CMSC 474, Game Theory

7. Incomplete-Information Games

Dana Nau
University of Maryland

Introduction

- All the kinds of games we've looked at so far have assumed that everything relevant about the game being played is common knowledge to all the players:
  - the number of players
  - the actions available to each
  - the payoff vector associated with each action vector
- True even for imperfect-information games
  - The actual moves aren't common knowledge, but the game is
- We'll now consider games of *incomplete* information
  - Players are uncertain about the game being played

Example

- Consider the payoff matrix shown here
  - ε is a small positive constant; Agent 1 knows its value
  - Agent 1 doesn't know the values of *a, b, c, d*
- The matrix represents a **set** of games, *G*
  - Agent 1 doesn't know which game in *G* is the one being played

	L	R
T	100, a	$1-\varepsilon$, b
B	2, c	1, d

- What kind of strategy makes sense?
  - So far, we've seen two possibilities
    - *maxmin* strategy: maximize worst-case expected utility
    - *minimax regret* strategy: minimize worst-case regret
- Suppose we have a probability distribution on the games in *G* ...

Bayesian Games

	L	R
T	100, a	$1-\varepsilon$, b
B	2, c	1, d

- *Bayesian Game*: a set of games G that satisfies two fundamental conditions:
  - Condition 1: same strategy space
  - Condition 2: common prior

- **Condition 1: same strategy space**.
  - Each game in G has the same number of agents
  - For each agent i, each game in G has the same *strategy space*
    - same set of possible strategies (hence same set of actions A_i)
  - Only difference is in the payoffs

- This condition isn't very restrictive
  - Can often reformulate problems to fit it

Example

- Suppose we don't know whether player 2 only has strategies L and R, or also an additional strategy C:

Game G_1

	L	R
U	1, 1	1, 3
D	0, 5	1, 13

Game G_2

	L	C	R
U	1, 1	0, 2	1, 3
D	0, 5	2, 8	1, 13

- Having no strategy C is equivalent to having a strategy C that's strictly dominated by the other strategies

Game G_1'

	L	C	R
U	1, 1	0, –100	1, 3
D	0, 5	2, –100	1, 13

has same Nash equilibria as G_1

- We've reduced the problem to this:
  - Which payoffs does player 2 have:
  - The ones in G_1', or the ones in G_2?

Bayesian Games

- **Condition 2: common prior**. The agents have common knowledge of a prior probability distribution over the games in G
  - *prior*: what an agent knows before it learns additional information
- The agents' individual beliefs are *posterior probabilities*
  - Combine the common prior distribution with individual "private signals" (what's "revealed" to the individual players)

- This rules out whole families of games, but greatly simplifies the theory
  - So most work on incomplete-information games uses it
- Later: some examples of games that don't satisfy Condition 2

Definitions of Bayesian Games

- The book discusses three different ways to define Bayesian games
  - All are
    - equivalent (ignoring a few subtleties)
    - useful in some settings
    - intuitive in their own way

- The first definition (Section 7.1.1) is based on information sets
- A Bayesian game consists of
  - a set of games that differ only in their payoffs
  - a common (i.e., known to all players) prior distribution over them
  - for each agent, a partition structure (set of information sets) over the games
- Formal definition on the next page

7.1.1 Definition based on Information Sets

- A *Bayesian game* is a 4-tuple (N,G,P,I) where:
  - N is a set of agents
  - G is a set of N-agent games
  - For every agent i, every game in G has the same strategy space
  - P is a *common prior* over G
    - *common*: common knowledge (known to all the agents)
    - *prior*: probability before learning any additional info
  - $I = (I_1, \ldots, I_N)$ is a tuple of partitions of G, one for each agent
    - Information sets

- **Example:**

  G = {Matching Pennies (MP),
  Prisoner's Dilemma (PD),
  Coordination (Crd),
  Battle of the Sexes (BoS)}

$I_{2,1}$ $I_{2,2}$

$I_{1,1}$

MP $(p = 0.3)$

	L	R
U	2, 0	0, 2
D	0, 2	2, 0

PD $(p = 0.1)$

	L	R
U	2, 2	0, 3
D	3, 0	1, 1

$I_{1,2}$

Crd $(p=0.2)$

	L	R
U	2, 2	0, 0
D	0, 0	1, 1

BoS $(p = 0.4)$

	L	R
U	2, 1	0, 0
D	0, 0	1, 2

Example (Continued)

- G = {MP, PD, Crd, BoS}
  - Suppose the randomly chosen game is MP
- Agent 1's information set is $I_{1,1}$
  - 1 knows the game is MP or PD
- 1 can infer *posterior* probabilities for MP and PD

$$\Pr[\mathrm{MP} \mid I_{1,1}] = \frac{\Pr[\mathrm{MP} \wedge I_{1,1}]}{\Pr[I_{1,1}]} = \frac{\Pr[\mathrm{MP}]}{\Pr[\mathrm{MP}] + \Pr[\mathrm{PD}]}$$

$$= \frac{0.3}{0.3 + 0.1} = \frac{3}{4}$$

$$\Pr[\mathrm{PD} \mid I_{1,1}] = \frac{\Pr[\mathrm{PD}]}{\Pr[\mathrm{MP}] + \Pr[\mathrm{PD}]} = \frac{0.1}{0.3 + 0.1} = \frac{1}{4}$$

- Agent 2's information set is $I_{2,1}$

$$\Pr[\mathrm{MP} \mid I_{2,1}] = \frac{\Pr[\mathrm{MP}]}{\Pr[\mathrm{MP}] + \Pr[\mathrm{CrD}]} = \frac{0.3}{0.3 + 0.2} = \frac{3}{5}$$

$$\Pr[\mathrm{Crd} \mid I_{2,1}] = \frac{\Pr[\mathrm{Crd}]}{\Pr[\mathrm{MP}] + \Pr[\mathrm{CrD}]} = \frac{0.2}{0.3 + 0.2} = \frac{2}{5}$$

$I_{2,1}$ | $I_{2,2}$

$I_{1,1}$

MP $(p = 0.3)$

	L	R
U	2, 0	0, 2
D	0, 2	2, 0

PD $(p = 0.1)$

	L	R
U	2, 2	0, 3
D	3, 0	1, 1

$I_{1,2}$

Crd $(p=0.2)$

	L	R
U	2, 2	0, 0
D	0, 0	1, 1

BoS $(p = 0.4)$

	L	R
U	2, 1	0, 0
D	0, 0	1, 2

7.1.2 Definition Based on Chance Moves

- **Extensive form with Chance Moves**
  - The book gives a description, but not a formal definition
- Hypothesize a special agent, *Nature*
  - Nature has no utility function
- At the start of the game, Nature makes a probabilistic choice according to the common prior
- Agents receive *individual signals* about Nature's choice
  - Some choices are "revealed" to some players, others to other players
  - The players receive **no** other information

