# CMSC 474, Game Theory

## 2. Analyzing Normal-Form Games

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Chapter 2 of the textbook,
plus several related topics

# How to reason about games?

- In single-agent decision theory, look at an **optimal** strategy
  - Maximize the agent’s expected payoff in its environment

- What is your optimal strategy if you’re interacting with other agents?
  - Depends on both your choices and theirs
- Identify certain subsets of outcomes called **solution concepts**

- Chapter 2 discusses two solution concepts:
  - Pareto optimality
  - Nash equilibrium
- Chapter 3 will discuss several others

# Pareto Optimality

- Vilfredo Pareto (1848–1923)

- A strategy profile $\mathbf{s}$ **Pareto dominates** a strategy profile $\mathbf{s}'$ if
  - every agent does at least as well with $\mathbf{s}$ as with $\mathbf{s}'$
    i.e., $u_i(\mathbf{s}) \geq u_i(\mathbf{s}')$ for all $i$ ,
  - at least one agent does better with $\mathbf{s}$ than with $\mathbf{s}'$,
    i.e., $u_i(\mathbf{s}) > u_i(\mathbf{s}')$ for at least one $i$

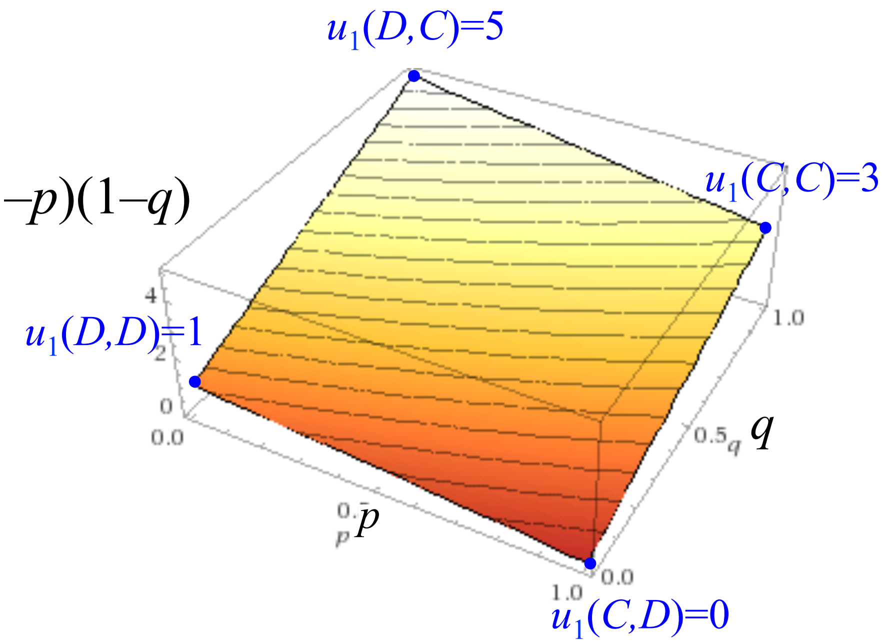
- $\mathbf{s}$ is **Pareto optimal** (or **Pareto efficient**) if there's no other strategy profile that Pareto dominates $\mathbf{s}$
  - Every game has at least one Pareto optimal profile
  - Always at least one Pareto optimal profile in which the strategies are pure

# Examples

**Prisoner's Dilemma**

| | $C$ | $D$ |
|---|---|---|
| $C$ | 3, 3 | 0, 5 |
| $D$ | 5, 0 | 1, 1 |

- $(D,D)$ isn't Pareto optimal
  - $(C,C)$ Pareto dominates it
- $(D,C)$ is Pareto optimal
  - For all other strategy profiles, $u_1$ is lower
  - Let $s_1 = \{(p,\ C),\ (1\!-\!p,\ D)\}$ and $s_2 = \{(q,\ C),\ (1\!-\!q,\ D)\}$
  - $u_1(s_1, s_2) = 3pq + 5(1\!-\!p)q + 0 + (1\!-\!p)(1\!-\!q)$
- $(C,D)$ is Pareto optimal
  - For other strategy profiles, $u_2$ is smaller
- $(C,C)$ is Pareto optimal
  - In all other strategy profiles, either $u_1$ is smaller or $u_2$ is smaller
- $(s_1, s_2)$ is Pareto optimal for every $(s_1, s_2)$ except $(D,D)$



# Examples

- Which Side of the Road
  - Pareto optimal: (Left,Left) and (Right,Right)

| | Left | Right |
|---|---|---|
| Left | 1, 1 | 0, 0 |
| Right | 0, 0 | 1, 1 |

- In common-payoff games, all Pareto optimal strategy profiles have the same payoffs
  - If not, the one with lower payoffs wouldn’t be Pareto optimal

- **Poll 2.1**: Here’s a common-payoff game. Which strategy profiles are Pareto optimal?

