A Description of the Generalized Table of Free Weights

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1 Introduction

I'm going to consider a more general function than the one you've described in your PDF document. First, notice that we can let the initial tape $b(k)$ equal

$$b(k) = \begin{cases} 1 & k \text{ is even and negative} \\ 0 & k \text{ is odd or nonnegative} \end{cases}$$

and we have fundamentally the same tape; we have just divided all the values of your function by 4. This simplifies things slightly. Now, define $B_{x,y}(k,n)$ by

$$B_{x,y}(k,0) = \begin{cases} 1 & k \equiv 0 \pmod 2 \text{ and } k \leq 0 \\ 0 & k \equiv 1 \pmod 2 \text{ or } k > 0 \end{cases} \tag{1}$$

and

$$B_{x,y}(k,n+1) = xB_{x,y}(k-1,n) + yB_{x,y}(k+1,n). \tag{2}$$

Note that the precise function you defined is $B_{1,3}(k,n)$ with n starting at 1 rather than 0 (so your definition for (1) would change similarly); I allow n to be zero as this doesn't change much other than making things simpler in many cases. One of the more noticeable changes is that, later on, your sequence $B_{1,3}(0,2n-1)/4^{2n-1}$ will become $B_{1,3}(0,2n)/4^{2n}$, with n starting at 0 rather than 1.

I'm going to prove everything that follows in the general case, for $B_{x,y}$. I will have a brief discussion at the end about your conjectures if you would just like to skip to the end. The proofs are not very enlightening or enjoyable to read; I essentially use induction to prove everything. I will give a brief description of what each theorem proves here, to make things easier to interpret.

2 Determination of the Table Values

This next result lemma says that, (a), "The value of $B_{x,y}$ to the left of, and on, the diagonal $(-k,k)$ equals $(x+y)^k$", for (b), "The value of $B_{x,y}$ to the right of the diagonal (k,k) equals zero", for (c), "The value of $B_{x,y}$ on the diagonal (k,k) equals x^k", and, for (d), "There is a checkerboard pattern of zeroes on the entire grid". This completely determines the values of $B_{x,y}$ everywhere except on the upside-down triangle centered at 0 on the infinite grid.

Lemma 2.1. We have

(a) If $k,j \geq 0$, then $B_{x,y}(-k-2j,k) = (x+y)^k$.

(b) If $k \geq 0$ and $\ell \geq 1$, then $B_{x,y}(k+\ell, k) = 0$.

(c) If $k \geq 0$, then $B_{x,y}(k,k) = x^k$.

(d) If $j, k \in \mathbb{Z}$, then $B_{x,y}(2k+1, 2j) = B_{x,y}(2k, 2j+1) = 0$.

Proof. We proceed by induction on k. For $k = 0$, we have that, by definition, $B_{x,y}(-2j, 0) = 1 = (x+y)^0$. Letting $j = 0$ proves the base case of (c). For the base case of (b), we have $B(\ell, 0) = 0$. Now fix $k \in \mathbb{N}$ and suppose (a), (b), and (c) hold for k. Then, using (2),

$$\begin{aligned} B_{x,y}(-[k+1]-2j, k+1) &= xB_{x,y}(-k-2j-2, k) + yB_{x,y}(-k-2j, k) \\ &= xB_{x,y}(-k-2(j+1), k) + yB_{x,y}(-k-2j, k) \\ &= x(x+y)^k + y(x+y)^k \\ &= (x+y)^{k+1}, \end{aligned}$$

which proves (a). For (b),

$$\begin{aligned} B_{x,y}([k+1]+\ell, k+1) &= xB_{x,y}(k+\ell, k) + yB_{x,y}(k+[\ell+2], k) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