Example

- Same example as before, but translated into extensive form
  - Game tree of depth 3
- Nature randomly chooses MP
  - Nature sends signal $I_{1,1}$ to Agent 1
  - Nature sends signal $I_{2,1}$ to Agent 2
- Each agent chooses independently

$I_{2,1}$ $I_{2,2}$

$I_{1,1}$

MP ($p = 0.3$)

	L	R
U	2, 0	0, 2
D	0, 2	2, 0

PD ($p = 0.1$)

	L	R
U	2, 2	0, 3
D	3, 0	1, 1

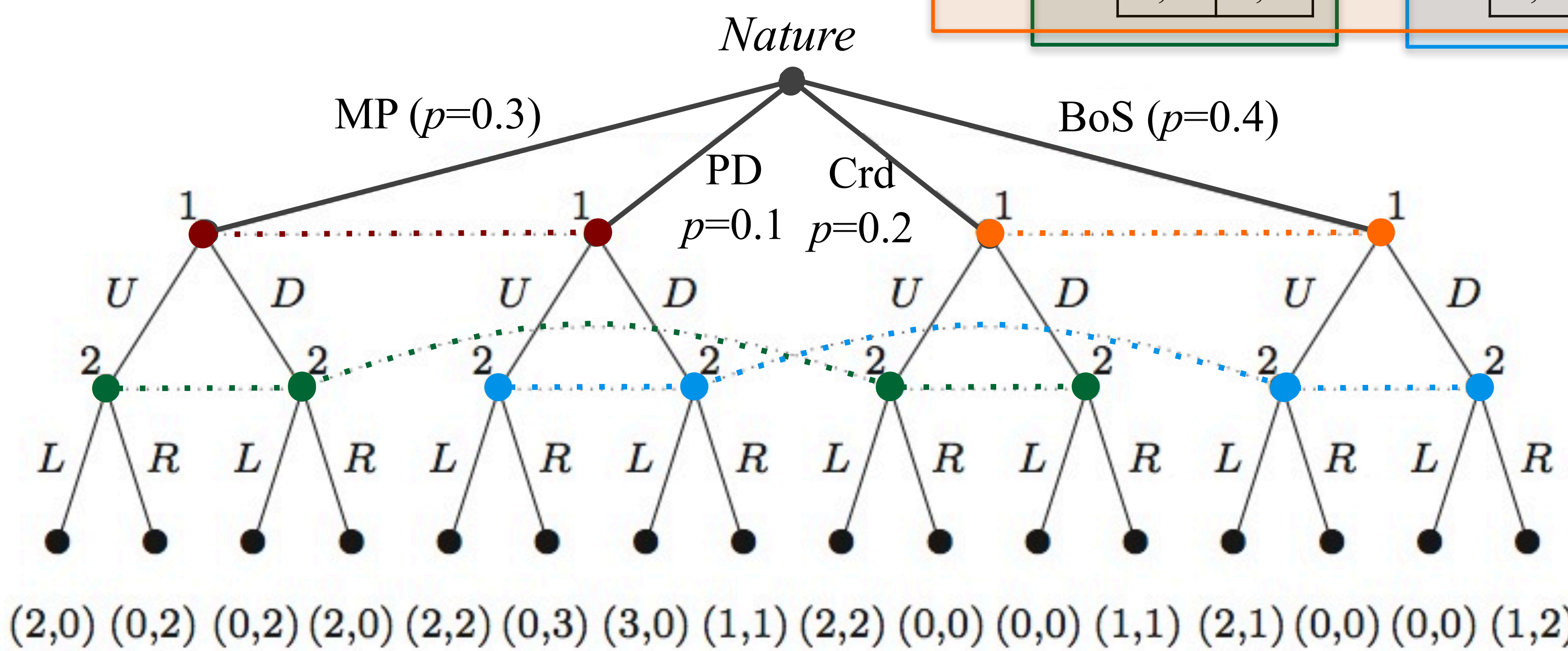
$I_{1,2}$

Crd (p=0.2)

	L	R
U	2, 2	0, 0
D	0, 0	1, 1

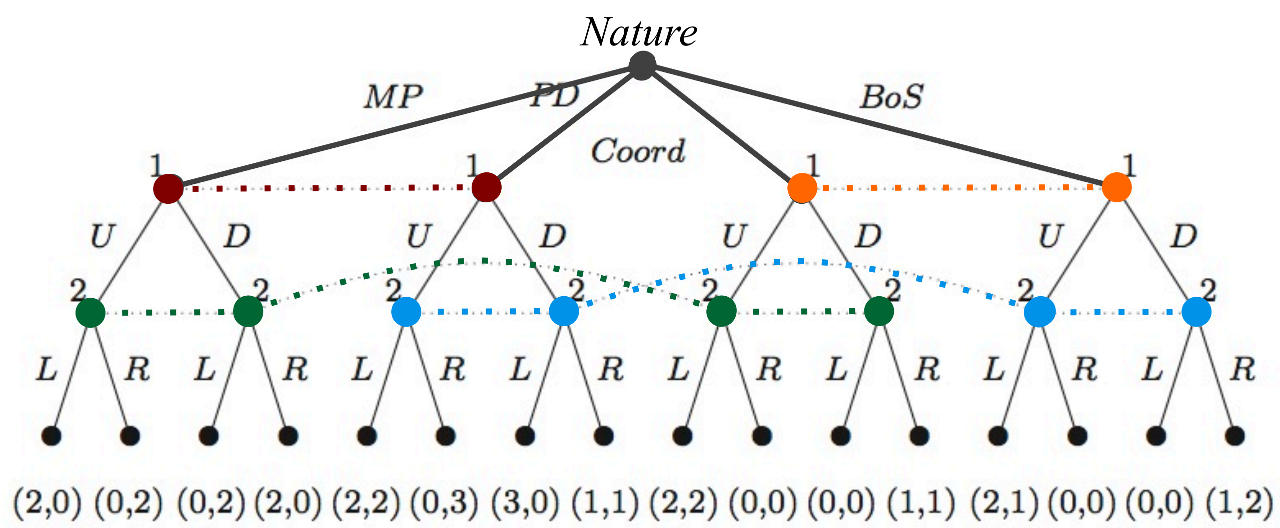
BoS ($p = 0.4$)

