Barrier Function-Based Adaptive Lyapunov Redesign for Systems Without *A Priori* Bounded Perturbations

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***Abstract*—The problem of an adaptive Lyapunov redesign is revisited for a class of systems without *a priori* knowledge of the function majoring nonlinear uncertainties and disturbances. An adaptive barrier function-based gain for unit control is proposed, ensuring an arbitrary *a priori* predefined uniform ultimate bound for solutions despite the presence of uncertainties and disturbances. The usage of positive semi-definite barrier function generates a continuous control signal adjusting the chattering, when the perturbations are decreasing to zero.**

***Index Terms*—Adaptive control, barrier function, Lyapunov redesign, sliding modes, uncertain systems.**

I. Introduction

A. Classical Lyapunov Redesign

The term Lyapunov redesign, coined in [1], refers to the classical approach developed by Gutman [2] and Leitmann [3] for the stabilization of perturbed systems with *a priori* knowledge of the upper bound of uncertainties and disturbances (UBUD). In order to deal with the uncertainties/disturbances, Lyapunov redesign approach adds a unit control term to the nominal control with a switching manifold of relative degree one. This manifold is obtained from the time derivative of the nominal Lyapunov function on the trajectories of perturbed system. Such a choice of the switching manifold ensures that the time derivative of Lyapunov function calculated on the trajectories of perturbed system coincides with that of nominal system, whenever the gain of the discontinuous term is chosen larger than the UBUD. Consequently, for an asymptotically stable nominal system, asymptotic stability of origin is ensured even in the presence of perturbations: parametric uncertainties and disturbances. Later, under the same assumptions, the compensating discontinuous control term was substituted by its continuous approximation [4]–[8], ensuring convergence of the system's solutions to a predefined vicinity of the origin.

Usually, UBUD is unknown or overestimated. That is why the gain of the discontinuous terms needs to be adapted. Such adaptation should solve two contradictory problems.

1) After finite-time, the system's trajectories should enter into an *a priori* predefined vicinity of the origin and never leave it, despite the presence of perturbations: uncertainties and disturbances.
2) The discontinuous unit control gains should be adapted to follow the perturbation variations without being overestimated.

B. Adaptive Sliding-Mode Control

Similarly to Lyapunov redesign, adaptive sliding-mode control has shown its high efficiency compensating matched perturbations. Mostly dealing with *a priori* bounded uncertainties and disturbances, it focuses on the design of update rules for the gain of controllers to enforce ideal or real sliding modes [9]–[25]. However, the latter lacks of a straightforward methodology to predefine a behavior for the perturbed system's trajectories.

The paper [26] proposes the barrier function-based adaptation for the scalar systems, subject to disturbances with constant but unknown upper bound, having the properties 1) and 2). Extensions for a chain of integrators have been tackled by means of dual-layer high-order sliding-mode control in [24] and for SISO systems in regular form by using monitoring functions in [25] but without complying with property 1).

C. Adaptive Lyapunov Redesign

Following the pioneering work of Gutman and Leitmann, some authors [27]–[31] used adaptive Lyapunov redesign to deal with parametrical uncertainties.[1] Nonetheless, it cannot be ensured that properties 1) and 2) simultaneously hold. A trade-off

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[1] Those are upper bounded by a function of time and state that also depends in an known manner from a vector of uncertain parameters. This upper bound takes into account vanishing and *a priori* bounded (nonvanishing) terms depending on state and time, respectively (see the early works in adaptive control [32] and references therein).

exists between the overestimation of the gain of a continuous approximation of unit control and an attainable residual set for the system's trajectories.

In the paper [33], it is shown that barrier function-based adaptation of Lyapunov redesign is not straightforward, even for systems with known control matrix and constant but unknown upper-bound for disturbances, because to ensure the property 1) it is necessary to show that the system's trajectories converge into the *a priori* predefined vicinity of origin for all system's states. Moreover, an estimation of the radius for this vicinity in terms of the barrier width is needed.

D. Contribution of the Article

This article proposes a new approach to a classical stabilization problem: barrier function-based adaptive Lyapunov redesign for a class of MIMO nonlinear systems with parametrical perturbations: disturbances and unknown control matrix but without an *a priori* known UBUD. First, according to a usual step in Lyapunov redesign, a *sliding manifold* based on the derivative of the nominal Lyapunov function should be chosen. Then, the behavior of system's trajectories driven by the barrier function-based adaptation of Lyapunov redesign consists of three phases. During the first phase, the so-called reaching phase (RP), adaptation law increases a unit control gain until the value allowing a compensation of the uncertainties and disturbances, and, consequently, the system's trajectories converge into the interior of the vicinity of the *sliding manifold* with an *a priori* given radius: a barrier width. During the second stage termed barrier width dynamics (BWD), the system's trajectories are following stable sliding dynamics being in barrier width. Finally, a predefined phase (PdP) starts, where the system's trajectories will belong to the predefined vicinity of the origin of the system states while staying in the barrier width of the *sliding manifold*.

The main advantages of the barrier function-based adaptive Lyapunov redesign can be summarized as follows.

- It is ensured that the system's trajectories will enter into an *a priori* predefined vicinity of the origin in finite-time and never leave it.
- If the solution converges to the origin, the adaptive gain converges to perturbation norm, i.e., energy consumption will be diminished.
- The usage of a positive semi-definite barrier function (PSBF) produces a continuous control signal adjusting the chattering.

The rest of this article is organized as follows. In Section II, the problem formulation is presented. Section III gives a reaching phase method to drive the solution of the system to the interior of the barrier width. Based on the barrier function approach, the predefined advance ultimate bound is introduced in Section IV. The global main result and a simple algorithm for the designer are presented in Section V. A case of study regarding to trajectory tracking of a fully actuated robot manipulator is given in Section VI. Discussion about advantages and disadvantages of proposed result and its comparison with existing literature is provided in Section VII. Conclusions are drawn in Section VIII. Stability proofs are found in Appendix X and some algebraic manipulations for the case study are given in Appendix XII.

Notation: The absolute value and Euclidean vector norm are denoted by $|\cdot|$ and $\|\cdot\|$, respectively. $\mathbb{R}^+$ denotes the nonnegative real numbers, $\mathbb{Z}$ denotes the set of integer numbers. $\mathcal{C}^n$ denotes space of n-times continuously differentiable functions. $\mathbf{I}_m$ denotes an $m \times m$ identity matrix. Let A be a symmetric matrix $A = A^T$, and $\lambda_M(A) > \lambda_m(A)$ denote its maximum and minimum eigenvalues, respectively. Tombstone $\square$ (respectively, $\blacksquare$) is used to terminate assumptions (respectively, proofs). Arguments of functions are omitted when they are understood from the context.

II. Problem Formulation

A. System Under Consideration

Consider the nonlinear perturbed system class:

$$\dot{\mathbf{x}} = f(t,\mathbf{x}) + B(t,\mathbf{x})\left\{[\mathbf{I}_m + \Delta B_m(t,\mathbf{x})]\,\mathbf{u} + \Delta f_m(t,\mathbf{x})\right\} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is a vector of states, $\mathbf{u} \in \mathbb{R}^m$ is a vector of control inputs with $m \le n$. The functions $f(\cdot)$, $B(\cdot)$, $\Delta B_m(\cdot)$, $\Delta f_m(\cdot)$ piece-wise continuous in t and Lipschitz in $\mathbf{x}$ for all $(t,\mathbf{x}) \in \mathbb{R}^n \times \mathbb{R}^+$, f and B are known; the matrix function $B \in \mathbb{R}^{n\times m}$ is the control input matrix whose columns are m-linearly independent vector fields $b_i(t,\mathbf{x})$, $i = 1,2,\ldots,m$. The uncertain control-input matrix and the unknown perturbation, respectively, $\Delta B_m \in \mathbb{R}^{m\times m}$ and $\Delta f_m \in \mathbb{R}^m$ are unknown measurable functions in t, for all $\mathbf{x} \in \mathbb{R}^n$, and continuous functions in $\mathbf{x}$, for almost all $t \in \mathbb{R}^+$. In this article, the solutions for all dynamical systems are defined in sense of Filippov [34].

Assumption 1: The uncertainty $\Delta B_m(t,\mathbf{x})$ and disturbance $\Delta f_m(t,\mathbf{x})$ are bounded by unknown constant $\epsilon_b \in \mathbb{R}^+$ and function $\rho_m(t,\mathbf{x}) \in \mathbb{R}^+$, i.e., $\|\Delta B_m(t,\mathbf{x})\| < \epsilon_b < 1$ and $\|\Delta f_m(t,\mathbf{x})\| \le \rho_m(t,\mathbf{x})$. $\square$

Assumption 2: For a fixed $N \ge 2$, $N \in \mathbb{Z}$, $-1 < \lambda_m(\frac{1}{2}(\Delta B_m + \Delta B_m^T)) < N-1$, for all $(t,\mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^n$. $\square$

Remark 1: Assumption 2 is not a restriction; it ensures that ΔB_m could also reduce the control effort, when $\lambda_m(\frac{1}{2}(\Delta B_m + \Delta B_m^T)) > 0$.

B. Classical Lyapunov Redesign

Assumption 3: The known *nominal nonlinear system* associated to (1)

$$\dot{\mathbf{x}}(t) = f(t,\mathbf{x}) + B(t,\mathbf{x})\mathbf{u}(t) \tag{2}$$

is globally asymptotically stabilizable with a *nominal control law* $\mathbf{u} = \psi(t,\mathbf{x})$. Moreover, there exists a nonempty set $\mathcal{V}_0$ of two-times continuously differentiable Lyapunov functions, such that for any $V_0(\cdot) \in \mathcal{V}_0$

$$\alpha_1(\|\mathbf{x}\|) \le V_0(t,\mathbf{x}) \le \alpha_2(\|\mathbf{x}\|) \tag{3}$$

$$\dot{V}_0 = \frac{\partial V_0}{\partial t} + \frac{\partial V}{\partial \mathbf{x}}\left[f(t,\mathbf{x}) + B(t,\mathbf{x})\psi(t,\mathbf{x})\right] \le -\alpha_3(\|\mathbf{x}\|) \tag{4}$$

with class $\mathcal{K}_\infty$ functions α_i, $i = 1,2$ and class $\mathcal{K}$ function α_3. $\square$

Differentiating $V_0(t,\mathbf{x})$ along the trajectories of perturbed system (1) yields

$$\dot{V}_0 = \frac{\partial V_0}{\partial t} + \frac{\partial V_0}{\partial \mathbf{x}}\Big[f(t,\mathbf{x}) + B(t,\mathbf{x}) \times \left\{[\mathbf{I}_m + \Delta B_m(t,\mathbf{x})]\,\mathbf{u} + \Delta f_m(t,\mathbf{x})\right\}\Big].$$

Notice that under Assumption 3, the predefined qualitative behavior of the known nominal plant can be disrupted under the presence of the unknown terms $\Delta B_m(t,\mathbf{x})$ and $\Delta f_m(t,\mathbf{x})$. Consider a compensation term $\mathbf{v}$ added in the control law $\mathbf{u} = \psi(t,\mathbf{x}) + \mathbf{v}$. Hence,

$$\dot{V}_0 \leq -\alpha_3(\|\mathbf{x}\|) + \mathbf{w}^T \left[(\mathbf{I}_m + \Delta B_m(t,\mathbf{x}))\,\mathbf{v} + \Delta F_m(t,\mathbf{x})\right] \tag{5}$$

where

$$\mathbf{w}^T(t,\mathbf{x}) := \tfrac{\partial V_0(t,\mathbf{x})}{\partial \mathbf{x}} B(t,\mathbf{x}) \tag{6}$$

is a *sliding variable* and $\Delta F_m(t,\mathbf{x}) := \Delta B_m(t,\mathbf{x})\psi(t,\mathbf{x}) + \Delta f_m(t,\mathbf{x})$.

