CMSC 474, Game Theory

1. Introduction

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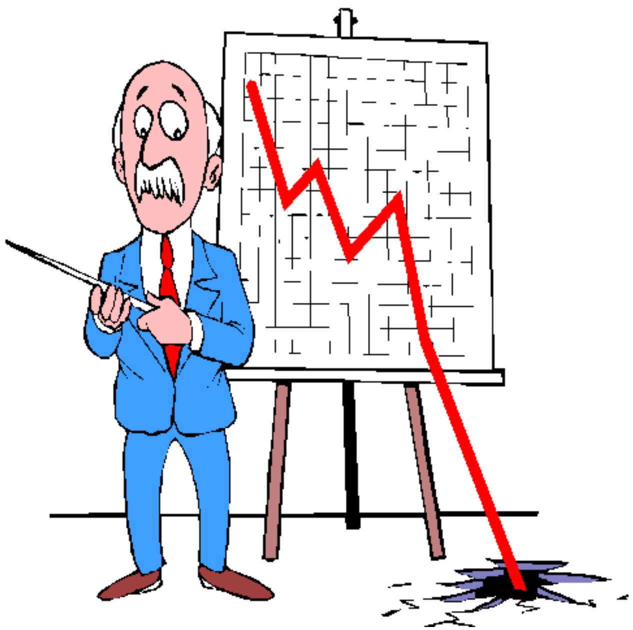
This lecture covers Chapter 1 of the textbook, plus several related topics

What is Game Theory?

- Game theory is about interactions among **agents** (or **individuals** or **players**) that are **self-interested**:
  - Different agents have different **preferences**
  - They like some outcomes more than others

Fields where Game Theory is Used

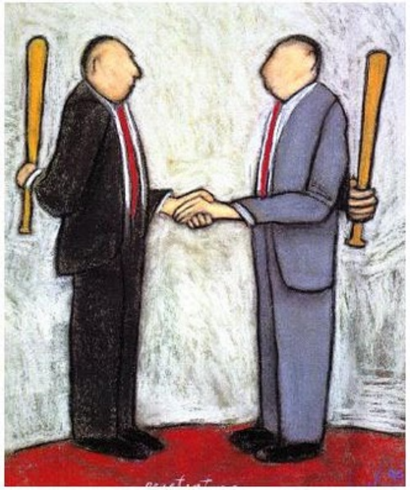
- Studied mainly by mathematicians and economists
  - Businesses, markets, auctions, economic predictions, bargaining, fair division

