CMSC 474, Game Theory

5. Imperfect-Information Games

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Motivation

- So far, we've assumed that players in an extensive-form game always know what node they're at
  - Know all prior choices
    - Both theirs and the others'
  - Thus "perfect information" games
- But sometimes players
  - Don't know all the actions the others took or
  - Don't recall all their past actions
- Extend extensive-form game representation to include this

Definition

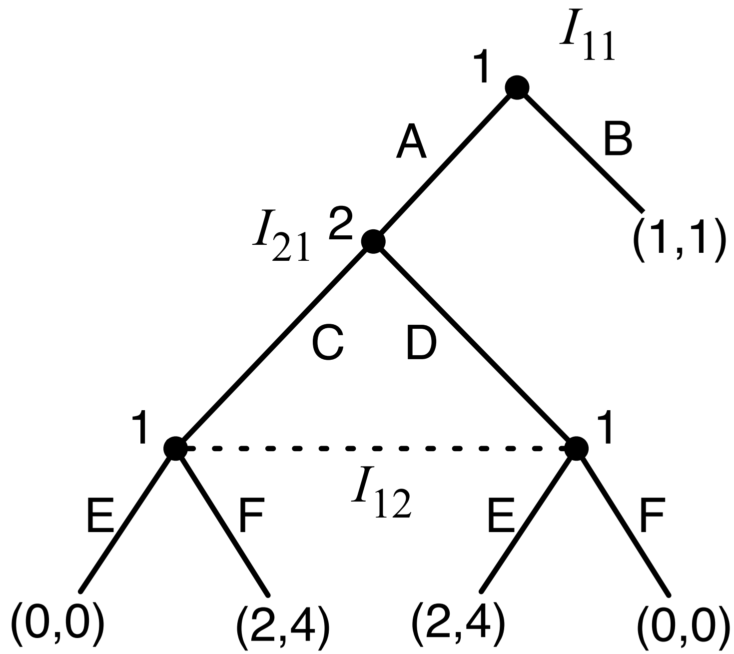
- **Imperfect-information** game: extensive-form game in which each agent's choice nodes are partitioned into **information sets**
  - An information set = {all choice nodes an agent ***might*** be at}
- Let H = {all nodes where it's agent i's move}
  - Agent i's information sets are $I_{i1}, \ldots, I_{im}$ for some m, where
    - $I_{i1} \cup \ \ldots \cup I_{im} = H$
    - $I_{ij} \cap I_{ik} = \varnothing$ for all $j \neq k$

What is this called?

- If h and h' are in the same information set, they are **indistinguishable** to i
  - So they must have the same set of actions
    - $\chi(h) = \chi(h')$
  - But the actions may have different outcomes
- A perfect-information game is a special case
  - Each I_{ij} contains just one node

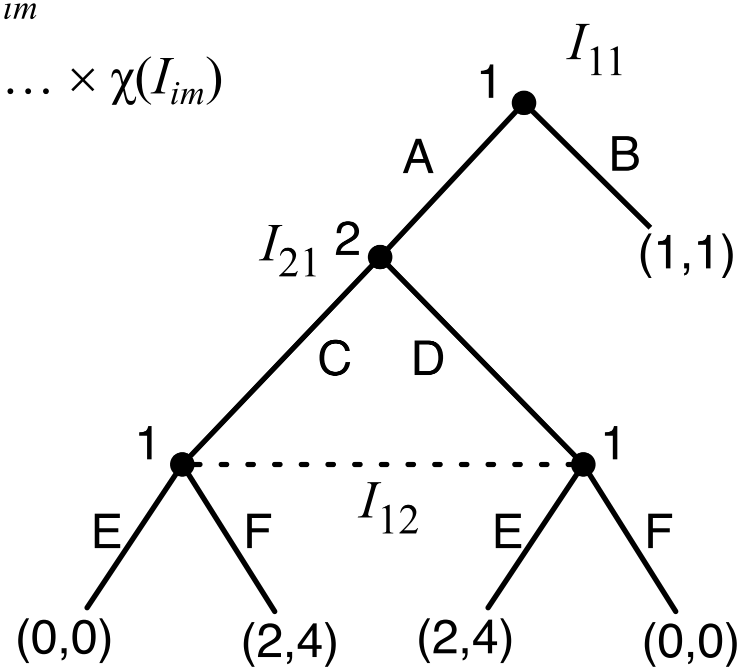
Example

- Agent 1 has two information sets
  - I_{11} and I_{12}
- In I_{12} , agent 1 doesn't know whether agent 2 chose C or D
- Agent 2 has just one information set
  - I_{21}