	L	R
U	2, 1	0, 0
D	0, 0	1, 2



Discussion

- Can we represent a real game this way?
- For n players, always get an imperfect-information game tree of depth n+1
  - 2 players → depth 3
- Root node: choice node for *Nature*
  - Nature makes probabilistic choice according to the common prior
- Nodes at depth i: information sets for player i
  - i's strategy s_i maps the information sets into actions

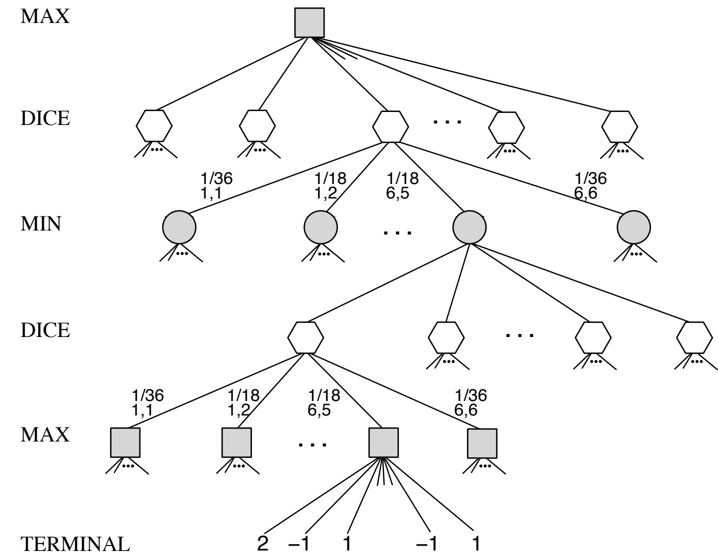


Example

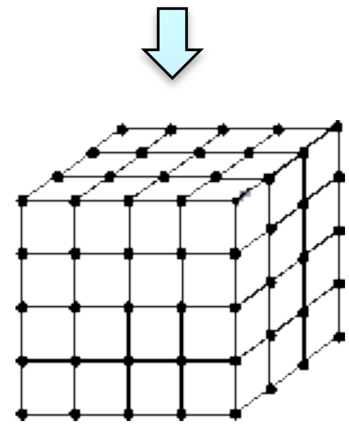
Translate Backgammon to a Bayesian game?

- Nature makes choices throughout the game
  - ➢ Dice rolls have random outcomes
  - ➢ Players see the outcomes
- Each player makes moves throughout the game
  - ➢ Both players see all moves
- Translate to normal form game
  - ➢ 3D matrix
  - ➢ For each player, a huge number of possible strategies
  - ➢ From the normal form, construct the depth-3 tree
- *Not practical*

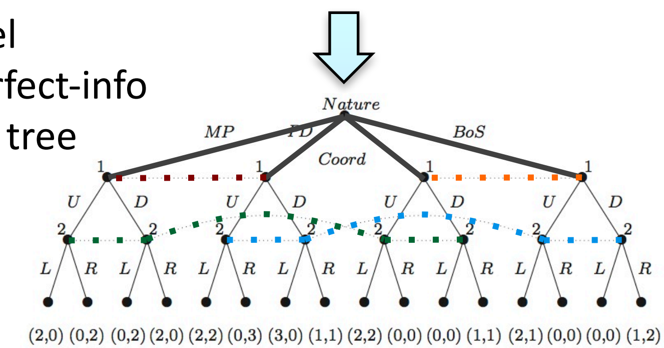
game tree



payoff matrix

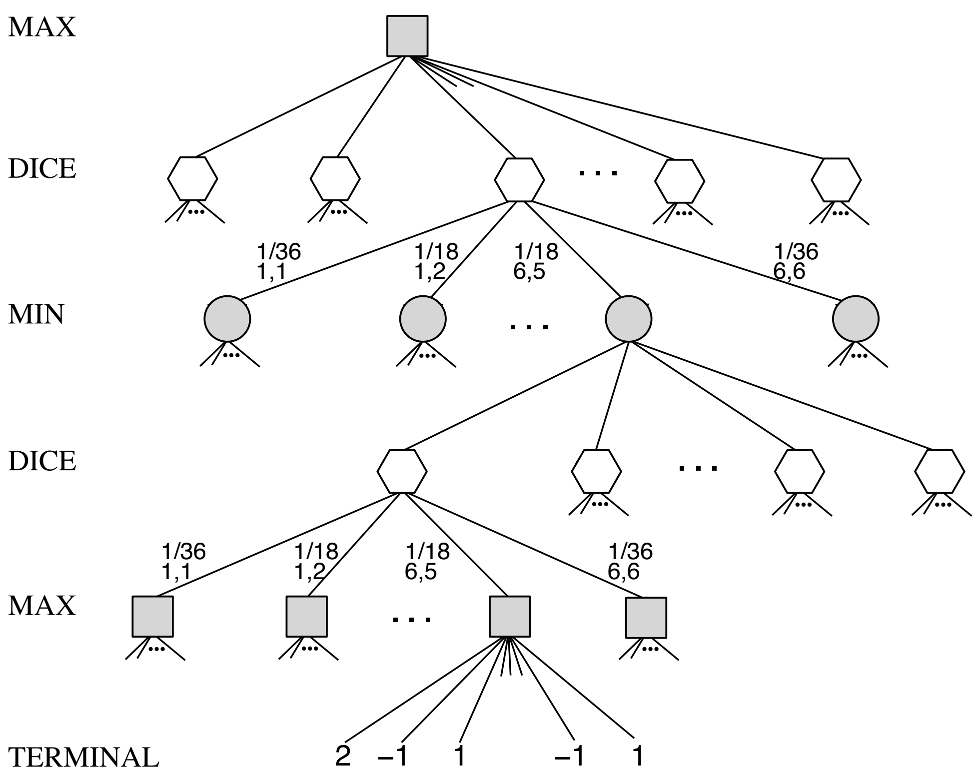


3-level imperfect-info game tree



Extending the Definition

- Could extend the definition to include
  - Players sometimes get information about each other's moves
  - Nature makes choices and sends signals throughout the game
- To model backgammon, bridge, …
  - Just use their usual game trees



