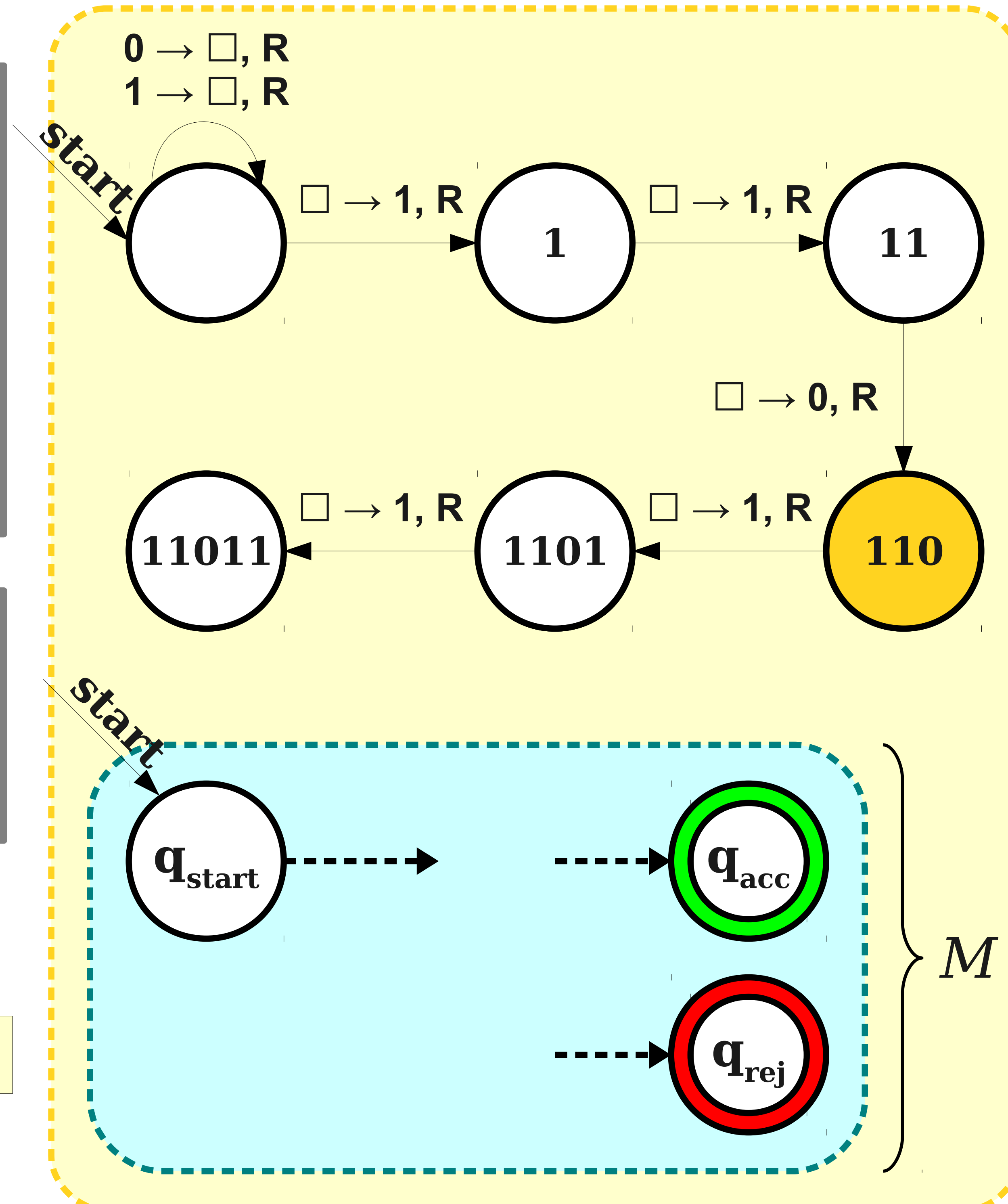


N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

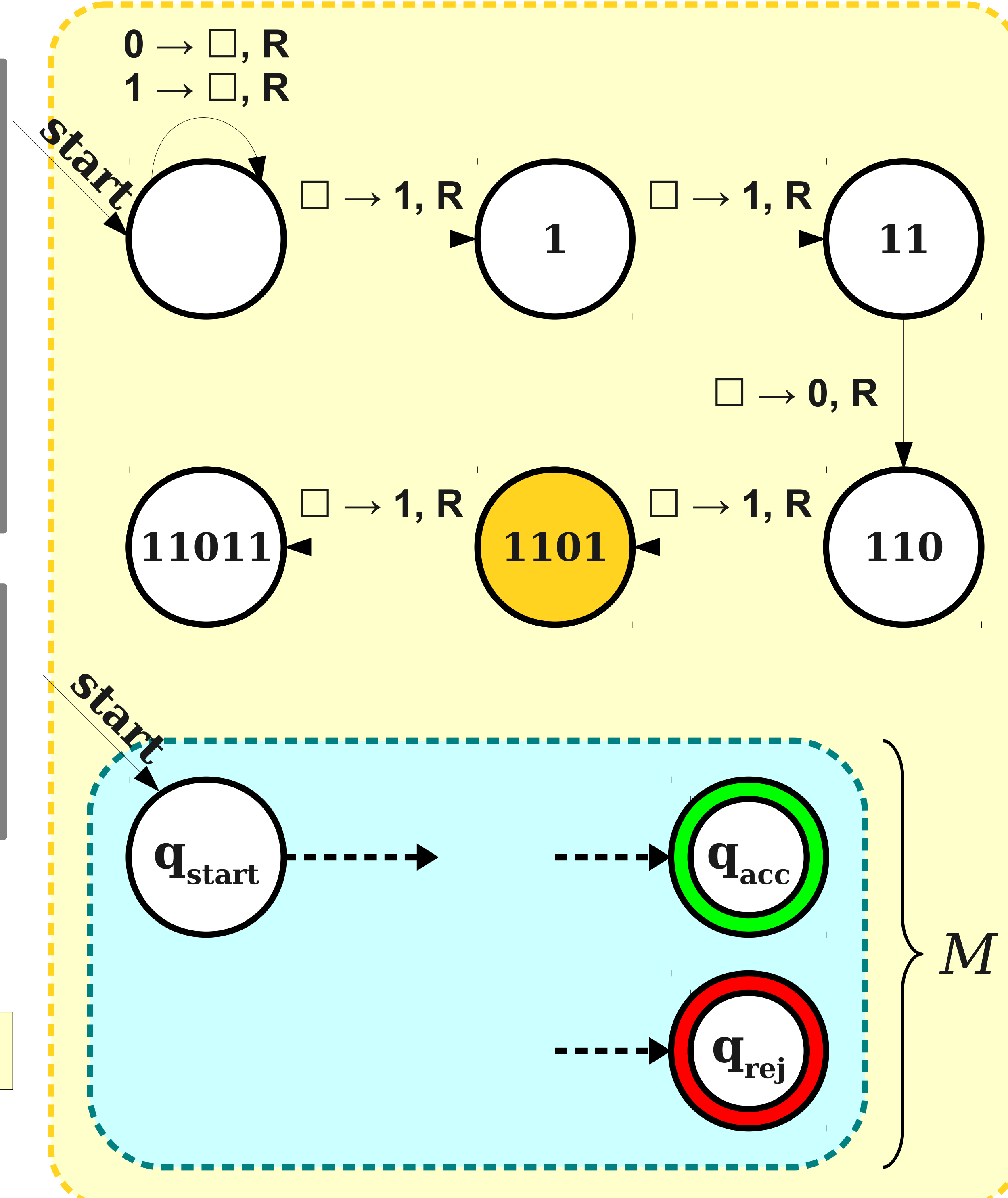
Hypothetically, assume that w is the string **11011**.



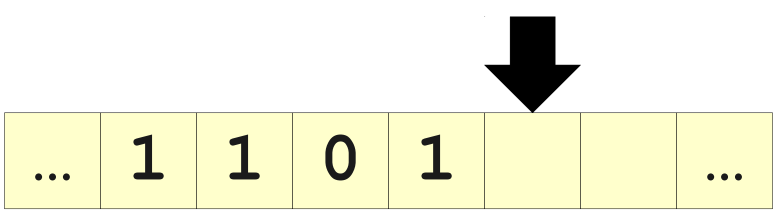


N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."