| | Left | Right |
|---|---|---|
| Left | 1, 1 | 2,2 |
| Right | 0, 0 | 2,2 |

# Best Response

- Suppose agent $i$ knows how the others are going to play
- Then $i$ has an ordinary optimization problem:
  - Maximize $i$'s expected utility

- We'll use $\mathbf{s}_{-i}$ to mean a strategy profile for all of the agents except $i$

$$\mathbf{s}_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$$

- *Notation*: if $s_i$ is a strategy for agent $i$, then

$$(s_i, \mathbf{s}_{-i}) = (s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n)$$

- $s_i$ is a **best response** to $\mathbf{s}_{-i}$ if for every strategy $s_i'$ available to agent $i$,

$$u_i(s_i, \mathbf{s}_{-i}) \geq u_i(s_i', \mathbf{s}_{-i})$$

- There is always at least one best response

# Examples

- **Poll 2.2**: Suppose 1's strategy is Left
  - What are 2's best responses?

| | Left | Right |
|---|---|---|
| Left | 1, 1 | 0, 0 |
| Right | 0, 0 | 1, 1 |

- **Poll 2.3**: Suppose 1's strategy is {(½,Left), (½,Right)}
  - What are 2's best responses?

- **Poll 2.4**: Suppose 1's strategy is {($^{2}/_{3}$,Left), ($^{1}/_{3}$,Right)}
  - What are 2's best responses?

# Announcements (Sept 6)

- I'm recovering from an illness
- Ali Shafahi's office hours: Wednesdays & Thursdays, 11am–12pm
- What we've covered so far:
  - Pareto dominance, Pareto optimality
    - Prisoner's Dilemma: all strategy profiles are Pareto optimal except (D,D)
    - Common-payoff games: all Pareto optimal strategy profiles have same payoffs
  - Notation: $\mathbf{s}_{-i}$
    - $(s_i, \mathbf{s}_{-i})$
  - Best response:
    - Given $\mathbf{s}_{-i}$ , a strategy that maximizes $u(s_i, \mathbf{s}_{-i})$
    - How does $i$ know what $_{-i}$ is?

| | Left | Right |
|---|---|---|
| Left | 1, 1 | 0, 0 |
| Right | 0, 0 | 1, 1 |

# Best Response

**Theorem.** Given $\mathbf{s}_{-i}$ , there are only two possibilities:

(1) $i$ has a pure strategy $s_i$ that is the *only* best response to $\mathbf{s}_{-i}$

(2) $i$ has *infinitely many* best responses to $\mathbf{s}_{-i}$

*Proof strategy*: Show that if (1) is false then (2) must be true.

**Proof.** Let $s_i$ be a best response to $\mathbf{s}_{-i}$ , and suppose (1) is false.

- Either $s_i$ isn't the only best response to $\mathbf{s}_{-i}$ or $s_i$ isn't pure
- *Case 1:* $s_i$ isn't the only best response to $\mathbf{s}_{-i}$
  - The others must have the same expected utility as $s_i$
  - Thus every mixture of them is a best response, so (2) holds.
- *Case 2:* $s_i$ isn't pure. It's a mixture of at least two pure strategies
  - Each of them must have the same expected utility as $s_i$
  - They're both best responses, so this reduces to Case 1.

# Nash Equilibrium

- $\mathbf{s} = (s_1, \ldots, s_n)$ is a **Nash equilibrium** if for every $i$, $s_i$ is a best response to $\mathbf{s}_{-i}$
  - Every agent's strategy is a best response to the other agents' strategies
  - No agent can do better by *unilaterally* changing his/her strategy

- **Theorem (Nash, 1951)**: Every game with finitely many agents and action profiles has at least one Nash equilibrium

- Prisoner's Dilemma: $(D,D)$

| | $C$ | $D$ |
|---|---|---|
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# Nash Equilibrium

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  - Every agent's strategy is a best response to the other agents' strategies
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- Prisoner's Dilemma: $(D,D)$

| | $C$ | $D$ |
|---|---|---|
| $C$ | 3, 3 | 0, 5 |
| $D$ | 5, 0 | 1, 1 |

- Modified Prisoner's Dilemma:
  - $(s,D)$, $s$ is any mixture of $D$ and $E$

| | $C$ | $D$ |
|---|---|---|
| $C$ | 3, 3 | 0, 5 |
| $D$ | 5, 0 | 1, 1 |
| $E$ | 5, 0 | 1, 1 |

# Strict and Weak Nash Equilibria

- Let $\mathbf{s} = (s_1, \ldots, s_n)$ be a Nash equilibrium
  - $\mathbf{s}$ is **strict** if each $s_i$ in $\mathbf{s}$ is the *only* best response to $\mathbf{s}_{-i}$
    - any agent who unilaterally changes strategy will do worse
  - Otherwise $\mathbf{s}$ is **weak**