Strategies

- **Pure strategy** for agent i
  - a function s_i telling what action to take in each of i's information sets
  - $s_i(I)$ = agent i's action in information set I
- Suppose i has information sets $I_{i1}, \ldots, I_{im}$
  - {all pure strategies for i} = $\chi(I_{i1}) \times \ldots \times \chi(I_{im})$

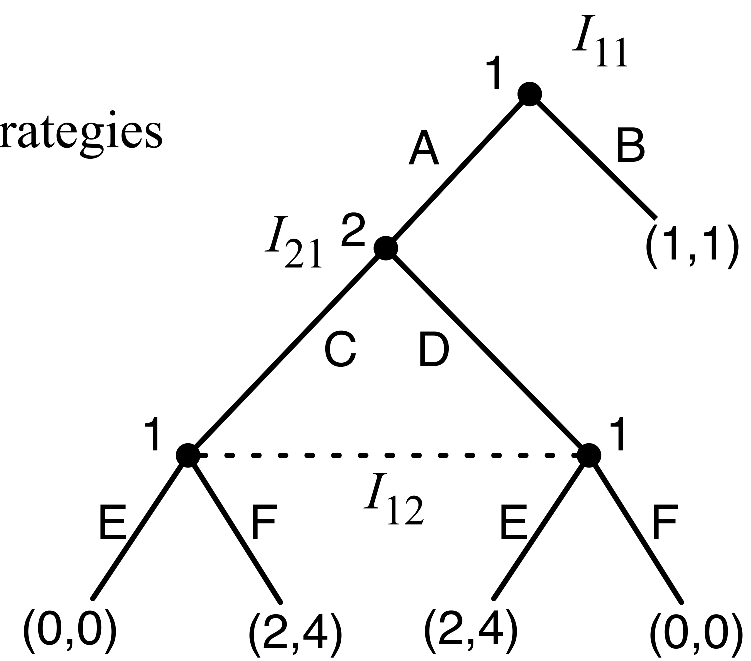


- Agent 1's pure strategies:
  - $\{A,B\} \times \{E, F\}$ =
    $\{(A,E), (A,F), (B,E), (B,F)\}$
- Agent 2's pure strategies: {C, D}

Extensive Form → Normal Form

- Can transform any extensive-form imperfect-information game into an equivalent normal-form game
  - Same strategies and same payoffs
  - Thus same Nash equilibria, same Pareto optimal strategy profiles, etc.
- Just like we did it for perfect-information games
  - n-dimensional payoff matrix
  - i'th dimension ⇔ agent i's pure strategies

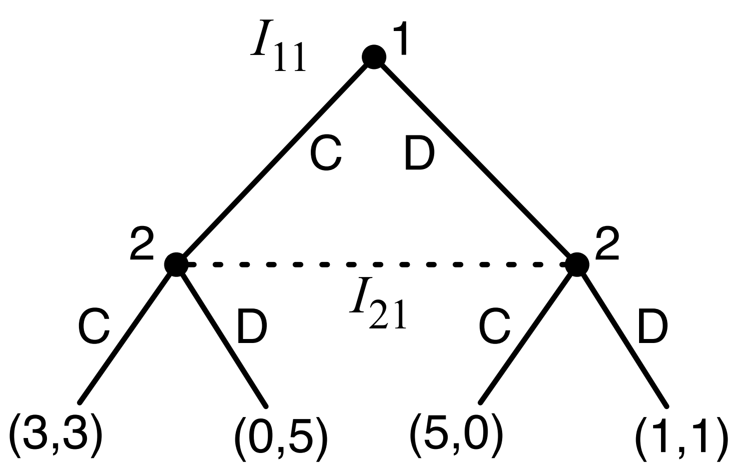
	C	D
(A,E)	0, 0	2, 4
(A,F)	2, 4	0, 0
(B,E)	1, 1	1, 1
(B,F)	1, 1	1, 1

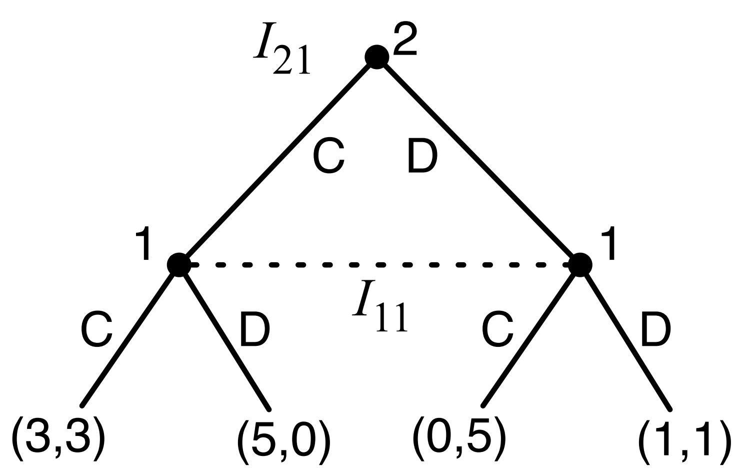


Normal Form → Extensive Form

- Can translate any normal-form game into an equivalent extensive-form imperfect-information game
  - n-level game tree, one level for each agent
  - each agent has exactly one information set
- Same strategies, payoffs, Nash equilibria, Pareto optimal strategy profiles, etc.
- Example: Prisoner's Dilemma
  - Two equivalent game trees