Finally, for (c),

$$\begin{aligned} B_{x,y}(k+1, k+1) &= xB_{x,y}(k,k) + yB_{x,y}(k+2, k) \\ &= xB_{x,y}(k,k) + yB_{x,y}([k+1]+1, [k+1]-1) \\ &\overset{(*)}{=} x(x^k) + 0 \\ &= x^{k+1}, \end{aligned}$$

where $(*)$ follows by (b). To prove (d), we do forward-backward induction on j. If $j = 0$, we have

$$B_{x,y}(2k+1, 0) = 0$$

by definition, and

$$B_{x,y}(2k, 1) = xB_{x,y}(2k-1, 0) + yB_{x,y}(2k+1, 0) = 0 + 0 = 0$$

by the previous case. Now, fix $j \in \mathbb{N}$ and suppose (d) holds for this k. Then, for $j+1$, we have

$$\begin{aligned} B_{x,y}(2k+1, 2j+2) &= xB_{x,y}(2k, 2j+1) + yB_{x,y}(2[k+1], 2j+1) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} B_{x,y}(2k, 2j+3) &= xB_{x,y}(2k-1, 2[j+1]) + yB_{x,y}(2k+1, 2[j+1]) \\ &= 0 + 0 \\ &= 0, \end{aligned}$$

by the previous case. For $j-1$, we have

$$\begin{aligned} B_{x,y}(2k+1,2j-2) &= xB_{x,y}(2k,2j-3) + yB_{x,y}(2[k+1],2j-3) \\ &= 0+0 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} B_{x,y}(2k,2j-1) &= xB_{x,y}(2k-1,2[j-1]) + yB_{x,y}(2k+1,2[j-1]) \\ &= 0+0 \\ &= 0, \end{aligned}$$

again by the previous case. Hence, (d) holds for all $j,k \in \mathbb{Z}$. □

Theorem 2.2. Let $k \geq 1$ and $j \geq 0$. Then

$$B_{x,y}(k-j,k+j) = \sum_{\ell=0}^{j} \binom{k+j}{\ell} x^{k+j-\ell} y^{\ell} = (x+y)^{k+j} - \sum_{\ell=j+1}^{k+j} \binom{k+j}{\ell} x^{k+j-\ell} y^{\ell}. \tag{3}$$

The second equality follows from the binomial theorem

Proof. This proof is by induction on k and j. The fact that the theorem holds for k and $j=0$ is part (c) of the previous lemma. Now fix j and suppose that the theorem is true for all k for the fixed j; we get an induction hypothesis $(\star)$. To prove that the theorem is true for $j+1$, we do induction on k. For $k=1$, we have (using the second formulation of (3))

$$\begin{aligned} B_{x,y}(1-[j+1],1+[j+1]) &= xB(-[j+1],[j+1]) + yB(1-j,1+j) \\ &= x(x+y)^{j+1} + y\left[(x+y)^{j+1} - \sum_{\ell=j+1}^{j+1} \binom{j+1}{\ell} x^{j+1-\ell} y^{\ell}\right] \\ &= x(x+y)^{j+1} + y(x+y)^{j+1} - y(y^{j+1}) \\ &= (x+y)^{j+2} - y^{j+2} \\ &= \sum_{\ell=0}^{1+j} \binom{1+(j+1)}{\ell} x^{1+(j+1)-\ell} y^{\ell}. \end{aligned}$$

Now, fix k and suppose the theorem holds for this k and $j+1$; we get an induction hypothesis $(*)$. Then for $k+1$ and $j+1$, we have

$$B_{x,y}([k+1]-[j+1],[k+1]+[j+1]) = xB_{x,y}(k-[j+1],k+[j+1]) + yB_{x,y}([k+1]-j,[k+1]+j)$$

If we use the induction hypothesis $(*)$ on the first term and the induction hypothesis $(\star)$ on the second, we get

$$x\left((x+y)^{k+j+1}-\sum_{\ell=j+1}^{k+j+1}\binom{k+j+1}{\ell}x^{k+j+1-\ell}y^{\ell}\right)+$$
$$y\left((x+y)^{k+j+1}-\sum_{\ell=j}^{k+j+1}\binom{k+j+1}{\ell}x^{k+j+1-\ell}y^{\ell}\right).$$

First, we change the index of summation of the second term by replacing ℓ with $\ell-1$ in the series. Then the bounds of the second sum become $\ell=j+1$ to $\ell=k+j+2$. After distributing the x and y into the summation, this gives us

$$(x+y)^{k+j+2}-\left[\sum_{\ell=j+1}^{k+j+1}\binom{k+j+1}{\ell}x^{k+j+2-\ell}y^{\ell}+\sum_{\ell=j+1}^{k+j+2}\binom{k+j+1}{\ell-1}x^{k+j+2-\ell}y^{\ell}\right],$$