| | $C$ | $D$ |
|---|---|---|
| $C$ | 3, 3 | 0, 5 |
| $D$ | 5, 0 | 1, 1 |

- If $\mathbf{s}$ includes a mixed* strategy $s_i$, then $\mathbf{s}$ is weak
  - e.g., $(s, D)$ where $s = \{(½, D), (½, E)\}$

| | $C$ | $D$ |
|---|---|---|
| $C$ | 3, 3 | 0, 5 |
| $D$ | 5, 0 | 1, 1 |
| $E$ | 5, 0 | 1, 1 |

- For $\mathbf{s}$ to be strict, all strategies in $\mathbf{s}$ must be pure

- **Poll 2.5**: if all strategies in $\mathbf{s}$ are pure, is $\mathbf{s}$ guaranteed to be strict?

# Strict and Weak Nash Equilibria

- Weak Nash equilibria often are less stable than strict Nash equilibria
  - If **s** is weak, at least one agent has infinitely many best responses and only one of them is in **s**

- Example: $s_1 = s_2 = \{(½, \text{Left}), (½, \text{Right})\}$
  - For 2, Left is also a best response to $s_1$
  - If 2 unilaterally switches to Left, 1's strategy is no longer a best response

| | Left | Right |
|---|---|---|
| Left | 1, 1 | 0, 0 |
| Right | 0, 0 | 1, 1 |

# Finding Mixed*-Strategy Nash Equilibria

- Difficult in general
  - Easier if we can identify the support of the equilibrium strategies

- In 2x2 games, it's easy
  - Each agent has two actions, support must include both of them
- If there's a mixed*-strategy Nash equilibrium $(s_1, s_2)$ then
  - $s_1$ is a mixture of actions $a$ and $a'$ that have same expected utility given $s_2$
  - $s_2$ is a mixture of actions $b$ and $b'$ that have same expected utility given $s_1$

- Look for $s_1$ and $s_2$ that make those things true
  - Solve linear equations

# Finding Mixed*-Strategy Nash Equilibria

- **Example: Battle of the Sexes**
  - $s_1 = \{(p, A), (1-p, B)\}$
  - $s_2 = \{(q, A), (1-q, B)\}$
- If 2's strategy isn't pure, 2's actions must have same expected utility
  - $u_2(s_1,A) = u_2(s_1,B)$
    - $p = 2/3$
  - $s_1 = \{(2/3, A), (1/3, B)\}$

| | $A$ | $B$ |
|---|---|---|
| $A$ | 2, $\boxed{1}$ | 0, $\boxed{0}$ |
| $B$ | 0, $\boxed{0}$ | 1, $\boxed{2}$ |

$1p + 0(1-p) \qquad 0p + 2(1-p)$

$$p = 2(1-p)$$

# Finding Mixed*-Strategy Nash Equilibria

- **Example: Battle of the Sexes**
  - $s_1 = \{(p, A), (1-p, B)\}$
  - $s_2 = \{(q, A), (1-q, B)\}$
- If 2's strategy isn't pure, 2's actions must have same expected utility
  - $u_2(s_1,A) = u_2(s_1,B)$
    - $p = 2/3$
  - $s_1 = \{(2/3, A), (1/3, B)\}$

  - **Poll 2.6**: what is $q$?

| | $A$ | $B$ |
|---|---|---|
| $A$ | 2, $\boxed{1}$ | 0, $\boxed{0}$ |
| $B$ | 0, $\boxed{0}$ | 1, $\boxed{2}$ |

$1p + 0(1-p)$ $\quad$ $0p + 2(1-p)$

$$p = 2(1-p)$$

# Finding Mixed*-Strategy Nash Equilibria

- **Example: Battle of the Sexes**
  - $s_1 = \{(p, A), (1-p, B)\}$
  - $s_2 = \{(q, A), (1-q, B)\}$
- If 2's strategy isn't pure, 2's actions must have same expected utility
  - $u_2(s_1,A) = u_2(s_1,B)$
    - $p = 2/3$
  - $s_1 = \{(2/3, A), (1/3, B)\}$
- If 1's strategy isn't pure, 1's actions must have same expected utility
  - $u_1(A, s_2) = u_1(B, s_2)$
    - $q = 1/3$
  - $s_2 = \{(1/3, A), (2/3, B)\}$

| | $A$ | $B$ |
|---|---|---|
| $A$ | 2, $\boxed{1}$ | 0, $\boxed{0}$ |
| $B$ | 0, $\boxed{0}$ | 1, $\boxed{2}$ |

$1p + 0(1-p)$ $\quad$ $0p + 2(1-p)$

$$p = 2(1-p)$$

| | $A$ | $B$ |
|---|---|---|
| $A$ | $\boxed{2}$, 1 | $\boxed{0}$, 0 |
| $B$ | $\boxed{0}$, 0 | $\boxed{1}$, 2 |