	C	D
C	3, 3	0, 5
D	5, 0	1, 1



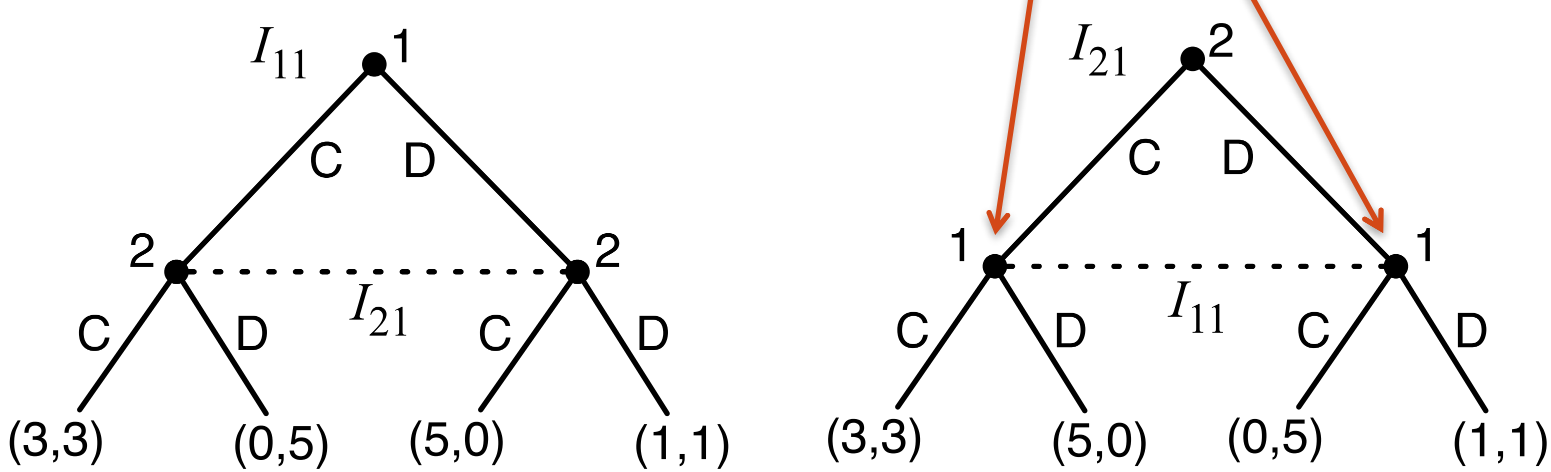


Different Kinds of Strategies

- **Pure strategy**:
  - for each information set an agent makes the same move at all nodes in the information set
  - E.g., agent 1 choosing D in the normal form game
    - same as choosing D at both of these nodes

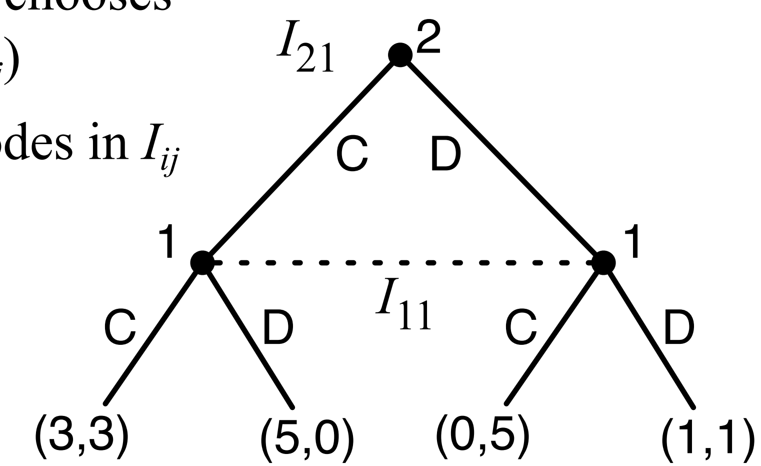
	C	D
C	3, 3	0, 5
D	5, 0	1, 1

- **Mixed strategy**:
  - Does the agent make the same move at all nodes in the information set?