7.1.3 Definition Based on Epistemic Types

- Recall, we can assume the only thing players are uncertain about are the other players' utility functions
  - Define uncertainty directly over the utility functions
- **Definition 7.1.2**: a *Bayesian game* is a tuple $(N, \mathbf{A}, \Theta, \Pr, \mathbf{u})$ where:
  - N is a set of agents
  - $\mathbf{A} = A_1 \times \ldots \times A_n$, where A_i is player i's set of possible actions
  - $\boldsymbol{\Theta} = \Theta_1 \times \ldots \times \Theta_n$, where Θ_i is player i's set of possible *types*
  - $\Pr : \boldsymbol{\Theta} \to [0,1]$ is a common prior distribution over types
  - $\mathbf{u} = (u_1, \ldots, u_n)$, where u_i is player i's utility function
    - $u_i : \mathbf{A} \times \boldsymbol{\Theta} \to \mathcal{R}$ *i.e.*, $u_i(a_1,\ldots,a_n, \theta_1, \ldots, \theta_n) = x$

- All of the above is common knowledge among the players
- Agent i's *type* is the information i has that isn't common knowledge
  - i knows i's type, but not what the other agents' types are

Types

- θ_i : all information *i* has that *isn't* common knowledge, e.g.,
  - *i*'s actual payoff function
  - *i*'s beliefs about other agents' payoff functions,
  - *i*'s beliefs about *their* beliefs about his/her own payoff function
  - Any other higher-order beliefs

Example

- Agent 1's possible types: $\Theta_1 = \{\theta_{1,1}, \theta_{1,2}\}$
  - 1's type is $\theta_{1,j}$ ⇔ 1's info set is $I_{1,j}$
- Agent 2's possible types: $\Theta_2 = \{\theta_{2,1}, \theta_{2,2}\}$
  - 2's type is $\theta_{2,j}$ ⇔ 2's info set is $I_{1,j}$

$\theta_{2,1}$ | $\theta_{2,2}$

$\theta_{1,1}$

MP ($p = 0.3$)

	L	R
U	2, 0	0, 2
D	0, 2	2, 0

PD ($p = 0.1$)

	L	R
U	2, 2	0, 3
D	3, 0	1, 1

$\theta_{1,2}$

Crd (p=0.2)

	L	R
U	2, 2	0, 0
D	0, 0	1, 1

BoS ($p = 0.4$)

	L	R
U	2, 1	0, 0
D	0, 0	1, 2

- Joint distribution on the types:

$$\Pr[\theta_{1,1}, \theta_{2,1}] = 0.3; \quad \Pr[\theta_{1,1}, \theta_{2,2}] = 0.1$$

$$\Pr[\theta_{1,2}, \theta_{2,1}] = 0.2; \quad \Pr[\theta_{1,2}, \theta_{2,2}] = 0.4$$

- Conditional probabilities for agent 1:
  - $\Pr[\theta_{2,1} \mid \theta_{1,1}] = 0.3/(0.3 + 0.1) = 3/4; \quad \Pr[\theta_{2,2} \mid \theta_{1,1}] = 0.1/(0.3 + 0.1) = 1/4$
  - $\Pr[\theta_{2,1} \mid \theta_{1,2}] = 0.2/(0.2 + 0.4) = 1/3; \quad \Pr[\theta_{2,2} \mid \theta_{1,2}] = 0.4/(0.2 + 0.4) = 2/3$

Example (continued)

- $u_i(a_1,\ldots,a_n,\theta_1,\ldots,\theta_n)$
  - depends on both the types and the actions
  - the types determine what game it is
  - the actions determine the payoff within that game

$\theta_{2,1}$ $\theta_{2,2}$

$\theta_{1,1}$

MP ($p = 0.3$)

	L	R
U	2, 0	0, 2
D	0, 2	2, 0

PD ($p = 0.1$)

	L	R
U	2, 2	0, 3
D	3, 0	1, 1

$\theta_{1,2}$

Crd (p=0.2)

	L	R
U	2, 2	0, 0
D	0, 0	1, 1

BoS ($p = 0.4$)

	L	R
U	2, 1	0, 0
D	0, 0	1, 2

a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

Strategies

- In principle, we could use any of the three definitions of a Bayesian game
- The book uses the 3rd one (epistemic types)
- *Pure strategy* for player i
  - function that maps each of i's types to an action
  - what i would play if i had that type
- *Mixed strategy* s_i
  - probability distribution over pure strategies
  - $s_i(a_i \mid \theta_j) = \Pr[i \text{ plays action } a_j \mid i\text{'s type is } \theta_j]$
- Three kinds of expected utility, depending on what we know about the players' types
  - *ex ante*: before we know anything other than the common prior
  - *ex post*: after we know everyone's type
  - *ex interim*: know only agent i's type
    - i.e., the game from i's point of view

Expected Utility

- The players' expected utilities depend on both their strategies and their types
- *Type profile*: a vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ of types, one for each agent
  - $\boldsymbol{\theta}_{-i} = (\theta_1, \theta_2, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$
  - $\boldsymbol{\theta} = (\theta_i, \boldsymbol{\theta}_{-i})$
- Agent i's *ex post* expected utility (know what $\boldsymbol{\theta}$ is):

$$EU_i(\mathbf{s},\boldsymbol{\theta}) = \sum_{\mathbf{a}} \Pr[\mathbf{a} \mid \mathbf{s},\boldsymbol{\theta}]\ u_i(\mathbf{a},\boldsymbol{\theta})$$

- Agent i's *ex ante* expected utility (only know the common prior):

$$EU_i(\mathbf{s}) = \sum_{\boldsymbol{\theta}} \Pr[\boldsymbol{\theta}]\ \boxed{EU_i(\mathbf{s},\boldsymbol{\theta})}$$

- Agent i's *ex interim* expected utility (know θ_i and the common prior)

$$EU_i(\mathbf{s},\theta_i) = \sum_{\boldsymbol{\theta}_{-i}} \Pr[\boldsymbol{\theta}_{-i} \mid \theta_i]\ \boxed{EU_i(\mathbf{s},(\theta_i,\boldsymbol{\theta}_{-i}))}$$

Bayes-Nash Equilibria

- Just like the definition of a Nash equilibrium, except that we're using
  - Bayesian-game strategies
  - *ex ante* expected utilities

- Given a strategy profile $\mathbf{s}_{-i}$
  - a *best response* for agent i is a strategy s_i^* such that
    - $EU_i\,(s_i^*, \mathbf{s}_{-i}) = \max_{s_i} EU_i\,(s_i, \mathbf{s}_{-i})$