$2q + 0(1-q)$

$0q + 1(1-q)$

$$2q = 1 - q$$

# Finding Mixed*-Strategy Nash Equilibria

- **Example: Battle of the Sexes**
  - $s_1 = \{(p, A), (1-p, B)\}$
  - $s_2 = \{(q, A), (1-q, B)\}$
- If 2's strategy isn't pure, 2's actions must have same expected utility
  - $u_2(s_1,A) = u_2(s_1,B)$
    - $p = 2/3$
  - $s_1 = \{(2/3, A), (1/3, B)\}$
- If 1's strategy isn't pure, 1's actions must have same expected utility
  - $u_1(A, s_2) = u_1(B, s_2)$
    - $q = 1/3$
  - $s_2 = \{(1/3, A), (2/3, B)\}$
- What will happen if there's no mixed*-strategy equilibrium?

| | $A$ | $B$ |
|---|---|---|
| $A$ | 2, $\boxed{1}$ | 0, $\boxed{0}$ |
| $B$ | 0, $\boxed{0}$ | 1, $\boxed{2}$ |

$1p + 0(1-p)$ $\quad$ $0p + 2(1-p)$

$$p = 2(1-p)$$

| | $A$ | $B$ | |
|---|---|---|---|
| $A$ | $\boxed{2}$, 1 | $\boxed{0}$, 0 | $2q + 0(1-q)$ |
| $B$ | $\boxed{0}$, 0 | $\boxed{1}$, 2 | $0q + 1(1-q)$ |

$$2q = 1 - q$$

# Finding Mixed*-Strategy Nash Equilibria

**Matching Pennies**

- No pure-strategy Nash equilibrium
  - In every case, one of the agents can do better by changing strategy

| | Heads | Tails |
|---|---|---|
| Heads | 1, –1 | –1, 1 |
| Tails | –1, 1 | 1, –1 |

- There's a mixed-strategy equilibrium
  - Get it the same way as in the Battle of the Sexes
    - Result is $(s,s)$, where $s = \{(½, \text{Heads}), (½, \text{Tails})\}$
  - More about this in Chapter 3

# Another Interpretation of Mixed Strategies

- Suppose agent *i* has a deterministic method for picking a strategy, but it depends on things that aren't part of the game itself
  - If *i* plays a game several times, *i* may pick different strategies
- If the other players don't know how *i* picks a strategy, they'll be uncertain what *i*'s strategy will be
  - Agent *i*'s mixed strategy is **everyone else's assessment** of how likely *i* is to play each pure strategy
- Example:
  - In a series of soccer penalty kicks, use a pseudo-random number generator to decide whether to kick left or right

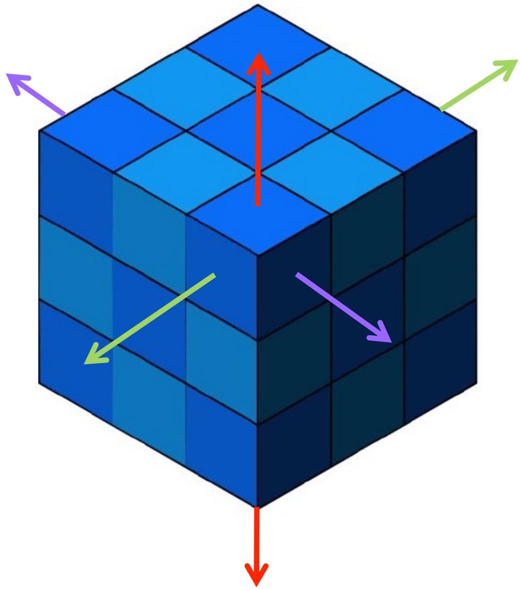
# Finding Nash Equilibria

|  | $b$ | $b'$ |
|---|---|---|
| $a$ | $u_1, v_1$ | $u_2, v_2$ |
| $a'$ | $u_3, v_3$ | $u_4, v_4$ |

- For 2x2 games:
  - Pure-strategy equilibria
    - Look for cells where neither player can do better by switching to the other action
  - Mixed-strategy equilibria
    - Write 1's strategy as $\{(p, a), (1–p, a')\}$
      - Look for $p$ such that 2 gets same expected utility for $b$ and $b'$
    - Write 2's strategy as $\{(q, b), (1–q, b')\}$
      - Look for $q$ such that 1 gets same expected utility for $a$ and $a'$
  - Equilibria where one strategy is pure and the other is mixed*
    - If there is a weak pure-strategy equilibrium, then look for mixed-strategy best-responses
- What about the general case?

# Finding Nash Equilibria

- General case (not in the book):
  - $n$ players, $m_i$ actions for player $i$
  - size of payoff matrix: $m_1 m_2 \ldots m_n$
- Brute-force approach:
  - Pure-strategy equilibria
    - Look for cells where no player can do better by unilaterally choosing a different action
    - Time is polynomial in the size of the matrix
  - Equilibria in which one or more strategies are mixed*
    - For every possible combination of supports for $s_1, \ldots, s_n$
      - Solve sets of simultaneous equations
    - Exponentially many combinations of supports → exponential time
- Can it be done more quickly?



