CMSC 474, Game Theory

3a. More about Normal-Form Games

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First part of Chapter 3,
plus related topics

Outline

- Chapter 2 discussed two solution concepts:
  - Pareto optimality and Nash equilibrium
- Chapter 3 discusses several more

- Lecture 3a:
  - First part of Chapter 3
    - Maxmin and Minmax
    - Minimax Regret
    - Dominant strategies
    - Iterated Elimination of Strictly Dominated Strategies

Worst-Case Expected Utility

- The **worst-case** expected utility of i's strategy s_i is i's min utility over every possible strategy profile for the other agents:

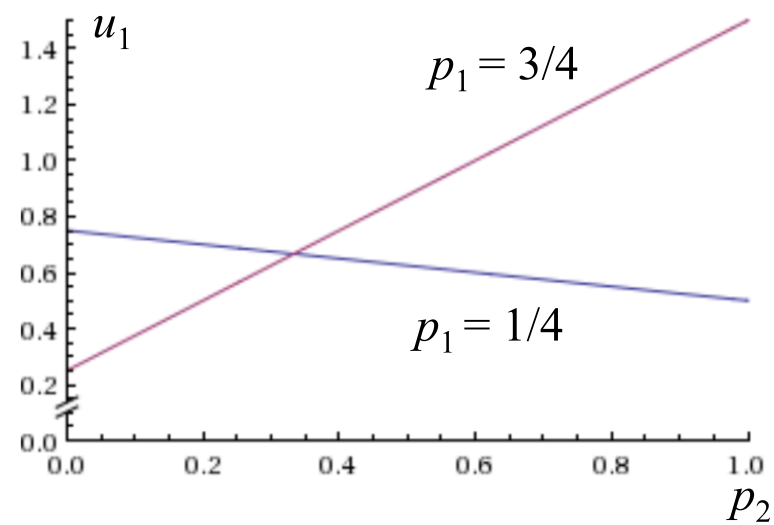
$$\min_{\mathbf{s}_{-i}} u_i\left(s_i, \mathbf{s}_{-i}\right)$$

	A	B
A	2, 1	0, 0
B	0, 0	1, 2

- **Example: Battle of the Sexes**
  - $s_1 = \{(p_1, \text{A}), (1 - p_1, \text{B})\}$
  - $s_2 = \{(p_2, \text{A}), (1 - p_2, \text{B})\}$

Why did I write $u_1(p_1, p_2)$ instead of $u_1(s_1, s_2)$?

  - $u_1(p_1, p_2) = 2p_1 p_2 + (1 - p_1)(1 - p_2)$
  - Given p_1, u_1 is linear in p_2
  - $0 \le p_2 \le 1$, so min u_1 is at $p_2 = 0$ or $p_2 = 1$
    - $u_1(p_1,0) = 1 - p_1$
    - $u_1(p_1,1) = 2p_1$
  - $\min_{p_2} u_1\left(p_1, p_2\right) = \min(1 - p_1, 2p_1)$



Discussion

- Suppose an agent i has no information about the other agents
    - whether they are rational
    - what their payoffs are
    - whether they draw their action choices from known distributions

  and suppose i wants to be cautious
- Or
    - Suppose i believes the other agents are hostile
        - believes they want to minimize i's expected utility

- i might want to choose a strategy with *highest* worst-case expected utility
    - **maxmin strategy**

Maxmin Strategies

maximin

- A **maxmin** strategy for agent i
  - A strategy s_i^+ that maximizes i's worst-case expected utility:
    $$s_i^+ = \arg\max_{s_i} \min_{\mathbf{s}_{-i}} u_i\left(s_i, \mathbf{s}_{-i}\right)$$
  - Not necessarily unique
  - Often mixed

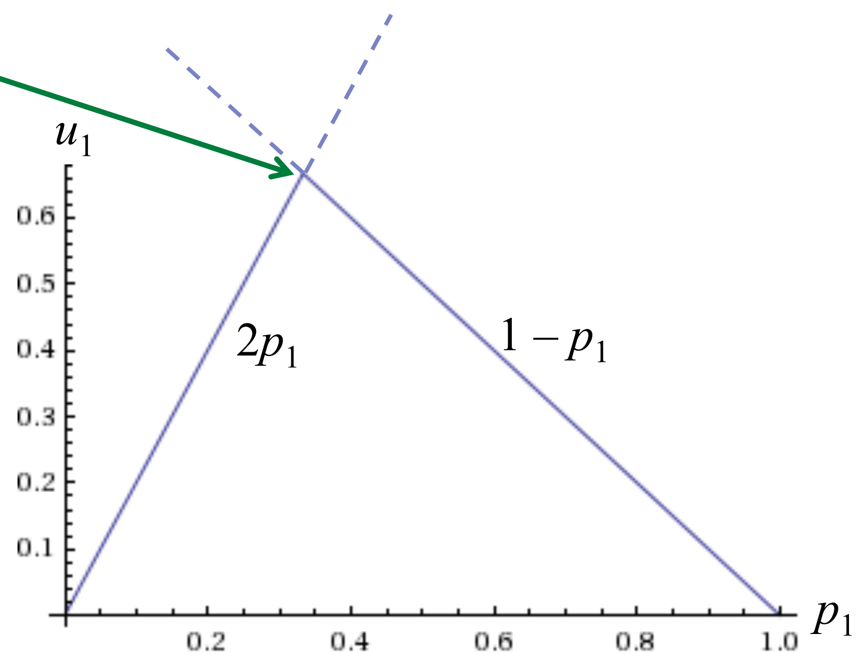
- Agent i's **maxmin value**, or **security level** is
  i's maximum expected utility against the worst possible $\mathbf{s}_{-i}$

$$u_i^+ = \max_{s_i} \min_{\mathbf{s}_{-i}} u_i\left(s_i, \mathbf{s}_{-i}\right)$$

Example

	A	B
A	2, 1	0, 0
B	0, 0	1, 2

- Strategies:
  - $s_1 = \{(p_1, \text{A}), (1 - p_1, \text{B})\}$
  - $s_2 = \{(p_2, \text{A}), (1 - p_2, \text{B})\}$
- 1's min expected utility given p_1:
  - $\min_{p_2} u_1(p_1, p_2) = \min[u_1(p_1, \text{A}), u_1(p_1, \text{B})] = \min(2p_1, 1 - p_1)$
  - Find p_1 that maximizes it
- Max is at $2p_1 = 1 - p_1 \quad \rightarrow \quad p_1 = 1/3$
  - $u_1^+ = 1 - p_1 = 2/3$
  - $s_1^+ = \{(1/3, \text{A}), (2/3, \text{B})\}$



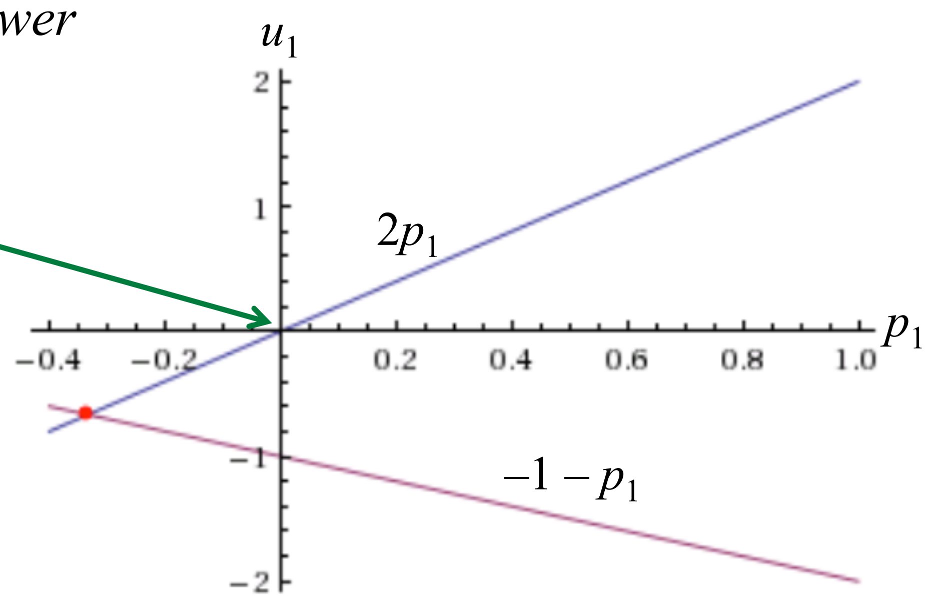
- Similarly,
  - 2's maxmin value $u_2^+ = 2/3$
  - 2's maxmin strategy is $s_2^+ = \{(2/3, \text{A}), (1/3, \text{B})\}$

Example

- Strategies:
  - $s_1 = \{(p_1, \text{A}), (1 - p_1, \text{B})\}$
  - $s_2 = \{(p_2, \text{A}), (1 - p_2, \text{B})\}$

	A	B
A	2, 1	–2, 0
B	0, 0	–1, 2

- 1's min expected utility given p_1:
  - $\min_{p_2} u_1(p_1, p_2) = \min[u_1(p_1, \text{A}), u_1(p_1, \text{B})] = \min(2p_1, -1 - p_1)$
  - Find p_1 that maximizes it
- $2p_1 = -1 - p_1 \rightarrow p_1 = -1/3$, *not the answer*
  - Not a probability
    - smallest possible p_1 is 0
- For 1, B is **dominant**
  - payoff always higher than A
- $s_1{}^+ = \text{B}$
- $u_1{}^+ = -1$

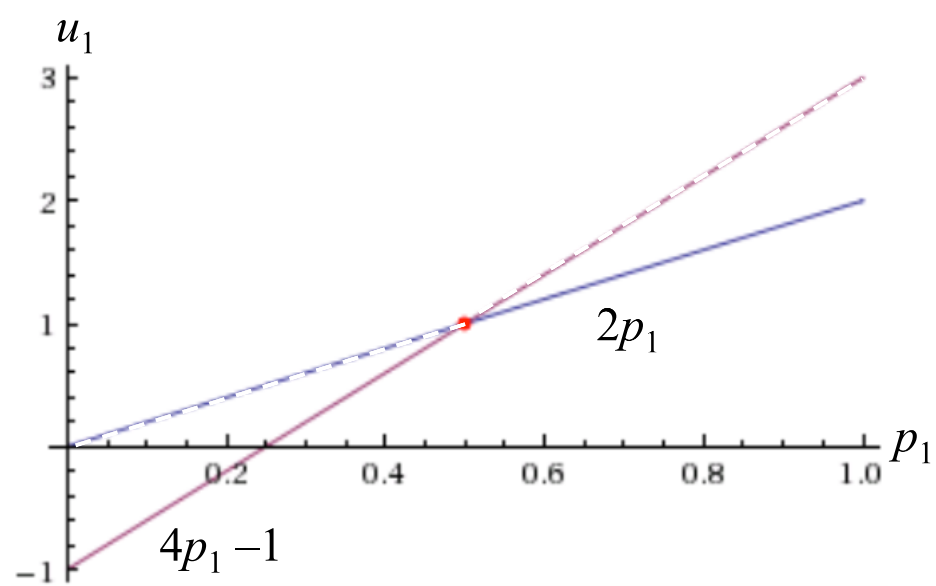


Example

- Strategies:
  - $s_1 = \{(p_1, \text{A}), (1 - p_1, \text{B})\}$
  - $s_2 = \{(p_2, \text{A}), (1 - p_2, \text{B})\}$

	A	B
A	2, 1	3, 0
B	0, 0	–1, 2

- 1's min expected utility given p_1:
  - $\min_{p_2} u_1(p_1, p_2) = \min[u_1(p_1, \text{A}), u_1(p_1, \text{B})] = \min(2p_1, 4p_1 - 1)$
  - Find p_1 that maximizes it
- $2p_1 = 4p_1 - 1 \rightarrow p_1 = ½$, *not the answer*
  - Both lines have positive slope
- maxmin strategy is at $p_1 = 1$
  - $s_1^+ = \text{A}$
  - $u_1^+ = 2$



- What if both lines had negative slope?