Bringing out the $\ell=k+j+2$ term, we have

$$(x+y)^{k+j+2}-\left[\sum_{\ell=j+1}^{k+j+1}\binom{k+j+1}{\ell}x^{k+j+2-\ell}y^{\ell}+\sum_{\ell=j+1}^{k+j+1}\binom{k+j+1}{\ell-1}x^{k+j+1-\ell}y^{\ell}+y^{k+j+2}\right].$$

Combining the summations yields

$$(x+y)^{k+j+2}-\left\{\sum_{\ell=j+1}^{k+j+1}\left[\binom{k+j+1}{\ell}+\binom{k+j+1}{\ell-1}\right]x^{k+j+2-\ell}y^{\ell}+y^{k+j+2}\right\}$$

Now we use the fact that $\binom{n-1}{k}+\binom{n-1}{k-1}=\binom{n}{k}$ to get

$$(x+y)^{k+j+2}-\left[\sum_{\ell=j+1}^{k+j+1}\binom{k+j+2}{\ell}x^{k+j+2-\ell}y^{\ell}+y^{k+j+2}\right]$$

which simply equals

$$(x+y)^{k+j+2}-\left[\sum_{\ell=j+1}^{k+j+2}\binom{k+j+2}{\ell}x^{k+j+2-\ell}y^{\ell}\right],$$

as desired. Thus, the theorem holds for all k and $j+1$ by induction, so we conclude it holds for all j,k with $j\geq 0$ and $k\geq 1$ by induction. $\square$

Remark. This completely determines all values of $B_{x,y}(k,n)$. If (n,k) have opposite parity, then $B_{x,y}(k,n)=0$. If they have the same parity, they lie on a diagonal. Specifically, if If $|k|<n$, then $\alpha=(n+k)/2>0$ (and is a natural number, since n and k have the same parity) and $\delta=n-\alpha=\alpha-k$. Thus, we have $k=\alpha-\delta$, $n=\alpha+\delta$ and we apply the previous theorem to get the value of $B_{x,y}(\alpha-\delta,\alpha+\delta)$. If $|k|=n$, then this is either case (a) or (c). If $|k|>n$, then this is either case (a) or case (b).

Now, we define $\|B_{x,y}(k,n)\|$ by

$$\|B_{x,y}(k,n)\| = \frac{B_{x,y}(k,n)}{(x+y)^n}.$$

3 Description of the Sequence

Theorem 3.1. Suppose $x, y > 0$. Then

(a) If $x + y > 1$, then $\|B_{x,y}(k,n)\| \leq 1$ for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}_0$.

(b) If $k \geq 2$, then $a_k = \|B_{x,y}(0,2k)\|$ is a monotonically decreasing sequence with $a_k \geq a_{k+1}$ so long as
$$\frac{y}{x} \geq \left(1 - \frac{1}{k+1}\right).$$
Thus, if $y \geq x > 0$, the sequence is monotonically decreasing with limit 0.

Proof. For (a), this is clear since $B_{x,y}(n,k)$ is either $0, (x+y)^n$, or $(x+y)^n$ minus a portion of it's binomial expansion — a positive quantity (since x and y are assumed to be positive).

For (b), the expression $a_{k+1} = \|B_{x,y}(0, 2(k+1))\| \leq \|B_{x,y}(0,2k)\| = a_k$ is equivalent to

$$B_{x,y}(0, 2(k+1)) \leq (x+y)^2 B_{x,y}(0,2k). \tag{4}$$

If we expand out $B_{x,y}(0, 2(k+1))$ via the definition and expand/distribute the RHS, we have

$$x^2 B_{x,y}(-2,2k) + 2xyB_{x,y}(0,2k) + y^2 B_{x,y}(2,2k) \leq x^2 B(0,2k) + 2xyB(0,2k) + y^2 B(0,2k),$$

thus, (4) is equivalent to the inequality

$$x^2 B_{x,y}(-2,2k) + y^2 B_{x,y}(2,2k) \leq x^2 B_{x,y}(0,2k) + y^2 B_{x,y}(0,2k). \tag{5}$$

