CMSC 474, Game Theory

8. Coalitional Game Theory

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Introduction

- **Coalition**: group of agents that cooperate with each other
- **Coalitional game**: agents may choose to form into coalitions
  - How well can possible coalition do for itself?
    - payoff for the group
  - Not concerned with
    - how the agents make individual choices within a coalition,
    - how they coordinate, or
    - any other such detail
- **Transferable utility** assumption
  - Payoffs to a coalition may be freely redistributed among its members
  - Satisfied whenever there is a universal **currency** that's used for exchange in the system
  - Implies that each coalition can be assigned a single value as its payoff

Introduction

- **Coalitional game with transferable utility**: a pair $G = (N,v)$
  - N = {1, 2, …, n} is a finite set of players
  - $v : 2^N \rightarrow \Re$ is the **characteristic function**
    - For each coalition $S \subseteq N$, an amount $v(S)$ that the coalition members can distribute among themselves
      - $v(S)$ is the coalition's **payoff** or **worth**
    - Assume $v(\varnothing) = 0$
- Coalitional game theory is normally concerned with two questions

  (1) Which coalition will form?

  (2) How should that coalition divide its payoff among its members?
- The answer to (1) is often "the grand coalition" (all of the agents)
  - But that can depend on making the right choice about (2)

Example: A Voting Game

- Consider a parliament with 100 representatives from four political parties:
    - A (45 reps.), B (25 reps.), C (15 reps.), D (15 reps.)
  - Vote on whether to pass a $100,000,000 spending bill
    - and how much of it should be controlled by each party
  - Need a majority ($\geq$ 51 votes) to pass legislation
    - If the bill doesn't pass, then every party gets 0
- More generally, a **voting game** would include
  - A set of agents N
  - A set of *winning* coalitions $W \subseteq 2^N$
    - In the example, all coalitions that have enough votes to pass the bill
  - If $S \in W$ then $v(S) = 1$; otherwise $v(S) = 0$
  - Or equivalently, use any fixed amount other than 1
    - If $S \in W$ then $v(S) = \$100M$; otherwise $v(S) = \$0$

Superadditive Games

- A coalitional game $G = (N,v)$ is **superadditive** if the union of two disjoint coalitions is worth at least the sum of its members' worths
  - for all $S, T \subseteq N$, if $S \cap T = \varnothing$, then $v(S \cup T) \geq v(S) + v(T)$
  - By working together, coalitions can accomplish as much or more than they could accomplish separately
- If G is superadditive, the grand coalition always has the highest possible payoff
  - For any $S \neq N$, $v(N) \geq v(S) + v(N–S) \geq v(S)$
- The parliament example is superadditive
  - Why?

Superadditive Games

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  - ➢ By working together, coalitions can accomplish as much or more than they could accomplish separately

- If G is superadditive, the grand coalition always has the highest possible payoff
  - ➢ For any $S \neq N$, $v(N) \geq v(S) + v(N\text{–}S) \geq v(S)$

- The parliament example is superadditive
  - ➢ If $S \cap T = \varnothing$, then at least one of the coalitions (say, T) is worth 0
    - Case 1: $v(S) = v(T) = 0$. Then $v(S \cup T) \geq 0$
    - Case 2: $v(S) = 1$ and $v(T) = 0$. Then $v(S \cup T) = 1$

Additive and Constant-Sum Games

- $G = (N, v)$ is **additive** (or **inessential**) if combining disjoint coalitions produces no advantage, no disadvantage
  - if $S, T \subseteq N$ and $S \cap T = \varnothing$, then $v(S \cup T) = v(S) + v(T)$
- G is **constant-sum** if the worth of N (the grand coalition) equals the sum of the worths of any two coalitions that partition N
  - $v(S) + v(N - S) = v(N)$, for every $S \subseteq N$

- Every additive game is constant-sum
  - additive => $v(S) + v(N - S) = v(S \cup (N - S)) = v(N)$
- But not every constant-sum game is additive (see next slide)

Relationships:

additive => constant-sum (but not vice versa)