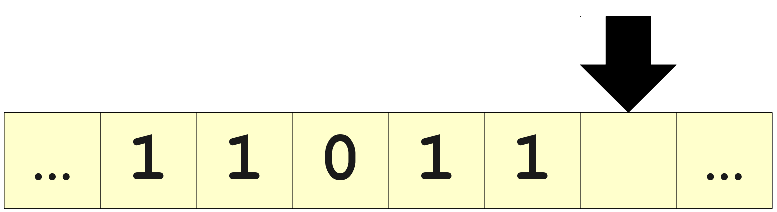
Hypothetically, assume that w is the string **11011**.

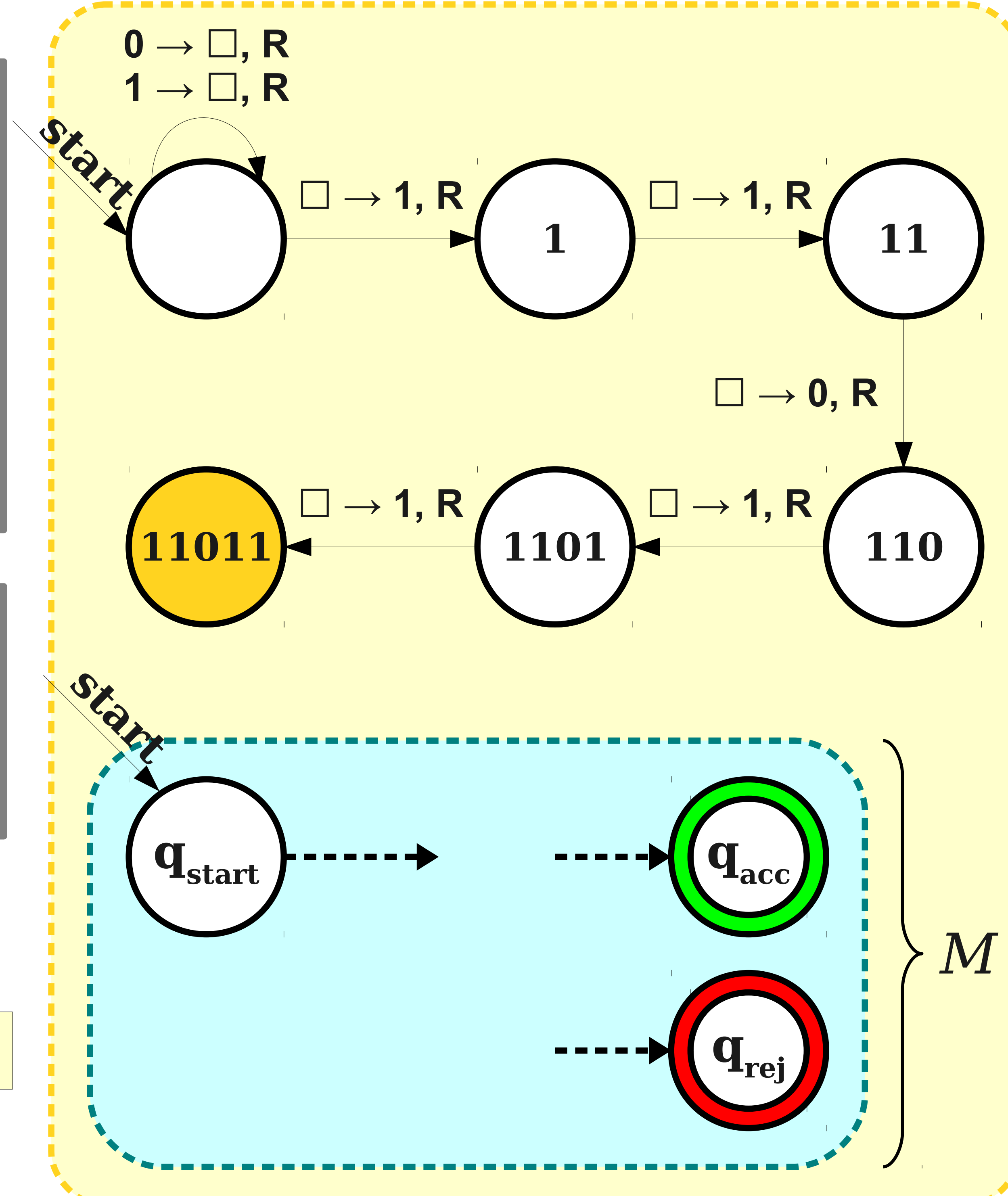


N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

Hypothetically, assume that w is the string **11011**.