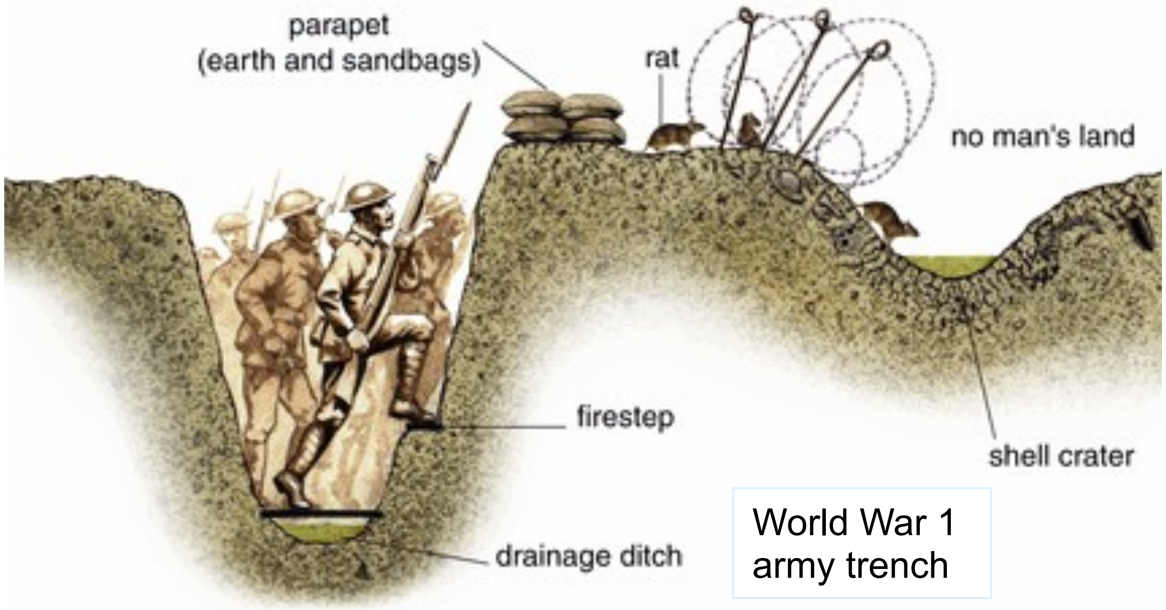


- Increasingly useful in other areas

Fields where Game Theory is Used

- Government, politics, military
  - Negotiations
  - Voting systems
  - International relations
  - Conflicts



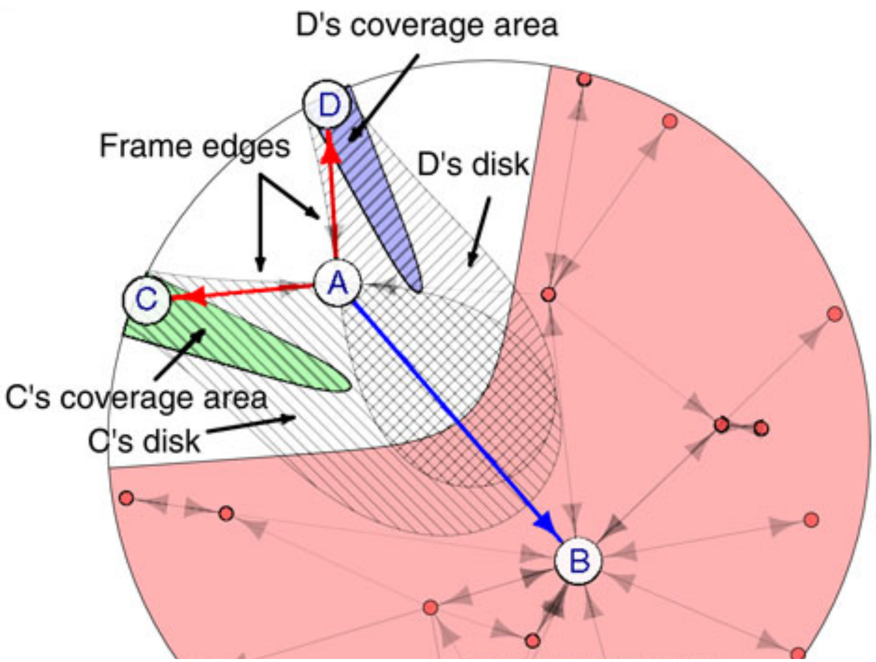


Fields where Game Theory is Used

- Biology, psychology, sociology
  - Population ratios, territoriality
  - Parasitism, symbiosis
  - Social behavior

Fields where Game Theory is Used

- Engineering, computer science
  - Computer game programs
  - Multi-agent systems
  - Communication networks, computer networks, road networks



Example

- I need two volunteers to play a game
  - Two people who don't know each other

Example

- I need two volunteers to play a game
  - Two people who don't know each other

- Instructions
  - Don't talk to each other
  - Come to the front of the room
  - Face opposite directions

- The rest of you:
  - Get out your computer or smartphone, and login to Piazza
  - In a moment I'm going to ask you to do a poll

Example

- I need two volunteers to play a game
  - Two people who don’t know each other

- Choose one of these actions, but keep your choice secret
  - *Take*: take 1 chocolate to keep for yourself
  - *Give*: take 3 chocolates to give to the other player

Games in Normal Form

- A (finite, n-person) **normal-form game**:
  1. An ordered set $N = (1, 2, 3, \ldots, n)$ of **agents** or **players**:
  2. For each agent i, a finite set A_i of possible actions
     - An **action profile** is an n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, where each $a_i \in A_i$
     - The set of all possible action profiles is $\mathbf{A} = A_1 \times \cdots \times A_n$
  3. For each agent i, a real-valued **utility** (or **payoff**) function