A Constant-Sum Game That Isn't Additive

- Consider a game with $N = \{p\ q,r\}$ and the following coalition worths:
  - $v(\varnothing) = 0$
  - $v(\{p\}) = v(\{q\}) = v(\{r\}) = 3$
  - $v(\{p,q\}) = v(\{p,r\}) = v(\{q,r\}) = 9$
  - $v(\{p,q,r\}) = 12$
- Constant sum:
  - $v(N) + \mathrm{v}(\varnothing) = 12 + 0 = 12$
  - $v(\{p,q\}) + \mathrm{v}(\{r\}) = 9 + 3 = 12$
  - $v(\{p,r\}) + \mathrm{v}(\{q\}) = 9 + 3 = 12$
  - $v(\{q,r\}) + \mathrm{v}(\{p\}) = 9 + 3 = 12$
- Not additive:
  - $v(\{p,q\}) = 9$
  - $v(\{p\}) + v(\{q\}) = 6$

Convex Games

- G is **convex** if for all $S, T \subseteq N$,
    - $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$

- Recall the definition of a superadditive game:
    - for all $S, T \subseteq N$, if $S \cap T = \varnothing$, then
        - $v(S \cup T) \geq v(S) + v(T)$
- Thus every convex game is superadditive

Relationships:

additive => constant-sum (but not vice versa)

convex => superadditive

Simple Games

- $G = (N, v)$ is **simple** for every coalition S,
  - either $v(S) = 1$ (i.e., S **wins**) or $v(S) = 0$ (i.e., S **loses**)
  - Used to model voting
- Often add a requirement that if S wins, all supersets of S would also win:
  - if $v(S) = 1$, then for all $T \supseteq S$, $v(T) = 1$

- Parliament game is simple and superadditive
  - Is every simple game superadditive?

- No
  - Consider a voting game G in which 50% of the votes is sufficient to pass a bill
  - Two coalitions S and T, each is exactly 50% N
  - $v(S) = 1$ and $v(T) = 1$, but $v(S \cup T) \neq 2$

Proper-Simple Games

- G is a **proper simple game** if it is both simple and constant-sum
    - simple → $v(S) \in \{0,1\}$
    - constant-sum → $v(S) + v(N - S) = v(N)$
- Properties
    - If S is a losing coalition, then
        - either $N - S$ is a winning coalition, or else **all** coalitions lose
    - If S is a winning coalition, then $N - S$ is a losing coalition

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Relationships:

additive => constant-sum (but not vice versa)

convex => superadditive

proper simple = simple *and* constant-sum

Analyzing Coalitional Games

- Main question in coalitional game theory: how to divide the payoff to the grand coalition?

- Why focus on the grand coalition?
  - Many widely studied games are super-additive
    - Expect the grand coalition to form because it has the highest payoff
  - Grand coalition may be the only acceptable option
    - E.g., public projects that are legally bound to include all participants
- Given a coalitional game $G = (N, v)$, where $N = \{1, \ldots, n\}$
  - We'll want to look at the agents' shares in the grand coalition's payoff
  - Notation
    - Payoff profile $\mathbf{x} = (x_1, \ldots, x_n)$
    - $\psi(N,v)$ = payoff profile for the grand coalition
    - $\psi_i(N,v)$ = agent i's payoff in the grand coalition

Terminology

- Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a payoff profile
- $\mathbf{x}$ is **feasible** if it doesn't distribute more than the worth of the grand coalition
  - $x_1 + x_2 + \ldots + x_n \leq v(N)$
- $\mathbf{x}$ is a **pre-imputation** if it is feasible and **efficient** (distributes the entire worth of the grand coalition)
  - $x_1 + x_2 + \ldots + x_n = v(N)$
- A pre-imputation is an **imputation** if each agent gets at least what he/she would get by going alone (i.e., forming a singleton coalition)
  - $\forall i \in N, x_i \geq v(\{i\})$
  - If $\psi(N,v)$ is an imputation, it would be reasonable for the grand coalition to form

im•pute: verb [trans.] represent as being done, caused, or possessed by someone; attribute *: the crimes **imputed to** Richard.*

Fairness Axioms: 1. Symmetry

- What is a **fair** division of the payoffs?
  - Three axioms describing fairness
    - *Symmetry* axiom
    - *Dummy player* axiom
    - *Additivity* axiom

- Definition: agents i and j are **interchangeable** if they always contribute the same amount to every coalition of the other agents
  - for every S that contains neither i nor j , $v(S \cup \{i\}) = v(S \cup \{j\})$
- **Symmetry axiom**:
  - In a fair distribution of the payoffs, interchangeable agents should receive the same payments
  - If i and j are interchangeable and $(x_1, \ldots, x_n)$ is the payoff profile, then $x_i = x_j$

Example

- The parliamentary voting game again
  - Parties A, B, C, and D have 45, 25, 15, and 15 representatives
  - A simple majority (51 votes) is required to pass the $100M bill
- Every coalition with ≥ 51 members has value 1; other coalitions have value
- Consider whether B and C are interchangeable
  - Here are all coalitions of the other agents:
    - ∅, {A}, {D}, and {A,D}
- **Poll 8.1**: much value does B add to each of them?