# Complexity of Finding Nash Equilibria

- 2 players:
  - Lemke & Howson (1964): solve a set of simultaneous equations that includes all possible sets of supports
    - Some of the equations are quadratic => worst-case exponential time
  - Porter, Nudelman, & Shoham (2004)
    - AI methods (constraint programming)
  - Sandholm, Gilpin, & Conitzer (2005)
    - Mixed Integer Programming (MIP) problem
- *n*-player games
  - van der Laan, Talma, & van der Heyden (1987)
  - Govindan, Wilson (2004)
  - Porter, Nudelman, & Shoham (2004)
- Worst-case running time still is exponential in the size of the payoff matrix

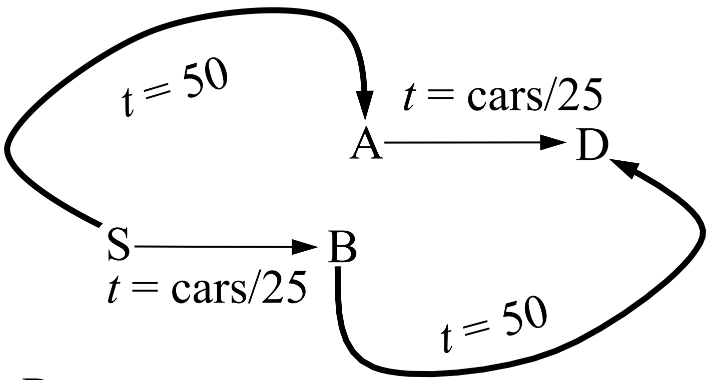
# Complexity of Finding Nash Equilibria

- For the general case,
  - It's unknown whether there are polynomial-time algorithms to do it
  - It's unknown whether there are polynomial-time algorithms to compute approximations
- One of the most important open problems in computational complexity theory

- Some special cases can be done in polynomial time
  - Finding pure-strategy Nash equilibria
    - Check each square of the payoff matrix
  - Finding Nash equilibria in zero-sum games
    - Linear programming

# Problem Structure

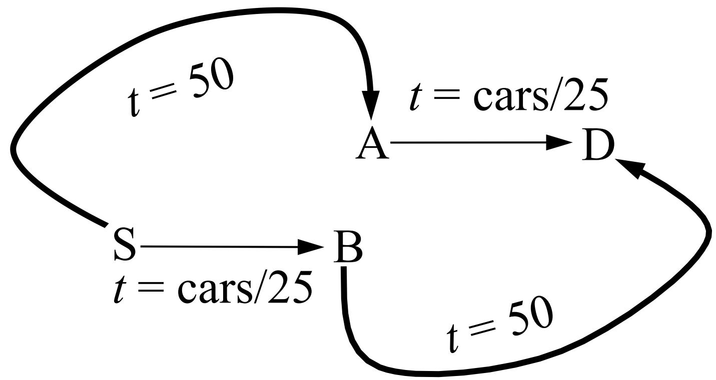
- In some classes of problems, can reason about problem structure
- Example (not in the book):
  - Suppose 1,000 drivers want to go from $S$ (start) to $D$ (destination)
  - Two routes: $S \to A \to D$ and $S \to B \to D$
  - $S \to A$ and $B \to D$ are long and wide
    - $t = 50$ minutes, no matter how many cars
  - $A \to D$ and $S \to B$ are short, but narrow
    - $t = $ (number of cars)/25




- Assume each driver's utility is $-t$
- Huge payoff matrix:
  - $2^{1000}$ action profiles
- But we don't need to write the matrix

# Problem Structure

- Nash equilibrium:
  - 500 cars go through A
  - 500 cars through B
- Everyone's expected time:
  50 + 20 = 70 minutes

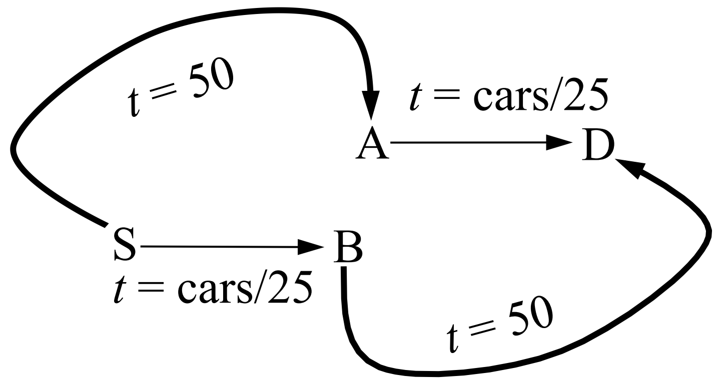



- Consider a driver whose strategy is $s = S \rightarrow A \rightarrow D$
  - Suppose the driver changes unilaterally to $S \rightarrow B \rightarrow D$
    - 501 cars on $S \rightarrow B \rightarrow D$
    - Expected travel time: 50 + 501/25 = 70.04