Minmax Strategies (in 2-Player Games)

or **minimax**

- Suppose agent 1 wants to punish agent 2
  - regardless of how it affects agent 1's own payoff
- Agent 1's **minmax strategy** (against agent 2)
  - A strategy s_1^- that minimizes 2's maximum expected utility

expected utility of 2's best response to s_1

$$s_1^- = \arg\min_{s_1} \max_{s_2} \; u_2(s_1, s_2)$$

- Agent 2's **minmax value**
  - 2's maximum expected utility if agent 1 uses s_1^-

$$u_2^- = \min_{s_1} \max_{s_2} \; u_2(s_1, s_2)$$

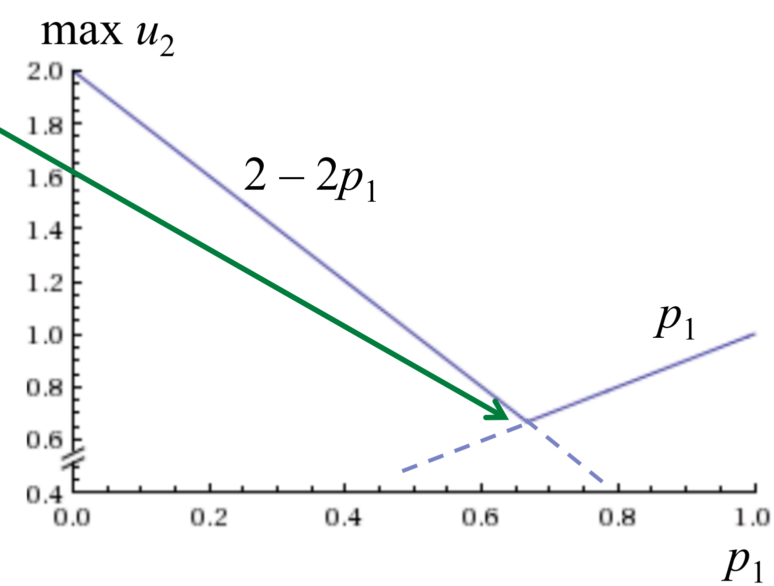
- **Minmax strategy profile**: both players use their minmax strategies

Example

- $s_1 = \{(p_1, \text{A}), (1 - p_1, \text{B})\}$
- $s_2 = \{(p_2, \text{A}), (1 - p_2, \text{B})\}$

	A	B
A	2, 1	0, 0
B	0, 0	1, 2

- $u_2(p_1, p_2) = p_1 p_2 + 2(1 - p_1)(1 - p_2)$
  - Given p_1, find max value of $u_2(p_1, p_2)$
  - u_2 is linear in p_2, so max will be at $p_2 = 0$ or $p_2 = 1$
    - $\max_{p_2} u_2(p_1, p_2) = \max\big(u_2(p_1, 0), u_2(p_1, 1)\big) = \max(2 - 2p_1, p_1)$
  - player 1 chooses p_1 to minimize it
    - smallest at $2 - 2p_1 = p_1 \;\Rightarrow\; p_1 = 2/3$



- 1's minmax strategy is
  $s_1^- = \{(2/3, \text{A}), (1/3, \text{B})\}$
  - 2's minmax value is $u_2^- = 2/3$
- Similarly, 2's minmax strategy is
  $s_2^- = \{(1/3, \text{A}), (2/3, \text{B})\}$
  - 1's minmax value is $u_1^- = 2/3$
- As before, only works if $0 \le \text{p1} \le 1$ and lines slope in opposite directions

Minmax Strategies in *n*-Agent Games

- For $n > 2$, agent i usually can't minimize agent j's payoff by acting unilaterally
- But suppose all the agents "gang up" on agent j
  - Let $\mathbf{s}_{-j}^{-}$ be a mixed-strategy profile that minimizes j's maximum payoff

$$\mathbf{s}_{-j}^{-} = \underset{\mathbf{s}_{-j}}{\operatorname{argmin}} \left(\max_{s_j} \, u_j\left(s_j, \mathbf{s}_{-j}\right) \right)$$

  - For every agent $i \neq j$, i's component of $\mathbf{s}_{-j}^{-}$ is a **minmax strategy for *i* against *j***

- **Agent *j*'s minmax value** is j's maximum payoff against $\mathbf{s}_{-j}^{-}$

$$u_j^{-} = \min_{\mathbf{s}_{-j}} \max_{s_j} \, u_j\left(s_j, \mathbf{s}_{-j}\right)$$

- **Poll 3.2**: what strategy should j use in this case?

Maxmin, Minmax, and Nash

- In the Battle of the Sexes, each agent's maxmin, minmax, and Nash equilibrium payoffs were the same
  - That doesn't always happen
    - homework problem

- There are some special cases where it does happen
  - Zero-sum games

Minimax Theorem (von Neumann, 1928)

- **Theorem.** Let G be any finite two-player zero-sum game. For each player i,
  - i's expected utility in any Nash equilibrium
    = i's maxmin value
    = i's minmax value
  - In other words, for every Nash equilibrium (s_1^*, s_2^*),

$$u_1(s_1^*, s_2^*) = \min_{s_1} \max_{s_2} u_1(s_1, s_2) = \max_{s_1} \min_{s_2} u_1(s_1, s_2)$$

    - Why doesn't the equation mention u_2?

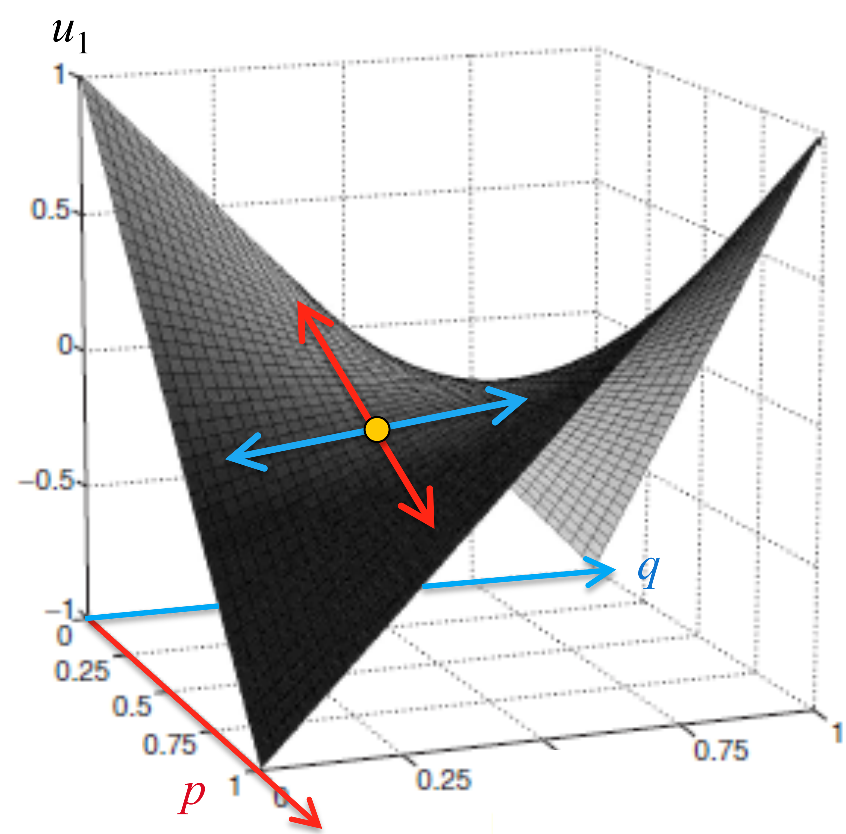
- **Corollary.** {Nash equilibria} = {maxmin strategy profiles}
  = {minmax strategy profiles}

- Terminology: the **value** (or **minmax value**) of G is agent 1's minmax value

Example: Matching Pennies

	Heads	Tails
Heads	1, –1	–1, 1
Tails	–1, 1	1, –1

- Write the players' strategies as
  $s_1 = \{(p, \text{Heads}), (1–p, \text{Tails})\}$
  $s_2 = \{(q, \text{Heads}), (1–q, \text{Tails})\}$
- $u_1(s_1, s_2) = pq + (1–p)(1–q) – p(1–q) – q(1–p) = 1 – 2p – 2q + 4pq$
  - ➢ Saddle surface, cross-sections are straight lines
  - ➢ For each p, u_1 is linear in q
  - ➢ For each q, u_1 is linear in p
- The only Nash equilibrium strategy profile: $p = ½$, $q = ½$
  - ➢ also the only maxmin strategies
  - ➢ also the only minmax strategies
- Two ways to see this:
  - ➢ Solve simultaneous equations
  - ➢ Look at the graph