Example

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- Consider whether B and C are interchangeable
  - Here are all coalitions of the other agents:
    - ∅, {A}, {D}, and {A,D}
- **Poll 8.1**: much value does B add to each of them?
  - Answer: 0, 1, 0, and 0
- **Poll 8.2**: much value does C add to each of them?

Example

- The parliamentary voting game again
  - Parties A, B, C, and D have 45, 25, 15, and 15 representatives
  - A simple majority (51 votes) is required to pass the $100M bill
- Every coalition with ≥ 51 members has value 1; other coalitions have value
- Consider whether B and C are interchangeable
  - Here are all coalitions of the other agents:
    - ∅, {A}, {D}, and {A,D}
- **Poll 8.1**: much value does B add to each of them?
  - Answer: 0, 1, 0, and 0
- **Poll 8.2**: much value does C add to each of them?
  - Answer: 0, 1, 0, and 0
- Same for D: 0, 1, 0, and 0

- B, C, and D are interchangeable
  - Fairness axiom says they should each get the same amount

Fairness Axioms: 2. Dummy Players

- Agent i is a **dummy player** if i's contribution to a coalition is always the same amount that i can achieve alone
  - for every S that doesn't contain i,
    $v(S \cup \{i\}) = v(S) + v(\{i\})$

- **Dummy player axiom**
  - In a fair distribution of the payoffs, dummy players should receive the same amount they can achieve on their own
  - If $(x_1, \ldots, x_n)$ is the payoff profile, then for every dummy player i,
    - $x_i = v(\{i\})$

Example

- Agent i is a **dummy player** if i's contribution to a coalition is always the same amount that i can achieve alone
  - for every S that doesn't contain i,
    $v(S \cup \{i\}) = v(S) + v(\{i\})$
- Example: the parliamentary voting game again
  - Parties A, B, C, and D have 45, 25, 15, and 15 representatives
  - A simple majority (51 votes) is required to pass the $100M bill
- Every coalition with ≥ 51 members has value 1; other coalitions have value 0
- **Poll 8.3:** How many dummy players are there?

Example

- Agent i is a **dummy player** if i's contribution to a coalition is always the same amount that i can achieve alone
  - for every S that doesn't contain i,
    $v(S \cup \{i\}) = v(S) + v(\{i\})$

- Example: the parliamentary voting game again
  - Parties A, B, C, and D have 45, 25, 15, and 15 representatives
  - A simple majority (51 votes) is required to pass the $100M bill
- Every coalition with ≥ 51 members has value 1; other coalitions have value 0

- **Poll 8.3:** How many dummy players are there?
  - Answer: none

Fairness Axioms: 3. Additivity

- Let $G_1 = (N,v_1)$ and $G_2 = (N,v_2)$ be two coalitional games with the same agents
- Consider the combined game $G = (N, v_1 + v_2)$, where
  - $(v_1 + v_2)(S) = v_1(S) + v_2(S)$

- **Additivity axiom**
  - In a fair distribution of payoffs for G, the agents should get the sum of what they would get in the two separate games
  - For each player i, $\psi_i(N, v_1 + v_2) = \psi_i(N, v_1) + \psi_i(N, v_2)$

Shapley Values

- Recall that a pre-imputation is a payoff division that is both feasible and efficient
  - Distributes exactly the worth of the grand coalition
- **Theorem.** Given a coalitional game (N,v), there's a unique pre-imputation $\varphi(N,v)$ that satisfies the Symmetry, Dummy player, and Additivity axioms. For each player i, i's share of $\varphi(N,v)$ is

$$\varphi_i(N,v) = \frac{1}{|N|!} \sum_{S \subseteq N - \{i\}} |S|!\ (|N| - |S| - 1)!\ (v(S \cup \{i\}) - v(S))$$