Different Kinds of Strategies

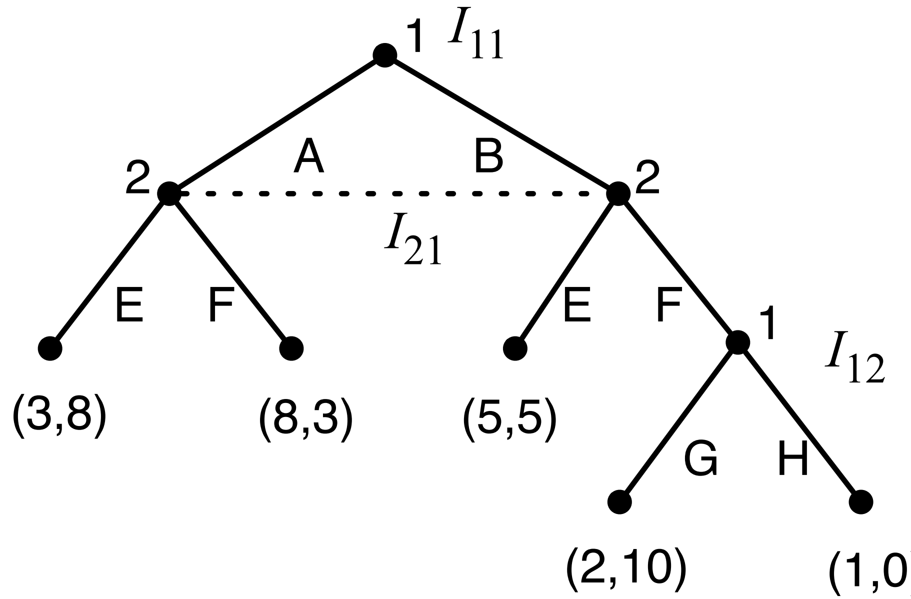
- New class of strategies: **behavioral strategies**
  - Suppose agent i has behavioral strategy s_i
  - Each time i is in information set I_{ij}, he/she chooses from the same probability distribution $s_i(I_{ij})$
    - *independently* of i's choices at other nodes in I_{ij}
  - E.g., flip coin every time you're in I_{ij}



- In some games, every behavioral strategy has an **equivalent** mixed strategy, and vice versa
    - Same probability distributions over outcomes
  - Example
    - Behavioral strategy for Agent 1: (I_{11}, {(0.3, C), (0.7, D)})
    - Mixed strategy is basically the same: {(0.3, C), (0.7, D)}
  - More examples later

Example 2

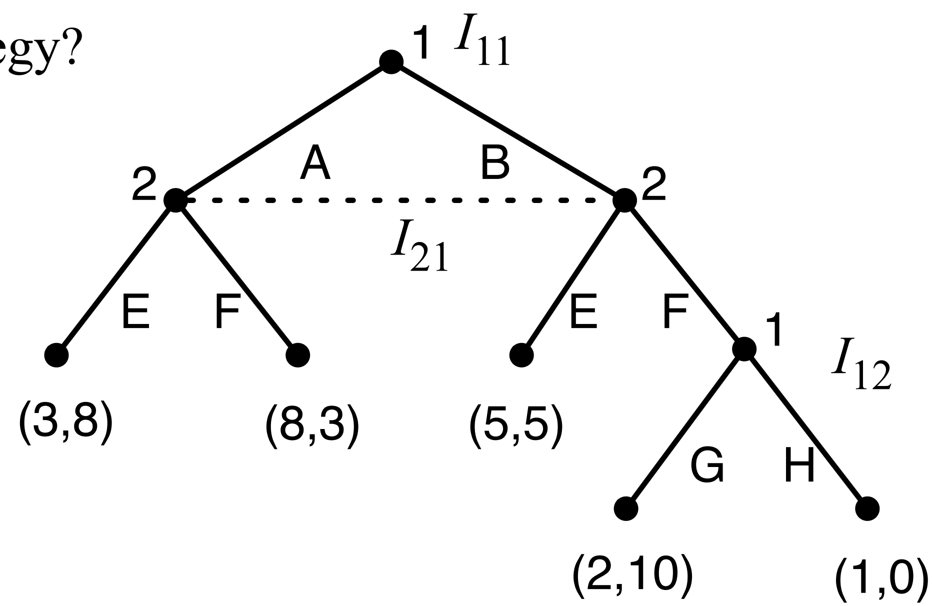
- A behavioral strategy for Agent 1:
  - In I_{11}, {(0.4, A), (0.6, B)}
  - In I_{12}, {(0.3, G), (0.7, H)}



- An equivalent mixed strategy:
  - {(0.12, (A,G)),
    (0.28, (A,H)),
    (0.18, (B,G)),
    (0.42, (B,H))}
  - How did I get those numbers?