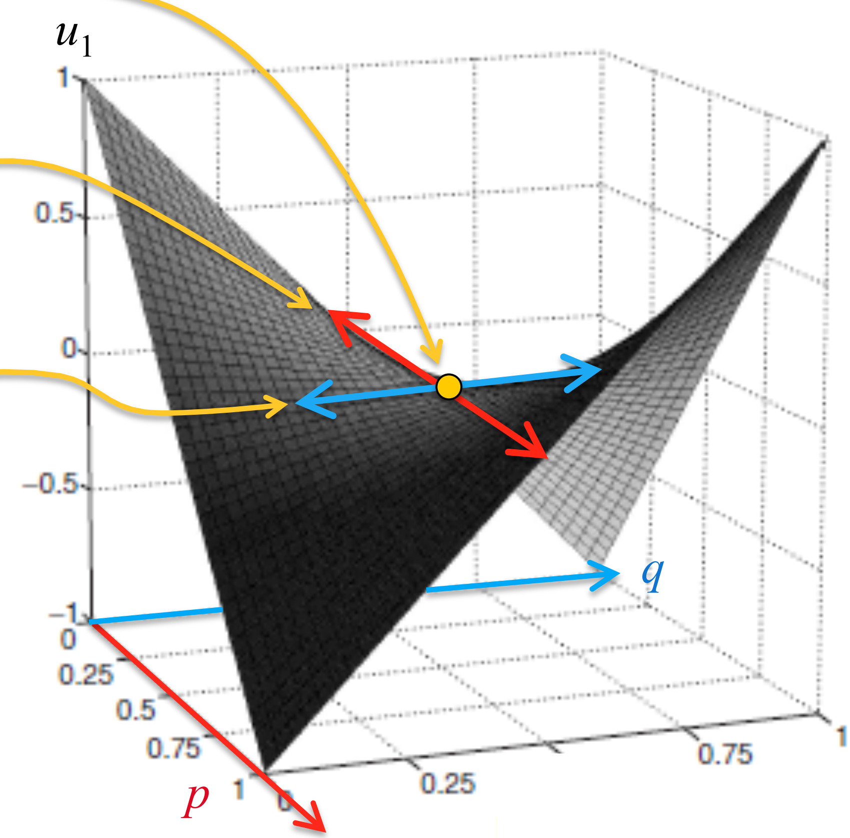
Example: Matching Pennies

	Heads	Tails
Heads	1, –1	–1, 1
Tails	–1, 1	1, –1

- **Find Nash equilibria**

$$u_1(s_1, s_2) = 1 - 2p - 2q + 4pq$$
$$u_2(s_1, s_2) = -u_1(s_1, s_2)$$

- If $p = q = ½$
  - $u_1 = u_2 = 0$
- If agent 1 unilaterally changes to $p \neq ½$
  - $u_1(p, ½) = 1 - 2p - 1 + 2p = 0$
- If agent 2 unilaterally changes to $q \neq ½$
  - $u_2(½, q) = -(1 - 2q - 1 + 2q) = 0$
- Thus $p = q = ½$ is a Nash equilibrium

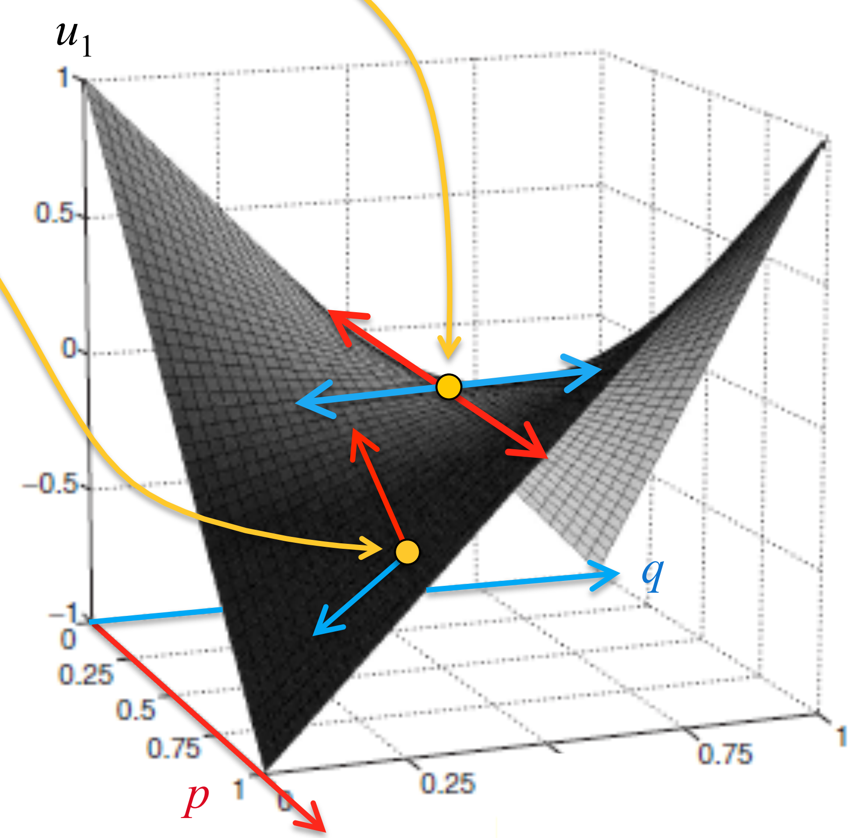


- Are there any others?

Example: Matching Pennies

	Heads	Tails
Heads	1, –1	–1, 1
Tails	–1, 1	1, –1

- $p = q = ½$ is the **only** Nash equilibrium
  - ➢ It's the only case in which both lines are flat
- Consider any strategy profile in which $p \neq ½$ or $q \neq ½$ or both
  - ➢ In each case, either
    - 1 can increase u_1 by changing p,
    - 2 can increase u_2 by changing q,
    - or usually both



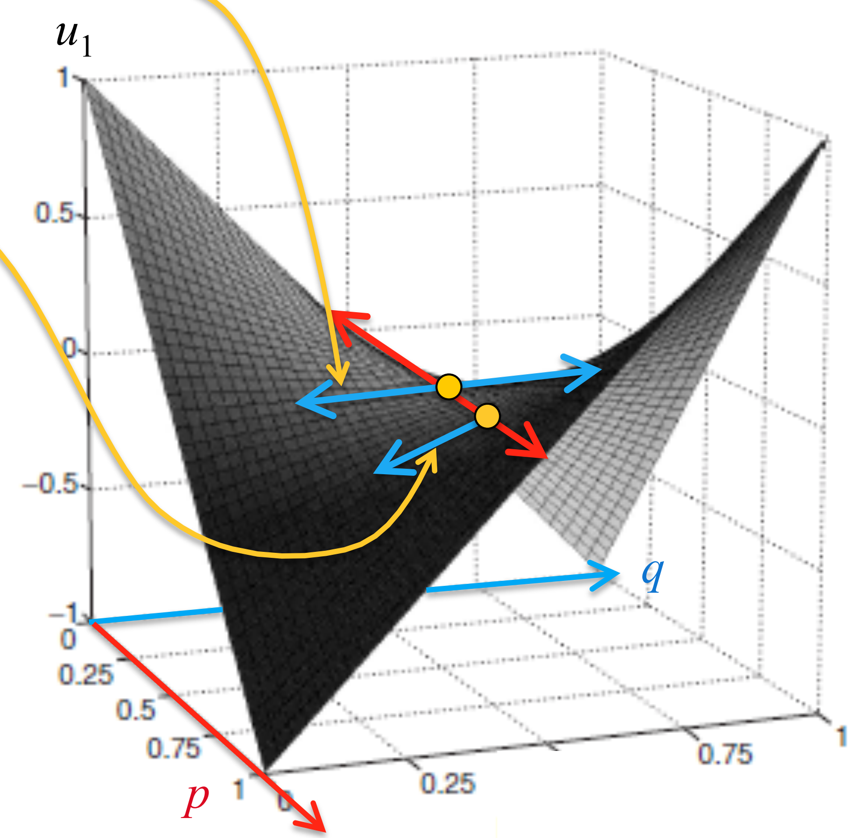
Example: Matching Pennies

	Heads	Tails
Heads	1, –1	–1, 1
Tails	–1, 1	1, –1

- **Find all maxmin strategy profiles**

- How can agent 1 maximize 1's minimum utility?
  - If 1 uses $p = ½$
    - $u_1 = 0$ regardless of what q is
  - If 1 uses $p > ½$
    - $u_1 < 0$ whenever $q < ½$
  - If 1 uses $p < ½$
    - $u_1 < 0$ whenever $q > ½$
- So 1's maximin strategy is $p = ½$

- Similarly,
  agent 2's maximin strategy is $q = ½$



Example: Matching Pennies

	Heads	Tails
Heads	1, –1	–1, 1
Tails	–1, 1	1, –1

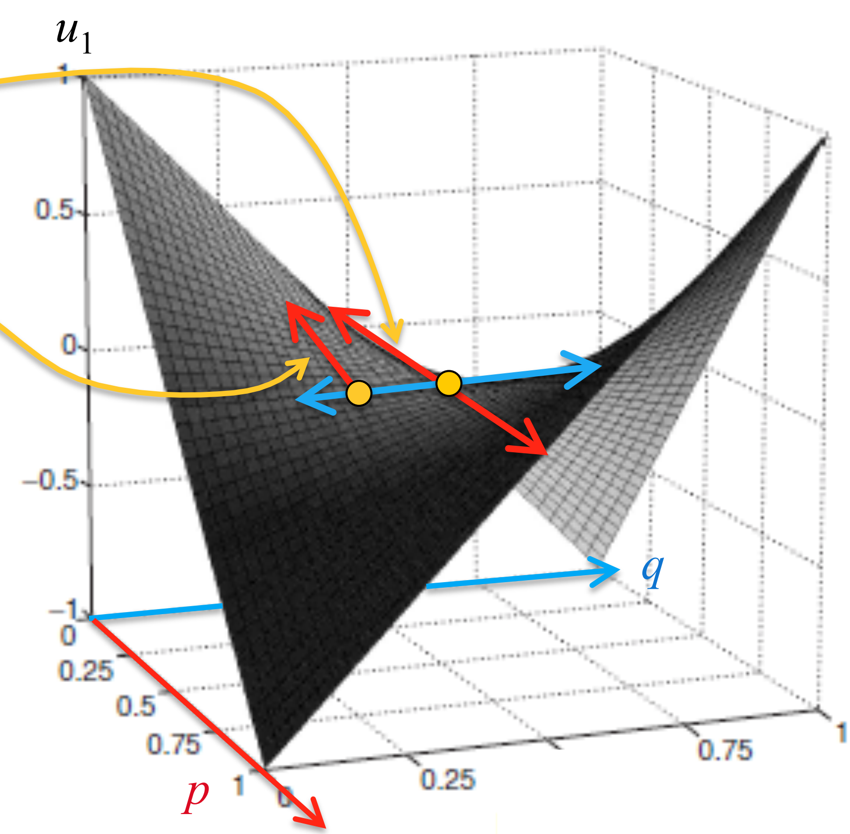
- **Find all minmax strategy profiles**

$$u_1(p, q) = 1 - 2p - 2q + 4pq$$

$$u_2(p, q) = -u_1(p, q)$$

- How can 2 minimize 1's maximum utility?
  - If $q = ½$
    - $u_1 = 0$ regardless of what p is
  - If $q < ½$
    - $u_1 > 0$ whenever $p < ½$
  - If $q > ½$
    - $u_1 > 0$ whenever $p > ½$
- So agent 2's minmax strategy is $q = ½$