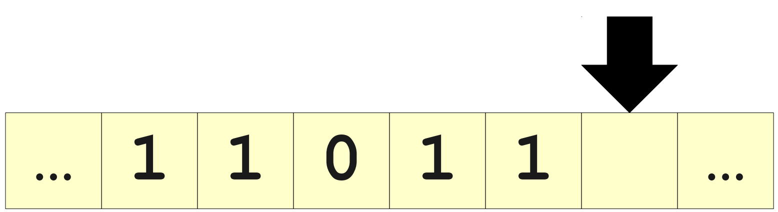


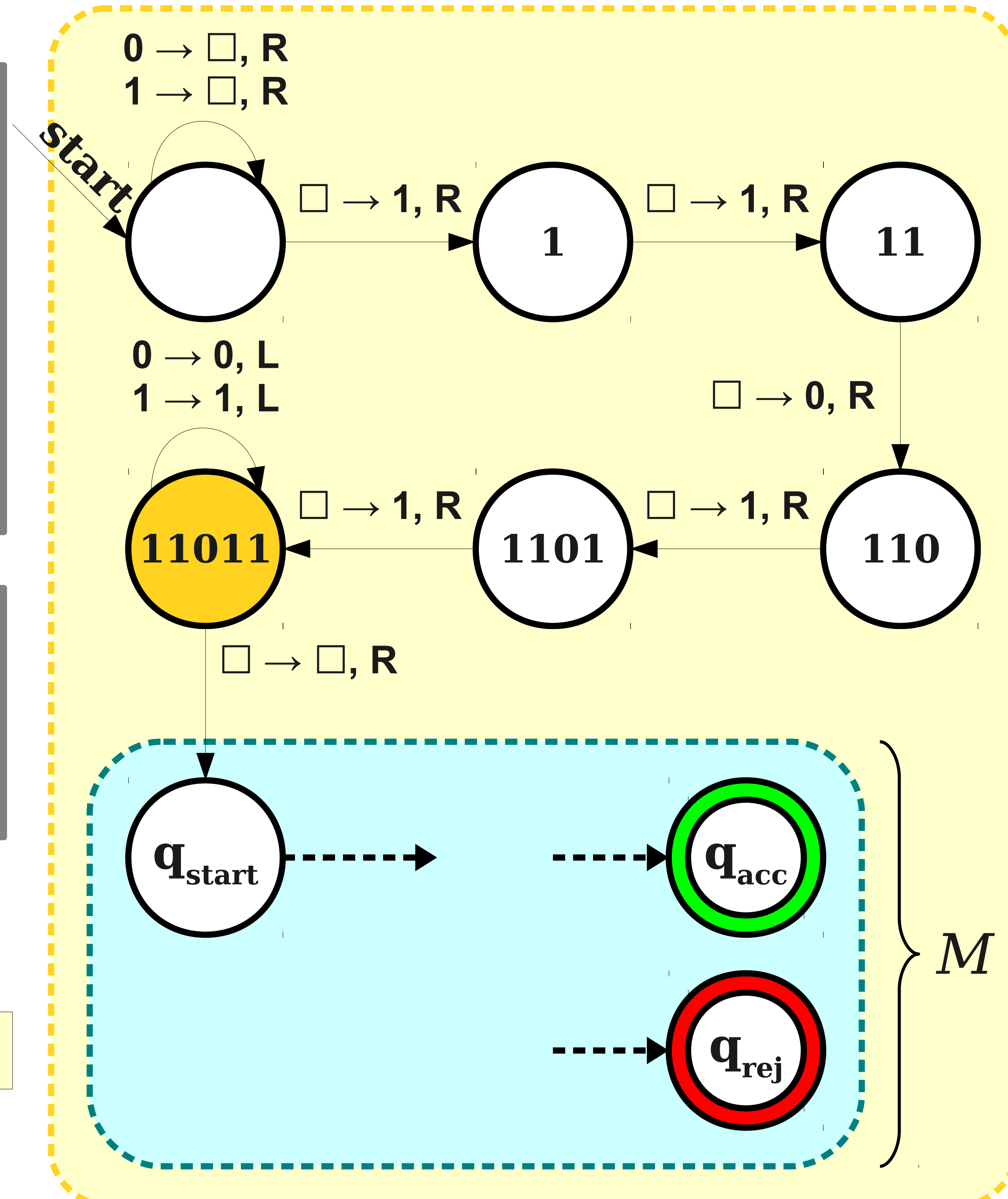


N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

Hypothetically, assume that w is the string **11011**.