     $u_i\,(a_1, \ldots, a_n) = i$'s payoff if the action profile is $(a_1, \ldots, a_n)$
- Usually represented by an n-dimensional **payoff** (or **utility**) **matrix**
  - for each action profile, shows the utilities of all the agents
- Most other game representations can be reduced to normal form

	give	take
give	3, 3	0, 4
take	4, 0	1, 1

The Chocolate Dilemma

- Actions:
  - *Take*: take 1 to keep for yourself
  - *Give*: take 3 to give to the other player
- Payoff matrix:

		Player 2: give	Player 2: take
Player 1:	give	3, 3	0, 4
	take	4, 0	1, 1

- http://theoryclass.wordpress.com/2010/03/05/the-chocolate-dilemma/

Poll 1.1

- Actions:
  - *Take*: take 1 to keep for yourself
  - *Give*: take 3 to give to the other player

		Player 2: give	Player 2: take
Player 1:	give	3, 3	0, 4
	take	4, 0	1, 1

- Go to piazza.com and answer the following poll:
  - Suppose you're player 1. Which action will maximize the number of chocolates you get?
    - A. give
    - B. take
    - C. depends on which action player 2 chooses

Game-Theoretic Answer

- Regardless of what the other player does,
take gets you one more chocolate than *give* does
  - *take* is a **dominant strategy**

	give	take
give	3, 3	0, 4
take	4, 0	1, 1

- Suppose that—
  - Both players are decision-theoretically rational
  - The ***only*** thing each player cares about is to get as many chocolates as possible
  - Those things are common knowledge* to both players
- Then each player will choose *take*
  - If they can talk to each other beforehand, they'll still choose *take*
  - Repeat any fixed number of times => they'll still choose *take*
  - Repeat an unbounded number of times => they might choose *give*
- Is this realistic?

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*Complicated topics; I'll discuss later

Chocolate-Dilemma Survey Results

31 people answered these survey questions in Fall 2014

	In each of the following circumstances, which action would you choose?	*% Take*	*% Give*
1	The other player is a stranger whom you'll never meet again.	68	32
2	The other player is an enemy.	90	10
3	The other player is a friend.	10	90
4	The other player is a computer program instead of a human.	94	6
5	You haven't eaten in two days.	97	3
6	*Take* means you take two chocolates instead of just one.	87	13
7	You and the other player can discuss what choices to make.	19	81
8	You will be playing the game repeatedly with the same person.	23	77
9	Thousands of people are playing the game anonymously. Nobody will ever know which of the others is the one they're playing the game with.	74	26
10	Thousands of people are playing the game anonymously. *Give* means the three chocolates go to a collection that will be divided equally among everyone.	23	77
11	The bag is filled with money. *Take* means you take $2500 and keep it. *Give* means you take $3000 to give to the other player.	100	0

The Prisoner's Dilemma

- Scenario: The police are holding two prisoners as suspects for committing a crime
  - For each prisoner
    - The police have enough evidence for a 1 year prison sentence
    - They want to get enough evidence for a 4 year prison sentence
  - They tell each prisoner,
    - "If you testify against the other prisoner, we'll reduce your prison sentence by 1 years"
  - *C* = *Cooperate* (with the other prisoner): refuse to testify against him/her
  - *D* = *Defect*: testify against the other prisoner
  - Both prisoners cooperate => both go to prison for 1 year
  - One defects, other cooperates => defector goes free; cooperator goes to prison for 4 years
  - Both prisoners defect => both go to prison for 4 – 1 = 3 years

	C	*D*
C	–1, –1	–4, 0
D	0, –4	–3, –3

Prisoner's Dilemma

- The numbers we used:

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

- Chocolate dilemma
- Equivalent, just add 4

	give	take
give	3, 3	0, 4
take	4, 0	1, 1

- Commonly used numbers:
  - Still equivalent

	C	D
C	3, 3	0, 5
D	5, 0	1, 1

- General form:

  $t > r > p > s$

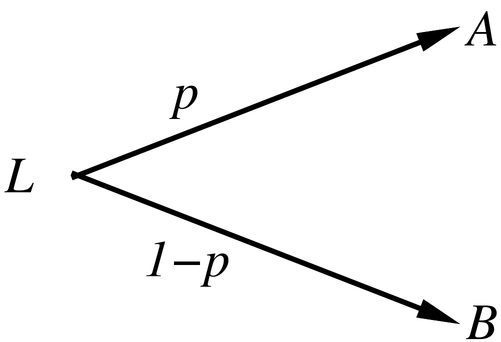
  $2r > s + t$

	C	D
C	r, r	s, t
D	t, s	p, p

Preferences

- Game-theoretic utilities are based on **preferences**
- Consider an agent that can choose among
  - **prizes** (A, B, etc.), and
  - **lotteries** (situations with uncertain prizes)

- Lottery $L = \{(p, A), (1-p, B)\}$
  - Probability p of getting prize A,
  - Probability $1-p$ of getting prize B



- Notation:
  - $A \succ B$ agent prefers A to B
  - $A \sim B$ agent is indifferent between A and B
  - $A \succeq B$ $A \succ B$ *or* $A \sim B$

Rational Preferences

- Idea: the preferences of a rational agent must obey some constraints
- Agent's choices are based on rational preferences
  $\Rightarrow$ agent's behavior is describable as maximization of expected utility
- Constraints:

Orderability (sometimes called **Completeness**):

$$(A \succ B) \lor (B \succ A) \lor (A \sim B)$$

Transitivity:

$$(A \succ B) \land (B \succ C) \Rightarrow (A \succ C)$$

Continuity:

$$A \succ B \succ C \Rightarrow \exists p \;\; B \sim \{(p, A), (1-p, C)\}$$

Substitutability (sometimes called **Independence**):

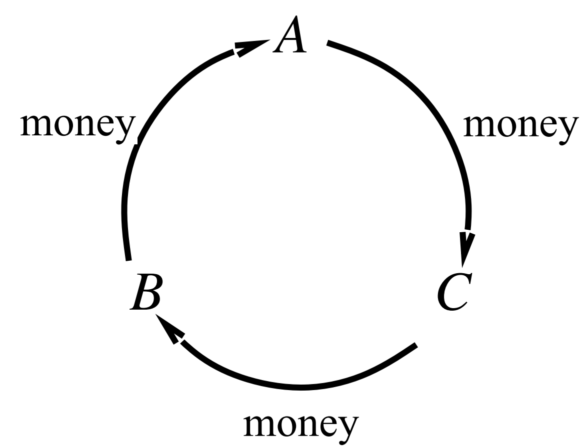
$$A \sim \mathrm{B} \Rightarrow \{(p, A), (1-p, C)\} \sim \{(p, B), (1-p, C)\}$$

Monotonicity:

$$A \succ B \Rightarrow (p \geq q \Leftrightarrow \{(p, A), (1-p, B)\} \succeq \{(q, A), (1-q, B)\}$$

Rational Preferences

- What happens if the constraints are violated?
- Example: ***intransitive preferences***
  - Suppose an agent's preferences are
    - $B \succ C, \ A \succ B, \ C \succ A$
  - If agent has C, will trade C and some money to get B
  - If agent has B, will trade B and some money to get A
  - If agent has A, will trade A and some money to get C