- Similarly,
  agent 1's minmax strategy is $p = ½$



Finding Strategies for Zero-Sum Games

- In zero-sum games, minmax/maxmin strategies are Nash equilibrium strategies
  - So just look for Nash equilibria

- **Example: Morra**
- Thousands of years old – goes back to ancient Rome
  - Still played in southern Italy
- Players hold up various numbers of fingers, and count up the total
  - Can be used for decision making or for betting
- Several different versions
  - I'll discuss two simple ones

Two-Finger Morra, Win/Loss Scoring

- Two players: *Odd* and *Even*
- Each player holds up 1 or 2 fingers
  - If the total is odd, Odd wins and Even loses
  - If the total is even, Even wins and Odd loses

	1	2
1	–1, 1	1, –1
2	1, –1	–1, 1

- Equivalent to Matching Pennies
  - No pure-strategy Nash equilibria
  - Mixed-strategy Nash equilibrium is $(p = ½, q = ½)$
    - where
      - p = Pr[Odd plays *one*]
      - q = Pr[Even plays *one*]

Two-Finger Morra, Numeric Payoffs

- From Russell & Norvig, *Artificial Intelligence: A Modern Approach*

- Each player holds up 1 or 2 fingers
  - If the total is odd
    - Odd gets that many points
    - Even loses that many points
  - If the total is even
    - Even gets that many points
    - Odd loses that many points
- As before, no pure-strategy Nash equilibria
- Mixed-strategy Nash equilibrium:
  - As before, let
    - p = Pr[Odd plays *one*]
    - q = Pr[Even plays *one*]

…

	1	2
1	–2, 2	3, –3
2	3, –3	–4, 4

Two-Finger Morra, Numeric Payoffs

- Odd plays 1 => expected utility $-2q + 3(1-q) = 3 - 5q$
- Odd plays 2 => expected utility $3q - 4(1-q) = 7q - 4$
- At equilibrium, the two must be equal
  - $3 - 5q = 7q - 4$, so $q = 7/12$
- Odd's expected utility is $3 - 5(7/12) = 1/12$

	1	2
1	–2, 2	3, –3
2	3, –3	–4, 4

- Even plays 1 => expected utility is $2p - 3(1-p) = 5p - 3$
- Even plays 2 => expected utility is $-3p + 4(1-p) = 4 - 7p$
- At equilibrium, the two must be equal
  - $5p - 3 = 4 - 7p$, so $p = 7/12$
- Even's expected utility is $5(7/12) - 3 = -1/12$

Two-Finger Morra, Numeric Payoffs

- In Russell & Norvig's version, Odd has a built-in advantage

- Morra with numeric payoffs is usually done in a different way
  - Each player displays some number of figures, and simultaneously guesses the total number of fingers that both players will display
  - If one player guesses correctly and the other doesn't, the one who guessed correctly wins the amount that he/she guessed
- Does either player have an advantage?

What we've covered in this chapter

- Maxmin strategies
- Minmax strategies
- Minimax Theorem
  - finite two-player zero-sum games

- General idea is to solve simultaneous equations
  - But there are two cases where you have to do something different
    - For maxmin, I've added examples of both
  - Same cases can occur for minmax

Minimax Regret – Motivating Example

- Consider the payoff matrix shown here
  - ε is a small positive constant; Agent 1 knows its value
  - Agent 1 doesn't know the values of *a, b, c, d*
- Suppose Agent 1 doesn't think Agent 2 is malicious
- Agent 1 might reason as follows:
  - If Agent 2 plays *R*, then 1's strategy changes 1's payoff by only a small amount
    - Payoff is 1 or 1–ε, difference is only *ε*
  - If Agent 2 plays *L*, then 1's strategy changes 1's payoff by a much bigger amount
    - Either 100 or 2, difference is 98
  - If Agent 1 chooses *T*, this will minimize 1's maximum **regret**
    - Maximum difference between the payoff of the chosen action and the payoff of the other action

	L	*R*
T	100, *a*	1–ε, *b*
B	2, *c*	1, *d*

Minimax Regret

- Suppose i plays action a_i , other agent(s) play $\mathbf{a}_{-i}$
  - **regret** for a_i: what i lost by not playing best response to $\mathbf{a}_{-i}$

$$\text{regret}_i(a_i, \mathbf{a}_{-i}) = \left[\max_{a_i' \in A_i} u_i\left(a_i', \mathbf{a}_{-i}\right)\right] - u_i\left(a_i, \mathbf{a}_{-i}\right)$$

	L	R
T	100, a	$1-\varepsilon$, b
B	2, c	1, d

- i doesn't know what $\mathbf{a}_{-i}$ will be
  - **maximum** regret for a_i

$$\max_{\mathbf{a}_{-i} \in \mathbf{A}_{-i}} \left(\left[\max_{a_i' \in A_i} u_i\left(a_i', \mathbf{a}_{-i}\right)\right] - u_i\left(a_i, \mathbf{a}_{-i}\right)\right)$$

	L	R
T	$\text{regret}_1(T,L) = -98$	$\text{regret}_1(T,R) = \varepsilon$
B	$\text{regret}_1(B,L) = 98$	$\text{regret}_1(B,R) = -\varepsilon$

1's max regret for $T = \varepsilon$
1's max regret for $B = 98$

- **Minimax regret action**: smallest maximum regret

$$\underset{a_i \in A_i}{\text{argmin}} \max_{\mathbf{a}_{-i} \in \mathbf{A}_{-i}} \left(\left[\max_{a_i' \in A_i} u_i\left(a_i', \mathbf{a}_{-i}\right)\right] - u_i\left(a_i, \mathbf{a}_{-i}\right)\right)$$

Minimax Regret

	L	R
U	3, 1	0, 1
M	1, 1	1, 1
D	0, 1	4, 1

- Suppose i plays action a_i , other agent(s) play $\mathbf{a}_{-i}$
  - **regret** for a_i: what i lost by not playing best response to $\mathbf{a}_{-i}$

$$\text{regret}_i(a_i, \mathbf{a}_{-i}) = \left[\max_{a_i' \in A_i} u_i(a_i', \mathbf{a}_{-i})\right] - u_i(a_i, \mathbf{a}_{-i})$$

	L	R
U	$\text{regret}_1(U,L) = -2$	$\text{regret}_1(U,R) = 4$
M	$\text{regret}_1(M,L) = 2$	$\text{regret}_1(M,R) = 3$
D	$\text{regret}_1(D,L) = 3$	$\text{regret}_1(D,R) = -3$

- i doesn't know what $\mathbf{a}_{-i}$ will be
  - **maximum** regret for a_i

$$\max_{\mathbf{a}_{-i} \in \mathbf{A}_{-i}} \left(\left[\max_{a_i' \in A_i} u_i(a_i', \mathbf{a}_{-i})\right] - u_i(a_i, \mathbf{a}_{-i}) \right)$$

1's max regret for $U = 4$
1's max regret for $M = 3$
1's max regret for $D = 3$

- **Minimax regret action**: smallest maximum regret

$$\operatorname*{argmin}_{a_i \in A_i} \max_{\mathbf{a}_{-i} \in \mathbf{A}_{-i}} \left(\left[\max_{a_i' \in A_i} u_i(a_i', \mathbf{a}_{-i})\right] - u_i(a_i, \mathbf{a}_{-i}) \right)$$

Dominant Strategies

- Let s_i and s_i' be two strategies for agent i
  - s_i dominates s_i' if agent i does better with s_i than with s_i' regardless of what the other agents do
    - e.g., D in the Prisoner's Dilemma
- Three gradations of dominance:
  - s_i **strictly dominates** s_i' if

$$u_i\,(s_i, \mathbf{s}_{-i}) > u_i\,(s_i', \mathbf{s}_{-i}) \text{ for every } \mathbf{s}_{-i}$$

  - s_i **weakly dominates** s_i' if

$$u_i\,(s_i, \mathbf{s}_{-i}) \geq u_i\,(s_i', \mathbf{s}_{-i}) \text{ for every } \mathbf{s}_{-i}$$

$$\text{and} \quad u_i\,(s_i, \mathbf{s}_{-i}) > u_i\,(s_i', \mathbf{s}_{-i}) \text{ for at least one } \mathbf{s}_{-i}$$

  - s_i **very weakly dominates** s_i' if

$$u_i\,(s_i, \mathbf{s}_{-i}\,) \geq u_i\,(s_i', \mathbf{s}_{-i}) \text{ for every } \mathbf{s}_{-i}$$

	C	D
C	3, 3	0, 5
D	**5**, 0	**1**, 1

- **Poll 3.3**: what is the relationship between dominance and max regret?

Dominant Strategy Equilibria

- A strategy is **strictly** (resp., **weakly**, **very weakly**) **dominant** for an agent if it strictly (weakly, very weakly) dominates any other strategy for that agent
- A strategy profile $(s_1, \ldots, s_n)$ in which every s_i is dominant for agent i (strictly, weakly, or very weakly) is a Nash equilibrium
  - Why?
  - Called an equilibrium in **strictly (weakly, very weakly) dominant strategies**
- E.g., Prisoner's Dilemma
  - (D,D) is an equilibrium in strictly dominant strateiges

	C	D
C	3, 3	0, 5
D	5, 0	1, 1

Another Example

- British TV game: *Golden Balls*
- Last round: "Split or Steal?"
  - Prize: large amount of money
  - Each contestant chooses "Split" or "Steal"
  - If both choose "Split", money is split equally
  - If one chooses "Split", one chooses "Steal"
    - Stealer takes all, Splitter gets none
  - If both choose "Steal", both get nothing
- Compare with Prisoner's Dilemma

	Split	*Steal*
Split	½, ½	0, 1
Steal	1, 0	0, 0

	C	*D*
C	3, 3	0, 5
D	5, 0	1, 1

Another Example

- British TV game: *Golden Balls*
- Last round: “Split or Steal?”
  - Prize: large amount of money
  - Each contestant chooses “Split” or “Steal”
  - If both choose “Split”, money is split equally
  - If one chooses “Split”, one chooses “Steal”
    - Stealer takes all, Splitter gets none
  - If both choose “Steal”, both get nothing
- Compare with Prisoner’s Dilemma
- Many episodes on Youtube
  - Contestants talk to each other before choosing
  - Usually they each promise to split
    - Sometimes they split, sometimes they don’t
- http://www.youtube.com/watch?v=S0qjK3TWZE8

	Split	*Steal*
Split	½, ½	0, 1
Steal	1, 0	0, 0

	C	*D*
C	3, 3	0, 5
D	5, 0	1, 1

Strictly Dominated Strategies

- A strategy s_i is **strictly** (**weakly**, **very weakly**) **dominated** for an agent i if some other strategy s_i' strictly (weakly, very weakly) dominates s_i