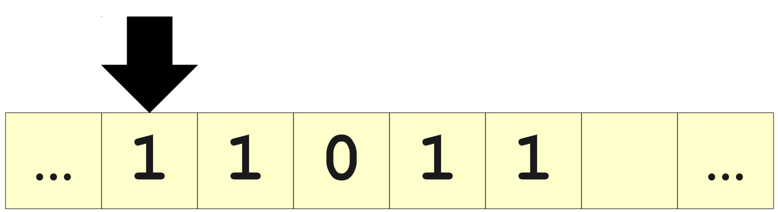


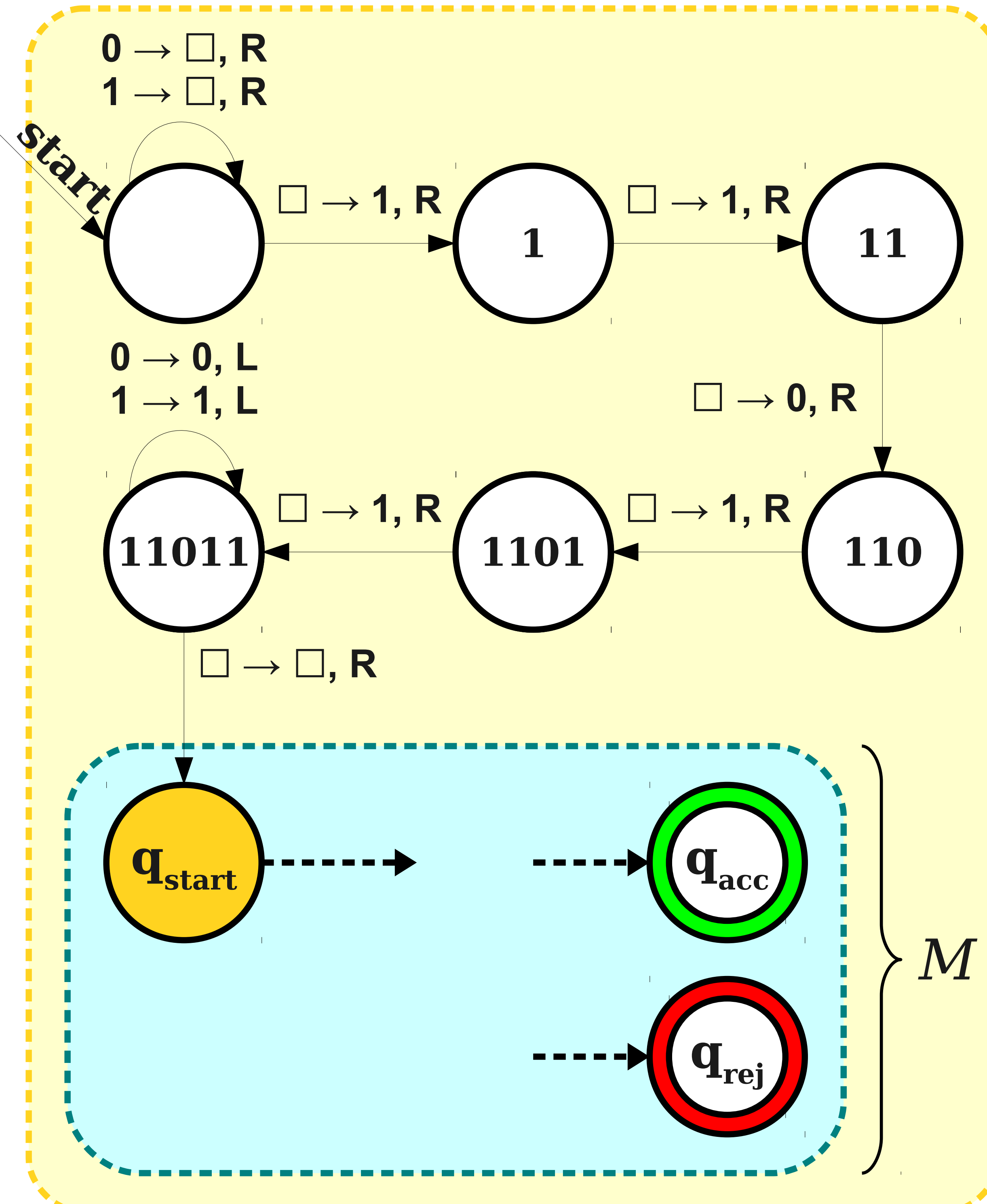


N = "On input x:

- Ignore x.
- Run M on w.
- If M accepts w, then N accepts x.
- If M rejects w, then N rejects x."

Hypothetically, assume that w is the string **11011**.





The Takeaway Point

- Turing machines can embed TMs inside of other TMs.
- TMs of the following form are legal:

H = "On input $\langle M, w\rangle$, where M is a TM:
- Construct N = "On input x:
    - Do something with x.
    - Run M on w.
    - …"