Now, using Theorem 2.2 and that

$$\begin{aligned} B_{x,y}(0,2k) &= B_{x,y}(k-k, k+k) \\ B_{x,y}(-2,2k) &= B_{x,y}([k-1]-[k+1], [k-1]+[k+1]) \\ B_{x,y}(2,2k) &= B_{x,y}([k+1]-[k-1], [k+1]+[k-1]), \end{aligned}$$

we expand everything out into its closed form

$$\begin{aligned} x^2 B_{x,y}(0,2k) &= \sum_{\ell=0}^{k} \binom{2k}{\ell} x^{2k+2-\ell} y^{\ell} \\ y^2 B_{x,y}(0,2k) &= \sum_{\ell=0}^{k} \binom{2k}{\ell} x^{2k-\ell} y^{\ell+2} \\ x^2 B_{x,y}(-2,2k) &= \sum_{\ell=0}^{k+1} \binom{2k}{\ell} x^{2k+2-\ell} y^{\ell} \\ y^2 B_{x,y}(2,2k) &= \sum_{\ell=0}^{k-1} \binom{2k}{\ell} x^{2k-\ell} y^{\ell+2}. \end{aligned}$$

It is here that we use the assumption that $k \geq 2$, so that we can indeed apply Theorem 2.2. This shows us that, in fact, the LHS of (5) is equivalent to

$$\left[\binom{2k}{k+1}x^{k+1}y^{k+1} + \sum_{\ell=0}^{k}\binom{2k}{\ell}x^{2k+2-\ell}y^{\ell}\right] + \left[\sum_{\ell=0}^{k}\binom{2k}{\ell}x^{2k-\ell}y^{\ell+2} - \binom{2k}{k}x^{k}y^{k+2}\right].$$

Because the middle two sums are equal to the RHS of (5), it follows that (5) is equivalent to the inequality (after dividing by $x^{k+1}y^{k+1}$)

$$\begin{aligned} 0 &\geq \binom{2k}{k+1} - \binom{2k}{k}\left(\frac{y}{x}\right) \\ &= -C_n + \left(1 - \frac{y}{x}\right)\binom{2k}{k} \\ &= -\frac{1}{k+1}\binom{2k}{k} + \left(1 - \frac{y}{x}\right)\binom{2k}{k}, \end{aligned}$$

where C_n are the Catalan numbers. After dividing by $\binom{2k}{k}$, this is equivalent to

$$\frac{1}{k+1} \geq \left(1 - \frac{y}{x}\right),$$

as desired. □

4 Specific Cases and Conjectures

In this section, $B(k, n)$ refers to the function you defined in your paper, and $B_{1,3}(k, n)$ refers to the slightly different one I've defined here. First, the most notable result is that the closed form expression for the middle sequence (without the norm) $a_k = B(0, 2k-1)$ you defined is given by (in terms of my functions)

$$a_k = 4B_{1,3}(0, 2k) = 4\sum_{\ell=0}^{k}\binom{2k}{\ell}1^{2k-\ell}3^{\ell} = 4\sum_{\ell=0}^{k}\binom{2k}{\ell}3^{\ell}.$$

Lemma 2.1 says that $B(k-2j, k+1) = (1+3)^k = 4^k$ for all $j, k \geq 0$, $B(k, n) = 0$ if k and n have opposite parity, $B(k, k+1) = 1^k = 1$, and $B(k+j, k+1) = 0$ for all $j > 0$. All of the other terms can be found in a somewhat roundabout way by applying Theorem 2.2 and using the remark.

Unfortunately, most of the conjectures fail. The prime number conjecture fails at $B_{1,3}(-3, 5) = 781$ (or $B(-3, 6) = 781 \times 4$ with your original function), since $781 = 11 \times 71$, none of which appear in the prime factorization of previous numbers.

The conjecture 2.2a fails because the sequence is monotonically decreasing (as Theorem 3.1 shows; we have that $3 \geq 1$). Theorem 3.1 shows that conjecture 2.2b holds, because it holds for every number on the table. It seems you had the idea that the sequence was bounded between 7/16 and 1, but this is not the case, which means that 2.2c fails as well.