- A strictly dominated strategy can't be a best response to any move, so we can eliminate it (remove it from the payoff matrix)
  - This gives a **reduced** game
  - Other strategies may now be strictly dominated, even if they weren't dominated before

	L	R
~~U~~	~~3, 3~~	~~0, 5~~
D	5, 1	1, 0

	L	~~R~~
D	5, 1	~~1, 0~~

- **IESDS (Iterated Elimination of Strictly Dominated Strategies)**:
  - Repeat until no more eliminations are possible
  - Result: the **maximal reduction** of the game

	L
D	5, 1

IESDS

- If you eliminate a strictly dominated strategy, the reduced game has the same Nash equilibria as the original one
- Thus

  {Nash equilibria of the original game}

  = {Nash equilibria of the maximally reduced game}

	L	R
~~U~~	~~3, 3~~	~~0, 5~~
D	5, 1	1, 0

- Use this technique to simplify finding Nash equilibria
  - Look for Nash equilibria on the maximally reduced game

	L	~~R~~
D	5, 1	~~1, 0~~

- In the example, we ended up with a single cell
  - The single cell *must* be a unique Nash equilibrium in all three of the games

	L
D	5, 1

IESDS

- Even if s_i isn't strictly dominated by a pure strategy, it may be strictly dominated by a mixed strategy
- **Example**: the three games shown at right
  - 1st game:
    - R is strictly dominated by L (and by C)
  - 2nd game:
    - Neither U nor D dominates M
    - $\{(½, U), (½, D)\}$ strictly dominates M
      - This wasn't true before we removed R
    - Eliminate it, get 3rd game
  - Maximally reduced

	L	C	R
U	3, 1	0, 1	~~0, 0~~
M	1, 1	1, 1	~~5, 0~~
D	0, 1	4, 1	~~0, 0~~

	L	C
U	3, 1	0, 1
~~M~~	~~1, 1~~	~~1, 1~~
D	0, 1	4, 1

	L	C
U	3, 1	0, 1
D	0, 1	4, 1

IEWDS

- If you eliminate a *weakly* dominated strategy, every Nash equilibrium of the reduced game is a Nash equilibrium of the original one
- Thus

  {Nash equilibria of the original game}

  $\subseteq$ {Nash equilibria of the maximally reduced game}

- Depending on the order in which you remove strategies, you might get different Nash equilibria

	L	*R*
U	1, 1	0, 0
M	1, 1	2, 1
D	0, 0	2, 1

	L	*R*
U	~~1, 1~~	~~0, 0~~
M	~~1, 1~~	2, 1
D	~~0, 0~~	2, 1

	L	*R*
U	1, 1	~~0, 0~~
M	1, 1	~~2, 1~~
D	~~0, 0~~	~~2, 1~~

IEWDS

- If you eliminate a *weakly* dominated strategy, every Nash equilibrium of the reduced game is a Nash equilibrium of the original one

  {Nash equilibria of the original game}

  $\subseteq$ {Nash equilibria of the maximally reduced game}

- Depending on the order in which you remove strategies, you might get different Nash equilibria
  - ➢ All of them are Nash equilibria of the original game

	L	*R*
U	1, 1	0, 0
M	1, 1	2, 1
D	0, 0	2, 1

	L	*R*
U	~~1, 1~~	~~0, 0~~
M	1, 1	2, 1
D	0, 0	2, 1

	L	*R*
U	1, 1	0, 0
M	1, 1	2, 1
D	~~0, 0~~	~~2, 1~~

	L	*R*
M	1, 1	2, 1
D	0, 0	2, 1

	L	*R*
U	1, 1	0, 0
M	1, 1	2, 1

The *p*-Beauty Contest

(not in the book)

- Recall that I asked you to play the following game:

> - Choose a number in the range $0 \le x \le 100$
> - Write your choice on a piece of paper
> - Fold the paper so nobody else can see your number
> - Pass the paper to the front of the room
>
> - The winner(s) will be those whose number is closest to 2/3 of the average of all the numbers

- The ***p*-beauty contest** (usually $p = 2/3$)
  - Famous among economists and game theorists

IESDS in the *p*-Beauty Contest

- Previous times I taught this course, I used this argument:
  - All numbers ≤ 100 => 2/3 of the average < 67
    - Eliminate numbers ≥ 67
    - Eliminate mixed strategies whose support includes a number ≥ 67
  - The remaining strategies are confined to $[0,67)$
  - All numbers < 67 => 2/3 of the average < 45
    - Eliminate numbers ≥ 45
    - Eliminate mixed strategies whose support includes a number ≥ 45
  - The remaining strategies are confined to $[0,45)$
  - All numbers < 45 => 2/3 of the average < 30

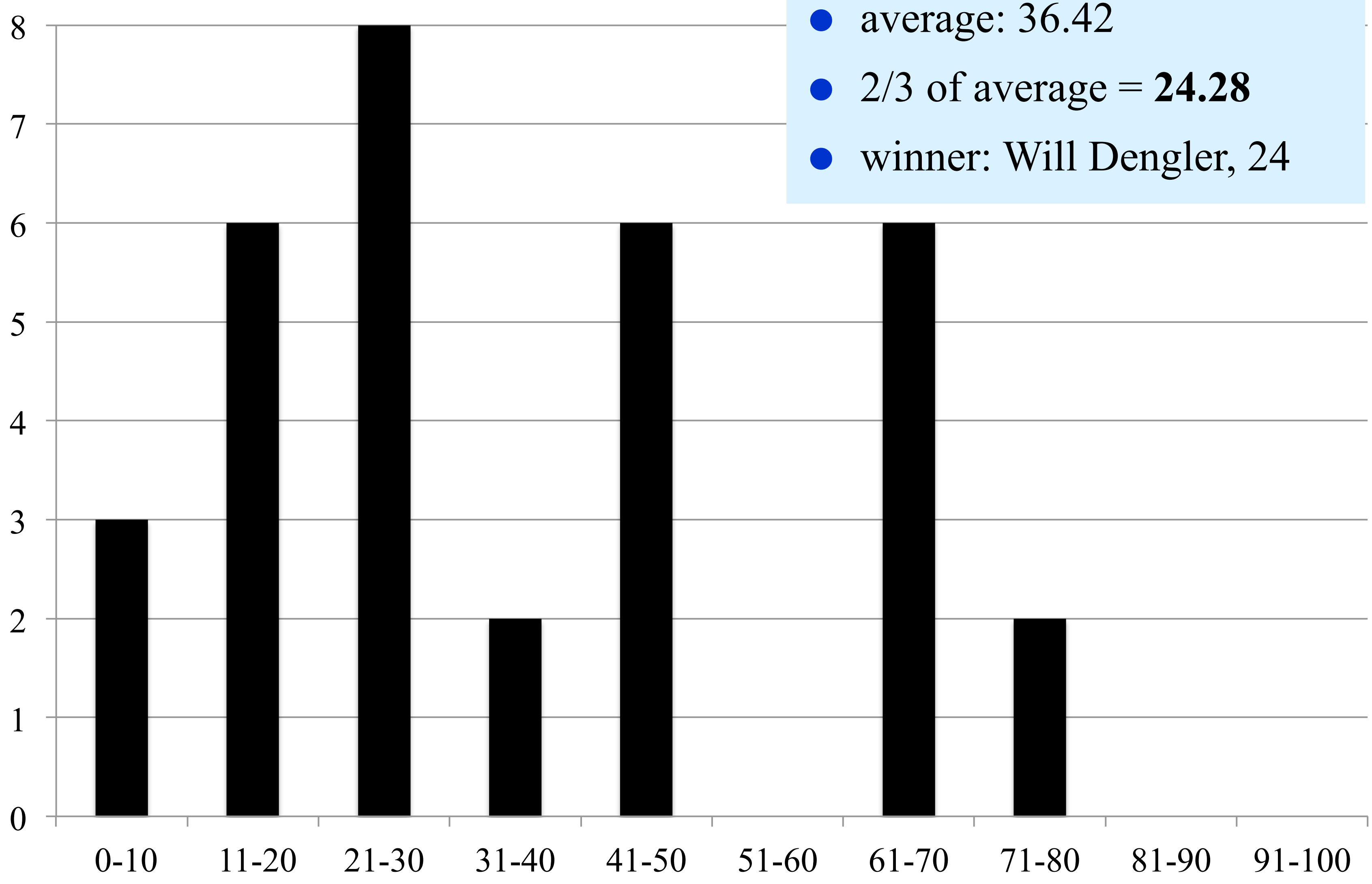
    . . .
  - The only strategy that survives is to choose 0
    - The only Nash equilibrium

IESDS in the *p*-Beauty Contest

- Problem
  - I removed some *weakly* dominated strategies

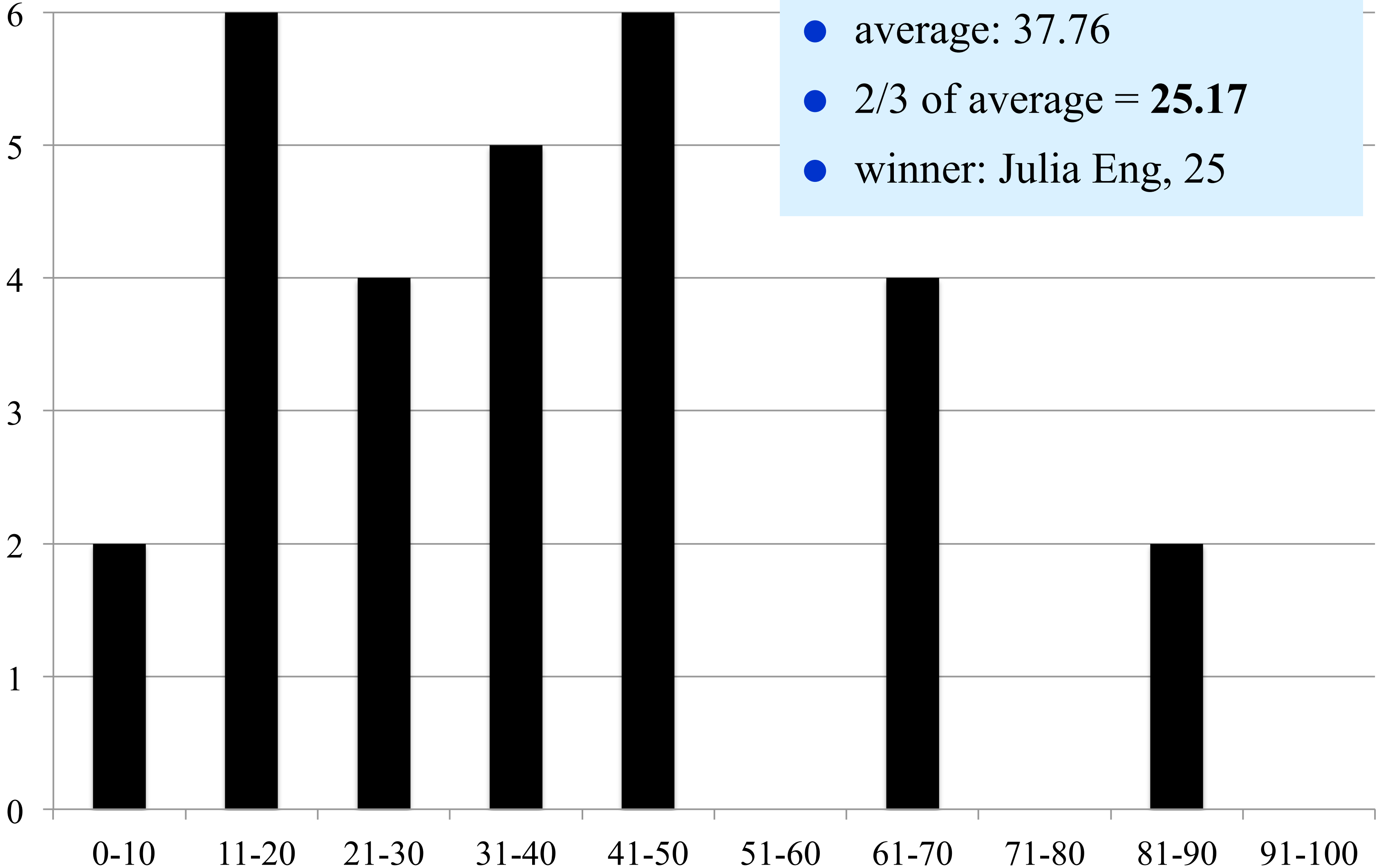
- There are ways to fix this
  - A little complicated, I won't do it here
  - If you figure it out, you can discuss it on Piazza