Rational Preferences

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- Example: ***intransitive preferences***
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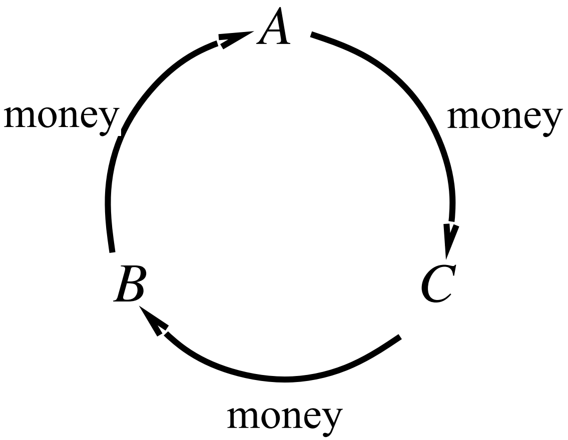
  - If agent has C, will trade C and some money to get B
  - If agent has B, will trade B and some money to get A
  - If agent has A, will trade A and some money to get C
- Self-evident irrationality
  - The agent can be induced to give away all its money



Utility Functions

- **Theorem** (Ramsey, 1931; von Neumann and Morgenstern, 1944).
- If the preferences satisfy the constraints, then there is a real-valued **utility function** u such that

$$u(A) \geq u(B) \;\Leftrightarrow\; A \succeq B$$

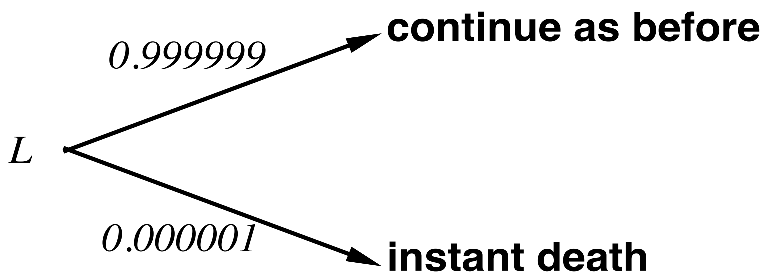
$$u(\{(p_1, A_1), \;\ldots, (p_n, A_n)\}) \;=\; \Sigma_i \, p_i \, u(A_i)$$

- **Maximum Expected Utility (MEU) principle**:
  - If an agent's choices are based on rational preferences, then its behavior is describable as maximization of expected utility
- An agent can maximize the expected utility without ever representing or manipulating utilities and probabilities
  - E.g., a lookup table to play tic-tac-toe perfectly

Human Utilities

- Standard approach to assessing human utilities:
  - Compare a given state s to a **standard lottery** L_p that has
    - best possible outcome $u_{\max}$ with probability p
    - worst possible outcome $u_{\min}$ with probability $1 - p$
  - Adjust lottery probability p until $s \sim L_p$
- How much would you pay to avoid a 1/1,000,000 chance of death?
- State s: continue as before

Lottery:



- **Poll 1.2:** how much would you be willing to pay to avoid the lottery?
  - \$10?
  - \$100?
  - \$1000?
  - \$10,000?
  - \$100,000?
  - more?

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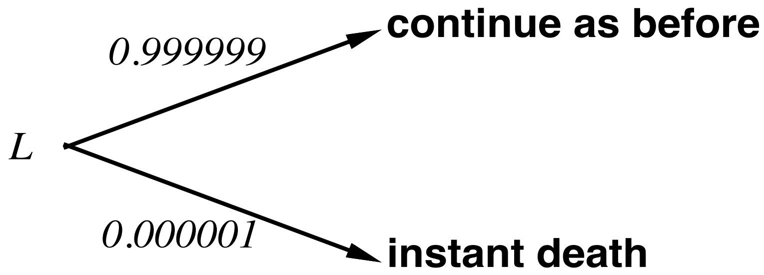
Lottery:



- 1/1,000,000 chance of death = one **micromort**

  ≈ Probability of accidental death in 230 miles of car travel

  ≈ Probability of accidental death in 6000 miles of train travel
- Judging from people's actions, they will pay about \$50 to avoid it

What we've covered so far

- Basic concepts:
  - normal form, pure strategies, mixed strategies, expected utility
- Not in book:
  - How utilities relate to rational preferences
  - Relationship to human decision making

Decision Making Under Risk

- **Poll 1.3:**
- Which lottery would you choose?
  - A: 100% chance of getting $3000
  - B: 80% chance of getting $4000; 20% chance of getting nothing