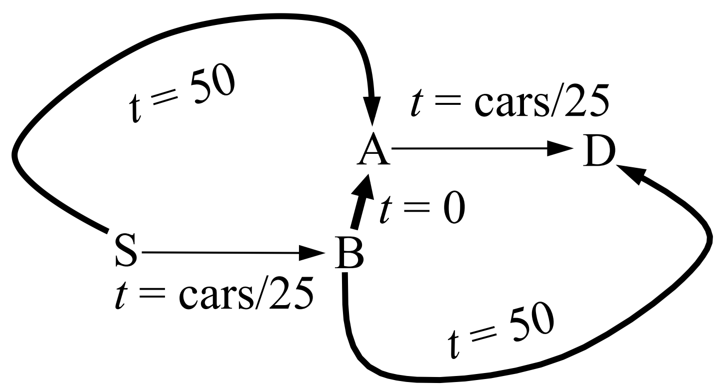
- Can generalize to the case where $s = \{(p, S \rightarrow A \rightarrow D\ ), (1–p, S \rightarrow B \rightarrow D)\}$

# Problem Structure

- Nash equilibrium:
  - On average, 500 cars go $S \rightarrow A \rightarrow D$ and 500 cars go $S \rightarrow B \rightarrow D$
- Everyone's expected travel time: 50 + 20 = 70 minutes



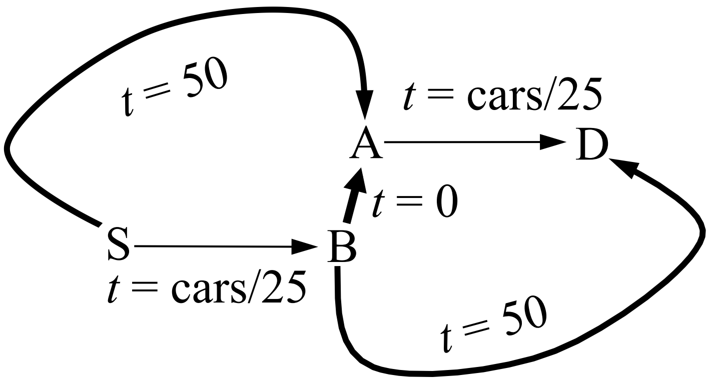

- Modify the network
  - Add a road from B to A that's *very short* and *very wide*
  - 0 minutes, regardless of how many cars




- Three possible routes:
  - $S \rightarrow A \rightarrow D$
  - $S \rightarrow B \rightarrow D$
  - S→B→A→D
- **Poll 2.7**: which one would you take?

# Braess's Paradox

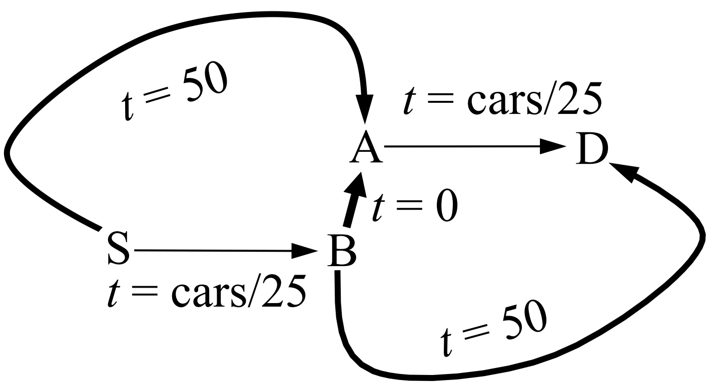
- Nash equilibrium: all cars go S→B→A→D
  - time for S→B is 1000/25 = 40
  - same for A→D
- Total time is 40 + 40 = 80 minutes




- To see that it's a Nash equilibrium:
  - If a driver unilaterally switches to $S \to A \to D$ or $S \to B \to D$
    - driving time is 50 + 40 = 90 minutes
  - If a driver unilaterally switches to
    - $s = \{(p, S \to B \to A \to D), (q, S \to A \to D), (1-p-q, \mathrm{S} \to \mathrm{B} \to \mathrm{D})\}$ with $p < 1$
    - What happens?