- $\varphi(N,v)$ is called the **Shapley value**
  - Lloyd Shapley introduced it in 1953
- It captures agent i's **average marginal contribution**
  - The average contribution that i makes to the coalition, averaged over every possible sequence in which the grand coalition can be built up from the empty coalition

Shapley Values

- Suppose agents join the grand coalition one by one, all sequences equally likely
- Let
  - S = {agents that joined before i}
  - T = {agents that joined after i} $= N - (S \cup \{i\})$
- i's marginal contribution is $v(S \cup \{i\}) - v(S)$
  - independent of how S is ordered, independent of how T is ordered
- Pr[S, then i, then T]

  = (# of sequences that include S then i then T) / (total # of sequences)

  $= |S|!\ |T|!\ /\ |N|! \ = |S|!\ (|T| - |S| - 1)!\ /\ |N|!$
- Shapley's formula:
  - $\varphi_i(N,v)$

    $= \sum_{S \in N-\{i\}}$ Pr[S, then i, then T] × (i's marginal contribution when i joins)

    $$= \frac{1}{|N|!} \sum_{S \subseteq N-\{i\}} |S|!\ (|N| - |S| - 1)!\ (v(S \cup \{i\}) - v(S))$$

Example 1

- The parliamentary voting game again
  - Parties A, B, C, and D have 45, 25, 15, and 15 representatives
  - A simple majority (51 votes) is required to pass the \$100M bill

- Let's compute $\varphi_A(N,v_1)$ = fair payoff for A in the grand coalition
- $N = \{A,B,C,D\}$, so S may be any of the following:
  - $\varnothing$, {B}, {C}, {D}, {B,C}, {B,D}, {C,D}, {B,C,D}
- For each, we need to compute two things:
  - i's marginal contribution when i joins $= v_1(S \cup \{i\}) - v_1(S)$
  - $\Pr[S, \text{then } i, \text{then } T] = |S|!\ (|T| - |S| - 1)!\ /\ |N|!$

Example 1

A: 45 representatives
B: 25 representatives
C: 15 representatives
D: 15 representatives

Need 51 to win

$$\varphi_{i,S} = \frac{|S|!\,(|N|-|S|-1)!}{|N|!}\big(v(S \cup \{i\}) - v(S)\big)$$

$S=\varnothing$:	$v_1(\{\mathrm{A}\} \cup S) - v_1(S) = 0 - 0 = 0$	$\Pr[S,A,T] = 0!\,3!\,/4! = 1/4$
$S=\{\mathrm{B}\}$:	$v_1(\{\mathrm{A}\} \cup S) - v_1(S) = 100\mathrm{M} - 0 = 100\mathrm{M}$	$\Pr[S,A,T] = (1!\,2!\,/4!) = {}^1/_{12}$
$S=\{\mathrm{C}\}$:	$v_1(\{\mathrm{A}\} \cup S) - v_1(S) = 100\mathrm{M} - 0 = 100\mathrm{M}$	$\Pr[S,A,T] = (1!\,2!\,/4!) = {}^1/_{12}$
$S=\{\mathrm{D}\}$:	$v_1(\{\mathrm{A}\} \cup S) - v_1(S) = 100\mathrm{M} - 0 = 100\mathrm{M}$	$\Pr[S,A,T] = (1!\,2!\,/4!) = {}^1/_{12}$
$S=\{\mathrm{B,C}\}$:	$v_1(\{\mathrm{A}\} \cup S) - v_1(S) = 100\mathrm{M} - 0 = 100\mathrm{M}$	$\Pr[S,A,T] = (2!\,1!\,/4!) = {}^1/_{12}$
$S=\{\mathrm{B,D}\}$:	$v_1(\{\mathrm{A}\} \cup S) - v_1(S) = 100\mathrm{M} - 0 = 100\mathrm{M}$	$\Pr[S,A,T] = (2!\,1!\,/4!) = {}^1/_{12}$
$S=\{\mathrm{C,D}\}$:	$v_1(\{\mathrm{A}\} \cup S) - v_1(S) = 100\mathrm{M} - 0 = 100\mathrm{M}$	$\Pr[S,A,T] = (2!\,1!\,/4!) = {}^1/_{12}$
$S=\{\mathrm{B,C,D}\}$:	$v_1(\{\mathrm{A}\} \cup S) - v_1(S) = 100\mathrm{M} - 1 = 0$	$\Pr[S,A,T] = (3!\,0!\,/4!) = 1/4$