Decision Making Under Risk

- **Poll 1.4:**
- Which lottery would you choose?
  - C: 100% chance of losing $3000
  - D: 80% chance of losing $4000; 20% chance of losing nothing

Decision Making Under Risk

- Kahneman & Tversky, 1979
  - http://www.econport.org/econport/request?page=man_ru_advanced_prospect
- A: 100% chance of receiving \$3000
- B: 80% chance of getting \$4000; 20% chance of getting nothing
  - EV(A) = \$3000 < EV(B) = \$3200, but most people would choose A
  - For prospects involving gains, we're **risk-averse**
- C: 100% chance of losing \$3000
- D: 80% chance of losing \$4000; 20% chance of losing nothing
  - EV(C) = –\$3000 > EV(D) = –\$3200, but most people would choose D
  - For prospects involving losses, we're **risk-prone**
- Either money isn't a utility function, or our preferences aren't rational, or both

Anchoring

- Influence of irrelevant information on human judgment

    [D. Kahneman and A. Tversky (1974). Judgment under Uncertainty: Heuristics and Biases. *Science* **185**:4157, 1124–1131.]

    - Each subject first spun a wheel that supposedly would stop at random on any number between 1 and 100.
    - Then the subject was asked what percentage of African countries belong to the United Nations.

- For one group of subjects, the wheel was rigged to stop on 10.
    - On average, these subjects guessed 25%
- For a second group, the wheel was rigged to stop on on 65.
    - On average, these subjects guessed 45%

Utility Scales

- Rational preferences are invariant with respect to **positive affine** (or **positive linear**) transformations
- Let

$$u'(x) = c\,u(x) + d$$

c and d are constants, $c > 0$

  - Then u' models the same set of preferences that u does

- **Normalized utilities:**
  - define u such that $u_{\max} = 1$ and $u_{\min} = 0$

Utility Scales for Games

- Suppose all the agents have rational preferences, and that this is common knowledge*
- Then games are insensitive to positive affine transformations of the payoffs
  - Let c and d be constants, $c > 0$
  - For one or more agents, replace each payoff x with $cx + d$
  - Both players still have the same preferences

	b_1	b_2
a_1	$x_1,\ y_1$	$x_2,\ y_2$
a_2	$x_3,\ y_3$	$x_4,\ y_4$

	b_1	b_2
a_1	$cx_1+d,\ y_1$	$cx_2+d,\ y_2$
a_2	$cx_3+d,\ y_3$	$cx_4+d,\ y_4$

	b_1	b_2
a_1	$cx_1+d,\ ey_1+f$	$cx_2+d,\ ey_2+f$
a_2	$cx_3+d,\ ey_3+f$	$cx_4+d,\ ey_4+f$

*Complicated topic; I'll discuss later

Examples

- Is this a positive affine transformation?

	C	D
C	–2, –2	–5, 0
D	0, –5	–3, –3

→ add 5 to every payoff →

	C	D
C	3, 3	0, 5
D	5, 0	1, 1

- Is this?

	C	D
C	3, 3	0, 4
D	4, 0	1, 1

→ change 4 to 5 →

	C	D
C	3, 3	0, 5
D	5, 0	1, 1

Several different kinds of games

- Classified by their payoff matrices

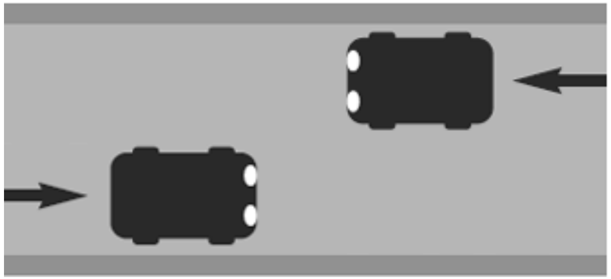
Common-payoff Games

- **Common-payoff game**:
  - For every action profile, all agents have the same payoff

	C	D
A	w, w	x, x
B	y, y	z, z

- Also called a **pure coordination** game or a **team game**
  - Need to coordinate on an action that is maximally beneficial to all