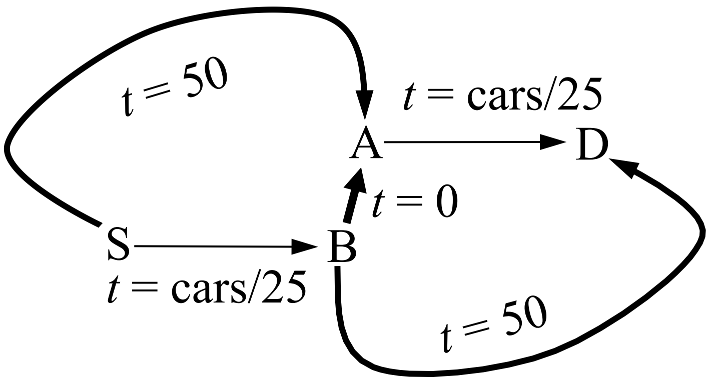
# Braess's Paradox

- To see that S→B→A→D is the ***only*** Nash equilibrium:
- Let
  - $a$ = expected # of cars S→A→D
  - $b$ = expected # of cars S→B→D
  - $0 < a + b \leq 1000$
- Times:
  - time to go S→A→D:
    - $50 + (1000 - b)/25 = 90 - b/25$
  - time to go S→B→D:
    - $50 + (1000 - a)/25 = 90 - a/25$
  - time to go S→B→A→D:
    - $(1000 - a)/25 + (1000 - b)/25 = 80 - a/25 - b/25$
  - Any driver that goes S→A→D or S→B→D
    can get lower travel time by switching to S→B→A→D



# Discussion

- Travel time
  - 70 minutes before adding the road
  - 80 minutes after
- Suggests that sometimes adding road capacity can hurt



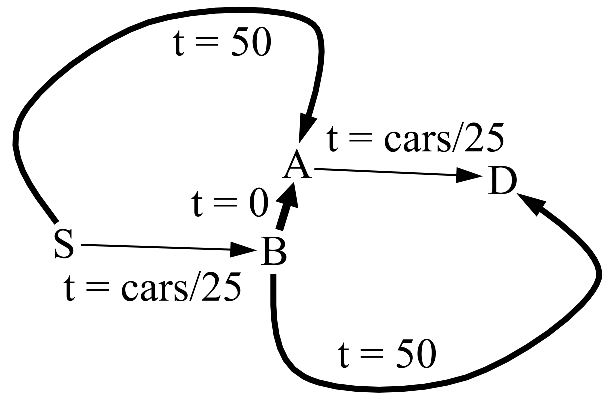

- We assumed
  - $t = 0$ regardless of how many cars
  - $t = 50$ regardless of how many cars
  - $t = \text{cars}/25$
- Is that realistic?
- Can this really happen in practice?

# Braess's Paradox in Practice

- 1969, Stuttgart, Germany – when a new road to the city center was opened, traffic got worse; didn't improve until the road was closed
- 1990, Earth day, New York – closing 42nd street improved traffic flow
- 1999, Seoul, South Korea – closing a tunnel improved traffic flow
- 2003, Seoul, South Korea – traffic flow was improved by closing a 6-lane motorway and replacing it with a 5-mile-long park
- 2010, New York – closing parts of Broadway has improved traffic flow
- Sources
  - http://www.umassmag.com/transportationandenergy.htm
  - http://www.cs.caltech.edu/~adamw/courses/241/lectures/brayes-j.pdf
  - http://www.guardian.co.uk/environment/2006/nov/01/society.travelsenvironmentalimpact
  - http://www.scientificamerican.com/article.cfm?id=removing-roads-and-traffic-lights
  - http://www.lionhrtpub.com/orms/orms-6-00/nagurney.html

# Discussion

- Nash equilibrium:
  - All 1000 cars go S→B→A→D
  - Total time is 80 minutes




- Compare with the Prisoner's Dilemma

| | *C* | *D* |
|---|---|---|
| *C* | –1, –1 | –4, 0 |
| *D* | 0, –4 | –3, –3 |

# Comments

- Braess's paradox can also occur in other kinds of networks
  - Queueing networks
  - Communication networks
- In principle, it can occur in Internet traffic
  - I don't know enough about this topic to know how much of a problem it is

# Here's Another Game

- All of you can play!
  - Choose a number in the range $0 \leq x \leq 100$
  - Write your choice on a piece of paper
  - Fold the paper so nobody else can see your number
  - Pass the paper to the front of the room

- The winner(s) will be those whose number is closest to 2/3 of the average of all the numbers

- I'll present the results next time

# Announcements (Sept 8)

- What we covered last time:
  - Best response
  - Nash equilibrium
    - strict, weak
    - finding mixed-strategy Nash equilibria
    - problem structure (Braess's paradox)
  - Game:
    - choose a number between 0 and 100
    - winner(s): whoever chooses a number that's closest to 2/3 of the average

# The Price of Anarchy

(not in the book)

- In the Prisoner's dilemma, recall that
  - (C,C) is the action profile that provides the best outcome for everyone
  - If we assume each payer acts to maximize his/her utility without regard to the other, we get (D,D)
  - By choosing (C,C), each player could have gotten 3 times as much
- Let's generalize "best outcome for everyone"

| | *C* | *D* |
|---|---|---|
| *C* | $\boxed{3, 3}$ | 0, 5 |
| *D* | 5, 0 | 1, 1 |

| | *C* | *D* |
|---|---|---|
| *C* | 3, 3 | 0, 5 |
| *D* | 5, 0 | $\boxed{1, 1}$ |