CMSC 474, September 2016



- average: 36.42
- 2/3 of average = **24.28**
- winner: Will Dengler, 24

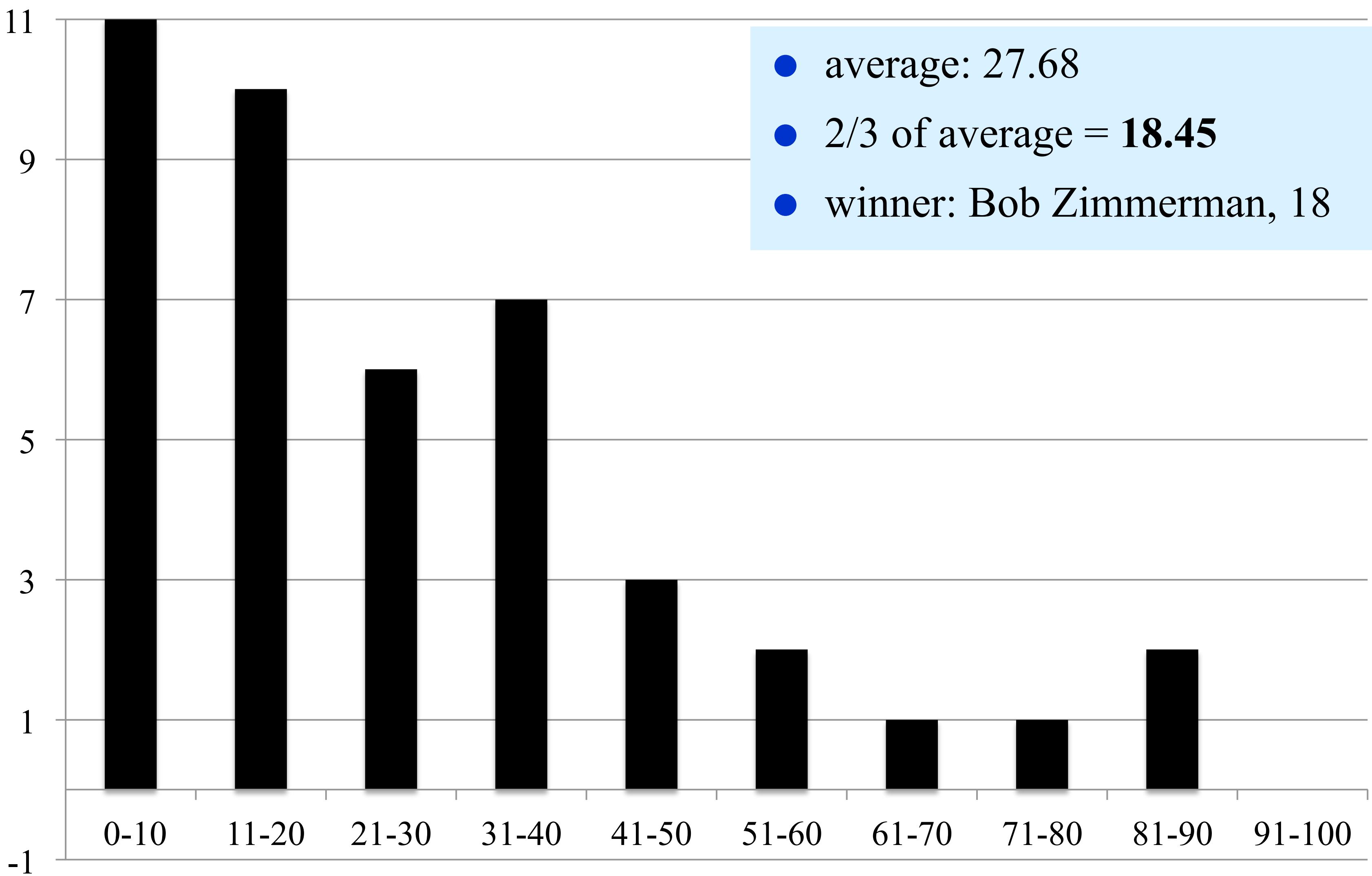
CMSC 474, September 2014



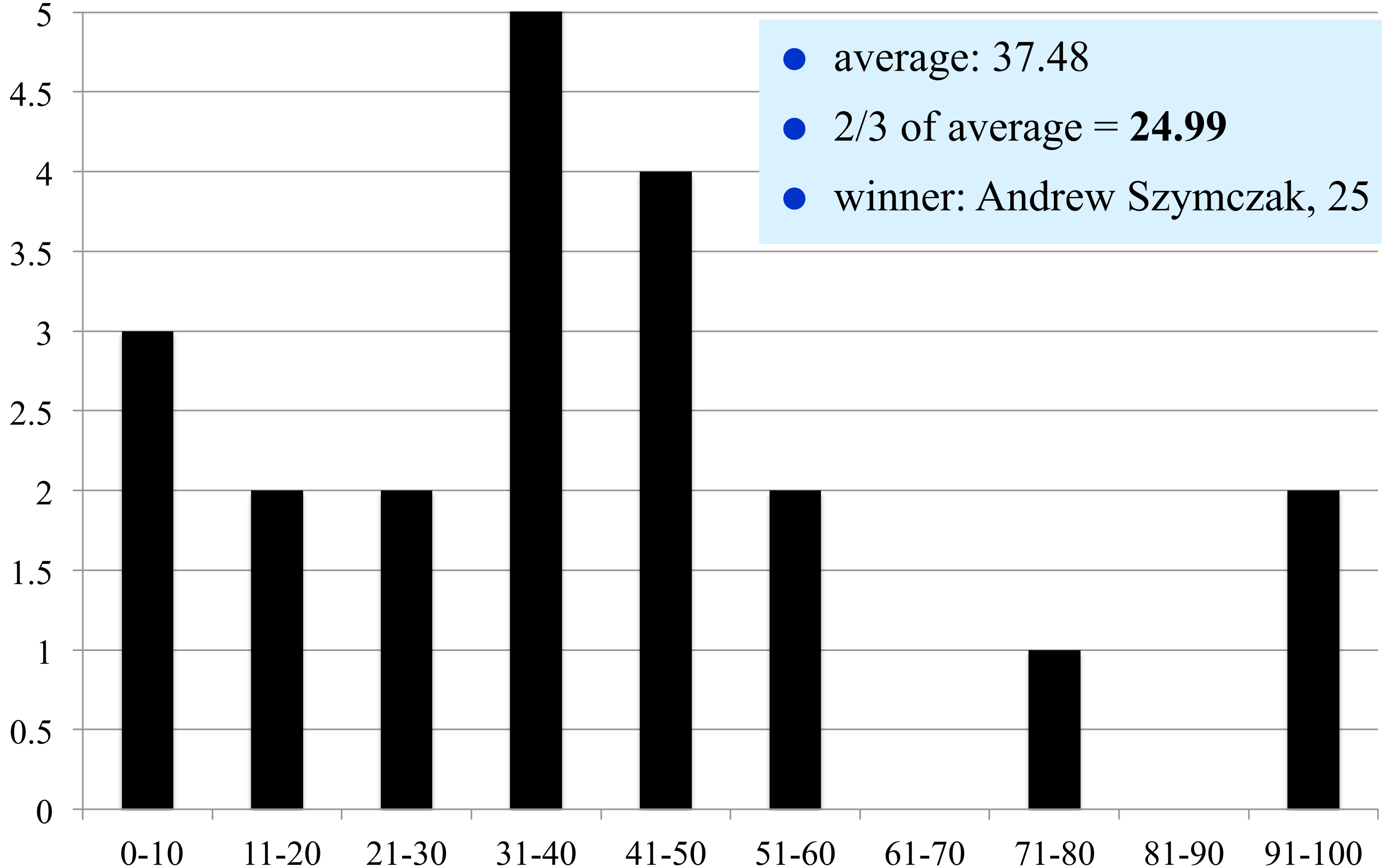
- average: 37.76
- 2/3 of average = **25.17**
- winner: Julia Eng, 25