- *Bayes-Nash* equilibrium
  - a strategy profile $\mathbf{s}$ such that
    - for every s_i in $\mathbf{s}$, s_i is a best response to $\mathbf{s}_{-i}$

Computing Bayes-Nash Equilibria

- Basic idea
  - Construct a payoff matrix for the entire Bayesian game
  - Find equilibria on that matrix

	$\theta_{2,1}$	$\theta_{2,2}$
$\theta_{1,1}$	MP ($p = 0.3$)	PD ($p = 0.1$)
$\theta_{1,2}$	Crd (p=0.2)	BoS ($p = 0.4$)

MP ($p = 0.3$)

	L	R
U	2, 0	0, 2
D	0, 2	2, 0

PD ($p = 0.1$)

	L	R
U	2, 2	0, 3
D	3, 0	1, 1

Crd (p=0.2)

	L	R
U	2, 2	0, 0
D	0, 0	1, 1

BoS ($p = 0.4$)

	L	R
U	2, 1	0, 0
D	0, 0	1, 2

- First, write each pure strategy as a list of actions, one action for each type

- Agent 1's pure strategies:
  - UU: U if type $\theta_{1,1}$, U if type $\theta_{1,2}$
  - UD: U if type $\theta_{1,1}$, D if type $\theta_{1,2}$
  - DU: D if type $\theta_{1,1}$, U if type $\theta_{1,2}$
  - DD: D if type $\theta_{1,1}$, D if type $\theta_{1,2}$

- Agent 2's pure strategies:
  - LL: L if type $\theta_{2,1}$, L if type $\theta_{2,2}$
  - LR: L if type $\theta_{2,1}$, R if type $\theta_{2,2}$
  - RL: R if type $\theta_{2,1}$, L if type $\theta_{2,2}$
  - RR: R if type $\theta_{2,1}$, R if type $\theta_{2,2}$

Computing Bayes-Nash Equilibria (continued)

- Next, compute the *ex ante* expected utility for each pure-strategy profile
  - e.g.,

$$EU_2(UU,LL) = \sum_{\boldsymbol{\theta}} \Pr[\boldsymbol{\theta}]\, u_2(U,L,\boldsymbol{\theta})$$

$$= \Pr[\theta_{1,1},\theta_{2,1}]u_2(U,L,\theta_{1,1},\theta_{2,1})$$
$$+\Pr[\theta_{1,1},\theta_{2,2}]u_2(U,L,\theta_{1,1},\theta_{2,2})$$
$$+\Pr[\theta_{1,2},\theta_{2,1}]u_2(U,L,\theta_{1,2},\theta_{2,1})$$
$$+\Pr[\theta_{1,2},\theta_{2,2}]u_2(U,L,\theta_{1,2},\theta_{2,2})$$
$$= 0.3\times 0 + 0.1\times 2 + 0.2\times 2 + 0.4\times 1$$
$$= 1$$

$\theta_{2,1}$ $\theta_{2,2}$

$\theta_{1,1}$

MP $(p = 0.3)$

	L	R
U	2, 0	0, 2
D	0, 2	2, 0

PD $(p = 0.1)$

	L	R
U	2, 2	0, 3
D	3, 0	1, 1

$\theta_{1,2}$

Crd $(p{=}0.2)$

	L	R
U	2, 2	0, 0
D	0, 0	1, 1

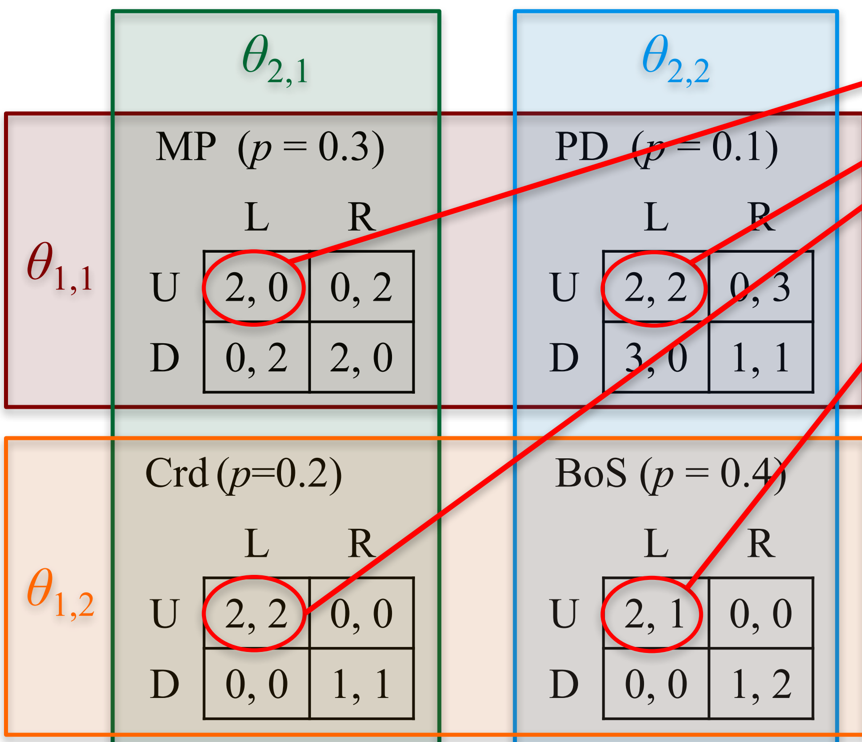
BoS $(p = 0.4)$

	L	R
U	2, 1	0, 0
D	0, 0	1, 2

$$EU_1(UU,LL)$$
$$= 0.3\times 2 + 0.1\times 2 + 0.2\times 2 + 0.4\times 2$$
$$= 2$$

Computing Bayes-Nash Equilibria (continued)