- **Which Side of the Road?**
  - 2 people driving toward each other in a country with no traffic rules
  - Each driver independently decides whether to stay on the left or the right
  - Need to coordinate your action with the action of the other driver



	Left	Right
Left	1, 1	0, 0
Right	0, 0	1, 1

A Brief Digression

	Left	Right
Left	2, 2	0, 0
Right	0, 0	1, 1

	Left	Right
Left	1, 1	0, 0
Right	0, 0	2, 2

- **Mechanism design**: design the rules and payoffs to give the agents an incentive to choose a desired outcome
- E.g., the law says what side of the road to drive on
  - Sweden, September 3, 1967:

Zero-sum Games

- These games are purely competitive

- **Constant-sum** game:
  - For every action profile, the sum of the payoffs is the same, i.e.,
  - there is a constant c such for every action profile $\mathbf{a} = (a_1, \ldots, a_n)$,
    - $u_1(\mathbf{a}) + \ldots + u_n(\mathbf{a}) = c$

- Every constant-sum game is equivalent to a game in which $c = 0$
  - Positive affine transformation: subtract c/n from every payoff

- Thus constant-sum games are usually called **zero-sum** games

Examples

- **Matching Pennies**
  - Two agents, each has a penny
  - Each independently chooses to display Heads or Tails
    - If same, agent 1 gets both pennies
    - Otherwise agent 2 gets both pennies

	Heads	Tails
Heads	1, –1	–1, 1
Tails	–1, 1	1, –1

- **Rock, Paper, Scissors**
  - Each agent independently chooses to display a symbol for rock, paper, or scissors

	Rock	Paper	Scissors
Rock	0, 0	–1, 1	1, –1
Paper	1, –1	0, 0	–1, 1
Scissors	–1, 1	1, –1	0, 0

Examples

- **Soccer penalty kicks**
  - A kicker and a goalie
  - Kicker can kick left or right
  - Goalie can jump to left or right
  - Kicker scores if he/she kicks to one side and goalie jumps to the other

	Left	Right
Left	1, 0	0, 1
Right	0, 1	1, 0

- Let's ignore whether the goalie can predict the kick from the kicker's motions

- Positive affine transformation into Matching Pennies

Nonzero-Sum Games

- A game is **nonconstant-sum** (usually called **nonzero-sum**) if there are action profiles **a** and **b** such that
    - $u_1(\mathbf{a}) + \ldots + u_n(\mathbf{a}) \neq u_1(\mathbf{b}) + \ldots + u_n(\mathbf{b})$
  - e.g., the Prisoner's Dilemma

	C	D
C	3, 3	0, 5
D	5, 0	1, 1

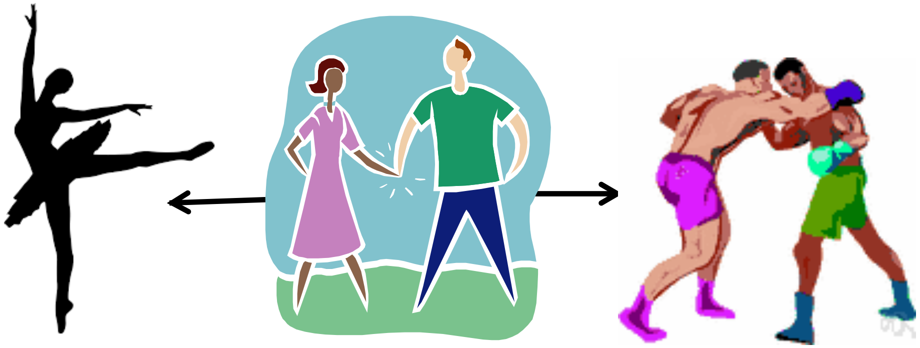
Nonzero-Sum Games

- **Battle of the Sexes**
  - Two agents need to coordinate their actions, have different preferences
  - Example:
    - Two nations must act together to deal with an international crisis, and they prefer different solutions

	A	*B*
A	2, 1	0, 0
B	0, 0	1, 2

- Why it's called Battle of the Sexes
  - Original scenario (1957): where to go for the evening?
  - Alice prefers ballet, Bob prefers boxing match
  - If they can't coordinate, neither will enjoy themselves