The key idea in Lyapunov redesign is to recover the value of the derivative of the nominal Lyapunov function (4) over the trajectories of the perturbed system (1). In classical Lyapunov redesign, it is supposed that there exists a function $M(t,\mathbf{x})$ as an upper bound of the disturbance $\|\Delta F_m(t,\mathbf{x})\| \leq M(t,\mathbf{x})$, such that if one chooses the unit control law

$$\mathbf{v} = -k_0(t,\mathbf{x})\tfrac{\mathbf{w}}{\|\mathbf{w}\|} \tag{7}$$

with gain $k_0(t,\mathbf{x}) > M(t,\mathbf{x})/(1+\lambda_0)$, $\lambda_0 := \lambda_m(\frac{1}{2}(\Delta B_m + \Delta B_m^T))$ (cf. [2], [3]), it is ensured that

$$\begin{aligned}\dot{V}_0 &\leq -\alpha_3(\|\mathbf{x}\|) - ((1+\lambda_0)k_0(t,\mathbf{x}) - M(t,\mathbf{x}))\|\mathbf{w}\| \\ &\leq -\alpha_3(\|\mathbf{x}\|).\end{aligned} \tag{8}$$

However, the unit control gain depends of $M(t,\mathbf{x})$ and λ_0, which are *a priori* unknown. Moreover, it has not been shown when $\mathbf{x}$ reaches the manifold $\mathbf{w} = 0$, which is crucial to compensate the perturbations at the origin.

C. Problem Under Consideration

In this article, the function $\mathbf{w}$, given in (6), will be considered as the *sliding variable*, and a set $\mathcal{W} = \{(t,\mathbf{x}) \in [t_0,\infty) \times \mathbb{R}^n : \mathbf{w}(t,\mathbf{x}) = 0\}$ as a *sliding manifold*. The control gain for $\mathbf{v}$ is designed, ensuring that following the sliding dynamics on $\mathbf{w}$, system (1) trajectories lying in the barrier width will reach some predefined vicinity of origin after finite-time, despite the presence of the perturbations ΔB_m and ΔF_m with unknown UBUD.

For this purpose, the time derivative of $\mathbf{w}$ along the trajectories of (1) should be calculated

$$\dot{\mathbf{w}} = \bar{F}(t,\mathbf{x}) + \bar{B}(t,\mathbf{x})\left[(\mathbf{I}_m + \Delta B_m(t,\mathbf{x}))\,\mathbf{v} + \Delta F_m(t,\mathbf{x})\right] \tag{9}$$

taking into account that $V_0 \in \mathcal{C}^2$, define $\bar{F}(t,\mathbf{x}) := \frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{w}}{\partial \mathbf{x}}\bar{f}$, $\bar{B}(t,\mathbf{x}) := \frac{\partial \mathbf{w}}{\partial \mathbf{x}}B(t,\mathbf{x})$ and $\bar{f}(t,\mathbf{x}) := f(t,\mathbf{x}) + B(t,\mathbf{x})\psi(t,\mathbf{x})$.

Assumption 4: The vector relative degree of $\mathbf{w}(t,\mathbf{x})$ is $\mathbf{r} = \mathbf{1}_m^T$, *i.e.*, $\bar{B}(t,\mathbf{x}) \in \mathbb{R}^{m\times m}$ is invertible for all $(t,\mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^n$. □

Assumption 5: For all $(t,\mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^n$, the following inequality holds $\lambda_m^{1/2}(\bar{B}\bar{B}^T)\lambda_m(\frac{1}{2}(\bar{B}^{-1}(\mathbf{I} + \Delta B_m) + (\mathbf{I} + \Delta B_m)^T\bar{B}^{-T})) \geq 1 + \lambda_m(\frac{1}{2}(\Delta B_m + \Delta B_m^T)) > 0$. □

Remark 2: Assumption 4 is commonly encountered in sliding-mode control literature; together with Assumption 1, it ensures the existence of equivalent control once in the sliding mode. Assumption 5 is a sufficient condition needed for the computation of the prescribed ultimate upper bound of the uncertain system's solutions.

With Assumption 4, the dynamics of the *sliding* variable can be further reduced to

$$\dot{\mathbf{w}} = \bar{B}(t,\mathbf{x})\left[(\mathbf{I}_m + \Delta B_m(t,\mathbf{x}))\,\mathbf{v} + \overline{\Delta F}_m(t,\mathbf{x})\right] \tag{10}$$

where $\overline{\Delta F}_m(t,\mathbf{x}) := \bar{B}^{-1}(t,\mathbf{x})\bar{F}(t,\mathbf{x}) + \Delta F_m(t,\mathbf{x})$ is the bounded matched disturbance satisfying the following commonly accepted conditions for the parametric UBUD (*cf.* [27]).

Assumption 6: There exist an *unknown* function $\overline{M} : \mathbb{R}^+ \times \mathbb{R}^n$ such that for all $(t,\mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^n$,

$$\|\overline{\Delta F}_m(t,\mathbf{x})\| \leq \overline{M}(t,\mathbf{x}). \tag{11}$$

Moreover, there exist an unknown constant parameter vector $\mathbf{b} \in \mathbb{R}^p$ and a known function $\bar{\Gamma} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^+$ such that for all $(t,\mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^n$

- The bound $\bar{M}(\cdot)$ in (11) depends in a known manner on an unknown parameter $\mathbf{b} \in \mathbb{R}^p$ with positive entries, i.e.,

$$\bar{M}(t,\mathbf{x}) = b_0\bar{\Gamma}(t,\mathbf{x},\mathbf{b}) \tag{12}$$

  where b_0 is an unknown positive constant.
- The function $\bar{\Gamma}(t,\mathbf{x},\cdot) : \mathbb{R}^p \to \mathbb{R}^+$ is $\mathcal{C}^1$, and nondecreasing with respect to $\mathbf{b}$. Moreover, $-\bar{\Gamma}(t,\mathbf{x},\cdot)$ is convex, i.e.,

$$-\bar{\Gamma}(t,\mathbf{x},\mathbf{b}) + \bar{\Gamma}(t,\mathbf{x},\hat{\mathbf{b}}) + \tfrac{\partial\bar{\Gamma}(t,\mathbf{x},\hat{\mathbf{b}})}{\partial \mathbf{b}}(\mathbf{b} - \hat{\mathbf{b}}) \geq 0 \tag{13}$$

  for all $\mathbf{b}, \hat{\mathbf{b}} \in \mathbb{R}^p$. □

Remark 3: For an unknown vector of parameters $\mathbf{b}$, a class of functions $\bar{\Gamma}$ satisfying Assumption 6 is that given by

$$\bar{\Gamma}(t,\mathbf{x},\mathbf{b}) = \kappa(t,\mathbf{x})^T\mathbf{b} \tag{14}$$

where the term $\kappa : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^p$ is a known function. In this case, $\bar{\Gamma}$ takes into account vanishing and nonvanishing terms that depend on state and time, respectively. For the class of robotic systems with rotational joints with $\kappa(t,\mathbf{x}) = (1\ \|\mathbf{x}\|\ \|\mathbf{x}\|^2)^T$, each entry of the unknown vector $\mathbf{b} = (b_1\, b_2\, b_3)^T$ represents the coefficient of the upper bound of external time-dependent matched disturbances, gravity and friction forces, and Coriolis forces, respectively [35]. See Section VI and references [27]–[31] for concrete examples on systems without a priori bounded perturbations.

In this case, the robustifying control law for (10) will be chosen as

$$\mathbf{v} = -k(t,\mathbf{x})\tfrac{\bar{\mathbf{w}}}{\|\bar{\mathbf{w}}\|} \tag{15}$$

with $\bar{\mathbf{w}} := \bar{B}^T(t,\mathbf{x})\mathbf{w}$ and $k(t,\mathbf{x})$ denotes the unit control gain. Before continuing with the problem formulation, suppose that Assumptions 1–4 are fulfilled. Select the feedback control law in the form $\mathbf{u} = \psi(t,\mathbf{x}) + \mathbf{v}$ with $\mathbf{v}$ from (15), with the complete knowledge of the upper bound in (11), the following result holds.

Proposition 1: If the gain is selected as $k(t,\mathbf{x}) = \frac{\overline{M}(t,\mathbf{x})+\rho_0}{1+\lambda_0}$, $\rho_0 > 0$. Then, the trajectories of the closed-loop system reach the sliding manifold $\mathcal{W}$ in a time no greater than

$$T_1 = \frac{\|\mathbf{w}(\mathbf{x}(t_0),t_0)\|}{\rho_0\lambda_m^{1/2}(\bar{B}(t,\mathbf{x})\bar{B}^T(t,\mathbf{x}))} + t_0. \tag{16}$$

Equivalently, global uniform asymptotic stability of $\mathbf{x} = 0$ is ensured.

Proof: The proof is given in Appendix A. ■

The above proposition formalizes the idea of Lyapunov redesign from the point of view of sliding-mode control; however, the gain selection depends on $\bar{M}$ and λ_0 which are unfortunately unknown (see Assumption 6). Let us recall the following definition.

Definition 1: [1] A solution $\mathbf{x}(t)$ of system (1) is said to be globally uniformly ultimately bounded with ultimate bound b, if there exists a positive constant b independent of t_0 and for every arbitrarily large positive constant c, there is $T(c, b) \geq 0$, independent of t_0, such that

$$\|\mathbf{x}(t_0)\| \leq c \quad \Rightarrow \quad \|\mathbf{x}(t)\| \leq b, \quad \forall\, t \geq t_0 + T. \tag{17}$$

Taking into account the above definition, the following problem is addressed.

Problem: Design a robustifying control law $\mathbf{u} = \psi(t, \mathbf{x}) + \mathbf{v}(\mathbf{x}, \mathbf{p})$, where $\mathbf{p} \in \mathbb{R}^{\eta}$ is vector of tunable parameters, independent from the UBUD for $\Delta F_m(t, \mathbf{x})$ and $\Delta B_m(t, \mathbf{x})$, ensuring uniform ultimate boundedness of $\mathbf{x}(t)$ with an ultimate bound $\gamma = \gamma(\mathbf{p})$, despite the presence of uncertainties and disturbances.

Since the UBUD for terms $M(\cdot)$ (respectively, $\bar{M}(\cdot)$) and λ_0 are unknown, the solution of the problem is proposed via adaption of the gain $k(t, \mathbf{x})$ in (15).

The gain adaptation consists of two steps. First, during the RP, the gain increases till a value ensuring the convergence of the system's trajectories into the barrier width. The adaptation during the second step exploits the properties of the barrier function-based approach. It allows to ensure that the system's trajectories starting in barrier width will never leave it. Moreover, a reasonable selection of Lyapunov redesign-based *sliding variable* and the size of barrier width ensures that, staying inside barrier width, the system's trajectories will reach PdP in finite time despite the UBUD is unknown.