- $\varphi_A(N,v_1) = \frac{1}{4}(0) + 6({}^1/_{12})(100\mathrm{M}) + \frac{1}{4}(0) = 50\mathrm{M}$

Example 1

A: 45 representatives
B: 25 representatives
C: 15 representatives
D: 15 representatives

Need 51 to win

- $\varphi_A(N,v) = ½$
- Similarly, $\varphi_B(N,v) = \varphi_C(N,v) = \varphi_D(N,v) = 16^2/_3$M
  - The text calculates these using Shapley's formula
- Here's another way to get them:
  - If A gets ½, then the other ½ will be divided among B, C, and D
  - B, C, and D are interchangeable
    - Divide the amount equally among them
- So distribute the money as follows:
  - A gets ½ (100M) = 50M
  - B, C, D each get $^1/_3$ (50M) = 16 $^2/_3$ M

Example 2

- In addition to the spending bill in Example 1, suppose there's a 2nd one:
  - As before, parties A, B, C, D have 45, 25, 15, and 15 representatives
  - \$50M bill, and needs a ¾ majority (75 votes)

- Every coalition with ≥ 75 members has value 1; other coalitions have value 0
  - Consider whether B and C are interchangeable
- Here are all coalitions of the other agents:
  - ∅, {A}, {D}, and {A,D}
- How much value does B add to each of them?
  - 0, 0, 0, and 1
- Same for C, so B and C are interchangeable

- Like before, B, C, and D are all interchangeable

Example 2

A: 45 representatives
B: 25 representatives
C: 15 representatives
D: 15 representatives

Need 51 to win

$$\varphi_{i,S} = \frac{|S|!\,(|N|-|S|-1)!}{|N|!}(v(S \cup \{i\}) - v(S))$$

$S=\varnothing$: $v_2(\{A\} \cup S) - v_2(S) = 0 - 0 = 0$ $\quad \Pr[S,A,T] = 0!\,3!\,/4! = 1/4$

$S=\{B\}$: $v_2(\{A\} \cup S) - v_2(S) = 0 - 0 = 0$ $\quad \Pr[S,A,T] = (1!\,2!\,/4!) = 1/12$

$S=\{C\}$: $v_2(\{A\} \cup S) - v_2(S) = 0 - 0 = 0$ $\quad \Pr[S,A,T] = (1!\,2!\,/4!) = 1/12$

$S=\{D\}$: $v_2(\{A\} \cup S) - v_2(S) = 0 - 0 = 0$ $\quad \Pr[S,A,T] = (1!\,2!\,/4!) = 1/12$

$S=\{B,C\}$: $v_2(\{A\} \cup S) - v_2(S) = 50M - 0 = 50M$ $\quad \Pr[S,A,T] = (2!\,1!\,/4!) = 1/12$

$S=\{B,D\}$: $v_2(\{A\} \cup S) - v_2(S) = 50M - 0 = 50M$ $\quad \Pr[S,A,T] = (2!\,1!\,/4!) = 1/12$

$S=\{C,D\}$: $v_2(\{A\} \cup S) - v_2(S) = 50M - 0 = 50M$ $\quad \Pr[S,A,T] = (2!\,1!\,/4!) = 1/12$

$S=\{B,C,D\}$: $v_2(\{A\} \cup S) - v_2(S) = 50M - 0 = 50M$ $\quad \Pr[S,A,T] = (3!\,0!\,/4!) = 1/4$

- $\varphi_A(N,v_2) = \frac{1}{4}(0) + 3(1/12)(50M) + \frac{1}{4}(50M) = \frac{1}{2}(50M) = 25M$

Example 2

- $\varphi_A(N,v_2) = ½$
- B, C, and D are interchangeable
  - Fair division: divide the other ½ among them equally
- Each gets $^1/_3$ (25M) = $8^1/_3$M

- Let v_1 be the value function in Example 1
  - We had $\varphi_A(N,v_1) = 50M$, and $\varphi_B(N,v_1) = \varphi_C(N,v_1) = \varphi_D(N,v_1) = 16\,^2/_3$ M
- Combined game:
  - Grand coalition: parliament decides to pass both spending bills
- *Additivity* axiom: in a fair division, each party gets the sum of what it would get for the two bills individually
  - $\varphi_A(N,v_1+v_2) = 50M + 25M = 75M$
  - $\varphi_B(N,v_1+v_2) = \varphi_C(N,v_1+v_2) = \varphi_D(N,v_1+v_2) = 16\,^2/_3\,M + 8^1/_3M = 25M$

