The mystery of the table of free weights

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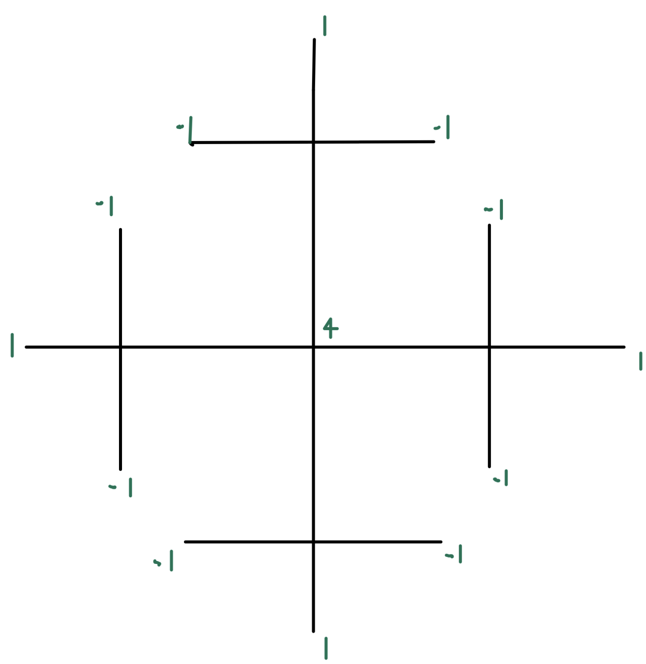

1 Introducing the table of free weights

1.1 The free weight cellular automaton

Define the **free weight cellular automaton** C as follows. The mechanism C takes as input a biinfinite tape with integers in the cells. Mathematically, such a tape is a function $a : \mathbb{Z} \to \mathbb{Z}$. It outputs a new tape $\mathsf{C}[a] : \mathbb{Z} \to \mathbb{Z}$ defined by setting the value of $\mathsf{C}[a]$ at some coordinate $k \in \mathbb{Z}$ on the output tape to be given by adding the value one coordinate to the left from the input tape with three times the value one coordinate to the right from the input tape. Mathematically, this means

$$\mathsf{C}[a](k) = a(k-1) + 3a(k+1)$$

This cellular automaton has never been studied in detail by anyone other than me. I have good reason to believe it has connections to many of the hardest problems in theoretical mathematics and computer science.

1.2 Constructing the table

Define an initial input tape $b : \mathbb{Z} \to \mathbb{Z}$ by

$$b(k) = \begin{cases} 4 & \text{if } k \text{ is even and nonpositive} \\ 0 & \text{if } k \text{ is odd or positive} \end{cases}$$

We can consider the full evolution of b under the cellular automaton consisting of the sequence of tapes

$$b \implies \mathsf{C}[b] \implies \mathsf{C}[\mathsf{C}[b]] \implies \mathsf{C}[\mathsf{C}[\mathsf{C}[b]]] \cdots \implies \underbrace{\mathsf{C}[\cdots \mathsf{C}[b] \cdots]}_{n \text{ applications of } \mathsf{C}} \implies \cdots$$

This is equivalent to defining an infinite sequence of tapes $(b_n)_{n=1}^{\infty}$ recursively by setting $b_1 = b$ and $b_{n+1} = \mathsf{C}[b_n]$ for $n \in \mathbb{N}$. By stacking these tapes one on top of each other going down we obtain a grid of numbers

$$\{B(k, n) : k \in \mathbb{Z}, n \in \mathbb{N}\}$$

which is biinfinite in the horizontal coordinate and one-sided infinite down in the vertical coordinate and satisfies

$$B(k, n+1) = B(k-1, n) + 3B(k+1, n)$$

for all $k \in \mathbb{Z}$ and all $n \in \mathbb{N}$. Mathematically, this just means $B(k, n) = b_n(k)$. We refer to this grid as the **table of free weights**

1.3 Reading the table

We define a lexicographic order $\prec$ on this grid in the natural way, by stipulating that

$$\big[(m, \ell) \prec (k, n)\big] \iff \big[m < k \text{ or } (m = k \text{ and } \ell < n)\big]$$

for all $m, k \in \mathbb{Z}$ and all $\ell, n \in \mathbb{N}$. Therefore we read the table first right along the rows and then going down one row at a time, with the top edge of the grid being b where the 4's extend left. We are able to start and finish reading each row even though they are infinite using the observation that the n^{th} row has a one-sided infinite sequence consisting of zero and 4^n at the left and an infinite sequence of zeroes at the right. Once we see only zeroes at the end of a row, we jump one row down and start reading right again from the first number other than 4^n.