		Bob:	
		Ballet	Boxing
Alice:	Ballet	2, 1	0, 0
	Boxing	0, 0	1, 2



Symmetric Games

- **Symmetric** game: every agent has the same actions and payoffs
  - ➢ If we interchange any pair of agents, the payoff matrix stays the same
- 2x2 symmetric game
  - ➢ For every action profile (a_1,a_2),
    - $u_1(a_1,a_2) = u_2(a_2,a_1)$
- In the payoff matrix of a symmetric game, we only need to display u_1
  - ➢ If you want to know i's payoff, interchange agents i and 1

Which Side of the Road

	Left	Right
Left	1, 1	0, 0
Right	0, 0	1, 1

	a	b
a	w, w	x, y
b	y, x	z, z

	a	b
a	w	x
b	y	z

Strategies in Normal-Form Games

- **Pure strategy**: select a single action and play it
  - Each row or column of a payoff matrix represents both an action and a pure strategy

	b_1	b_2
a_1	$x_1,\ y_1$	$x_2,\ y_2$
a_2	$x_3,\ y_3$	$x_4,\ y_4$

- **Mixed strategy**: randomize over the set of available actions according to some probability distribution
  - $s_i(a_j)$ = probability that action a_j will be played in mixed strategy s_i
- s_i's **support**: {actions that have probability > 0 in s_i}
  - A pure strategy is a mixed strategy whose support is a single action
  - But I'll often use "mixed strategy" to mean one that isn't pure
- A strategy s_i is **fully mixed** if its support is A_i
  - all of agent i's actions have nonzero probability
- **Strategy profile**: an n-tuple of strategies $\mathbf{s} = (s_1, \ldots, s_n)$
  - s_i is agent i's strategy

Some Comments

- The normal-form game representation is very restricted
  - No such thing as a conditional strategy (e.g., cross the bay if the temperature is above 70)
  - No temperature or anything else to observe
- Only two kinds of strategies:
  - **Pure strategy**: a single action
  - **Mixed strategy**: probability distribution over pure strategies

	C	*D*
C	3, 3	0, 5
D	5, 0	1, 1

- Much more complicated games can be mapped into normal-form games
  - Pure strategy: a complete description of what you'll do in *every* situation you might ever encounter
- Examples in Chapter 4

Expected Utility

- A payoff matrix only shows the payoffs for pure-strategy profiles
- For mixed strategies, use expected utility
- Utility of a strategy profile $\mathbf{s} = (s_1, \ldots, s_n)$
  - Sum, over all action profiles $\mathbf{a} = (a_1, \ldots, a_n)$,
    - utility of **a** × probability of **a**
  - $u_i(\mathbf{s}) = \sum_{\mathbf{a} \in \mathbf{A}} u_i(\mathbf{a}) \Pr[\mathbf{a} \mid \mathbf{s}]$
- Important assumption: each mixed strategy is independent of the other agents' strategies
  - $\Pr[(a_1, \ldots, a_n) \mid \mathbf{s}] = \Pr[a_1 \mid s_1] \Pr[a_2 \mid s_2] \ldots \Pr[a_n \mid s_n] = \prod_{j=1}^{n} \Pr(a_j \mid s_i)$

  so

  - $u_i(\mathbf{s}) = \sum_{(a1, \ldots, an)} u_i(a_1, \ldots, a_n) \prod_{j=1}^{n} \Pr(a_j \mid s_i)$

Summary

- Basic concepts:
  - normal form, pure strategies, mixed strategies, expected utility
- How utilities relate to rational preferences (not in the book)
- Some classifications of games based on their payoffs
  - Zero-sum
    - Rock-paper-scissors, Matching Pennies
  - Non-zero-sum
    - Chocolate Dilemma, Prisoner's Dilemma,
      Which Side of the Road, Battle of the Sexes
  - Common-payoff
    - Which Side of the Road
  - Symmetric
    - all of the above except Battle of the Sexes