# The Price of Anarchy

- **Social welfare function** $w(\mathbf{s})$
    - measures the players' welfare given $\mathbf{s}$
  - **Utilitarian** welfare function: $w(\mathbf{s})$ = *average* expected utility
  - **Egalitarian** welfare function: $w(\mathbf{s})$ = *minimum* expected utility
- **Social optimum**: benevolent dictator chooses $\mathbf{s}^*$ that optimizes $w$
  - $\mathbf{s}^* = \arg\max_{\mathbf{s}} w(\mathbf{s})$
  - Is this Pareto optimal?

- **Anarchy**: no dictator; every player selfishly tries to optimize his/her own expected utility, disregarding the welfare of the other players
  - Get a strategy profile $\mathbf{s}$ (e.g., a Nash equilibrium)
  - In general, $w(\mathbf{s}) \leq w(\mathbf{s}^*)$

- **Price of anarchy** $= w(\mathbf{s}^*) / w(\mathbf{s})$

# The Price of Anarchy

- Example: the Prisoner's Dilemma
  - Utilitarian welfare function:
    $w(\mathbf{s})$ = average expected utility

| | $C$ | $D$ |
|---|---|---|
| $C$ | $\boxed{3, 3}$ | 0, 5 |
| $D$ | 5, 0 | 1, 1 |

- Social optimum: $\mathbf{s}^* = (C,C)$
  - $w(\mathbf{s}^*) = 3$

| | $C$ | $D$ |
|---|---|---|
| $C$ | 3, 3 | 0, 5 |
| $D$ | 5, 0 | $\boxed{1, 1}$ |

- Anarchy: $\mathbf{s} = (D,D)$
  - $w(\mathbf{s}) = 1$

- Price of anarchy

  $= w(\mathbf{s}^*) / w(\mathbf{s}) = 3/1 = 3$

- What would the answer be if we used the egalitarian welfare function?

# The Price of Anarchy

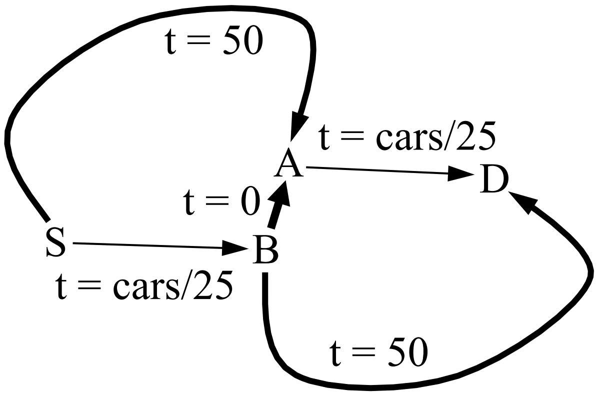
- Sometimes instead of maximizing a welfare function $w$, we want to minimize a cost function $c$
  - Utilitarian function: $c(\mathbf{s})$ = avg. expected cost
  - Egalitarian function: $c(\mathbf{s})$ = max. expected cost

- Need to adjust the definitions
  - ***Social optimum***: $\mathbf{s}^* = \arg\min_{\mathbf{s}} c(\mathbf{s})$
  - ***Anarchy***: every player selfishly tries to minimize his/her own cost, disregarding the costs of the other players
    - Get a strategy profile $\mathbf{s}$ (e.g., a Nash equilibrium)
    - In general, $c(\mathbf{s}) \geq c(\mathbf{s}^*)$
  - ***Price of anarchy*** = $c(\mathbf{s}) / c(\mathbf{s}^*)$
    - reciprocal of what we had before

# The Price of Anarchy

- Example: Braess's Paradox
  - Utilitarian cost function: $c(\mathbf{s})$ = average expected cost

- Social optimum:
  - $\mathbf{s}^*$ = [500 go S→A→D; 500 go S→B→D]
  - $c(\mathbf{s}^*) = 70$
- Anarchy: $\mathbf{s}$ = [1000 drivers go S→B→A→D]
  - $c(\mathbf{s}) = 80$
- Price of anarchy = $c(\mathbf{s}) / c(\mathbf{s}^*) = 8/7$




- What would the answer be if we used the egalitarian cost function?

- This can be generalized

# Summary

- Pareto optimality
  - Prisoner's Dilemma, Which Side of the Road
- Best responses and Nash equilibria
  - Battle of the Sexes, Matching Pennies
- Finding pure-strategy and mixed-strategy Nash equilibria
  - Methods for special cases
- Not in the book:
  - Brief discussion of computational complexity
- Road-network example (not in the book)
  - Braess's paradox
- Price of anarchy (not in the book)
  - Prisoner's dilemma, road networks