III. Reaching Phase Design for Parametric Perturbation Terms

If the initial condition does not belong to barrier width, it is necessary to design an adaptive control law ensuring that in finite time the trajectories will reach it. Such adaptation determines the RP, allowing the gain in the unit control to reach certain value that compensates the perturbations. This means that the system (1) trajectories converge into the interior of barrier width of the *sliding manifold*. Consider the following adaptive redesigned control law (15) and gain $k(t, \mathbf{x})$ given as follows:

$$k(t, \mathbf{x}) = \|\bar{\mathbf{w}}\| + \bar{\Gamma}(t, \mathbf{x}, \hat{\mathbf{b}}) + \tfrac{\hat{\rho}}{\|\bar{\mathbf{w}}\|} \tag{18}$$

with the adaptive set of rules

$$\dot{\hat{\mathbf{b}}} = L\left[\frac{\partial \bar{\Gamma}(t, \mathbf{x}, \hat{\mathbf{b}})}{\partial \mathbf{b}}^{T} \|\bar{\mathbf{w}}\| - \Xi \hat{\mathbf{b}}\right], \quad \hat{\mathbf{b}}(t_0) = \hat{\mathbf{b}}_{t_0} \tag{19}$$

$$\dot{\hat{\rho}} = \ell - \hat{\rho}, \quad \hat{\rho}(t_0) > \ell \tag{20}$$

where $\hat{\mathbf{b}}(t_0) \in \mathbb{R}^p$ has positive elements, $\hat{\rho}(t_0) \in \mathbb{R}^+$, $\Xi \in \mathbb{R}^{p\times p}$, and $L \in \mathbb{R}^{p\times p}$ are diagonal matrices with positive entries and $\ell \gg 1$. Thus, RP is formalized as follows.

Proposition 2: Consider the perturbed system (1). Let Assumptions 1–6 be fulfilled. Set the feedback control law $\mathbf{u} = \psi(t, \mathbf{x}) + \mathbf{v}$ with $\mathbf{v}$ in (15), gain $k(t, \mathbf{x})$ in (18), and adaptive set of rules (19) and (20). Given $\varepsilon > 0$, then for any $\mathbf{x}(t_0)$, there exists a finite-time $\bar{t}$ such that the solutions of (1) reach the set $\|\mathbf{w}\| < \varepsilon$ for all $t \geq \bar{t} + t_0$.

Proof: The detailed proof is given in Appendix B. ■

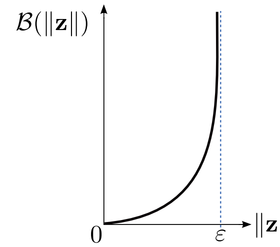

Fig. 1. A multivariable PSBF.

IV. Barrier Width Dynamics

In this section, the PSBF-based adaptation of unit control gain is presented. It is ensured that the trajectories of system (1) starting in barrier width of *sliding manifold* $\mathbf{w} = 0$ will never leave it. Also, while staying in barrier width, the solutions of system (1) converge in finite time into a predefined vicinity of the origin and stay in there during PdP. Moreover, the size of this vicinity is estimated in terms of the barrier width. The choice of PSBF ensures the continuity of control signal and adjusts the chattering, when the perturbations are decreasing to zero.

A. Barrier Functions

Definition 2: Given $\varepsilon > 0$, $\mathbf{z} \in \mathbb{R}^q$ and $\|\mathbf{z}\| < \varepsilon$, the multivariable barrier functions $\mathcal{B} : [0, \varepsilon) \to [\bar{\mathcal{B}}, \infty)$ are defined as the class of strictly increasing functions in the barrier width $[0, \varepsilon)$, with vertical asymptote $\lim_{\|\mathbf{z}\|\to\varepsilon^-} \mathcal{B}(\|\mathbf{z}\|) = +\infty$, and a unique global minimum at zero, i.e., $\mathcal{B}(0) = \bar{\mathcal{B}} \geq 0$.

Remark 4: Consider any positive function $\Delta : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ and let $\|\mathbf{z}\| < \varepsilon$. For any barrier function $\mathcal{B}(\|\mathbf{z}\|)$, if $s := s(\varepsilon, \bar{\mathcal{B}}, \Delta)$ is a root of $\mathcal{B}(s) - \Delta(t, \mathbf{z}) = 0$ such that $s < \varepsilon$. Then, $\mathcal{B}(\|\mathbf{z}\|) - \Delta(t, \mathbf{z}) > 0$ whenever $s < \|\mathbf{z}\| < \varepsilon$. This is a consequence of $\mathcal{B}$ being a strictly increasing function, hence for all $s < \|\mathbf{z}\| < \varepsilon$, $\mathcal{B}(\|\mathbf{z}\|) > \mathcal{B}(s) = \Delta(t, \mathbf{z}) > 0$.

Example 1: A PSBF depicted in Fig. 1 is given as follows:

$$\mathcal{B}(\|\mathbf{z}\|) = \tfrac{\|\mathbf{z}\|}{\varepsilon - \|\mathbf{z}\|} \tag{21}$$

according to Remark 4,

$$s = \tfrac{\Delta}{1+\Delta}\varepsilon \tag{22}$$

is a root of $\mathcal{B}(s) - \Delta(t, \mathbf{z}) = 0$ for $\Delta(t, \mathbf{z}) > 0$. It should be mentioned that by using as gain the PSBF in (21) for unit control (15), the produced control signal is continuous since PSBF is only zero at $\mathbf{z} = 0$.

B. Barrier Function-Based Compensation Method

The class of barrier functions that fulfill Definition 2 and Remark 4 can be used to ensure that the system trajectories of a perturbed system (1) are confined within an arbitrary ε-neighborhood of the sliding manifold despite the presence of any uncertain term ΔB_m and disturbance $\Delta F_m(t, \mathbf{x})$, even when UBUD is unknown. However, the trajectories must start within this same neighborhood. As a result, it is possible to predefine an ultimate bound for the systems solutions following stable sliding dynamics. To illustrate these ideas, consider the following example.

Example 2: Consider system

$$\dot{x} = -x + (1 + x^2)(v + d(t, x)), \quad |x(t_0)| = a < \varepsilon \tag{23}$$

where v and d denote a control input and a growing disturbance. On the one hand, the origin of the nominal system, i.e., system (23) has the form $\dot{x} = -x$, is asymptotically stable. It could be checked by using a Lyapunov function $V_0 = x^2/2$ that its derivative yields $\dot{V}_0 = -x^2$. On the other hand, the derivative of the function V_0 along system (23) trajectories has the following form:

$$\dot{V}_0 = -x^2 + w(v + d) \tag{24}$$

where $w := x(1 + x^2) = 0$ is the *sliding manifold*. The dynamics of the *sliding variable* satisfy the equation $\dot{w} = (1 + 3x^2)(1 + x^2)(v + \Delta), \Delta = d - \frac{x}{1+x^2}$. Assume that $|\Delta| \le \bar{M}(t, x)$ with $\bar{M}(t, x) > 0$. Selecting a control law $v = -\mathcal{B}(|w|)\frac{\bar{w}}{|\bar{w}|}$, $\mathcal{B}(|w|) = \frac{|w|}{\varepsilon - |w|}$, $\bar{w} := (1 + 3x^2)(1 + x^2)w$, then the time derivative of $V_s = w^2/2 + \mathcal{B}^2(|w|)/2$ is negative in the set $\{s_1 := \frac{\bar{M}}{1+\bar{M}}\varepsilon < |w| < \varepsilon\}$, which also holds in the set $\{s_1 < |w| < \frac{N\bar{M}}{1+N\bar{M}} =: s_N\}$, for all $N \ge 2$. From the choice of $\bar{M}$ and s_N, it follows that $\dot{V}_0 \le -x^2 - (\frac{s_1^2}{\varepsilon - s_1} - \bar{M}s_N) \le -x^2 + \frac{N-1}{N}\varepsilon$. This means that $\dot{V}_0 \le -x^2/2$ for all $|x| > \sqrt{\varepsilon}$, $N = 2$. If we pick $R > 0$ such that $\frac{\varepsilon}{2} < \frac{R^2}{2}$, then $\dot{V}_0 < 0$ in $\{\frac{\varepsilon}{2} \le V_0 \le \frac{R^2}{2}\}$. Then, the solution is uniformly ultimately bounded with predefined ultimate bound $|x| \le \sqrt{\varepsilon} < R$. This means that for any $R > 0$, there exists such an ε for the barrier width ensuring that $|x(t)| \le R$ for all $t > 0$.

Suppose that Assumptions 1–6 are fulfilled. Select the feedback control law in the form $\mathbf{u} = \psi(t, \mathbf{x}) + \mathbf{v}$ with $\mathbf{v}$ from (15), with the gain $k(t, \mathbf{x}) = \mathcal{B}(\|\mathbf{w}\|)$ chosen as the PSBF in (21) satisfying Remark 4 with $\mathbf{z} = \mathbf{w}$ and $\mathbf{w}$ given in (6). Then, the underlying BWD and PdP are resumed in the following result.

Lemma 1: For any $\varepsilon > 0$ and an integer number $N \ge 2$, the solution of system (1) starting inside the set $\{\|\mathbf{w}(\mathbf{x}(t))\| < \varepsilon\}$ will be contained therein for all future times. Moreover, the upper bound of $\mathbf{x}(t)$ has the form

$$\|\mathbf{x}(t)\| \le \alpha_1^{-1}(\alpha_2(\alpha_3^{-1}((N-1)\varepsilon))), \quad \forall\, t \ge t_0. \tag{25}$$

Proof: The proof is given in Appendices C. ■

Remark 5: In Lemma 1, the solution is maintained within the barrier width $\|\mathbf{w}(t, \mathbf{x})\| < \varepsilon$ where the barrier function attains a finite value. Notice that under Assumption 6, the ultimate bound (25) holds by construction of the barrier width, the stability of the nominal system, and the properties of the nominal Lyapunov function. However, the solution of system (1) will belong to a rather small set given in terms of the barrier width $\|\mathbf{w}(t, \mathbf{x})\| < s_1 < \varepsilon$, where s_1 is the root of the equation given in Remark 4.

The number N in (25) can be interpreted as a balance of control effort, that is, larger (respectively, smaller) values of N reduce (respectively, increase) the control gain (see Remarks 1 and 2); however, the control effort in this set is determined by the barrier function into consideration since the solution is contained in the barrier width.

V. Barrier Function-Based Lypunov Redesign: Main Results

Theorem 1: Consider the perturbed system (1). Let Assumptions 1–6 be fulfilled. Set the feedback control law $\mathbf{u} = \psi(t, \mathbf{x}) + \mathbf{v}$ with $\mathbf{v}$ in (15), gain

$$k(t, \mathbf{x}) = \begin{cases} \|\bar{\mathbf{w}}\| + \bar{\Gamma}(t, \mathbf{x}, \hat{\mathbf{b}}) + \frac{\hat{\rho}}{\|\bar{\mathbf{w}}\|}, & \text{if } \|\mathbf{w}\| > \varepsilon/2 \\ \mathcal{B}(\|\mathbf{w}\|), & \text{if } \|\mathbf{w}\| \le \varepsilon/2 \end{cases} \tag{26}$$

where the barrier function $\mathcal{B}(\|\mathbf{w}\|)$ defined in (21) with $\mathbf{z} = \mathbf{w}$, and adaptive set of rules (19) and (20). Given $\varepsilon > 0, N \ge 2$, then for any $\mathbf{x}(t_0)$, there exists a finite-time $\bar{t}$ such that the solutions of (1) reach the set $\|\mathbf{w}\| \le \varepsilon$ for all $t \ge t_0 + \bar{t}$. Moreover, there is $T \ge 0$ such that the solution of the closed-loop system (1) satisfies

$$\|\mathbf{x}\| \le \beta(\|\mathbf{x}(t_0)\|, t - t_0), \quad \forall\, t_0 \le t < t_0 + \bar{t} + T$$
$$\|\mathbf{x}\| \le \gamma(\varepsilon), \qquad \forall\, t \ge t_0 + \bar{t} + T$$

where β is a class $\mathcal{KL}$ function and

$$\gamma(\varepsilon) := \alpha_1^{-1}(\alpha_2(\alpha_3^{-1}(\varepsilon(N-1)))) \tag{27}$$

is a class $\mathcal{K}$ function.