2 Table of free weights conjectures

My initial computer experiments suggest the following conjectures are true. More evidence is desirable, and any contribution to that will be well received by the entire community of professional mathematicians. If you find some nontrivial phenomenon while looking at the experiments, you'll be guaranteed a full scholarship to any math Ph.D. program in the US. You are the first people to see these conjectures and no intelligence in this part of the galaxy knows the answers yet, but the answers are likely related to the fundamental structure of numbers in some unknown way. If you want to use your laptop to investigate something related to the fundamental structure of numbers that nobody has ever investigated before, try to test these and send me the results at

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It's really easy but I'm currently focused on higher-level things related to these issues so I don't have time to program more computer experiments myself. The starting place is just to get the computer to output the table on the screen with all entries in their prime factorizations and stare at it looking for patterns.

2.1 Appearances of new primes

Conjecture 2.1 (Table of free weights conjecture I)**.** *For all $(k, n) \in \mathbb{Z} \times \mathbb{N}$ we have that the prime factorization of $B(k, n)$ contains at most prime number which does not appear in the prime factorization of $B(m, \ell)$ for some $(m, \ell) \in \mathbb{Z} \times \mathbb{N}$ with $(m, \ell) \prec (k, n)$. In other words, if we disregard all primes which have previously appeared as factors of entries in the table, at most one new prime at a time appears in the factorization of each entry we read.*

2.2 Randomness of normalized odd central sequence

The next conjecture concerns the normalized odd central sequence

$$\begin{gathered} \frac{B(0,1)}{4} \\ \frac{B(0,3)}{64} \\ \frac{B(0,5)}{1024} \\ \vdots \\ \frac{B(0,2j-1)}{4^{2j-1}} \\ \vdots \end{gathered}$$

and related conjectures may be made for other sequences obtained by fixing the horizontal coordinate in the table.

Conjecture 2.2 (Table of free weights conjecture II)**. (a)** *We have*

$$\frac{7}{16} = \frac{B(0,3)}{4^3} \leq \frac{B(0,2j-1)}{4^{2j-1}}$$

for all $j \in \mathbb{N}$.

(b) *We have*

$$1 = \frac{B(0,1)}{4^1} \geq \frac{B(0,2j-1)}{4^{2j-1}}$$

for all $j \in \mathbb{N}$.

(c) *Given* $a, b \in \mathbb{R}$ *with* $a < b$ *define the indicator function* $\mathbf{1}_{a,b} : \mathbb{R} \to \{0,1\}$ *of the interval* $[a,b]$ *to be given for a real number* x *by*

$$\mathbf{1}_{a,b}(x) = \begin{cases} 1 & \text{if } a \leq x \leq b \\ 0 & \text{if } x < a \text{ or } b < x \end{cases}$$

Then any pair of numbers $a, b \in \mathbb{R}$ *with* $\frac{7}{16} \leq a < b \leq 1$ *we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}_{a,b}\left(\frac{B(0,2j-1)}{4^{2j-1}}\right) = \frac{16(b-a)}{9}$$

Part (c) of Conjecture II asserts that the normalized odd central sequence behaves like a sample from the uniform distribution Unif $\left(\frac{7}{16}, 1\right)$. In other words, it is equidistributed in the interval $\left[\frac{7}{16}, 1\right]$ because the average number of terms of the sequence falling in a subinterval of $\left[\frac{7}{16}, 1\right]$ is equal to the proportion of the total length occupied by that interval.