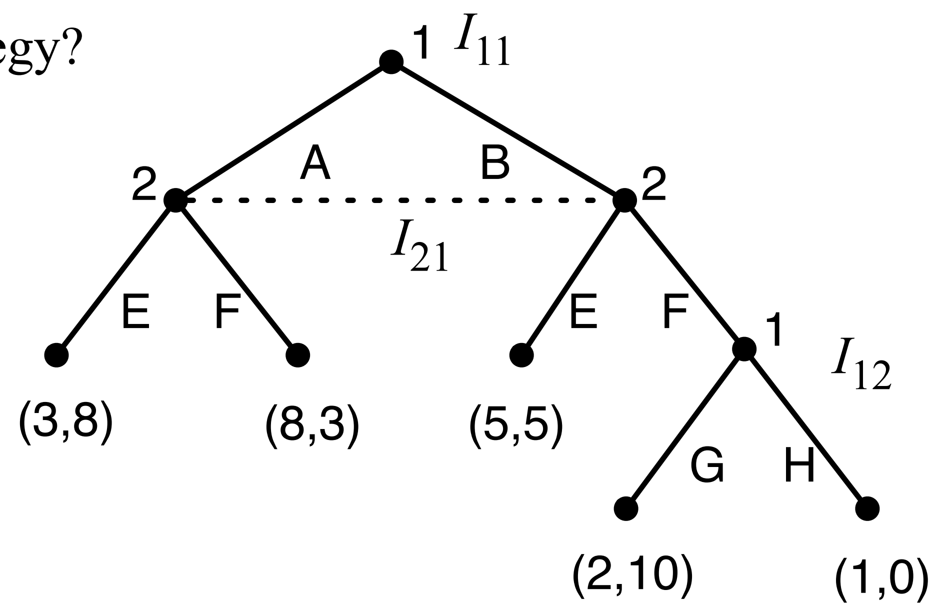
Example 3

- A mixed strategy for agent 1:
  - {(0.6, (A,G)), (0.4, (B,H))}
- The choices in the two information sets aren't independent
  - Choose A in I_{11} ⇔ choose G in I_{12}
  - Choose B in I_{11} ⇔ choose H in I_{12}
- Is there an equivalent behavioral strategy?



Example 3

- A mixed strategy for agent 1:
  - {(0.6, (A,G)), (0.4, (B,H))}
- The choices in the two information sets aren't independent
  - Choose A in I_{11} ⇔ choose G in I_{12}
  - Choose B in I_{11} ⇔ choose H in I_{12}
- Is there an equivalent behavioral strategy?
  - In I_{11}, {(p, A), ($1-p$, B)}
  - In I_{12}, {(q, G), ($1-q$, H)}
- Look for p and q that give the same probabilities of outcomes
  - $\Pr[A] = p = 0.6$
  - $\Pr[(B,G)] = (1-p)\, q = 0$
  - $\Pr[(B,H)] = (1-p)\,(1-q) = 0.4$

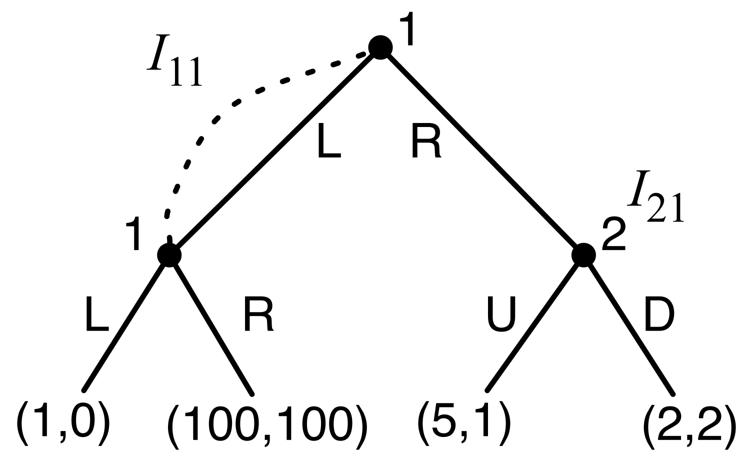


Behavioral vs. Mixed Strategies

- In some games, there are
  - mixed strategies that have no equivalent behavioral strategy
  - behavioral strategies that have no equivalent mixed strategy

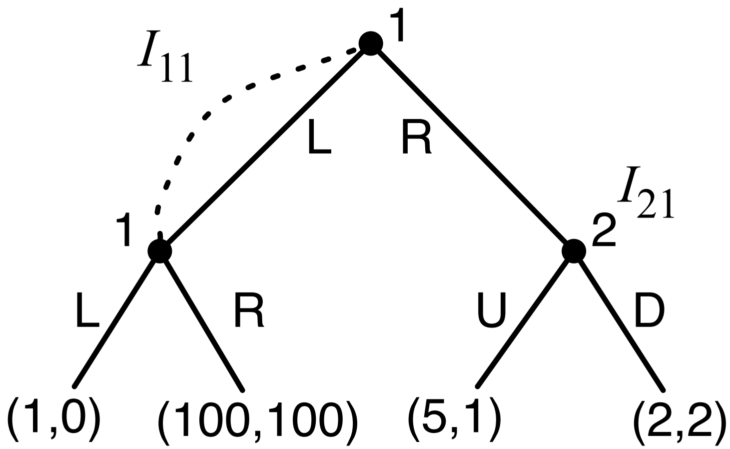
- Thus mixed and behavioral strategies can produce different Nash equilibria

- Example:
  - Both of Agent 1's choice nodes are in the same information set, I_{11}
    - How could this ever happen?