Proof: The proof follows by applying Proposition 2 for $t_0 \le t \le t_0 + \bar{t}$ (RP) and Lemma 1 for $t_0 + \bar{t} < t < t_0 + \bar{t} + T$ (BWD). Finally, PdP is given for all $t \ge t_0 + \bar{t} + T$, and existence of T follows from the application of Theorem 4.18 in [1] during BWD (cf. end of proof in Lemma 1).

Corollary 1: Suppose that the assumptions in Theorem 1 are fulfilled. Moreover, if uncertainty $\Delta B_m = 0$ and the gain $k(t, \mathbf{x})$ is selected as in (26) with adaptive set of rules (19) and (20). Then, for any $\varepsilon > 0$ and $N \ge 2$, the ultimate bound $\|\mathbf{x}\| \le \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\varepsilon)$ holds for all $t \ge t_0 + \bar{t}$.

Remark 6: The main result in [33] follows from Corollary 1 if $\hat{\rho} = 0$, $\Xi = 0$ and $\bar{\Gamma} = b_1 \in \mathbb{R}^+$. Compared to Theorem 1 and Corollary 1, the adaptive controller in [33] increases its gain more than it is needed during RP. Moreover, the predefined ultimate bound in [33] is larger than the one presented in Corollary 1.

Remark 7: Consider the nominal Lyapunov function in Assumption 3 and define $\nu_1 := \varepsilon(N-1)$, $\nu_2 := \varepsilon$, and $\nu_3 := 2\varepsilon + h$ for all $\varepsilon, h > 0, N \ge 2$; let $\{\|\mathbf{x}\| \ge \alpha_3^{-1}(\nu_i)\}, i = 1, 2, 3$ denote the sets where asymptotic stability is recovered, respectively, in Theorem 1, Corollary 1, and the Lemma in [28]. By taking into account that the main result reported in [28] does not consider input matrix uncertainty, the predefined ultimate bound in (27) is less conservative than the one obtained in [28], i.e., $\nu_1 - \nu_3 = (N-3)\varepsilon - h < 0$ for $N = 2, 3$. In equal conditions, the ultimate bound in Corollary 1 is smaller than the one obtained in [28], i.e., $\nu_2 - \nu_3 = -(\varepsilon + h) < 0$ for all $N \ge 2$. The above comments still hold true for the case reported in [8] where the UBUD is assumed to be known.

Remark 8: Note that the ultimate bound in (27) is predefined, that is, it is uniform with respect to t_0, the UBUD for matched perturbation ΔF_m, and input matrix uncertainty ΔB_m. The following algorithm provides a strategy to the designer to predefine an ultimate bound for system (1).

Algorithm 1: Algorithm Design of Predefined Ultimate Bound.

Step 1. Select a radius of predefined vicinity $R > 0$ of the state $\mathbf{x}$

Step 2. Select the *sliding variable* $\mathbf{w}$ from (6)

Step 3. Fix $N \ge 2$ in accordance with Assumption 2 and select $\varepsilon < (\alpha_3 \circ \alpha_2^{-1} \circ \alpha_1(R))/(N-1)$

Step 4. Use barrier function in Theorem 1 with BW of size ε defined in Step 3.

VI. Case of Study: 2-DoF Robot

Consider the 2-degrees of freedom robot presented in [35]

$$\tilde{J}(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \tau + \varphi(t) \tag{28}$$

where $\mathbf{q} = [q_1\, q_2]^T$, $\tau = [\tau_1\, \tau_2]^T$ and the matrices $\tilde{J} := J(\mathbf{q}) + \Delta(\mathbf{q})$

$$J(\mathbf{q}) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, \quad \Delta(\mathbf{q}) = \begin{bmatrix} \delta_1 & \delta_2 \\ \delta_2 & \delta_3 \end{bmatrix} \tag{29}$$

$$\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}, \quad C(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} -h\dot{q}_2 & -h(\dot{q}_1 + \dot{q}_2) \\ h\dot{q}_1 & 0 \end{bmatrix} \tag{30}$$

with

$$\begin{aligned}
J_{11} &= m_1 l_{c1}^2 + I_1 + m_2[l_1^2 + l_{c2}^2 + 2l_1 l_{c2}\cos(q_2)] + I_2 \\
J_{22} &= m_2 l_{c2}^2 + I_2 \\
J_{12} &= J_{21} = m_2 l_1 l_{c2}\cos(q_2) + m_2 l_{c2}^2 + I_2 \\
\delta_1 &= \bar{m}_1 l_{c1}^2 + \bar{m}_2\left(l_1^2 + l_{c2}^2 + 2l_1 l_{c2}\cos(q_2)\right) \\
\delta_2 &= \bar{m}_2\left(l_1 l_{c2}\cos(q_2) + l_{c2}^2\right) \\
\delta_3 &= \bar{m}_2 l_{c2}^2 \\
h &= m_2 l_1 l_{c2}\sin(q_2) \\
g_1 &= m_1 l_{c1} g \sin(q_1) + m_2 g[l_{c2}\sin(q_1 + q_2) + l_1\sin(q_1)] \\
g_2 &= m_2 l_{c2} g \sin(q_1 + q_2).
\end{aligned}$$

Let δ_i denote uncertainty in the masses for $i = 1, 2, 3$ and $\varphi(t)$ denotes an external time-dependent disturbance. Set variables $x_1 = \mathbf{q} - \mathbf{q}_d$ and $x_2 = \dot{\mathbf{q}} - \dot{\mathbf{q}}_d$, let $\mathbf{q}_d \in \mathcal{C}^2$ denote the desired trajectory. Then, the system can be rewritten in the desired compact form (see details in Appendix XII)

$$\dot{\mathbf{x}} = A\mathbf{x} + B\left[G(x_1)\left(\mathbf{I} + \Delta_G(x_1)\right)\tau + h(t, \mathbf{x})\right] \tag{31}$$

where

$$A = \begin{bmatrix} 0 & \mathbf{I}_2 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ \mathbf{I}_2 \end{bmatrix}, \ G(x_1) = [J(x_1)]^{-1}$$

$$\Delta_G(x_1) = \left(J(x_1)(\tilde{J}(x_1))^{-1} - \mathbf{I}\right)$$

$$\begin{aligned}
h(t, \mathbf{x}) &= G(x_1)\left(\mathbf{I} + \Delta_G(x_1)\right) \times \\
&\quad \times \left(-C(x_1, x_2)x_2 - \mathbf{g}(x_1) + \varphi(t) - \tilde{J}(x_1)\ddot{q}_d\right)
\end{aligned}$$

and $\mathbf{x} = [x_1\, x_2]^T$, $x_1 = [x_{11}\, x_{12}]^T$, $x_2 = [x_{21}\, x_{22}]^T$.

The control objective is to stabilize the error $\mathbf{x}$ in the presence of matched perturbations: uncertainties and disturbances. This is achieved in a two-step design: a nominal control design is given first and later on a redesigned control law is obtained according to Theorem 1.

Nominal control design: Consider the nominal system, i.e., $\delta_i = 0$ for $i = 1, 2, 3$ and $\mathbf{g}(\cdot) = C(\cdot, \cdot) = \varphi = \ddot{q}_d = 0$, as

$$\dot{x} = A\mathbf{x} + BG(x_1)\tau$$

and consider the control $\tau = \psi(t, \mathbf{x}) = -\varrho B^T P\mathbf{x}$, with $\varrho \geq \frac{1}{2}\lambda_M(J(\mathbf{x}))$, where P is the solution of the algebraic Riccati equation

$$PA + A^T P - \varrho PBB^T P + 2\mathbf{I} = 0\,. \tag{32}$$

Then, by means of the nominal Lyapunov function, $V_0 = \mathbf{x}^T P\mathbf{x}$, it can be proved that $\dot{V}_0 \leq -\|\mathbf{x}\|^2$.

Control redesign: To deal with the uncertain case, set the redesigned control $\tau = \psi(t, \mathbf{x}) + \mathbf{v}$ and the variable $\mathbf{w} = B^T P\mathbf{x}$, with its time derivative given by

$$\begin{aligned}
\dot{\mathbf{w}} = B^T P\bar{A}\mathbf{x} + \bar{B}(t, \mathbf{x})\left[(\mathbf{I} + \Delta_G(x_1))\,\mathbf{v}\right. \\
\left. + \varrho\Delta_G(x_1)\mathbf{w} + h(t, \mathbf{x})\right]
\end{aligned} \tag{33}$$

where $\bar{A} = A - \varrho BG(x_1)B^T P$ and with $\bar{B}(t, \mathbf{x}) = B^T PBG(x_1)$. Recalling that $\psi(t, \mathbf{x}) = -\varrho\mathbf{w}$, we define the matched uncertainty term as

$$\overline{\Delta F}_m(t, \mathbf{x}) = \bar{B}(t, \mathbf{x})^{-1}B^T P\bar{A}\mathbf{x} + \varrho\Delta_G(x_1)\mathbf{w} + h(t, \mathbf{x})\,. \tag{34}$$

From the properties of manipulator robots with only rotative joints, gravitational and Coriolis forces are bounded by $\|g(x_1)\| \leq k'$ and $\|C(x_1, x_2)x_2\| \leq k_{C1}\|x_2\|^2$ (see [35]), then

$$\|h(t, \mathbf{x})\| \leq b_1 + b_3\|\mathbf{x}\|^2$$

$$\|(\bar{B}(t, \mathbf{x}))^{-1}B^T P\bar{A}\mathbf{x} + \varrho\Delta_G(x_1)\mathbf{w}\| \leq b_2\|\mathbf{x}\|\,.$$

such that $\|\overline{\Delta F}_m(t, \mathbf{x})\| \leq b_1 + b_2\|\mathbf{x}\| + b_3\|\mathbf{x}\|^2$, where the parameter vector $\mathbf{b} = [b_1,\, b_2,\, b_3]$ is unknown. From the above derivation and according to Proposition 2, $\bar{\Gamma}(t, \mathbf{x}, \hat{\mathbf{b}}) := \hat{b}_1 + \hat{b}_2\|\mathbf{x}\| + \hat{b}_3\|\mathbf{x}\|^2$ and $\bar{\mathbf{w}} = (B^T PBG(x_1))\mathbf{w}$.