- Sanity check: 75M + 3 × 25M = 150M = total of the two bills

Stability of the Grand Coalition

- Agents have incentive to form the grand coalition iff there aren't any smaller coalitions in which they could get higher payoffs
- Sometimes a subset of the agents may prefer a smaller coalition
- Example: the parliamentary voting game again
  - Parties A, B, C, and D have 45, 25, 15, and 15 representatives
  - A simple majority (51 votes) is required to pass the $100M bill
- Every coalition with ≥ 51 members has value 1; other coalitions have value 0
- Shapley values: A gets $50M; B, C, D each get $16⅔M
  - A on its own can't do better
  - But {A, B} have incentive to deviate (leave and form their own coalition) and divide the $100M between themselves
    - e.g., $75M for A and $25M for B
- What payment divisions would make the agents want to join the grand coalition?

The Core

- The **core** of a coalitional game includes every payoff vector **x** that gives every sub-coalition S at least as much in the grand coalition as S could get by itself
  - All feasible payoff vectors $\mathbf{x} = (x_1, \ldots, x_n)$ such that
    - for every $S \subseteq N$, $\sum_{i \in S} x_i \geq v(S)$
- For every payoff vector **x** in the core, no S has any incentive to **deviate** from the grand coalition
  - i.e., form their own coalition, excluding the others
- It follows immediately that if **x** is in the core then **x** is an imputation
  - Why?

Analogy to Nash Equilibria

- Nash equilibrium in a noncooperative game
  - No agent can do better by deviating from the equilibrium
- Core in a coalitional game
  - No *set* of agents can do better by deviating from the grand coalition
- Unlike the set of Nash equilibria, the core may sometimes be empty
  - In some cases, no matter what the payoff vector is, some agent or group of agents has incentive to deviate

Example of an Empty Core

- Consider the voting example again:
  - Shapley values are \$50M to A, and \$16.33M each to B, C, D
- {B,C,D} can achieve 51 votes without A
  - If the sum of the payoffs to B, C, and D is < \$100M, they have incentive to deviate from the grand coalition
  - Thus if **x** is in the core, **x** must allocate \$100M to {B, C, D}
- But if B, C, and D get the entire \$100M, then A gets \$0
  - At least one party in {B,C,D} got less than \$34M
  - That party and A have incentive to form their own coalition
    - e.g., form a coalition {A,D} without the others
  - So if **x** allocates the entire \$100M to {B,C,D} then **x** can't be in the core
- So the core is empty

Simple Games

- Several situations in which there are guarantees whether the core exists
  - The first two involve simple games
- Recall: G is simple for every coalition S, either $v(S) = 1$ or $v(S) = 0$
- Player i is a **veto player** if $v(N - \{i\}) = 0$

- **Theorem**. In a simple game, the core is empty iff there is no veto player
- Example: previous slide

- **Theorem**. In a simple game in which there are veto players, the core is {All payoff profiles in which non-veto players get 0}

- **Example**: voting game, modified to require 80% majority
  - Recall that A, B, C, and D have 45, 25, 15, and 15 representatives
  - Winning coalitions: {A, B, C}, {A, B, D} and {A, B, C, D}
  - A and B are veto players; all winning coalitions include both of them
  - The core includes all distributions of the $100M among A and B

Non-Additive Constant-Sum Games

- Recall that
  - $G = (N,v)$ is **additive** if combining disjoint coalitions adds their worths:
    - if $S, T \subseteq N$ and $S \cap T = \emptyset$, then $v(S \cup T) = v(S) + v(T)$
  - G is **constant-sum** if the worth of N (the grand coalition) equals the sum of the worths of any two coalitions that partition N
    - $v(S) + v(N - S) = v(N)$, for every $S \subseteq N$