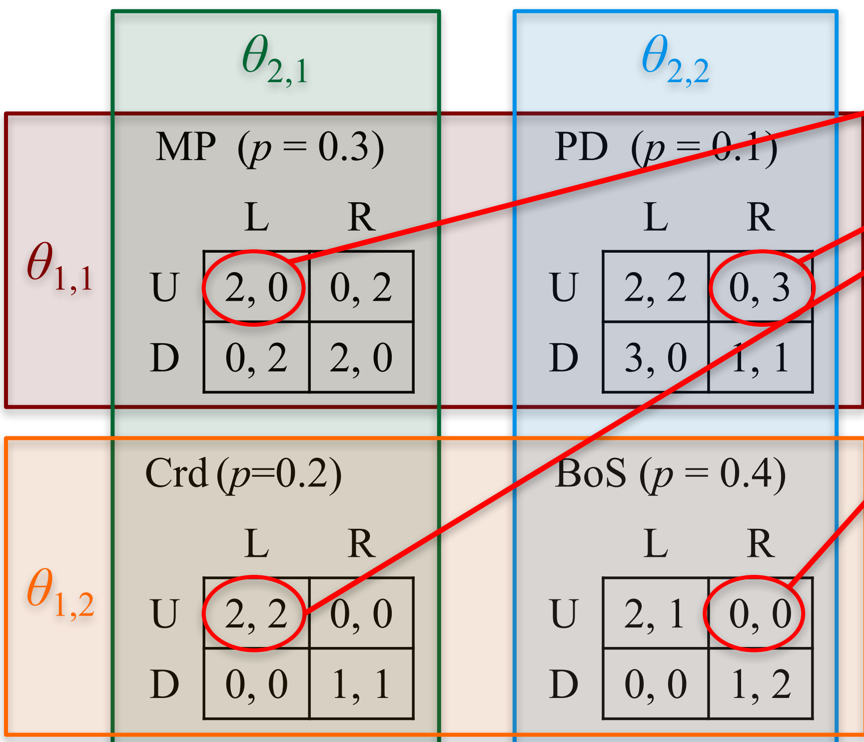
- Put all of the *ex ante* expected utilities into a payoff matrix
  - e.g., $EU_1(UU,LL)$, $EU_2(UU,LL)$
- Now we can compute best responses and Nash equilibria



	LL	LR	RL	RR
UU	2, 1	1, 0.7	1, 1.2	0, 0.9
UD	0.8, 0.2	1, 1.1	0.4, 1	0.6, 1.9
DU	1.5, 1.4	0.5, 1.1	1.7, 0.4	0.7, 0.1
DD	0.3, 0.6	0.5, 1.5	1.1, 0.2	1.3, 1.1

Computing Bayes-Nash Equilibria (continued)

- Put all of the *ex ante* expected utilities into a payoff matrix
  - e.g., $EU_1(UU,LL)$, $EU_2(UU,LL)$
- Now we can compute best responses and Nash equilibria



	LL	LR	RL	RR
UU	2, 1	1, 0.7	1, 1.2	0, 0.9
UD	0.8, 0.2	1, 1.1	0.4, 1	0.6, 1.9
DU	1.5, 1.4	0.5, 1.1	1.7, 0.4	0.7, 0.1
DD	0.3, 0.6	0.5, 1.5	1.1, 0.2	1.3, 1.1

Ex Interim Payoff Matrix

- Suppose we learn agent 1's type is $\theta_{1,1}$
- Recompute the expected payoffs using the posterior probabilities
  - $\Pr[\mathrm{MP} \mid \theta_{1,1}] = 3/4$, $\Pr[\mathrm{PD} \mid \theta_{1,1}] = 1/4$
  - $u_2(UU,LL \mid \theta_{1,1}) = 3/4\,(0) + 1/4\,(2) = 0.5$

- *Ex interim* payoff matrix when agent 1's type is $\theta_{1,1}$
- Can't use it to compute equilibria, since $\theta_{1,1}$ isn't common knowledge

$\theta_{1,1}$

$\theta_{2,1}$: MP ($p = 0.3$)

	L	R
U	2, 0	0, 2
D	0, 2	2, 0

$\theta_{2,2}$: PD ($p = 0.1$)

	L	R
U	2, 2	0, 3
D	3, 0	1, 1

	LL	*LR*	*RL*	*RR*
UU	2, 0.5	1.5, 0.75	0.5, 2	0, 2.25
UD	2, 0.5	1.5, 0.75	0.5, 2	0, 2.25
DU	0.75, 1.5	0.25, 1.75	2.25, 0	1.75, 0.25
DD	0.75, 1.5	0.25, 1.75	2.25, 0	1.75, 0.25

Ex Post Equilibria

- An *ex post* equilibrium: a strategy profile **s** such that for every s_i in **s** and for *every* type profile θ,
  - s_i is a best response to $\mathbf{s}_{-i}$ given θ
    - *i.e.*, $EU_i((s_i, \mathbf{s}_{-i}), \theta) = \max_{s_i'}(EU_i((s_i', \mathbf{s}_{-i}), \theta))$
- Doesn't say that *i* knows the other agents' types
- It says that *regardless* of what *i* knows about the other agents' types, *i* wouldn't want to switch to a different strategy
  - Not even if *i* had inaccurate information
  - Not even if *i* believed the others had inaccurate information
- A little like a dominant strategy equilibrium
  - Not guaranteed to exist
- Many dominant strategy equilibria are *ex post* equilibria, but not always

Example: Auctions
(this material isn't in the book)

- An auction is a way (other than bargaining) to sell a fixed supply of a *commodity* (an item to be sold) for which there is no well-established ongoing market
- *Bidders* make *bids*
  - proposals to pay various amounts of money for the commodity
- The commodity is sold to the bidder who makes the largest bid
- Example applications
  - Real estate, art, oil leases, electromagnetic spectrum, electricity, eBay, google ads
- Several kinds of auctions are incomplete-information, and can be modeled as Bayesian games

Types of Auctions

- Classify according to how the commodity is valued:
  - **Private-value auctions**
    - Each bidder may have a different *bidder value (BV)*, i.e., how much the commodity is worth to that bidder
    - A bidder's BV is his/her private information, not known to others
    - E.g., flowers, art, antiques
  - **Common-value auctions**
    - The ultimate value of the item is the same for all bidders, but bidders are unsure what that ultimate value is
    - E.g., oil leases, Olympic broadcast rights
  - **Affiliated (correlated) value auctions**
    - These are somewhere between private and common-value auctions
    - BVs for the auctioned item(s) are correlated, but not necessarily the same for all

Types of Auctions

- Classify according to the rules for bidding
  - English
  - Dutch
  - First price sealed bid
  - Vickrey
  - many others
- I'll describe several of these and will analyze their equilibria

- Possible problem: *collusion* (secret agreements for fraudulent purposes)
  - Groups of bidders who won't bid against each other, to keep the price low
  - Bidders who place phony (phantom) bids to raise the price (hence the auctioneer's profit)
- If there's collusion, the equilibrium analysis is no longer valid

English Auction

- The name comes from oral auctions in English-speaking countries, but I think this kind of auction was also used in ancient Rome
  - Used for antiques, artworks, cattle, horses, wholesale fruits and vegetables, old books, etc.
- Typical rules:
  - Auctioneer solicits an opening bid from the group
  - Anyone who wants to bid should call out a new price at least c higher than the previous high bid (e.g., $c = 1$ dollar)
  - The bidding continues until all bidders but one have dropped out
  - The highest bidder gets the object, for a price equal to his/her final bid
- For each bidder i, let
  - v_i = i's valuation of the commodity (private information)
  - B_i = i's final bid
- If i wins, then i's profit is $\pi_i = v_i - B_i$ and everyone else's profit = 0

English Auction (continued)

- Nash equilibrium:
  - Each bidder i participates until the bidding reaches v_i then drops out
    - So assuming rationality, $B_i < v_i$

Why not $B_i \leq v_i$?