Behavioral vs. Mixed Strategies

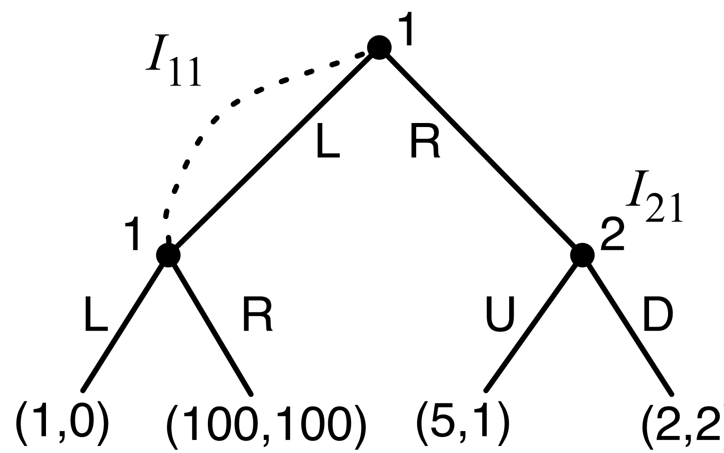
- Mixed strategy $\{(p, \text{L}), (1-p, \text{R})\}$
  - agent 1 chooses L or R randomly, but commits to it
  - ➢ Choose L ⇒ history ⟨L,L⟩
  - ➢ Choose R ⇒ history ⟨R,U⟩ or ⟨R,D⟩
  - ➢ **never** ⟨L,R⟩



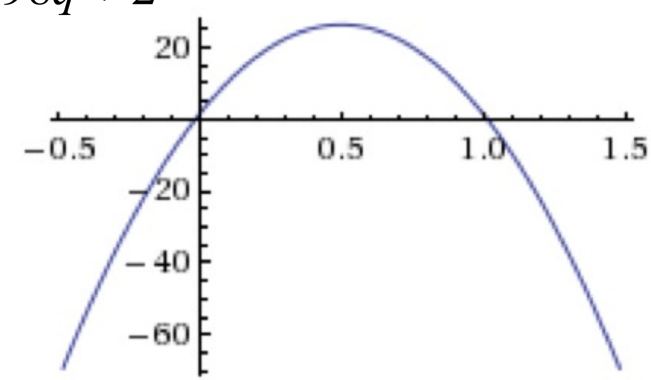
- Nash equilibrium in mixed strategies:
  - For agent 1, R is strictly dominant
  - For agent 2, D is strictly dominant
  - ➢ So (R,D) is the unique Nash equilibrium

Behavioral vs. Mixed Strategies

- Behavioral strategy $(I_{11}, \{(q, \text{L}), (1-q, \text{R})\})$
    - Remake the choice each time agent 1 is in I_{11}
  - If $p = q = 0$, have the pure strategy L
  - If $p = q = 1$, have the pure strategy R
  - In all other cases, $\Pr[\langle \text{L,R} \rangle] = q(1-q) > 0$



- Nash equilibrium in behavioral strategies:
  - For 2, D is strictly dominant
    - Find 1's best response among behavioral strategies
  - Suppose 1's behavioral strategy is $(I_{11}, \{(q, L), (1-q, \text{R})\})$
    - $u_1 = 1\,q^2 + 100\,q(1-q) + 2\,(1-q) = -99q^2 + 98q + 2$
    - Maximum is where $du_1/dq = 0$
      - $-198q + 98 + 0 = 0$
      - $q = 49/99$



- Equilibrium is $(\{(49/99, \text{L}), (50/99, \text{R})\}, \text{D})$

Games of Perfect Recall

- The reason the strategies weren't equivalent was because agent 1 could be in the same information set more than once
  - Mixed strategy ⇒ agent 1 will make the same move every time
  - Behavioral strategy ⇒ agent 1 may make a different move each time
    - Like mixed strategy + faulty memory

- Look at games where agents have perfect memories
  - Agent *i* has **perfect recall** if *i* never forgets anything *i* knew earlier
  - *G* is a **game of perfect recall** if every agent in *G* has perfect recall