- Do something with N."

Theorem: $\overline{\mathrm{A}}_{\mathrm{TM}} \leq_{\mathrm{M}} L_{\mathrm{e}}$.

Proof: We exhibit a mapping reduction from $\overline{\mathrm{A}}_{\mathrm{TM}}$ to L_{e}.

For any TM/string pair $\langle M, w\rangle$, let $f(\langle M, w\rangle) = \langle N\rangle$, where $\langle N\rangle$ is defined in terms of M and w as follows:

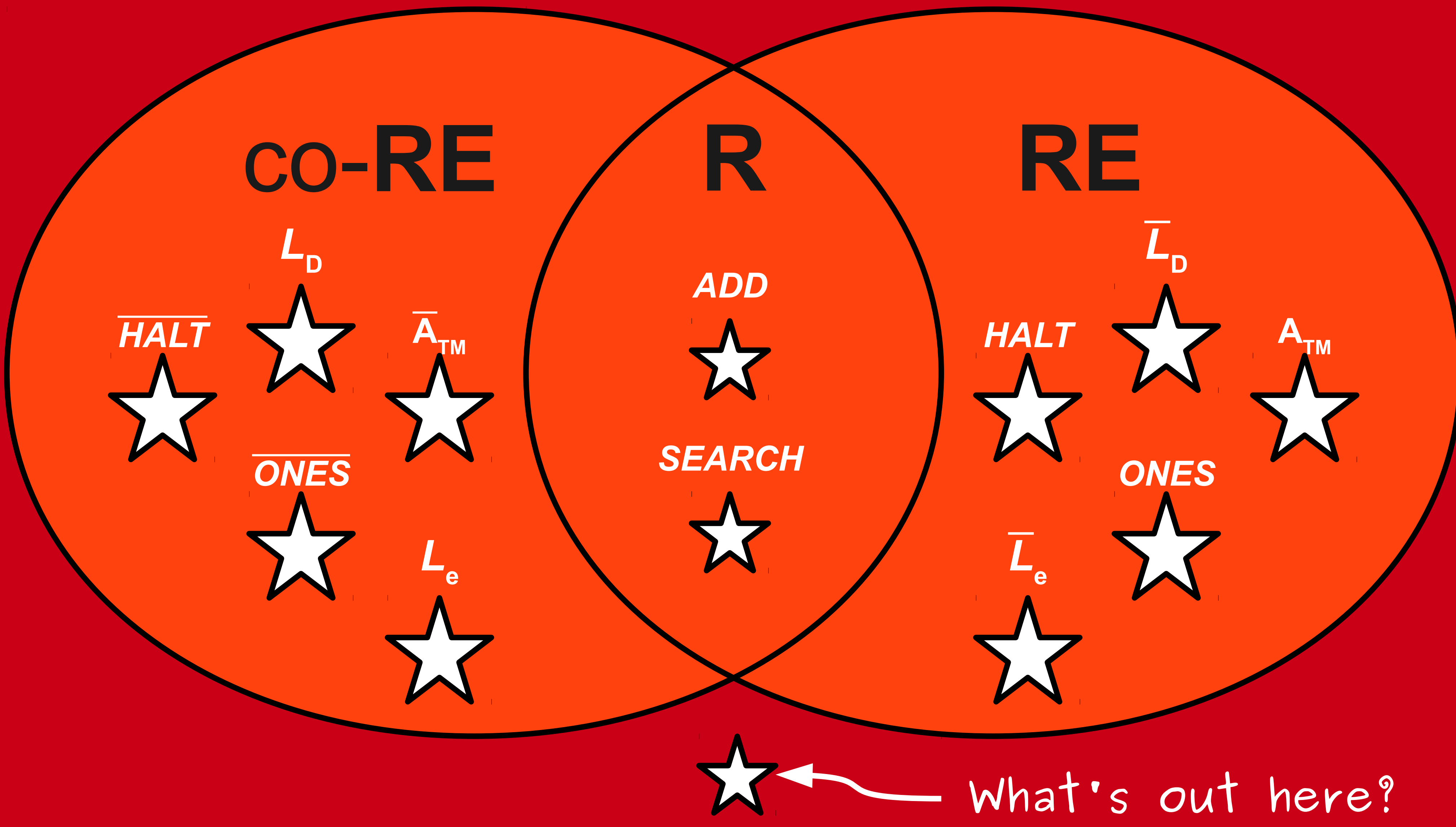
N = "On input x:
Ignore x.
Run M on w.
If M accepts w, then N accepts x.
If M rejects w, then N rejects x."