  - The highest bidder, i, gets the object, at price $B_i < v_i$, so $\pi_i = B_i - v_i > 0$
    - B_i is close to the second highest bidder's valuation
  - For every bidder $j \neq i$, $\pi_j = 0$
- Why is this an equilibrium?
- Suppose bidder j deviates and none of the other bidders deviate
  - If j deviates by dropping out earlier,
    - Then j's profit will be 0, no better than before
  - If j deviates by bidding $B_j > v_j$, then either
    - someone else bids higher and wins the auction, so j's profit is still 0
    - j win's the auction but j's profit is $v_j - B_j \leq 0$, worse than before
- Which kind of equilibrium is this?

English Auction (continued)

- If there is a large range of bidder valuations, then the difference between the highest and 2nd-highest valuations may be large
  - Thus if there's wide disagreement about the item's value, the winner might be able to get it for much less than his/her valuation
- Let *n* be the number of bidders
  - The higher *n* is, the more likely it is that the highest and 2nd-highest valuations are close
    - Thus, the more likely it is that the winner pays close to his/her valuation

Example

- Auction a 20-dollar bill
  - It will be sold to the highest bidder, who must pay the amount of his/her bid
  - The second-highest bidder must also pay his/her bid, but gets nothing
  - No collusion
  - The minimum increment for a new bid is 10 cents

Example

That's **not** an English auction!

- Auction a 20-dollar bill
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  - No collusion
  - The minimum increment for a new bid is 10 cents
- This is called an *escalation auction*

All-Pay Auction

- **Swoopo**
  - Used to be a web site that auctioned items
  - Now defunct
  - URL redirects to another site that does something similar
- In ordinary auctions, bids cost nothing
  - But Swoopo required bidders to pay 60 cents/bid for each of their bids
- Bidders didn't pick the price they bid
  - Swoopo would increment the last offer by a fixed amount—a penny, 6 cents, 12, cents—that was determined before the start of the auction.
- Each time someone placed a bid, the auction got extended by 20 seconds

- Related to a lottery or a raffle
  - Main differences:

    pay a fixed fee, rather than a bid; winner chosen at random

Swoopo Example

- From http://poojanblog.com/blog/2010/01/swoopo-psychology-game-theory-and-regulation
  - Swoopo auctioned an ounce of gold (worth about $1,100)
  - Selling price was $203.13
  - Increment was 1 cent => 20,313 bids
    - At 60 cents per bid, Swoopo got $12,187.80 in revenue
      - Swoopo netted about $11,000
  - Winner's total price:
    - Selling price, plus the price of his/her bids
    - Probably about $600

First-Price Sealed-Bid Auctions

- Examples:
  - construction contracts (lowest bidder)
  - real estate
  - art treasures
- Typical rules
  - Bidders write their bids for the object and their names on slips of paper and deliver them to the auctioneer
  - The auctioneer opens the bid and finds the highest bidder
  - The highest bidder gets the object being sold, for a price equal to his/her own bid
  - Winner's profit = BV– price paid
  - Everyone else's profit = 0

First-Price Sealed-Bid (continued)

- Suppose that
  - There are n bidders
  - Each bidder i has a BV, v_i, which is private information
  - But a probability distribution for v_i is common knowledge
    - e.g., let's say every v_i is uniformly distributed over [0,100]
  - Let B_i denote the bid of player i
  - Let π_i denote the profit of player i

- What is an equilibrium bidding strategy for the players?

- First we'll look at the case where $n = 2$

First-Price Sealed-Bid (continued)

- Let B_i be agent i's bid, and π_i be agent i's profit
- If $B_i \geq v_i$, then $\pi_i \leq 0$
  - So assuming rationality, $B_i < v_i$

Why not $B_i \leq v_i$?

- Thus
  - $\pi_i = 0$ if $B_i \neq \max_j \{B_j\}$
  - $\pi_i = v_i - B_i$ if $B_i = \max_j \{B_j\}$
- How much below v_i should your bid be?
- The smaller B_i is,
  - the less likely that i will win the object
  - the more profit i will make if i wins the object

First-Price Sealed-Bid (case $n = 2$)

- Suppose your BV is v and your bid is B
- Let x be the other bidder's BV and αx be his/her bid, where $0 < \alpha < 1$
  - You don't know the values of x and α
- Your expected profit from bidding B is
  - $E(\pi) = \Pr[\text{your bid is higher}](v-B) + \Pr[\text{your bid is lower}](0)$
- x is uniformly distributed over $[0,100]$
  - $\Pr[\text{your bid is higher}] = P[\alpha x < B] = \Pr[x < B/\alpha]$
    $= (1/100)\ (B/\alpha) = B/100\alpha$
- So $E(\pi) = \Pr[\text{your bid is higher}](v-B)$
  $= (B/100\alpha)\ (v - B) = (Bv - B^2)/100\alpha$
- $E(\pi)$ is maximized when derivative is 0
  - $v - 2B = 0 \;\; \Rightarrow \;\; B = v/2$
- To maximize your expected profit, bid ½ of what the item is worth to you!