Theorem: For every history in a game of perfect recall, no agent will have the same information set more than once

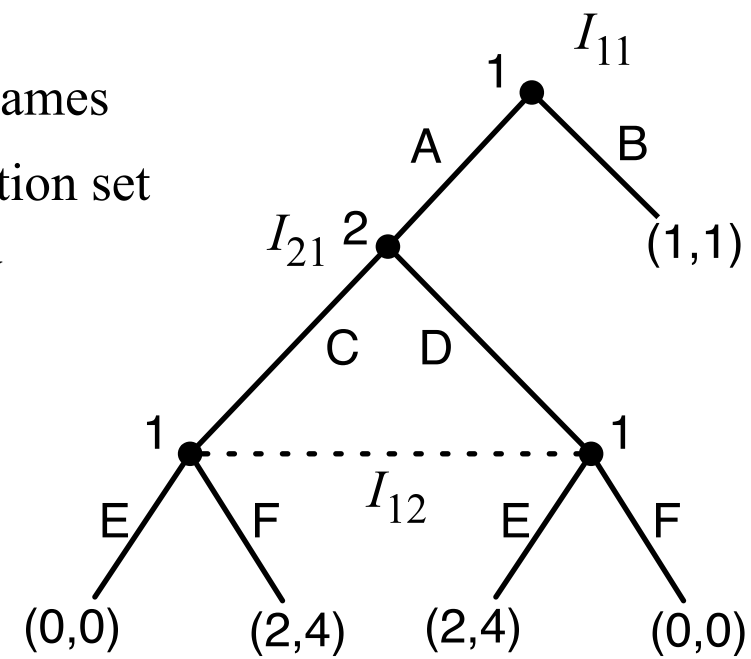
Games of Perfect Recall

- **Theorem:** For every history in a game of perfect recall, no agent will have the same information set more than once
- **Proof:** Let h be any history for G. Suppose that
    - At one point in h, i's information set is I
    - At another point later in h, i's information set is J
  - Then i must have made at least one move in between
  - If i remembers all his/her moves, then
    - At J, i remembers a longer sequence of moves than at I
    - Thus I and J are different information sets

- **Theorem** (Kuhn, 1953). In a game of perfect recall, for every mixed strategy there is an equivalent behavioral strategy, and vice versa
- **Corollary:** In a game of perfect recall, the set of Nash equilibria doesn't change if we consider behavioral strategies instead of mixed strategies

Sequential Equilibrium

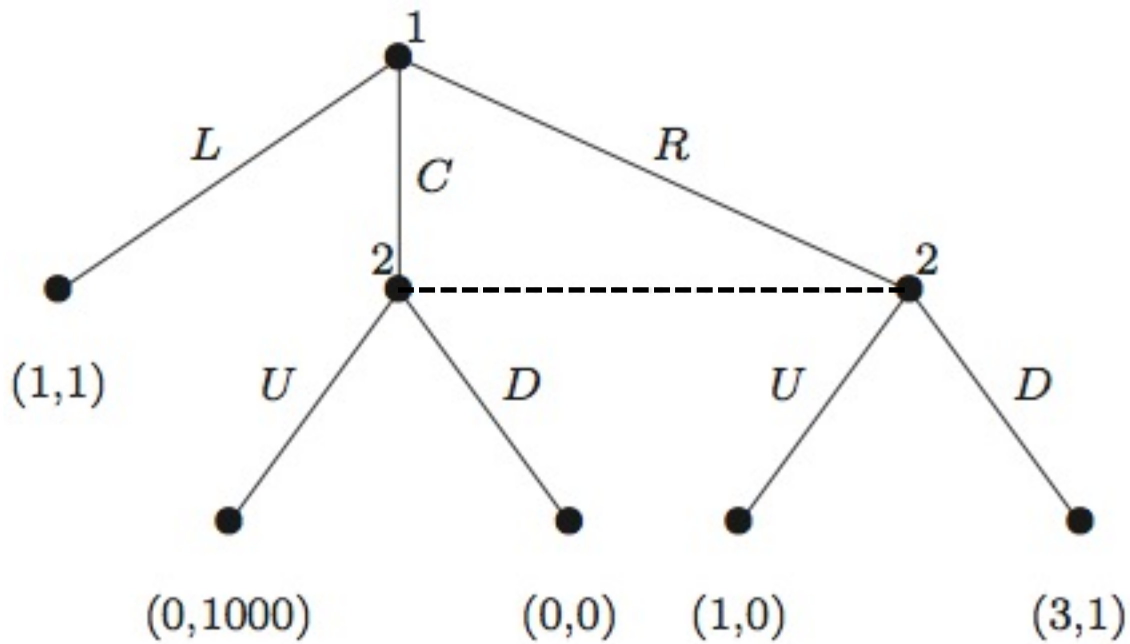
- For perfect-information games, subgame-perfect equilibria were useful
  - Avoided non-credible threats; could be computed more easily
  - Each agent's strategy must be a best response in every subgame
- Generalize to imperfect-information games?
- Information set ⇔ a set of possible subgames
  - One for each element of the information set
  - Could we require an agent's strategy to be a best response in all of the subgames?