We state without proof that N is computable. We further claim that $\langle M, w\rangle \in \overline{\mathrm{A}}_{\mathrm{TM}}$ iff $f(\langle M, w\rangle) \in L_{\mathrm{e}}$. To see this, note that $f(\langle M, w\rangle) = N \in L_{\mathrm{e}}$ iff N does not accept any strings. We claim that N does not accept any strings iff M does not accept w. To see this, note that M does not accept w iff M loops on w or M rejects w. By construction, if M loops on w, then N loops on all strings, and if M rejects w, then N rejects all strings. Thus N does not accept any strings iff M does not accept w. Finally, M does not accept w iff $\langle M, w\rangle \in \overline{\mathrm{A}}_{\mathrm{TM}}$. Thus $\langle M, w\rangle \in \overline{\mathrm{A}}_{\mathrm{TM}}$ iff $f(\langle M, w\rangle) \in L_{\mathrm{e}}$, so f is a mapping reduction from $\overline{\mathrm{A}}_{\mathrm{TM}}$ to L_{e}, and so $\overline{\mathrm{A}}_{\mathrm{TM}} \leq_{\mathrm{M}} L_{\mathrm{e}}$, as required. ■

A Math Joke

Recitation Sections

The Limits of Computability



RE $\cup$ co-**RE** is Not Everything

- Using the same reasoning as the first day of lecture, we can show that there must be problems that are neither **RE** nor co-**RE**.
- There are more sets of strings than TMs.
- There are more sets of strings than twice the number of TMs.
- What do these languages look like?

An Extremely Hard Problem

- Recall: All regular languages are also **RE**.
- This means that some TMs accept regular languages and some TMs do not.
- Let $\text{REGULAR}_{\text{TM}}$ be the language of all TM descriptions that accept regular languages:

$$\mathbf{REGULAR_{TM} = \{ \langle \mathit{M} \rangle \mid \mathscr{L}(\mathit{M}) \text{ is regular } \}}$$

- Is $\text{REGULAR}_{\text{TM}} \in \mathbf{R}$? How about **RE**? How about co-**RE**?

Building an Intuition

- If you were *convinced* that a TM had a regular language, how would you mechanically verify that?
- If you were *convinced* that a TM had a nonregular language, how would you mechanically verify that?
- Both of these seem difficult, if not impossible. Chances are $\text{REGULAR}_{\text{TM}}$ is neither **RE** nor co-**RE**.

$\text{REGULAR}_{\text{TM}} \notin \mathbf{RE}$

- It turns out that $\text{REGULAR}_{\text{TM}}$ is unrecognizable, meaning that there is no computer program that can confirm that another TM's language is regular!
- To do this, we'll do a reduction from L_{D} and prove that $L_{\text{D}} \leq_{\text{M}} \text{REGULAR}_{\text{TM}}$.