First-Price Sealed-Bid (continued)

- With n bidders, if your bid is B, then
  - If all other bidders have BVs uniformly distributed over [0,100]
    - Before, Pr[your bid is highest] was $B/100\alpha$
    - Now it's $(B/100\alpha)^{n-1}$
- Expected profit is $E(\pi) = (B^{n-1}/100\alpha)\,(v-B) = (vB^{n-1}-B^n)/100\alpha$
  - $E(\pi)$ is maximized when derivative = 0
    - $(n-1)B^{n-2}v - nB^{n-1} = 0$
    - $(n-1)v - nB = 0$
    - $B = v(n-1)/n$
- As n increases, $B \to v$
  - Increased competition drives bids close to the valuations
- Bayes-Nash Equilibrium for 1^{st}-price sealed-bid auctions:
  - each bidder i bids the expected highest value among i's rivals, conditional on i's own value being higher than all of the rivals' values

Dutch Auctions

- Examples: flowers in the Netherlands, fish market in England and Israel, tobacco market in Canada
- Typical rules
  - Auctioneer starts with a high price
  - Auctioneer lowers the price gradually, until some buyer shouts “Mine!”
  - The first buyer to shout “Mine!” gets the object at the price the auctioneer just called
  - Winner’s profit = BV – price
  - Everyone else’s profit = 0
- Game-theoretically equivalent to first-price, sealed-bid auctions
  - The object goes to the highest bidder at the highest price
  - A bidder must choose a bid without knowing the bids of any other bidders
  - The optimal bidding strategies are the same

Auction Design

- Two of the possible goals:

  (1) Pareto efficiency (Pareto optimal outcome):

  - The commodity should to the bidder i with the highest v_i
    - Suppose it goes to another bidder j with $v_j < v_i$
    - Then can make both i and j better off as follows:
      - Transfer the commodity from i to j
      - Have j pay i an amount between v_j and v_i

  (2) Profit maximization:

  - Highest expected profit to seller

Auction Design

- English auction does well at achieving both goals
  - Main drawback: bidders must make a long sequence of bids
  - Impractical in many cases
- Sealed-bid first price auction:
  - No buyer knows other buyers' valuations
  - Bidder with the highest valuation may bid too low and lose to another bidder

    => not Pareto efficient
- Dutch auction:
  - No buyer knows other buyers' valuations
  - Bidders don't want to claim the prize too early
  - Bidder with the highest valuation may delay too long and lose to another bidder

    => not Pareto efficient

Sealed-Bid, Second-Price Auctions

- Proposed by Vickrey in 1961; usually called a **Vickrey auction**
  - Same idea has been used in stamp collectors' auctions since 1893
- Other auctions that come close:
  - US Treasury's long-term bonds
  - eBay proxy bidding
- Typical rules
  - Bidders write their bids for the object and their names on slips of paper and deliver them to the auctioneer
  - The auctioneer opens the bid and finds the highest bidder
  - The highest bidder gets the object being sold, for a price equal to the *second highest* bid
- Winner's profit = BV – price
- Everyone else's profit = 0

Sealed-Bid, Second-Price (continued)

- Equilibrium bidding strategy: bid your true value

- **Proof**: show that bidding your true value is a weakly dominant strategy
- Let
  - v = your valuation of the object
  - X = the highest bid by anyone else
  - s_v = the strategy of bidding v
    - π_v = your profit when using s_v
  - s_B = a strategy that bids some $B \neq v$
    - π_B = your profit when using s_B
- Show that $\pi_B \leq \pi_v$ for all B, v, X
  - There are $3! = 6$ possible numeric orderings of B, v, and X
  - For each one, show that $\pi_B \leq \pi_v$

Sealed-Bid, Second-Price (continued)

- The 6 possible numeric orderings of B, v, and X:
  - $v < B < X$: you don't get the commodity, so $\pi_B = \pi_v = 0$.
  - $v < X < B$: $\pi_v = 0$, but $\pi_B = v - X < 0$
  - $X < v < B$: you pay X rather than your bid, so $\pi_B = \pi_v = v - X > 0$
  - $X < B < v$: you pay X rather than your bid, so $\pi_B = \pi_v = v - X > 0$
  - $B < X < v$: $\pi_B = 0$, but $\pi_v = v - X > 0$
  - $B < v < X$: you don't get the commodity, so $\pi_B = \pi_v = 0$
- $\pi_B \leq \pi_v$ in every case, so s_v is weakly dominant

- Equilibrium: everyone bids their true value
  - What kind of equilibrium is this?
    - Bayes-Nash? *ex post?*
- Equilibrium outcome is nearly the same as in English auctions
  - The object goes to the highest bidder
  - Price is the second highest BV

A Practical Problem

- Vickrey auctions don’t always go as planned
  - New Zealand, 1990 auction of electromagnetic spectrum
- One case:
  - Highest bid: NZ$100,000
  - Second-highest bid: NZ$6
- Another case:
  - Highest bid: NZ$7 million
  - Second-highest bid: NZ$5,000
- Why?
  - Only a few bidders, no minimum bid => poor profit for seller
- New Zealand's government has since amended its auction rules

Reserve Price

- Suppose there are 2 bidders for a commodity
  - Each buyer's valuation is \$20 with probability ½; \$50 with probability ½
- Probability ¼ for each of the following value profiles:
  - (\$20, \$20), (\$20, \$50), (\$50, \$20) and (\$50, \$50).

- English auction, minimum increment \$1, no reserve value
  - With probability ¼ each, winning bids will be \$20, \$21, \$21 and \$50
  - Seller's expected revenue is (\$20 + \$21 + \$21 + \$50)/4 = \$28

- English auction, minimum increment \$1, reserve value \$50
  - Probability 1/4 of no sale
  - Probability ¾ that the winning bid will be \$50
- Seller's expected revenue = ¾ (50) + ¼ (0) = \$37.5
- Probability ¼ of no trade => loss of Pareto efficiency

The Winner's Curse

- Consider a **common-value** auction
  - Item's ultimate value is the same for all bidders, but bidders are unsure what that ultimate value is
- Each bidder estimates the value and bids accordingly
  - Some overestimate, some underestimate
  - The largest overestimate ends up winning the auction
- A possible example: FCC 1996 spectrum auction
  - Largest bidder: $4.2 million, NextWave Personal Communications Inc
  - In January 1998 they went bankrupt – unable to pay their bills
- Optimal strategy:
  - Bid less than what you think the item is worth
  - How much less?

Summary

- Incomplete information vs. imperfect information
- Incomplete information vs. uncertainty about payoffs
- Bayesian games (three different definitions)
  - Changing uncertainty about games into uncertainty about payoffs
  - *Ex ante*, *ex interim*, and *ex post* utilities
  - Bayes-Nash equilibria
- Bayesian-game interpretations of Bridge and Backgammon
- Auctions and their equilibria
  - English, Dutch, sealed bid first price, sealed bid second price (Vickrey)
- Auction design
  - Pareto efficiency, profit maximization
  - Reserve price, winner's curse