A. Simulation Scenarios

In numerical simulation, the proposed redesigned control law in Theorem 1 is compared with those given in Theorem 4.1 in [27] and Lemma in [28]. These approaches are denoted as PSBF, CLA, and BTA, respectively. For the PSBF, consider the following parameters:

$$L = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \Xi = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}$$

with initial conditions $\hat{\mathbf{b}}(0) = \begin{bmatrix} 0.01 & 0.01 & 0.01 \end{bmatrix}^T$, $\hat{\rho}(0) = 2$, $x_1(0) = [1\ -1]^T$, and $x_2(0) = [0\quad 0]^T$. For CLA and BTA, set $\Gamma(t, x, \hat{\mathbf{a}}) = \bar{\Gamma}$, with their corresponding update rules, the same matrix L, and $\epsilon(0) = 10$ (cf. with [27], [31]).

Moreover, consider the external perturbation defined as $\varphi(t) = \eta(t)\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ with

$$\eta(t) = \begin{cases} 2\sin(4t) & \text{if } t < 4\pi \\ 5\sin(4t) & \text{if } 4\pi \leq t < 8\pi \\ 0.5\sin(4t) & \text{if } t \geq 8\pi \end{cases}$$

the mass uncertainty defined in Appendix XII and the desired trajectory $q_d = [\sin(t)\ \cos(t)]^T$. The remaining simulation parameters can be found in Tables I and II.

B. Analysis of the Results

The norm of the state $\mathbf{x}$ and the *sliding variable* $\mathbf{w}$ in the three approaches are shown in Fig. 2. To show the advantages of PSBF over CLA and BTA, an external perturbation $\varphi(t)$ changes its amplitude in two stages: increases at $t = 4\pi$ and decreases at $t = 8\pi$. For CLA [Fig. 2(a)], the convergence to zero is lost with the sudden increment of the perturbation amplitude at $t =$

TABLE I
PARAMETERS OF THE MODEL [35]

Notation	value	Notation	value
l_1	0.45 m	l_2	0.45 m
l_{c1}	0.091 m	l_{c2}	0.048 m
m_1	23.902 kg	m_2	3.880 kg
g	9.81 kg/s^2	I_1	1.266 kg m^2
I_2	0.093 kg m^2	$\bar{m}_1$	0 kg
$\bar{m}_2$	2 kg		

TABLE II
SIMULATION PARAMETERS

Notation	value	Notation	value
ϱ	5.03	k'	70.3013
k_{C1}	0.336	ε	0.05
$\bar{\ell}$	0.1	b_0	0.50
ℓ	10	h	0.001
$\lambda_m(P)$	0.7321	$\lambda_M(P)$	2.7321

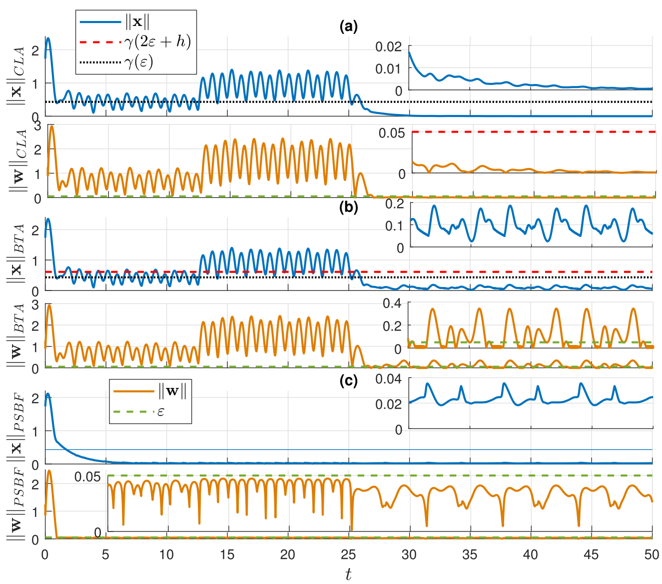

Fig. 2. Norm of the state $\mathbf{x}$ and sliding variable $\mathbf{w}$.

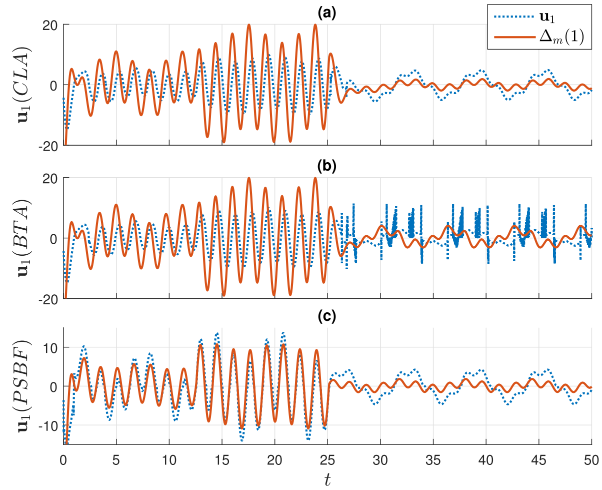

Fig. 3. First component of control signal and matched perturbation.

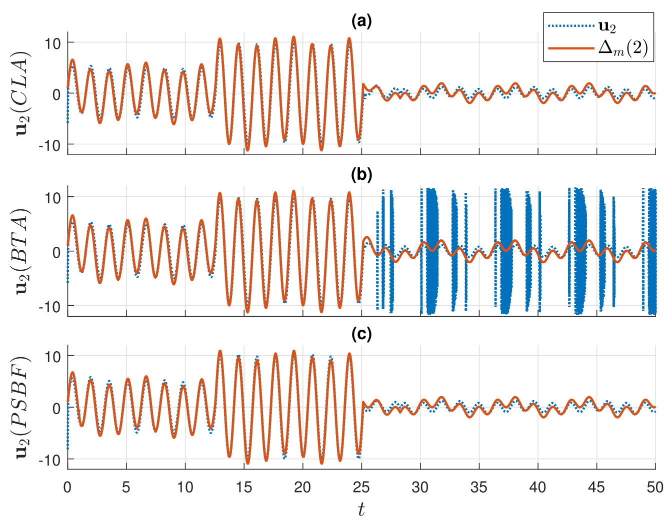

Fig. 4. Second component of control signal and matched perturbation.

4π, it surpasses the ultimate bounds of BTA and PSBF, and when the perturbation diminishes its magnitude at $t = 8\pi$, $\mathbf{x}(t)$ converges asymptotically to zero. For the BTA [Fig. 2(b)], the behavior is similar to CLA; note from $t = 4\pi$ that it actually surpasses its own predefined ultimate bound $\gamma(0.101) = 0.614$, but hysteresis maintains $\|\mathbf{x}\|$ in a region that clearly depends on UBUD. This is clearly a consequence that the barrier width $\|\mathbf{w}\| < 0.05$ is exceeded in the approaches CLA and BTA. In contrast, for the PSBF [Fig. 2(c)] after the RP, the solution $\mathbf{x}$ enters in the BWD with $\|\mathbf{w}\|_{\mathrm{PSBF}} < 0.05$, the barrier function maintains the bound of the solution far below the predefined ultimate bound $\gamma(0.05) = 0.432$ in (27) with $N = 2$, and hence the solution stays in PdP for all time on.

Being able to maintain the *sliding variable* inside an arbitrary ε-vicinity of the *sliding manifold* (compare odd subplots in Fig. 2) gives the best transient system response.

In Figs. 3 and 4, the control signals for three of the approaches are presented. From Figs. 3(c) and 4(c), the continuous control signals follow the perturbation variations and they are always close to the perturbation component. For CLA, Fig. 3(a) shows the perturbation's first component is bigger than the control signal, which explains why the norm of $\mathbf{w}$ in CLA is bigger. In Fig. 4(a), the second component of the control signal for CLA is shown, which appears to be close to the perturbation for this channel. In a similar way, Figs. 3(b) and 4(b) show that the control signal for BTA behaves similarly to the CLA scenario. However, it is important to remark that both of the control signals in CLA and (most evident) in BTA have numerical errors in the simulations due to the hysteresis-like function [28] and the smoothed approximation [27] of unit control, respectively.

On the other hand, Fig. 5 shows the gains in the CLA, BTA, and PSBF scenarios. It can be seen that the gain of the PSBF increases at the beginning in the RP, then it diminishes/increases its magnitude when the perturbation decreases/increases during BWD and PdP, which is a valuable property for low energy consumption in the actuator. In the case of CLA, the gain increases in order to overestimate the UBUD and maintain the state bounded near the origin. Also, for the BTA, gain increases; however, it can be seen that the magnitude with respect to CLA

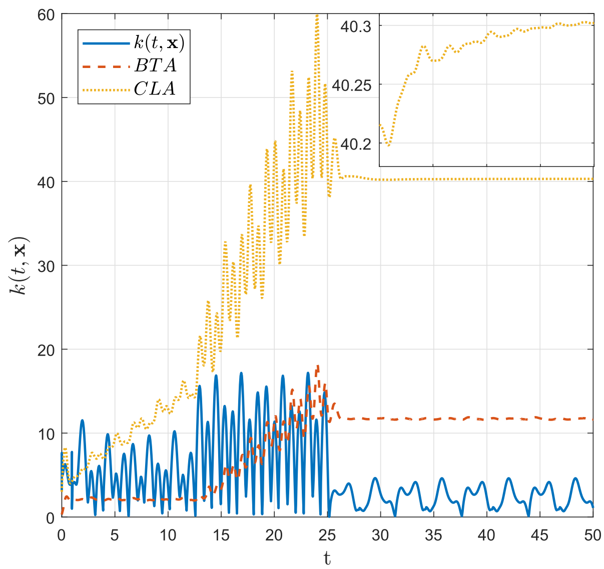

Fig. 5. Adaptive gains for the simulation scenarios.

is smaller, but again greater than the PSBF gain. It is important to remark that the gain related to the CLA and BTA continues to increase (in BTA at a smaller rate) and does not decrease as the external perturbation decreases its amplitude from $t = 4\pi$.

VII. Discussion

In this section, the differences between the obtained result and those in the existing literature are presented. Besides the advantages, also the disadvantages of the proposed approach are highlighted.

A. Class of Systems

According to the type of destabilizing perturbations of general nature, the class of systems considered in this article is broader than those considered without input matrix uncertainty in [28]–[31] and comply with the more general class of systems considered in [27].

B. Adaptive Behavior

Compared to the adaptive gain in [27]–[29], the adaptation presented in this article for RP does not increase the gain more than it is needed; once the solution enters to an arbitrary barrier width, the gain stops increasing and it is switched to a PSBF that follows perturbation variations. In contrast, the adaptive gains in [27]–[29] do not cease to grow and cannot follow the perturbation variations (e.g., see Fig. 5).

C. Predefined Phase

In [27]–[31], the gain is increased until the system's solutions reach the predefined neighborhood of the origin or residual set of the state-space, then the gain is fixed at this value, ensuring that system's solutions are contained inside the set for some interval. However, one cannot be sure that the perturbation can grow in any moment. If the disturbance grows, the system solutions will leave any predefined set, so the gain increases to reach it again. Hence, one cannot estimate *a priori* the size of the region in which system's solutions will escape from predefined neighborhood of the origin [e.g., see Fig. 2(a) and (b)]. In contrast, in our approach, the PdP is *a priori* given and reached in finite-time [see Fig. 2(c)].

D. Limitations

The barrier function-based adaptive Lyapunov redesign presented in this article has two disadvantages.