Example

- No strategy is a best response to both C and R

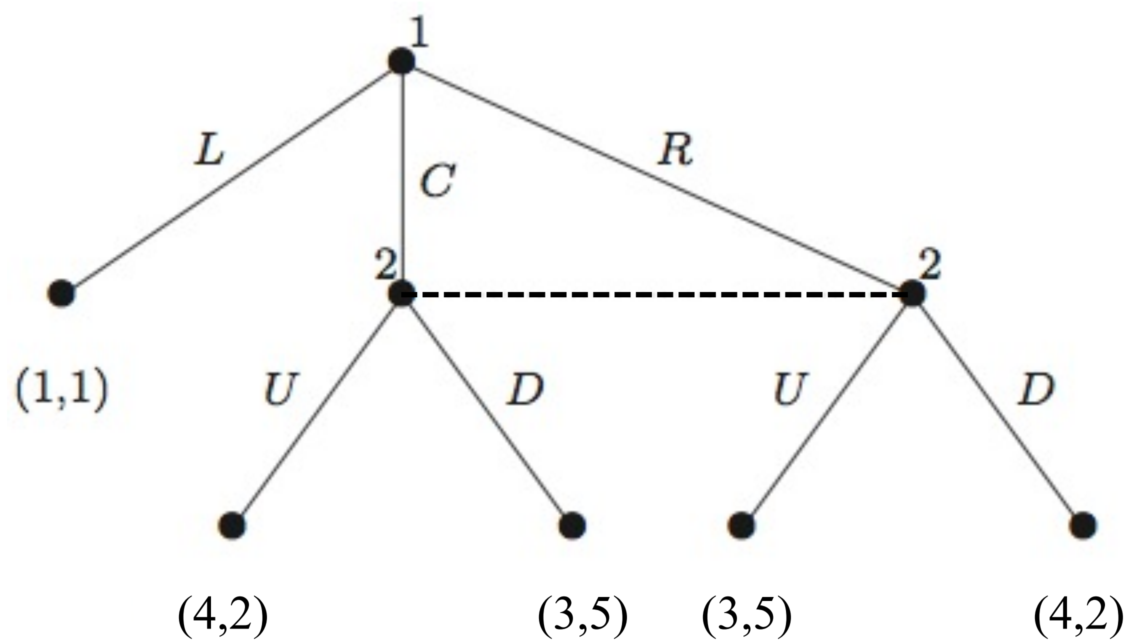
- Assume common knowledge of rationality
  - 1 will never choose C
  - 2 only needs a best response to R



Example

- No strategy is a best response to both C and R
- Assume common knowledge of rationality
  - 1 will never choose L
- Suppose the agents' mixed strategies are
  - $s_1 = \{(p, C), (1\!-\!p, R)\}$

    and
  - $s_2 = \{(q, U), (1\!-\!q, D)\}$
- Can show there is one Nash equilibrium, at $p = q = ½$
  - But $q = ½$ is not a best response to either C or R



Sequential Equilibrium

- In general, need Bayesian reasoning about the players' strategy profiles
- This leads to a complicated solution concept called **sequential equilibrium**
  - A little like a trembling-hand perfect equilibrium, but with additional complications to deal with the tree structure

Definition 5.3.1 (Sequential equilibrium). *A strategy profile S is a* sequential equilibrium *of an extensive-form game G if there exist probability distributions $\mu(h)$ for each information set h in G, such that the following two conditions hold:*

1. $(S, \mu) = \lim_{n\to\infty}(S^n, \mu^n)$ *for some sequence* $(S^1, \mu^1), (S^2, \mu^2), \ldots$, *where S^n is fully mixed, and μ^n is consistent with S^n (in fact, since S^n is fully mixed, μ^n is uniquely determined by S^n); and*
2. *For any information set h belonging to agent i, and any alternative strategy S'_i of i, we have* $S \mid h, \mu(h)) \geq u_i((S', S_{-i}) \mid h, \mu(h))$.

- Every finite game of perfect recall has a sequential equilibrium
- Every subgame-perfect equilibrium is a sequential equilibrium, but not vice versa
- I won't discuss it further

Summary

- Topics covered:
  - information sets
  - behavioral vs. mixed strategies
  - games of perfect recall
    - equivalence between behavioral and mixed strategies in such games
  - very brief discussion of sequential equilibrium