CMSC 474, January 2014

- average: 27.68
- 2/3 of average = **18.45**
- winner: Bob Zimmerman, 18

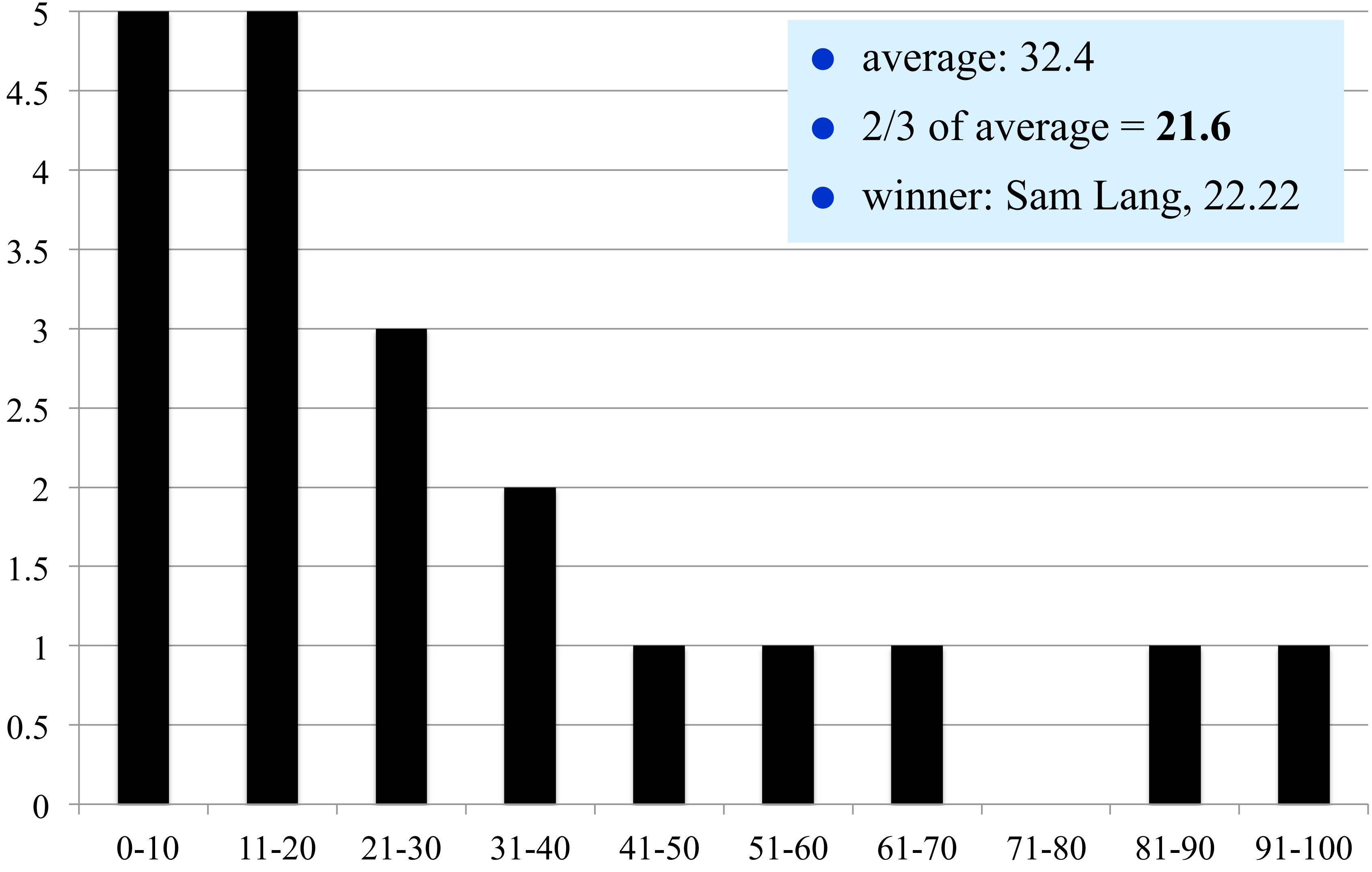


CMSC 498T, September 2011

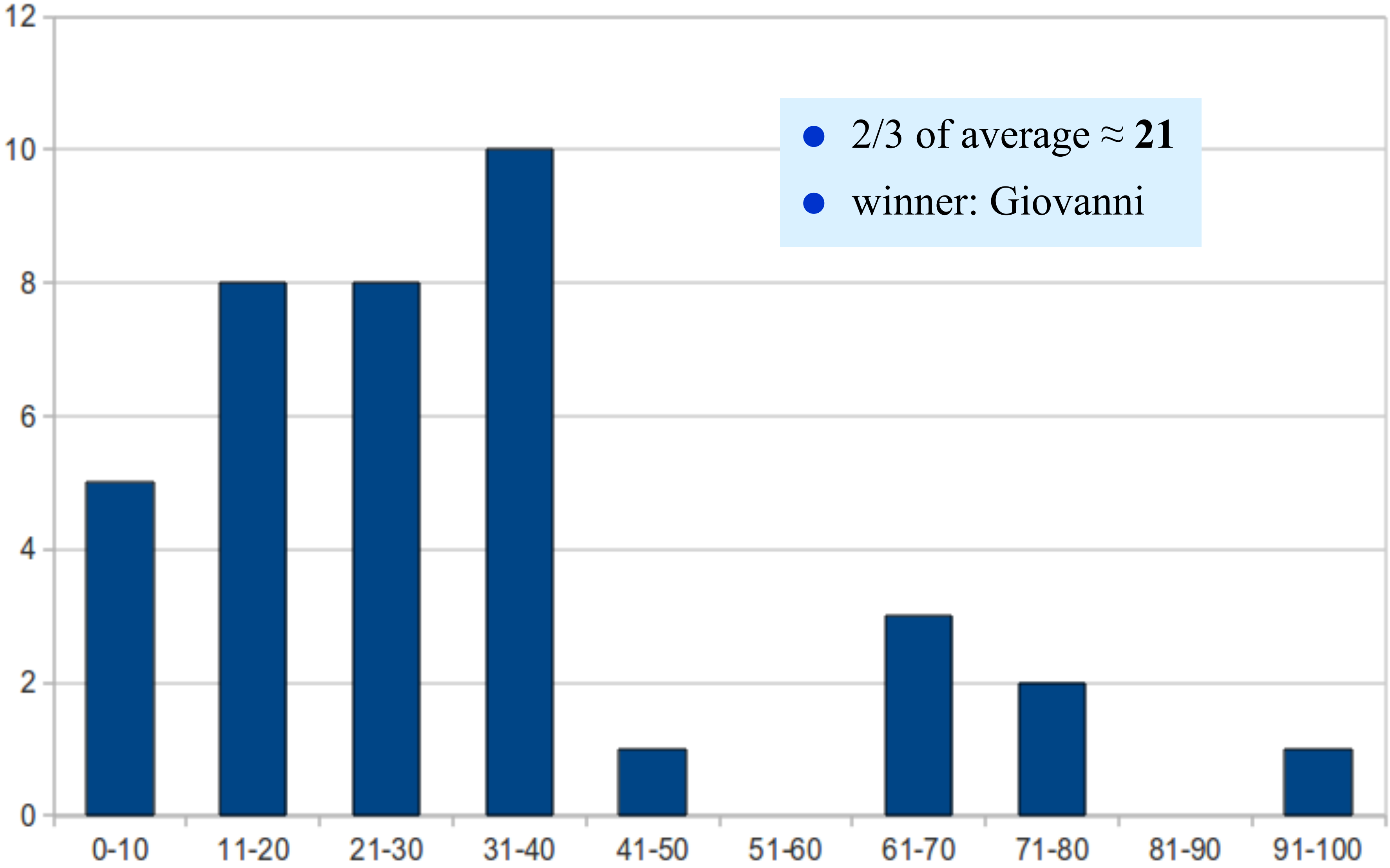


- average: 37.48
- 2/3 of average = **24.99**
- winner: Andrew Szymczak, 25

CMSC 498T, January 2011

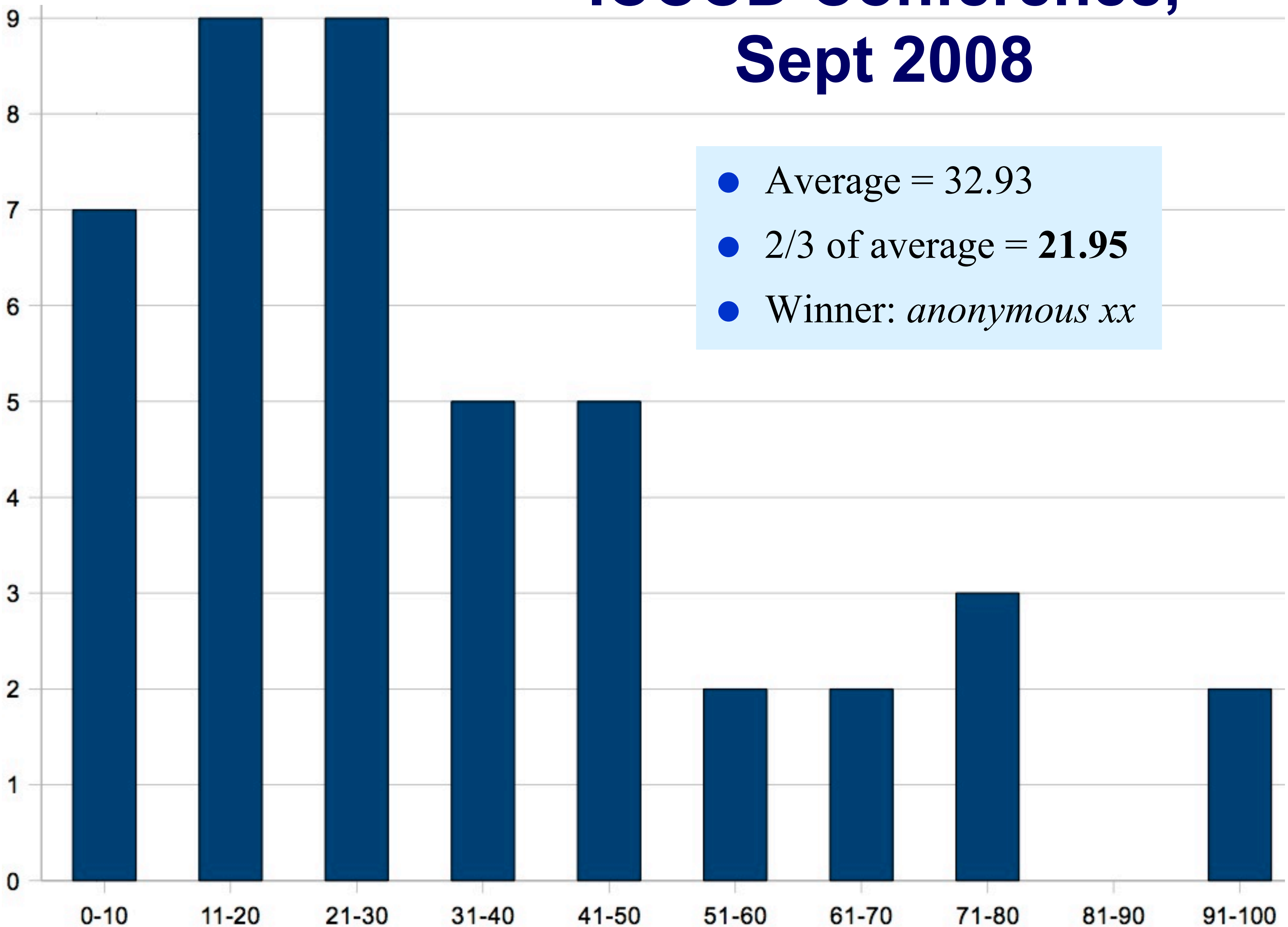


University of Brescia (Italy), March 2010



ICCCD Conference, Sept 2008

- Average = 32.93
- 2/3 of average = **21.95**
- Winner: *anonymous xx*



Empirical Result

- Average is often around 33
  - Approximately 50 * 2/3
  - 2/3 of average is around 22

- Why didn’t we choose a Nash equilibrium?