- Barrier function-based adaptation requires a theoretically unbounded control signal that does not look feasible. On the other hand, for the stabilization of any system, an actuator with a known capacity should be selected. Moreover, even the UBUD is unknown; it should be supposed that the actuator with known capacity can compensate for the perturbation. An example of such compensation with saturated barrier function is presented in [36].
- The proposed algorithm has certain tolerance margin with respect to the noises. Since convergence to the origin is not needed, i.e., the solution should only be contained inside the barrier width. It caused that the barrier function gain can be even less than the norm of perturbation, and for sure smaller than the actuator capacity. This difference between the maximum capacity of the actuator and real gain forms a tolerance margin with respect to the noise (see Remark 5).

VIII. Conclusion

This article addressed a barrier function adaptation of Lyapunov redesign for the global practical stabilization problem with predefined upper bound of system's solutions for the wide class of nonlinear systems with parametrical uncertainties and disturbances without an *a priori* known UBUD. First, according to a usual step in Lyapunov redesign, a *sliding manifold* based on the derivative of the nominal Lyapunov function should be chosen. Then, the behavior of system's trajectories driven by the barrier function-based adaptation of Lyapunov redesign evolves in three phases. During the first phase, the so-called RP, by increasing the gain until the value allowing the uncertainties and disturbances compensation, the system's trajectories converge into the interior of the vicinity of the *sliding manifold* with a precalculated radius: a barrier width. During the second stage called BWD, the system's trajectories are following stable sliding dynamics being in the barrier width. Finally, a PdP starts, where the system's trajectories will belong to the predefined origin of the system states while staying in the barrier width of the *sliding manifold*.

The barrier function-based adaptive Lyapunov redesign offers the following advantages.

- It is ensured that the system's trajectories converge not only into a given vicinity of the sliding manifold, but also they will enter into an *a priori* predefined vicinity of origin in finite-time and never leave it.
- If the solution converges to the origin, the adaptive control signal converges to perturbation norm, i.e., the gain is not overestimated and energy consumption required to maintain the system in the predefined vicinity of origin will be even less than the norm of perturbation.
- The usage of PSBF produces a continuous control signal and reduces the chattering.

APPENDIX A
TECHNICAL PROOFS

A. Proof of Proposition 1

Consider the Lyapunov function $V_{\rm smc} = \frac{1}{2}\|\mathbf{w}\|^2$. The time derivative of $V_{\rm smc}$ along the solutions of system (10) with (15) gives

$$\dot{V}_{\rm smc} = -k(t,\mathbf{x})\|\bar{\mathbf{w}}\| - k(t,\mathbf{x})\bar{\mathbf{w}}^T\Delta B_m\frac{\bar{\mathbf{w}}}{\|\bar{\mathbf{w}}\|} + \bar{\mathbf{w}}^T\overline{\Delta F}_m.$$

Since $\bar{\mathbf{w}}^T\Delta B_m\bar{\mathbf{w}} = \bar{\mathbf{w}}^T\Delta B_m^T\bar{\mathbf{w}}$, it follows from Assumption 2 and the knowledge of the upper bound in (11) that

$$\begin{aligned}\dot{V}_{\rm smc} &\leq -k(t,\mathbf{x})\|\bar{\mathbf{w}}\| - k(t,\mathbf{x})\frac{\bar{\mathbf{w}}^T(\frac{1}{2}(\Delta B_m+\Delta B_m^T)\bar{\mathbf{w}}}{\|\bar{\mathbf{w}}\|} + \bar{M}\|\bar{\mathbf{w}}\| \\ &\leq -k(t,\mathbf{x})b_0\|\bar{\mathbf{w}}\| + \bar{M}\|\bar{\mathbf{w}}\|\end{aligned}$$

where we applied the Rayleigh–Ritz and Cauchy–Schwartz inequalities, and considered $b_0 = 1+\lambda_0$, $\lambda_0 = \lambda_m(\frac{1}{2}(\Delta B_m + \Delta B_m^T))$. Take the unit control gain as $k(t,\mathbf{x}) = \frac{\bar{M}(t,\mathbf{x})+\rho_0}{1+\lambda_0}$, $\rho_0 > 0$, after some algebraic manipulations[2]

$$\dot{V}_{\rm smc} \leq -\rho_0\|\bar{\mathbf{w}}\| \leq -\rho_0\lambda_m^{1/2}(\bar{B}\bar{B})^T\|\mathbf{w}\| \leq -\rho V_{\rm smc}^{1/2}$$

where $\rho = \rho_0\lambda_m^{1/2}(\bar{B}\bar{B}^T)\sqrt{2}$. Hence, the solutions of uncertain system reach the sliding manifold $\mathcal{W}$ in a finite time T_1 given in (16) and stay therein for all future times. Finally, after time T_1, the result follows from Assumption 3 by checking that the second term in (5) equals zero when $\mathbf{x} \in \mathcal{W}$. ∎

B. Proof of Proposition 2

It is shown that the sliding variable $\mathbf{w}$ reaches the closed ball $\|\mathbf{w}\| \leq \varepsilon/2$ for the first time in a finite-time $\bar{t}(c,t_0)$ with $c > 0$. Consider the redesigned control law in (15) with the adaptive gain $k(t,\mathbf{x})$ in (18). Thus, the closed-loop system of the sliding variable $\mathbf{w}$ is given by

$$\dot{\mathbf{w}} = \bar{B}(t,\mathbf{x})\left[(\mathbf{I}_m + \Delta B_m)(-k(t,\mathbf{x})\frac{\bar{\mathbf{w}}}{\|\bar{\mathbf{w}}\|}) + \overline{\Delta F}_m\right]. \quad (35)$$

Moreover, consider the following candidate LF:

$$V_T = V_{T_1} + V_{T_2} + V_{T_3} \quad (36)$$

$$V_{T_1} = \tfrac{1}{2}\|\mathbf{w}\|^2 \quad (37)$$

$$V_{T_2} = \tfrac{b_0}{2}(\hat{\mathbf{b}}(t)-\mathbf{b})^TL^{-1}(\hat{\mathbf{b}}(t)-\mathbf{b}) \quad (38)$$

$$V_{T_3} = \tfrac{b_0}{2\theta}(\hat{\rho}-\ell)^2 \quad (39)$$

where $b_0 := (1+\lambda_0) > 0$, $\lambda_0 = \lambda_m(\frac{1}{2}(\Delta B_m + \Delta B_m^T))$ (see Assumption 2), $L > 0$, ℓ, $\theta > 0$; the time derivative of expression (36) yields $\dot{V}_T = \dot{V}_{T_1} + \dot{V}_{T_2} + \dot{V}_{T_3}$.

- For $\dot{V}_{T_1}$, it follows from (35) and (37) that

$$\begin{aligned}\dot{V}_{T_1} &= \mathbf{w}^T\bar{B}(t,\mathbf{x}) \\ &\times\left[(\mathbf{I}_m + \Delta B_m)(-k(t,\mathbf{x})\frac{\bar{\mathbf{w}}}{\|\bar{\mathbf{w}}\|}) + \overline{\Delta F}_m\right]. \quad (40)\end{aligned}$$

By using adaptive gain $k(t,\mathbf{x})$ in (18) for all $\|\mathbf{w}\| > \varepsilon$ gives the following upper bound for (40):

$$\begin{aligned}\dot{V}_{T_1} &\leq -b_0\|\bar{\mathbf{w}}\|^2 - b_0\bar{\Gamma}(t,x,\hat{\mathbf{b}})\|\bar{\mathbf{w}}\| - b_0\hat{\rho} + \bar{M}(t,\mathbf{x})\|\bar{\mathbf{w}}\| \\ &= -b_0\|\bar{\mathbf{w}}\|^2 - b_0\left(\bar{\Gamma}(t,x,\hat{\mathbf{b}}) - \tfrac{\bar{M}(t,\mathbf{x})}{b_0}\right)\|\bar{\mathbf{w}}\| - b_0\hat{\rho} \\ &= -b_0\|\bar{\mathbf{w}}\|^2 + b_0(\bar{\Gamma}(t,x,\mathbf{b}) - \bar{\Gamma}(t,x,\hat{\mathbf{b}}))\|\bar{\mathbf{w}}\| - b_0\hat{\rho}\end{aligned} \quad (41)$$

where $\bar{M}(t,\mathbf{x})/b_0 = \bar{\Gamma}(t,\mathbf{x},\mathbf{b})$ due to (12). Given the fact that $-\bar{\Gamma}(t,\mathbf{x},\cdot)$ is convex, from (13), it follows that

$$\bar{\Gamma}(t,\mathbf{x},\mathbf{b}) - \bar{\Gamma}(t,\mathbf{x},\hat{\mathbf{b}}) \leq -\tfrac{\partial\bar{\Gamma}(t,\mathbf{x},\hat{b})}{\partial b}(\hat{\mathbf{b}}-\mathbf{b}). \quad (42)$$

Substituting (42) in the upper bound in (41) yields

$$\dot{V}_{T_1} \leq -b_0\|\bar{\mathbf{w}}\|^2 - b_0(\hat{\mathbf{b}}-\mathbf{b})^T\tfrac{\partial\bar{\Gamma}(t,\mathbf{x},\hat{\mathbf{b}})}{\partial\mathbf{b}}^T\|\bar{\mathbf{w}}\| - b_0\hat{\rho}. \quad (43)$$

- For $\dot{V}_{T_2}$, it follows from (38) and (19) that

$$\begin{aligned}\dot{V}_{T_2} &= b_0(\hat{\mathbf{b}}-\mathbf{b})^TL^{-1}\dot{\hat{\mathbf{b}}} \\ &= b_0(\hat{\mathbf{b}}-\mathbf{b})^T\tfrac{\partial\bar{\Gamma}(t,\mathbf{x},\hat{\mathbf{b}})}{\partial\mathbf{b}}^T\|\bar{\mathbf{w}}\| - b_0(\hat{\mathbf{b}}-\mathbf{b})^T\Xi\hat{\mathbf{b}}\end{aligned} \quad (44)$$

It is not hard to see that if $\Xi = \Xi^T$, then

$$\begin{aligned}&(\hat{\mathbf{b}}-\mathbf{b})^T\Xi\hat{\mathbf{b}} \\ &\quad= \tfrac{1}{2}(\hat{\mathbf{b}}-\mathbf{b})^T\Xi(\hat{\mathbf{b}}-\mathbf{b}) + \tfrac{1}{2}\hat{\mathbf{b}}^T\Xi\hat{\mathbf{b}} - \tfrac{1}{2}\mathbf{b}^T\Xi\mathbf{b}. \quad (45)\end{aligned}$$

Substituting (45) into (44) yields

$$\begin{aligned}\dot{V}_{T_2} &= b_0(\hat{\mathbf{b}}-\mathbf{b})^T\tfrac{\partial\bar{\Gamma}(t,\mathbf{x},\hat{\mathbf{b}})}{\partial\mathbf{b}}^T\|\bar{\mathbf{w}}\| \\ &\quad- \tfrac{b_0}{2}(\hat{\mathbf{b}}-\mathbf{b})^T\Xi(\hat{\mathbf{b}}-\mathbf{b}) - \tfrac{b_0}{2}\hat{\mathbf{b}}^T\Xi\hat{\mathbf{b}} + \tfrac{b_0}{2}\mathbf{b}^T\Xi\mathbf{b} \\ &\leq b_0(\hat{\mathbf{b}}-\mathbf{b})^T\tfrac{\partial\bar{\Gamma}(t,\mathbf{x},\hat{\mathbf{b}})}{\partial\mathbf{b}}^T\|\bar{\mathbf{w}}\| - \lambda_{\rm m}(\Xi)\tfrac{b_0}{2}\|\hat{\mathbf{b}}-\mathbf{b}\|^2 \\ &\quad+ \lambda_{\rm M}(\Xi)\tfrac{b_0}{2}\|\mathbf{b}\|^2.\end{aligned} \quad (46)$$

- For $\dot{V}_{T_3}$, it follows from (39) and (20) that

$$\begin{aligned}\dot{V}_{T_3} &= \tfrac{b_0}{\theta}(\hat{\rho}-\ell)\dot{\hat{\rho}} = \tfrac{b_0\ell}{\theta}(\hat{\rho}-\ell) - \tfrac{b_0}{\theta}(\hat{\rho}-\ell)\hat{\rho} \\ &\leq \tfrac{b_0\ell}{\theta}(\hat{\rho}-\ell) - \tfrac{b_0}{2\theta}(\hat{\rho}-\ell)^2 + \tfrac{b_0}{2\theta}\ell^2\end{aligned} \quad (47)$$

where the fact that $2(\hat{\rho}-\ell)\hat{\rho} = (\hat{\rho}-\ell)^2 + \hat{\rho}^2 - \ell^2$ was used.