- **Theorem**. Every non-additive constant-sum game has an empty core

Example

- **Theorem**. Every non-additive constant-sum game has an empty core

- **Example:** recall this example of a non-additive constant-sum game:
  - $v(\{p\}) = v(\{q\}) = v(\{r\}) = 3$
  - $v(\{p,q\}) = v(\{p,r\}) = v(\{q,r\}) = 9$
  - $v(\{p,q,r\}) = 12$
- Consider $\mathbf{x} = (4, 4, 4)$
  - $v(\{p,q\}) = 9$
  - If $\{p,q\}$ deviate, they can allocate (4.5, 4.5)
- To keep $\{p,q\}$ from deviating, suppose we use $\mathbf{x} = (4.5, 4.5, 3)$
  - $v(\{p,r\}) = 9$
  - If $\{p,r\}$ deviate, they can allocate (5, 4)

Convex Games

- Recall:
  - G is **convex** if for all $S, T \subseteq N$,
    $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$
- **Theorem**. Every convex game has a nonempty core
- **Theorem**. In every convex game, the Shapley value is in the core

Example

- Modify the previous game:
  - $v(\{p\}) = v(\{q\}) = v(\{r\}) = 3$
  - $v(\{p,q\}) = v(\{p,r\}) = v(\{q,r\}) = 9$
  - $v(\{p,q,r\}) = 18$
- Is it convex?
  - G is convex if for all $S,T \subseteq N$, $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$
- All three players are interchangeable
  - So the Shapley values are (6,6,6)
- Consider $\mathbf{x} = (6, 6, 6)$
  - $v(\{p\}) = 3$, so no incentive to deviate
    - Same for $\{q\}$ and for $\{r\}$
  - $v(\{p,q\}) = 9$, can only allocate (4.5, 4.5) so no incentive to deviate
    - Same for $\{p,r\}$ and for $\{q,r\}$

Modified Parliament Example

A: 45 representatives
B: 25 representatives
C: 15 representatives
D: 15 representatives

- Suppose any coalition of parties can approve a spending bill worth \$1K times the number of representatives in the coalition:

$$v(S) = \sum_{i \in S} \$1000 \times \text{size}(i)$$

- Is the game convex?

Modified Parliament Example

- Each party's Shapley value is the average value it adds to the grand coalition, averaged over all 24 of the possible sequences in which the coalition might be formed:

A: 45 representatives
B: 25 representatives
C: 15 representatives
D: 15 representatives

A, B, C, D;   A, B, D, C;   A, C, B, D;   A, C, D, B;   . . .

- In every sequence, every party adds exactly \$1K times its size
- Thus every party's Shapley value is \$1K times its size:
  - φ_A = \$45K, φ_B = \$25K, φ_C = \$15K, φ_D = \$15K

Modified Parliament Example

A: 45 representatives
B: 25 representatives
C: 15 representatives
D: 15 representatives

- Suppose we distribute $v(N)$ by giving each party its Shapley value
- Does any party or group of parties have an incentive to leave and form a smaller coalition T?
  - $v(T)$ = \$1K times the number of representatives in T
    = the sum of the Shapley values of the parties in T
  - If each party in T gets its Shapley value, it does no better in T than in N
  - If some party in T gets more than its Shapley value, then another party in T will get less than its Shapley value
- No case in which every party in T does better in T than in N
- No case in which all of the parties in T will have an incentive to leave N and join T

- Thus the Shapley value is in the core

Two More Examples

A: 45 representatives
B: 25 representatives
C: 15 representatives
D: 15 representatives

- Suppose each coalition S can approve a spending bill worth this amount:

$$v(S) = \sum_{i \in S} (\$1000 \times \text{size}(i)) - \$1000$$

  - Is the game convex?
  - What is each party's Shapley value?
  - What is the game's core?

- What if we have this instead?

$$v(S) = \sum_{i \in S} (\$1000 \times \text{size}(i)) + \$1000$$

Schedule for the rest of the semester

- Tues. Nov 22: no class, you can leave for Thanksgiving early
- Tues. Nov 29: How to use AI planning in game programs
- Thur. Dec 1: Guest lecture by Bill Gasarch
- Tues. Dec 6: Guest lecture by VS Subrahmanian
- Thur. Dec 8: Last class, review for the final exam
- Tues. Dec 20, 10:30–12:30: Final exam

- Homework 7: I'll add some problems for Chapter 8, and postpone the due date
    - There were 2 problems, there will be 5 instead
    - I'll post it after class today
  - New due date: Tues. Dec 6
  - New late date: Thur. Dec 8