$L_D \leq_M \text{REGULAR}_{TM}$

- We want to find a computable function f such that

$$\boldsymbol{\langle M \rangle \in L_D \quad \textbf{iff} \quad f(\langle M \rangle) \in \textbf{REGULAR}_{TM}.}$$

- We need to choose N such that $f(\langle M \rangle) = \langle N \rangle$ for some TM N. Then

$$\boldsymbol{\langle M \rangle \in L_D \quad \textbf{iff} \quad f(\langle M \rangle) \in \textbf{REGULAR}_{TM}}$$

$$\boldsymbol{\langle M \rangle \in L_D \quad \textbf{iff} \quad \langle N \rangle \in \textbf{REGULAR}_{TM}}$$

$$\boldsymbol{\langle M \rangle \notin \mathscr{L}(M) \quad \textbf{iff} \quad \mathscr{L}(N) \textbf{ is regular.}}$$

- Question: How do we pick N?

$L_D \leq_M \text{REGULAR}_{TM}$

- We want to construct some N out of M such that
  - If $\langle M \rangle \in \mathscr{L}(M)$, then $\mathscr{L}(N)$ is not regular.
  - If $\langle M \rangle \notin \mathscr{L}(M)$, then $\mathscr{L}(N)$ is regular.
- One option: choose two languages, one regular and one nonregular, then construct N so its language switches from regular to nonregular based on whether $\langle M \rangle \notin \mathscr{L}(M)$.
  - If $\langle M \rangle \in \mathscr{L}(M)$, then $\mathscr{L}(N) = \{ \mathtt{0}^n\mathtt{1}^n \mid n \in \mathbb{N} \}$
  - If $\langle M \rangle \notin \mathscr{L}(M)$, then $\mathscr{L}(N) = \emptyset$

The Reduction

- We want to build N from M such that
  - If $\langle M \rangle \in \mathscr{L}(M)$, then $\mathscr{L}(N) = \{ \mathtt{0}^n\mathtt{1}^n \mid n \in \mathbb{N} \}$
  - If $\langle M \rangle \notin \mathscr{L}(M)$, then $\mathscr{L}(N) = \emptyset$
- Here is one way to do this:

N = "On input x:

If x does not have the form $\mathtt{0}^n\mathtt{1}^n$, reject.

Run M on $\langle M \rangle$.

If M accepts, accept x.

If M rejects, reject x."

Theorem: $L_D \leq_M \text{REGULAR}_{TM}$.

Proof: We exhibit a mapping reduction from L_D to REGULAR_{TM}.

For any TM M, let $f(\langle M \rangle) = \langle N \rangle$, where N is defined in terms of M as follows:

N = "On input x:

If x does not have the form 0^n1^n, then N rejects x.

Run M on $\langle M \rangle$.

If M accepts $\langle M \rangle$, then N accepts x.

If M rejects $\langle M \rangle$, then N rejects x."

We claim f is computable and omit the details from this proof. We further claim that $\langle M \rangle \in L_D$ iff $f(\langle M \rangle) \in \text{REGULAR}_{TM}$. To see this, note that $f(\langle M \rangle) = \langle N \rangle \in \text{REGULAR}_{TM}$ iff $\mathscr{L}(N)$ is regular. We claim that $\mathscr{L}(N)$ is regular iff $\langle M \rangle \notin \mathscr{L}(M)$. To see this, note that if $\langle M \rangle \notin \mathscr{L}(M)$, then N never accepts any strings. Thus $\mathscr{L}(N) = \emptyset$, which is regular. Otherwise, if $\langle M \rangle \in \mathscr{L}(M)$, then N accepts all strings of the form 0^n1^n, so we have that $\mathscr{L}(N) = \{ 0^n1^n \mid n \in \mathbb{N} \}$, which is not regular. Finally, $\langle M \rangle \notin \mathscr{L}(\langle M \rangle)$ iff $\langle M \rangle \in L_D$. Thus $\langle M \rangle \in L_D$ iff $f(\langle M \rangle) \in \text{REGULAR}_{TM}$, so f is a mapping reduction from L_D to REGULAR_{TM}. Therefore, $L_D \leq_M \text{REGULAR}_{TM}$. ■