Reasons for not choosing a Nash equilibrium

(1) Wrong payoff matrix

- Might not encode agents' actual preferences
- It's common to take an external measure (money, points, etc.) and assume it's the only thing (or the main thing) an agent cares about
  - Not always correct
- An agent may consider other things more important
  - Being helpful
  - Curiosity, exploration
  - Frustration, spite
  - Creating mischief

Reasons for not choosing a Nash equilibrium

(2) Irrational preferences (see Lecture 1)

(3) Limitations in reasoning

- Limited by
  - the information the individuals have
  - their cognitive limitations
  - the amount of time they have to make a decision
- Might not know whether the other individuals are rational
- Might not know what the other individuals' preferences are
- Might not know all of the effects of various actions or action profiles
- Might not be able to calculate the Nash equilibrium correctly
- Might not even know the concept

Reasons for not choosing a Nash equilibrium

(4) Beliefs about other agents' likely actions

- A Nash equilibrium strategy is best for you if the other agents also use their Nash equilibrium strategies
- If they don't, and if you can guess what strategies they'll use
  - Compute your best response
  - Good guess ⇒ you may do much better than the Nash equilibrium
  - Bad guess ⇒ you may do much worse
- More about this later

Rationalizability

- A strategy is **rationalizable** if a *perfectly rational agent* could justifiably play it against *perfectly rational opponents*
  - The formal definition is complicated
- Informally:
  - A strategy for agent *i* is rationalizable if it's a best response to strategies that *i* could *reasonably* believe the other agents have
  - To be reasonable, *i*'s beliefs must take into account *i*'s knowledge of the rationality of the others. This incorporates
    - the other agents' knowledge of *i*'s rationality,
    - their knowledge of *i*'s knowledge of *their* rationality,
    - and so on *ad infinitum*
- A **rationalizable strategy profile** is a strategy profile that consists only of rationalizable strategies

Rationalizability

	Left	Right
Left	1, 1	0, 0
Right	0, 0	1, 1

- Every Nash equilibrium is composed of rationalizable strategies

Example: Which Side of the Road

- For Agent 1, the pure strategy $s_1 = \textit{Left}$ is rationalizable because
  - $s_1 = \textit{Left}$ is 1's best response if 2 uses $s_2 = \textit{Left}$,
  - and 1 can reasonably believe 2 would rationally use $s_2 = \textit{Left}$, because
    - $s_2 = \textit{Left}$ is 2's best response if 1 uses $s_1 = \textit{Left}$,
    - and 2 can reasonably believe 1 would rationally use $s_1 = \textit{Left}$, because
      - $s_1 = \textit{Left}$ is 1's best response if 2 uses $s_2 = \textit{Left}$,
      - and 1 can reasonably believe 2 would rationally use $s_2 = \textit{Left}$, because
        - … and so on, *ad infinitum* …

Rationalizability

	Heads	Tails
Heads	1, –1	–1, 1
Tails	–1, 1	1, –1

- Some rationalizable strategies are not part of any Nash equilibrium

Example: Matching Pennies

- For Agent 1, the pure strategy s_1 = *Heads* is rationalizable because
  - s_1 = *Heads* is 1's best response if 2 uses s_2 = *Heads*,
  - and 1 can reasonably believe 2 would rationally use s_2 = *Heads*, because
    - s_2 = *Heads* is 2's best response if 1 uses s_1 = *Tails*,
    - and 2 can reasonably believe 1 would rationally use s_1 = *Tails*, because
      - s_1 = *Tails* is 1's best response if 2 uses s_2 = *Tails*,
      - and 1 can reasonably believe 2 would rationally use s_2 = *Tails*, because
        - … and so on, *ad infinitum* …

Strategies that Aren't Rationalizable

Prisoner's Dilemma

- Strategy *C* isn't rationalizable for agent 1
- It isn't a best response to any of agent 2's strategies

	C	*D*
C	3, 3	0, 5
D	5, 0	1, 1

The 3x3 game we used earlier

- *M* is not a rationalizable strategy for agent 1
- It *is* a best response to one of agent 2's strategies, namely *R*
- But 1 cannot reasonably believe that 2 would rationally play *R*
  - *R* isn't 2's best response to any of agent 1's strategies
  - *L* and *C* will always give 2 a bigger payoff

	L	*C*	*R*
U	3, 1	0, 1	0, 0
M	1, 1	1, 1	5, 0
D	0, 1	4, 1	0, 0

Comments

- The formal definition of rationalizability is complicated because of the infinite regress
  - But we can say some intuitive things about rationalizable strategies
- Nash equilibrium strategies are always rationalizable
  - So the set of rationalizable strategies (and strategy profiles) is always nonempty
- A strategy that's strictly dominated will never be a best response to any strategy profile of the other agents
  - So if a strategy doesn't survive IESDS, it isn't rationalizable

Comments

- In two-player games:
  - Perform IESDS
  - The rationalizable strategies are the ones that remain afterwards

- In *n*-player games:
  - Iteratively remove all strategies that are never a best response to any strategy profile by the other agents
    - this removes strictly dominated strategies
    - it may also remove some others
  - The rationalizable strategies are the ones that remain afterwards

Common Knowledge

- Rationalizability is closely related to the idea of ***common knowledge***
- Important for Nash equilibria
  - Can't expect agents to converge on a Nash equilibrium unless they all have rational preferences and their rationality is common knowledge
- The book mentions common knowledge in at least 5 different places, but doesn't define what it means

- Definition is analogous to the definition of rationalizability
- Among a group of individuals, a property p is *common knowledge* if
  - They all know p
  - They all know that they all know p
  - They all know that they all know that they all know p
  - …

Example of Common Knowledge

- http://en.wikipedia.org/wiki/Common_knowledge_(logic)#Example
- On an island, there are k people who have blue eyes, and the rest of the people have green eyes. There is at least one blue-eyed person on the island ($k \geq 1$).
- If a person ever knows they have blue eyes, they must leave the island at dawn the next day. Each person can see every other person's eye color, but there are no mirrors, and there is no discussion of eye color.
  - So nobody knows their own eye color.
- At some point, an outsider comes to the island, calls all the people together, and makes the following public announcement: "at least one of you has blue eyes". This makes the fact common knowledge:
  - they all know it, they all know that they all know it, and so on.
- Everyone on the island (including the outsider) is truthful, and can do arbitrarily complex logical reasoning. This also is common knowledge.
  - What will happen?

Example of Common Knowledge

- On the kth dawn after the announcement, all the blue-eyed people will leave the island.
- Case $k = 1$. Let's call the blue-eyed person A.
  - Before the announcement, A didn't know anyone had blue eyes.
  - After the announcement, A knows at least one person has blue eyes.
  - A can see that it isn't anyone else, so it must be A.
  - So A leaves at the 1^{st} dawn.

Example of Common Knowledge

- Case $k = 2$. Let's call the blue-eyed people A and B.
  - Before the announcement, B knows the island has at least one blue-eyed person, but doesn't know **whether they all know**.
    - If B didn't have blue eyes then A wouldn't see anyone with blue eyes, and wouldn't know whether anyone on the island is blue-eyed.
  - After the announcement, B knows that they all know, **including *A***.
  - B knows that if B doesn't have blue eyes, then A will know A's eyes must be blue, and will leave at the 1st dawn.
  - When that doesn't happen, it tells B that A can't be the only blue-eyed person.
  - Seeing no blue-eyed people other than A, B infers that the other blue-eyed person is B.
  - Thus B leaves at the 2nd dawn.
  - Using similar reasoning, A also leaves at the 2nd dawn.

Example of Common Knowledge

- Case *k*=3. Let's call the blue-eyed people *A*, *B*, and *C*.
  - Before the announcement, *C* can infer that they all know there are blue-eyed people.
    - Even if *C* didn't have blue eyes, *A* and *B* would each see one other blue-eyed person.
  - But *C* doesn't know **whether they all know that they all know**.
    - For example, if *C* didn't have blue eyes, then (see case *k*=2 on the previous page) it would be possible for *B* to think that *A* doesn't know whether anyone has blue eyes.
  - After the announcement, *C* knows that if *A* and *B* are the **only** blue-eyed people then they'll leave at the 2nd dawn.
  - They don't do so, which tells *C* there must be a 3rd blue-eyed person. *C* sees only 2 blue-eyed people, and infers that *C* is the 3rd one.
  - Thus *C* leaves at the 3rd dawn.
  - Using similar reasoning, *A* and *B* also leave at the 3rd dawn.

Example of Common Knowledge

- For $k > 1$, the outsider is only telling the people what they already know: that there are blue-eyed people among them.
- But before the announcement, that isn't common knowledge.

  Either they don't all know that they all know,

  - or they don't all know that they all know that they all know,
  - or they don't all know that they all know that they all know that they all know, …

  - How many levels deep depends on how big k is
- Once it becomes common knowledge, each blue-eyed person has enough information to infer that if there were only k–1 blue-eyed people, they all would leave by the k–1$^{\text{th}}$ dawn.
- When this doesn't happen, each blue-eyed person knows there must be k blue-eyed people—and seeing only k–1 of them, realizes he/she must be the k^{th} one.

Common Knowledge and Nash Equilibria

- When can we expect a group of individuals to choose a Nash equilibrium?
  - The payoff matrix has a single Nash equilibrium
  - The payoff matrix accurately models the individuals' preferences
  - Rationality is common knowledge
    - The individuals all are rational
    - They all know that they all are rational
    - They all know that they all know that they all are rational
    - …

- What if there *isn't* common knowledge of rationality?

More about Common Knowledge

- Role of common knowledge in language and communication
  - http://www.youtube.com/watch?v=3-son3EJTrU

Summary

- Maxmin and minmax strategies, and the Minimax Theorem
  - Matching Pennies, Two-Finger Morra
- Minimax regret
- Dominant strategies
  - Prisoner’s Dilemma, Which Side of the Road, Matching Pennies
  - Comparison to the Chocolate Dilemma survey
  - Iterated elimination of strictly dominated strategies (IESDS)
  - Iterated elimination of weakly dominated strategies (IEWDS)
  - *p*-beauty contest
- Rationalizability
  - relation to IESDS
- Common knowledge
  - “island example”
  - role in language and communication