Let $\tilde{\mathbf{b}} := \hat{\mathbf{b}} - \mathbf{b}$, $\tilde{\rho} := \hat{\rho} - \ell$ and define the function $W(\cdot) : \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}_+ \to \mathbb{R}_+$ as

$$W(\bar{\mathbf{w}},\tilde{\mathbf{b}},\tilde{\rho}) := b_0\|\bar{\mathbf{w}}\|^2 + \lambda_{\rm m}(\Xi)\tfrac{b_0}{2}\|\tilde{\mathbf{b}}\|^2 + \tfrac{b_0}{2\theta}(\tilde{\rho})^2. \quad (48)$$

Substituting (43), (46), (47), and (48) into $\dot{V}_T$, it holds that

$$\begin{aligned}\dot{V}_T &\leq -W - b_0\hat{\rho} + \lambda_{\rm M}(\Xi)\tfrac{b_0}{2}\|\mathbf{b}\|^2 + \tfrac{b_0\ell}{\theta}\tilde{\rho} + \tfrac{b_0}{2\theta}\ell^2 \\ &= -W - b_0(\hat{\rho} - \tfrac{\lambda_{\rm M}(\Xi)}{2}\|\mathbf{b}\|^2 - \tfrac{\ell^2}{2\theta}) + \tfrac{b_0\ell}{\theta}\tilde{\rho} + b_0(\ell-\ell) \\ &= -W - b_0(1-\tfrac{\ell}{\theta})\tilde{\rho} - b_0(\ell - \tfrac{\lambda_{\rm M}(\Xi)}{2}\|\mathbf{b}\|^2 - \tfrac{\ell^2}{2\theta}).\end{aligned} \quad (49)$$

[2]Notice from definition $\bar{\mathbf{w}} = \bar{B}^T(t,\mathbf{x})\mathbf{w}$, it holds that $\|\bar{\mathbf{w}}\|^2 = \|\bar{B}^T\mathbf{w}\|^2 = (\mathbf{w}^T\bar{B}\bar{B}^T\mathbf{w})$, since $\bar{B}$ is invertible, then $\bar{B}\bar{B}^T > 0$ (see Assumption 4) and $-\rho_0\|\bar{\mathbf{w}}\| \leq -\rho_0\lambda_m^{1/2}(\bar{B}\bar{B}^T)\|\mathbf{w}\|$.

Let $d := \frac{\lambda_{\mathrm{M}}(\Xi)}{2}\|\mathbf{b}\|^2 + \frac{\ell^2}{2\theta}$ and note that there always exists a $\theta > \ell > d$ such that $(1 - \frac{\ell}{\theta}) > 0$ and $(\ell - d) > 0$, thus

$$\dot{V}_T \leq -W - b_0(1 - \tfrac{\ell}{\theta})|\tilde{\rho}| - b_0|\ell - d| \leq -W(\bar{\mathbf{w}}, \tilde{\mathbf{b}}, \tilde{\rho})$$

taking into account that $\hat{\rho} \geq \ell$ for all t. Hence, the equilibrium $(\mathbf{w}, \tilde{\mathbf{b}}, \tilde{\rho}) = 0$ is globally asymptotically stable.[3] Since $\mathbf{w}$ converges asymptotically to zero, there exists a function $\beta_2 \in \mathcal{KL}$ such that for $t \geq t_0$

$$\|\mathbf{w}\| \leq \beta_2(\|\mathbf{w}(t_0, \mathbf{x}(t_0))\|, t - t_0) \leq \beta_2(c, t - t_0) \tag{50}$$

due to $\mathbf{w}$ is bounded with $c > 0$. Finally, since $\beta_2(c, t - t_0) \to 0$ as t grows unbounded, given $\varepsilon > 0$, there exists $T > 0$ such that $\beta_2(c, t - t_0) < \varepsilon/2$, whenever $t \geq T + t_0$. Then, take $\bar{t} \geq T$ and from (50), it follows that $\|\mathbf{w}(t, \mathbf{x}(t))\| < \varepsilon/2$ for all $t \geq \bar{t} + t_0$. ■

C. Proof of Lemma 1

The proof will be performed in two steps. On the one hand, it is proved that the solution of a closed-loop system (1) is restricted to the set $\|\mathbf{w}(t, \mathbf{x}(\mathbf{t}))\| < \varepsilon$ after a finite-time interval. On the other hand, the predefined ultimate bound of the solution of the closed-loop system (1) is obtained.

1) First Step: Let $t = t_0$ denote the first time instant such that $\|\mathbf{w}(t, \mathbf{x}(t))\| \leq \varepsilon/2$. Consider the following LF candidate for $t \geq t_0$:

$$V_S = \tfrac{1}{2}\|\mathbf{w}\|^2 + \tfrac{1}{2}(\mathcal{B}(\|\mathbf{w}\|))^2 \tag{51}$$

with $\mathcal{B} := \mathcal{B}(\|\mathbf{w}\|)$ defined in (21). Note that the time derivative of the barrier function $\mathcal{B}$ is given by

$$\dot{\mathcal{B}}(\|\mathbf{w}\|) = \tfrac{\varepsilon}{(\varepsilon - \|\mathbf{w}\|)^2}\tfrac{\mathbf{w}^T\dot{\mathbf{w}}}{\|\mathbf{w}\|}. \tag{52}$$

Consider the unit control (15) and $k = \mathcal{B}(\|\mathbf{w}\|)$, the time derivative of V_S along the trajectories of (52) and (10) is given by

$$\begin{aligned}\dot{V}_S = {} & \bar{\mathbf{w}}^T\left[(\mathbf{I}_m + \Delta B_m(t, \mathbf{x}))(-\mathcal{B}\tfrac{\bar{\mathbf{w}}}{\|\bar{\mathbf{w}}\|}) + \overline{\Delta F}_m(t, \mathbf{x})\right] \\ & + \zeta\mathcal{B}\tfrac{\bar{\mathbf{w}}^T}{\|\mathbf{w}\|}\left[(\mathbf{I}_m + \Delta B_m(t, \mathbf{x}))(-\mathcal{B}\tfrac{\bar{\mathbf{w}}}{\|\bar{\mathbf{w}}\|}) + \overline{\Delta F}_m(t, \mathbf{x})\right]\end{aligned} \tag{53}$$

where $\zeta := \varepsilon/(\varepsilon - \|\mathbf{w}\|)^2$ is positive. From Assumptions 1 and 2 and (11), the following inequality for $\dot{V}_S$ is fulfilled:

$$\begin{aligned}\dot{V}_S \leq {} & -\mathcal{B}(\|\mathbf{w}\|)(\|\bar{\mathbf{w}}\| + \tfrac{\bar{\mathbf{w}}^T\Delta B_m(t,x)\bar{\mathbf{w}}}{\|\bar{\mathbf{w}}\|}) + \bar{M}\|\bar{\mathbf{w}}\| \\ & - \zeta\mathcal{B}^2(\|\mathbf{w}\|)(\tfrac{\|\bar{\mathbf{w}}\|}{\|\mathbf{w}\|} + \tfrac{\bar{\mathbf{w}}^T\Delta B_m\bar{\mathbf{w}}}{\|\mathbf{w}\|\|\bar{\mathbf{w}}\|}) + \zeta\mathcal{B}(\|\mathbf{w}\|)\bar{M}\|\bar{\mathbf{w}}\|.\end{aligned} \tag{54}$$

From Rayleigh–Ritz inequality and the fact that $\mathbf{w}^T\Delta B_m\mathbf{w} = \mathbf{w}^T\Delta B_m^T\mathbf{w}$, it follows

$$\begin{aligned}\dot{V}_S \leq {} & -((1 + \lambda_0)\mathcal{B} - \bar{M})\|\bar{\mathbf{w}}\| - \zeta\mathcal{B}\cdot((1 + \lambda_0)\mathcal{B} - \bar{M})\tfrac{\|\bar{\mathbf{w}}\|}{\|\mathbf{w}\|} \\ \leq {} & -b_0(\mathcal{B} - \tfrac{\bar{M}(t,\mathbf{x})}{b_0})\|\bar{\mathbf{w}}\| - \zeta b_0\mathcal{B}(\mathcal{B} - \tfrac{\bar{M}(t,\mathbf{x})}{b_0})\tfrac{\|\bar{\mathbf{w}}\|}{\|\mathbf{w}\|}\end{aligned} \tag{55}$$

where $b_0 := 1 + \lambda_0 > 0$, $\lambda_0 := \lambda_m(\frac{1}{2}(\Delta B_m + \Delta B_m^T))$. Suppose that $\|\mathbf{w}(t, \mathbf{x}(t)\| > s_1$, where s_1 is defined in (22) (see Remark 4) with $\Delta = \frac{\bar{M}(t,\mathbf{x})}{b_0}$. Due to the fact that $\mathcal{B}$ is an increasing function, then $\mathcal{B}(\|\mathbf{w}\|) > \mathcal{B}(s_1) = \Delta$, and

$$\dot{V}_S \leq -b_0\zeta_s\|\bar{\mathbf{w}}\| - \zeta b_0\mathcal{B}\zeta_s\tfrac{\|\bar{\mathbf{w}}\|}{\|\mathbf{w}\|} \tag{56}$$

where $\zeta_s := (\mathcal{B} - \Delta) > 0$. $\bar{B}(t, \mathbf{x})$ is an invertible matrix, then $\bar{B}(t, \mathbf{x})\bar{B}^T(t, \mathbf{x}) > 0$. Now, by using the Rayleigh–Ritz inequality, the derivative of V_S can be estimated as

$$\begin{aligned}\dot{V}_S & \leq -b_s\|\mathbf{w}\| - b_s\zeta\mathcal{B}(\|\mathbf{w}\|) = -b_s\sqrt{2}\left(\tfrac{\|\mathbf{w}\|}{\sqrt{2}} + \tfrac{\zeta}{\sqrt{2}}|\mathcal{B}|\right) \\ & \leq -b_s\sqrt{2}\min\{1, \zeta\}\left(\tfrac{\|\mathbf{w}\|}{\sqrt{2}} + \tfrac{|\mathcal{B}(\|\mathbf{w}\|)|}{\sqrt{2}}\right) \leq -c_1 V_S^{\frac{1}{2}}\end{aligned}$$

with $b_s = b_0\zeta_s\lambda_m^{1/2}(\bar{B}\bar{B}^T)$, $c_1 := \sqrt{2}b_s\min\{1, \zeta\}$. Thus, finite-time convergence to the domain $\|\mathbf{w}\| \leq s_1$ is ensured. Once trajectories converge to the domain described by $\|\mathbf{w}\| \leq s_1$, it is left to prove that $\mathbf{x}$ stays within the domain $\|\mathbf{w}\| < \varepsilon$. After a short period $T_1 > 0$ has elapsed, such that $t \geq t_0 + T_0 + T_1$, the sign of $\dot{V}_S$ is indefinite (cf. with the previous bound). Hence, $\mathbf{x}$ increases until it reaches the boundary, i.e., $\mathbf{x} \in \{\|\mathbf{w}(t_0 + T_0 + T_1, \mathbf{x}(t_0 + T_0 + T_1)\| = s_1\}$, given that $\dot{V}_0 \leq 0$ on the boundary, then for all $t \geq t_0 + T_0 + T_1$, the inequality $\|\mathbf{w}(t, \mathbf{x}(\mathbf{t}))\| \leq s_1$ holds.

Finally, by construction $s_1 < \varepsilon$, it immediately follows $\|\mathbf{w}(t, \mathbf{x}(\mathbf{t})\| < \varepsilon$ always holds from $t \geq t_0$ since $\dot{V}_S$ is negative in the set $\{s_1 < \|\mathbf{w}\| < \varepsilon\}$ for all $t \geq t_0$, and the solution is uniformly bounded $\|\mathbf{w}(t, \mathbf{x})\| < \varepsilon$ for all $t \geq t_0$.

2) Second Step: According to the last step, the solution $\mathbf{w}$ is uniformly bounded. Moreover, by choosing any number $N \in \mathbb{Z}_+\backslash\{1\}$ and $s_N = \varepsilon\frac{N\Delta}{N\Delta + 1}$ such that $s_1 < s_N < \varepsilon$, then $\dot{V}_2$ will be negative within the set $\mathcal{S} = \{s_1 < \|\mathbf{w}\| \leq s_N\}$ for all $t \geq t_0$. On the one hand, note that the time derivative of the nominal Lyapunov function V_0 of the closed-loop system (1) with $\mathbf{u} = \psi(t, \mathbf{x}) + \mathbf{v}$ and $\mathbf{v} = -\mathcal{B}(\|\mathbf{w}\|)\frac{\bar{\mathbf{w}}}{\|\bar{\mathbf{w}}\|}$ satisfies the following bound on $\mathcal{S}$:

$$\begin{aligned}\dot{V}_0 & \leq -\alpha_3(\|\mathbf{x}\|) + \mathbf{w}^T(\mathbf{I}_m + \Delta B_m)\mathbf{v} + \mathbf{w}^T\Delta F_m \\ & \leq -\alpha_3(\|\mathbf{x}\|) - \mathcal{B}\mathbf{w}^T(\mathbf{I}_m + \Delta B_m)\tfrac{\bar{\mathbf{w}}}{\|\bar{\mathbf{w}}\|} + \|\Delta F_m\|\|\mathbf{w}\|.\end{aligned}$$

From Assumption 4, it holds that $\mathbf{w}^T = \bar{\mathbf{w}}^T\bar{B}^{-1}$. Since $\bar{\mathbf{w}}^T\bar{B}^{-1}(\mathbf{I}_m + \Delta B_m)\bar{\mathbf{w}} = \bar{\mathbf{w}}^T(\mathbf{I}_m + \Delta B_m)^T\bar{B}^{-T}\bar{\mathbf{w}}$ and using Assumption 5,

$$\begin{aligned}\dot{V}_0 & \leq -\alpha_3(\|\mathbf{x}\|) - \mathcal{B}\tfrac{\bar{\mathbf{w}}^T\bar{B}^{-1}(\mathbf{I}_m + \Delta B_m)\bar{\mathbf{w}}}{\|\bar{\mathbf{w}}\|} + \bar{M}(t, \mathbf{x})\|\mathbf{w}\| \\ & \leq -\alpha_3(\|\mathbf{x}\|) - \mathcal{B}c_0\|\mathbf{w}\| + \bar{M}(t, \mathbf{x})\|\mathbf{w}\| \\ & \leq -\alpha_3(\|\mathbf{x}\|) - (c_0 - b_0)\mathcal{B}\|\mathbf{w}\| - b_0(\mathcal{B} - \tfrac{\bar{M}}{b_0})\|\mathbf{w}\| \\ & \leq -\alpha_3(\|\mathbf{x}\|) - b_0(\mathcal{B} - \tfrac{\bar{M}}{b_0})\|\mathbf{w}\|\end{aligned}$$

where $c_0 > b_0$, $c_0 := \lambda_m(\frac{1}{2}(\bar{B}^{-1}(\mathbf{I}_m + \Delta B_m) + (\mathbf{I}_m + \Delta B_m)^T\bar{B}^{-T}))\lambda_m^{1/2}(\bar{B}\bar{B}^T)$, and we used the facts that $\|\bar{\mathbf{w}}\| = \|\bar{B}^T\mathbf{w}\| = (\mathbf{w}^T\bar{B}\bar{B}^T\mathbf{w})^{1/2}$ and $\|\Delta F_m\| \leq \bar{M}(t, \mathbf{x})$ (see Assumption 6). On the other hand, recalling that $\Delta = \bar{M}/b_0$ and by using the upper and lower bounds s_1 and s_N for $\mathbf{w} \in \mathcal{S}$,

[3] Since $-b_0\|\bar{\mathbf{w}}\| \leq -b_0\lambda_m^{1/2}(\bar{B}\bar{B}^T)\|\mathbf{w}\|$. Hence, $\dot{V}_T \leq -W < 0$.

the above bound can be rewritten as follows:

$$\begin{aligned}\dot{V}_0 &\le -\alpha_3(\|\mathbf{x}\|) - b_0\left(\tfrac{s_1^2}{\varepsilon - s_1} - \Delta\cdot s_N\right)\\ &\le -\alpha_3(\|\mathbf{x}\|) - b_0\left(\tfrac{\Delta^2}{\Delta+1}\varepsilon - \tfrac{N\Delta^2}{N\Delta+1}\varepsilon\right)\\ &\le -\alpha_3(\|\mathbf{x}\|) + b_0(N-1)\tfrac{\Delta}{\Delta+1}\tfrac{\Delta}{N\Delta+1}\varepsilon\\ &\le -\alpha_3(\|\mathbf{x}\|) + b_0(N-1)\tfrac{\Delta}{\Delta+1}\tfrac{\Delta}{N\Delta+1}\varepsilon \qquad (57)\end{aligned}$$

for any $N \ge 2$ subject to Assumption 2. Noticing that $\frac{\Delta}{\Delta+1} < 1$ and $\frac{\Delta}{N\Delta+1} < \frac{1}{N}$, it holds that

$$\dot{V}_0 \le -\alpha_3(\|\mathbf{x}\|) + b_0\tfrac{N-1}{N}\varepsilon$$

for all $\forall t \ge t_0$ and $N \ge 2$. Recalling that $b_0 = 1 + \lambda_0$ and $\lambda_0 = \lambda_m(\frac{1}{2}(\Delta B_m + \Delta B_m^T))$, in view of Assumption 2, it is possible to chose a number $\kappa \in (N-1, N)$ such that $b_0 < \kappa < N$, then

$$\begin{aligned}\dot{V}_0 &\le -\alpha_3(\|\mathbf{x}\|) + \kappa\tfrac{N-1}{N}\varepsilon\\ &\le -\tfrac{N-\kappa}{N}\alpha_3(\|\mathbf{x}\|) - \tfrac{\kappa}{N}\alpha_3(\|\mathbf{x}\|) + \kappa\tfrac{N-1}{N}\varepsilon\\ &\le -\tfrac{N-\kappa}{N}\alpha_3(\|\mathbf{x}\|) - \tfrac{\kappa}{N}\alpha_3(\|\mathbf{x}\|) + (N-1)\varepsilon.\end{aligned}$$

For any r arbitrarily large, choose $\varepsilon < \frac{1}{N-1}\alpha_3(\alpha_2^{-1}(\alpha_1(r)))$ and set $\mu = \alpha_3^{-1}(\varepsilon(N-1)) < \alpha_2^{-1}(\alpha_1(r))$. Then,

$$\dot{V}_0 \le -\tfrac{N-\kappa}{N}\alpha_3(\|\mathbf{x}\|), \quad \forall \mu \le \|\mathbf{x}\| < r.$$

From the application of Theorem 4.18 in [1], it is concluded that the solutions $\mathbf{x}$ of the closed-loop system are uniformly ultimately bounded by a class $\mathcal{K}$ function in (25) that depends on the *a priori* defined ε. ■

Appendix B
Uncertainty in Control Matrix

Let m_i denote the nominal mass. Consider the uncertainty in the mass of the robot model (28), such that the inertia matrix is given by

$$\tilde{J}(x_1) = \begin{bmatrix}\tilde{j}_{11} & \tilde{j}_{12}\\ \tilde{j}_{12} & \tilde{j}_{22}\end{bmatrix}$$

where $\tilde{j}_{11} = J_{11} + \delta_1(x_1)$, $\tilde{j}_{22} = J_{22} + \delta_3(x_1)$, $\tilde{j}_{12} = J_{12} + \delta_2(x_1)$, and δ_i, $i = 1, 2, 3$ defined as follows:

$$\delta_1 = \bar{m}_1 l_{c1}^2 + \bar{m}_2\left(l_1^2 + l_{c2}^2 + 2l_1 l_{c2}\cos(x_{12})\right)$$
$$\delta_2 = \bar{m}_2\left(l_1 l_{c2}\cos(x_{12}) + l_{c2}^2\right), \ \delta_3 = \bar{m}_2 l_{c2}^2$$

with $\bar{m}_i$ as an uncertainty in the mass fulfilling the relation $\tilde{m}_i = m_i + \bar{m}_i$. Letting

$$\tilde{J}(x_1) = J(x_1) + \Delta(x_1), \quad \Delta(x_1) = \begin{bmatrix}\delta_1 & \delta_2\\ \delta_2 & \delta_3\end{bmatrix}$$

we can rewrite $\tilde{J}(x_1) = [\mathbf{I} + \Delta(x_1)(J(x_1))^{-1}]J(x_1)$ such that the error dynamics can be given as follows:

$$\dot{x}_1 = x_2, \ \dot{x}_2 = \left[\tilde{J}(x_1)\right]^{-1}\left(\bar{h}(t,\mathbf{x}) + \boldsymbol{\tau}\right).$$

where $\bar{h}(t,\mathbf{x}) := -C(x_1,x_2)x_2 - \mathbf{g}(x_1) + \varphi(t) - \tilde{J}(x_1)\ddot{q}_d$. By adding and subtracting $J(x_1)$, then

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \left[(\tilde{J}(x_1))^{-1} + (J(x_1))^{-1} - (J(x_1))^{-1}\right]\left(\bar{h}(t,\mathbf{x}) + \tau\right)\end{aligned}$$

yields

$$\dot{x}_2 = (J(x_1))^{-1}\left[\mathbf{I} + \Delta_G\right]\left(\bar{h}(t,\mathbf{x}) + \tau\right)$$

where $\Delta_G := (J(x_1)(\tilde{J}(x_1))^{-1} - \mathbf{I})$, the error dynamics is in the required form.

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